R406: Applied Economic Modelling with Python

Difference Equations of Second and Higher Order

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Lecture Contents

Introduction

Introduction

- Dependence on past or future values is not limited to one period only in the general case
- We first review second-order difference equations using our knowledge on how to solve a quadratic equation by hand
- Then we'll generalize to higher order

Second-order difference equations

Second-order difference equations

- The general form of second-order difference equations is:

$$y_{t+2} = f(t, y_t, y_{t+1}), \quad t = 0, 1, 2, \ldots$$

- There are infinitely many solutions to this equation if no other information is supplied
- Let the first two values of y_t (y_0 and y_1) be fixed and known
- Then y_0 and y_1 uniquely determine the solution

Linear equations

- The general form of a linear second-order difference equation is:

$$y_{t+2} + a_t y_{t+1} + b_t y_t = c_t \quad (*)$$

where a_t, b_t, and c_t are known (given) functions of t

- If $c_t = 0$ then the equation is **homogeneous**:

$$y_{t+2} + a_t y_{t+1} + b_t y_t = 0 \quad (**)$$

Two theorems

Theorem 1

The solution of the homogeneous equation $(**)$ *is:*

$$y_t = Au_t^{(1)} + Bu_t^{(2)}$$

where $u_t^{(1)}$ *and* $u_t^{(2)}$ *are two linearly independent solutions and* A *and* B *are arbitrary constants.*

Two theorems (2)

Theorem 2

The solution of the non-homogeneous equation $(*)$ *is:*

$$y_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

where $Au_t^{(1)} + Bu_t^{(2)}$ *is the general solution to* $(**)$ *and* u_t^* *is any particular solution of* $(*)$.

- It can be shown that if we manage to find two linearly independent solutions to the homogeneous equation, then we are able to also find the general solution to $(*)$

Constant coefficients

- Take $(**)$ and let $a_t = a$ and $b_t = b \neq 0$ be arbitrary constants
- This makes the equation a constant-coefficients one:

$$y_{t+2} + ay_{t+1} + by_t = 0$$

- Note first that if we set $y_t = m^t$ and ignore the trivial case $m = 0$, then we would have $y_{t+1} = m^{t+1} = m^t m$ and $y_{t+2} = m^{t+2} = m^t m^2$
- Plug these expressions in the equation:

$$m^t m^2 + am^t m + bm^t = 0$$

- This can also be written as:

$$m^t (m^2 + am + b) = 0$$

Constant coefficients (2)

- Since we have $m \neq 0$, we can divide both sides by m^t to get:

$$m^2 + am + b = 0$$

- This is the **characteristic equation** of the difference equation
- Its solutions will provide us with the values of m that can be used to construct the general solution of $(**)$

The solutions of the characteristic equation

- Using high-school algebra, we find that the roots of the quadratic equation in our case equal:

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- Having also the necessary knowledge on complex numbers, you already know that the solutions are well defined for all values of the discriminant
- The solutions of the difference equation in the three possible cases for the value of the discriminant $D = a^2 - 4b$ are given in the following Theorem

The solutions of the characteristic equation (2)

Theorem 3

The general solution of $(**)$ *when* $b \neq 0$ *is as follows:*

1. *If* $D > 0$ *(two distinct real roots of the characteristic equation) then:*

$$y_t = Am_1^t + Bm_2^t, \quad m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

2. *If* $D = 0$ *(one double real root) then:*

$$y_t = (A + Bt)m^t, \quad m = -\frac{a}{2}$$

3. *If* $D < 0$ *(two complex conjugate roots) then:*

$$y_t = R^t(A\cos(\theta t) + B\sin(\theta t)), \quad R = \sqrt{b}, \cos(\theta) = -\frac{a}{2\sqrt{b}}, \theta \in [0, \pi]$$

The solutions of the characteristic equation (3)

- First, we note that in order to obtain a particular solution to the difference equation, we need two initial values of y_t
- If those two values are given, say y_0 and y_1, then the constants A and B are uniquely determined
- In the case when the solutions are complex conjugates, the fact that they contain sines and cosines implies that they characterize cyclical behaviour (oscillations)
- The value of the radius (modulus) R determines the type of cyclical fluctuations:
    1. When $|R| < 1$, the oscillations are damped (their amplitude decreases over time)
    2. When $|R| > 1$, the oscillations are explosive (increasing amplitude)
    3. $|R| = 1$, oscillations remain with unchanged amplitude over time

The non-homogeneous case

- General form:

$$y_{t+2} + ay_{t+1} + by_t = c_t, \quad b \neq 0 \quad (\spadesuit)$$

- Recall that according to Theorem 2 the solution is:

$$y_t = Au_t^{(1)} + Bu_t^{(2)} + u_t^*$$

  where u_t^* is a particular solution of ($\spadesuit$)
- It turns out that finding u_t^* is a difficult task even if c_t is a relatively simple function

The non-homogeneous case (2)

- An easier case: $c_t = c$, i.e. a constant
- Then (♠) becomes:

$$y_{t+2} + ay_{t+1} + by_t = c, \quad b \neq 0 \quad (\heartsuit)$$

- So, we try to find a solution of the form: $y_t = C$, where $C = const$
- If $y_t = C$, then $y_{t+1} = y_{t+2} = C$. Substitute all these in $(\heartsuit)$ to get:

$$C + aC + bC = c \Leftrightarrow C(1 + a + b) = c$$

- Therefore, if $1 + a + b \neq 0$:

$$C = \frac{c}{1 + a + b}$$

- Then, the particular solution is:

$$u_t^* = \frac{c}{1 + a + b}$$

The non-homogeneous case (3)

- What if $1 + a + b = 0$?
- Then there is no constant function that can satisfy $(\heartsuit)$
- In such a case we can write $b = -(1 + a)$ and substitute this in $(\heartsuit)$:

$$y_{t+2} + ay_{t+1} - (1 + a)y_t = c$$

- In this case, a constant function would solve only the homogeneous function, so we look for another particular solution

The non-homogeneous case (4)

- Try with $u_t^* = Dt$:

$$\begin{aligned} u_{t+2}^* + a u_{t+1}^* - (1+a) u_t^* &= D(t+2) + aD(t+1) - (1+a)Dt = \\ &= Dt + 2D + aDt + aD - Dt - aDt = \\ &= (a+2)D \end{aligned}$$

- So, if $a \neq -2$, then $D = \dfrac{c}{a+2}$, and the particular solution is:

$$u_t^* = \frac{ct}{a+2}$$

The non-homogeneous case (5)

- Now, what if in addition to $1 + a + b = 0$ we have also $a = -2$? Then $(\heartsuit)$ becomes:

$$y_{t+2} - 2y_{t+1} + y_t = c$$

- We try then to find a solution of the form $u_t^* = Dt^2$. With this, we have:

$$\begin{aligned} u_{t+2}^* - 2u_{t+1}^* + u_t^* &= D(t+2)^2 - 2D(t+1)^2 + Dt^2 = \\ &= Dt^2 + 4Dt + 4D - 2Dt^2 - 4Dt - 2D + Dt^2 = \\ &= 2D \Rightarrow D = \frac{c}{2} \end{aligned}$$

- The particular solution is:

$$u_t^* = \frac{ct^2}{2}$$

Stability of solutions

- Intuitively speaking, a discrete dynamic system is **stable** if whenever small changes are made to the initial conditions, typically around an equilibrium point, the solution remains "near" the equilibrium point
- Otherwise (i.e. when even small changes might lead to large differences in long-term behaviour) the system is **unstable**
- Returning to $(\heartsuit)$, it is called **globally asymptotically stable** if the solution of its associated homogeneous equation tends to 0 as $t \to \infty$

Theorem 4

The equation $(\heartsuit)$ is globally asymptotically stable iff the following two equivalent conditions are satisfied:

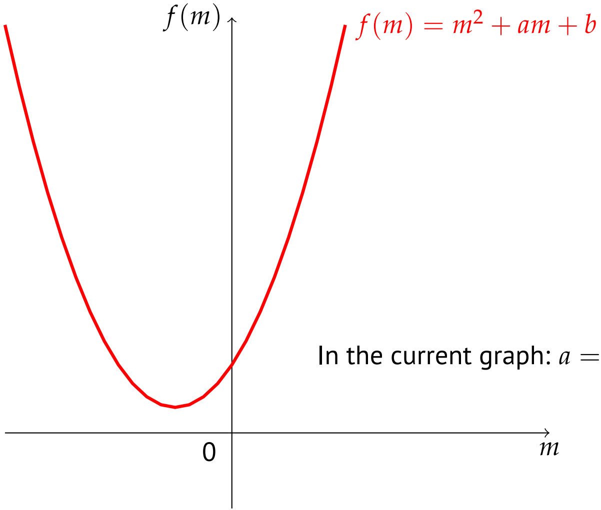
(A) *The moduli of the roots of the characteristic equation $m^2 + am + b = 0$ are strictly less than 1*

(B) $|a| < 1 + b$ *and* $|b| < 1$

Proof of Theorem 4

- (This proof is provided exceptionally because the equivalence of (A) and (B) is not so obvious)
- We will prove first that $(B) \Rightarrow (A)$
- Consider first the case in which the characteristic equation has two complex conjugate roots, i.e. $a^2 - 4b < 0 \Leftrightarrow b > \dfrac{a^2}{4}$
- Note that the latter implies that $b > 0$
- Both roots have moduli equal to $\sqrt{b}$
- If $b < 1$ (and obviously $|a| < 1 + b$), then $\sqrt{b} < 1$. This proves $(B) \Rightarrow (A)$
- Now, in order to prove $(A) \Rightarrow (B)$, look at the following graph

Proof of Theorem 4 (2)



In the current graph: $a = 1.5, b = 0.9$

Proof of Theorem 4 (3)

- From the graph it is visible that the parabola never crosses the horizontal axis (this is the same as the fact that none of the roots is real)
- In other words, no matter what the value of m, we have $f(m) > 0$
- Take m to equal in turns -1 and 1; then:

$$
\begin{aligned}
f(-1) = 1 - a + b > 0 \Rightarrow a < 1 + b \\
f(1) = 1 + a + b > 0 \Rightarrow -a < 1 + b
\end{aligned}
$$

- But these two are equivalent to $|a| < 1 + b$
- From the fact that the moduli of the roots are strictly less than one directly follows that $\sqrt{b} < 1$, and therefore $b < 1$
- The above leads to $(A) \Rightarrow (B)$
- This completes the proof for complex roots

Proof of Theorem 4 (4)

- In the case of real roots, the discriminant is non-negative: $a^2 - 4b \geq 0$
- This is equivalent to $b \leq \frac{a^2}{4}$
- The two roots are:

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- In the real case, that their moduli are strictly less than 1 means that their absolute values should be less than 1
- For m_1 this means (check that the same result is obtained for m_2):

$$-1 < \frac{-a + \sqrt{a^2 - 4b}}{2} < 1 \Rightarrow -2 + a < \sqrt{a^2 - 4b} < 2 + a$$

Proof of Theorem 4 (5)

- Square all parts of the last double inequality to get:

$$a^2 - 4a + 4 < a^2 - 4b < a^2 + 4a + 4,$$

or:

$$-a + 1 < -b < a + 1,$$

or:

$$-a < b + 1 < a,$$

which is the same as $|a| < b + 1$

- The latter can also be obtained from the fact that $f(-1) > 0$ and $f(1) > 0$

Proof of Theorem 4 (6)

- Note also that in those two points the signs of the first derivative of $f(m)$, $f'(m) = 2m + a$, are known:

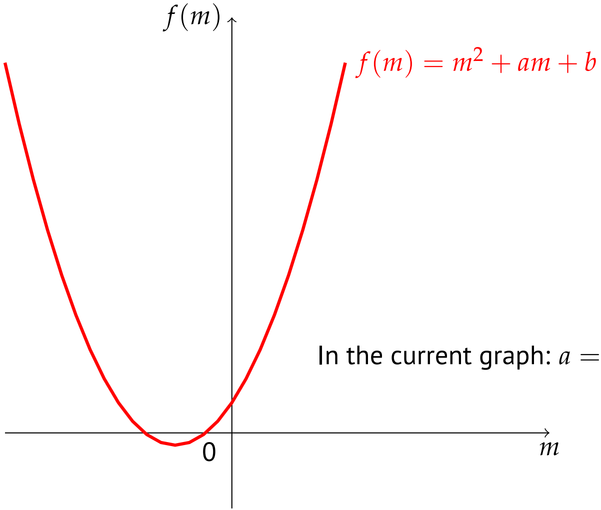
$$f'(-1) = -2 + a < 0$$
$$f'(1) = 2 + a > 0$$

- From the latter follows that $|a| < 2$
- Combine this with $b \leq \dfrac{a^2}{4}$ to find that:

$$b \leq \frac{a^2}{4} < \frac{4}{4} = 1$$

- This proves $(A) \Rightarrow (B)$

Proof of Theorem 4 (7)



In the current graph: $a = 1.5, b = 0.4$

Proof of Theorem 4 (8)

- To prove equivalence in the reverse direction, first note that if $|a| < 1 + b$ and $b < 1$, obviously $|a| < 2$
- The latter is equivalent to $-2 < a < 2$, or $2 + a > 0$ and $-2 + a < 0$
- We can also see that $2 + a$ and $-2 + a$ are the values of $f'(m)$ respectively at 1 and -1
- Using $|a| < 1 + b$, we can consecutively write:

$$\begin{aligned}
-a < 1 + b < a \quad & \Leftrightarrow \quad -a + 1 < -b < a + 1 \Leftrightarrow \\
& \Leftrightarrow \quad -4a + 4 < -4b < 4a + 4 \Leftrightarrow \\
& \Leftrightarrow \quad a^2 - 4a + 4 < a^2 - 4b < a^2 + 4a + 4 \Leftrightarrow \\
& \Leftrightarrow \quad (a - 2)^2 < a^2 - 4b < (a + 2)^2
\end{aligned}$$

- From this point onwards, establishing that the roots of the characteristic equation lie in $(-1, 1)$ is straightforward

Example: The multiplier-accelerator model

- Keynesian business cycle model, due to Samuelson (1939)
- We consider a slightly modified version
- Model equations:

$$\begin{aligned}
C_t &= a + bY_{t-1} \\
I_t &= v(Y_{t-1} - Y_{t-2}) \\
G_t &= \overline{G}, \quad \forall t \\
E_t &= C_t + I_t + G_t \\
Y_t &= E_t
\end{aligned}$$

- Combine all equations to get the following second-order non-homogeneous difference equation:

$$Y_t - (b+v)Y_{t-1} + vY_{t-2} = a + \overline{G}$$

Example: The multiplier-accelerator model (2)

- To find a particular solution, set $Y_t = Y^* = const$, i.e.:

$$Y^* - (b+v)Y^* + vY^* = a + \overline{G}$$

- After rearrangement, we have:

$$Y^* = \frac{a + \overline{G}}{1 - b}$$

- The latter is interpreted in the following way: equilibrium income corresponds to the result from the simple Keynesian multiplier model

Example: The multiplier-accelerator model (3)

- The homogeneous equation that corresponds to this example is:

$$Y_t - (b+v)Y_{t-1} + vY_{t-2} = 0$$

- The roots of its characteristic equation are as follows:

$$m_{1,2} = \frac{(b+v) \pm \sqrt{(b+v)^2 - 4v}}{2}$$

- Three cases emerge again:
  1. Two distinct real roots
  2. One double real root
  3. Two complex conjugate roots

Example: The multiplier-accelerator model (4)

- Case 1: Two distinct real roots, i. e. $(b+v)^2 - 4v > 0$
- In order to be able to analyse the dynamics implied by the difference equation, it is a good idea to use the Vieta formulae which define the relationships between the two roots:

$$\begin{aligned} m_1 + m_2 &= b + v \\ m_1 m_2 &= v \end{aligned}$$

- We can use these two to find that:

$$\begin{aligned} (1-m_1)(1-m_2) &= 1 - m_2 - m_1 + m_1 m_2 = \\ &= 1 - (b+v) + v = \\ &= 1 - b \end{aligned}$$

- As b is interpreted as MPC, $b \in (0,1)$. The latter implies that also $(1-m_1)(1-m_2) \in (0,1)$

Example: The multiplier-accelerator model (5)

- In this case, the general solution is given by:

$$Y_t = Am_1^t + Bm_2^t + Y^*$$

- The larger of the two roots (say this is m_1 in our example) determines the development path of Y_t[1]
- From $b > 0$ and $v > 0$ follows that $m_1 m_2 = v > 0$; this implies that m_1 and m_2 should either be both positive or both negative
- But because of the fact that $m_1 + m_2 = b + v > 0$, the option that the two roots are both negative is ruled out; therefore $m_1 > 0$ and $m_2 > 0$
- This means that Y_t is not characterized with oscillations
- Two possibilities arise with respect to the magnitude of the larger root

[1] This is valid in general for any polynomial: the root having the largest modulus dominates the remaining ones.

Example: The multiplier-accelerator model (6)

- If $m_1 > 1$, then we should also have $m_2 > 1$ (otherwise the condition $(1 - m_1)(1 - m_2) \in (0, 1)$ will be violated)
- With $m_1 > m_2 > 1$, Y_t has an explosive path
- The above also implies that $m_1 m_2 = v > 1$, i. e. the accelerator coefficient is greater than 1
- If $m_1 < 1$, then $0 < m_2 < m_1 < 1$. From this follows first that $v \in (0, 1)$ and second, that the dynamics is damped towards the equilibrium
- Note that the roots cannot equal 1 since otherwise $(1 - m_1)(1 - m_2)$ would not be positive but would also equal zero!

Example: The multiplier-accelerator model (7)

- Case 2: One double real root, i. e. $(b+v)^2 - 4v = 0$
- The root equals:

$$m_{1,2} = \frac{b+v}{2} = m$$

- Since $m^2 = v$, we have $v \geq 0$; but m cannot be zero because $b > 0$, therefore $v > 0$ (although this should be an obvious assumption from the very beginning); finally, this means that $m > 0$
- Again, two possibilities
- First, if $0 < m < 1$, then we should have $0 < v < 1$ and a damped path for income
- Second, if $m > 1$, then $v > 1$ and the dynamics of Y_t is explosive
- By the same reasoning as above, m cannot be equal to one

Example: The multiplier-accelerator model (8)

- Case 3: Two complex conjugate roots, i. e. $(b+v)^2 - 4v < 0$
- The roots equal:

$$\begin{aligned} m_1 &= \alpha + i\beta \\ m_2 &= \alpha - i\beta \end{aligned}$$

where $\alpha = \dfrac{b+v}{2}$ and $\beta = \dfrac{\sqrt{4v-(b+v)^2}}{2}$

- The general solution to the difference equation is:

$$Y_t = R^t[A\cos(\theta t) + B\sin(\theta t)] + Y^*,$$

where $R = \sqrt{\alpha^2+\beta^2} = \sqrt{\dfrac{(b+v)^2+4v-(b+v)^2}{4}} = \sqrt{v}$,
$\cos(\theta) = \dfrac{b+v}{2\sqrt{v}}$, and $\sin(\theta) = \dfrac{\sqrt{4v-(b+v)^2}}{2\sqrt{v}}$

Higher-order equations

Higher-order equations

- In general, we can have pth-order difference equations

$$y_{t+p} = f(t, y_t, y_{t+1}, \ldots, y_{t+p-1}), \quad t = 0, 1, 2, \ldots$$

- In order to have a uniquely defined solution, p initial values are needed
- The general solution of such an equation is a function $y_t = g(t; C_1, \ldots, C_p)$, where $C_1, \ldots, C_p$ are arbitrary constants
- For each given set of values of $C_1, \ldots, C_p$, we can obtain the corresponding solution of the equation

Higher-order equations: The linear case

Theorem 5

The pth-order linear homogeneous difference equation:

$$y_{t+p} + a_{1,t}\, y_{t+p-1} + \ldots + a_{p-1,t}\, y_{t+1} + a_{p,t}\, y_t = 0, \quad a_{p,t} \neq 0$$

has the following solution:

$$y_t = C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)}$$

where $u_t^{(1)}, \ldots, u_t^{(p)}$ are p linearly independent solutions of the equation, and $C_1, \ldots, C_p$ are arbitrary constants.

Higher-order equations: The linear case (2)

Theorem 6

The pth-order linear non-homogeneous difference equation:

$$y_{t+p} + a_{1,t}\, y_{t+p-1} + \ldots + a_{p-1,t}\, y_{t+1} + a_{p,t}\, y_t = b_t, \quad a_{p,t} \neq 0$$

has the following solution:

$$y_t = C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)} + u_t^*$$

where $C_1 u_t^{(1)} + \ldots + C_p u_t^{(p)}$ is the general solution of the homogeneous equation, and u_t^* is a particular solution of the non-homogeneous equation

Linear higher-order equations with constant coefficients

- Homogeneous case:

$$y_{t+p} + a_1 y_{t+p-1} + \ldots + a_{p-1} y_{t+1} + a_p y_t = 0, \quad t = 0, 1, 2, \ldots$$

- Non-homogeneous case:

$$y_{t+p} + a_1 y_{t+p-1} + \ldots + a_{p-1} y_{t+1} + a_p y_t = b_t, \quad t = 0, 1, 2, \ldots$$

- Solutions having the form $y_t = m^t$ are sought, as we did in the second-order equations case
- This leads to the following characteristic equation:

$$m^p + a_1 m^{p-1} + \ldots + a_{p-1} m + a_p = 0$$

Linear higher-order equations with constant coefficients (2)

- This is a polynomial equation, and it has as many roots as the degree of the polynomial (p in the current case)
- Those p roots can either be different, or there could be multiple roots (as with double roots in second-order equations)
- Also, there could be complex roots, and they always come in pairs of conjugates

Linear higher-order equations with constant coefficients (3)

The general rules that are followed in finding the roots are as follows:

1. If a real root m_i is unique, then it provides one solution m_i^t
2. If a real root m_j is repeated k times, then it provides k solutions $m_j^t, tm_j^t, t^2m_j^t, \ldots, t^{k-1}m_j^t$
3. A pair of complex conjugates $\alpha \pm i\beta$ which is encountered only once among the list of roots provides two solutions: $R^t \cos(\theta t)$ and $R^t \sin(\theta t)$
4. A pair of complex conjugates $\alpha \pm i\beta$ which is encountered l times among the list of root provides $2l$ solutions: $u, v, tu, tv, , \ldots, t^{l-1}u, t^{l-1}v$, where $u = R^t \cos(\theta t)$ and $v = R^t \sin(\theta t)$

For the non-homogeneous equation, a particular solution u_t^* needs also to be found.

Stability conditions for higher-order equations

Theorem 7

A necessary and sufficient condition for the pth-order linear difference equation to be globally asymptotically stable is that the roots of the characteristic polynomial all lie within the unit circle, i.e. all have moduli less than 1.

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