R401: Statistical and Mathematical Foundations
Lecture 20: Infinite-Horizon Stochastic Optimal Control in Discrete Time

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Introduction

Introduction

- Economic decision-making is intrinsically done in an uncertain environment.

- A realistic modelling framework should therefore take uncertainty on board in some manner.

- A standard approach to modelling decision-making under uncertainty is to use dynamic optimization problems with a stochastic element.

- The popular dynamic stochastic general equilibrium (DSGE) models are a prominent example of this approach.

- The presence of randomness creates differences in comparison with deterministic problems. In particular, the concept of an optimal sequence of controls has to be replaced by the concept of an optimal decision rule (policy).

Introduction

- In constructing and solving a dynamic stochastic model or, specifically, a DSGE model, one typically goes through several stages:
  - model formulation – goal $\rightarrow$ mechanisms, shocks (structural interpretation)

  - deriving NCs for optimality

  - computing a steady state – deterministic steady states

  - linear approximations of the model around the steady state – log-linearization

  - calibration/estimation

  - forecasts, simulations

Introducing randomness in an optimization problem with explicit controls

Suppose that the state equations also include a random variable or vector:

$$x_{t+1} = f(x_t, u_t, \epsilon_t), \quad x_0 - \text{given} \tag{1}$$

Then the state variables $\{x_t\}$ also become random, therefore $F(x_t, u_t)$ will be random.

This requires a modification of the objective functional. We need to take its expectation (or, more precisely, conditional expectation given information available at the respective moment). Assume that that we are at $t = 0$:

$$\max_{\{u_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t F(x_t, u_t) \tag{2}$$

Introducing randomness in an optimization problem with explicit controls

- Sequencing of events within a given time period:

$$x_t \quad \rightarrow \quad u_t \quad \rightarrow \quad \epsilon_t \quad \Longrightarrow \quad x_{t+1} \quad \rightarrow \quad \cdots$$

This means that, when making the decision, we are not sure where we'll be next period.
Some sources write $x_{t+1} = f(x_t, u_t, \epsilon_{t+1})$ to make this sequencing more explicit.

- Unlike deterministic models, the solution here cannot be a predetermined path $u_0^*, u_1^*, \ldots$ For instance, it is impossible to determine the optimal u_5 when we are at time 0, since we do not know the value of x_5 (recall that it is now random).

- For this reason the optimal mode of behaviour is given as a *reaction function (rule, policy function)* $u_t = \mu(x_t)$, which depends on the state x_t of the system.

Introducing randomness in an optimization problem with explicit controls

- Under this structure of the model, decisions are taken only for the current moment.

- Everything else is a plan for the future which may or may not be carried out as envisaged.

- As time advances, the model is re-solved over and over for the new conditions that materialize as a result of the combination of past decisions and random disturbances.

- This is reflected in the way the conditional mathematical expectation is applied when computing NCs for optimality.

Necessary conditions

Necessary conditions for optimality

General formulation (algorithmic solution)

We have the following problem

$$\max_{u_{s+t}} E_s \sum_{t=0}^{\infty} \beta^t F(x_{s+t}, u_{s+t})$$

s.t.

$$x_{s+t+1} = f_{s+t}(x_{s+t}, u_{s+t}, \epsilon_{s+t}), \quad x_s - \text{given},$$

where $x_{s+t} \in \mathbb{R}^n$ and $u_{s+t} \in \mathbb{R}^m$.

As usual, we construct the Lagrangian

$$\mathcal{L} = E_s \sum_{t=0}^{\infty} \beta^t \left\{ F(x_{s+t}, u_{s+t}) + \lambda'_{s+t}(f_{s+t}(x_{s+t}, u_{s+t}, \epsilon_{s+t}) - x_{s+t+1}) \right\}$$

Necessary conditions for optimality

General formulation (algorithmic solution)

Under certain conditions, which are fulfilled for typical economic problems, the solution will satisfy

$$\frac{\partial \mathcal{L}}{\partial u_{s+t}^i} = 0,\ i = 1, \ldots, m; \quad \frac{\partial \mathcal{L}}{\partial x_{s+t+1}^j} = 0,\ j = 1, \ldots, n,\ \forall t \geq 0,\ \text{i.e.}$$

$$E_s \left\{ \frac{\partial}{\partial u_{s+t}^i} F(x_{s+t}, u_{s+t}) + \sum_{k=1}^n \lambda_{s+t}^k \frac{\partial}{\partial u_{s+t}^i} f_{s+t}^k(x_{s+t}, u_{s+t}, \epsilon_{s+t}) \right\} = 0,$$

$$\beta E_s \left\{ \frac{\partial}{\partial x_{s+t+1}^j} F(x_{s+t+1}, u_{s+t+1}) + \right.$$

$$\left. \sum_{k=1}^n \lambda_{s+t+1}^k \frac{\partial}{\partial x_{s+t+1}^j} f_{s+t}^k(x_{s+t+1}, u_{s+t+1}, \epsilon_{s+t+1}) \right\} = E_s \left\{ \lambda_{s+t}^j \right\}$$

Necessary conditions for optimality

Practical advice and comments

- It is easier to writh the problem for the case $s = 0$. Standard problems are invariant with respect to time and we lose no generality.

- It is convenient to apply the expectation operator at the end of the computation, i.e. after differentiation and the subsequent transformations.

- We differentiate w.r.t. all controls at an arbitrary moment t and w.r.t. to the state variables at $t + 1$, then set the results equal to zero.

- When we apply the conditional expectation operator, it is applied at time t.

Necessary conditions for optimality

Practical advice and comments

- In addition to the above conditions, a solution to the model also has to satisfy a stochastic version of the transversality condition. We ignore this here but bear in mind that it is another requirement that has to be checked for a given model.

- Sometimes the state equations are written in the form

$$x_t = f_t(x_{t-1}, u_t, \epsilon_t).$$

(This can be interpreted as measuring the state variables at the end of each period rather than the beginning of the next one, i.e. 31 December instead of 1 January.) In such cases we have to differentiate w.r.t. the state variables at time t instead of $t + 1$.

Examples

Example I

Problem:

$$\max_{\{C_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$A_{t+1} = (1+r)A_t + Y_t - C_t, \ Y_t = \bar{Y} - \epsilon_t$$

C_t – consumption, A_t – assets, Y_t – income, r – interest rate

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[(1+r)A_t + Y_t - C_t - A_{t+1} \right] \right\}$$

Example I

Necessary conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \{C_t^{-\sigma} - \lambda_t\} = 0 \quad \implies \quad C_t^{-\sigma} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial A_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}(1+r) = 0 \quad \implies \quad \lambda_t = \beta(1+r)\lambda_{t+1}$$

We apply the expectation operator E_t:

$$E_t\{C_t^{-\sigma}\} = E_t\{\lambda_t\} \quad \implies \quad C_t^{-\sigma} = \lambda_t$$

$$E_t\{\lambda_t\} = E_t\{\beta(1+r)\lambda_{t+1}\} \quad \implies \quad \lambda_t = \beta(1+r)E_t\{\lambda_{t+1}\}$$

Substituting $\lambda_t = C_t^{-\sigma}$ in the last expression:

$$C_t^{-\sigma} = \beta(1+r)E_t\left\{C_{t+1}^{-\sigma}\right\}$$

Example I

Necessary conditions:

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Substituting $\lambda_t = C_t^{-\sigma}$ in the last expression:

$$C_t^{-\sigma} = \beta(1+r)E_t\left\{C_{t+1}^{-\sigma}\right\}$$

Homework: Solve Example I for the case when the state equation is presented according to the convention $A_t = (1+r)A_{t-1} + Y_t - C_t$.

Example II

Problem:

$$\max_{\{C_t\},\{N_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

$$Q_t B_t = B_{t-1} + W_t N_t - P_t C_t - T_t$$

C_t – consumption, N_t – labour (hours worked), B_t – discount bonds purchased at time t and maturing at $t+1$, Q_t – bond price, P_t – price of consumer good, W_t – wages, T_t – taxes or transfers (depending on sign)

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(C_t, N_t) + \lambda_t \left[B_{t-1} + W_t N_t - P_t C_t - T_t - Q_t B_t \right] \right\}$$

Example II

Define $U_{c,t} := \frac{\partial}{\partial C} U(C_t, N_t), U_{n,t} := \frac{\partial}{\partial N} U(C_t, N_t)$.

Differentiate w.r.t. the controls:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t U_{c,t} - \beta^t P_t \lambda_t = 0 \quad \implies \quad \lambda_t = \frac{U_{c,t}}{P_t}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t U_{n,t} + \beta^t \lambda_t W_t = 0 \quad \implies \quad \lambda_t W_t = -U_{n,t}$$

Combining the previous two equations, we obtain

$$\frac{W_t}{P_t} = -\frac{U_{n,t}}{U_{c,t}}.$$

Example II

Differentiate w.r.t. the state variable (at time t in this case!):

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\beta^t Q_t \lambda_t + \beta^{t+1} \lambda_{t+1} = 0 \implies Q_t \lambda_t = \beta E_t \{\lambda_{t+1}\}$$

Substitute the expression for λ for times t and $t+1$ to obtain

$$Q_t = \beta E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\}.$$

We ultimately get

$$\frac{W_t}{P_t} = -\frac{U_{n,t}}{U_{c,t}}, \quad Q_t = \beta E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\} \text{ for } t = 0, 1, 2, \dots$$

Homework: What do the above NCs look like for the case when the utility function has the form $U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$, $\sigma > 0$, $\varphi \geq 1$?

Linear approximation and log-linearization

Linear approximations

General remarks

- When solving a dynamic economic model it is often convenient to work with an approximation to the true equations of the model.

- We are usually interested in the properties of the model in a neighbourhood of an equilibrium point (steady state), therefore the approximation is taken around such a point.

- These approximations are typically linear: we transform the original, nonlinear equation in a linear one using some technique.

Linear approximations
Ways to construct a linear approximation

- Taking logs (for positive variables):

$$\frac{W_t}{P_t} = C_t^\sigma N_t^\varphi \quad \implies \quad w_t - p_t = \sigma c_t + \varphi n_t,$$

where $w_t := \ln W_t$, $p_t := \ln P_t$ etc.

- Direct linearization using Taylor's formula (we'll work it out as needed)

- Log-linearization (we'll study that next)

Log-linearization

Let $x_t \neq 0$ and let $\bar{x}$ be one of the possible values x can take (it is usually a steady state value).

The *logarithmic deviation (log-deviation)* of x_t from $\bar{x}$ is defined as:

$$\hat{x}_t := \ln\left(\frac{x_t}{\bar{x}}\right).$$

In view of

$$\ln(1+z) \approx \ln(1+z_0) + \frac{1}{1+z_0}(z - z_0),$$

taking $z_0 = 0$ we obtain $\ln(1+z) \approx z$ and therefore

$$\ln\left(\frac{x_t}{\bar{x}}\right) = \ln\left(1 + \frac{x_t - \bar{x}}{\bar{x}}\right) \approx \frac{x_t - \bar{x}}{\bar{x}}.$$

Log-linearization

Let $y_t = f(x_t, z_t)$, where f is differentiable.

Then

$$\hat{y}_t = \frac{f_x(\bar{x}, \bar{z})\bar{x}}{f(\bar{x}, \bar{z})}\hat{x}_t + \frac{f_z(\bar{x}, \bar{z})\bar{z}}{f(\bar{x}, \bar{z})}\hat{z}_t. \tag{3}$$

We have:

1. $y_t = x_t z_t \quad \Rightarrow \quad \hat{y}_t = \hat{x}_t + \hat{z}_t,$

2. $y_t = x_t / z_t \quad \Rightarrow \quad \hat{y}_t = \hat{x}_t - \hat{z}_t,$

3. $y_t = x_t + z_t \quad \Rightarrow \quad \hat{y}_t = \frac{\bar{x}}{\bar{y}}\hat{x}_t + \frac{\bar{z}}{\bar{y}}\hat{z}_t,$

4. $g(x_t, y_t) = 0 \quad \Rightarrow \quad \hat{y}_t = -\frac{g_x(\bar{x}, \bar{y})\bar{x}}{g_y(\bar{x}, \bar{y})\bar{y}}\hat{x}_t.$

Log-linearization: an example

Log-linearize the equation

$$A_{t+1} = (1 + r_t)A_t + Y_t - P_tC_t.$$

Log-linearization: an example

Log-linearize the equation

$$A_{t+1} = (1 + r_t)A_t + Y_t - P_t C_t.$$

We have

$$\hat{A}_{t+1} = \frac{(1+r)A}{A}(\widehat{1+r_t})A_t + \frac{Y}{A}\hat{Y}_t - \frac{PC}{A}\widehat{P_t C_t} \Rightarrow$$

$$\hat{A}_{t+1} = (1+r)((\widehat{1+r_t}) + \hat{A}_t) + \frac{Y}{A}\hat{Y}_t - \frac{PC}{A}(\hat{P}_t + \hat{C}_t) \Rightarrow$$

$$\hat{A}_{t+1} = (1+r)\left(\frac{r}{1+r}\hat{r}_t + \hat{A}_t\right) + \frac{Y}{A}\hat{Y}_t - \frac{PC}{A}(\hat{P}_t + \hat{C}_t) \Rightarrow$$

$$\hat{A}_{t+1} = r\hat{r}_t + (1+r)\hat{A}_t + \frac{Y}{A}\hat{Y}_t - \frac{PC}{A}(\hat{P}_t + \hat{C}_t).$$

Readings

Additional readings:

Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 12.

John Stachurski. *Economic Dynamics – Theory and Computation*. MIT Press, 2009 (Not easy but good mix between theory and computational applications. Contains an introduction to Python + code.)

For the brave:
Stokey and Lucas. *Recursive methods in economic dynamics*. Part III.

D. Bertsekas, S. Shreve. *Stochastic Optimal Control: The Discrete-Time Case*. New York, Academic Press, 1978.