R401: Statistical and Mathematical Foundations

Lecture 19: Infinite-Horizon Deterministic Optimal Control in Discrete Time

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Lecture Contents

Introduction

Switching to discrete time

- You are already familiar with a number of dynamic optimization problems in which time is continuous.
- In many applications, however, it is natural to work in discrete time.
- This provides a bridge to validating/calibrating models with data or estimating them, among others.
- We therefore need to develop the counterpart of the continuous-time optimal control framework for the case of discrete time.

Switching to discrete time

- Some of the details of such a transition are easily predictable:
  - Differential equations will be replaced by difference equations.
  - The objective functional will involve a series instead of an integral.
- Other details need to be specified further. In particular, there exist two broad classes of dynamic optimization problems in discrete time:
  - Problems in variational form
  - Problems with explicit controls
- Both classes can be used to address a wide variety of problems.
- However, problems with explicit controls are a bit more transparent in terms of their structure.

Specific assumptions

- We will work in an infinite-horizon setup. Finite-horizon formulations for discrete-time problems exist but are less common in economic applications.
- We sacrifice some generality from the outset by assuming a specific structure of the problems:
    - Special (time) discounting in the objective functionals.
    - Autonomous difference equations describing the evolution of the system that is being modelled.

Problems in variational form

Formulation

The Euler equation

Sufficiency

Example: NCs for a problem in variational form

Compute the NCs for the problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_{t+1} = (1-\delta)k_t + y_t - c_t, \quad k_0 > 0 \text{ – given}$$

$$y_t = Ak_t^{\alpha}, \quad 0 < \underline{\epsilon} \le c_t \le \bar{\epsilon}, \quad \alpha, \beta \in (0,1), \quad A > 0$$

NCs:

$$\sum_{t=0}^{\infty} \beta^t \underbrace{\ln((1-\delta)k_t + Ak_t^{\alpha} - k_{t+1})}_{=F(x_t, x_{t+1})}.$$

Euler equation:

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0$$

Example: NCs for a problem in variational form

Applying the Euler equation, we get:

$$\frac{-1}{(1-\delta)k_t + Ak_t^{\alpha} - k_{t+1}} + \beta \frac{(1-\delta) + \alpha A k_{t+1}^{\alpha-1}}{(1-\delta)k_{t+1} + Ak_{t+1}^{\alpha} - k_{t+2}} = 0$$

The last result has the form

$$G(x_{t+2}, x_{t+1}, x_t) = 0,$$

i.e. a nonlinear second-order difference equation.

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Question: Can you compute the solution of the equation for $\beta = 0.95,\ A = 1,\ \delta = 0.05,\ \alpha = 0.5$ and $k_0 = 2$?

Example: NCs for a problem in variational form

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Question: Can you compute the solution of the equation for $\beta = 0.95,\ A = 1,\ \delta = 0.05,\ \alpha = 0.5$ and $k_0 = 2$? If not, what else do you need?

Problems with explicit controls

Formulation of the problem with explicit controls

$X \subset \mathbb{R}^n$ – *state space*, the state variables are $x = (x^1, \ldots, x^n)$.

We assume that $\forall x \in X,\ \exists U(x) \subset \mathbb{R}^m, U(x) \neq \emptyset$. The elements of $u = (u^1, \ldots, u^m)$ are our *controls*.

Objective function (instantaneous): $f^0(x, u)\ \ x \in X, u \in U(x)$

State equations:

$$x_{t+1} = f(x_t, u_t), \quad x_0 \text{ – given} \tag{1}$$

where $f(x, u)$ is a vector function taking values in X, for $x \in X, u \in U(x)$.

Formulation of the problem with explicit controls

We need to find a sequence of admissible controls $\mathbf{u} = \{u_t\}, t = 0, 1, 2, \ldots,$ which determine a sequence of state variables $\{x_{t+1}\}, t = 0, 1, 2, \ldots$ via (1) for which

$$j(x_0, \mathbf{u}) = \sum_{t=0}^{\infty} \beta^t f^0(x_t, u_t) \qquad (2)$$

attains a maximum

$$v(x_0) = \sup_{\mathbf{u}} j(x_0, \mathbf{u}). \qquad (3)$$

We refer to the above problem as *Problem A*.

Formulation of the problem with explicit controls

The number $\beta \in (0,1)$ is the *discount factor* in the model.

Denote by $\mathrm{FC}(x_0)$ the set of all feasible control sequences $\{u_t\}_{t=0}^{\infty}$ for initial $x_0 \in X$, i.e. x_{t+1} satisfies (1) for $u_t \in U(x_t)$, $t = 0, 1, 2, \ldots$, and a given x_0.

We denote the optimal sequence of pairs of state variables and controls for Problem A by $\{x_{t+1}^*, u_t^*\},\ t = 0, 1, 2, \ldots$, i.e. $\{u_t^*\} \in \mathrm{FC}(x_0)$, and

$$v(x_0) = j(x_0, \mathbf{u}^*),\ \mathbf{u}^* = \{u_t^*\}.$$

Necessary conditions for optimality

A variation of the familiar approach for problems of this type is the following:

1. Construct the Lagrangian

$$\mathcal{L}(x_1, x_2, \ldots, u_0, u_1, \ldots) = \sum_{t=0}^{\infty} \beta^t \left[f^0(x_t, u_t) + \lambda_t' \cdot [f(x_t, u_t) - x_{t+1}] \right],$$

where $\lambda_t = (\lambda_t^1, \ldots, \lambda_t^n), t = 0, 1, 2, \ldots$, are the Lagrange multipliers and the dot $(\cdot)$ denotes a matrix product or a scalar product:

$$\lambda_t' \cdot [f(x_t, u_t) - x_{t+1}] = \sum_{i=1}^{n} \lambda_t^i [f^i(x_t, u_t) - x_{t+1}^i].$$

Necessary conditions for optimality

2. Differentiate $\mathcal{L}$ w.r.t. x_t and u_t, set the resulting expressions equal to zero and obtain first-order necessary conditions for optimality:

$$\begin{aligned}\beta\left[f^0_{x^k_t}(x_t,u_t)+\sum_{i=1}^{n}\lambda^i_t f^i_{x^k_t}(x_t,u_t)\right] &= \lambda^k_{t-1},\ k=1,\dots,n,\\ f^0_{u^j_t}(x_t,u_t)+\sum_{i=1}^{n}\lambda^i_t f^i_{u^j_t}(x_t,u_t) &= 0,\ j=1,\dots,m.\end{aligned} \tag{4}$$

3. Equations (1) and (4) are solved as a system and we obtain a candidate solution $\{u_t\}_{t=0}^{\infty}$ or, more precisely, a sequence $\{x_{t+1},u_t\}_{t=0}^{\infty}$. (It is common to find a stationary point of the system (1) and (4), and work with a linearised version of the system around that point.)

Necessary conditions for optimality

In matrix notation the above takes that form:

$$\mathcal{L}_x = \beta^t f_x^0(x_t, u_t) + \beta^t f_x'(x_t, u_t) \cdot \lambda_t - \beta^{t-1}\lambda_{t-1} = 0 \Rightarrow$$

$$\beta(f_x^0(x_t, u_t) + f_x'(x_t, u_t) \cdot \lambda_t) = \lambda_{t-1}. \tag{5}$$

$$\mathcal{L}_u = \beta^t f_u^0(x_t, u_t) + \beta^t f_u'(x_t, u_t) \cdot \lambda_t = 0 \Rightarrow$$

$$f_u^0(x_t, u_t) + f_u'(x_t, u_t) \cdot \lambda_t = 0 \tag{6}$$

Necessary conditions for optimality

Why is this algorithm valid?

The value function for Problem A satisfies a version of the Bellman equation:

$$v(x) = \sup_{u \in U(x)} \left[f^0(x, u) + \beta v(f(x, u)) \right]. \tag{7}$$

Let the supremum in (7) be attained on the interior of the set $U(x)$. Denote this point by $u = \nu(x)$ and assume that all objects used below are differentiable.

Necessary conditions for optimality

We have

$$v(x) = f^0(x, \nu(x)) + \beta v(f(x, \nu(x))). \tag{8}$$

Also the extremum condition is

$$f_u^0(x, \nu(x)) + \beta f_u'(x, \nu(x)) \cdot v_x(f(x, \nu(x))) = 0. \tag{9}$$

Differentiating (8) w.r.t. x, we obtain

$$\begin{aligned} v_x(x) = & f_x^0(x, \nu(x)) + \nu_x'(x) \cdot f_u^0(x, \nu(x)) + \\ & \beta \left[f_x'(x, \nu(x)) + \nu_x'(x) \cdot f_u'(x, \nu(x)) \right] \cdot v_x(f(x, \nu(x))) \\ = & f_x^0(x, \nu(x)) + \beta f_x'(x, \nu(x)) \cdot v_x(f(x, \nu(x))) + \\ & \underbrace{\nu_x'(x) \cdot f_u^0(x, \nu(x)) + \beta \nu_x'(x) \cdot f_u'(x, \nu(x)) \cdot v_x(f(x, \nu(x)))}_{=0 \text{ in view of (9)}}. \end{aligned}$$

Necessary conditions for optimality

We thus get

$$v_x(x) = f_x^0(x, \nu(x)) + \beta f_x'(x, \nu(x)) \cdot v_x(f(x, \nu(x))). \tag{10}$$

For $x = x_t^*$ and $u_t^* = \nu(x_t^*)$, equations (9) and (10) take the form

$$f_u^0(x_t^*, u_t^*) + \beta f_u'(x_t^*, u_t^*) \cdot v_x(x_{t+1}^*) = 0, \tag{11}$$

$$v_x(x_t^*) = f_x^0(x_t^*, u_t^*) + \beta f_x'(x_t^*, u_t^*) \cdot v_x(x_{t+1}^*). \tag{12}$$

Set $\lambda_t := \beta v_x(x_{t+1}^*)$ in (11) and (12), to obtain precisely (5) and (6).

Sufficient conditions

Fact 1

Let $\{\lambda_t\}$ and $\{x^*_{t+1}, u^*_t\}, t = 0, 1, 2, \ldots,$ be obtained by using (1) and (4). If

1. The functions $f^0(x, u)$ and $f(x, u)$ are concave in (x, u),
2. The Lagrange multipliers $\lambda^1_t, \ldots, \lambda^n_t,\ t = 0, 1, 2, \ldots$ are nonnegative,
3. The state space X is a subset of $\mathbb{R}^n_+$ and the following transversality condition is valid
$$\lim_{T\to\infty} \beta^T \lambda'_T \cdot x^*_{T+1} = 0,$$

then the sequence $\{x^*_{t+1}, u^*_t\}$ (for a given x_0) is optimal for Problem A.

Sufficient conditions

We shall verify the validity of Fact 1.

Recall that (4) in matrix terms is given by (5) and (6).
Consider

$$\mathcal{L}_T(x_t, u_t) = \sum_{t=0}^{T} \beta^t \left\{ f^0(x_t, u_t) + \lambda_t' \cdot [f(x_t, u_t) - x_{t+1}] \right\}.$$

We have

$$\begin{aligned} D := \mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) = \sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \\ \sum_{t=0}^{T} \beta^t [f^0(x_t, u_t) + \lambda_t' \cdot f(x_t, u_t) - f^0(x_t^*, u_t^*) - \lambda_t' \cdot f(x_t^*, u_t^*)]. \end{aligned} \tag{13}$$

Sufficient conditions

Then, in view of concavity, we get

$$\mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) \ \ (= D) \ \ \leq \sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) +$$

$$\sum_{t=0}^{T} \beta^t [f_x^{0'}(x_t^*, u_t^*) \cdot (x_t - x_t^*) + f_u^{0'}(x_t^*, u_t^*) \cdot (u_t - u_t^*) +$$

$$\lambda_t' \cdot [f_x(x_t^*, u_t^*) \cdot (x_t - x_t^*) + f_u(x_t^*, u_t^*) \cdot (u_t - u_t^*)]] =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T} \beta^t \underbrace{[f_x^{0'}(x_t^*, u_t^*) + \lambda_t' \cdot f_x(x_t^*, u_t^*)]}_{= \frac{\lambda_{t-1}'}{\beta} \text{ in view of (5)}} \cdot (x_t - x_t^*)$$

$$+ \sum_{t=0}^{T} \beta^t \underbrace{[f_u^{0'}(x_t^*, u_t^*) + \lambda_t' \cdot f_u(x_t^*, u_t^*)]}_{= \mathbf{0}' \text{ in view of (6)}} \cdot (u_t - u_t^*).$$

Sufficient conditions

I.e.

$$D \leq \sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T} \beta^t \frac{\lambda_{t-1}'}{\beta} \cdot \underbrace{(x_t - x_t^*)}_{\text{N.B.: } x_0 = x_0^*} =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=1}^{T} \beta^{t-1} \frac{\lambda_{t-1}'}{\beta} \cdot (x_t - x_t^*) =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda_t'}{\beta} \cdot (x_{t+1} - x_{t+1}^*) =$$

$$\underbrace{\sum_{t=0}^{T-1} \beta^t \lambda_t' \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda_t'}{\beta} \cdot (x_{t+1} - x_{t+1}^*)}_{=0} +$$

$$\beta^T \lambda_T' \cdot (x_{T+1}^* - x_{T+1}) \underbrace{\leq}_{\text{since } \lambda_t, x_t \geq \mathbf{0}} \beta^T \lambda_T' \cdot x_{T+1}^*.$$

Sufficient conditions

In view of the transversality condition, we have:

$$D \leq \beta^T \lambda_T' \cdot x_{T+1}^* \underset{T \to \infty}{\longrightarrow} 0,$$

i.e.

$$\mathcal{L}_T(x_t^*, u_t^*) - \mathcal{L}_T(x_t, u_t) \geq 0,$$

which proves the optimality of the sequence $\{x_{t+1}^*, u_t^*\}$.

Homework

Readings

Main references:Additional readings:

Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 12.