

# R406: Applied Economic Modelling with Python

## Ordinary Differential Equations of First, Second, and Higher Order

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# Introduction

# What is a Differential Equation?

## Definition 1

A differential equation is an equation that relates an unknown function with one or more of its derivatives.

- Note that a differential equation relates to an *unknown function* not to an *unknown number*
- Note that such an equation can have one or more variables, constants, etc.
- Given two variables  $x$  and  $y$ , the general form of a differential equation in those two variables could be:

$$\frac{dy}{dx} = f(x, y)$$

- We will be considering only variables that are functions of time, therefore we can speak equivalently of *dynamic systems*
- Time itself (denoted by  $t$ ) is considered a continuous variable

# Ordinary Differential Equations

## Definition 2

An ordinary differential equation (ODE) is a differential equation that includes the derivatives of only one function of time.

- Example:

$$\dot{x} = ax + b$$

This is not an ODE:

$$\dot{x} = ax + b + \dot{y} + cy$$

- Note that we will be using Newton's *dot notation* of derivatives
- Just to be aware: equations including the partial derivatives of unknown functions of more than one variable are called *partial differential equations*

# ODE Order

## Definition 3

The order of an ODE is defined by the order of the highest derivative present in the equation.

- Examples:

$$\dot{x} = ax \text{ (first-order ODE)}$$

$$\ddot{x} = \dot{x} + z \text{ (second-order ODE)}$$

etc.

# Linear vs. Non-linear ODE

## Definition 4

A linear ODE satisfies the following:

- The dependent variable and its derivatives appear to the power of one only
- There are no products of the dependent variable and/or its derivatives in the equation
- There are no transcendental functions<sup>a</sup> of the dependent variable and/or its derivatives

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<sup>a</sup>Functions that cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.

- A non-linear ODE is one not fitting the above definition
- Example:

$$\dot{x} = \frac{1}{t} - 2 \cos(x)$$

# ODE: Constant vs. Variable Coefficients

- Consider the general form of an  $n$ th-order linear ODE:

$$a_0(t) \frac{d^n x}{dt^n} + a_1(t) \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_n(t) x = g(t) \quad (*)$$

where  $g(t)$  is some function of time

- Some equations might imply coefficient which themselves are functions of  $t$
- Those are equations with *variable coefficients*
- With some exceptions, we will usually work with *constant-coefficients equations* which means that all  $a_i = \text{const}, i = 0, \dots, n$



# Homogeneous vs. Non-homogeneous ODE

## Definition 5

In (\*), if  $g(t) \equiv 0$ , then the equation is called *homogeneous*.

- Example:

$$\dot{x} = ax$$

- Otherwise, the equation is called *non-homogeneous*

- Example:

$$\dot{x} = ax + b, \quad b \neq 0$$

# Autonomous vs. Non-autonomous ODE

## Definition 6

*Autonomous* ODE are equations in which time is not explicitly present as an independent variable.

- Therefore, such equations are also popular as *time-invariant* ODE
- The equations in the previous slide, for example, are autonomous
- A non-autonomous one would be for instance:

$$\dot{x} = ax + be^{-t}, \quad b \neq 0$$

# Solutions to Linear ODE

- Take (\*) once again

## Definition 7

A solution to this equation (if it exists) in an interval  $I = \{t : a < t < b\}$  is the  $n$ -times differentiable function  $\phi(t)$  which is defined on  $I$  and is such that when substituted in (\*), satisfies it exactly over the whole interval.

- 1 The graph of the solution of (\*) is called the *solution curve* (or *integral curve*)
- 2 The set of all solutions is called the *general solution*
- 3 Any specific solution that satisfies the differential equation is called a *particular solution*

# Explicit and Implicit Solutions; Graphical Solutions

- Some equations have *explicit solutions*, i.e. the unknown functions can be expressed directly in terms of the independent variable(s)
- Sometimes, however, this is impossible, therefore solutions could also be present as *implicit* ones
- Graphical solution can be *quantitative* if the exact values of the function are known
- If the exact values are unknown but could be reasonably approximated, the solutions are called *qualitative*

# Initial-Value Problems

- As far as solutions imply integration, finding the unknown function(s) is in fact finding a family of functions
- Often we are interested in a particular member of this family that passes through a specified point
- The latter specifies the *initial-value problem*
- Initial-value problems are common in economics (example: the path of the capital stock)

# Qualitative Theory

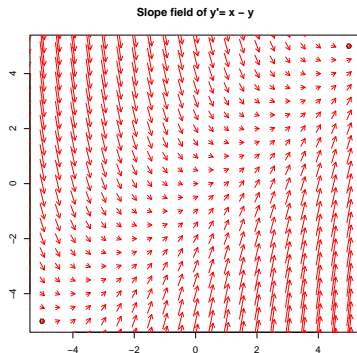
- Not all differential equations have explicit solutions
- Such are not always needed, however
- It is enough to know some of the properties of the dynamic system
- Qualitative theory offers the necessary existence and uniqueness theorems, etc.

# Slope Fields

- Also called *direction diagrams*
- They are stylized graphical representations of all solutions to a dynamic system
- For example, for the equation

$$\dot{y} = x - y$$

we have



# Separable Equations



# Separable Equations

- Suppose we have the following first-order ODE:

$$\dot{x} = F(x, t)$$

- Suppose also that  $F(x, t)$  can be written as the product of two functions, one of  $t$  only, and one of  $x$  only:

$$\dot{x} = f(t) \cdot g(x) \quad (**)$$

## Definition 8

Equations that can be represented as the product of such functions are called *separable*.

# Separable Equations (2)

- Examples:

$$\dot{x} = -2tx^2 \quad (\text{separable})$$

$$\dot{x} = -2t + x^2 \quad (\text{non - separable})$$

- If  $g(x)$  has a zero at  $x = a$  (i.e.  $g(a) = 0$ ), then  $x(t) \equiv a$  is a particular solution to (\*\*)
- Why? Because  $\dot{x}(t) = \dot{a} = 0$  (LHS) and also  $f(t) \cdot g(x) = f(t) \cdot g(a) = 0$  (RHS)

# Separable Equations: Solution Steps

- ① Write  $(**)$  as follows:

$$\frac{dx}{dt} = f(t) \cdot g(x)$$

- ② Separate variables:

$$\frac{dx}{g(x)} = f(t)dt$$

- ③ Integrate:

$$\int \frac{dx}{g(x)} = \int f(t)dt$$

- ④ Evaluate integral, find solution (explicit or implicit)

- ⑤ Every zero  $x = a$  of  $g(x)$  gives a particular solution  $x(t) \equiv a$   
(We will solve some examples.)

# Solved Examples of Separable Equations

- **Example 1:** Solve the equation:

$$\frac{dx}{dt} = \frac{1}{x^2}$$

- **Solution:**

$$x^2 dx = dt$$

$$\int x^2 dx = \int dt$$

$$\frac{x^3}{3} = t + C \Rightarrow x = \sqrt[3]{3(t + C)}$$

$$\text{Check: } \frac{dx}{dt} = \frac{1}{3} \cdot [3(t + C)]^{-2/3} \cdot 3 = \frac{1}{\left(\sqrt[3]{3(t + C)}\right)^2}$$

## Solved Examples of Separable Equations (2)

- **Example 2:**<sup>1</sup> Radioactive decay of atomic particles:

$$\frac{dn}{dt} = -\lambda n, \quad \lambda > 0$$

where  $\frac{dn}{dt}$  is the number of atoms that degenerate per unit of time, and  $\lambda$  is the decay constant

- **Solution:**

$$\frac{dn}{n} = -\lambda dt$$

$$\int \frac{dn}{n} = -\lambda \int dt$$

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<sup>1</sup>Borrowed from Shone (2002), p. 48.

## Solved Examples of Separable Equations (3)

- Solution (cont'd):

$$\ln n = -\lambda t + C$$

$$n = e^{-\lambda t + C}$$

This can be written also as follows:

$$n = C_1 e^{-\lambda t}$$

where  $C_1 = e^{-C}$

(This result is relevant to calculating the so-called *half-life*.)

# First-order Linear ODE

# First-order Linear ODE

- First-order linear ODE have the following general representation:

$$\dot{x} + a(t)x = b(t)$$

where  $a(t)$  and  $b(t)$  are continuous functions of time in a specified interval

- The simplest version is when the equation has constant coefficients:

$$\dot{x} + ax = b, \quad a \neq 0$$

- Multiply both sides by  $e^{at}$  (called *integrating factor*):

$$\dot{x}e^{at} + axe^{at} = be^{at}$$



# First-order Linear ODE (2)

- Observe that the LHS is the derivative of  $xe^{at}$ , i.e. the equation can also be written as:

$$\frac{d}{dt}(xe^{at}) = be^{at}$$

- Integrate:

$$xe^{at} = \int be^{at} dt + C \Rightarrow xe^{at} = \frac{b}{a}e^{at} + C$$

where  $C$  is an arbitrary constant

- Multiply both sides by  $e^{-at}$ :

$$x = \frac{b}{a} + Ce^{-at}$$

- The latter is the solution of the ODE

# First-order Linear ODE (3)

- If  $C = 0$ , then  $x = \frac{b}{a}$
- The latter is called the *equilibrium/stationary/steady state* of the equation
- This solution is obtained also if we set  $\dot{x} = 0$
- If  $a > 0$ , then

$$\lim_{t \rightarrow \infty} x = \lim_{t \rightarrow \infty} \left( \frac{b}{a} + Ce^{-at} \right) = \frac{b}{a}$$

- In this case the steady state is *stable*

# Variable RHS

- In this case the equation has the following form:

$$\dot{x} + ax = b(t)$$

- Multiply again by  $e^{at}$ :

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at}$$

- Use again the fact that the LHS is the derivative of  $xe^{at}$ :

$$\frac{d}{dt}(xe^{at}) = b(t)e^{at}$$

- Integrate:

$$xe^{at} = \int b(t)e^{at} dt + C$$

- Multiply both sides by  $e^{-at}$  to yield the solution:

$$x = e^{-at} \int b(t)e^{at} dt + Ce^{-at}$$

# The General Case

- The ODE has the following form:

$$\dot{x} + a(t)x = b(t)$$

- Multiply by an appropriately chosen integrating factor  $e^{A(t)}$ :

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}$$

- $A(t)$  has to be such that the LHS is the derivative of  $xe^{A(t)}$
- The derivative of  $xe^{A(t)}$  is:

$$\frac{d}{dt}(xe^{A(t)}) = \dot{x}e^{A(t)} + x\dot{A}(t)e^{A(t)}$$

## The General Case (2)

- The latter implies that  $\dot{A}(t) = a(t)$
- Therefore  $A(t)$  can be chosen in the following way:

$$A(t) = \int a(t) dt$$

- Thus,

$$\frac{d}{dt}(xe^{A(t)}) = b(t)e^{A(t)}$$

- Integrate to yield:

$$xe^{A(t)} = \int b(t)e^{A(t)} dt + C$$

## The General Case (3)

- Multiply by  $e^{-A(t)}$  to yield the solution:

$$x = e^{-A(t)} \int b(t)e^{A(t)} dt + Ce^{-A(t)}$$

where  $A(t) = \int a(t) dt$

# Qualitative Theory and Stability

# Qualitative Theory and Stability

- Recall that there are many cases in which explicit solutions cannot be found
- Nevertheless, the properties of solutions can be often studied reasonably well
- We will discuss some results concerning autonomous (time-invariant) equations
- They can be represented as a special case of the equation

$$\dot{x} = F(x, t),$$

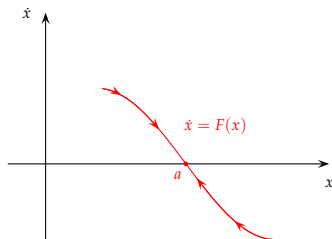
namely

$$\dot{x} = F(x)$$



# Autonomous Equations and Stability

- Take a look at the following phase diagram (a two-dimensional graph where the derivative is on the  $y$  axis, and the variable – on the horizontal one):

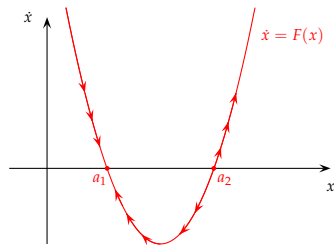


- To the left of  $a$ ,  $\dot{x} > 0$ , therefore  $x$  is increasing; to the right of it,  $\dot{x} < 0$ , therefore  $x$  is decreasing

# Autonomous Equations and Stability (2)

- In the latter, the point  $a$  is an equilibrium point as  $F(a) = 0$
- In other words,  $x$  is not changing at this point, it is stationary there
- The graph displays a case of *globally asymptotically stable* equilibrium
- Explanation:  $x$  will always return to equilibrium no matter where it starts from
- In the following graph, we have two different types of equilibria

# Autonomous Equations and Stability (3)



- Here,  $a_1$  is a *locally stable equilibrium*, while  $a_2$  is *unstable*

## A summary of stability conditions

- $F(a) = 0$  and  $F'(a) < 0 \Rightarrow a$  is locally stable
- $F(a) = 0$  and  $F'(a) > 0 \Rightarrow a$  is unstable
- $F(a) = 0$  and  $F'(a) = 0 \Rightarrow$  inconclusive

# Existence and Uniqueness

# Existence and Uniqueness

## Theorem 1

*Given the first-order ODE:*

$$\dot{x} = F(t, x),$$

*suppose that  $F(t, x)$  and  $F'_x(t, x)$  are continuous in an open set  $A$  in the  $tx$ -plane. Take an arbitrary point  $(t_0, x_0) \in A$ . Then there exists a unique “local” solution that passes through  $(t_0, x_0)$ .*

- “local” means that existence of solution is guaranteed for a small neighbourhood of  $t_0$  only

# Global Existence and Uniqueness

## Theorem 2

*Consider the initial-value problem:*

$$\dot{x} = F(t, x), \quad x(t_0) = x_0$$

*Suppose that  $F(t, x)$  and  $F'_x(t, x)$  are continuous  $\forall (t, x)$ . Suppose also there exist continuous functions  $a(t)$  and  $b(t)$  such that*

$$|F(t, x)| \leq a(t)|x| + b(t), \quad \forall (t, x)$$

*For an arbitrary point  $(t_0, x_0)$  there exists a unique solution defined on  $(-\infty, \infty)$ . If the above condition is weakened to*

$$xF(t, x) \leq a(t)|x|^2 + b(t), \quad \forall x \text{ and } \forall t \geq t_0$$

*then there is a unique solution of the problem on  $[t_0, +\infty)$ .*

# Second-Order Equations

# Second-Order Equations

- Typical second-order equations have the following form:

$$\ddot{x} = F(t, x, \dot{x})$$

- A solution to that equation on an interval  $I$  is a  $C^2$  function that satisfies it
- A very simple example:

$$\ddot{x}(t) = k, \quad k = \text{const}$$

- The solution with respect to the unknown function  $x(t)$  is obtained after integrating the equation twice



# Special Cases Where $x$ or $t$ is Missing

- **Case 1:**  $x$  is missing in the RHS

$$\ddot{x} = F(\dot{x}, t)$$

- Solution: introduce a variable  $u = \dot{x}$
- Then, the equation can be reduced to a first-order one, i.e.

$$\dot{u} = F(u, t)$$

- After solving this for the unknown function  $u(t)$ , then we go to solve the first-order equation

$$\dot{x} = u$$

**Example:**  $\ddot{x} = \dot{x} + t$

## Special Cases Where $x$ or $t$ is Missing (2)

- **Case 2:**  $t$  is missing in the RHS
- In such a case, the equation is autonomous
- It can be transformed into an equation in terms of  $t$  and then again reduced to a first-order equation
- Examples are left to your curiosity (see e.g. Problem 6 on p. 225)

# Linear Second-Order ODE

- General form:

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t)$$

- Unlike first-order equations, there is no explicit solution in the general case
- Nevertheless, the general structure of the solution can still be characterized
- Start with the homogeneous version of the above equation:

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0$$

- By the *superposition principle*<sup>2</sup>, if  $u_1 = u_1(t)$  and  $u_2 = u_2(t)$  both satisfy the homogeneous equation, then  $x = Au_1 + Bu_2$  will also satisfy it,  $\forall A, B$

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<sup>2</sup><http://mathworld.wolfram.com/SuperpositionPrinciple.html>

# Linear Second-Order ODE (2)

- Indeed, since  $\dot{x} = A\dot{u}_1 + B\dot{u}_2$  and  $\ddot{x} = A\ddot{u}_1 + B\ddot{u}_2$ :

$$\begin{aligned}\ddot{x} + a(t)\dot{x} + b(t)x &= A\ddot{u}_1 + B\ddot{u}_2 + a(t)(A\dot{u}_1 + B\dot{u}_2) + b(t)(Au_1 + Bu_2) = \\ &= A[\ddot{u}_1 + a(t)\dot{u}_1 + b(t)u_1] + B[\ddot{u}_2 + a(t)\dot{u}_2 + b(t)u_2]\end{aligned}$$

- But since  $u_1$  and  $u_2$  satisfy the equation, then it is clear that the RHS of the latter equals 0
- This directly shows that  $x = Au_1 + Bu_2$  also satisfies it,  $\forall A, B$
- However,  $u_1$  and  $u_2$  must not be proportional to each other

# Linear Second-Order ODE (3)

- Suppose that a *particular solution*  $u^* = u^*(t)$  to the non-homogeneous equation can be found
- If  $x(t)$  is an arbitrary solution of the non-homogeneous equation, then  $x(t) - u^*(t)$  will be a solution to its homogeneous counterpart
- To see this, set  $v = v(t) = x(t) - u^*(t)$
- Then,  $\dot{v} = \dot{x} - \dot{u}^*$  and  $\ddot{v} = \ddot{x} - \ddot{u}^*$ , so that

$$\begin{aligned}
 \ddot{v} + a(t)\dot{v} + b(t)v &= \ddot{x} - \ddot{u}^* + a(t)(\dot{x} - \dot{u}^*) + b(t)(x - u^*) = \\
 &= \ddot{x} + a(t)\dot{x} + b(t)x - [\ddot{u}^* + a(t)\dot{u}^* + b(t)u^*] = \\
 &= f(t) - f(t) = 0
 \end{aligned}$$

# Linear Second-Order ODE (4)

- The latter showed that  $x(t) - u(t)^*$  is a solution to the homogeneous equation
- But then we can write (using the above arguments) that:

$$x(t) - u(t)^* = Au_1(t) + Bu_2(t)$$

where  $u_1(t)$  and  $u_2(t)$  are two non-proportional solutions of the homogeneous equation, and  $A$  and  $B$  are two arbitrary constants

- The latter implies that if  $x(t) - u(t)^*$  is a solution to the homogeneous equation,  $x(t)$  is a solution to the non-homogeneous one

# Linear Second-Order ODE (5)

## Theorem 3

(a) The **general solution** of the homogeneous equation is

$$x(t) = Au_1(t) + Bu_2(t)$$

where  $u_1(t)$  and  $u_2(t)$  are two non-proportional (linearly independent) solutions, and  $A$  and  $B$  are two arbitrary constants

(b) The **general solution** of the non-homogeneous equation is

$$x(t) = Au_1(t) + Bu_2(t) + u^*(t)$$

where  $Au_1(t) + Bu_2(t)$  is the general solution of the homogeneous equation, and  $u^*(t)$  is any **particular solution** of the non-homogeneous equation

# Constant Coefficients

- The homogeneous equation has the following form:

$$\ddot{x} + a\dot{x} + bx = 0$$

- Using the theorem, we look for two non-proportional solutions  $u_1(t)$  and  $u_2(t)$
- Since the coefficients are constants, we try solutions such that  $x$ ,  $\dot{x}$ , and  $\ddot{x}$  are constant multiples of each other
- The function that has such a property is  $x = e^{rt}$  (because  $\dot{x} = re^{rt} = rx$  and  $\ddot{x} = r^2e^{rt} = r^2x$ )



## Constant Coefficients (2)

- Substitute it in the equation:

$$r^2 e^{rt} + a r e^{rt} + b e^{rt} = 0$$

- Divide both sides by  $e^{rt} \neq 0$  to get

$$r^2 + ar + b = 0$$

- The latter is the *characteristic equation* of the differential equation
- Finding the values of  $r$  from it allows to find the solutions  $x = e^{rt}$  that satisfy the homogeneous equation

# Constant Coefficients (3)

## Theorem 4

The **general solution** of the constant-coefficients homogeneous equation is determined by the roots of its characteristic equation as follows:

(I) If the discriminant  $a^2 - 4b > 0$ , there are two distinct real roots, and:

$$x = Ae^{r_1 t} + Be^{r_2 t}, \quad r_{1,2} = -\frac{a \pm \sqrt{a^2 - 4b}}{2}$$

(II) If  $a^2 - 4b = 0$ , then there is a double real root, and:

$$x = (A + Bt)e^{rt}, \quad r = -\frac{a}{2}$$

(III) If  $a^2 - 4b < 0$ , then there are two complex conjugate roots  $r_{1,2} = \alpha \pm \beta i$ , and:

$$x = e^{\alpha t} (A \cos \beta t + B \sin \beta t), \quad \alpha = -\frac{a}{2}, \quad \beta = \frac{\sqrt{4b - a^2}}{2}$$

# The Non-Homogeneous Equation

- Has the following form:

$$\ddot{x} + a\dot{x} + bx = f(t)$$

where  $f(t)$  is an arbitrary continuous function

- According to Theorem 3 (b), its general solution is:

$$x(t) = Au_1(t) + Bu_2(t) + u^*(t)$$

- Finding two solutions  $u_1(t)$  and  $u_2(t)$  is clear from the homogeneous case
- What's needed is a method to find  $u^*(t)$

# The Non-Homogeneous Equation (2)

- One way is to use the *method of undetermined coefficients*
- If  $b = 0$ , then the equation can be transformed into a first order one (after setting  $u = \dot{x}$ )
- Assuming  $b \neq 0$ , four special cases are discussed here
- **Case 1:**  $f(t) = A = \text{const}$
- The first step is to check whether the particular solution is a constant, i.e.  $u^* = c$
- If so, then  $\dot{u}^* = \ddot{u}^* = 0$ , and the equation reduces to  $bc = A \Rightarrow c = A/b$
- Therefore, for  $b \neq 0$  the particular solution of

$$\ddot{x} + a\dot{x} + bx = A$$

has the particular solution

$$u^* = A/b$$

# The Non-Homogeneous Equation (3)

- **Case 2:**  $f(t)$  is a polynomial of degree  $n$
- Then it is reasonable to assume that the solution is also a polynomial of degree  $n$ :

$$u^* = A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$$

- The coefficients  $A_n, A_{n-1}, \dots, A_0$  are determined so that  $u^*$  is required to satisfy the non-homogeneous equation and the coefficients of like powers of  $t$  are equated
- **Example:** solve  $\ddot{x} - 4\dot{x} + 4x = t^2 + 2$

# The Non-Homogeneous Equation (4)

- **Case 3:**  $f(t) = pe^{qt}$
- A solution of the form  $u^* = Ae^{qt}$  is tried
- With this,  $\dot{u}^* = Aqe^{qt}$  and  $\ddot{u}^* = Aq^2e^{qt}$
- Substituting the latter two into the equation leads to:

$$Ae^{qt}(q^2 + aq + b) = pe^{qt}$$

- Three possibilities exist:
  - ①  $q^2 + aq + b \neq 0$ , i.e.  $q$  is not a solution of the characteristic equation (in other words,  $e^{qt}$  is not a solution of the homogeneous equation)
  - ②  $q$  is a simple root of  $q^2 + aq + b = 0$
  - ③  $q$  is a double root of  $q^2 + aq + b = 0$

# The Non-Homogeneous Equation (5)

- In case 3.1, the particular solution of the equation  $\ddot{x} + a\dot{x} + bx = pe^{qt}$  is

$$u^* = \frac{p}{q^2 + aq + b} e^{qt}$$

- In case 3.2, a constant  $B$  is sought such that  $Bte^{qt}$  is a solution
- In case 3.3, a constant  $C$  is sought such that  $Ct^2e^{qt}$  is a solution

# The Non-Homogeneous Equation (6)

- **Case 4:**  $f(t) = p \sin(rt) + q \cos(rt)$
- The method of undetermined coefficients is used again
- Let  $u^* = A \sin(rt) + B \cos(rt)$
- The constants  $A$  and  $B$  are adjusted so that the coefficients of  $\sin rt$  and  $\cos rt$  match
- If  $f(t)$  is itself a solution of the homogeneous equation, then for suitable choices of  $A$  and  $B$ , the particular solution equals:

$$u^* = At \sin(rt) + Bt \cos(rt)$$



# Stability

## Stability

The equation

$$\ddot{x} + a\dot{x} + bx = f(t)$$

is *globally asymptotically stable* iff both roots of its characteristic equation have negative real parts.

- Note that this result extends to equations of higher order
- That an equation is globally asymptotically stable means that every solution  $Au_1(t) + Bu_2(t)$  of the associated homogeneous equation tends to 0 as  $t \rightarrow \infty, \forall A, B$

# Systems of Differential Equations

# Systems of Differential Equations

- *Normal form* for systems of differential equations:

$$\begin{array}{lcl} \dot{x}_1 & = & f_1(t, x_1, x_2, \dots, x_n) \\ \dot{x}_2 & = & f_2(t, x_1, x_2, \dots, x_n) \\ & \dots & \\ \dot{x}_n & = & f_n(x_1, x_2, \dots, x_n) \end{array} \quad (\spadesuit)$$

- In other words, derivatives should be located only in the LHSs of equations
- Also, there is one derivative per equation
- Finally, all derivatives are first-order only

# Systems of Differential Equations (2)

- Example of a system of differential equations:

$$\begin{cases} \ddot{x}_1 &= F_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) \\ \ddot{x}_2 &= F_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2) \end{cases}$$

- This system is *not* in normal form!
- To transform it to a normal form, introduce new variables:

$$u_1 = x_1, \quad u_2 = x_2, \quad u_3 = \dot{x}_1, \quad u_4 = \dot{x}_2$$

- The system becomes

$$\begin{cases} \dot{u}_1 &= u_3 \\ \dot{u}_3 &= f_1(t, u_1, u_2, u_3, u_4) \\ \dot{u}_2 &= u_4 \\ \dot{u}_4 &= f_2(t, u_1, u_2, u_3, u_4) \end{cases}$$

# Systems of Differential Equations (3)

- A set of functions  $x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t)$  that satisfies the system (♠) is called a *solution*
- In  $\mathbb{R}^n$  the graph of the solution is a surface
- The vector  $\dot{\mathbf{x}}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dots, \dot{x}_n(t))$  is called the *velocity vector*
- The space spanned by all vectors  $(x_1(t), x_2(t), \dots, x_n(t))$  is called the *phase space*
- If all functions  $f_i(t, x_1(t), x_2(t), \dots, x_n(t))$  are collected in a vector  $\mathbf{F}(t, \mathbf{x}(t))$ , then the system of differential equations can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t))$$

# Systems of Differential Equations (4)

- In economics the functions  $x_1(t), x_2(t), \dots, x_n(t)$  are called *state variables* as they describe the state of the economy at each time
- If the state of the economy at some time  $t_0$  is given, i.e.

$$\mathbf{x}(t_0) = (x_1(t_0), x_2(t_0), \dots, x_n(t_0))$$

is known, the specific solution of the system passing through this point could be found

- Existence and uniqueness of the solution is guaranteed by the condition that  $f_i$  and  $\frac{\partial f_i}{\partial x_j}$  are all continuous for  $i, j = 1, 2, \dots, n$

# Linear Systems

- General form:

$$\left| \begin{array}{lcl} \dot{x}_1(t) & = & a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \dots + a_{1n}(t)x_n(t) + b_1(t) \\ \dot{x}_2(t) & = & a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \dots + a_{2n}(t)x_n(t) + b_2(t) \\ \dots\dots\dots & & \dots\dots\dots \\ \dot{x}_n(t) & = & a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \dots + a_{nn}(t)x_n(t) + b_n(t) \end{array} \right.$$

- Using matrix notation, the latter can also be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$$

- If the coefficients  $a_{ij}$ ,  $i, j = 1, 2, \dots, n$  are all constants, then the equation becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \quad (\diamond)$$

- Equations of this type can always be solved explicitly

# Linear Systems (2)

- The system defined by ( $\diamond$ ) is globally asymptotically stable if and only if all the eigenvalues of  $\mathbf{A}$  have negative real parts
- Assume that the system is autonomous, i.e.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$$

- To solve it based on eigenvalues, first consider its homogeneous counterpart:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

- The goal is to find numbers  $\lambda$  and  $v_1, v_2, \dots, v_n$  such that the vector function  $\mathbf{x} = \mathbf{v}e^{\lambda t}$  satisfies the homogeneous system



# Linear Systems (3)

- Differentiate  $\mathbf{x} = \mathbf{v}e^{\lambda t}$  with respect to  $t$ :

$$\dot{\mathbf{x}}(t) = \lambda \mathbf{v}e^{\lambda t}$$

- Then the homogeneous system becomes

$$\lambda \mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} \Rightarrow \mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

- The latter implies that any non-zero solution is an eigenvector of  $\mathbf{A}$  with a corresponding eigenvalue  $\lambda$
- If  $\mathbf{A}$  has  $n$  different eigenvalues then the general solution of the homogeneous system is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 + \dots + C_n e^{\lambda_n t} \mathbf{v}_n$$

# Linear Systems (4)

- Concerning the solution of the non-homogeneous system, if  $\mathbf{x}^0$  is an equilibrium point, then

$$\mathbf{A}\mathbf{x}^0 + \mathbf{b} = \mathbf{0}$$

- Define the deviation of  $\mathbf{x}(t)$  from this equilibrium point as

$$\mathbf{w}(t) = \mathbf{x}(t) - \mathbf{x}^0$$

- Clearly,  $\dot{\mathbf{w}} = \dot{\mathbf{x}}$ , and therefore

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \Leftrightarrow \dot{\mathbf{w}}(t) = \mathbf{A}(\mathbf{w}(t) + \mathbf{x}^0) + \mathbf{b}$$

- Uncover the parentheses in the far RHS:

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \underbrace{\mathbf{A}\mathbf{x}^0 + \mathbf{b}}_{=0} = \mathbf{A}\mathbf{w}(t)$$

i.e. the non-homogeneous system can be transformed into a homogeneous one

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