Economic models with strategic interactions

Andrey Vassilev

Lecture Contents

Strategic interactions and game theory

What is a game?

- You are already familiar with the idea that rational economic agents will pursue well-defined objectives under certain constraints.
- This is typically formalized as an optimization problem of some sort (e.g. a static constrained optimization problem or an optimal control problem).
- In simpler setups, it is assumed that the decision maker controls (optimizes over) certain variables and takes others as completely exogenous. Thus, the outcome will (rather, seems to) depend only on their own decisions.
- In some situations, however, it is more relevant to take into account that the outcome also depends on the decisions of other persons/entities, each having its own objective.
- When there are recognized multiperson interactions, there is potential for *strategic interdependence*.

What is a game?

- Strategic interdependence implies that we can reason about how other agents will take into account the way everybody plans their actions, and adjust our behaviour accordingly.
- Everybody else is of course doing the same.
- **Such situations are called games.** They are the object of study of *game theory*.
- Contrast this with the theory of competitive equilibrium, where agents are not directly interested in one another's actions but in certain environment variables (e.g. prices), even though these environment variables are the outcome of everybody's decisions.

Example 1

A first look at a game (1)

- Simple games can often be represented in the form of a table.
- Consider two persons, Player 1 and Player 2, each having at their disposal two possible actions: $\{U, D\}$ for Player 1 and $\{L, R\}$ for Player 2.
- They choose simultaneously and independently an action, knowing that the outcome will be determined by the combination of their choices.
- The outcome takes the form of monetary payoffs to each player and is summarized in the table shown on the next slide.

Example 1

A first look at a game (2)

		Player 2	
		L	R
Player 1	U	$5, 6$	$0, 9$
	D	$10, 1$	$3, 2$

Here is a preview of some of the considerations arising in game theoretic situations:

- Suppose that Player 1 played U and Player 2 played L. Will Player 1 regret her choice after the outcome has been revealed? Would she change her mind if given the chance to amend her choice?
- How about Player 2?
- What if Player 1 chooses U and Player 2 chooses R?

Games in strategic form

General formulation of games in strategic form

- Even though games in strategic form are often presented in tabular form (as we have already seen), this is not the most general form.
- A strategic-form game is defined by the following elements:
  - The set of players $N = \{1, 2, \ldots, n\}$.
  - The set S_i of strategies available to player $i \in N$.
    The set of all possible strategy vectors is denoted $S = S_1 \times S_2 \times \cdots \times S_n$. An element of S is sometimes called a *strategy profile*.
  - A function $u_i : S \to \mathbb{R}$ which gives the payoff to player $i \in N$ associated with a vector of strategies $s \in S$.
- Note that this definition does not require the sets of strategies S_i to be finite.
- *Finite games*, defined by the fact that the players' strategy sets are finite, are a special case that is representable in tabular form.

Nash equilibria

Motivation

- One approach to the analysis of a game is to look for some sort of stable outcome.
- A possible definition of "stability" of an outcome is based on the idea that in a stable outcome no-one should have incentives to unilaterally deviate from his chosen strategy if given that choice.
- This idea forms the basis of the so-called *Nash equilibrium*.

Nash equilibria

Definition

We introduce the following notation: if s_i is the strategy of player i in a strategy vector s, then the strategies of all the other players in s will be denoted by s_{-i}.

Definition 1 (Nash equilibrium)

A strategy vector $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium* if for each player $i \in N$ and each strategy $s_i \in S_i$ the following is satisfied:

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*). \tag{1}$$

The payoff vector $u(s^*) = (u_1(s*), \ldots, u_n(s^*))$ is the equilibrium payoff corresponding to Nash equilibrium s^*.

Nash equilibria

Existence and uniqueness of Nash equilibria

- Nash equilibria need not exist for an arbitrary game.
- Their existence is guaranteed under special technical assumptions. We shall not go into the details of the existence results but shall confine ourselves to cases where Nash equilibria exist.
- In cases where there is a Nash equilibrium, it need not be unique.

Best replies and Nash equilibria (1)

- An equivalent way of defining a Nash equilibrium is through the concept of a *best reply*.
- It is sometimes more convenient to work with best replies since they provide a natural approach to computing Nash equilibria.

Definition 2 (Best reply)

Let s_{-i} be a strategy vector of all the players except player i. Player i's strategy s_i is called a *best reply* to s_{-i} if

$$u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i}). \tag{2}$$

Best replies and Nash equilibria (2)

The following definition of a Nash equilibrium can be shown to be **equivalent** to Definition 1:

Definition 3 (Nash equilibrium)

A strategy vector $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium* if s_i^* is a best reply to s_{-i}^* for every player $i \in N$.

Example 2

An obvious Nash equilibrium

Player 2

Player 1		X	Y
	A	$5,5$	$1,1$
	B	$1,1$	$0,0$

- It is obvious that (A, X) is an equilibrium: it yields the highest possible payoffs for both players and there are no incentives to deviate from it.
- The interests of both players coincide and there is no element of conflict in the game.
- In that sense this game is trivial.

Example 3

Prisoner's Dilemma (1)

- Two persons are arrested on suspicion of committing a serious crime. The sentence for this crime is 10 years in prison.
- However, evidence is insufficient and they can only be sentenced to 1 year in prison for a lesser crime.
- The suspects are held in separate cells and cannot communicate.
- The prosecutor offers each of them a deal: if they tell on their friend (defect, strategy D), they will receive immunity as a state witness (i.e. 0 years in prison) and the friend gets the full 10 years.

Example 3

Prisoner's Dilemma (2)

- If they cooperate with each other and remain silent (strategy C), they get 1 year in prison each for the small crime.
- If both suspects defect, they get an intermediate sentence of 6 years in prison, since their confession is counted as mitigating behaviour in court.
- The prisoners are asked to make their choices on the prosecutor's offer independently and simultaneously. No information is exchanged in the process.

Example 3

Prisoner's Dilemma (3)

		Player 2 D	Player 2 C
Player 1	D	$6, 6$	$0, 10$
Player 1	C	$10, 0$	$1, 1$

The Prisoner's Dilemma: outcomes presented as years in prison.

- This game has one Nash equilibrium, (D, D), with a payoff profile of $(6, 6)$.
- It is easy to verify that, whatever the choice of the other player, a player who played C would wish he had played D instead.
- Indeed, C is a dominated strategy in this game.

Nash equilibria and Pareto optimality

- The example of the Prisoner's Dilemma shows that a Nash equilibrium need not be Pareto optimal.
- The outcome $(6, 6)$ is Pareto-dominated by $(1, 1)$, obtainable if both players choose to cooperate.
- Thus, strategic incentives may lead to a deviation from social optimality.

Duopoly theory

Example 4

Bertrand duopoly competition (1)

- In some cases we need to go beyond the framework of finite games and work with an infinite pure strategy set.
- The Bertrand model provides a simple example.
- There are two firms operating in a common market and producing the same good.
- Market demand, represented by a relation giving quantity q as a function of the price $p \geq 0$, is
$$q(p) = \max\{0, 2 - p\}.$$
- Both firms have constant returns to scale technologies with the same cost per unit $c > 0$, with c sufficiently small (more on that later).

Example 4

Bertrand duopoly competition (2)

- The two firms simultaneously set their prices p_1 and p_2, i.e. their strategy sets are the admissible values of $p_i, i = 1, 2$.
- Buyers then choose the firm with the lower price to execute their orders.
- Quantity sold for Firm 1:

$$q_1(p_1, p_2) = \begin{cases} q(p_1) & \text{if } p_1 < p_2 \\ \frac{1}{2} q(p_1) & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2. \end{cases}$$

- Quantity sold for Firm 2:

$$q_2(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 < p_2 \\ \frac{1}{2} q(p_2) & \text{if } p_1 = p_2 \\ q(p_2) & \text{if } p_1 > p_2. \end{cases}$$

Example 4

Bertrand duopoly competition (3)

- The payoff for Firm i is its profit, given by

$$u_i(p_1, p_2) = p_i q_i(p_1, p_2) - c q_i(p_1, p_2) = (p_i - c) q_i(p_1, p_2).$$

- The requirement of c being “sufficiently small” should now be clear – we want to avoid a situation where payoffs are nonpositive over the entire strategy set of a player.
- What are the Nash equilibria of this game?
- First, observe that $p_i < c$ cannot be a part of a Nash equilibrium profile because it is dominated by $p_i = c$.
- Next, note that $p_i = c,\ p_j > c$ is not a Nash equilibrium.
    - Choosing $p_i = c$ yields zero payoff, which is dominated by any $p_i + \varepsilon(p_j - c)$, $\varepsilon \in (0, 1)$.

Example 4

Bertrand duopoly competition (4)

- Then, observe that $p_i > c$ and $p_j > c$ is not a Nash equilibrium.
  - Suppose, for instance, that $p_1 \leq p_2$.
  - Firm 2 can make at most $(p_1 - c)\frac{1}{2}q(p_1)$ (when $p_1 = p_2$) or 0 when $p_1 < p_2$.
  - If it changed its price to $p_1 - \varepsilon$ for $\varepsilon > 0$, its profit would be $(p_1 - \varepsilon - c)q(p_1 - \varepsilon)$, which is greater than $\frac{1}{2}(p_1 - c)q(p_1)$ for ε small enough.
- Finally, $p_1 = p_2 = c$ is a Nash equilibrium, associated with zero profits.
  - Increasing p_i above c doesn't change anything.
  - Lowering p_i below c brings about negative profit.
- Thus, the standard Bertrand model reproduces the competitive equilibrium outcome.

Example 5

Cournot duopoly competition (1)

- The Cournot model provides a modification of the Bertrand setup for the case where competitors choose quantities instead of prices.
- There are two firms operating on a market for a single good. Firm i produces quantity $q_i \geq 0, i = 1, 2$. Thus, the total quantity supplied is $q = q_1 + q_2$.
- The price of the good is determined by demand and can be represented as a function of the total quantity supplied through the relation[1]

$$p = 2 - q = 2 - q_1 - q_2.$$

- The cost per unit of the good is $c_i > 0$ for Firm i.

[1] Here we use a simpler formulation for convenience instead of $p = \max\{0, 2 - q\}$.

Example 5

Cournot duopoly competition (2)

- The payoff for Player 1 is his profit, given by

$$u_1(q_1, q_2) = q_1(2 - q_1 - q_2) - q_1 c_1 = -q_1^2 + (2 - c_1 - q_2)q_1$$

and the payoff for Player 2 is analogously

$$u_2(q_1, q_2) = q_2(2 - q_1 - q_2) - q_2 c_2 = -q_2^2 + (2 - c_2 - q_1)q_2.$$

- For a fixed q_2, Player 1's best reply is given by the maximizing value of q_1, which is

$$q_1 = \frac{2 - c_1 - q_2}{2}. \tag{3}$$

The best reply of Player 2 to a given q_1 is similarly

$$q_2 = \frac{2 - c_2 - q_1}{2}. \tag{4}$$

Example 5

Cournot duopoly competition (3)

- Solving the system of best replies (3) and (4), we obtain the Nash equilibrium

$$q_1^* = \frac{2 - 2c_1 + c_2}{3}, \qquad q_2^* = \frac{2 - 2c_2 + c_1}{3}.$$

- In equilibrium the respective payoffs can be shown to be

$$u_1(q_1^*, q_2^*) = (q_1^*)^2$$

and

$$u_2(q_1^*, q_2^*) = (q_2^*)^2.$$