R406: Applied Economic Modelling with Python

Difference Equations of Second and Higher Order

Kaloyan Ganev (main author)
Andrey Vassilev (minor modifications)

Lecture Contents

Introduction

Introduction

- Dependence on past or future values is not limited to one period only in the general case
- We first review second-order difference equations using our knowledge on how to solve a quadratic equation by hand
- Then we'll generalize to higher order

Second-order difference equations

Second-order difference equations

- The general form of second-order difference equations is:

$$y_{t+2} = f(t, y_t, y_{t+1}), \quad t = 0, 1, 2, \ldots$$

- There are infinitely many solutions to this equation if no other information is supplied
- Let the first two values of y_t (y_0 and y_1) be fixed and known
- Then y_0 and y_1 uniquely determine the solution

Linear equations

- The general form of a linear second-order difference equation is:

$$y_{t+2} + a_t y_{t+1} + b_t y_t = c_t \quad (*)$$

  where a_t, b_t, and c_t are known (given) functions of t
- If $c_t = 0$ then the equation is **homogeneous**:

$$y_{t+2} + a_t y_{t+1} + b_t y_t = 0 \quad (**)$$

Two theorems

Theorem 1

The solution of the homogeneous equation $(**)$ *is:*

$$y_t = Au_t^{(1)} + Bu_t^{(2)}$$

where $u_t^{(1)}$ *and* $u_t^{(2)}$ *are two linearly independent solutions and* A *and* B *are arbitrary constants.*

Two theorems (2)

Theorem 2

The solution of the non-homogeneous equation $(*)$ *is:*

$$y_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

where $Au_t^{(1)} + Bu_t^{(2)}$ *is the general solution to* $(**)$ *and* u_t^* *is any particular solution of* $(*)$.

- It can be shown that if we manage to find two linearly independent solutions to the homogeneous equation, then we are able to also find the general solution to $(*)$

Constant coefficients

- Take $(**)$ and let $a_t = a$ and $b_t = b \neq 0$ be arbitrary constants
- This makes the equation a constant-coefficients one:

$$y_{t+2} + ay_{t+1} + by_t = 0$$

- Note first that if we set $y_t = e^{rt}$, then we would have $y_{t+1} = e^{r(t+1)} = e^{rt}e^r$ and $y_{t+2} = e^{r(t+2)} = e^{rt}e^{2r}$
- Plug these expressions in the equation:

$$e^{rt}e^{2r} + ae^{rt}e^r + be^{rt} = 0$$

- This can also be written as:

$$e^{rt}(e^{2r} + ae^r + b) = 0$$

Constant coefficients (2)

- e^{rt} is always positive, so we can divide both sides by it to get:

$$e^{2r} + ae^{r} + b = 0$$

- If we put $m = e^{r}$, then we need to solve practically the following quadratic equation:

$$m^2 + am + b = 0$$

- This is the **characteristic equation** of the difference equation
- Its solutions will provide us with the values of m that make e^{rt} also a solution

The solutions of the characteristic equation

- Using high-school algebra, we find that the roots of the quadratic equation in our case equal:

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- Having also the necessary knowledge on complex numbers, you already know that the solutions are well defined for all values of the discriminant
- The solutions of the difference equation in the three possible cases for the value of the discriminant $D = a^2 - 4b$ are given in the following Theorem

The solutions of the characteristic equation (2)

Theorem 3

The general solution of $(**)$ *when* $b \neq 0$ *is as follows:*

1. *If* $D > 0$ *(two distinct real roots of the characteristic equation) then:*

$$y_t = Am_1^t + Bm_2^t, \quad m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

2. *If* $D = 0$ *(one double real root) then:*

$$y_t = (A + Bt)m^t, \quad m = -\frac{a}{2}$$

3. *If* $D < 0$ *(two complex conjugate roots) then:*

$$y_t = R^t(A\cos(\theta t) + B\sin(\theta t)), \quad R = \sqrt{b}, \cos(\theta) = -\frac{a}{2\sqrt{b}}, \theta \in [0, \pi]$$

The solutions of the characteristic equation (3)

- First, we note that we need two initial values of y_t to solve the equation
- If those two values are given, say y_0 and y_1, then the constants A and B are uniquely determined
- In the case when the solutions are complex conjugates, the fact that they contain sines and cosines implies that they characterize cyclical behaviour (oscillations)
- The value of the radius (modulus) R determines the type of cyclical fluctuations:
    1. When $|R| < 1$, the oscillations are damped (their amplitude decreases over time)
    2. When $|R| > 1$, the oscillations are explosive (increasing amplitude)
    3. $|R| = 1$, oscillations remain with unchanged amplitude over time

The non-homogeneous case

- General form:

$$y_{t+2} + ay_{t+1} + by_t = c_t, \quad b \neq 0 \quad (\spadesuit)$$

- Recall that according to Theorem 2 the solution is:

$$y_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

  where u_t^* is a particular solution of ($\spadesuit$)
- It turns out that finding u_t^* is a very difficult task even if c_t is a relatively simple function

The non-homogeneous case (2)

- An easier case: $c_t = c$, i.e. a constant
- Then (♠) becomes:

$$y_{t+2} + ay_{t+1} + by_t = c, \quad b \neq 0 \quad (\heartsuit)$$

- So, we have to find a solution of the form: $y_t = C$, where $C = const$
- If $y_t = C$, then $y_{t+1} = y_{t+2} = C$. Substitute all these in $(\heartsuit)$ to get:

$$C + aC + bC = c \Leftrightarrow C(1 + a + b) = c$$

- Therefore, if $1 + a + b \neq 0$:

$$C = \frac{c}{1 + a + b}$$

- Then, the particular solution is:

$$u_t^* = \frac{c}{1 + a + b}$$

The non-homogeneous case (3)

- What if $1 + a + b = 0$?
- Then there is no constant function that can satisfy $(\heartsuit)$
- In such a case we can write $b = -(1 + a)$ and substitute this in $(\heartsuit)$:

$$y_{t+2} + ay_{t+1} - (1 + a)y_t = c$$

- In this case, a constant function would solve only the homogeneous function, so we look for another particular solution

The non-homogeneous case (4)

- Try with $u_t^* = Dt$:

$$\begin{aligned} u_{t+2}^* + au_{t+1}^* - (1+a)u_t^* &= D(t+2) + aD(t+1) - (1+a)Dt = \\ &= Dt + 2D + aDt + aD - Dt - aDt = \\ &= (a+2)D \end{aligned}$$

- So, if $a \neq -2$, then $D = \dfrac{c}{a+2}$, and the particular solution is:

$$u_t^* = \frac{ct}{a+2}$$

The non-homogeneous case (5)

- Now, what if in addition to $1 + a + b = 0$ we have also $a = -2$? Then $(\heartsuit)$ becomes:

$$y_{t+2} - 2y_{t+1} + y_t = c$$

- We try then to find a solution of the form $u_t^* = Dt^2$. With this, we have:

$$\begin{aligned} u_{t+2}^* - 2u_{t+1}^* + u_t^* &= D(t+2)^2 - 2D(t+1)^2 + Dt^2 = \\ &= Dt^2 + 4Dt + 4D - 2Dt^2 - 4Dt - 2D + Dt^2 = \\ &= 2D \Rightarrow D = \frac{c}{2} \end{aligned}$$

- The particular solution is:

$$u_t^* = \frac{ct^2}{2}$$

Stability of solutions

- Generally speaking, a discrete dynamic system is **stable** if whatever changes are made to the initial conditions, eventually their effect vanishes as $t \to \infty$
- Otherwise (i.e. when even small changes might lead to large differences in long-term behaviour) the system is **unstable**
- Returning to $(\heartsuit)$, it is called **globally asymptotically stable** if the solution of its associated homogeneous equation tends to 0 as $t \to \infty$

Theorem 4

The equation $(\heartsuit)$ is globally asymptotically stable iff the following two equivalent conditions are satisfied:

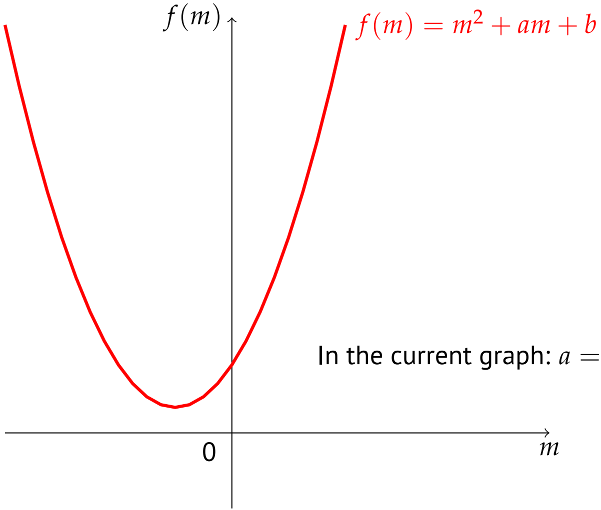
(A) *The moduli of the roots of the characteristic equation $m^2 + am + b = 0$ are strictly lower than 1*

(B) $|a| < 1 + b$ *and* $|b| < 1$

Proof of Theorem 4

- (This proof is provided exceptionally because the equivalence of (A) and (B) is not so obvious)
- We will prove first that $(B) \Rightarrow (A)$
- Consider first the case in which the characteristic equation has two complex conjugate roots, i.e. $a^2 - 4b < 0 \Leftrightarrow b > \dfrac{a^2}{4}$
- Note that the latter implies that $b > 0$
- Both roots have moduli equal to $\sqrt{b}$
- If $b < 1$ (and obviously $|a| < 1 + b$), then $\sqrt{b} < 1$. This proves $(B) \Rightarrow (A)$
- Now, in order to prove $(A) \Rightarrow (B)$, look at the following graph

Proof of Theorem 4 (2)



In the current graph: $a = 1.5, b = 0.9$

Proof of Theorem 4 (3)

- From the graph it is visible that the parabola never crosses the horizontal axis (this is the same as the fact that none of the roots is real)
- In other words, no matter what the value of m, we have $f(m) > 0$
- Take m to equal in turns -1 and 1; then:

$$\begin{gathered} f(-1) = 1 - a + b > 0 \Rightarrow a < 1 + b \\ f(1) = 1 + a + b > 0 \Rightarrow -a < 1 + b \end{gathered}$$

- But these two are equivalent to $|a| < 1 + b$
- From the fact that the moduli of the roots are strictly less than one directly follows that $\sqrt{b} < 1$, and therefore $b < 1$
- The above leads to $(A) \Rightarrow (B)$
- This completes the proof for complex roots

Proof of Theorem 4 (4)

- In the case of real roots, the discriminant is non-negative: $a^2 - 4b \geq 0$
- This is equivalent to $b \leq \frac{a^2}{4}$
- The two roots are:

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- In the real case, that their moduli are strictly less than 1 means that their absolute values should be less than 1
- For m_1 this means (check that the same result is obtained for m_2):

$$-1 < \frac{-a + \sqrt{a^2 - 4b}}{2} < 1 \Rightarrow -2 + a < \sqrt{a^2 - 4b} < 2 + a$$

Proof of Theorem 4 (5)

- Square all parts of the last double inequality to get:

$$a^2 - 4a + 4 < a^2 - 4b < a^2 + 4a + 4,$$

or:

$$-a + 1 < -b < a + 1,$$

or:

$$-a < b + 1 < a,$$

which is the same as $|a| < b + 1$

- The latter can also be obtained from the fact that $f(-1) > 0$ and $f(1) > 0$

Proof of Theorem 4 (6)

- Note also that in those two points the signs of the first derivative of $f(m)$, $f'(m) = 2m + a$, are known:

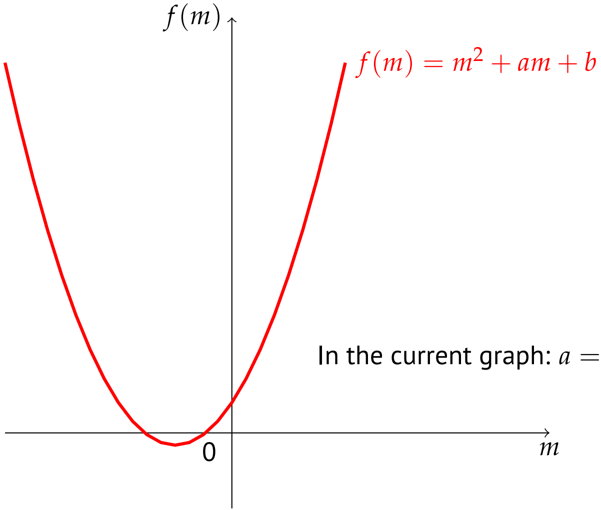
$$
\begin{aligned}
f'(-1) &= -2 + a < 0 \\
f'(1) &= 2 + a > 0
\end{aligned}
$$

- From the latter follows that $|a| < 2$
- Combine this with $b \leq \dfrac{a^2}{4}$ to find that:

$$b \leq \frac{a^2}{4} < \frac{4}{4} = 1$$

- This proves $(A) \Rightarrow (B)$

Proof of Theorem 4 (7)



In the current graph: $a = 1.5, b = 0.4$

Proof of Theorem 4 (8)

- To prove equivalence in the reverse direction, first note that if $|a| < 1 + b$ and $b < 1$, obviously $|a| < 2$
- The latter is equivalent to $-2 < a < 2$, or $2 + a > 0$ and $-2 + a < 0$
- We can also see that $2 + a$ and $-2 + a$ are the values of $f'(m)$ respectively at 1 and -1
- Using $|a| < 1 + b$, we can consecutively write:

$$
\begin{aligned}
-a < 1 + b < a \quad & \Leftrightarrow \quad -a + 1 < -b < a + 1 \Leftrightarrow \\
& \Leftrightarrow \quad -4a + 4 < -4b < 4a + 4 \Leftrightarrow \\
& \Leftrightarrow \quad a^2 - 4a + 4 < a^2 - 4b < a^2 + 4a + 4 \Leftrightarrow \\
& \Leftrightarrow \quad (a - 2)^2 < a^2 - 4b < (a + 2)^2
\end{aligned}
$$

- From this point onwards, establishing that the roots of the characteristic equation lie in $(-1, 1)$ is straightforward

Example: The multiplier-accelerator model

- Keynesian business cycle model, due to Samuelson (1939)
- We consider a slightly modified version
- Model equations:

$$
\begin{aligned}
C_t &= a + bY_{t-1} \\
I_t &= v(Y_{t-1} - Y_{t-2}) \\
G_t &= \overline{G}, \quad \forall t \\
E_t &= C_t + I_t + G_t \\
Y_t &= E_t
\end{aligned}
$$

- Combine all equations to get the following second-order non-homogeneous difference equation:

$$Y_t - (b+v)Y_{t-1} + vY_{t-2} = a + \overline{G}$$

Example: The multiplier-accelerator model (2)

- To find a particular solution, set $Y_t = Y^* = const$, i.e.:

$$Y^* - (b + v)Y^* + vY^* = a + \overline{G}$$

- After rearrangement, we have:

$$Y^* = \frac{a + \overline{G}}{1 - b}$$

- The latter is interpreted in the following way: equilibrium income corresponds to the result from the simple Keynesian multiplier model

Example: The multiplier-accelerator model (3)

- The homogeneous equation that corresponds to this example is:

$$Y_t - (b+v)Y_{t-1} + vY_{t-2} = 0$$

- The roots of its characteristic equation are as follows:

$$m_{1,2} = \frac{(b+v) \pm \sqrt{(b+v)^2 - 4v}}{2}$$

- Three cases emerge again:
  1. Two distinct real roots
  2. One double real root
  3. Two complex conjugate roots

Example: The multiplier-accelerator model (4)

- Case 1: Two distinct real roots, i. e. $(b+v)^2 - 4v > 0$
- In order to be able to analyse the dynamics implied by the difference equation, it is a good idea to use the Vieta formulae which define the relationships between the two roots:

$$
\begin{aligned}
m_1 + m_2 &= b + v \\
m_1 m_2 &= v
\end{aligned}
$$

- We can use these two to find that:

$$
\begin{aligned}
(1-m_1)(1-m_2) &= 1 - m_2 - m_1 + m_1 m_2 = \\
&= 1 - (b+v) + v = \\
&= 1 - b
\end{aligned}
$$

- As b is interpreted as MPC, $b \in (0,1)$. The latter implies that also $(1-m_1)(1-m_2) \in (0,1)$

Example: The multiplier-accelerator model (5)

- In this case, the general solution is given by:

$$Y_t = Am_1^t + Bm_2^t + Y^*$$

- The larger of the two roots (say this is m_1 in our example) determines the development path of Y_t[1]
- From $b > 0$ and $v > 0$ follows that $m_1 m_2 = v > 0$; this implies that m_1 and m_2 should either be both positive or both negative
- But because of the fact that $m_1 + m_2 = b + v > 0$, the option that the two roots are both negative is ruled out; therefore $m_1 > 0$ and $m_2 > 0$
- This means that Y_t is not characterized with oscillations
- Two possibilities arise with respect to the magnitude of the larger root

[1] This is valid in general for any polynomial: the root having the largest modulus dominates the remaining ones.

Example: The multiplier-accelerator model (6)

- If $m_1 > 1$, then we should also have $m_2 > 1$ (otherwise the condition $(1 - m_1)(1 - m_2) \in (0, 1)$ will be violated)
- With $m_1 > m_2 > 1$, Y_t has an explosive path
- The above also implies that $m_1 m_2 = v > 1$, i. e. the accelerator coefficient is greater than 1
- If $m_1 < 1$, then $0 < m_2 < m_1 < 1$. From this follows first that $v \in (0, 1)$ and second, that the dynamics is damped towards the equilibrium
- Note that the roots cannot equal 1 since otherwise $(1 - m_1)(1 - m_2)$ would not be positive but would also equal zero!

Example: The multiplier-accelerator model (7)

- Case 2: One double real root, i. e. $(b+v)^2 - 4v = 0$
- The root equals:

$$m_{1,2} = \frac{b+v}{2} = m$$

- Since $m^2 = v$, we have $v \geq 0$; but m cannot be zero because $b > 0$, therefore $v > 0$ (although this should be an obvious assumption from the very beginning); finally, this means that $m > 0$
- Again, two possibilities
- First, if $0 < m < 1$, then we should have $0 < v < 1$ and a damped path for income
- Second, if $m > 1$, then $v > 1$ and the dynamics of Y_t is explosive
- By the same reasoning as above, m cannot be equal to one

Example: The multiplier-accelerator model (8)

- Case 3: Two complex conjugate roots, i. e. $(b+v)^2 - 4v < 0$
- The roots equal:

$$m_1 = \alpha + i\beta$$
$$m_2 = \alpha - i\beta$$

where $\alpha = \dfrac{b+v}{2}$ and $\beta = \dfrac{\sqrt{4v-(b+v)^2}}{2}$

- The general solution to the difference equation is:

$$Y_t = R^t[A\cos(\theta t) + B\sin(\theta t)] + Y^*,$$

where $R = \sqrt{\alpha^2+\beta^2} = \sqrt{\dfrac{(b+v)^2 + 4v - (b+v)^2}{4}} = \sqrt{v}$,
$\cos(\theta) = \dfrac{b+v}{2\sqrt{v}}$, and $\sin(\theta) = \dfrac{\sqrt{4v-(b+v)^2}}{2\sqrt{v}}$

Higher-order equations

Higher-order equations

- In general, we can have pth-order difference equations

$$y_{t+p} = f(t, y_t, y_{t+1}, \ldots, y_{t+p-1}), \quad t = 0, 1, 2, \ldots$$

- In order to have a uniquely defined solution, p initial values are needed
- The general solution of such an equation is a function $y_t = g(t; C_1, \ldots, C_p)$, where $C_1, \ldots, C_p$ are arbitrary constants
- For each given set of values of $C_1, \ldots, C_p$, we can obtain the corresponding solution of the equation

Higher-order equations: The linear case

Theorem 5

The pth-order linear homogeneous difference equation:

$$y_{t+p} + a_1(t)y_{t+p-1} + \ldots + a_{p-1}(t)y_{t+1} + a_p(t)y_t = 0, \quad a_p(t) \neq 0$$

has the following solution:

$$y_t = C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)}$$

where $u_t^{(1)}, \ldots, u_t^{(p)}$ are p linearly independent solutions of the characteristic equation, and $C_1, \ldots, C_p$ are arbitrary constants.

Higher-order equations: The linear case (2)

Theorem 6

The pth-order linear non-homogeneous difference equation:

$$y_{t+p} + a_1(t)y_{t+p-1} + \ldots + a_{p-1}(t)y_{t+1} + a_p(t)y_t = b_t, \quad a_p(t) \neq 0$$

has the following solution:

$$y_t = C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)} + u_t^*$$

where $C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)}$ is the general solution of the homogeneous equation, and u_t^* is a particular solution of the non-homogeneous equation

Linear higher-order equations with constant coefficients

- Homogeneous case:

$$y_{t+p} + a_1 y_{t+p-1} + \ldots + a_{p-1} y_{t+1} + a_p y_t = 0, \quad t = 0, 1, 2, \ldots$$

- Non-homogeneous case:

$$y_{t+p} + a_1 y_{t+p-1} + \ldots + a_{p-1} y_{t+1} + a_p y_t = b_t, \quad t = 0, 1, 2, \ldots$$

- Solutions having the form $y_t = m^t$ are sought, as we did in the second-order equations case
- This leads to the following characteristic equation:

$$m^p + a_1 m^{p-1} + \ldots + a_{p-1} m + a_p = 0$$

Linear higher-order equations with constant coefficients (2)

- This is a polynomial equation, and it has as many roots as the degree of the polynomial (p in the current case)
- Those p roots can either be different, or there could be multiple roots (as with double roots in second-order equations)
- Also, there could be complex roots, and they always come in pairs of conjugates

Linear higher-order equations with constant coefficients (3)

The general rules that are followed in finding the roots are as follows:

1. If a real root m_i is unique, then it provides one solution m_i^t
2. If a real root m_j is repeated k times, then it provides k solutions $m_j^t, tm_j^t, t^2m_j^t, \ldots, t^{k-1}m_j^t$
3. A pair of complex conjugates $\alpha \pm i\beta$ which is encountered only once among the list of roots provides two solutions: $R^t \cos(\theta t)$ and $R^t \sin(\theta t)$
4. A pair of complex conjugates $\alpha \pm i\beta$ which is encountered l times among the list of root provides $2l$ solutions: $u, v, tu, tv, , \ldots, t^{l-1}u, t^{l-1}v$, where $u = R^t \cos(\theta t)$ and $v = R^t \sin(\theta t)$

For the non-homogeneous equation, a particular solution u_t^* needs also to be found.

Stability conditions for higher-order equations

Theorem 7

A necessary and sufficient condition for the pth order linear difference equation is that the roots of the characteristic polynomial all lie within the unit circle, i. e. all have moduli less than 1.

References

- Chiang, A. and K. Wainwright (2005): *Fundamental Methods of Mathematical Economics*, McGraw-Hill, Ch. 18
- Elaydi, S. (2005): *An Introduction to Difference Equations*, Springer, Ch. 1
- Shone, R. (2002): *Economic Dynamics: Phase Diagrams and Their Economic Application*, Cambridge University Press, Ch. 3
- Sydsæter, K., P. Hammond, and A. Strøm (2008): *Further Mathematics for Economic Analysis*, Prentice Hall, Ch.11