ODE_Problems

January 30, 2019

1 Problems solved in Sydsaeter et. al (2008), Chapters 5 and 6

1.1 Load necessary Python stuff

```
In [5]: import numpy as np
        from matplotlib import pyplot as plt
        import sympy as sym
        sym.init_printing(use_unicode=True)
```

1.2 Example 1 (p. 195)

Solve the differential equation

$$\frac{dx}{dt} = -2tx^2$$

and find the integral curve that passes through $(t, x) = (1, -1)$.

1.2.1 Solution (analytical):

This is a separable equation so

$$-\frac{dx}{x^2} = 2t\,dt$$

Integrate:

$$-\int \frac{dx}{x^2} = \int 2t\,dt$$

$$\frac{1}{x} = t^2 + C \quad \Rightarrow \quad x = \frac{1}{t^2 + C}$$

The curve that passes through $(1, -1)$:

$$-1 = \frac{1}{1^2 + C} \quad \Rightarrow \quad C = -2$$

1.2.2 Solution (using Sympy):

```
In [2]: t = sym.symbols('t')
        x = sym.symbols('x', cls=sym.Function)
        sol = sym.dsolve(x(t).diff(t) + 2*t*pow(x(t),2), x(t))
        sol
```

Out[2]:

$$x(t) = \frac{1}{C_1 + t^2}$$

```
In [3]: C1 = sym.symbols('C1')
        constant = sym.solve([sol.args[1].subs(t,1) + 1], C1)
        constant
```

Out[3]:

$$\{C_1 : -2\}$$

```
In [4]: final_sol = sol.subs(constant)
        final_sol
```

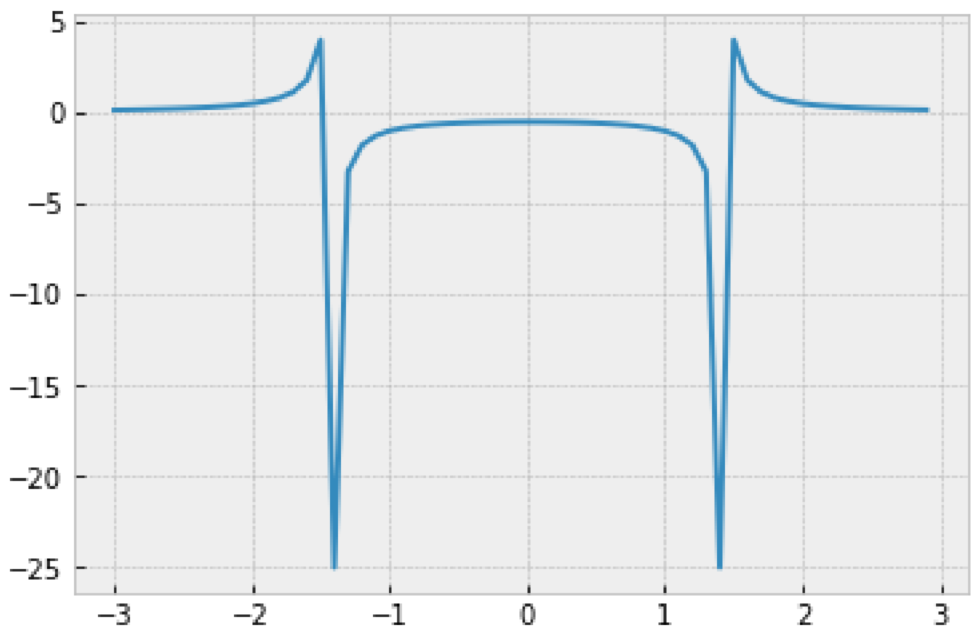
Out[4]:

$$x(t) = \frac{1}{t^2 - 2}$$

1.2.3 Plot solution

```
In [6]: func = sym.lambdify(t, final_sol.rhs,'numpy')
        xvals = np.arange(-3,3,0.1)
        yvals = func(xvals)

        with plt.style.context('bmh'):
            plt.figure()
            plt.plot(xvals, yvals)
```



1.2.4 Solve the problem

$$\dot{x}(t) + 5x(t) = 11t, \quad x(0) = 3$$

In [19]:
```
t = sym.symbols('t')
x = sym.symbols('x', cls=sym.Function)
sol = sym.dsolve(x(t).diff(t) + 5*x(t) - 11*t, x(t))
sol
```

Out[19]:

$$x(t) = \left(C_1 + \frac{11\,(5t-1)\,e^{5t}}{25}\right)e^{-5t}$$

In [20]:
```
C1 = sym.symbols('C1')
constant = sym.solve([sol.args[1].subs(t,0) - 3], C1)
constant
```

Out[20]:

$$\left\{C_1 : \frac{86}{25}\right\}$$

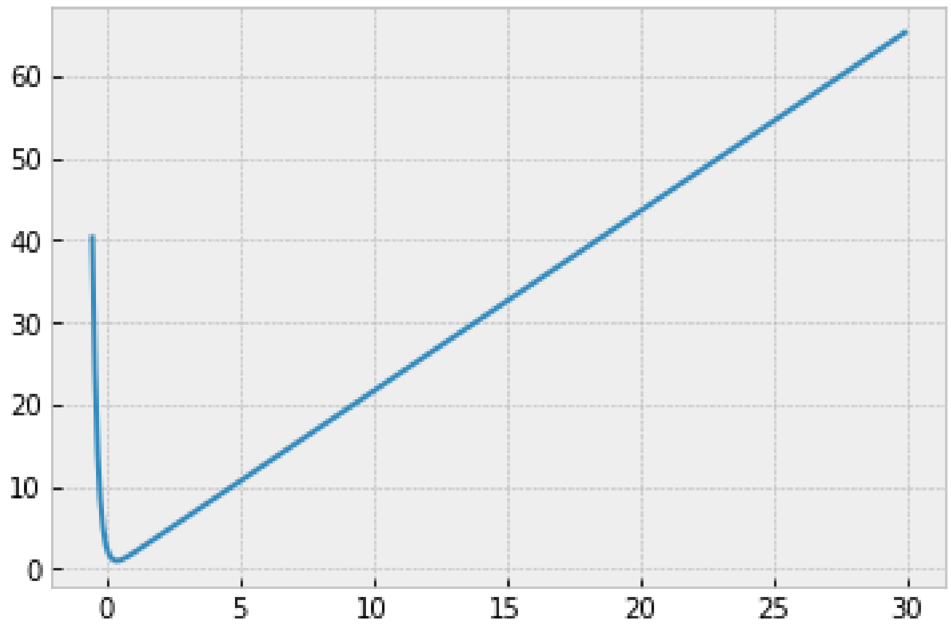
In [21]:
```
final_sol = sol.subs(constant)
final_sol
```

Out[21]:

$$x(t) = \left(\frac{11\,(5t-1)\,e^{5t}}{25} + \frac{86}{25}\right)e^{-5t}$$

In [24]:
```
func = sym.lambdify(t, final_sol.rhs,'numpy')
xvals = np.arange(-0.5,30,0.1)
yvals = func(xvals)

with plt.style.context('bmh'):
    plt.figure()
    plt.plot(xvals, yvals)
```



1.3 Second-order differential equations

Solve the problem

$$\ddot{x}(t) = 5$$

1.3.1 Solution ("pen and paper")

Integrate once

$$\int \ddot{x}(t)\,dt = \dot{x}(t) = 5t + C_1$$

Integrate once more

$$\int \dot{x}(t) = x(t) = 5\frac{t^2}{2} + C_1 t + C_2$$

1.3.2 Solution (Sympy)

```
In [6]: t = sym.symbols('t')
        x = sym.symbols('x', cls=sym.Function)
        sol = sym.dsolve(x(t).diff(t,2) - 5, x(t))
        sol
```

Out[6]:

$$x(t) = C_1 + C_2 t + \frac{5t^2}{2}$$

1.3.3 Example: x missing (Example 2 on p. 224)

Solve

$$\ddot{x} = \dot{x} + t$$

1.3.4 Solution

Set $\dot{x} = u$. The equation becomes

$$\dot{u} = u + t$$

The solution is (check):

$$u(t) = C_1 e^t - t - 1$$

Now

$$\dot{x} = C_1 e^t - t - 1$$

Integrate:

$$\int \dot{x}\, dx = x(t) = C_1 e^t - \frac{t^2}{2} - t + C_2$$

In [7]:
```
# Sympy
t = sym.symbols('t')
x = sym.symbols('x', cls=sym.Function)
sol = sym.dsolve(x(t).diff(t,2) - x(t).diff(t) - t, x(t))
sol
```

Out[7]:

$$x(t) = C_1 + C_2 e^t - \frac{t^2}{2} - t$$

1.4 Linear second-order ODE

1.4.1 Homogeneous case

Solve

$$\ddot{x} - 7\dot{x} + 12x = 0$$

1.4.2 Solution

Characteristic equation:

$$r^2 - 7r + 12 = 0$$

Discriminant: $D = 49 - 48 = 1; \quad \sqrt{D} = 1$ Roots:

$$r_{1,2} = \frac{7 \pm 1}{2}$$

ODE Solution:

$$x(t) = C_1 e^{4t} + C_2 e^{3t}$$

In [14]:
```python
# Sympy
t = sym.symbols('t')
x = sym.symbols('x', cls=sym.Function)
sol = sym.dsolve(x(t).diff(t,2) - 7*x(t).diff(t) + 12*x(t), x(t))
sol
```

Out[14]:

$$x(t) = \left(C_1 + C_2 e^{t}\right) e^{3t}$$

1.4.3 Still the homogeneous case

Solve

$$\ddot{x} - 6\dot{x} + 12x = 0$$

1.4.4 Solution

Characteristic equation:

$$r^2 - 6r + 12 = 0$$

Discriminant: $D = 36 - 48 = -12 < 0;\quad \sqrt{D} = -2i\sqrt{3}$ Roots:

$$r_{1,2} = \frac{6 \pm 2i\sqrt{3}}{2}$$

$\alpha = -\frac{-6}{2} = 3; \beta = \sqrt{12 - \frac{(-6)^2}{4}} = \sqrt{3}$
ODE Solution:

$$x(t) = e^{3t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right)$$

In [18]:
```python
# Sympy
t = sym.symbols('t')
x = sym.symbols('x', cls=sym.Function)
sol = sym.dsolve(x(t).diff(t,2) - 6*x(t).diff(t) + 12*x(t), x(t))
sol
```

Out[18]:

$$x(t) = \left(C_1 \sin\left(\sqrt{3}t\right) + C_2 \cos\left(\sqrt{3}t\right)\right) e^{3t}$$