R406: Applied Economic Modelling with Python

Unconstrained Optimization. Static Optimization with Equality Constraints. Lagrange Multipliers

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Lecture Contents

Warm-up: Basic Unconstrained Optimization in $\mathbb{R}^1$

Warm-up: Basic Unconstrained Optimization in $\mathbb{R}^1$

Fact 1

For a function $f : \mathbb{R} \to \mathbb{R}$ differentiable at a point x, a necessary condition for a local extreme point (i.e. a maximum or a minimum) at x is

$$f'(x) = 0.$$

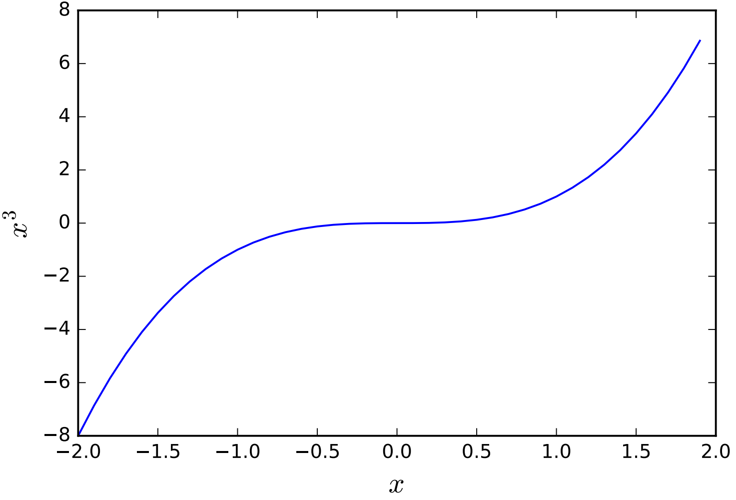
Example 1

If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ and the condition $f'(x) = 0$ yields the familiar $x = -\frac{b}{2a}$ (recall your high-school days). Depending on the sign of a, this is a maximum or a minimum (What is the relationship?).

Example 2

If $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(x) = 0 \Rightarrow x = 0$.
Does the function attain a maximum or a minimum at $x = 0$?

Warm-up: Basic Unconstrained Optimization in $\mathbb{R}^1$



Warm-up: Basic Unconstrained Optimization in $\mathbb{R}^1$

Example 3 (cont.)

The answer is "neither"! The point $x = 0$ is not a local extreme point of $f(x) = x^3$.

This illustrates the pitfalls of using necessary conditions – they supply only candidates that need to be checked further.

The above examples generalize in the following manner:

Fact 2

Let a function f be n times differentiable at a point x and

$$f'(x) = f''(x) = \ldots = f^{(n-1)}(x) = 0, \qquad f^{(n)} \neq 0.$$

1. *If n is odd, the point x is not an extreme point of $f(x)$.*
2. *If n is even and $f^{(n)}(x) > 0$, the point x is a minimum.*
3. *If n is even and $f^{(n)}(x) < 0$, the point x is a maximum.*

Unconstrained Optimization in $\mathbb{R}^n$

Unconstrained Optimization in $\mathbb{R}^n$

Necessary conditions

Fact 3

For a function $f : \mathbb{R}^n \to \mathbb{R}$, differentiable at a point $\mathbf{x}$, a necessary condition for $\mathbf{x}$ to be a local extreme point is

$$f'(\mathbf{x}) = \mathbf{0},$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } f'(\mathbf{x}) = \begin{pmatrix} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x_1, \ldots, x_n)}{\partial x_n} \end{pmatrix} (= \nabla f(\mathbf{x}))$$

Note: A point where the gradient of a function f vanishes is called a *critical point* or a *stationary point*. This also applies to functions on $\mathbb{R}^1$.

Unconstrained Optimization in $\mathbb{R}^n$

Example 4

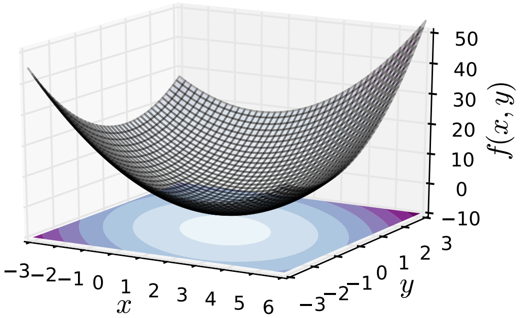
$$f(x,y) = x^2 + 2y^2 - 3x + xy$$

$$\frac{\partial f}{\partial x} = 2x - 3 + y = 0 \quad \Rightarrow \quad x = \frac{3-y}{2}$$

$$\frac{\partial f}{\partial y} = 4y + x = 0 \quad \Rightarrow \quad y = -\frac{x}{4}$$

$$x = \frac{12}{7},\ y = -\frac{3}{7}$$

Unconstrained Optimization in $\mathbb{R}^n$



Unconstrained Optimization in $\mathbb{R}^n$

The necessity of the condition $f'(\mathbf{x}) = \mathbf{0}$ has implications that are similar to the univariate case:

Example 5

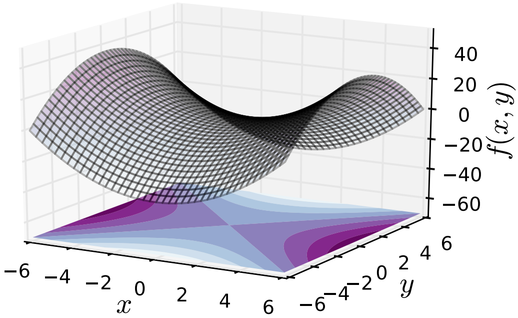
Consider the function $f(x, y) = x^2 - y^2$. The NCs yield the following candidate:

$$\frac{\partial f}{\partial x} = 2x = 0 \quad \Rightarrow \quad x = 0,$$

$$\frac{\partial f}{\partial y} = -2y = 0 \quad \Rightarrow \quad y = 0.$$

Let's look at the graph of the function in a neighbourhood of the point $(0, 0)'$.

Unconstrained Optimization in $\mathbb{R}^n$



Unconstrained Optimization in $\mathbb{R}^n$

Example 6 (cont.)

The critical point $\mathbf{x} = (0, 0)'$ is an example of a *saddle point*. The function f (obviously) does not attain an extremum at $\mathbf{x}$.

Example 5 illustrates the need to refine the approach for checking candidate points in the n-dimensional case. To this end, we have to review several concepts.

A symmetric square matrix A is called *positive semidefinite* if, for any vector $\mathbf{x}$, we have

$$\mathbf{x}' A \mathbf{x} \geq 0.$$

If the inequality is strict for any non-zero vector $\mathbf{x}$, the matrix is called *positive definite*.
Similarly, a symmetric square matrix A is called *negative semidefinite* if, for any vector $\mathbf{x}$, we have $\mathbf{x}' A \mathbf{x} \leq 0$, and *negative definite* in case of strict inequality for $\mathbf{x} \neq \mathbf{0}$.

Unconstrained Optimization in $\mathbb{R}^n$

Incidentally, for a given square symmetric matrix A, the function $Q(\mathbf{x}) = \mathbf{x}' A\mathbf{x}$ is called a *quadratic form*. Quadratic forms are also referred to as "positive/negative (semi)definite", depending on the properties of the respective matrix.

Recall that, for an $n \times n$ matrix A, a *principal minor* of order k ($1 \leq k \leq n$), denoted by Δ_k, is the determinant of the submatrix obtained by deleting $n - k$ rows of the matrix and the correspondingly numbered columns, e.g.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

Note: The notation Δ_k does not identify a unique principal minor of order k.

Unconstrained Optimization in $\mathbb{R}^n$

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Note: The notation Δ_k does not identify a unique principal minor of order k.

Unconstrained Optimization in $\mathbb{R}^n$

The k-th *leading principal minor* of a matrix A ($1 \leq k \leq n$), denoted by D_k, is the determinant of the submatrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,k} \end{pmatrix},$$

i.e. the principal minor obtained by deleting the last $n-k$ rows and columns and, respectively, keeping the first k.

Unconstrained Optimization in $\mathbb{R}^n$

Fact 4 (Sylvester's criterion)

Let A be a symmetric matrix. Then:

1. *A is positive definite if and only if $D_k > 0,\ k = 1, \ldots, n$.*
2. *A is positive semidefinite if and only if $\Delta_k \geq 0$ for all principal minors of order $k = 1, \ldots, n$.*
3. *A is negative definite if and only if $(-1)^k D_k > 0,\ k = 1, \ldots, n$.*
4. *A is negative semidefinite if and only if $(-1)^k \Delta_k \geq 0$ for all principal minors of order $k = 1, \ldots, n$.*

Note that the necessary and sufficient conditions for “semidefiniteness” involve all principal minors (and hence are cumbersome to check), not just the leading principal minors.

Unconstrained Optimization in $\mathbb{R}^n$

Let a function $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ be twice differentiable. The matrix of second partial derivatives, evaluated at a point $\mathbf{x}$, i.e.

$$\begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \ldots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \ldots & \ldots & \ddots & \ldots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \ldots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

is called the *Hessian (matrix)* of f at $\mathbf{x}$.

Unconstrained Optimization in $\mathbb{R}^n$

- The Hessian is denoted $\mathbf{f}''(\mathbf{x})$.
- The Hessian is symmetric.
- Sometimes the partial derivative $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$ is written as $f''_{ij}(\mathbf{x})$.
- A leading principal minor of order k of the Hessian is denoted $D_k(\mathbf{x})$.
- An arbitrary principal minor of order k of the Hessian is denoted $\Delta_k(\mathbf{x})$.

Unconstrained Optimization in $\mathbb{R}^n$

Fact 5

Let a (twice) differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ *have a critical point at* $\mathbf{x}^*$.

1. *If the Hessian* $\mathbf{f}''(\mathbf{x}^*)$ *is positive definite or, equivalently,* $D_k(\mathbf{x}^*) > 0,\ k = 1, \ldots, n$, *then* $\mathbf{x}^*$ *is a local minimum point.*
2. *If the Hessian* $\mathbf{f}''(\mathbf{x}^*)$ *is negative definite or, equivalently,* $(-1)^k D_k(\mathbf{x}^*) > 0,\ k = 1, \ldots, n$, *then* $\mathbf{x}^*$ *is a local maximum point.*
3. *If* $D_n(\mathbf{x}^*) \neq 0$ *and neither 1), nor 2) is satisfied, then* $\mathbf{x}^*$ *is a saddle point.*

Unconstrained Optimization in $\mathbb{R}^n$

Fact 6

Let a (twice) differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ *have an extreme point at* $\mathbf{x}^*$.

1. *If* $\mathbf{x}^*$ *is a local minimum point, then the Hessian* $\mathbf{f}''(\mathbf{x}^*)$ *is positive semidefinite or, equivalently,* $\Delta_k(\mathbf{x}^*) \geq 0$ *for all principal minors of order* $k = 1, \ldots, n$.
2. *If* $\mathbf{x}^*$ *is a local maximum point, then the Hessian* $\mathbf{f}''(\mathbf{x}^*)$ *is negative semidefinite or, equivalently,* $(-1)^k \Delta_k(\mathbf{x}^*) \geq 0$ *for all principal minors of order* $k = 1, \ldots, n$.

Unconstrained Optimization in $\mathbb{R}^n$

Example 7 (Verification of Example 4)

Recall that:

$$f(x,y) = x^2 + 2y^2 - 3x + xy$$

$$\frac{\partial f}{\partial x} = 2x - 3 + y, \quad \frac{\partial f}{\partial y} = 4y + x.$$

We now have:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial y \partial x} = 1.$$

$$D_1 = \det(2) = 2 > 0, \quad D_2 = \det \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = 2 \cdot 4 - 1 \cdot 1 = 7 > 0.$$

Since $D_1 > 0$, $D_2 > 0$, the critical point $x = \frac{12}{7}$, $y = -\frac{3}{7}$ is a minimum.

Static Optimization with Equality Constraints. Lagrange Multipliers

Static Optimization with Equality Constraints

Formulation

Now we look at problems of the form

$$f(x_1, \ldots, x_n) \to \min(\max) \tag{1}$$

s.t.

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= b_1 \\ g_2(x_1, \ldots, x_n) &= b_2 \\ \ldots \\ g_m(x_1, \ldots, x_n) &= b_m \end{aligned} \tag{2}$$

where $m < n$. (Can you explain the last requirement?)

Note: In what follows, all required properties of the objects in (1) and (2) like differentiability are implicitly assumed.

Static Optimization with Equality Constraints

Formulation

Using vector notation for compactness, the objective function is:

$$f(\mathbf{x}) \to \min(\max)$$

We introduce

$$\mathbf{g}(\mathbf{x}) := (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))', \quad \mathbf{b} = (b_1, \ldots, b_m)'$$

and the constraints are written as

$$\mathbf{g}(\mathbf{x}) = \mathbf{b}.$$

Static Optimization with Equality Constraints

The Lagrangian

The standard approach to solving (1)-(2) starts by defining a *Lagrangian*:

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \lambda_1(g_1(\mathbf{x}) - b_1) - \cdots - \lambda_m(g_m(\mathbf{x}) - b_m).$$

The numbers $\lambda_1, \ldots, \lambda_m$ are called *Lagrange multipliers*.

This can also be written in vector notation:

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \boldsymbol{\lambda}'(\mathbf{g}(\mathbf{x}) - \mathbf{b}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)'$ is the vector of Lagrange multipliers. While, strictly speaking, $\mathcal{L}$ depends on $\boldsymbol{\lambda}$, we'll omit this dependence to lighten notation.

Note: Here we follow the economics sign convention and subtract the constraint terms in the Lagrangian. Math textbooks typically use a "+" in front of λ; both lead to the same optimality conditions, with multipliers differing only by sign.

Static Optimization with Equality Constraints

We can use the Lagrangian to produce candidates for optimality in the following manner:

Algorithm

1. Form the Lagrangian as above
2. Differentiate it w.r.t. the variables we are optimizing over, i.e.

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i}, \; i = 1, \ldots, n$$

3. Set the resulting derivatives equal to zero, i.e. construct the first-order conditions (FOCs)

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \; i = 1, \ldots, n \tag{3}$$

4. The FOCs in the preceding step, together with the constraints (2), form a system of $n + m$ equations which is solved for the unknowns x_i and λ_j

Static Optimization with Equality Constraints

Remarks

- Sometimes the Lagrangian is formulated as

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \cdots - \lambda_m g_m(\mathbf{x}).$$

  It obviously makes no difference in the differentiation step and produces the same result.
- One modification of the algorithm additionally requires differentiating the Lagrangian w.r.t. λ_j and setting the resulting derivatives equal to zero. This simply reproduces the constraints (2) and is covered by the last step of our algorithm.
- A Lagrange multiplier is interpreted as a *shadow price*, i.e. the gain (or loss) arising from relaxing the associated constraint.
- The Lagrangian algorithm produces necessary conditions for optimality only under additional assumptions.

Static Optimization with Equality Constraints

Necessary conditions for optimality

Fact 7

Suppose that the functions f and $g_1, \ldots, g_m$ are defined on a set S in $\mathbb{R}^n$, and that $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ is an interior point of S that solves problem (1)-(2). Suppose further that f and $g_1, \ldots, g_m$ are C^1 in a ball around $\mathbf{x}^*$, and that the $m \times n$ matrix of partial derivatives of the constraint functions

$$\mathbf{g}'(\mathbf{x}^*) = \begin{pmatrix} \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_n} \end{pmatrix}$$

has rank m. Then there exist unique numbers $\lambda_1, \ldots, \lambda_m$ such that the solution $\mathbf{x}^*$ satisfies the FOCs (3).

Static Optimization with Equality Constraints

Sufficient conditions for optimality

Fact 8

If there exist numbers $\lambda_1, \dots, \lambda_m$ and an admissible $\mathbf{x}^*$ which together satisfy the first-order conditions in the Lagrangian algorithm, and if the Lagrangian $\mathcal{L}(\mathbf{x})$ is concave (convex) in $\mathbf{x}$, and if S is convex, then $\mathbf{x}^*$ solves the maximization (minimization) problem (1)-(2).

Static Optimization with Equality Constraints

Local second-order conditions

- The sufficient conditions in Fact 8 are global and therefore often turn out to be too restrictive. A more widely applicable route to establishing optimality is given by *local second-order conditions*.
- Local second-order conditions extend the unconstrained second-order test by replacing the Hessian of the objective function with an appropriate modification that incorporates the equality constraints.

Static Optimization with Equality Constraints

Local second-order conditions: bordered Hessians

For problem (1)-(2) and $r = m+1, \ldots, n$ we define the **bordered Hessian determinant** as follows:

$$B_r(\mathbf{x}^*) = \begin{vmatrix} 0 & \cdots & 0 & \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_r} \\ \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_1} & \dfrac{\partial^2 \mathcal{L}(\mathbf{x}^*)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 \mathcal{L}(\mathbf{x}^*)}{\partial x_1 \partial x_r} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_1(\mathbf{x}^*)}{\partial x_r} & \cdots & \dfrac{\partial g_m(\mathbf{x}^*)}{\partial x_r} & \dfrac{\partial^2 \mathcal{L}(\mathbf{x}^*)}{\partial x_r \partial x_1} & \cdots & \dfrac{\partial^2 \mathcal{L}(\mathbf{x}^*}{\partial x_r^2} \end{vmatrix}$$

Static Optimization with Equality Constraints

Remarks on bordered Hessians

- Some expositions directly define the full bordered Hessian matrix (the matrix under the determinant sign in the previous slide in the case $r = n$) and then discuss the last $n - m$ leading principal minors of this matrix, which of course are our objects $B_r(\mathbf{x}^*), r = m + 1, \ldots, n$.

- One way of compactly describing the matrix underlying a bordered Hessian determinant is to say it has the following block structure:

$$\begin{pmatrix} \mathbf{0}_{m\times m} & J_{m\times r} \\ J'_{r\times m} & H_{r\times r} \end{pmatrix},$$

  where J denotes the (partial) Jacobian of the vector function specifying the constraints, H is the (partial) Hessian of the Lagrangian function and the subscripts indicate the dimensions of the corresponding matrices.

- We may need to re-label variables to ensure that the first m columns of the $J_{m\times r}$ matrix are linearly independent. This is assumed in what follows.

Static Optimization with Equality Constraints

Local second-order conditions: the general case

The following second-derivative test provides "local" sufficient conditions for optimality:

Fact 9

Suppose the functions f and $g_1, \ldots, g_m$ are defined on a set $S \subset \mathbb{R}^n$, and let $\mathbf{x}^*$ be an interior point of S satisfying the necessary conditions in Fact 7. Suppose further that f and $g_1, \ldots, g_m$ are C^2 in a ball around $\mathbf{x}^*$. With the determinants $B_r(\mathbf{x}^*)$ defined as above, we have:

1. *If $(-1)^m B_r(\mathbf{x}^*) > 0$ for $r = m+1, \ldots, n$, then $\mathbf{x}^*$ solves the local minimization problem in (1)-(2).*
2. *If $(-1)^r B_r(\mathbf{x}^*) > 0$ for $r = m+1, \ldots, n$, then $\mathbf{x}^*$ solves the local maximization problem in (1)-(2).*

Note that the first condition requires the bordered Hessian determinants to keep the same sign, while the second one requires that the signs alternate.

Static Optimization with Equality Constraints

Local second-order conditions: one constraint (setup)

Consider the common case of a **single** equality constraint ($m = 1$):

$$f(\mathbf{x}) \to \max\ (\min) \quad \text{s.t.} \quad g(\mathbf{x}) = b, \qquad \mathbf{x} \in S \subset \mathbb{R}^n.$$

For $r = 2, \ldots, n$, the **bordered Hessian determinant** (restricted to $x_1, \ldots, x_r$) takes the form

$$B_r(\mathbf{x}^*) = \begin{vmatrix} 0 & \nabla g_r(\mathbf{x}^*)' \\ \nabla g_r(\mathbf{x}^*) & H_r(\mathbf{x}^*) \end{vmatrix},$$

where

$$\nabla g_r(\mathbf{x}^*) = \begin{pmatrix} \dfrac{\partial g(\mathbf{x}^*)}{\partial x_1} \\ \vdots \\ \dfrac{\partial g(\mathbf{x}^*)}{\partial x_r} \end{pmatrix}, \qquad H_r(\mathbf{x}^*) = \left[\frac{\partial^2 \mathcal{L}(\mathbf{x}^*)}{\partial x_i \, \partial x_j} \right]_{i,j=1,\ldots,r}.$$

Static Optimization with Equality Constraints

Local second-order conditions: one constraint (test)

Suppose $\mathbf{x}^*$ is an admissible point satisfying the first-order conditions for the Lagrangian method with one equality constraint, and assume the required C^2 regularity in a neighborhood of $\mathbf{x}^*$.

Second-derivative test ($m = 1$)

For the determinants $B_r(\mathbf{x}^*)$ from the previous slide:

- **Local maximum:** $(-1)^r \, B_r(\mathbf{x}^*) > 0$ for $r = 2, \ldots, n$.
- **Local minimum:** $-B_r(\mathbf{x}^*) > 0$ for $r = 2, \ldots, n$ (equivalently, $B_r(\mathbf{x}^*) < 0$ for all r).

Remark: The local maximum condition requires alternating signs across r, while the local minimum condition requires the same sign across r.

Static Optimization with Equality Constraints

Example 8 (Basic intertemporal optimization)

- An economic agent lives for two periods and supplies a fixed amount of labour in the first period of his life in exchange for monetary payment y.
- In period 1 the agent consumes c_1 units of a good out of his income and saves the remaining $y - c_1$. (For convenience we assume there is no inflation and the price of the good is normalized to one.)
- Savings are remunerated at an interest rate r. Thus, in the second period the agent has at his disposal

$$(y - c_1)(1 + r)$$

  to finance consumption, denoted c_2.
- The agent obtains utility from consumption according to the utility function

$$u(c_1, c_2) = \ln c_1 + \beta \ln c_2, \ \beta \in (0, 1).$$

- The agent seeks to maximize utility w.r.t. c_1, c_2.

Static Optimization with Equality Constraints

Example 8 (cont.)

The above problem can be formalized as

$$\max_{c_1, c_2} u(c_1, c_2)$$

s.t.

$$c_2 = (y - c_1)(1 + r).$$

Notice that the constraint can be written equivalently as

$$c_1 + \frac{c_2}{1 + r} = y$$

to conform to the $\mathbf{g}(\mathbf{x}) = \mathbf{b}$ convention. (Can you interpret the last equation in terms of discounting to period 1 quantities?)
The Lagrangian for this problem is

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 - \lambda \left(c_1 + \frac{c_2}{1 + r} - y \right).$$

Static Optimization with Equality Constraints

Example 8 (cont.)

The solution algorithm yields

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \quad \Rightarrow \quad c_1 = \frac{1}{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \frac{\lambda}{1+r} = 0 \quad \Rightarrow \quad c_2 = \frac{\beta(1+r)}{\lambda}$$

Combining the above equations to eliminate λ, we obtain

$$c_2 = \beta(1+r)c_1.$$

Substitute the last expression in the budget constraint:

$$c_1 + \frac{\beta(1+r)c_1}{1+r} = y \quad \Rightarrow \quad c_1^* = \frac{y}{1+\beta}.$$

Static Optimization with Equality Constraints

Example 8 (cont.)

We then have

$$c_2^* = \beta(1+r)c_1 = (1+r)\frac{\beta}{1+\beta}y.$$

Second-order condition (bordered Hessian)

The constraint is $g(c_1, c_2) = c_1 + \dfrac{c_2}{1+r} = y$, hence

$$\frac{\partial g}{\partial c_1} = 1, \qquad \frac{\partial g}{\partial c_2} = \frac{1}{1+r}.$$

The Hessian of the Lagrangian w.r.t. (c_1, c_2) is

$$H = \begin{pmatrix} \dfrac{\partial^2 \mathcal{L}}{\partial c_1^2} & \dfrac{\partial^2 \mathcal{L}}{\partial c_1 \partial c_2} \\ \dfrac{\partial^2 \mathcal{L}}{\partial c_2 \partial c_1} & \dfrac{\partial^2 \mathcal{L}}{\partial c_2^2} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{c_1^2} & 0 \\ 0 & -\dfrac{\beta}{c_2^2} \end{pmatrix}.$$

Static Optimization with Equality Constraints

Example 8 (cont.)

Since $n = 2$ and $m = 1$, we only need B_2:

$$B_2(\mathbf{x}^*) = \begin{vmatrix} 0 & 1 & \dfrac{1}{1+r} \\ 1 & -\dfrac{1}{(c_1^*)^2} & 0 \\ \dfrac{1}{1+r} & 0 & -\dfrac{\beta}{(c_2^*)^2} \end{vmatrix} = \frac{\beta}{(c_2^*)^2} + \frac{1}{(1+r)^2}\frac{1}{(c_1^*)^2} \; > \; 0.$$

Therefore $(-1)^2 B_2(\mathbf{x}^*) > 0$, so (c_1^*, c_2^*) is a **local maximum**.

Static Optimization with Equality Constraints

Example 8 (cont.)

Let us check how the optimal value of the utility function $u^* = u(c_1^*, c_2^*)$ changes with income:

$$\begin{aligned}
\frac{\partial u^*}{\partial y} =& \frac{\partial}{\partial y}\left(\ln\frac{y}{1+\beta} + \beta\ln\frac{(1+r)\beta y}{1+\beta}\right) \\
=& \frac{1+\beta}{y}\frac{1}{1+\beta} + \beta\frac{1+\beta}{(1+r)\beta y}\frac{(1+r)\beta}{1+\beta} \\
=& \frac{1}{y} + \frac{\beta}{y} = \frac{1+\beta}{y}
\end{aligned}$$

Static Optimization with Equality Constraints

Example 8 (cont.)

Let us check how the optimal value of the utility function $u^* = u(c_1^*, c_2^*)$ changes with income:

$$
\begin{aligned}
\frac{\partial u^*}{\partial y} =& \frac{\partial}{\partial y}\left(\ln \frac{y}{1+\beta} + \beta \ln \frac{(1+r)\beta y}{1+\beta}\right) \\
=& \frac{1+\beta}{y}\frac{1}{1+\beta} + \beta \frac{1+\beta}{(1+r)\beta y}\frac{(1+r)\beta}{1+\beta} \\
=& \frac{1}{y} + \frac{\beta}{y} = \frac{1+\beta}{y} = \frac{1}{c_1^*} = \lambda. \quad \text{Interpretation?}
\end{aligned}
$$

Readings

Main references:
Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 3.

Additional readings:
Simon and Blume. *Mathematics for economists*. Chapters 17 and 18.