R406: Applied Economic Modelling with Python

First-order Difference Equations

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Lecture Contents

Introduction

Discrete Time

- In many cases a variable of interest can be determined from its past values (as well as potentially other quantities)
- Examples
  - The end-period balance of a bank account can be computed from the starting balance plus deposits/withdrawals and interest
  - The stock of capital at a point in time is determined by the stock from the previous period, investment and depreciation
- Here we consider the case of discrete time: t takes only integer values:

$$t = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

What Are Difference Equations?

- In general a difference equation can be stated as a function relating the variable to its own past values
- Formally written,

$$y_{t+1} = f(y_t, y_{t-1}, \ldots)$$

- We call this a *recursive relationship*
- Every next value is obtained from this relationship; for example, in the relationship $y_{t+1} = f(y_t)$:

$$y_{t+2} = f(y_{t+1}) = f(f(y_t)) = f^2(y_t)$$

- From this follows:

$$y_{t+k} = f^k(y_t)$$

Note: k is not interpreted as a power!

What Are Difference Equations? (2)

- Sometimes difference equations are written in more general, **implicit** form as
$$F(y_{t+1}, y_t, y_{t-1}, \ldots) = 0.$$
We shall ignore such cases and work with **explicit** difference equations, i.e. ones that can be solved for the variable with the largest index
- The order of a difference equation is given by the difference between the largest and the smallest time indexes
- For example, the equation
$$y_{t+1} = f(y_t, y_{t-1}, y_{t-2})$$
is a third-order equation because $(t+1) - (t-2) = 3$

What Are Difference Equations? (3)

- A difference equation, e.g.
$$y_{t+1} = f(y_t),$$
can be re-written as
$$y_{t+1} - y_t = f(y_t) - y_t.$$
- We can define the (first) difference as $\Delta y_t := y_t - y_{t-1}$ and a new right-hand side $g(y) := f(y) - y$
- The above equation can then be written as
$$\Delta y_{t+1} = g(y_t),$$
hence the name *difference equation*
- While you may not see immediate benefits from writing the equation to feature an explicit difference operator Δ, this form becomes useful to motivate the transition to differential equations

Linear vs. Non-linear Equations

- Recursive relationships can be either linear or non-linear
- Therefore linear and non-linear difference equations
- (Note again that time and/or other variables could also appear as function arguments)
- We will stick most of the time to linear equations
- Linearity will be directly assumed, while non-linearity will be explicitly indicated when necessary

Constant-coefficients vs. Variable-coefficients Equations

- Take the equation:

$$y_{t+3} = ay_{t+2} + by_{t+1} + cy_t + d$$

  where a, b, c, d are constants
  This is a constant-coefficients difference equation

- Compare this to:

$$y_{t+3} = a_{t+2}y_{t+2} + b_{t+1}y_{t+1} + c_t y_t + d_t$$

  This is a variable-coefficients difference equation

Autonomous vs. Non-autonomous Difference Equations

- Autonomous (or time-invariant) equations: time does not enter independently as an argument of $f(\cdot)$; example:

$$y_t = 0.4y_{t-1} + 2$$

- Non-autonomous (or time-variant) equations: time enters independently as an argument:

$$y_t = 0.4y_{t-1} + 3t + 2$$

- The latter are much more difficult to study, and we will only occasionally mention them

Homogeneous vs. Non-homogeneous Equations

- Take the following equation where all instances of the y variable are on the left-hand side, and 'all the rest' is summarized on the right-hand side in $g(t)$:

$$y_{t+2} + ay_{t+1} + by_t = g(t)$$

- (A second-order equation in this case, but order here does not matter)
- If $g(t) = 0$, the equation is homogeneous[1]
- If $g(t) \neq 0$, the equation is non-homogeneous

[1] For such equations, if y_t is a solution, so is αy_t.

The Initial-value Problem

The Initial-value Problem

- Take the equation:

$$y_t = f(y_{t-1}, y_{t-2}, \dots, y_{t-p})$$

- In order to be able to solve for y_t, p starting values will be needed
- Otherwise it would be impossible to find the solution
- In the case of the first-order equation, one such value is needed

Equilibrium and Stability Conditions

Equilibrium

- We begin with the autonomous first-order-equation case so that the idea is grasped more easily
- We will consider stability conditions again when we get to higher-order equations
- Start from the definition of a function's fixed point:

Definition 1

Let $K \subset \mathbb{R}^n$ and let $\mathbf{f}$ be a function that maps $\mathbf{x} \in K$ into another point $\mathbf{f}(\mathbf{x}) \in K$ (i.e. the function maps K into itself). If the point $\mathbf{x}^*$ is such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$, then $\mathbf{x}^*$ is called a **fixed (or equilibrium) point**.

Equilibrium (2)

- In the current context K consists of all y_t's
- For example, in the difference equation $y_{t+1} = f(y_t)$, f maps y_t into y_{t+1}
- The definition then implies that y^* is an equilibrium point if and only if (iff):

$$y^* = f(y^*)$$

- Graphically, this means that equilibrium points always lie on the 45° line
- Example: Find the equilibrium points of $f(x) = x^3$; make a graphical sketch of your solution

Stability of Equilibrium

Let y^* be an equilibrium point of f. Some definitions follow.

Definition 2

The point y^* is **stable** if for any $\varepsilon > 0 \, \exists \, \delta > 0$:

$$|y_0 - y^*| < \delta \Rightarrow |f^n(y_0) - y^*| < \varepsilon, \, \forall n > 0.$$

Otherwise, y^* is **unstable**.

Definition 3

The point y^* is **repelling** if $\exists \, \varepsilon > 0$:

$$0 < |y_0 - y^*| < \varepsilon \Rightarrow |f(y_0) - y^*| > |y_0 - y^*|.$$

Stability of Equilibrium (2)

Definition 4

The point y^* is **attracting (asymptotically stable)** if:

1. It is stable
2. $\exists\, \eta > 0$:

$$|y_0 - y^*| < \eta \Rightarrow \lim_{t \to \infty} y_t = y^*.$$

If $\eta = \infty$, then y^* is **globally asymptotically stable**.

Stability of Equilibrium (3)

Theorem 1

Let y^* be a fixed point of $y_{t+1} = f(y_t)$, and let f be continuously differentiable at y^*. Then:

1. *If $|f'(y^*)| < 1$, then y^* is an attractor*
2. *If $|f'(y^*)| > 1$, then y^* is a repeller*
3. *If $|f'(y^*)| = 1$ and:*
    1. *if $f''(y^*) \neq 0$, then y^* is unstable*
    2. *if $f''(y^*) = 0$ and if $f'''(y^*) > 0$, then y^* is unstable*
    3. *if $f''(y^*) = 0$ and if $f'''(y^*) < 0$, then y^* is an attractor*
4. *If $f'(y^*) = -1$ and:*
    1. *if $-2f'''(y^*) - 3[f''(y^*)]^2 < 0$, then y^* is an attractor*
    2. *if $-2f'''(y^*) - 3[f''(y^*)]^2 > 0$, then y^* is unstable*

Note: Theorems are provided just for usage and reference, no proofs are given or required!

Stability of Equilibrium (4)

- There exist fixed points which are neither attractors, nor repellers
- Take for example:

$$y_{t+1} = -y_t + k$$

- If y_0 is the initial value, then

$$\begin{aligned} y_0 &= y_2 = y_4 = \dots \\ y_1 &= y_3 = y_5 = \dots \end{aligned}$$

- Although there is a fixed point $y^* = \frac{k}{2}$, if $y_0 \neq y^*$, the system cycles between y_0 and $-y_0 + k$ without getting nearer or farther from the fixed point over time

Stability of Equilibrium (5)

- A solution y_n is periodic if

$$y_{n+m} = y_n$$

for some fixed integer m and all n. The smallest integer m is called **the period of the solution**.

Definition 5

If the sequence $\{y_t\}$ has q repeating values, then $y_1, y_2, \ldots, y_q$ are called **period points**, and the set $\{y_1, y_2, \ldots, y_q\}$ is called a **periodic orbit**.

- For example, if the equation is $y_{t+1} = f(y_t)$, a k-periodic point is the y-coordinate of the intersection point of $f^k(y)$ and the 45°-line $y_{t+1} = y_t$

Stability of Equilibrium (6)

Theorem 2

Let b be a k-period point of f. Then b is:

1. *Stable if it is a stable fixed point of f^k*
2. *Asymptotically stable if it is an attracting point of f^k*
3. *Repelling if it is a repelling fixed point of f^k*

First-order Difference Equations

First-order Difference Equations: Problem Setup

- A first-order difference equation has the form:

$$y_{t+1} = f(y_t)$$

- Initial-value problem:

$$\left| \begin{array}{l} y_{t+1} = f(y_t) \\ y(0) = y_0 \end{array} \right.$$

  where y_0 is given initially
- It is easy to see that $y_1 = f(y_0)$, $y_2 = f(y_1)$, $y_3 = f(y_2)$, etc.
- In other words, it implies the following sequence of values:

$$y_0, f(y_0), f(f(y_0)), f(f(f(y_0))), \ldots$$

- As already noted, sometimes this is denoted more conveniently as follows:

$$y_0, f(y_0), f^2(y_0), f^3(y_0), \ldots$$

Solution

- Start with the linear non-homogeneous, constant-coefficients case:

$$\left| \begin{array}{l} y_{t+1} = ay_t + b \\ y(0) = y_0 \end{array} \right.$$

- Beginning with y_0, the next values are obtained through iteration:

$$\begin{array}{l} y_0 \\ y_1 = ay_0 + b \\ y_2 = ay_1 + b = a^2 y_0 + ab + b \\ \dots \\ y_t = a^t y_0 + b(a^{t-1} + a^{t-2} + \ldots + a + 1) \end{array}$$

Solution (2)

- We have the sum of a geometric series in the parentheses
- Using the standard formula, this sum equals:

$$\begin{cases} \dfrac{1-a^t}{1-a}, & a \neq 1 \\ t, & a = 1 \end{cases}$$

- Let $a \neq 1$. Then the solution to the equation is:

$$y_t = a^t \left(y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

- In the other case ($a = 1$), the solution is:

$$y_t = y_0 + tb$$

Solution (3)

- To find the equilibrium (fixed) point of this equation, we seek for a solution y^* such that $y_t = y^*$, $\forall t$ (i.e. if we start at this point, we stay there at all times)
- This means that it should satisfy in particular the following: $y_{t+1} = y_t = y^*$
- Plug this into the equation:

$$y^* = ay^* + b$$

- It follows that the fixed point is:

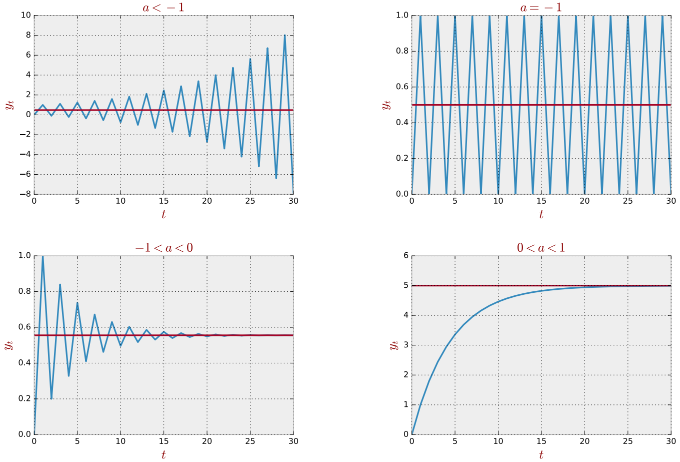
$$y^* = \frac{b}{1-a}, a \neq 1$$

- It is directly obvious that in the homogeneous case $y^* = 0$

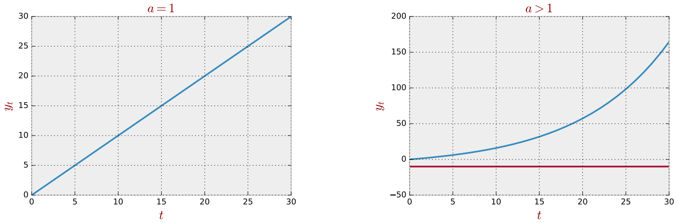
Fixed-point Stability

- Using Theorem 1 and Theorem 2, we can obtain the stability conditions very easily
- The first derivative of f with respect to y_t equals a; the higher-order derivatives are all zero
- Therefore:
  1. If $|a| < 1$, the equilibrium is stable
  2. If $|a| > 1$, the equilibrium is unstable
  3. If $a = 1$, the fixed point is undefined
  4. If $a = -1$, the fixed point equals $\frac{b}{2}$ but it is neither stable nor unstable; the solution is periodic and the system cycles between y_0 and $-y_0 + b$

Fixed-point Stability (2)



Fixed-point Stability (3)



Time-varying b

- Let b_t be an arbitrary but known function of t (i.e. the values of b_t are given beforehand)
- The equation becomes:

$$y_{t+1} = ay_t + b_t, \quad t = 0, 1, 2, \ldots$$

- Starting from a given y_0, recursive substitutions leads eventually to:

$$y_t = a^t y_0 + \sum_{k=0}^{t-1} a^{t-k-1} b_k, \quad t = 0, 1, 2, \ldots$$

- Stability conditions are much harder to specify
- However, equations of this kind are successfully used to describe special types of time series data
- Stability conditions are derived after plausible assumptions on b_t are made

Time-varying a

- If, in addition to b, the coefficient a also changes over time, then the equation becomes:

$$y_{t+1} = a_t y_t + b_t, \quad t = 0, 1, 2, \ldots$$

- The same line of reasoning as before can be applied finally leading to:

$$y_t = \left(\prod_{k=0}^{t-1} a_k \right) y_0 + \sum_{i=0}^{t-1} \left(\prod_{k=i+1}^{t-1} a_k \right) b_i$$

where all terms $\prod_{k=t}^{t-1} a_k$ are set to equal 1.

Example 1: Mortgage Repayment

- An amount equal to K is borrowed at time 0 to finance a mortgage at the fixed monthly interest rate r
- The mortgage is repaid monthly at amounts equal to a, and the number of monthly payments until full repayment equals T
- In each period the outstanding principal b_t satisfies the following relationship:

$$b_{t+1} = (1 + r)b_t - a$$

- Additionally, it is obvious from the problem that $b_0 = K$ and $b_T = 0$

Example 1: Mortgage Repayment (2)

- Assuming that $r > 0$, the fixed point is:

$$b^* = -\frac{a}{1-(1+r)} = \frac{a}{r}$$

- Therefore, the solution is:

$$b_t = (1+r)^t \left(b_0 - \frac{a}{r}\right) + \frac{a}{r}$$

or:

$$b_t = (1+r)^t \left(K - \frac{a}{r}\right) + \frac{a}{r}$$

Example 1: Mortgage Repayment (3)

- For $t = T$, the solution is:

$$0 = (1+r)^T \left(K - \frac{a}{r}\right) + \frac{a}{r}$$

- If we solve the latter for K, we get:

$$K = \frac{a}{r}\left(1 - \frac{1}{(1+r)^T}\right) = a\sum_{t=1}^{T} \frac{1}{(1+r)^t}$$

i.e. the borrowed amount equals the sum of PDVs of all monthly payments until full repayment

Example 1: Mortgage Repayment (4)

- We could also solve for a:

$$a = \frac{rK}{1-(1+r)^{-T}} = \frac{rK(1+r)^T}{(1+r)^T-1}$$

or, in other words, this result allows to calculate the monthly mortgage payment given the borrowed amount and the interest rate

Example 2: Compound Interest and PDVs with Constant Interest Rates

- Let w_t denote a person's wealth (assets) at time t
- If y_t and c_t are respectively this person's income and consumption
- Given a constant interest rate r, we have:

$$w_{t+1} = (1+r)w_t + y_{t+1} - c_{t+1}$$

- If initial wealth is w_0, then the solution is:

$$w_t = (1+r)^t w_0 + \sum_{k=0}^{t-1} (1+r)^{t-k-1} (y_{k+1} - c_{k+1})$$

Example 2: Compound Interest and PDVs with Constant Interest Rates (2)

- Divide both sides by $(1+r)^t$ to get:

$$(1+r)^{-t}w_t = w_0 + \sum_{k=0}^{t-1}(1+r)^{-k-1}(y_{k+1} - c_{k+1})$$

- The left-hand side is the PDV of w_t at time 0
- It equals the initial wealth plus the PDV of the flow of savings from time 1 to time t

Example 3: The Harrod-Domar Growth Model

- We consider the discrete-time version
- Saving is a fixed share of output/income:

$$S_t = sY_t, \quad 0 < s < 1$$

- Investment depends on output growth:

$$I_t = v(Y_t - Y_{t-1}), \quad v > 0$$

- Saving equals investment:

$$S_t = I_t$$

Example 3: The Harrod-Domar Growth Model (2)

- Substituting the first two equations into the third yields:

$$sY_t = v(Y_t - Y_{t-1}) \Rightarrow Y_t = \frac{v}{v-s} Y_{t-1}$$

- Solving the equation leads to:

$$Y_t = \left(\frac{v}{v-s} \right)^t Y_0,$$

  and the only fixed point is 0

- There are several possibilities for the relationship between s and v, leading to different conclusions on the stability of the system

Example 4: Hog Cycles and Cobwebs

- Based on the fact that for various commodities output and prices exhibit cyclical behaviour
- Cycles stem from the consecutive adjustments, which are manifested as movements between the supply and demand curve
- Also, from the technological perspective, there is a span of time required in order to produce a certain output (e.g. in agriculture)
- Generally speaking, the “cobweb phenomenon” stems from the presence of lagged effects on some economic variables
- Also, there is a mismatch between the time of the production decision and the time of purchase decisions by customers (demand)

Example 4: Hog Cycles and Cobwebs (2)

- Initially, the system (the market) is in equilibrium
- Each cycle begins with a demand or with a supply shock
- In this version of the model, only one-period lags matter
- Supply, demand, and equilibrium are described via the following three equations:

$$q_t^D = \alpha - \beta p_t$$

$$q_t^S = \gamma + \delta p_{t-1}$$

$$q_t^D \equiv q_t^S$$

where $\alpha > 0$, $\beta > 0$, and $\delta > 0$.

Example 4: Hog Cycles and Cobwebs (3)

- Substituting supply and demand in the equilibrium condition leads to the following difference equation:

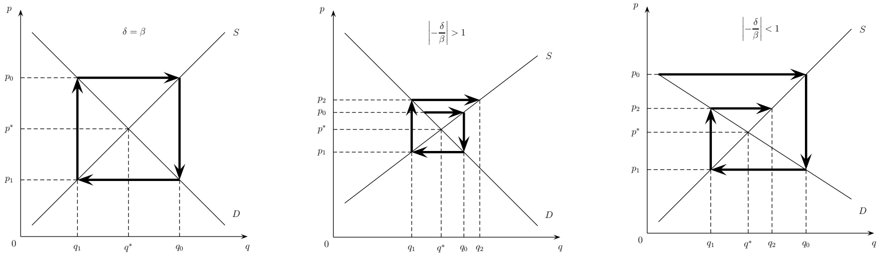
$$p_t = \frac{\alpha - \gamma}{\beta} - \frac{\delta}{\beta} p_{t-1},$$

- Its solution is:

$$p_t - p^* = \left(-\frac{\delta}{\beta}\right)^t (p_0 - p^*)$$

  where p^* is the equilibrium price
- There are three possible system behaviours depending on the values of δ and β

Example 4: Hog Cycles and Cobwebs (4)



References

- Sydsæter et al. (2008), ch. 11
- Chiang and Wainwright (2005), ch. 17
- Shone (2002), ch. 3