R401: Statistical and Mathematical Foundations

Lecture 18: Deterministic Optimal Control in Continuous Time: The Infinite Horizon Case

Andrey Vassilev

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Lecture Contents

Introduction

The rationale behind infinite horizons

- We now start studying an important class of optimal control problems for which there is no finite terminal time T. Thus, the objective functional will look like

$$\int_0^\infty F(x(t), u(t), t)\, dt$$

or a version thereof.

The rationale behind infinite horizons

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- **Why do we need the infinite planning horizon?**

The rationale behind infinite horizons

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- **Why do we need the infinite planning horizon?**

- After all, people are mortal and we'll stop planning one day...

The rationale behind infinite horizons

- There are two (related) economic reasons why an infinite-horizon formulation might be appropriate:
  1. Entities such as households and firms may exist indefinitely despite turnover in their composition (i.e. family members dying or moving, employees changing jobs etc.).
  2. Often there is uncertainty about the end of the planning horizon. This can be conveniently modelled as an infinite horizon, especially when it is reasonable to assume that the true, finite horizon is sufficiently distant.

- A technical complication with finite planning horizons arises when state variables represent economically valuable resources (wealth, capital). In these common cases we need to either:
  - exhaust the respective resource fully as required by optimality if there is no scrap value, which is often implausible, or
  - specify an appropriate scrap value term in the objective function, which may be difficult.

The rationale behind infinite horizons

- Apart from matters of interpretation, an infinite-horizon formulation eliminates some mathematical difficulties (one generally obtains simpler and cleaner expressions).
- However, this is not costless, as certain other complications arise.
- More specifically, we need to modify appropriately our definition of optimality to capture situations that arise in the case of an infinite horizon.

The basic problem

The problem we shall be studying is the following:

$$\begin{aligned}
&\max_{u(t)\in\Omega(t)} \int_0^\infty F(x(t),u(t),t)\,dt \\
&\text{s.t.} \\
&\dot{x}(t) = f(x(t),u(t),t), \quad x(0) = x_0.
\end{aligned} \tag{1}$$

Sometimes problem (1) is replaced by a simplified version that reflects the structure of typical economic problems:

$$\begin{aligned}
&\max_{u(t)\in\Omega(t)} \int_0^\infty e^{-\rho t}\phi(x(t),u(t))\,dt, \quad \rho > 0, \\
&\text{s.t.} \\
&\dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0.
\end{aligned} \tag{2}$$

Specifics of the infinite-horizon setting

- One obvious requirement in order to have a well-defined problem is for the improper integral forming the objective functional to converge for every admissible state-control pair.
- This leads to a relatively simple case, obtainable for instance for problem (2) when
$$|\phi(x(t), u(t))| \leq M, \ \forall (x, u).$$
- Unfortunately many problems do not admit such a clean characterization, necessitating a generalization of optimality criteria.

Readings

Main references:

Seierstad and Sydsæter [SS]. *Optimal control theory with economic applications*. Chapter 3.
Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 9.

Additional readings:
Sethi and Thompson [ST]. *Optimal control theory: applications to management science and economics*. Chapter 3.