R401: Statistical and Mathematical Foundations

Lecture 17: Deterministic Optimal Control in Continuous Time: The Finite Horizon Case

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Lecture Contents

Introduction to optimal control

From calculus of variations to optimal control

- We are already familiar with the basic variational problem

$$\max_{x(t)} \left(\min_{x(t)} \right) \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \qquad x(t_0) = x_0,\ x(t_1) = x_1.$$

- It involved choosing directly the function $x(t)$. Sometimes this is precisely the object we are interested in and we are happy to work with it directly.
- In many economic situations, however, we are unable to change the variables of interest directly but only through changing other variables.
- The calculus of variations setup is less suitable for handling this case and some modifications are required.

From calculus of variations to optimal control

- The modifications lead to the class of *optimal control* problems.
- There are different formulations of optimal control problems with varying levels of complexity but the basic ingredients are as follows.
- The variables that can be changed directly are called *control variables* or *controls* (we can control the evolution of the system through them).
- The variables that can be influenced only indirectly, via the controls, are called *state variables* or simply *states*. In standard formulations the state variables are described by differential or difference equations (depending on whether time is discrete or continuous) featuring the controls.
- Similar to the calculus of variations setup, the objective functional is typically in integral or series form and involves the states and the controls.

Note: In this lecture we shall work with the continuous time case.

From calculus of variations to optimal control

- Optimal control problems usually have constraints on the controls and, in more involved cases, constraints on the state variables or on functions of both the controls and the state variables.
- Sometimes state variables may be required to lie in a certain set at the end of the horizon.
- The horizon of an optimal control problem – i.e. the interval on which the respective functions are defined and over which we seek a solution – may be finite or infinite.
- In the finite horizon case the objective functional may have an additional term capturing the state of the system at the end of the horizon.

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We now turn to making these descriptions precise.

A basic optimal control problem

- Time is continuous and finite, represented by the interval $[0, T]$.
- The state of the system at time $t \in [0, T]$ is described by means of n variables and is denoted by $x(t) \in \mathbb{R}^n$.
- There are m control variables, denoted $u(t) \in \mathbb{R}^m$ at time t.
- Usually controls at time t are constrained to lie in some set: $u(t) \in \Omega(t) \subset \mathbb{R}^m$. Controls are assumed to be piecewise continuous.
- The state of the system at time 0 is given: $x(0) = x_0$.
- For a fixed admissible control $u(t),\ t \in [0, T]$, the evolution of the system is described by the (vector) differential equation (*state equation*)

$$\dot{x}(t) = f(x(t), u(t), t), \qquad x(0) = x_0. \tag{1}$$

The function f is assumed to be continuously differentiable.

A basic optimal control problem

- The objective functional to be optimized takes the form

$$J = \int_0^T F(x(t), u(t), t)\, dt + S[x(T), T]. \tag{2}$$

- The literature differs on what the functions F and S should be called. Depending on the problem, F may be called "instantaneous utility", "instantaneous profit" or "running cost", while S may be called "scrap value", "salvage value" or "terminal cost".

- In any case, the idea is that we would like to measure running performance (as captured by the integral term) but also take into account the final state of the system (as captured by the salvage value).

- The functions F and S are also assumed continuously differentiable.

- For the sake of brevity, we shall be working with the problem of maximizing the functional (2).

A basic optimal control problem

The basic optimal control problem is

$$
\begin{aligned}
&\max_{u(t)} J = \int_0^T F(x(t), u(t), t)\, dt + S[x(T), T] \\
&\text{s.t.} \\
&u(t) \in \Omega(t), \\
&\dot{x}(t) = f(x(t), u(t), t), \qquad x(0) = x_0.
\end{aligned}
\tag{3}
$$

An admissible control $u^*(t)$ that solves problem (3) is called an *optimal control*. The associated solution of the state equation $x^*(t)$ is called the *optimal trajectory* or the *optimal path*.

A basic optimal control problem

- When the optimal control problem is formulated using the functional (2), it is said to be in *Bolza form*.
- In the special case when $S \equiv 0$, i.e. there is no salvage value, the problem is said to be in *Lagrange form*.
- In the special case when $F \equiv 0$, i.e. there is no measure of running performance, the problem is said to be in *Mayer form*. If in addition S is linear in $x(T)$, i.e. $J = cx(T)$, the problem is said to be in *linear Mayer form*.

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It can be shown that all these forms can be converted to linear Mayer form (see ST, Chapter 2).

The maximum principle

General comments

- We now turn to stating optimality conditions for problem (3).
- There are two general approaches: via Pontryagin's maximum principle and via Hamilton-Jacobi-Bellman equations.
- Here we'll follow te first one as it is generally more tractable and is widely used in economic applications.
- Using the maximum principle has the added benefit of being similar in spirit to the familiar Lagrangian approaches.

The Hamiltonian

We now introduce an object similar to the Lagrangian, which is called the Hamiltonian of the optimal control problem.

The Hamiltonian is defined as

$$\begin{aligned} H(x,u,\lambda,t) =& F(x,u,t) + \lambda \cdot f(x,u,t) \\ =& F(x,u,t) + \sum_{i=1}^{n} \lambda_i f_i(x,u,t). \end{aligned} \tag{4}$$

The variables λ_i are called the *costate variables* or *adjoint variables*.

Necessary conditions for optimality

We can find candidates for optimality using the following procedure.

Algorithm (The maximum principle)

1. Form the Hamiltonian (4) for the problem.
2. Maximize the Hamiltonian w.r.t. u, i.e. find $u^*(t)$ such that
$$H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u(t, \lambda(t), t), \quad \forall u(t) \in \Omega(t),\ t \in [0, T].$$
3. Compute the adjoint equations together with the terminal boundary conditions:
$$\dot{\lambda} = -H'_x(x^*, u^*, \lambda, t), \quad \lambda(T) = S'_x(x^*(T), T).$$
4. Solve the two-point boundary value problem comprising the adjoint equations and terminal boundary conditions from step (3), and the state equations
$$\dot{x}^* = f(x^*, u^*, t), \quad x^*(0) = x_0.$$

Comments on the maximum principle

- Working with the Hamiltonian involves treating it as a function of four variables: x, u, λ, t, without taking into account the dependence of x, u, λ on t.
- In many economic applications it turns out that the maximum is attained at an interior point of the set of feasible controls. The Hamiltonian maximization condition then takes the form

$$\frac{\partial H}{\partial u} = 0.$$

- When it is unclear whether the maximum is attained at an interior point, a more general approach, e.g. using the Kuhn-Tucker conditions to solve the Hamiltonian maximization problem, is called for.

Comments on the maximum principle

- Once we have found u^*, we can substitute it in the adjoint and state equations to obtain the above-mentioned two-point boundary value problem:

$$\begin{aligned} \dot{\lambda} &= -H'_x(x^*, u^*, \lambda, t), \quad \lambda(T) = S'_x(x^*(T), T), \\ \dot{x}^* &= f(x^*, u^*, t), \quad x^*(0) = x_0. \end{aligned}$$

- This is a system in $2n$ equations with n initial conditions (for the xs) and n terminal conditions (for the λs).
- In realistic applications such a system is solved numerically.
- The conditions $\lambda(T) = S'_x(x^*(T), T)$ are known as the *transversality conditions*.

The maximum principle in action

Example 1

Find the NCs for the problem

$$\max \int_0^T (1 - tx(t) - u(t)^2)\, dt$$

s.t.

$$\dot{x}(t) = u(t), \quad x(0) = x_0,\ u(t) \in \mathbb{R}.$$

The maximum principle in action

Example 1

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s.t.

$$\dot{x}(t) = u(t), \quad x(0) = x_0,\ u(t) \in \mathbb{R}.$$

To solve the problem, we set up the Hamiltonian:

$$H = 1 - tx - u^2 + \lambda u.$$

Because of the form of the Hamiltonian (what is it?), its maximum can be found via

$$\frac{\partial H}{\partial u} = -2u + \lambda = 0 \quad \Rightarrow \quad u = \frac{\lambda}{2}.$$

The maximum principle in action

Example 1 (cont.)

Since $\partial H/\partial x = -t$, we obtain

$$\dot{\lambda} = t \quad \Rightarrow \quad \lambda(t) = \frac{1}{2}t^2 + C_1.$$

For this example the salvage value is $S \equiv 0$, thus the transversality condition takes the form $\lambda(T) = 0$.
This can be used to compute C_1:

$$\lambda(T) = \frac{1}{2}T^2 + C_1 = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{2}T^2.$$

Therefore, we have

$$\lambda(t) = \frac{1}{2}t^2 - \frac{1}{2}T^2.$$

The maximum principle in action

Example 1 (cont.)

The state equation takes the form

$$\dot{x} = \frac{1}{2}\left(\frac{1}{2}t^2 + C_1\right).$$

Integrating, we obtain

$$\begin{aligned} x(t) = \int \frac{1}{4}t^2\,dt + \int \frac{1}{2}C_1\,dt &= \frac{1}{4\cdot 3}t^3 - \frac{1}{2}\frac{T^2}{2}t + C_2 \\ &= \frac{1}{12}t^3 - \frac{T^2}{4}t + C_2. \end{aligned}$$

Since $x(0) = x_0$, we get $C_2 = x_0$ and

$$x(t) = \frac{1}{12}t^3 - \frac{T^2}{4}t + x_0.$$

The maximum principle in action

Example 2

Find the NCs for the problem

$$\max \int_0^1 -x\,dt$$

s.t.

$$\dot{x} = u, \quad x(0) = 1,\ u \in \Omega = [-1, 1].$$

The maximum principle in action

Example 2

Find the NCs for the problem

$$\max \int_0^1 -x\,dt$$

s.t.

$$\dot{x} = u, \quad x(0) = 1,\ u \in \Omega = [-1, 1].$$

The Hamiltonian is

$$H = -x + \lambda u.$$

The maximum principle in action

Example 2

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The Hamiltonian is

$$H = -x + \lambda u.$$

As the Hamiltonian is linear w.r.t. the control, its maximum will in general be attained at a corner point of the feasible set Ω.

The maximum principle in action

Example 2 (cont.)

Thus, the solution for u will depend on the sign of the adjoint variable λ as follows:

$$u^* = \begin{cases} 1 & \text{for} \quad \lambda > 0, \\ -1 & \text{for} \quad \lambda < 0, \\ \text{can take any value in } \Omega & \text{for} \quad \lambda = 0. \end{cases}$$

Controls that switch between the bounds of the feasible set are known as *bang-bang controls*.

The maximum principle in action

Example 2 (cont.)

We have $\partial H/\partial x = -1$, and hence the adjoint equation is

$$\dot{\lambda} = 1 \quad \Rightarrow \quad \lambda(t) = t + C_1.$$

In this example $S \equiv 0$, leading to the transversality condition

$$\lambda(1) = 0.$$

The transversality condition in turn implies that $0 = 1 + C_1$, thus $C_1 = -1$ and

$$\lambda(t) = t - 1.$$

Since $t \in [0, 1]$, it follows that $\lambda(t) \leq 0$.

current-value vs present value Hamiltonians
CoV problems as special cases of OC problems
Sufficiency
mixed constraints?

Readings

Main references:
Sethi and Thompson [ST]. Optimal control theory: applications to management science and economics. Chapters 1 and 2.

Additional readings:
Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapters 9 and 10.