

# R406: Applied Economic Modelling with Python

## First-order Difference Equations

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# Introduction

# Discrete Time

- In many cases a variable of interest can be determined from its past values (as well as potentially other quantities)
- Examples
  - The end-period balance of a bank account can be computed from the starting balance plus deposits/withdrawals and interest
  - The stock of capital at a point in time is determined by the stock from the previous period, investment and depreciation
- Here we consider the case of discrete time:  $t$  takes only integer values:

$$t = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

# What Are Difference Equations?

- In general a difference equation can be stated as a function relating the variable to its own past values
- Formally written,

$$y_{t+1} = f(y_t, y_{t-1}, \dots)$$

- We call this a *recursive relationship*
- Every next value is obtained from this relationship; for example, in the relationship  $y_{t+1} = f(y_t)$ :

$$y_{t+2} = f(y_{t+1}) = f(f(y_t)) = f^2(y_t)$$

- From this follows:

$$y_{t+k} = f^k(y_t)$$

Note:  $k$  is not interpreted as a power!

# What Are Difference Equations? (2)

- Sometimes difference equations are written in more general, **implicit** form as

$$F(y_{t+1}, y_t, y_{t-1}, \dots) = 0.$$

We shall ignore such cases and work with **explicit** difference equations, i.e. ones that can be solved for the variable with the largest index

- The order of a difference equation is given by the difference between the largest and the smallest time indexes
- For example, the equation

$$y_{t+1} = f(y_t, y_{t-1}, y_{t-2})$$

is a third-order equation because  $(t + 1) - (t - 2) = 3$

# What Are Difference Equations? (3)

- A difference equation, e.g.

$$y_{t+1} = f(y_t),$$

can be re-written as

$$y_{t+1} - y_t = f(y_t) - y_t.$$

- We can define the (first) difference as  $\Delta y_t := y_t - y_{t-1}$  and a new right-hand side  $g(y) := f(y) - y$
- The above equation can then be written as

$$\Delta y_{t+1} = g(y_t),$$

hence the name *difference equation*

- While you may not see immediate benefits from writing the equation to feature an explicit difference operator  $\Delta$ , this form becomes useful to motivate the transition to differential equations

# Linear vs. Non-linear Equations

- Recursive relationships can be either linear or non-linear
- Therefore linear and non-linear difference equations
- (Note again that time and/or other variables could also appear as function arguments)
- We will stick most of the time to linear equations
- Linearity will be directly assumed, while non-linearity will be explicitly indicated when necessary

# Constant-coefficients vs. Variable-coefficients Equations

- Take the equation:

$$y_{t+3} = ay_{t+2} + by_{t+1} + cy_t + d$$

where  $a, b, c, d$  are constants

This is a **constant-coefficients difference equation**

- Compare this to:

$$y_{t+3} = a_{t+2}y_{t+2} + b_{t+1}y_{t+1} + c_ty_t + d_t$$

This is a **variable-coefficients difference equation**

# Autonomous vs. Non-autonomous Difference Equations

- **Autonomous** (or time-invariant) equations: time does not enter independently as an argument of  $f(\cdot)$ ; example:

$$y_t = 0.4y_{t-1} + 2$$

- **Non-autonomous** (or time-variant) equations: time enters independently as an argument:

$$y_t = 0.4y_{t-1} + 3t + 2$$

- The latter are much more difficult to study, and we will only occasionally mention them

# Homogeneous vs. Non-homogeneous Equations

- Take the following equation where all instances of the  $y$  variable are on the left-hand side, and 'all the rest' is summarized on the right-hand side in  $g(t)$ :

$$y_{t+2} + ay_{t+1} + by_t = g(t)$$

- (A second-order equation in this case, but order here does not matter)
- If  $g(t) = 0$ , the equation is **homogeneous**<sup>1</sup>
- If  $g(t) \neq 0$ , the equation is **non-homogeneous**

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<sup>1</sup>For such equations, if  $y_t$  is a solution, so is  $\alpha y_t$ .

# The Initial-value Problem

# The Initial-value Problem

- Take the equation:

$$y_t = f(y_{t-1}, y_{t-2}, \dots, y_{t-p})$$

- In order to be able to solve for  $y_t$ ,  $p$  starting values will be needed
- Otherwise it would be impossible to find the solution
- In the case of the first-order equation, one such value is needed

# Equilibrium and Stability Conditions

# Equilibrium

- We begin with the autonomous first-order-equation case so that the idea is grasped more easily
- We will consider stability conditions again when we get to higher-order equations
- Start from the definition of a function's fixed point:

## Definition 1

Let  $K \subset \mathbb{R}^n$  and let  $f$  be a function that maps  $x \in K$  into another point  $f(x) \in K$  (i.e. the function maps  $K$  into itself). If the point  $x^*$  is such that  $f(x^*) = x^*$ , then  $x^*$  is called a **fixed (or equilibrium) point**.

# Equilibrium (2)

- In the current context  $K$  consists of all  $y_t$ 's
- For example, in the difference equation  $y_{t+1} = f(y_t)$ ,  $f$  maps  $y_t$  into  $y_{t+1}$
- The definition then implies that  $y^*$  is an equilibrium point if and only if (iff):

$$y^* = f(y^*)$$

- Graphically, this means that equilibrium points always lie on the  $45^\circ$  line
- **Example:** Find the equilibrium points of  $f(x) = x^3$ ; make a graphical sketch of your solution

# Stability of Equilibrium

Let  $y^*$  be an equilibrium point of  $f$ . Some definitions follow.

## Definition 2

The point  $y^*$  is **stable** if for any  $\varepsilon > 0 \exists \delta > 0$ :

$$|y_0 - y^*| < \delta \Rightarrow |f^n(y_0) - y^*| < \varepsilon, \forall n > 0.$$

Otherwise,  $y^*$  is **unstable**.

## Definition 3

The point  $y^*$  is **repelling** if  $\exists \varepsilon > 0$ :

$$0 < |y_0 - y^*| < \varepsilon \Rightarrow |f(y_0) - y^*| > |y_0 - y^*|.$$

# Stability of Equilibrium (2)

## Definition 4

The point  $y^*$  is **attracting (asymptotically stable)** if:

- ① It is stable
- ②  $\exists \eta > 0$ :

$$|y_0 - y^*| < \eta \Rightarrow \lim_{t \rightarrow \infty} y_t = y^*.$$

If  $\eta = \infty$ , then  $y^*$  is **globally asymptotically stable**.

# Stability of Equilibrium (3)

## Theorem 1

Let  $y^*$  be a fixed point of  $y_{t+1} = f(y_t)$ , and let  $f$  be continuously differentiable at  $y^*$ . Then:

- ① If  $|f'(y^*)| < 1$ , then  $y^*$  is an attractor
- ② If  $|f'(y^*)| > 1$ , then  $y^*$  is a repeller
- ③ If  $|f'(y^*)| = 1$  and:
  - ① if  $f''(y^*) \neq 0$ , then  $y^*$  is unstable
  - ② if  $f''(y^*) = 0$  and if  $f'''(y^*) > 0$ , then  $y^*$  is unstable
  - ③ if  $f''(y^*) = 0$  and if  $f'''(y^*) < 0$ , then  $y^*$  is an attractor
- ④ If  $f'(y^*) = -1$  and:
  - ① if  $-2f'''(y^*) - 3[f''(y^*)]^2 < 0$ , then  $y^*$  is an attractor
  - ② if  $-2f'''(y^*) - 3[f''(y^*)]^2 > 0$ , then  $y^*$  is unstable

Note: Theorems are provided just for usage and reference, no proofs are given or required!

# Stability of Equilibrium (4)

- There exist fixed points which are neither attractors nor repellers
- Take for example:

$$y_{t+1} = -y_t + k$$

- If  $y_0$  is the initial value, then

$$y_0 = y_2 = y_4 = \dots$$

$$y_1 = y_3 = y_5 = \dots$$

- Although there is a fixed point  $y^* = \frac{k}{2}$ , if  $y_0 \neq y^*$ , the system cycles between  $y_0$  and  $-y_0 + k$  without getting nearer or farther from the fixed point over time

# Stability of Equilibrium (5)

- A solution  $y_n$  is periodic if

$$y_{n+m} = y_n$$

for some fixed integer  $m$  and all  $n$ . The smallest integer  $m$  is called the **the period of the solution**.

## Definition 5

If the sequence  $\{y_t\}$  has  $q$  repeating values, then  $y_1, y_2, \dots, y_q$  are called **period points**, and the set  $\{y_1, y_2, \dots, y_q\}$  is called a **periodic orbit**.

- For example, if the equation is  $y_{t+1} = f(y_t)$ , a  $k$ -periodic point is the  $y$ -coordinate of the intersection point of  $f^k(y)$  and the  $45^\circ$ -line  $y_{t+1} = y_t$

# Stability of Equilibrium (6)

## Theorem 2

Let  $b$  be a  $k$ -period point of  $f$ . Then  $b$  is:

- ① Stable if it is a stable fixed point of  $f^k$
- ② Asymptotically stable if it is an attracting point of  $f^k$
- ③ Repelling if it is a repelling fixed point of  $f^k$

# First-order Difference Equations

# First-order Difference Equations: Problem Setup

- A first-order difference equation has the form:

$$y_{t+1} = f(y_t)$$

- Initial-value problem:

$$\left| \begin{array}{l} y_{t+1} = f(y_t) \\ y(0) = y_0 \end{array} \right.$$

where  $y_0$  is given initially

- It is easy to see that  $y_1 = f(y_0)$ ,  $y_2 = f(y_1)$ ,  $y_3 = f(y_2)$ , etc.
- In other words, it implies the following sequence of values:

$$y_0, f(y_0), f(f(y_0)), f(f(f(y_0))), \dots$$

- As already noted, sometimes this is denoted more conveniently as follows:

$$y_0, f(y_0), f^2(y_0), f^3(y_0), \dots$$

# Solution

- Start with the linear non-homogeneous, constant-coefficients case:

$$\begin{cases} y_{t+1} = ay_t + b \\ y(0) = y_0 \end{cases}$$

- Beginning with  $y_0$ , the next values are obtained through iteration:

$$y_0$$

$$y_1 = ay_0 + b$$

$$y_2 = ay_1 + b = a^2y_0 + ab + b$$

...

$$y_t = a^t y_0 + b(a^{t-1} + a^{t-2} + \dots + a + 1)$$

## Solution (2)

- We have the sum of a geometric series in the parentheses
- Using the standard formula, this sum equals:

$$\begin{cases} \frac{1-a^t}{1-a}, & a \neq 1 \\ t, & a = 1 \end{cases}$$

- Let  $a \neq 1$ . Then the solution to the equation is:

$$y_t = a^t \left( y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}$$

- In the other case ( $a = 1$ ), the solution is:

$$y_t = y_0 + tb$$

## Solution (3)

- To find the equilibrium (fixed) point of this equation, we seek for a solution  $y^*$  such that  $y_t = y^*, \forall t$  (i.e. if we start at this point, we stay there at all times)
- This means that it should satisfy in particular the following:  $y_{t+1} = y_t = y^*$
- Plug this into the equation:

$$y^* = ay^* + b$$

- It follows that the fixed point is:

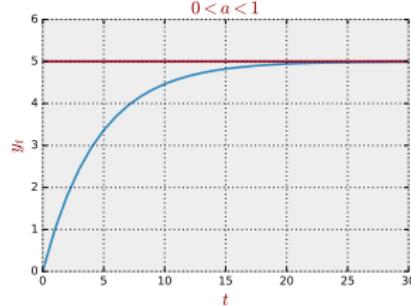
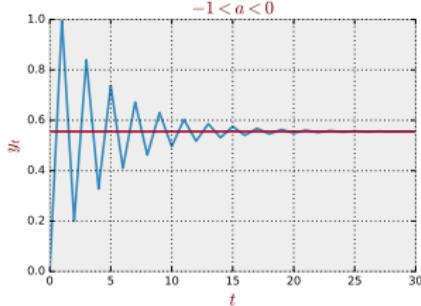
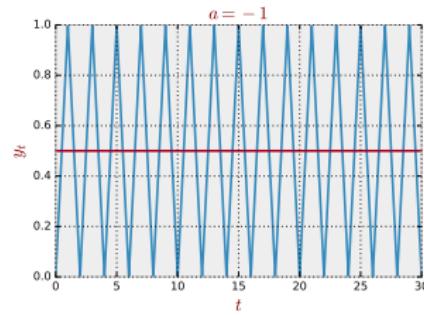
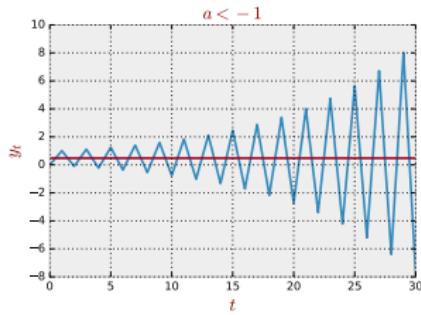
$$y^* = \frac{b}{1-a}, a \neq 1$$

- It is directly obvious that in the homogeneous case  $y^* = 0$

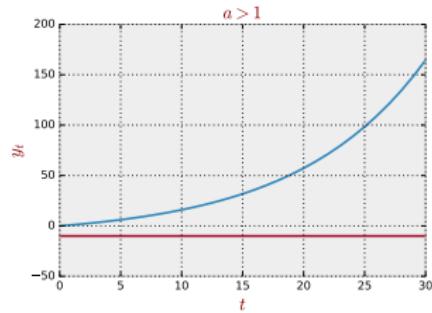
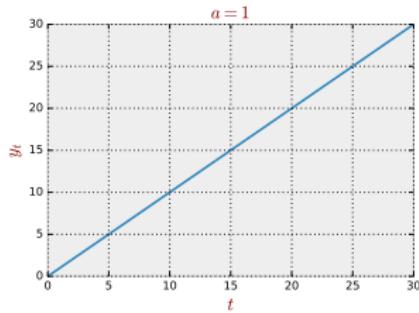
# Fixed-point Stability

- Using Theorem 1 and Theorem 2, we can obtain the stability conditions very easily
- The first derivative of  $f$  with respect to  $y_t$  equals  $a$ ; the higher-order derivatives are all zero
- Therefore:
  - ① If  $|a| < 1$ , the equilibrium is stable
  - ② If  $|a| > 1$ , the equilibrium is unstable
  - ③ If  $a = 1$ , the fixed point is undefined
  - ④ If  $a = -1$ , the fixed point equals  $\frac{b}{2}$  but it is neither stable nor unstable; the solution is periodic and the system cycles between  $y_0$  and  $-y_0 + b$

# Fixed-point Stability (2)



# Fixed-point Stability (3)



# Time-varying $b$

- Let  $b_t$  be an arbitrary but known function of  $t$  (i.e. the values of  $b_t$  are given beforehand)
- The equation becomes:

$$y_{t+1} = ay_t + b_t, \quad t = 0, 1, 2, \dots$$

- Starting from a given  $y_0$ , recursive substitutions leads eventually to:

$$y_t = a^t y_0 + \sum_{k=0}^{t-1} a^{t-k-1} b_k, \quad t = 0, 1, 2, \dots$$

- Stability conditions are much harder to specify
- However, equations of this kind are successfully used to describe special types of time series data
- Stability conditions are derived after plausible assumptions on  $b_t$  are made

# Time-varying $a$

- If, in addition to  $b$ , the coefficient  $a$  also changes over time, then the equation becomes:

$$y_{t+1} = a_t y_t + b_t, \quad t = 0, 1, 2, \dots$$

- The same line of reasoning as before can be applied finally leading to:

$$y_t = \left( \prod_{k=0}^{t-1} a_k \right) y_0 + \sum_{i=0}^{t-1} \left( \prod_{k=i+1}^{t-1} a_k \right) b_i$$

where all terms  $\prod_{k=t}^{t-1} a_k$  are set to equal 1.

# Example 1: Mortgage Repayment

- An amount equal to  $K$  is borrowed at time 0 to finance a mortgage at the fixed monthly interest rate  $r$
- The mortgage is repaid monthly at amounts equal to  $a$ , and the number of monthly payments until full repayment equals  $T$
- In each period the outstanding principal  $b_t$  satisfies the following relationship:

$$b_{t+1} = (1 + r)b_t - a$$

- Additionally, it is obvious from the problem that  $b_0 = K$  and  $b_T = 0$

# Example 1: Mortgage Repayment (2)

- Assuming that  $r > 0$ , the fixed point is:

$$b^* = -\frac{a}{1 - (1 + r)} = \frac{a}{r}$$

- Therefore, the solution is:

$$b_t = (1 + r)^t \left( b_0 - \frac{a}{r} \right) + \frac{a}{r}$$

or:

$$b_t = (1 + r)^t \left( K - \frac{a}{r} \right) + \frac{a}{r}$$

# Example 1: Mortgage Repayment (3)

- For  $t = T$ , the solution is:

$$0 = (1 + r)^T \left( K - \frac{a}{r} \right) + \frac{a}{r}$$

- If we solve the latter for  $K$ , we get:

$$K = \frac{a}{r} \left( 1 - \frac{1}{(1 + r)^T} \right) = a \sum_{t=1}^T \frac{1}{(1 + r)^t}$$

i.e. the borrowed amount equals the sum of PDVs of all monthly payments until full repayment

# Example 1: Mortgage Repayment (4)

- We could also solve for  $a$ :

$$a = \frac{rK}{1 - (1 + r)^{-T}} = \frac{rK(1 + r)^T}{(1 + r)^T - 1}$$

or, in other words, this result allows to calculate the monthly mortgage payment given the borrowed amount and the interest rate

## Example 2: Compound Interest and PDVs with Constant Interest Rates

- Let  $w_t$  denote a person's wealth (assets) at time  $t$
- If  $y_t$  and  $c_t$  are respectively this person's income and consumption
- Given a constant interest rate  $r$ , we have:

$$w_{t+1} = (1 + r)w_t + y_{t+1} - c_{t+1}$$

- If initial wealth is  $w_0$ , then the solution is:

$$w_t = (1 + r)^t w_0 + \sum_{k=0}^{t-1} (1 + r)^{t-k-1} (y_{k+1} - c_{k+1})$$

## Example 2: Compound Interest and PDVs with Constant Interest Rates (2)

- Divide both sides by  $(1 + r)^t$  to get:

$$(1 + r)^{-t} w_t = w_0 + \sum_{k=0}^{t-1} (1 + r)^{-k-1} (y_{k+1} - c_{k+1})$$

- The left-hand side is the PDV of  $w_t$  at time 0
- It equals the initial wealth plus the PDV of the flow of savings from time 1 to time  $t$

## Example 3: The Harrod-Domar Growth Model

- We consider the discrete-time version
- Saving is a fixed share of output/income:

$$S_t = sY_t, \quad 0 < s < 1$$

- Investment depends on output growth:

$$I_t = v(Y_t - Y_{t-1}), \quad v > 0$$

- Saving equals investment:

$$S_t = I_t$$

## Example 3: The Harrod-Domar Growth Model (2)

- Substituting the first two equations into the third yields:

$$sY_t = v(Y_t - Y_{t-1}) \Rightarrow Y_t = \frac{v}{v-s} Y_{t-1}$$

- Solving the equation leads to:

$$Y_t = \left( \frac{v}{v-s} \right)^t Y_0,$$

and the only fixed point is 0

- There are several possibilities for the relationship between  $s$  and  $v$ , leading to different conclusions on the stability of the system

## Example 4: Hog Cycles and Cobwebs

- Based on the fact that for various commodities output and prices exhibit cyclical behaviour
- Cycles stem from the consecutive adjustments, which are manifested as movements between the supply and demand curve
- Also, from the technological perspective, there is a span of time required in order to produce a certain output (e.g. in agriculture)
- Generally speaking, the “cobweb phenomenon” stems from the presence of lagged effects on some economic variables
- Also, there is a mismatch between the time of the production decision and the time of purchase decisions by customers (demand)

## Example 4: Hog Cycles and Cobwebs (2)

- Initially, the system (the market) is in equilibrium
- Each cycle begins with a demand or with a supply shock
- In this version of the model, only one-period lags matter
- Supply, demand, and equilibrium are described via the following three equations:

$$q_t^D = \alpha - \beta p_t$$

$$q_t^S = \gamma + \delta p_{t-1}$$

$$q_t^D \equiv q_t^S$$

where  $\alpha > 0$ ,  $\beta > 0$ , and  $\delta > 0$ .

## Example 4: Hog Cycles and Cobwebs (3)

- Substituting supply and demand in the equilibrium condition leads to the following difference equation:

$$p_t = \frac{\alpha - \gamma}{\beta} - \frac{\delta}{\beta} p_{t-1},$$

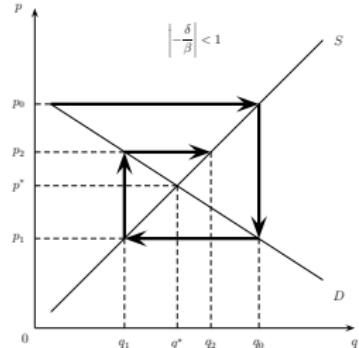
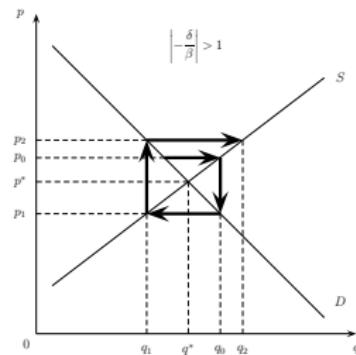
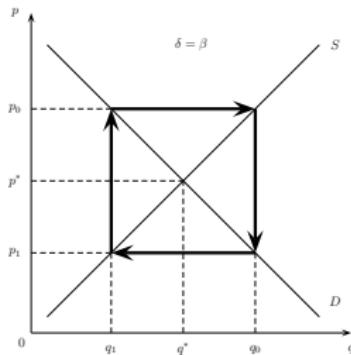
- Its solution is:

$$p_t - p^* = \left(-\frac{\delta}{\beta}\right)^t (p_0 - p^*)$$

where  $p^*$  is the equilibrium price

- There are three possible system behaviours depending on the values of  $\delta$  and  $\beta$

# Example 4: Hog Cycles and Cobwebs (4)



# References

- Sydsæter et al. (2008), ch. 11
- Chiang and Wainwright (2005), ch. 17
- Shone (2002), ch. 3