Spatial Description & Transformation

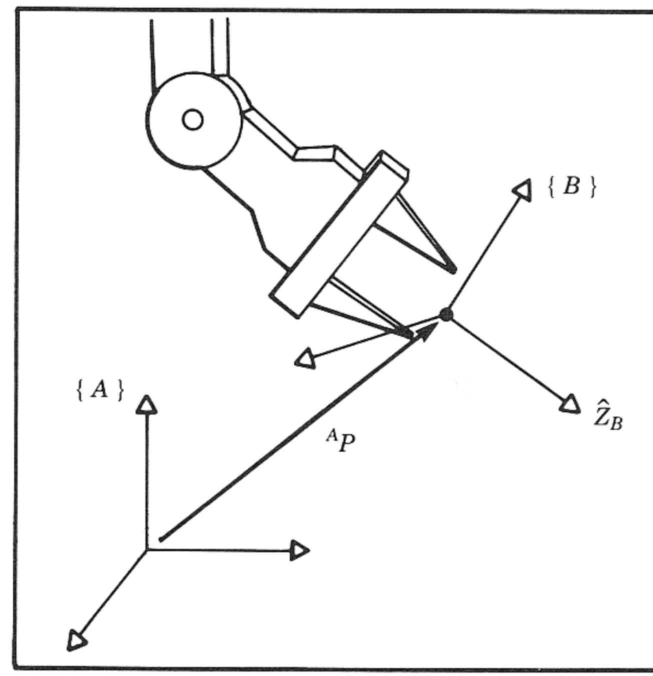
Central Topic

Problem

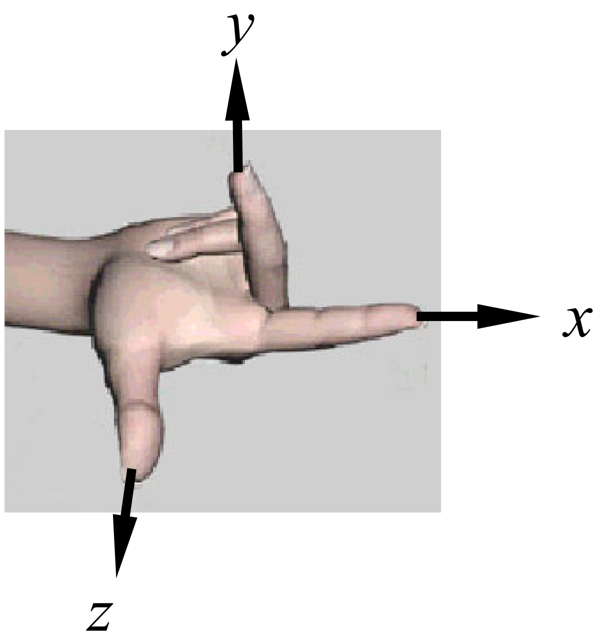
Robotic manipulation, by definition, implies that parts and tools will be moving around in space by the manipulator mechanism. This naturally leads to the need of representing positions and orientations of the parts, tools, and the mechanism it self.

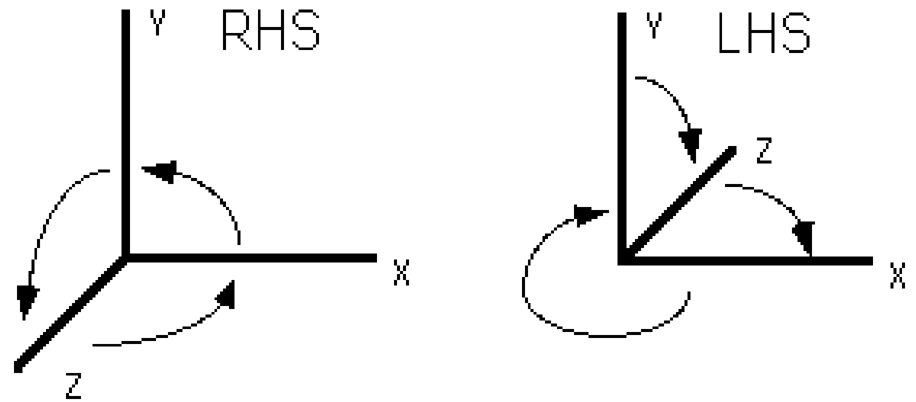
Solution

Mathematical tools for representing position and orientation of objects / frames in a 3D space.

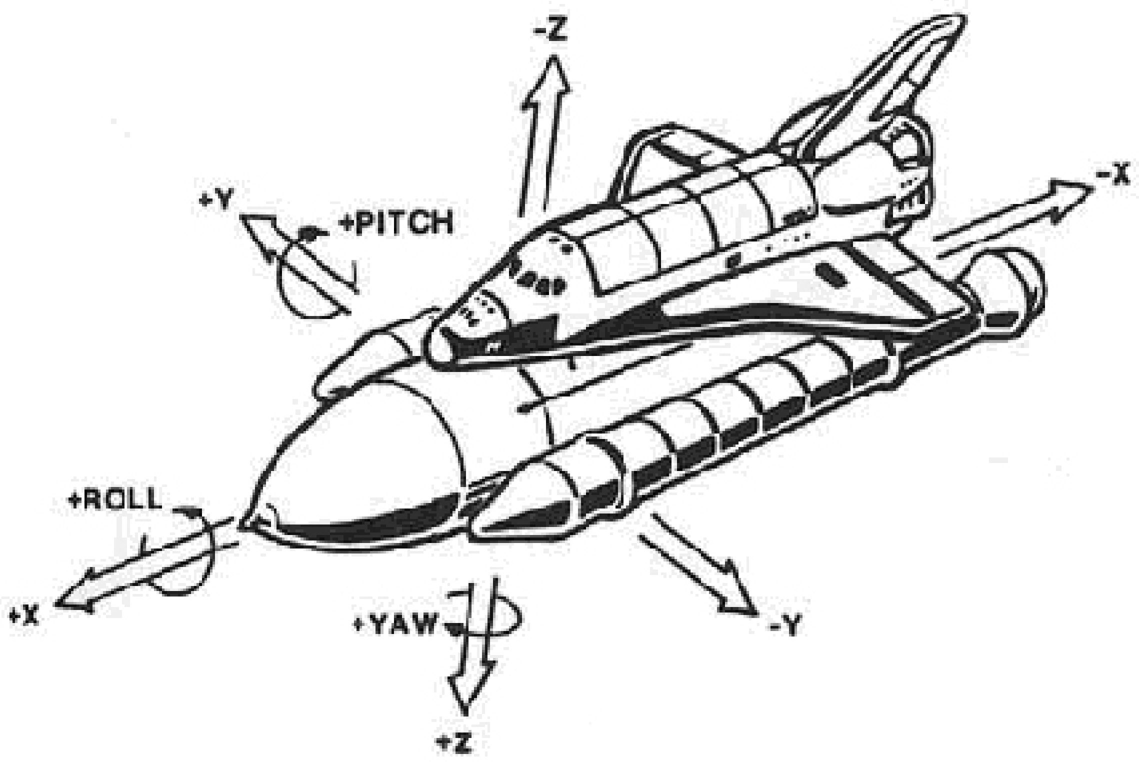


Coordinate System 1/2





Coordinate System 1/2



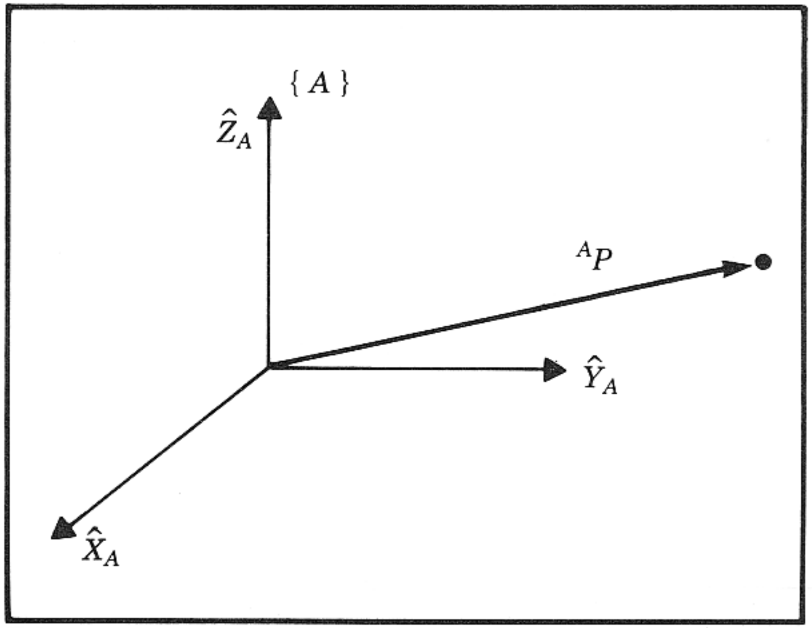
Description of a Position

The location of any point in can be described as a 3x1 ***position vector*** in a reference coordinate system

Coordinate System

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$

Position vector



Description of an Orientation

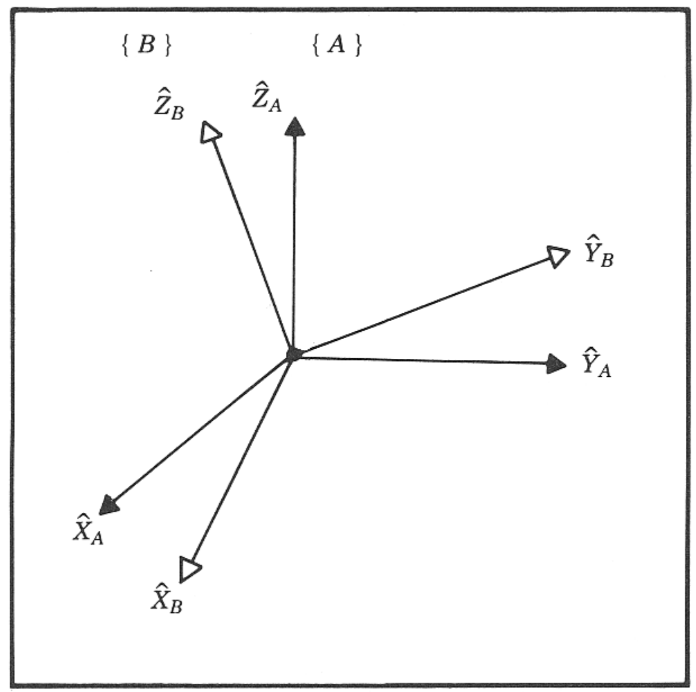
The orientation of a body is described by attaching a coordinate system to the body {B} and then defining the relationship between the body frame and the reference frame {A} using the rotation matrix.

The rotation matrix describing frame {B} relative to frame {A}

Reference Frame

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_{B}, {}^{A}\hat{Y}_{B}, {}^{A}\hat{Z}_{B}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Body Frame



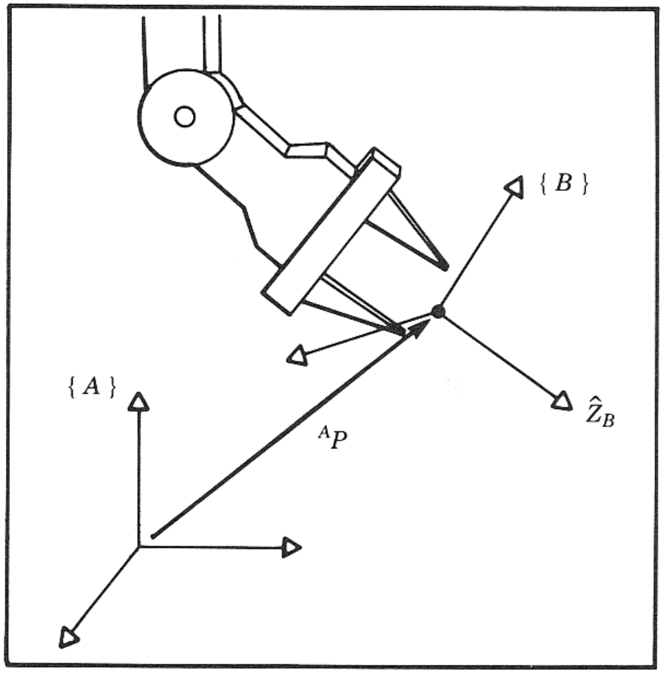
Description of an Frame

The information needed to completely specify where is the manipulator hand is a position and an orientation.

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$

The rotation matrix describing frame {B} relative to frame {A}

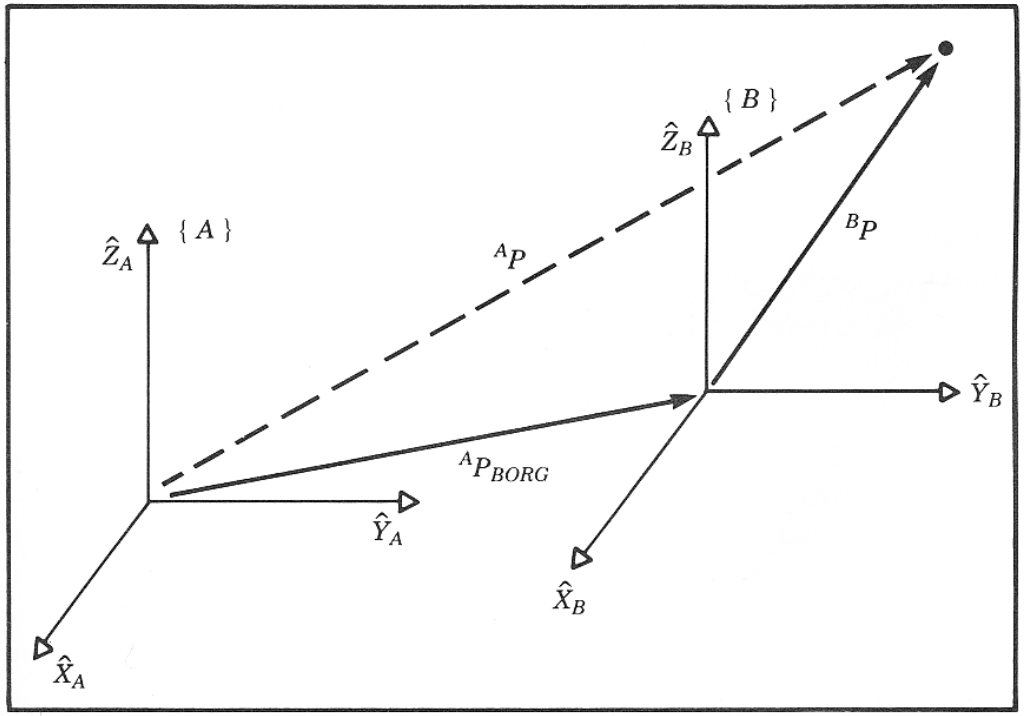
The origin of frame {B} relative to frame {A}



Mapping - Translated Frames

Assuming that frame {B} is only ***translated*** (not rotated) with respect frame {A}.
The position of the point can be expressed in frame {A} as follows.

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

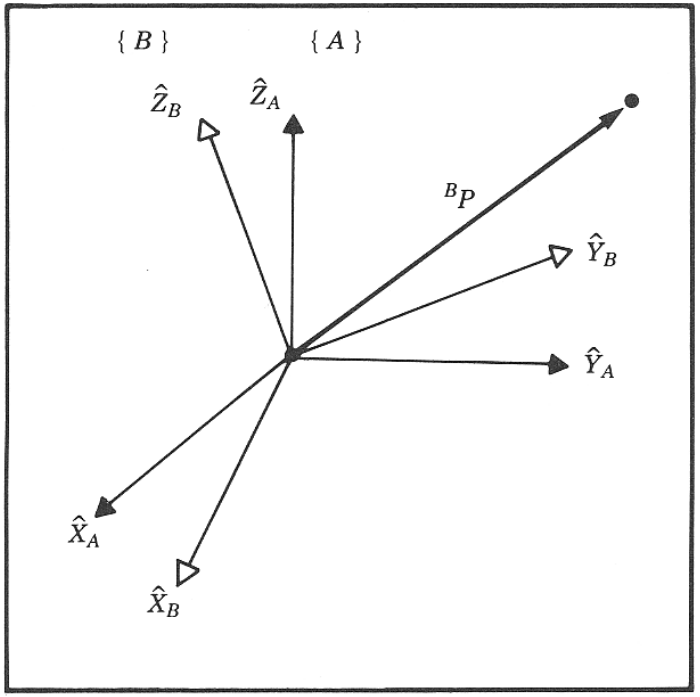


Mapping - Rotated Frames

Assuming that frame {B} is only ***rotated*** (not translated) with respect frame {A} (the origins of the two frames are located at the same point) the position of the point in frame {B} can be expressed in frame {A} using the rotation matrix as follows:

$$^{A}P = {}^{A}_{B}R \; {}^{B}P$$

$$^{B}P = {}^{B}_{A}R \; {}^{A}P$$



Mapping - Rotated Frames - Inversion

Given: The rotation matrix from frame {A} to frame {B} - ${}^{A}_{B}R$

Calculate: The rotation matrix from frame {B} to frame {A} ${}^{B}_{A}R$

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}P = {}^{A}_{B}R^{-1} \, {}^{A}_{B}R \; {}^{B}P $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}_{B}R = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ P = IP $$

$$ {}^{A}_{B}R^{-1} \, {}^{A}P = {}^{A}_{B}R^{-1} \, {}^{A}_{B}R \; {}^{B}P = I \, {}^{B}P = {}^{B}P $$

$$ {}^{B}P = {}^{A}_{B}R^{-1} \, {}^{A}P $$

$$ {}^{B}P = {}^{B}_{A}R \; {}^{A}P $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T} $$

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} $$

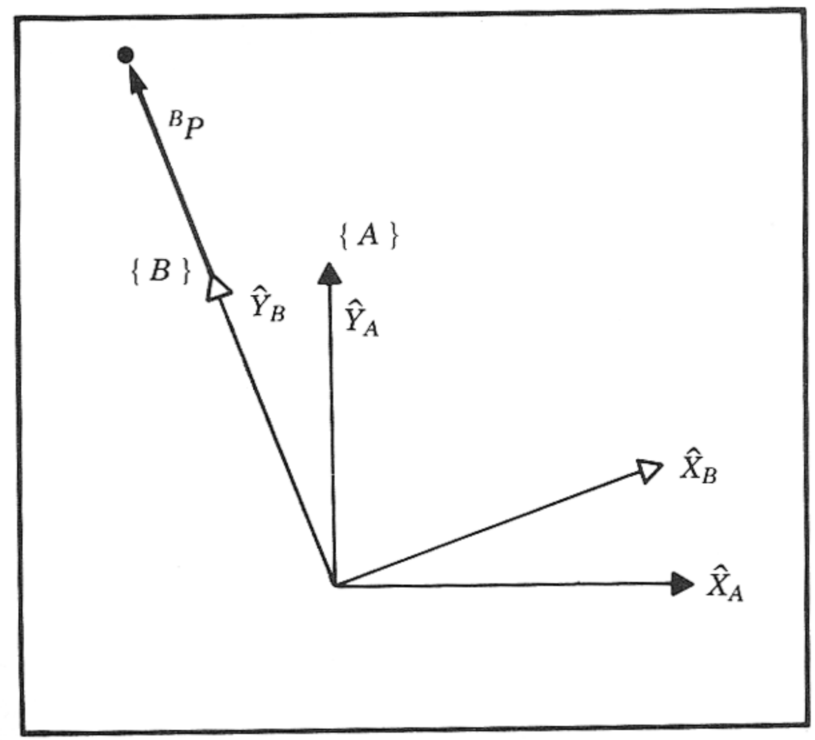
Orthogonal Coordinate system

Mapping - Rotated Frames - Example

Given :

$$^{B}P = \begin{bmatrix} 0 \\ ^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\theta = 30^{o}$$

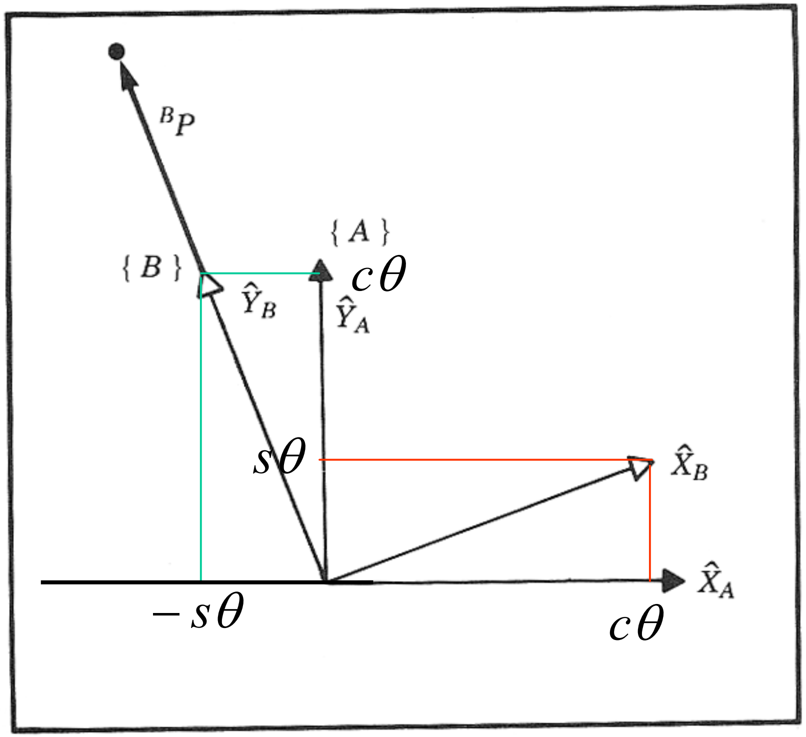


Compute: ^{A}P

Solution: $^{A}P = {}^{A}_{B}R \; {}^{B}P$

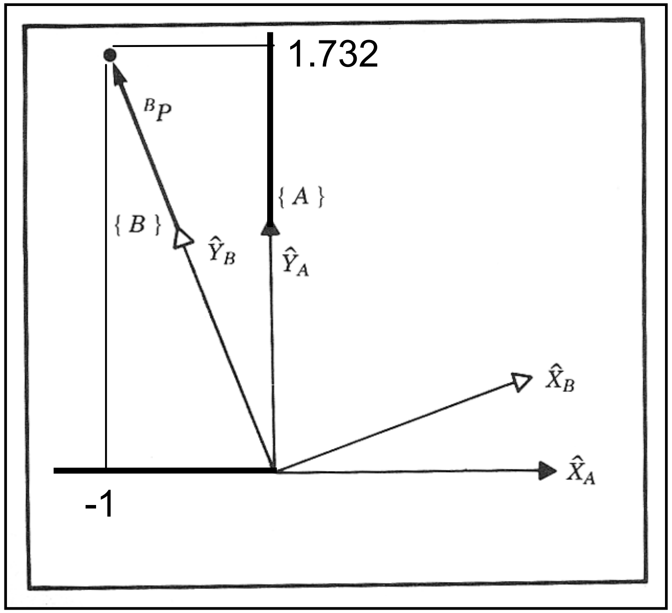
Mapping - Rotated Frames - Example

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_{B}, {}^{A}\hat{Y}_{B}, {}^{A}\hat{Z}_{B}] = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Mapping - Rotated Frames - Example

$$ {}^{A}P = {}^{A}_{B}R \; {}^{B}P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 2.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$



Mapping - Rotated Frames - General Notation

The rotation matrices with respect to the reference frame are defined as follows:

$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$R_Y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$$

$$R_Z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping - General Frames

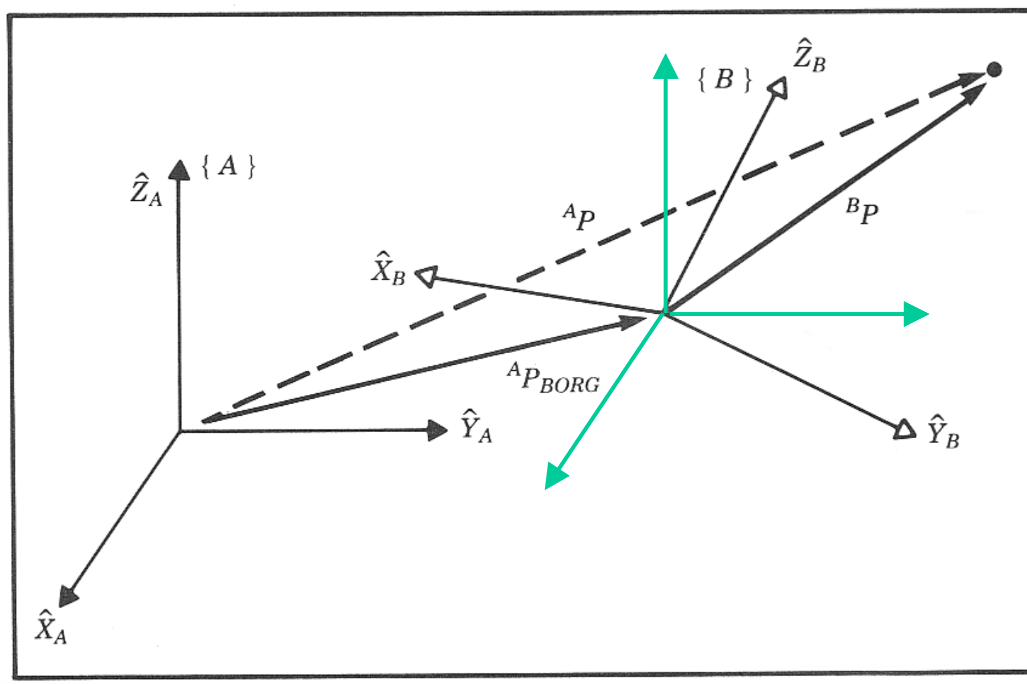
Assuming that frame {B} is both ***translated*** and ***rotated*** with respect frame {A}.

The position of the point expressed in frame {B} can be expressed in frame {A} as follows.

$$\{B\} = \{{}^A_B R, {}^A P_{BORG}\}$$

$$^A P = {}^A_B R \; {}^B P + {}^A P_{BORG}$$

$$^A P = {}^A_B T \, {}^B P$$



Mapping - Homogeneous Transform

The homogeneous transform is a 4x4 matrix casting the ***rotation*** and ***translation*** of a general transform into a single matrix. In other fields of study it can be used to compute perspective and scaling operations when the last raw is other then [0001] or the rotation matrix is not orthonormal.

$$^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BORG}$$

$$^{A}P = {}^{A}_{B}T \, {}^{B}P$$

$$\begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \left[\begin{array}{ccc:c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix}$$

Homogeneous Transform - Example (1/3)

Given:

$$ {}^{B}P = \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} $$

Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, and translated 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$

Calculate: The vector ${}^{A}P$ expressed in frame {A}.

Homogeneous Transform - Example (3/3)

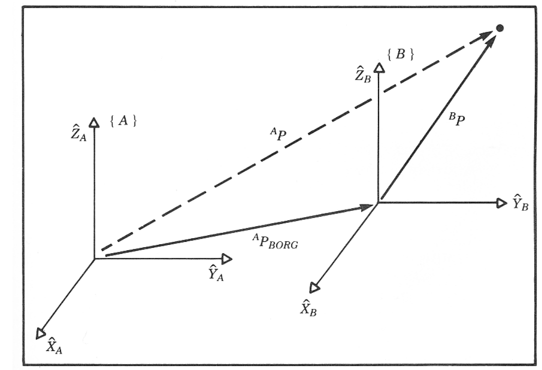
$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} {}^{A}P \\ \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ \\ 1 \end{bmatrix} $$

$$ {}^{A}P = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.0 \\ 1 \end{bmatrix} $$

Homogeneous Transform - Special Cases

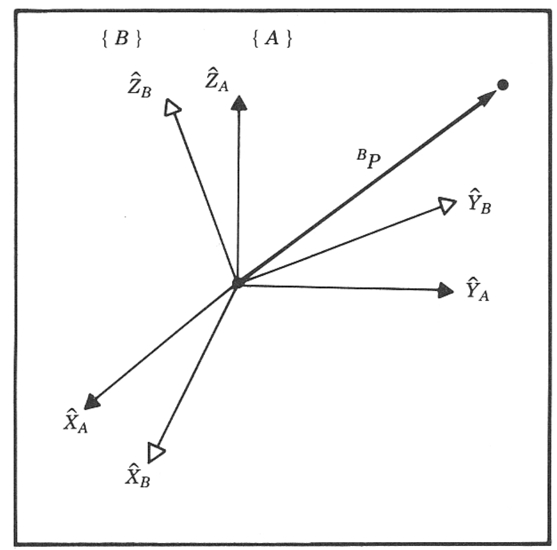
Translation

$$
{}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & {}^{A}P_{BORGx} \\ 0 & 1 & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Rotation

$$
{}^{A}_{B}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Operator - Rotating Vector

- Rotational Operator - Operates on a a vector ${}^{A}P_1$ and changes that vector to a new vector ${}^{B}P_1$, by means of a rotation R

$$ {}^{A}P_2 = R \; {}^{A}P_1 $$

- Note: The rotation matrix which rotates vectors through same the rotation R , is the same as the rotation which describes a frame rotated by R relative to the the reference frame

$$ {}^{A}P_2 = R \; {}^{A}P_1 \quad \Leftrightarrow \quad {}^{A}P = {}^{A}_{B}R \; {}^{B}P $$

Operator **Mapping**

Operator - Rotating Vector - Example

Given:

$$
{}^{A}P_1 = \begin{bmatrix} 0 \\ {}^{A}p_{1y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}
$$

Compute: The vector ${}^{B}P_1$ obtained by rotating this vector about $\hat{Z}$ by 30 degrees

Solution:

$$
{}^{A}P_1 = R(30^o)\ {}^{A}P_2 = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{A}p_{1y} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 2.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}
$$

Operator - Transforming Vector

- Transformation Operator - Operates on a a vector ${}^{A}P_1$ and changes that vector to a new vector ${}^{B}P_1$, by means of a rotation by R and translation by Q

$$ {}^{A}P_2 = T \; {}^{A}P_1 $$

- Note: The matrix of the transform operator T which rotates vectors by R and translation by Q , is the same as the transformation matrix which describes a frame rotated by R and translated by Q relative to the reference frame

$$ {}^{A}P_2 = T \; {}^{A}P_1 \quad \Longleftrightarrow \quad {}^{A}P = {}^{A}_{B}T \; {}^{B}P $$

Operator **Mapping**

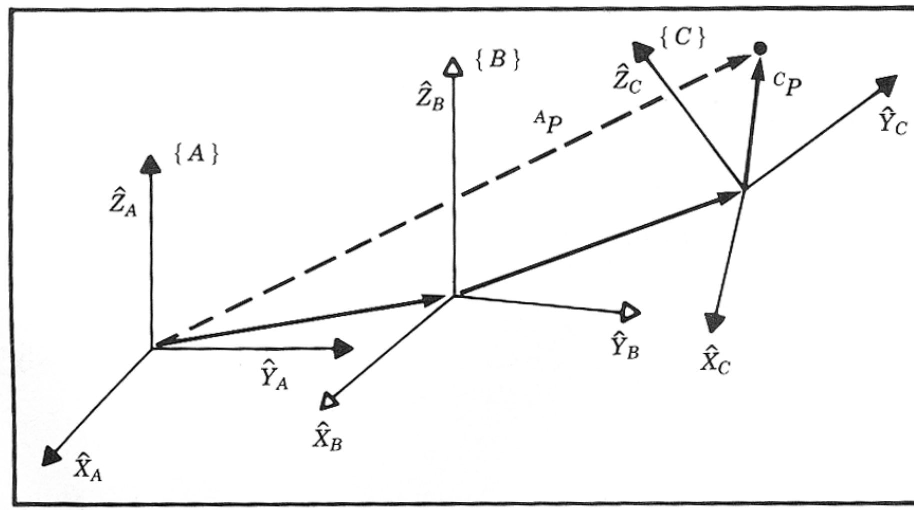
Transformation Arithmetic - Compound Transformations

Given: Vector ${}^{C}P$

Frame {C} is known relative to frame {B} - ${}^{B}_{C}T$

Frame {B} is known relative to frame {A} - ${}^{A}_{B}T$

Calculate: Vector ${}^{A}P$



$${}^{B}P = {}^{B}_{C}T\,{}^{C}P$$

$${}^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$${}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$

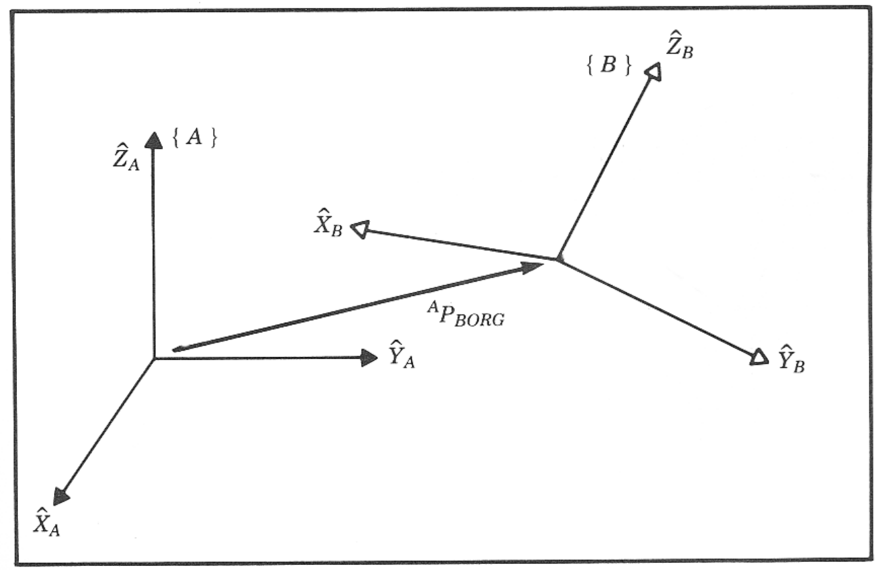
Transformation Arithmetic - Inverted Transformation

Given: Description of frame {B} relative to frame {A} - ${}^{A}_{B}T \quad ({}^{A}_{B}R, {}^{A}P_{BORG})$

Calculate: Description of frame {A} relative to frame {B} - Homogeneous Transform ${}^{B}_{A}T \quad ({}^{B}_{A}R, {}^{B}P_{AORG})$

$${}^{B}_{A}R = {}^{A}_{B}R^{T}$$

$${}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$



Note: ${}^{B}_{A}T = {}^{A}_{B}T^{-1}$

Inverted Transformation - Example (1/2)

Given: Description of frame {B} relative to frame {A} - ${}^{A}_{B}T \quad ({}^{A}_{B}R, {}^{A}P_{BORG})$
Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, and translated 4 units in $\hat{X}$, and 3 units in $\hat{Y}$

Calculate: Homogeneous Transform ${}^{B}_{A}T \;\; ({}^{B}_{A}R, {}^{B}P_{AORG})$

Inverted Transformation - Example (2/2)

$$
{}^{A}_{B}T = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & {}^{A}P_{BORGx} \\ s\theta & c\theta & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 4.000 \\ 0.500 & 0.866 & 0.000 & 3.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

$$
{}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
$$

Homogeneous Transform - Summary of Interpretation

- As a general tool to represent a frame we have introduced the ***homogeneous transformation,*** a 4x4 matrix containing orientation and position information.
- ***Three interpretation of the homogeneous transformation***

***1 .* Description of a frame -** ${}^A_B T$ describes the frame {B} relative to frame {A}

$$
{}^A_B T = \begin{bmatrix} \begin{matrix} & {}^A_B R & \end{matrix} & {}^A P_{BORG} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}
$$

2. **Transform mapping -** ${}^A_B T$ maps ${}^B P \rightarrow {}^A P$ $\quad {}^A P = {}^A_B T \ {}^B P$

3. Transform operator - T operates on ${}^A P_1$ to create ${}^A P_2$ $\quad {}^A P = {}^A_B T \ {}^B P$

Transform Equations

Given: ${}^{U}_{A}T,\ {}^{A}_{D}T,\ {}^{U}_{B}T,\ {}^{C}_{D}T$

Calculate: ${}^{B}_{C}T$

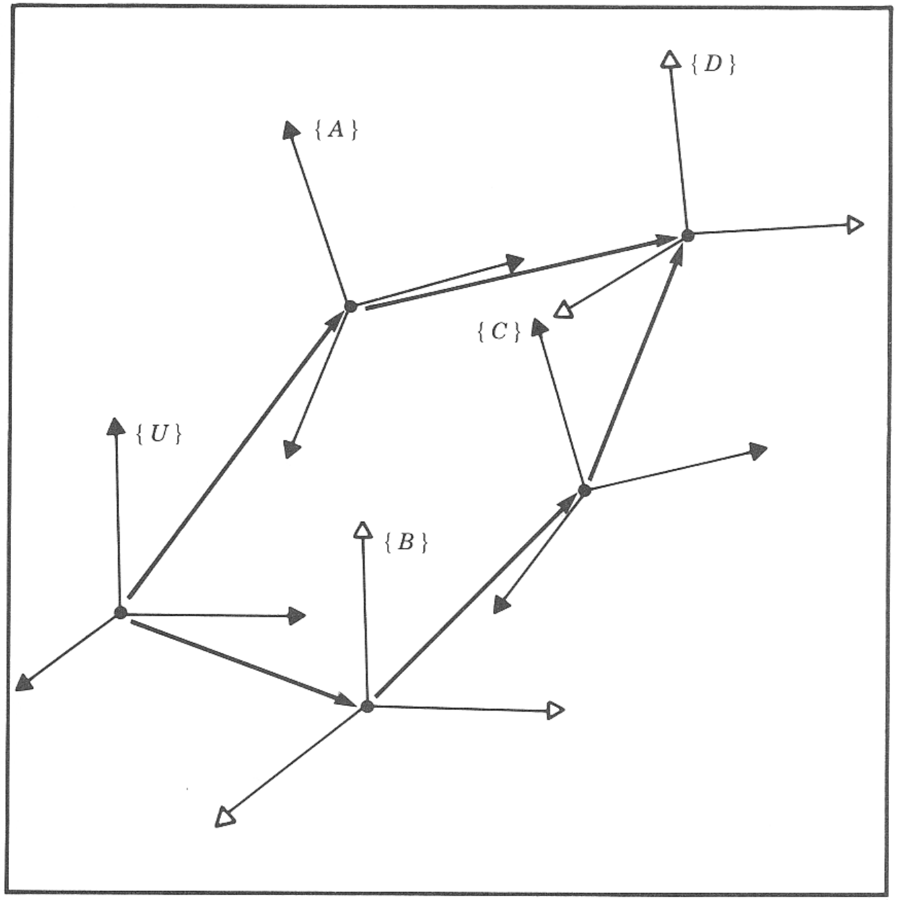
$$ {}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T $$

$$ {}^{U}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T $$

$$ {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T $$

$$ {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1} = {}^{U}_{B}T^{-1}\,{}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T\,{}^{C}_{D}T^{-1} $$

$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1} $$



Mapping - Rotated Frames - General Notation

The rotation matrices with respect to the reference frame are defined as follows:

$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$R_Y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$$

$$R_Z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping - Rotated Frames - Methods

- X-Y-Z Fixed Angles

  The rotations perform about an axis of a fixed reference frame

- Z-Y-X Euler Angles

  The rotations perform about an axis of a moving reference frame

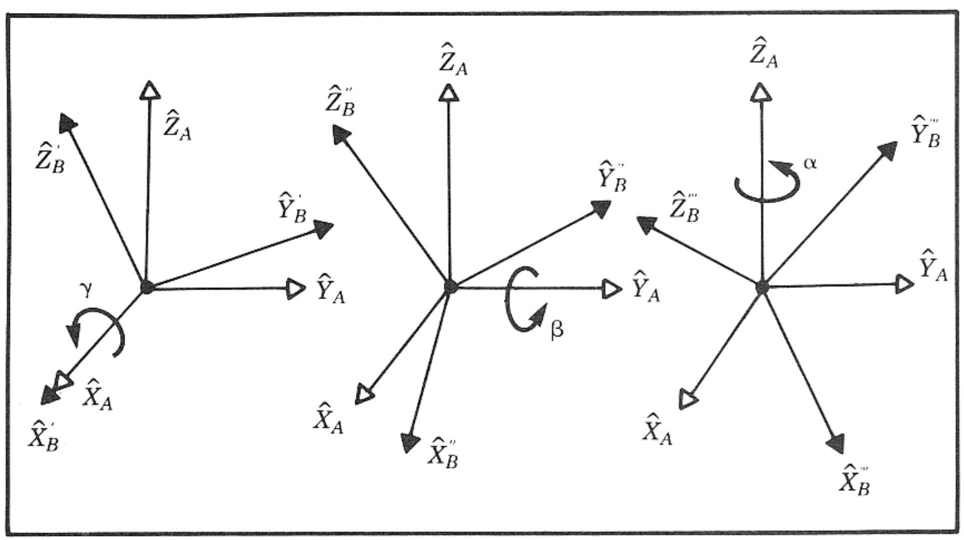
Mapping - Rotated Frames - X-Y-Z Fixed Angles

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{X}_A$ by an angle γ
- Rotate frame {B} about $\hat{Y}_A$ by an angle β
- Rotate frame {B} about $\hat{Z}_A$ by an angle α

Fixed Angles

Note - *Each of the three rotations takes place about an axis in the fixed reference frame {A}*



Mapping - Rotated Frames - X-Y-Z Fixed Angles

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

Mapping - Rotated Frames - X-Y-Z Fixed Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$\beta = \text{Atan2}(-r_{31},\sqrt{r_{11}^2 + r_{21}^2}) \qquad \text{for} \quad -90^{o} \leq \beta \leq 90^{o}$$

$$\alpha = \text{Atan2}(r_{21}/c\hat{a}, r_{11}/c\hat{a})$$

$$\gamma = \text{Atan2}(r_{32}/c\hat{a}, r_{33}/c\hat{a})$$

$$\beta = \pm 90^{o}$$

$$\alpha = 0$$

$$\gamma = \text{Atan2}(r_{12}, r_{22})$$

Atan2 - Definition

Four-quadrant inverse tangent (arctangent) in the range of $[-\pi \quad \pi]$

$$\mathrm{Atan}\,2(y,x) = \tan^{-1}(y/x)$$

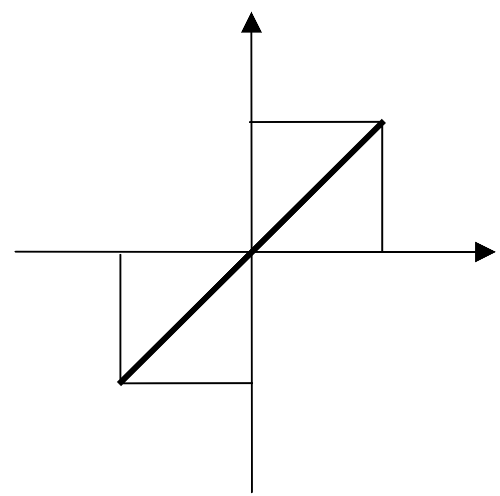
For example

$$\mathrm{Atan}(+1,+1) = 45^{o}$$

$$\mathrm{Atan}\,2(+1,+1) = 45^{o}$$

$$\mathrm{Atan}(-1,-1) = 45^{o}$$

$$\mathrm{Atan}\,2(-1,-1) = -135^{o}$$



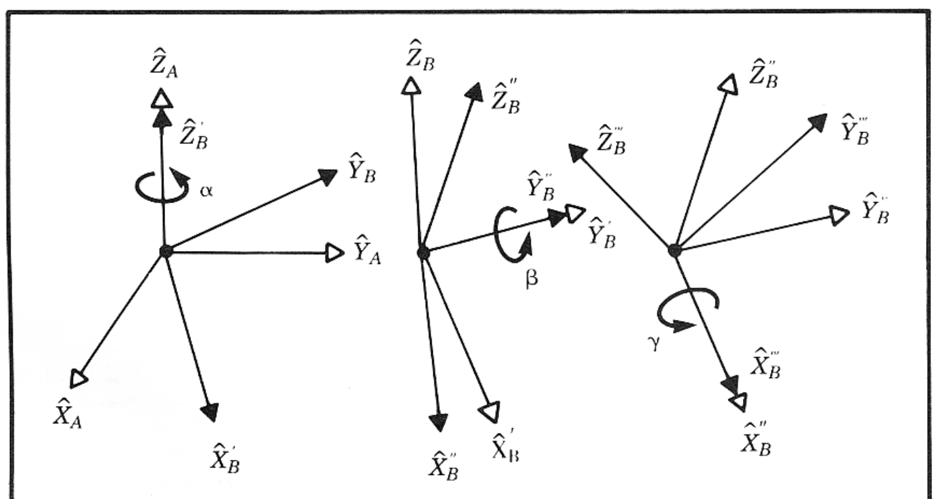
Mapping - Rotated Frames - Z-Y-X Euler Angles

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{Z}_A$ by an angle α
- Rotate frame {B} about $\hat{Y}_B$ by an angle β
- Rotate frame {B} about $\hat{X}_B$ by an angle γ

Euler Angles

Note - *Each rotation is preformed about an axis of the the moving reference frame {B}, rather then a fixed reference frame {A}.*



Mapping - Rotated Frames - X-Y-Z Euler Angles

$$
{}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
{}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

Mapping - Rotated Frames

Fixed Angles versus Euler Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) $$

Three rotations taken about fixed axes (Fixed Angles) yield the same final orientation as the same three rotation taken in an opposite order about the axes of the moving frame (Euler Angles)

Mapping - Rotated Frames

Fixed Angles versus Euler Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) $$

Why is that?

Mapping - Rotated Frames

Fixed Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

Euler Angles

$$ {}^{B}_{A}R_{Z'Y'X'}(\gamma,\beta,\alpha) = R_X(\gamma)R_Y(\beta)R_Z(\alpha) $$

with intermediate frames $\{B'\}$ and $\{B''\}$

$$
\begin{aligned}
{}^{B}_{A}R &= {}^{B}_{B''}R\,{}^{B''}_{B'}R\,{}^{B'}_{A}R \\
{}^{A}_{B}R &= {}^{B}_{A}R^{T} = \left({}^{B}_{B''}R\,{}^{B''}_{B'}R\,{}^{B'}_{A}R\right)^{T} \\
&= {}^{B'}_{A}R^{T}\left({}^{B}_{B''}R\,{}^{B''}_{B'}R\right)^{T} \\
&= {}^{B'}_{A}R^{T}\,{}^{B''}_{B'}R^{T}\,{}^{B}_{B''}R^{T} \\
&= {}^{A}_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R
\end{aligned}
$$