R406: Applied Economic Modelling with Python

Systems of Difference Equations

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Lecture Contents

Introduction

Introduction

- So far we considered single-variable equations
- Also, we stuck to the autonomous case; we shall do the same in the current topic
- Suppose that instead of having just one equation in one variable, we have the following set of two equations in two variables:

$$
\begin{aligned}
y_t &= ax_{t-1} + by_{t-1} \\
x_t &= cx_{t-1} + dy_{t-1}
\end{aligned}
$$

- Apparently, this is already a system of difference equations
- Such systems turn out to be very useful in describing relationships among economic variables which are not confined to a single direction of causality

A note on classification

- **Autonomous** vs. **non-autonomous** systems were already introduced on-the-fly
- **Non-homogeneous** systems are those in which at least one equation is non-homogeneous
- Likewise, **non-linear** systems of equations are the ones in which there is at least one non-linear equation
- As with the single-equation case, we will consider only linear systems

Analysis of linear systems of equations

Matrix form of systems

- The simple homogeneous system example given earlier can be written as follows:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} \quad (*)$$

- Using matrix shorthand, this is the same as:

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_{t-1}$$

  where $\mathbf{u}_t = [x_t, y_t]'$, and $\mathbf{A}$ is the coefficient matrix
- By analogy, the non-homogeneous counterpart to this system would look as follows:

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_{t-1} + \mathbf{b}$$

  where $\mathbf{b}$ is a vector of conformable size (can be time-dependent but let's forget about this for the time being)

Equilibrium

- Take $(*)$ are the reference point
- Equilibrium implies that $x_t = x^*$ and $y_t = y^*$, $\forall t$
- Therefore, in equilibrium:

$$\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix},$$

  or:

$$\mathbf{u}^* = \mathbf{A}\mathbf{u}^*$$

- Move everything to the LHS of the equation:

$$\mathbf{u}^* - \mathbf{A}\mathbf{u}^* = \mathbf{0} \Leftrightarrow (\mathbf{I} - \mathbf{A})\mathbf{u}^* = \mathbf{0}$$

Equilibrium (2)

- If $\mathbf{I} - \mathbf{A}$ is a non-singular matrix, the equilibrium corresponds to $\mathbf{u}^* = \mathbf{0}$
- Now, return to the non-homogeneous system; equilibrium again requires that $\mathbf{u}_t = \mathbf{u}^*$, $\forall t$, i. e.:

$$\mathbf{u}^* = \mathbf{A}\mathbf{u}^* + \mathbf{b}$$

- Move everything except $\mathbf{b}$ to the LHS:

$$\mathbf{u}^* - \mathbf{A}\mathbf{u}^* = \mathbf{b}$$

Equilibrium (3)

- This is the same as:

$$(\mathbf{I} - \mathbf{A})\mathbf{u}^* = \mathbf{b}$$

- If $\mathbf{I} - \mathbf{A}$ is a non-singular matrix, then the equilibrium of the system is found where:

$$\mathbf{u}^* = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

- This looks very much like the result for the single-equation case which shouldn't be surprising

Stability

- As in the single-equation case, in order to analyse stability, we need to solve the system
- The procedure that is followed is analogical
- Start with the homogeneous case:

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_{t-1} = \mathbf{A}(\mathbf{A}\mathbf{u}_{t-2}) = \mathbf{A}^2\mathbf{u}_{t-2} = \ldots = \mathbf{A}^t\mathbf{u}_0$$

- In the non-homogeneous case the solution is:

$$\mathbf{u}_t = \mathbf{A}^t\mathbf{u}_0 + (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots + \mathbf{A}^{t-1})\mathbf{b}$$

Stability (2)

- Note that a non-homogeneous system can be reduced to a homogeneous one by expressing it in deviations from equilibrium
- Therefore, after subtracting $\mathbf{u}^*$ from both sides of the matrix equation, we have:

$$\mathbf{u}_t - \mathbf{u}^* = \mathbf{A}(\mathbf{u}_{t-1} - \mathbf{u}^*)$$

- Denote $\mathbf{z}_t = \mathbf{u}_t - \mathbf{u}^*$; then:

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1}$$

- This is a first-order homogeneous system
- Therefore, from now on the discussion can proceed with homogeneous systems without losing generality with respect to the non-homogeneous case

Stability (3)

Theorem 1

If the matrix $\mathbf{A}$ *has two distinct eigenvalues* r *and* s*, then there exists a matrix* $\mathbf{V} = \begin{bmatrix}\mathbf{v}^r & \mathbf{v}^s\end{bmatrix}$ *composed of the eigenvectors corresponding to* r *and* s *such that:*

$$\mathbf{D} = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

- If we pre-multiply both sides of the Theorem's result by $\mathbf{V}$ and post-multiply them by $\mathbf{V}^{-1}$, we get:

$$\mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}\mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{V}^{-1} = \mathbf{A}$$

- It is also easy to see that $\mathbf{A}^t = \mathbf{V}\mathbf{D}^t\mathbf{V}^{-1}$

Stability (3)

- Using this, we can write:

$$\mathbf{u}_t = \mathbf{A}^t \mathbf{u}_0 = \mathbf{V}\mathbf{D}^t\mathbf{V}^{-1}\mathbf{u}_0$$

- But this can be stated explicitly as follows:

$$\mathbf{u}_t = \mathbf{V}\begin{bmatrix} r^t & 0 \\ 0 & s^t \end{bmatrix}\mathbf{V}^{-1}\mathbf{u}_0$$

Theorems

Theorem 2

A necessary and sufficient condition for the system of difference equations to be globally asymptotically stable is that all eigenvalues of the matrix $\mathbf{A}$ *have moduli strictly less than 1.*

Theorem 3

If all eigenvalues of $\mathbf{A}$ *have moduli strictly less than 1, the difference equation is globally asymptotically stable and every solution converges to the constant vector* $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$.

Transformation to autonomous form

- Consider the non-autonomous equation

$$y_{t+1} = f(t, y_t), \quad t = 0, 1, 2, \ldots$$

- We can introduce a variable x_t and consider the modified system

$$\begin{aligned} y_{t+1} &= f(x_t, y_t) \\ x_{t+1} &= x_t + 1 \end{aligned}$$

  together with the initial condition $x_0 = 0$

- The resulting system is autonomous, which has been achieved at the expense of an increase in the dimension of the original problem

- Sometimes such a transformation can facilitate the analysis of the system

Order reduction

- Consider the equation

$$y_{t+p} = f(y_t, y_{t+1}, \ldots, y_{t+p-1}), \quad t = 0, 1, 2, \ldots \tag{1}$$

- Introduce the following variables

$$\begin{aligned} x_{1,t} &= y_t \\ x_{2,t} &= y_{t+1} \\ x_{3,t} &= y_{t+2} \\ &\ldots \\ x_{p,t} &= y_{t+p-1} \end{aligned}$$

- Then we can write the system

$$\begin{aligned} x_{1,t+1} &= x_{2,t} \\ x_{2,t+1} &= x_{3,t} \\ &\ldots \\ x_{p-1,t+1} &= x_{p,t} \\ x_{p,t+1} &= f(x_{1,t}, x_{2,t}, \ldots, x_{p,t}) \end{aligned} \tag{2}$$

Order reduction (2)

- This approach trades the pth-order equation (1) for the first-order system of p equations (2)
- The representation of a problem as a system of equations can be more convenient to work with in certain cases