R408: Microeconomic Modelling

Topic XX: Introduction to Game Theory

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Lecture Contents

Fundamental concepts of game theory

What is a game?

Drivers manoeuvring in heavy traffic are playing a driving game. Bargain-hunters bidding on eBay are playing an auctioning game. A firm and a union negotiating next year's wage are playing a bargaining game. When opposing candidates choose their platform in an election, they are playing a political game. The owner of a grocery store deciding today's price for corn flakes is playing an economic game. In brief, a game is being played whenever human beings interact.

– Ken Binmore, "Game Theory: A Very Short Introduction", 2007

What is a game?

- You are already familiar with the idea that rational economic agents will pursue well-defined objectives under certain constraints.
- This is typically formalized as an optimization problem of some sort (e.g. a static constrained optimization problem or an optimal control problem).
- In simpler setups, it is assumed that the decision maker controls (optimizes over) certain variables and takes others as completely exogenous. Thus, the outcome will (rather, seems to) depend only on their own decisions.
- In some situations, however, it is more relevant to take into account that the outcome also depends on the decisions of other persons/entities, each having its own objective.
- When there are recognized multiperson interactions, there is potential for *strategic interdependence*.

What is a game?

- Strategic interdependence implies that we can reason about how other agents will take into account the way everybody plans their actions, and adjust our behaviour accordingly.
- Everybody else is of course doing the same.
- **Such situations are called games.** They are the object of study of *game theory*.
- Contrast this with the theory of competitive equilibrium, where agents are not directly interested in one another's actions but in certain environment variables (e.g. prices), even though these environment variables are the outcome of everybody's decisions.

Example 1

A first look at a game (1)

- Simple games can often be represented in the form of a table.
- Consider two persons, Player 1 and Player 2, each having at their disposal two possible actions: $\{U, D\}$ for Player 1 and $\{L, R\}$ for Player 2.
- They choose simultaneously and independently an action, knowing that the outcome will be determined by the combination of their choices.
- The outcome takes the form of monetary payoffs to each player and is summarized in the table shown on the next slide.

Example 1

A first look at a game (2)

		Player 2	
		L	R
Player 1	U	$5, 6$	$0, 9$
	D	$10, 1$	$3, 2$

Here is a preview of some of the considerations arising in game theoretic situations:

- Suppose that Player 1 played U and Player 2 played L. Will Player 1 regret her choice after the outcome has been revealed? Would she change her mind if given the chance to amend her choice?
- How about Player 2?
- What if Player 1 chooses U and Player 2 chooses R?

Basic elements of a game (1)

- The above example illustrates the main elements of a game theoretic situation.
- The entities participating in the game are called the *players*. These can be persons, firms, the government or even nature. In our applications there will be a finite number of players, though sometimes an infinity of players provides a useful abstraction.
- Each player has at their disposal different *actions*. The actions available will in general depend on the particular situation in the game.
- A game must have well-specified *rules*. These include e.g. who gets to play at a given stage, what they know, what actions are available to them, when the game should stop, what the outcome for each player will be.

Basic elements of a game (2)

- The outcomes for the players may be associated with utilities, monetary rewards or, in some cases, costs/expenses. Outcomes are commonly referred to as *payoffs*.
- Payoffs must obviously be ordered, i.e. the players must have preferences over them. The examples of payoffs given suggest the ordering:
  - maximize the monetary reward
  - minimize costs
- It is somewhat more typical for payoffs to be specified as rewards or utilities.

Cooperative and non-cooperative games

- In some games the basic decision-making entity is the individual player. This is the relevant choice when it is assumed that players are independent actors always acting at their own will.
- In other cases the basic decision-making entity is a group of players, acting as one. This is relevant when prior to the start of the game players can make binding agreements that guarantee the group will act as one entity.
- The first situation is the subject of *non-cooperative game theory*. The second setup is studied by *cooperative game theory*.
- We shall study exclusively non-cooperative games, reflecting the thrust of most research done in game theory over the past few decades.

Strategies (1)

- It is often more convenient to study the behaviour of players not in terms of specific actions but in terms of overall plans of behaviour.
- Such a plan must prescribe what a player should do in every imaginable situation that may arise in the course of a game. A plan like this is called a *strategy*.
- A strategy can be imagined as a sort of a rulebook: if A happens, do X; if B happens, do Y.
- Sometimes strategies are compared to the aircraft flight manuals used by pilots to apply emergency operating procedures.

Strategies (2)

- Strategies can be associated with random choice.
- One possibility is to choose between overall plans of action randomly. This is similar to choosing at random one of several possible rulebooks and then sticking to its instructions.
- This is known as a *mixed strategy*.

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- Another possibility is to randomize *within* the rulebook, as if tossing a coin or casting a die to decide what to do in particular situations arising in the course of the game.
- This is known as a *behaviour strategy*.

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- This is known as a *behaviour strategy*.
- We shall deal only with mixed strategies in this lecture. As a consolation, under appropriate information assumptions (perfect recall) mixed strategies and behaviour strategies are in a sense equivalent.

Game representations (1)

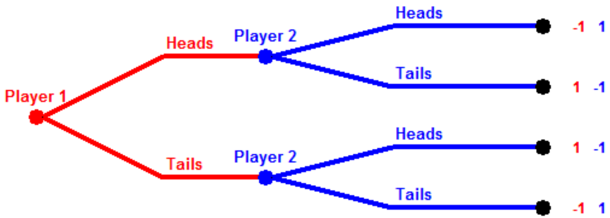
- It should be evident from the previous slides that a game can be a complex structure:
  - It can run over time, going through different stages.
  - Various players can take turns to make choices or act simultaneously.
  - The actions available to each player can vary in the course of the game.
  - Players can have different information at different stages.
  - ...
- If we want to take this complexity on board, then one way to represent a fairly broad class of games is by using special tree-like structures (a specific type of graph).
- This is known as the *extensive form* of a game.

Game representations (2)

- The players making a decision are represented as the nodes (vertices) of the game tree.
- The actions available to a player are represented as the edges emanating from a node.
- Terminal nodes represent the outcomes of the game and are associated with the payoffs for the players.

Game representations (3)

Example 2: Sequential version of the Matching Pennies game in extensive form



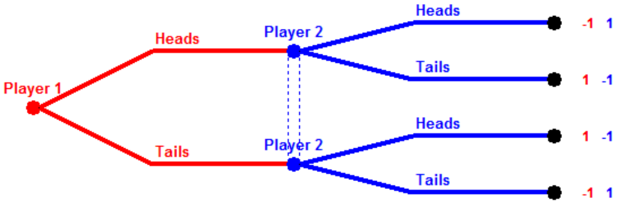
- Player 1 puts down a coin heads up or tails up.
- Player 2 observes this move and also puts down a coin heads up or tails up.
- If the faces of the two coins match, then Player 2 wins. If they are different, Player 1 wins. (Obviously skewed in favour of Player 2.)

Game representations (4)

- In certain cases players move simultaneously and neither can observe the others' decisions prior to their own action.
- In other cases players do not have enough information about the actions taken by others, even though these actions may have already occurred.
- The above situations are formalized through the concept of an *information set* of a player.
- Nodes belonging to the same information set are indistinguishable from the point of view of the respective player.
- To be truly indistinguishable, nodes have to satisfy additional requirements. In particular, the choices available must be the same at all nodes within the same information set and players should not forget what they once knew.

Game representations (5)

Example 3: Standard version of the Matching Pennies game in extensive form



- Both players put down a coin simultaneously. Or, equivalently, Player 1 puts down his coin first but his choice cannot be observed by Player 2.
- If the faces of the two coins match, Player 2 wins. If they are different, Player 1 wins.
- In any case Player 2 does not know Player 1's choice, thus the two blue nodes are in the same information set.

Game representations (6)

- Sometimes we are not interested in the details of the sequencing of events in a game but in comparing overall plans of action.
- Thus, we can study the different strategies available to players without focusing on the specific prescriptions given by a particular strategy.
- This approach, if deemed acceptable, can help reduce the complexity of the analysis and lead to simpler game representations. However, it also means that once a player has chosen a plan, she will not be allowed to reconsider and deviate from it in the course of the game.
- A game represented in terms of the strategies available to players is said to be in *strategic form* (or in *normal form*).
- In simpler cases this leads to a convenient representation of the game in tabular (matrix) form.

Game representations (7)

Example 4: Standard version of the Matching Pennies game in strategic form

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

- In this particular case strategies coincide with actions.

Game representations (8)

Example 5: Sequential version of the Matching Pennies game in strategic form

Player 2

		S1	S2	S3	S4
Player 1	H	$-1, 1$	$-1, 1$	$1, -1$	$1, -1$
	T	$1, -1$	$-1, 1$	$1, -1$	$-1, 1$

- Description of Player 2's strategies:
  - S1 = Play H if Player 1 plays H; play H if Player 1 plays T.
  - S2 = Play H if Player 1 plays H; play T if Player 1 plays T.
  - S3 = Play T if Player 1 plays H; play H if Player 1 plays T.
  - S4 = Play T if Player 1 plays H; play T if Player 1 plays T.
- Notice how each strategy constitutes a complete contingent plan.

Information and knowledge assumptions (1)

Perfect vs imperfect information

- If all players always know exactly where they are on the game tree, the game is one of *perfect information*.
- Formally, this means that all information sets contain exactly one node.
- Some implications of perfect information are:
  - Players know exactly the entire history of the game, including previous choices made by others and the outcomes of random events.
  - Simultaneous moves are incompatible with perfect information.
- As an example, the sequential version of Matching Pennies is a game of perfect information, while the standard version is not.

Information and knowledge assumptions (2)

Complete vs incomplete information

- As a first approximation, it is convenient to assume that the players are well-informed about one another. This includes the assumption that they know their payoffs.
- However, it is also reasonable to consider situations in which players do not fully know some of the characteristics of the other parties.
- Such games are called games of *incomplete information*.
- This stands in contrast to a game of *complete information*, where players are fully informed about the characteristics of others.
- As an example, decisions to enter a new market can be modelled as a game of incomplete information: the new entrant doesn't know the cost structure of firms that are already operating there.

Information and knowledge assumptions (3)

Perfect vs imperfect recall

- If a player does not forget any information that he knew in the past (e.g. his own moves, others' moves), then the player is said to have perfect recall.
- Perfect recall is a natural assumption that is often employed in economic applications of game theory, since a rational agent should not be expected to lose or forget information over time.
- However, it is possible to introduce imperfect recall in a game as a result of using a particular modelling approach. For instance, modelling a team of card players as a single player can lead to imperfect recall.

Games in strategic form

General formulation of games in strategic form

- Even though games in strategic form are often presented in tabular form (as we have already seen), this is not the most general form.
- A strategic-form game is defined by the following elements:
  - The set of players $N = \{1, 2, \dots, n\}$.
  - The set S_i of strategies available to player $i \in N$.
    The set of all possible strategy vectors is denoted $S = S_1 \times S_2 \times \cdots \times S_n$. An element of S is sometimes called a *strategy profile*.
  - A function $u_i : S \to \mathbb{R}$ which gives the payoff to player $i \in N$ associated with a vector of strategies $s \in S$.
- Note that this definition does not require the sets of strategies S_i to be finite.
- *Finite games*, defined by the fact that the players' strategy sets are finite, are a special case that is representable in tabular form.

Dominance (1)

- We now move forward in trying to develop some components of a theory of reasonable behaviour in games.
- The first idea we consider is that if, for every possible situation, some strategy leaves a player better off than another strategy, then the second strategy should not be used.
- Formally, the second strategy is a *dominated strategy*.

Dominance (2)

Example 6

To illustrate the idea of a dominated strategy, take the following game:

		Player 2		
		L	M	R
Player 1	T	$5,5$	$1,1$	$1,0$
	B	$1,1$	$5,5$	$5,0$

- Whatever Player 1 does, strategy M ensures that Player 2 will have a higher payoff compared to strategy R.
- Strategy R is called a *strictly dominated strategy*.
- It is possible to have strategies that are *weakly dominated*, i.e. they do worse compared to another strategy for some of the opponents' actions and do equally well for others.

A first look at Gambit (1)

- Gambit (http://www.gambit-project.org/) is a software that solves certain classes of game theory problems.
- It has a GUI-based version, which we shall use here, as well as the possibility to access its functionality programmatically via a set of command-line tools or a Python interface.
- It can work with games in strategic and in extensive form.
- One of its functionalities is related to finding dominated strategies.

A first look at Gambit (2)

Guided exercise

- Input the game from Example 6 in Gambit. We restate it here for convenience:

		Player 2		
		L	M	R
Player 1	T	5, 5	1, 1	1, 0
	B	1, 1	5, 5	5, 0

- What are the results when you study dominance via the Gambit interface?

Iterated elimination of dominated strategies (1)

- The existence of a dominated strategy effectively reduces the choices of the respective player.
- If players behave rationally they should not play a dominated strategy. Moreover, all players should know that everybody else is also rational, and all should know that all know ... (common knowledge).
- If the above is true, we can proceed to eliminate dominated strategies, arriving at a reduced game.
- Such a reduced game may in turn be analysed to further eliminate newly emerged dominated strategies, arriving at a still smaller game.
- This process can terminate at some intermediate stage or it can end when there is only one strategy left per player.
- In the latter case the resulting strategy profile can be regarded as a solution to the game.

Iterated elimination of dominated strategies (2)

Example 7

Take a look at the game shown below:

Player 2

Player 1		L	M	R
	U	$4,3$	$5,1$	$6,2$
	M	$2,1$	$8,4$	$3,6$
	D	$3,0$	$9,6$	$2,8$

- In this game, strategy M is strictly dominated by strategy R for Player 2.
- Therefore, Player 2 should not play strategy M, which results in a new, "smaller" game.

Iterated elimination of dominated strategies (3)

Example 7

The reduced game now is:

		Player 2	
		L	R
Player 1	U	$4,3$	$6,2$
	M	$2,1$	$3,6$
	D	$3,0$	$2,8$

- In this game strategy U will provide better outcomes for Player 1 than strategies M or D.
- Since strategies M and D are now strictly dominated (note that they weren't in the original game), they can be eliminated from consideration, resulting in an even smaller game.

Iterated elimination of dominated strategies (4)

Example 7

The game now takes the following form:

		Player 2	
		L	R
Player 1	U	$4,3$	$6,2$

- In this game Player 2 is better off playing L, i.e. strategy R is now dominated.
- Thus, we are left with the strategy profile (U, L), yielding payoffs 4 and 3 to Player 1 and Player 2, respectively.

Iterated elimination of dominated strategies (5)

- As our first example of a dominated strategy shows, a game is not always guaranteed to have a solution in terms of iterated elimination of dominated strategies.
- However, in the special case where each player has a *strictly dominant* strategy, i.e. one dominating all the other strategies for that player, the game will have a solution in strictly dominant strategies.
- Again, the assumption of common knowledge is crucial for the elimination procedure to work.

Exercise on iterated elimination of dominated strategies

Consider the following game:

		Player 2		
		L	M	R
Player 1	T	$1,0$	$1,2$	$0,1$
	B	$0,3$	$0,1$	$2,0$

- Input the game in Gambit.
- Study what happens when you repeatedly eliminate dominated strategies. Can you explain the outcomes?

Nash equilibria

Motivation

- Iterated elimination of dominated strategies provides one possibility of predicting the outcome of a game. Unfortunately, it is not applicable to many games.
- Another approach to the analysis of a game is to look for some sort of stable outcome.
- One definition of “stability” of an outcome is based on the idea that in a stable outcome no-one should have incentives to unilaterally deviate from his chosen strategy if given that choice.
- This idea form the basis of the so-called *Nash equilibrium*.

Nash equilibria

Definition

We introduce the following notation: if s_i is the strategy of player i in a strategy vector s, then the strategies of all the other players in s will be denoted by s_{-i}.

Definition 1 (Nash equilibrium)

A strategy vector $s^* = (s_1^*, \dots, s_n^*)$ is a *Nash equilibrium* if for each player $i \in N$ and each strategy $s_i \in S_i$ the following is satisfied:

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*). \tag{1}$$

The payoff vector $u(s^*) = (u_1(s*), \dots, u_n(s^*))$ is the equilibrium payoff corresponding to Nash equilibrium s^*.

Nash equilibria

Existence and uniqueness of Nash equilibria

- Nash equilibria need not exist for an arbitrary game.
- Their existence is guaranteed under special technical assumptions. We shall not go into the details of the existence results but shall confine ourselves to cases where Nash equilibria exist.
- It is useful to know from the start that we may need to resort to the notion of a mixed strategy (mentioned earlier) if we want to ensure existence.
- In cases where there is a Nash equilibrium, it need not be unique.

Best replies and Nash equilibria (1)

- An equivalent way of defining a Nash equilibrium is through the concept of a *best reply*.
- It is sometimes more convenient to work with best replies since they provide a natural approach to computing Nash equilibria.

Definition 2 (Best reply)

Let s_{-i} be a strategy vector of all the players except player i. Player i's strategy s_i is called a *best reply* to s_{-i} if

$$u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i}). \tag{2}$$

Best replies and Nash equilibria (2)

The following definition of a Nash equilibrium can be shown to be **equivalent** to Definition 1:

Definition 3 (Nash equilibrium)

A strategy vector $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium* if s_i^* is a best reply to s_{-i}^* for every player $i \in N$.

Example 8

An obvious Nash equilibrium

		Player 2	
		X	Y
Player 1	A	$5,5$	$1,1$
	B	$1,1$	$0,0$

- It is obvious that (A, X) is an equilibrium: it yields the highest possible payoffs for both players and there are no incentives to deviate from it.
- The interests of both players coincide and there is no element of conflict in the game.
- In that sense this game is trivial.
- Another way of looking at this game would be to notice that (A, X) is a solution in strictly dominant strategies for the game.

Example 9

Prisoner's Dilemma (1)

- Two persons are arrested on suspicion of committing a serious crime. The sentence for this crime is 10 years in prison.
- However, evidence is insufficient and they can only be sentenced to 1 year in prison for a lesser crime.
- The suspects are held in separate cells and cannot communicate.
- The prosecutor offers each of them a deal: if they tell on their friend (defect, strategy D), they will receive immunity as a state witness (i.e. 0 years in prison) and the friend gets the full 10 years.

Example 9

Prisoner's Dilemma (2)

- If they cooperate with each other and remain silent (strategy C), they get 1 year in prison each for the small crime.
- If both suspects defect, they get an intermediate sentence of 6 years in prison, since their confession is counted as mitigating behaviour in court.
- The prisoners are asked to make their choices on the prosecutor's offer independently and simultaneously. No information is exchanged in the process.

Example 9

Prisoner's Dilemma (3)

		Player 2	
		D	C
Player 1	D	$6, 6$	$0, 10$
	C	$10, 0$	$1, 1$

The Prisoner's Dilemma: outcomes presented as years in prison.

- This game has one Nash equilibrium, (D, D), with a payoff profile of $(6, 6)$.
- It is easy to verify that, whatever the choice of the other player, a player who played C would wish he had played D instead.
- Indeed, C is a dominated strategy in this game.

Nash equilibria and Pareto optimality

- The example of the Prisoner's Dilemma shows that a Nash equilibrium need not be Pareto optimal.
- The outcome $(6, 6)$ is Pareto-dominated by $(1, 1)$, obtainable if both players choose to cooperate.
- Thus, strategic incentives may lead to a deviation from social optimality.
- This stands in contrast to the case of competitive equilibrium, which is Pareto optimal by virtue of the First Fundamental Theorem of Welfare Economics.

The limitations of pure strategies

- So far we worked only with the elements of the strategy sets S_i directly, i.e. with pure strategies.
- Under this approach many interesting games do not have Nash equilibria.
- The standard version of Matching Pennies provides an illustration:

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

- For any pure strategy profile in this game, one of the players will find an alternative strategy yielding a higher payoff.

Mixed strategies (1)

- One answer to this limitation of pure strategies is to extend the notion of a strategy and consider the set of all distributions over the pure strategies of a player.
- These are the *mixed strategies* we mentioned previously.
- In the case of finite games mixed strategies are relatively simple objects.
- In this framework a pure strategy can be regarded as a special case of a mixed strategy placing probability 1 on a particular element.
- Mixed strategies are also convenient in terms of their interpretation. For example, if we treat a mixed strategy as randomization over pure strategies, then sometimes being unpredictable can provide an additional strategic advantage (think of a poker player who bluffs occasionally).
- **Warning:** Mixed strategies can have different interpretations. The direct interpretation as a deliberate randomization choice is only one possibility.

Mixed strategies (2)

Definition 4

If $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ is a finite strategic-form game, then the set

$$\Sigma_i := \left\{ \sigma_i : S_i \to [0,1] \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\} \tag{3}$$

is the set of mixed strategies for player i. The set of mixed strategy profiles is $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$.

For a finite set A, we can define the set

$$\Delta(A) := \left\{ p : A \to [0,1] \mid \sum_{a \in A} p(a) = 1 \right\},$$

which is called a *simplex* in $\mathbb{R}^{|A|}$. Thus, we have $\Sigma_i = \Delta(S_i)$.

The mixed extension of a game

Definition 5 (Mixed extension of G)

Let G be a finite strategic-form game. The *mixed extension* of G is the game

$$\Gamma := \left(N, \{\Sigma_i\}_{i \in N}, \{U_i\}_{i \in N}\right), \tag{4}$$

where the expected payoff functions $U_i : \Sigma \to \mathbb{R}$ are defined as

$$U_i(\sigma) = \mathbb{E}_\sigma[u_i(\sigma)] = \sum_{(s_1,\ldots,s_n) \in S} u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n). \tag{5}$$

Nash equilibria in mixed strategies

- The concept of Nash equilibrium can be applied to the mixed extension Γ of a finite strategic-form game G.
- It can be shown that a Nash equilibrium in mixed strategies always exists for a finite game (i.e. one with a finite number of players, each having a finite pure strategy set S_i).

Example 10

A pure coordination game (1)

Consider the following game:

		Player 2	
		X	Y
Player 1	A	$2,2$	$0,0$
	B	$0,0$	$2,2$

- Such a game is called a coordination game: players can benefit from matching (coordinating) their strategies.
- One can verify directly that the game has two pure strategy Nash equilibria: (A, X) and (B, Y)
- Can we recover them in a mixed strategy framework? Are there any other Nash equilibria in mixed strategies?

Example 10

A pure coordination game (2)

- Denote a mixed strategy for player i by (p_1^i, p_2^i).
- The expected payoff to Player 1 from a given mixed strategy profile is

$$(p_1^1, p_2^1)\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} p_1^2 \\ p_2^2 \end{pmatrix} = (p_1^1, 1 - p_1^1)\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} p_1^2 \\ 1 - p_1^2 \end{pmatrix}.$$

- The expected payoff to Player 2 is actually the same by virtue of the structure of the game.
- Such a payoff can be written as

$$2p_1^1 p_1^2 + 2(1 - p_1^1)(1 - p_1^2) = 2(2p_1^2 - 1)p_1^1 + 2(1 - p_1^2).$$

Example 10

A pure coordination game (3)

- This payoff is maximized for:
  - $p_1^1 = 0$ if $p_1^2 < \frac{1}{2}$,
  - $p_1^1 = 1$ if $p_1^2 > \frac{1}{2}$,
  - any p_1^1 if $p_1^2 = \frac{1}{2}$.
- A symmetric argument is valid for Player 2.
- Thus, *both* players' expected payoffs are maximized for the mixed strategy profiles $[(1,0);(1,0)]$ and $[(0,1);(0,1)]$, i.e. we have recovered the pure strategy equilibria.
- In addition, the profile $[(0.5,0.5);(0.5,0.5)]$ turns out to be a Nash equilibrium as well. It is an equilibrium in mixed strategies.

Example 11

Battle of the Sexes (1)

- A man and a woman have to decide how to spend an evening together.
- They have two options: going to a football game (F) or going to a concert (C).
- The man prefers the football game, while the woman prefers the concert.
- Yet, each would choose to go to the less preferable event with their partner, rather than going to their favourite event alone.

Example 11

Battle of the Sexes (2)

		Woman	
		F	C
Man	F	$2,1$	$0,0$
	C	$0,0$	$1,2$

- The game has several Nash equilibria.
- There are two equilibria in pure strategies: (F, F) and (C, C).
- There is also one equilibrium in mixed strategies with probabilities $(\frac{2}{3}, \frac{1}{3})$ for the man and probabilities $(\frac{1}{3}, \frac{2}{3})$ for the woman.

Exercise on the Battle of the Sexes

1. Input the game in Gambit.
2. Check if there are dominated strategies.
3. Compute all Nash equilibria and verify the claims from the previous slide.

Example 12

Rock-paper-scissors (1)

- This example formalizes the well-known hand game of the same name.
- There are two players.
- Each player chooses between three actions: *Rock*, *Paper* or *Scissors*. The choices are made simultaneously.
- The outcomes are determined by the following rules:
  - Identical choices, e.g. (Rock,Rock), result in a tie.
  - *Rock* wins over *Scissors* ("Rock breaks scissors")
  - *Scissors* wins over *Paper* ("Scissors cuts paper")
  - *Paper* wins over *Rock* ("Paper wraps rock")
- The game is zero-sum: a win for one player is a loss for the other.

Example 12

Rock-paper-scissors (2)

		Player 2 Rock	Player 2 Paper	Player 2 Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
Player 1	Paper	$1, -1$	$0, 0$	$-1, 1$
Player 1	Scissors	$-1, 1$	$1, -1$	$0, 0$

- It can be verified immediately that the game has no Nash equilibrium in pure strategies: for any strategy profile either the losing player or both players would like to unilaterally change their choice, if possible.
- The game has one equilibrium in mixed strategies: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for Player 1 and also $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for Player 2.

Exercise on Rock-paper-scissors

1. Input the game in Gambit.
2. Check if there are dominated strategies.
3. Compute all Nash equilibria and verify the claims from the previous slide.

Example 13

Cournot duopoly competition (1)

- In some cases we need to go beyond the framework of finite games and work with an infinite pure strategy set.
- The Cournot model provides a simple example.
- There are two firms operating on a market for a single good. Firm i produces quantity $q_i \geq 0, i = 1, 2$. Thus, the total quantity supplied is $q = q_1 + q_2$.
- The price of the good is determined by demand and can be represented as a function of the total quantity supplied through the relation[1]

$$p = 2 - q = 2 - q_1 - q_2.$$

- The cost per unit of the good is $c_i > 0$ for Firm i.

[1] Clearly, a more realistic formulation would be something like $p = \max\{0, 2 - q\}$ but we use the simpler one for convenience.

Example 13

Cournot duopoly competition (2)

- The payoff for Player 1 is his profit, given by

$$u_1(q_1, q_2) = q_1(2 - q_1 - q_2) - q_1 c_1 = -q_1^2 + (2 - c_1 - q_2)q_1$$

and the payoff for Player 2 is analogously

$$u_2(q_1, q_2) = q_2(2 - q_1 - q_2) - q_2 c_2 = -q_2^2 + (2 - c_2 - q_1)q_2.$$

- For a fixed q_2, Player 1's best reply is given by the maximizing value of q_1, which is

$$q_1 = \frac{2 - c_1 - q_2}{2}. \tag{6}$$

The best reply of Player 2 to a given q_1 is similarly

$$q_2 = \frac{2 - c_2 - q_1}{2}. \tag{7}$$

Example 13

Cournot duopoly competition (3)

- Solving the system of best replies (6) and (7), we obtain the Nash equilibrium

$$q_1^* = \frac{2 - 2c_1 + c_2}{3}, \qquad q_2^* = \frac{2 - 2c_2 + c_1}{3}.$$

- In equilibrium the respective payoffs can be shown to be

$$u_1(q_1^*, q_2^*) = (q_1^*)^2$$

and

$$u_2(q_1^*, q_2^*) = (q_2^*)^2.$$

Games in extensive form with perfect information

General considerations for games in extensive form

- If we want to trace explicitly the evolution of play in the course of a dynamic game, then we are back to the case of a game in extensive form.
- The concept of Nash equilibrium is valid for such games as well.
- We shall only look at the case of extensive-form games with perfect information and focus on pure strategies.

Example 14

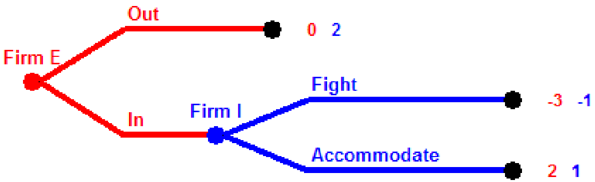
Market entry (1)

- Consider the decision of a firm (entrant, Firm E) to enter a new market.
- There is another firm (incumbent, Firm I) already operating on that market.
- If Firm E decides to stay out of the market (strategy Out), the status quo is preserved.
- If Firm E decides to enter the market (strategy In), Firm I has two choices:
  - Do nothing (strategy Accommodate), in which case the market is divided between the two firms.
  - Engage in a price war (strategy Fight), leading to heavy losses to both firms.

Example 14

Market entry (2)

The extensive-form version of the game is the following:



- There are two pure strategy Nash equilibria in the game.
- One equilibrium is (Out, Fight if Firm E chooses “In”)
- The other equilibrium is (In, Accommodate if Firm E chooses “In”)

Example 14

Market entry (3)

The Nash equilibria can be checked from the strategic-form representation of the game:

		Firm I	
		Fight if **E** plays "In"	Accommodate if **E** plays "In"
Firm E	Out	$0, 2$	$0, 2$
	In	$-3, -1$	$2, 1$

Question: Isn't (Out, Accommodate if Firm E chooses "In") a Nash equilibrium as well?

Example 14

Market entry (3)

The Nash equilibria can be checked from the strategic-form representation of the game:

		Firm I: Fight if **E** plays "In"	Firm I: Accommodate if **E** plays "In"
Firm E	Out	$0, 2$	$0, 2$
	In	$-3, -1$	$2, 1$

Question: Isn't (Out, Accommodate if Firm E chooses "In") a Nash equilibrium as well?

- However, one of the two pure strategy Nash equilibria, the equilibrium (Out, Fight if Firm E chooses "In"), isn't very plausible.
- This is because once Firm E has decided to enter, it is not optimal for Firm I to fight.
- Firm E can figure that out, so in real life it is likely to enter.

Example 14

Market entry (4)

- Effectively, what happens in this game is that the incumbent makes an empty threat and the concept of Nash equilibrium implies that the entrant will take it seriously.
- By virtue of the sequential nature of the game, under Nash equilibrium Player I can declare that it will do anything in the unreached branch of the game and its statement will affect the decision of the other player.
- This suggests that we may want to strengthen our criteria for optimality of play by taking on board the implications of the sequential structure of decision-making in a dynamic game.

The principle of sequential rationality

- The previous considerations suggest that equilibrium strategies should specify behaviour that is optimal not only in general but also in a “piecewise” manner, from any point in the game onward.
- This requirement is known as the *principle of sequential rationality*.
- Operationally, the principle of sequential rationality for games of perfect information is implemented by a procedure known as *backward induction*.

Backward induction and subgames

- Roughly, backward induction means that we solve that game starting from the end and reconstructing optimal behaviour by moving toward the initial node of the game tree.
- In order to do that we need a unit for which to consider optimality when going backward.
- A natural candidate for such a unit is a subset of the game tree that itself constitutes a game when taken in isolation.
- Such a subset is called a *subgame*.

The backward induction procedure

The backward induction procedure can be summarised as follows:

1. Find the minimal subgames containing the terminal nodes and compute their Nash equilibria.
2. Replace the subgames with the equilibrium payoffs associated with a Nash equilibrium and produce a reduced game. (In case of multiple Nash equilibria the procedure is repeated).
3. Go to step 1 and repeat the procedure for the new, reduced game. Continue until you have included the subgame containing the initial node.

The collections of moves obtained via the backward induction procedure define strategies that are consistent with the principle of sequential rationality.

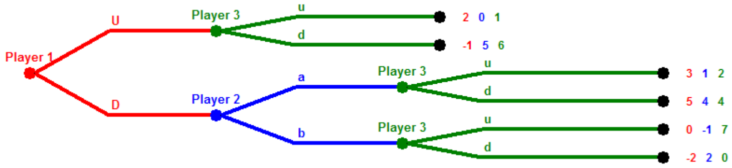
Subgame perfect Nash equilibria

- A profile of strategies that induces a Nash equilibrium for every subgame of a given extensive-form game is called a *subgame perfect Nash equilibrium*.
- Applying the backward induction procedure to an extensive-form game of perfect information by construction produces a subgame perfect Nash equilibrium.
- Every finite game of perfect information has a pure strategy subgame perfect Nash equilibrium. If no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

Example 15

Backward induction in action (1)

Consider the following game:

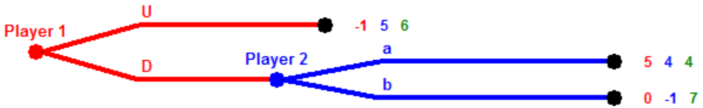


- The three subgames containing the terminal nodes all belong to Player 3 and are shown in green.
- Solving them (from top to bottom in the figure) reveals that Player 3 will play respectively d, d and u.
- We replace the subgames with the respective payoffs.

Example 15

Backward induction in action (2)

The resulting reduced game is:



- The next level subgame belongs to Player 2 and is shown in blue.
- The equilibrium in this subgame is a.
- We repeat the reduction procedure and replace the subgame with the resulting payoff.

Example 15

Backward induction in action (3)

The final reduced game is:

- It is optimal for Player 1 to choose D in this subgame.
- As this is the reduced game containing the initial node, this ends the backward induction procedure.
- The final set of payoffs is $(5, 4, 4)$ and it will be obtained by the sequence of moves $D \to a \to d$.

Example 15

Backward induction in action (4)

- Notice, however, that the strategies leading to the above sequence of moves are more complicated.
- These strategies are:
  - **Player 1:** Play D.
  - **Player 2:** Play a if Player 1 plays D.
  - **Player 3:** Play d if Player 1 plays U.
    Play d if Player 1 plays D and Player 2 plays a.
    Play u if Player 1 plays D and Player 2 plays b.
- The above strategy profile is a subgame perfect Nash equilibrium for the game.

Exercise on the game from Example 15

1. Input the game in Gambit.
2. Try to compute a Nash equilibrium for the game. Instruct Gambit to compute all Nash equilibria.
3. What happens when you explore dominance?

Useful topics not covered in this lecture

General remarks

- Game theory is a vast and proliferating subject. It is impossible to give a comprehensive overview of its current state.
- However, certain topics feature regularly in game theory courses and we can at least mention some of them for future reference.
- Even in this case we can only give an incomplete and highly stylized list that is necessarily subjective.
- Take a look at the sources in the References section for more information and additional ideas.

Equilibrium refinements

- Nash equilibrium is a very useful concept but it is not devoid of limitations.
- Outcomes that are legitimate Nash equilibria may be deemed unrealistic if assessed on common plausibility grounds, as the example with the market entry game shows.
- These considerations have spawned a branch of game theory focusing on selecting specific equilibria based on additional criteria.
- This is known as *equilibrium refinement*.
- There exist various equilibrium concepts refining the Nash equilibrium. (Subgame perfection is one example.) They are useful in different situations and there isn't one universally established refinement dominating the game-theoretic literature.

Incomplete information

- We mentioned games of incomplete information without actually studying them.
- This is not for lack of importance: many problems of practical relevance can be modelled as games of incomplete information.
- Examples of these include:
  - Market entry with no information on the incumbents' costs
  - Trade negotiations
  - Auctions

Repeated games

- In certain contexts one game may be played over and over by the same players.
- This situation is known as a *repeated game*.
- A repeated game enriches the potential for strategic interaction and allows strategies that would not be used by rational players in a one-shot game.
- As an example, in a repeated Prisoner's Dilemma cooperation can emerge as a viable strategy, unlike the case of the one-shot game.

Differential games (1)

- Recall that a continuous-time optimal control problem is a dynamic optimization problem in which the constraints are given by differential equations (the state equations).
- A simple version of a state equation may read

$$\dot{x}(t) = f(x(t), u(t)).$$

- Now suppose that there are two planning entities, each having their own objective functional and controls. The latter are denoted respectively by $u_1(t)$ and $u_2(t)$.
- Suppose further that the state equation is

$$\dot{x}(t) = f(x(t), u_1(t), u_2(t)),$$

i.e. it features both controls.

Differential games (2)

- Thus, the two entities are now interdependent, since they have a common state equation with partial control over it by virtue of having their own control.
- Such a situation is called a *differential game*.
- Differential games have various applications, e.g. in marketing, public finance, optimal resource exploitation and even military warfare.

Evolutionary games (1)

- Imagine a situation in which a game is played repeatedly in a population of individuals, who are matched randomly against one another in every round of play.
- There are different subsets of the population favouring different (fixed) strategies. This may be the result of rational deliberation or simply an innate preference.
- As the game progresses, groups employing more successful strategies grow in size, while groups using less successful strategies shrink.
- This is similar to a Darwinian survival-of-the-fittest process where evolution eliminates species that are unable to cope in their habitat.
- Such games are called *evolutionary games*.

Evolutionary games (2)

- Evolutionary games are important because they open the possibility that rational behaviour may emerge in the limit as the result of an adaptive process akin to learning.
- In particular, some Nash equilibria can materialize as stable outcomes in appropriately defined evolutionary games.
- This is more flexible and realistic compared to the assumption that individuals are perfectly rational from the start, which may imply almost superhuman reasoning powers of the participating players.

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