Modeling flow in a viscous continuously stratified fluid taking into account diffusivity effects

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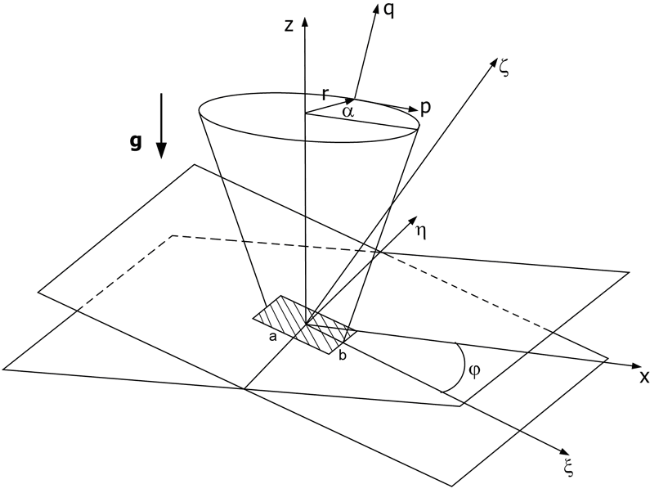
Waves and vortices in complex media

Review

1. J.V.S. Rayleigh
2. J. Lighthill
3. Yu.D. Chashechkin
4. V.A. Gorodtsov
5. T.N. Stevenson
6. D.G. Hurley, G.J. Keady
7. B.R. Sutherland
8. Yu.V. Kistovich
9. A.V. Kistovich
10. B. Voisin

Color schlirien images oscillation of disk

System coordinate frame for analytical calculation



Governing equations and boundary conditions

Governing equations

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\nabla\rho = 0, \quad \mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = \frac{1}{\rho}\nabla p + \nu\Delta v + \rho\mathbf{g}$$

$$\frac{\partial S}{\partial t} = \kappa_S \Delta S + \frac{v_z}{\Lambda}$$

Boundary conditions

$$\mathbf{v}\big|_\Gamma = \mathbf{u}_0 e^{-i\omega t}, \quad \kappa_S \left.\frac{\partial S}{\partial n}\right|_\Gamma = \frac{1}{\Lambda}\left.\frac{\partial z}{\partial n}\right|_\Gamma$$

$$v \to 0, \quad \rho \to \rho_0, \quad \partial P/\partial z \to \rho_0(z)\,g, r \to \infty$$

Toroidal-poloidal decomposition

$$\mathbf{v} = \nabla \times \mathbf{e}_z\Phi + \nabla \times (\nabla \times \mathbf{e}_z\Psi)$$

System equations for functions Φ and Ψ

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ \Phi = 0$$

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\ \Psi = 0$$

$$\left(\left(\frac{\partial}{\partial t} - D\Delta\right)\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta + N^2\Delta_\perp\right)\ S = 0$$

where

$$\Delta_\perp = \partial_x^2 + \partial_y^2 \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \quad N^2 = \sqrt{\frac{g}{\Lambda}}$$

Construction solution for Φ and Ψ in Fourier transform

$$\Phi = e^{-i\omega t} \sum_{j=1}^{3} \int_{-\infty}^{+\infty} A_j\left(k_\xi, k_\eta\right) E_j \; dk_\xi dk_\eta$$

$$S = -\frac{\rho_0}{\Lambda} e^{-i\omega t} \sum_{j=1}^{3} \int_{-\infty}^{+\infty} \frac{\left(k_\xi \cos\varphi - k_j \sin\varphi\right)^2 + {k_\eta}^2}{i\omega - \kappa_S \, k^2} A_j\left(k_\xi, k_\eta\right) E_j dk_\xi dk_\eta$$

$$\Psi = e^{-i\omega t} \int_{-\infty}^{+\infty} B\left(k_\xi, k_\eta\right) \; E_4 dk_\xi dk_\eta$$

where

$$E_j = \exp\left(ik_j\zeta + ik_\xi\xi + ik_\eta\eta\right), \quad k = \sqrt{k_j^2 + k_\xi^2 + k_\eta^2}$$

Viscous stratified fluid take into diffusion

Dispersion equation

$$\left(\nu\kappa_S\tilde{k}^6 - i\omega\left(\nu+\kappa_S\right)\tilde{k}^4 - \omega^2\tilde{k}^2 + N^2k_\perp^2\right)\left(\tilde{k}^2 + \frac{\omega}{i\nu}\right) = 0$$

$$\tilde{k}^2 = 2k_\zeta^2 + k_\perp^2, \quad k_\perp^2 = k_\xi^2 + k_\eta^2$$

Viscous stratified fluid take into diffusion

Regular solution(waves)

$$k_1 = \frac{k_\xi \sin\varphi \cos\varphi \pm \kappa \cos\theta}{\mu_\theta} \pm \delta_N^2 (1+\varepsilon) \frac{i \ \tan\theta \mu_\theta^4}{2\kappa\mu^4} + \dots$$

$$\mu = \sin^2\varphi - \sin^2\theta, \quad \mu_\theta = (k_\xi \sin\varphi \cos\varphi \pm \kappa \cos\varphi),$$

$$\varepsilon = Sc^{-1} = \frac{\kappa_S}{\nu}, \quad \delta_N = \sqrt{\frac{\nu}{N}}$$

Singular solution

$$k_{2,\,3} \approx \sqrt{\frac{i\omega(\varepsilon + 1 \pm \lambda_{\nu\kappa})}{\varepsilon}}, \quad \lambda_{\nu\kappa} = \frac{2}{\sin\theta}\sqrt{(1+\varepsilon)^2 - \frac{4\varepsilon\mu}{\sin^2\theta}}$$

$$k_4 = \sqrt{\frac{2i}{\delta_\nu^2} - k^2}, \quad \delta_\nu = \delta_N \sqrt{\frac{2}{\sin\theta}}, \quad \delta_\varphi = \delta_N \sqrt{\frac{2\sin\theta}{|\mu|}}, \quad \delta_\kappa = \delta_N \sqrt{\frac{2\varepsilon}{\sin\theta}}$$

Viscous stratified fluid take into diffusion. Vetrical component of the velocity

$$v_\zeta \approx \int\limits_{-\infty}^{+\infty} A_1 \left(k_\eta^2 \sin\varphi - k_\xi \beta_1\right) E_1 dk_\xi dk_\eta -$$

$$-ie^{\frac{i-1}{\delta_\nu}\zeta} \sin\varphi \int\limits_{-\infty}^{+\infty} B E_{\xi\eta}\, dk_\xi dk_\eta - \frac{i+1}{\delta_\varphi} e^{\frac{i-1}{\delta_\varphi}\zeta} \cos\varphi \int\limits_{-\infty}^{+\infty} A_2 k_\xi E_{\xi\eta} dk_\xi dk_\eta -$$

$$-\frac{1+i}{\delta_\kappa}\sqrt{\frac{\sin\theta}{2}} e^{-\frac{\sqrt{\sin\theta}}{\delta_\kappa\sqrt{2}}\zeta + \frac{i\zeta}{\delta_\kappa\sqrt{2}}} \cos\varphi \int\limits_{-\infty}^{+\infty} A_3 k_\xi E_{\xi\eta} dk_\xi dk_\eta$$

where

$$E_{\xi\eta} = exp\left(ik_\xi\xi + ik_\eta\eta\right)$$

The viscous stratified fluid. Friction source. Rectangle. 3D. $\varphi = 0$.

$$v_\xi = \int\limits_{-\infty}^{+\infty} k_\eta^2 L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi^2 L_1 \, dk_\xi dk_\eta,$$

$$v_\eta = \int\limits_{-\infty}^{+\infty} L_3 \, dk_\xi dk_\eta + \int\limits_{-\infty}^{+\infty} k_\xi k_\eta L_1 \, dk_\xi dk_\eta, \quad v_\zeta = \int\limits_{-\infty}^{+\infty} k_\eta k_\perp^2 L_2 \, dk_\xi dk_\eta$$

$$L_m = u_0 Q \frac{k_1^{2-m} \exp(ik_1\zeta) + k_2^{2-m} \exp(ik_2\zeta)}{\left(k_\eta^2 - k_\xi^2\right)(k_2 - k_1)}, \quad m = 1, 2,$$

$$L_3 = \frac{u_0 Q}{k_\eta^2 - k_\xi^2} \exp\left(-\frac{1-i}{\delta_\nu}\zeta\right)$$

The viscous stratified fluid. Friction source. Plate. 2D.

$$v_\xi = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_\xi \left(k_1 - k_2\right)} \sin \frac{k_\xi a}{2} \left(k_1 e^{ik_1\zeta} - k_2 e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$v_\eta = 0,$$

$$v_\zeta = \frac{u_0}{\pi^2} \int\limits_{-\infty}^{+\infty} \frac{1}{k_2 - k_1} \sin \frac{k_\xi a}{2} \left(e^{ik_1\zeta} + e^{ik_2\zeta}\right) e^{ik_\xi \xi} dk_\xi,$$

$$k_1 = k_\xi \cot\left(\varphi + \theta\right) \pm \frac{i k_\xi^3 \delta_N^2}{2 \cos\theta \sin^4\left(\varphi - \theta\right)}, \quad k_2 = \frac{1+i}{\delta_\varphi}$$

Why OpenFOAM?

Pluses:

1. OpenFOAM free and open source, under the GNU general public licence (GPL).
2. Support of community: `http://www.cfd-online.com/Forums/openfoam`, `http://openfoamwiki.net`, `http://stackoverflow.com/`
3. Support open-source Linux platform (openSUSE, Ubuntu, RHEL)

Minuses:

1. No GUI to create grids, but can use other applications such as: GMSH (**http://geuz.org/gmsh/**), Salome (**http://www.salome-platform.org/**) or commercial mesh generators such as: Icem CFD (**www.ansys.com**), Gambit (**www.ansys.com**), pro*star Star-CD (**www.cd-adapco.com**)

Hardware, software and workflow process

Platform: 8 cpu, 32 Gb
Software: CentOS 6,
OpenFOAM-2.2.2

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GitHub
(solvers, models)

Supercomputer
“Lomonosov”

Structure of the solver. IGWFoam.C

Navier - Stokes equations

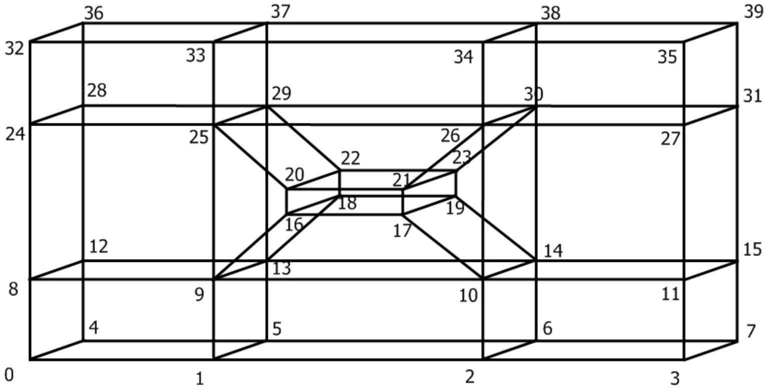
```
fvVectorMatrix UEqn (
   fvm::ddt(U) + fvm::div(phi, U)
 - fvm::laplacian(nu, U) - S*g
);
solve(UEqn = = -fvc::grad(p)/dens0);
```

Equations for salinity S and density *dens*

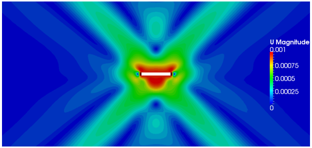
```
fvScalarMatrix SEqn (
   fvm::ddt(S) + fvm::div(phi, S)
 - fvm::laplacian(DS, S)
 - U.component(vector::Z)/Lambda
);
SEqn.solve();
dens = dens0*(1.0-Z/Lambda+S);
```

Create O-grid model

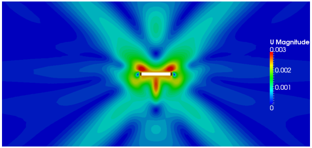
To construct the mesh used a standard utility of OpenFOAM **blockMesh** or **pyFoam** (Python for OpenFOAM)



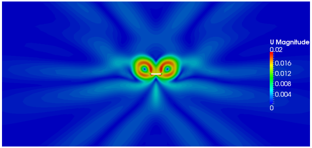
Different velocities $L_x = 1$ cm, $N = 0.9\ s^{-1}$, $\omega = 0.54\ s^{-1}$
Module of velocity. Source - horizontal plate. Type - piston



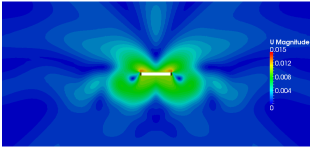
$u_0 = 0.001\ m\ s^{-1}$



$u_0 = 0.0025\ m\ s^{-1}$

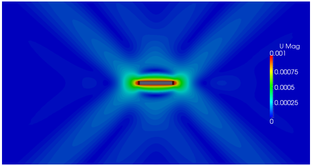


$u_0 = 0.025\ m\ s^{-1}$

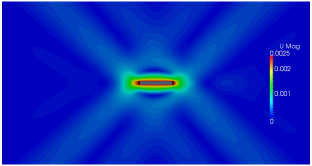


$u_0 = 0.01\ m\ s^{-1}$

Different velocities $L_x = 1$ cm, $N = 0.9\ s^{-1}$, $\omega = 0.54\ s^{-1}$
Module of velocity. Source - horizontal plate. Type - friction



$u_0 = 0.001\ m\ s^{-1}$



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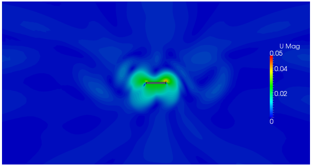
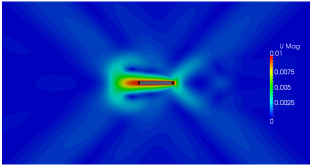
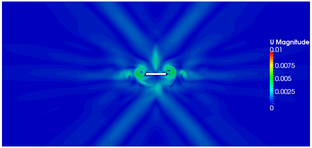


$u_0 = 0.025\ m\ s^{-1}$

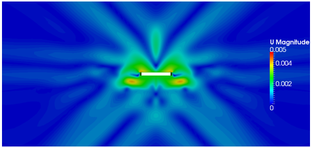


$u_0 = 0.01\ m\ s^{-1}$

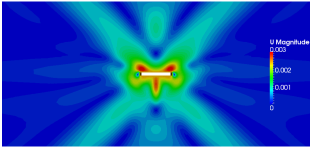
Different viscosity $L_x = 1$ cm, $N = 0.9\ s^{-1}$, $\omega = 0.54\ s^{-1}$
Module of velocity. Source - horizontal plate. Type - piston



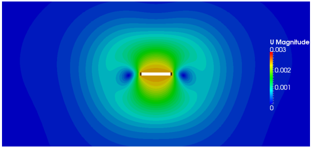
$\nu = 10^{-8}\ m^2\ s^{-1}$



$\nu = 10^{-7}\ m^2\ s^{-1}$

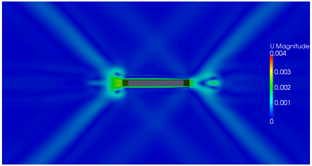


$\nu = 10^{-6}\ m^2\ s^{-1}$

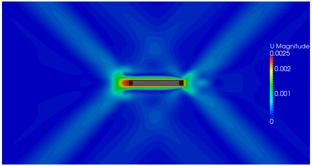


$\nu = 10^{-5}\ m^2\ s^{-1}$

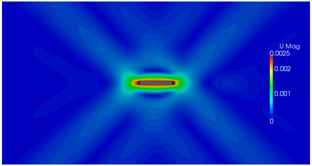
Different viscosity $L_x = 1$ cm, $N = 0.9\ s^{-1}$, $\omega = 0.54\ s^{-1}$
Module of velocity. Source - horizontal plate. Type - friction



$\nu = 10^{-8}\ m^2\ s^{-1}$



$\nu = 10^{-7}\ m^2\ s^{-1}$



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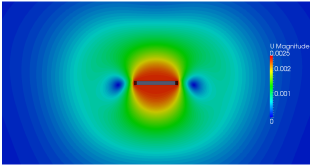
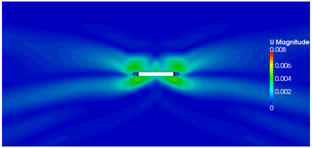
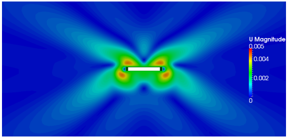


$\nu = 10^{-5}\ m^2\ s^{-1}$

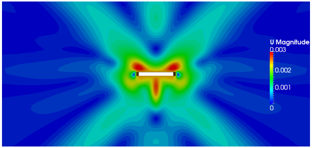
Different stratification $L_x = 1$ cm, $\nu = 10^{-2}\ cm^2\ s^{-1}$, $\omega = 0.54\ s^{-1}$ Module of velocity. Source - horizontal plate. Type - piston



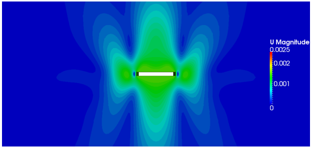
$N = 2.8\ s^{-1}$



$N = 1.46\ s^{-1}$

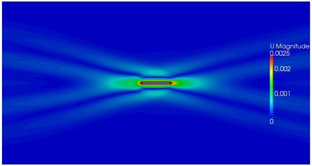


$N = 2.8\ s^{-1}$

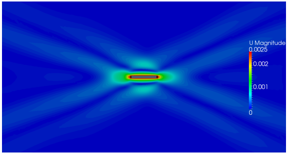


$N = 0.58\ s^{-1}$

Different stratification $L_x = 1$ cm, $\nu = 10^{-2}\ cm^2\ s^{-1}$, $\omega = 0.54\ s^{-1}$
Module of velocity. Source - horizontal plate. Type - friction



$N = 2.8\ s^{-1}$



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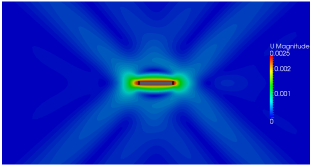

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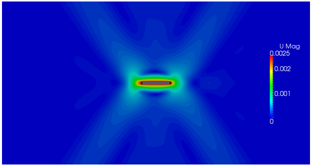

$N = 0.58\ s^{-1}$

Conclusion

1. In the general case, in a viscous stratified fluid there are two types of solutions: regular (waves) and three type singular solutions. Two of them don't have analogues in homogeneous fluid. Their properties are defined viscosity, stratification, diffusion and the geometry of the problem;
2. For a complete description of the flow of fluid you must consider all parameters (viscosity, stratification, diffusion);
3. Create solver for calculation of the internal gravity waves in a continuesly stratified fluid;
4. Calculations case horizontal plate for two types of the sources: friction and piston.

Thank you for your attention!
Any questions?