# R406: Applied Economic Modelling with Python

## Ordinary Differential Equations of First, Second, and Higher Order

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# Lecture Contents

# Introduction

# From difference to differential equations

- To provide an intuitive motivation for differential equations, consider a situation where we have a quantity of interest $y(t)$ that evolves in continuous time
- However, this quantity is observed at discrete points in time and we can choose the step between adjacent observations
- Given a time step $h$, our observations are recorded at times $\tau,\ \tau+h,\ \tau+2h,\ \tau+3h,\ \ldots$
- The above can always be re-labelled to "lose" the time step by setting

$$t+k=\tau+kh,\ k=0,1,2,\ldots$$

# From difference to differential equations (2)

- Assume that such a process can be described by a difference equation of the form

$$\frac{1}{h}(y_{s+1} - y_s) = f(y_s)$$

i.e.

$$\frac{1}{h}(y_{t+h} - y_t) = f(y_t)$$

- If we let the time step $h$ tend to 0, i.e. we observe with increasing frequency, the left-hand side will tend to the derivative $\dot{y}(t) := \frac{dy(t)}{dt}$
- In the limit, the recursive relationship will change to a relationship the form

$$\dot{y}(t) = f(y(t))$$

- The last expression is a relationship between an unknown function $y(t)$ and its first derivative

# What is a Differential Equation?

**Definition 1**

A differential equation is an equation that relates an unknown function with one or more of its derivatives.

- Note that a differential equation relates to an *unknown function* not to an *unknown number*
- Note that such an equation can have one or more variables, constants, etc.
- Given two variables $x$ and $y$, the general form of a differential equation in those two variables could be:

$$\frac{dy}{dx} = f(x, y)$$

- We will be considering only variables that are functions of time, therefore we can speak equivalently of *dynamic systems*
- Time itself (denoted by $t$) is considered a continuous variable

# Ordinary Differential Equations

**Definition 2**

An ordinary differential equation (ODE) is a differential equation that includes the derivatives of only one function of time.

- Example:

$$\dot{x} = ax + b$$

  This is not an ODE:

$$\dot{x} = ax + b + \dot{y} + cy$$

- Note that we will be using Newton's *dot notation* of derivatives
- Just to be aware: equations including the partial derivatives of unknown functions of more than one variable are called *partial differential equations*

# ODE Order

**Definition 3**

The order of an ODE is defined by the order of the highest derivative present in the equation.

- Examples:

$$\dot{x} = ax \text{ (first-order ODE)}$$

$$\ddot{x} = \dot{x} + z \text{ (second-order ODE)}$$

etc.

# Linear vs. Non-linear ODE

**Definition 4**

A linear ODE satisfies the following:

- The dependent variable and its derivatives appear to the power of one only
- There are no products of the dependent variable and/or its derivatives in the equation
- There are no transcendental functions[a] of the dependent variable and/or its derivatives

[a] Functions that cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.

- A non-linear ODE is one not fitting the above definition
- Example:

$$\dot{x} = \frac{1}{t} - 2\cos(x)$$

# ODE: Constant vs. Variable Coefficients

- Consider the general form of an $n$th-order linear ODE:

$$a_0(t)\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_n(t)x = g(t) \quad (*)$$

  where $g(t)$ is some function of time
- Some equations might involve coefficients which themselves are functions of $t$
- Those are equations with *variable coefficients*
- With some exceptions, we will usually work with *constant-coefficients equations* which means that all $a_i = const, i = 0, \ldots, n$

# Homogeneous vs. Non-homogeneous ODE

**Definition 5**

In $(*)$, if $g(t) \equiv 0$, then the equation is called *homogeneous*.

- Example:

$$\dot{x} = ax$$

- Otherwise, the equation is called *non-homogeneous*
- Example:

$$\dot{x} = ax + b, \quad b \neq 0$$

# Autonomous vs. Non-autonomous ODE

**Definition 6**

*Autonomous* ODE are equations in which time is not explicitly present as an independent variable.

- Therefore, such equations are also popular as *time-invariant* ODE
- The equations in the previous slide, for example, are autonomous
- A non-autonomous one would be for instance:

$$\dot{x} = ax + be^{-t}, \quad b \neq 0$$

# Solutions to Linear ODE

- Take $(*)$ once again

## Definition 7

A solution to this equation (if it exists) in an interval $I = \{t : a < t < b\}$ is the $n$-times differentiable function $\phi(t)$ which is defined on $I$ and is such that when substituted in $(*)$, satisfies it exactly over the whole interval.

1. The graph of the solution of $(*)$ is called the *solution curve* (or *integral curve*)
2. The set of all solutions is called the *general solution*
3. Any specific solution that satisfies the differential equation is called a *particular solution*

# Explicit and Implicit Solutions; Graphical Solutions

- Some equations have *explicit solutions*, i.e. the unknown functions can be expressed directly in terms of the independent variable(s)
- Sometimes, however, this is impossible, therefore solutions could also be present as *implicit* ones
- Graphical solution can be *quantitative* if the exact values of the function are known
- If the exact values are unknown but could be reasonably approximated, the solutions are called *qualitative*

# Initial-Value Problems

- As far as solutions imply integration, finding the unknown function(s) is in fact finding a family of functions
- Often we are interested in a particular member of this family that passes through a specified point
- The latter specifies the *initial-value problem*
- Initial-value problems are common in economics (example: the path of the capital stock)

# Qualitative Theory

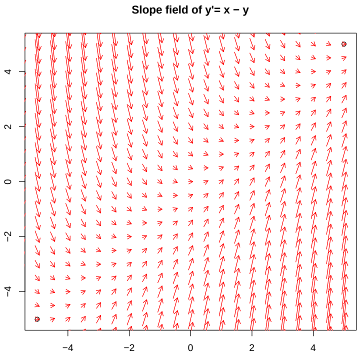
- Not all differential equations have explicit solutions
- Such are not always needed, however
- It is enough to know some of the properties of the dynamic system
- Qualitative theory offers the necessary existence and uniqueness theorems, etc.

# Slope Fields

- Also called *direction diagrams*
- They are stylized graphical representations of all solutions to a dynamic system
- For example, for the equation

$$dy/dx = x - y$$

we have



# Separable Equations

# Separable Equations

- Suppose we have the following first-order ODE:

$$\dot{x} = F(x, t)$$

- Suppose also that $F(x, t)$ can be written as the product of two functions, one of $t$ only, and one of $x$ only:

$$\dot{x} = f(t) \cdot g(x) \quad (**)$$

**Definition 8**

Equations that can be represented as the product of such functions are called *separable*.

# Separable Equations (2)

- Examples:

$$\dot{x} = -2tx^2 \quad (separable)$$

$$\dot{x} = -2t + x^2 \quad (non-separable)$$

- If $g(x)$ has a zero at $x = a$ (i.e. $g(a) = 0$), then $x(t) \equiv a$ is a particular solution to $(**)$
- Why? Because $\dot{x}(t) = \dot{a} = 0$ (LHS) and also $f(t) \cdot g(x) = f(t) \cdot g(a) = 0$ (RHS)

# Separable Equations: Solution Steps

1. Write $(**)$ as follows:
$$\frac{dx}{dt} = f(t) \cdot g(x)$$
2. Separate variables:
$$\frac{dx}{g(x)} = f(t)dt$$
3. Integrate:
$$\int \frac{dx}{g(x)} = \int f(t)dt$$
4. Evaluate integral, find solution (explicit or implicit)
5. Every zero $x = a$ of $g(x)$ gives a particular solution $x(t) \equiv a$

(We will solve some examples.)

# Solved Examples of Separable Equations

- Example 1: Solve the equation:

$$\frac{dx}{dt} = \frac{1}{x^2}$$

- Solution:

$$x^2\,dx = dt$$

$$\int x^2\,dx = \int dt$$

$$\frac{x^3}{3} = t + C \Rightarrow x = \sqrt[3]{3(t+C)}$$

Check: $\frac{dx}{dt} = \frac{1}{3} \cdot [3(t+C)]^{-2/3} \cdot 3 = \frac{1}{\left(\sqrt[3]{3(t+C)}\right)^2}$

# Solved Examples of Separable Equations (2)

- Example 2:[1] Radioactive decay of atomic particles:

$$\frac{dn}{dt} = -\lambda n, \quad \lambda > 0$$

where $\frac{dn}{dt}$ is the number of atoms that degenerate per unit of time, and $\lambda$ is the decay constant

- Solution:

$$\frac{dn}{n} = -\lambda \, dt$$

$$\int \frac{dn}{n} = -\lambda \int dt$$

[1] Borrowed from Shone (2002), p. 48.

# Solved Examples of Separable Equations (3)

- Solution (cont'd):

$$\ln n = -\lambda t + C$$

$$n = e^{-\lambda t + C}$$

This can be written also as follows:

$$n = C_1 e^{-\lambda t}$$

where $C_1 = e^{-C}$

(This result is relevant to calculating the so-called *half-life*.)

# First-order Linear ODE

# First-order Linear ODE

- First-order linear ODE have the following general representation:

$$\dot{x} + a(t)x = b(t)$$

  where $a(t)$ and $b(t)$ are continuous functions of time in a specified interval
- The simplest version is when the equation has constant coefficients:

$$\dot{x} + ax = b, \quad a \neq 0$$

- Multiply both sides by $e^{at}$ (called *integrating factor*):

$$\dot{x}e^{at} + axe^{at} = be^{at}$$

# First-order Linear ODE (2)

- Observe that the LHS is the derivative of $xe^{at}$, i.e. the equation can also be written as:

$$\frac{d}{dt}(xe^{at}) = be^{at}$$

- Integrate:

$$xe^{at} = \int be^{at}\,dt + C \Rightarrow xe^{at} = \frac{b}{a}e^{at} + C$$

  where $C$ is an arbitrary constant

- Multiply both sides by $e^{-at}$:

$$x = \frac{b}{a} + Ce^{-at}$$

- The latter is the solution of the ODE

# First-order Linear ODE (3)

- If $C = 0$, then $x = \dfrac{b}{a}$
- The latter is called the *equilibrium/stationary/steady state* of the equation
- This solution is obtained also if we set $\dot{x} = 0$
- If $a > 0$, then

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \left( \frac{b}{a} + Ce^{-at} \right) = \frac{b}{a}$$

- In this case the steady state is *stable*

# Variable RHS

- In this case the equation has the following form:

$$\dot{x} + ax = b(t)$$

- Multiply again by $e^{at}$:

$$\dot{x}e^{at} + axe^{at} = b(t)e^{at}$$

- Use again the fact that the LHS is the derivative of $xe^{at}$:

$$\frac{d}{dt}(xe^{at}) = b(t)e^{at}$$

- Integrate:

$$xe^{at} = \int b(t)e^{at}\,dt + C$$

- Multiply both sides by $e^{-at}$ to yield the solution:

$$x = e^{-at}\int b(t)e^{at}\,dt + Ce^{-at}$$

# The General Case

- The ODE has the following form:

$$\dot{x} + a(t)x = b(t)$$

- Multiply by an appropriately chosen integrating factor $e^{A(t)}$:

$$\dot{x}e^{A(t)} + a(t)xe^{A(t)} = b(t)e^{A(t)}$$

- $A(t)$ has to be such that the LHS is the derivative of $xe^{A(t)}$
- The derivative of $xe^{A(t)}$ is:

$$\frac{d}{dt}(xe^{A(t)}) = \dot{x}e^{A(t)} + x\dot{A}(t)e^{A(t)}$$

# The General Case (2)

- The latter implies that $\dot{A}(t) = a(t)$
- Therefore $A(t)$ can be chosen in the following way:

$$A(t) = \int a(t)\,dt$$

- Thus,

$$\frac{d}{dt}(xe^{A(t)}) = b(t)e^{A(t)}$$

- Integrate to yield:

$$xe^{A(t)} = \int b(t)e^{A(t)}\,dt + C$$

# The General Case (3)

- Multiply by $e^{-A(t)}$ to yield the solution:

$$x = e^{-A(t)} \int b(t) e^{A(t)} \, dt + C e^{-A(t)}$$

where $A(t) = \int a(t) \, dt$

# Qualitative Theory and Stability

# Qualitative Theory and Stability

- Recall that there are many cases in which explicit solutions cannot be found
- Nevertheless, the properties of solutions can be often studied reasonably well
- We will discuss some results concerning autonomous (time-invariant) equations
- They can be represented as a special case of the equation

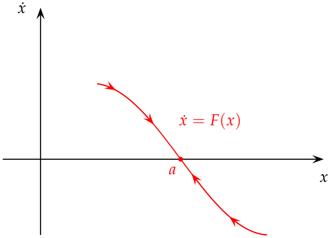
$$\dot{x} = F(x, t),$$

namely

$$\dot{x} = F(x)$$

# Autonomous Equations and Stability

- Take a look at the following phase diagram (a two-dimensional graph where the derivative is on the $y$ axis, and the variable – on the horizontal one):

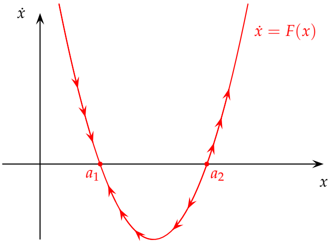



- To the left of $a$, $\dot{x} > 0$, therefore $x$ is increasing; to the right of it, $\dot{x} < 0$, therefore $x$ is decreasing

# Autonomous Equations and Stability (2)

- In the latter, the point $a$ is an equilibrium point as $F(a) = 0$
- In other words, $x$ is not changing at this point, it is stationary there
- The graph displays a case of *globally asymptotically stable* equilibrium
- Explanation: $x$ will always return to equilibrium no matter where it starts from
- In the following graph, we have two different types of equilibria

# Autonomous Equations and Stability (3)




- Here, $a_1$ is a *locally stable equilibrium*, while $a_2$ is *unstable*

## A summary of stability conditions

- $F(a)$ = 0 and $F'(a) < 0 \Rightarrow a$ is locally stable
- $F(a)$ = 0 and $F'(a) > 0 \Rightarrow a$ is unstable
- $F(a)$ = 0 and $F'(a) = 0 \Rightarrow$ inconclusive

# Existence and Uniqueness

# Existence and Uniqueness

**Theorem 1**

*Given the first-order ODE:*

$$\dot{x} = F(t, x),$$

*suppose that* $F(t, x)$ *and* $F'_x(t, x)$ *are continuous in an open set* $A$ *in the* $tx$*-plane. Take an arbitrary point* $(t_0, x_0) \in A$*. Then there exists a unique "local" solution that passes through* $(t_0, x_0)$*.*

- "local" means that existence of solution is guaranteed for a small neighbourhood of $t_0$ only

# Global Existence and Uniqueness

## Theorem 2

*Consider the initial-value problem:*

$$\dot{x} = F(t, x), \quad x(t_0) = x_0$$

*Suppose that* $F(t, x)$ *and* $F'_x(t, x)$ *are continuous* $\forall (t, x)$. *Suppose also there exist continuous functions* $a(t)$ *and* $b(t)$ *such that*

$$|F(t, x)| \leq a(t)|x| + b(t), \quad \forall (t, x)$$

*For an arbitrary point* $(t_0, x_0)$ *there exists a unique solution defined on* $(-\infty, \infty)$. *If the above condition is weakened to*

$$xF(t, x) \leq a(t)|x|^2 + b(t), \quad \forall x \text{ and } \forall t \geq t_0$$

*then there is a unique solution of the problem on* $[t_0, +\infty)$.

# Second-Order Equations

# Second-Order Equations

- Typical second-order equations have the following form:

$$\ddot{x} = F(t, x, \dot{x})$$

- A solution to that equation on an interval $I$ is a $C^2$ function that satisfies it
- A very simple example:

$$\ddot{x}(t) = k, \quad k = const$$

- The solution with respect to the unknown function $x(t)$ is obtained after integrating the equation twice

# Special Cases Where $x$ or $t$ is Missing

- Case 1: $x$ is missing in the RHS

$$\ddot{x} = F(\dot{x}, t)$$

- Solution: introduce a variable $u = \dot{x}$
- Then, the equation can be reduced to a first-order one, i.e.

$$\dot{u} = F(u, t)$$

- After solving this for the unknown function $u(t)$, then we go to solve the first-order equation

$$\dot{x} = u$$

Example: $\ddot{x} = \dot{x} + t$

# Special Cases Where $x$ or $t$ is Missing (2)

- Case 2: $t$ is missing in the RHS
- In such a case, the equation is autonomous
- It can be transformed into an equation in terms of $t$ and then again reduced to a first-order equation
- Examples are left to your curiosity (see e.g. Problem 6 on p. 225 in Sydsæter et al., 2008)

# Linear Second-Order ODE

- General form:

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t)$$

- Unlike first-order equations, there is no explicit solution in the general case
- Nevertheless, the general structure of the solution can still be characterized
- Start with the homogeneous version of the above equation:

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0$$

- By the *superposition principle*[2], if $u_1 = u_1(t)$ and $u_2 = u_2(t)$ both satisfy the homogeneous equation, then $x = Au_1 + Bu_2$ will also satisfy it, $\forall A, B$

[2] http://mathworld.wolfram.com/SuperpositionPrinciple.html

# Linear Second-Order ODE (2)

- Indeed, since $\dot{x} = A\dot{u}_1 + B\dot{u}_2$ and $\ddot{x} = A\ddot{u}_1 + B\ddot{u}_2$:

$$
\begin{aligned}
\ddot{x} + a(t)\dot{x} + b(t)x &= A\ddot{u}_1 + B\ddot{u}_2 + a(t)(A\dot{u}_1 + B\dot{u}_2) + b(t)(Au_1 + Bu_2) = \\
&= A[\ddot{u}_1 + a(t)\dot{u}_1 + b(t)u_1] + B[\ddot{u}_2 + a(t)\dot{u}_2 + b(t)u_2]
\end{aligned}
$$

- But since $u_1$ and $u_2$ satisfy the equation, then it is clear that the RHS of the latter equals 0
- This directly shows that $x = Au_1 + Bu_2$ also satisfies it, $\forall A, B$
- However, $u_1$ and $u_2$ must not be proportional to each other

# Linear Second-Order ODE (3)

- Suppose that a *particular solution* $u^* = u^*(t)$ to the non-homogeneous equation can be found
- If $x(t)$ is an arbitrary solution of the non-homogeneous equation, then $x(t) - u^*(t)$ will be a solution to its homogeneous counterpart
- To see this, set $v = v(t) = x(t) - u^*(t)$
- Then, $\dot{v} = \dot{x} - \dot{u}^*$ and $\ddot{v} = \ddot{x} - \ddot{u}^*$, so that

$$
\begin{aligned}
\ddot{v} + a(t)\dot{v} + b(t)v &= \ddot{x} - \ddot{u}^* + a(t)(\dot{x} - \dot{u}^*) + b(t)(x - u^*) = \\
&= \ddot{x} + a(t)\dot{x} + b(t)x - [\ddot{u}^* + a(t)\dot{u}^* + b(t)u^*] = \\
&= f(t) - f(t) = 0
\end{aligned}
$$

# Linear Second-Order ODE (4)

- The latter showed that $x(t) - u(t)^*$ is a solution to the homogeneous equation
- But then we can write (using the above arguments) that:

$$x(t) - u(t)^* = Au_1(t) + Bu_2(t)$$

  where $u_1(t)$ and $u_2(t)$ are two non-proportional solutions of the homogeneous equation, and $A$ and $B$ are two arbitrary constants
- The latter implies that if $x(t) - u(t)^*$ is a solution to the homogeneous equation, $x(t)$ is a solution to the non-homogeneous one

# Linear Second-Order ODE (5)

**Theorem 3**

(a) *The **general solution** of the homogeneous equation is*

$$x(t) = Au_1(t) + Bu_2(t)$$

*where $u_1(t)$ and $u_2(t)$ are two non-proportional (linearly independent) solutions, and $A$ and $B$ are two arbitrary constants*

(b) *The **general solution** of the non-homogeneous equation is*

$$x(t) = Au_1(t) + Bu_2(t) + u^*(t)$$

*where $Au_1(t) + Bu_2(t)$ is the general solution of the homogeneous equation, and $u^*(t)$ is any **particular solution** of the non-homogeneous equation*

# Constant Coefficients

- The homogeneous equation has the following form:

$$\ddot{x} + a\dot{x} + bx = 0$$

- Using the theorem, we look for two non-proportional solutions $u_1(t)$ and $u_2(t)$
- Since the coefficients are constants, we try solutions such that $x$, $\dot{x}$, and $\ddot{x}$ are constant multiples of each other
- The function that has such a property is $x = e^{rt}$ (because $\dot{x} = re^{rt} = rx$ and $\ddot{x} = r^2e^{rt} = r^2x$)

# Constant Coefficients (2)

- Substitute it in the equation:

$$r^2e^{rt} + are^{rt} + be^{rt} = 0$$

- Divide both sides by $e^{rt} \neq 0$ to get

$$r^2 + ar + b = 0$$

- The latter is the *characteristic equation* of the differential equation
- Finding the values of $r$ from it allows us to find the solutions $x = e^{rt}$ that satisfy the homogeneous equation

# Constant Coefficients (3)

## Theorem 4

*The **general solution** of the constant-coefficients homogeneous equation is determined by the roots of its characteristic equation as follows:*

(I) *If the discriminant* $a^2 - 4b > 0$*, there are two distinct real roots, and:*

$$x = Ae^{r_1 t} + Be^{r_2 t}, \quad r_{1,2} = -\frac{a \pm \sqrt{a^2 - 4b}}{2}$$

(II) *If* $a^2 - 4b = 0$*, then there is a double real root, and:*

$$x = (A + Bt)e^{rt}, \quad r = -\frac{a}{2}$$

(III) *If* $a^2 - 4b < 0$*, then there are two complex conjugate roots* $r_{1,2} = \alpha \pm \beta i$*, and:*

$$x = e^{\alpha t}(A \cos \beta t + B \sin \beta t), \quad \alpha = -\frac{a}{2}, \beta = \frac{\sqrt{4b - a^2}}{2}$$

# The Non-Homogeneous Equation

- Has the following form:
$$\ddot{x} + a\dot{x} + bx = f(t)$$
where $f(t)$ is an arbitrary continuous function
- According to Theorem 3 (b), its general solution is:
$$x(t) = Au_1(t) + Bu_2(t) + u^*(t)$$
- Finding two solutions $u_1(t)$ and $u_2(t)$ is clear from the homogeneous case
- What's needed is a method to find $u^*(t)$

# The Non-Homogeneous Equation (2)

- One way is to use the *method of undetermined coefficients*
- If $b = 0$, then the equation can be transformed into a first order one (after setting $u = \dot{x}$)
- Assuming $b \neq 0$, four special cases are discussed here
- **Case 1:** $f(t) = A = const$
- The first step is to check whether the particular solution is a constant, i.e. $u^* = c$
- If so, then $\dot{u}^* = \ddot{u}^* = 0$, and the equation reduces to $bc = A \Rightarrow c = A/b$
- Therefore, for $b \neq 0$ a particular solution of

$$\ddot{x} + a\dot{x} + bx = A$$

is the constant function

$$u^* = A/b$$

# The Non-Homogeneous Equation (3)

- **Case 2:** $f(t)$ is a polynomial of degree $n$
- Then it is reasonable to assume that the solution is also a polynomial of degree $n$:
$$u^* = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0$$
- The coefficients $A_n, A_{n-1}, \ldots, A_0$ are determined so that $u^*$ is required to satisfy the non-homogeneous equation and the coefficients of like powers of $t$ are equated
- Example: solve $\ddot{x} - 4\dot{x} + 4x = t^2 + 2$

# The Non-Homogeneous Equation (4)

- **Case 3:** $f(t) = pe^{qt}$
- A solution of the form $u^* = Ae^{qt}$ is tried
- With this, $\dot{u}^* = Aqe^{qt}$ and $\ddot{u}^* = Aq^2e^{qt}$
- Substituting the latter two into the equation leads to:

$$Ae^{qt}(q^2 + aq + b) = pe^{qt}$$

- Three possibilities exist:
  1. $q^2 + aq + b \neq 0$, i.e. $q$ is not a solution of the characteristic equation (in other words, $e^{qt}$ is not a solution of the homogeneous equation)
  2. $q$ is a simple root of $q^2 + aq + b = 0$
  3. $q$ is a double root of $q^2 + aq + b = 0$

# The Non-Homogeneous Equation (5)

- In case 3.1, the particular solution of the equation $\ddot{x} + a\dot{x} + bx = pe^{qt}$ is

$$u^* = \frac{p}{q^2 + aq + b} e^{qt}$$

- In case 3.2, a constant $B$ is sought such that $Bte^{qt}$ is a solution
- In case 3.3, a constant $C$ is sought such that $Ct^2e^{qt}$ is a solution

# The Non-Homogeneous Equation (6)

- **Case 4:** $f(t) = p\sin(rt) + q\cos(rt)$
- The method of undetermined coefficients is used again
- Let $u^* = A\sin(rt) + B\cos(rt)$
- The constants $A$ and $B$ are adjusted so that the coefficients of $\sin rt$ and $\cos rt$ match
- If $f(t)$ is itself a solution of the homogeneous equation, then for suitable choices of $A$ and $B$, the particular solution equals:

$$u^* = At\sin(rt) + Bt\cos(rt)$$

# Stability

## Stability

The equation

$$\ddot{x} + a\dot{x} + bx = f(t)$$

is *globally asymptotically stable* iff both roots of its characteristic equation have negative real parts.

- Note that this result extends to equations of higher order
- That an equation is globally asymptotically stable means that every solution $Au_1(t) + Bu_2(t)$ of the associated homogeneous equation tends to 0 as $t \to \infty, \forall A, B$

# Systems of Differential Equations

# Systems of Differential Equations

- *Normal form* for systems of differential equations:

$$\left|\begin{array}{lcl} \dot{x}_1 & = & f_1(t, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 & = & f_2(t, x_1, x_2, \ldots, x_n) \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \dot{x}_n & = & f_n(t, x_1, x_2, \ldots, x_n) \end{array}\right. \qquad (\spadesuit)$$

- In other words, derivatives should be located only in the LHSs of equations
- Also, there is one derivative per equation
- Finally, all derivatives are first-order only

# Systems of Differential Equations (2)

- Example of a system of differential equations:

$$\left| \begin{array}{rcl} \ddot{x}_1 & = & F_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2) \\ \ddot{x}_2 & = & F_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2) \end{array} \right.$$

- This system is *not* in normal form!
- To transform it to a normal form, introduce new variables:

$$u_1 = x_1, \quad u_2 = x_2, \quad u_3 = \dot{x}_1, \quad u_4 = \dot{x}_2$$

- The system becomes

$$\left| \begin{array}{rcl} \dot{u}_1 & = & u_3 \\ \dot{u}_3 & = & f_1(t, u_1, u_2, u_3, u_4) \\ \dot{u}_2 & = & u_4 \\ \dot{u}_4 & = & f_2(t, u_1, u_2, u_3, u_4) \end{array} \right.$$

# Systems of Differential Equations (3)

- A set of functions $x_1 = x_1(t), x_2 = x_2(t), \ldots, x_n = x_n(t)$ that satisfies the system ($\spadesuit$) is called a *solution*
- In $\mathbb{R}^n$ the graph of the solution is a surface
- The vector $\dot{\mathbf{x}}(t) = (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t))$ is called the *velocity vector*
- The space spanned by all vectors $(x_1(t), x_2(t), \ldots, x_n(t))$ is called the *phase space*
- If all functions $f_i(t, x_1(t), x_2(t), \ldots, x_n(t))$ are collected in a vector $\mathbf{F}(t, \mathbf{x}(t))$, then the system of differential equations can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t))$$

# Systems of Differential Equations (4)

- Sometimes the functions $x_1(t), x_2(t), \ldots, x_n(t)$ are called *state variables* as they describe the state of the system under consideration at each time
- If the state of the system at some time $t_0$ is given, i.e.

$$\mathbf{x}(t_0) = (x_1(t_0), x_2(t_0), \ldots, x_n(t_0))$$

  is known, the specific solution of the system passing through this point could be found
- Existence and uniqueness of the solution is guaranteed by the condition that $f_i$ and $\dfrac{\partial f_i}{\partial x_j}$ are all continuous for $i, j = 1, 2, \ldots, n$

# Linear Systems

- General form:

$$\left|\begin{array}{lcl} \dot{x}_1(t) & = & a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + b_1(t) \\ \dot{x}_2(t) & = & a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n(t) + b_2(t) \\ \multicolumn{3}{c}{\dotfill} \\ \dot{x}_n(t) & = & a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + b_n(t) \end{array}\right.$$

- Using matrix notation, the latter can also be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$$

- If the coefficients $a_{ij},\ i,j = 1,2,\ldots,n$ are all constants, then the equation becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \qquad (\diamondsuit)$$

- Equations of this type can always be solved explicitly

# Linear Systems (2)

- The system defined by $(\diamondsuit)$ is globally asymptotically stable if and only if all the eigenvalues of $\mathbf{A}$ have negative real parts
- Assume that the system is autonomous, i.e.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$$

- To solve it based on eigenvalues, first consider its homogeneous counterpart:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

- The goal is to find numbers $\lambda$ and $v_1, v_2, \ldots, v_n$ such that the vector function $\mathbf{x} = \mathbf{v}e^{\lambda t}$ satisfies the homogeneous system

# Linear Systems (3)

- Differentiate $\mathbf{x} = \mathbf{v}e^{\lambda t}$ with respect to $t$:

$$\dot{\mathbf{x}}(t) = \lambda \mathbf{v} e^{\lambda t}$$

- Then the homogeneous system becomes

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v} e^{\lambda t} \Rightarrow \mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

- The latter implies that any non-zero solution is an eigenvector of $\mathbf{A}$ with a corresponding eigenvalue $\lambda$
- If $\mathbf{A}$ has $n$ different eigenvalues then the general solution of the homogeneous system is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + C_n e^{\lambda_n t}\mathbf{v}_n$$

# Linear Systems (4)

- Concerning the solution of the non-homogeneous system, if $\mathbf{x}^0$ is an equilibrium point, then
$$\mathbf{A}\mathbf{x}^0 + \mathbf{b} = \mathbf{0}$$
- Define the deviation of $\mathbf{x}(t)$ from this equilibrium point as
$$\mathbf{w}(t) = \mathbf{x}(t) - \mathbf{x}^0$$
- Clearly, $\dot{\mathbf{w}} = \dot{\mathbf{x}}$, and therefore
$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \Leftrightarrow \dot{\mathbf{w}}(t) = \mathbf{A}(\mathbf{w}(t) + \mathbf{x}^0) + \mathbf{b}$$
- Uncover the parentheses in the far RHS:
$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \underbrace{\mathbf{A}\mathbf{x}^0 + \mathbf{b}}_{=\mathbf{0}} = \mathbf{A}\mathbf{w}(t)$$

  i.e. the non-homogeneous system can be transformed into a homogeneous one

# References


- Chiang, A., and K. Wainright (2004): *Fundamental Methods of Mathematical Economics*, McGraw-Hill, 4th ed., ch. 15, 16, 19
- Shone, R. (2002): *Economic Dynamics: Phase Diagrams and Their Economic Application*, Cambridge University Press, 2nd ed., ch. 2, 4
- Sydsæter, K., P. Hammond, A. Seierstad, and A. Strøm (2008): *Further Mathematics for Economic Analysis*, Prentice Hall, 2nd ed., ch. 5, 6, 7