R401: Statistical and Mathematical Foundations

Infinite-Horizon Deterministic Optimal Control in Discrete Time

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Lecture Contents

Introduction

Switching to discrete time

- You are already familiar with a number of dynamic optimization problems in which time is continuous.
- In many applications, however, it is natural to work in discrete time.
- This provides a bridge to validating/calibrating models with data or estimating them, among others.
- We therefore need to develop the counterpart of the continuous-time optimal control framework for the case of discrete time.

Switching to discrete time

- Some of the details of such a transition are easily predictable:
  - Differential equations will be replaced by difference equations.
  - The objective functional will involve a series instead of an integral.
- Other details need to be specified further. In particular, there exist two broad classes of dynamic optimization problems in discrete time:
  - Problems in variational form
  - Problems with explicit controls
- Both classes can be used to address a wide variety of problems.
- However, problems with explicit controls are a bit more transparent in terms of their structure.

Specific assumptions

- We will work in an infinite-horizon setup. Finite-horizon formulations for discrete-time problems exist but are less common in economic applications.
- We sacrifice some generality from the outset by assuming a specific structure of the problems:
  - Special (time) discounting in the objective functionals.
  - Autonomous difference equations describing the evolution of the system that is being modelled.

Problems in variational form

Formulation

A dynamic optimization problem in variational form is defined as follows:

$$\begin{aligned} \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t. } x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2 \ldots, \\ x_0 \in X \text{ – given} \end{aligned} \tag{1}$$

The problem is characterized by the following:

- We choose directly the sequence $\{x_t\}_{t=1}^{\infty}$. For any element x_t we have $x_t \in X$, where X is the set of states.
- At any point in time x_t defines a set $\Gamma(x_t)$ of admissible values for x in the following period.
- The number β is called the *discount factor* and $\beta \in (0, 1)$.
- We shall assume differentiability of the function F as needed.
- We write $\max$ everywhere with some sacrifice of mathematical precision.

Additional information on problems in variational form

The Bellman equation

- A problem of the form given in (1) has an associated equation of the form

$$v(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}, \quad \forall x \in X. \tag{2}$$

- The equation is called the *Bellman equation*.
- The Bellman equation is a *functional* equation: it involves finding an unknown function v.
- The Bellman equation may not have a solution or it may have multiple solutions.
- Notice that solving the Bellman equation (however it may be done) involves finding the maximizing value $y^* \in \Gamma(x)$.

Additional information on problems in variational form

The value function

- Denote the set of all feasible sequences $\{x_t\}_{t=0}^{\infty}$ starting from x_0 by $\Pi(x_0)$.
- Define the function $v^*(x_0)$ as

$$v^*(x_0) := \max_{\{x_t\} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}).$$

- This function is known as the *value function*.
- It can be shown that the value function is (one) solution to the Bellman equation.
- Moreover, v^* is the only solution to the Bellman equation that satisfies the boundedness condition

$$\lim_{t \to \infty} \beta^t v^*(x_t) = 0 \text{ for all } (x_0, x_1, \ldots) \in \Pi(x_0) \text{ and all } x_0 \in X.$$

Additional information on problems in variational form

- It can also be shown that an optimal sequence $\{x_t\}$ for problem (1) satisfies the relations

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \quad t = 0, 1, 2, \ldots$$

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- **The above statements are only indicative. The precise formulations require specific assumptions on the mathematical structure of the problem.**

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- **The above statements are only indicative. The precise formulations require specific assumptions on the mathematical structure of the problem.**
- The Bellman equation approach is quite general. However, it often leads to situations which are analytically intractable and computationally demanding.
- For this reason it is typical to resort to more restrictive but tractable approaches.

The Euler equations

- One approach that is essentially the discrete-time counterpart of the Euler equation from the calculus of variations can be used to provide necessary conditions for optimality.
- Because of this similarity, the resulting necessary conditions are also called *Euler equations*.
- They take the form

$$\nabla_y F(x_t, x_{t+1}) + \beta \nabla_x F(x_{t+1}, x_{t+2}) = 0. \tag{3}$$

- Notice that they lead to a second-order difference equation (or, more precisely, a system of second-order difference equations), just like the Euler equation for the continuous-time variational problem produced a second-order ODE.

Sufficiency

- The Euler equations can be supplemented with appropriate transversality conditions to obtain sufficient conditions for optimality.
- The precise formulation requires technical concepts and assumptions that are beyond this course (see Stokey and Lucas, Ch. 4).
- The main assumptions are that X is a convex subset of $\mathbb{R}^n_+$, $\Gamma(x)$ is nonempty and compact, F is bounded, concave, differentiable and strictly increasing in x_t. There are additional assumptions and qualifications.
- The essence of the sufficiency result is that, under the required assumptions, a feasible sequence $\{x_t^*\}_{t=0}^{\infty}$ satisfying the Euler equations (3) and the transversality condition

$$\lim_{t\to\infty} \beta^t \nabla_x F(x_t^*, x_{t+1}^*) x_t^* = 0$$

is optimal for problem (1).

Example: NCs for a problem in variational form

Compute the NCs for the problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_{t+1} = (1-\delta)k_t + y_t - c_t, \quad k_0 > 0 \text{ – given}$$

$$y_t = Ak_t^{\alpha}, \quad 0 < \underline{\epsilon} \leq c_t \leq \overline{\epsilon}, \quad \alpha, \beta \in (0,1), \quad A > 0$$

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To write the NCs we transform the problem in variational form:

$$\sum_{t=0}^{\infty} \beta^t \underbrace{\ln((1-\delta)k_t + Ak_t^{\alpha} - k_{t+1})}_{=F(x_t, x_{t+1})}.$$

Example: NCs for a problem in variational form

Applying the Euler equation, we get:

$$\frac{-1}{(1-\delta)k_t + Ak_t^{\alpha} - k_{t+1}} + \beta \frac{(1-\delta) + \alpha A k_{t+1}^{\alpha-1}}{(1-\delta)k_{t+1} + Ak_{t+1}^{\alpha} - k_{t+2}} = 0.$$

The last result has the form

$$G(x_{t+2}, x_{t+1}, x_t) = 0,$$

i.e. a nonlinear second-order difference equation.

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Question: Can you compute the solution of the equation for $\beta = 0.95,\ A = 1,\ \delta = 0.05,\ \alpha = 0.5$ and $k_0 = 2$?

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Question: Can you compute the solution of the equation for $\beta = 0.95,\ A = 1,\ \delta = 0.05,\ \alpha = 0.5$ and $k_0 = 2$? If not, what else do you need?

Problems with explicit controls

Formulation of the problem with explicit controls

It is also possible to introduce controls explicitly, as a direct counterpart of the continuous-time formulation. This is done as follows.

Let $X \subset \mathbb{R}^n$ be the *state space* for a model, where the state variables are $x = (x^1, \ldots, x^n)$.

We assume that $\forall x \in X,\ \exists \Omega(x) \subset \mathbb{R}^m, \Omega(x) \neq \emptyset$. The elements of $u = (u^1, \ldots, u^m)$ are our *controls*.

The (instantaneous) objective function is $F(x, u) \ \ x \in X, u \in \Omega(x)$

The state equations are

$$x_{t+1} = f(x_t, u_t), \quad x_0 \text{ – given} \tag{4}$$

where $f(x, u)$ is a vector function taking values in X, for $x \in X, u \in \Omega(x)$.

Formulation of the problem with explicit controls

We need to find a sequence of admissible controls $\mathbf{u} = \{u_t\}, t = 0, 1, 2, \ldots,$ which determine a sequence of state variables $\{x_{t+1}\}, t = 0, 1, 2, \ldots$ via (4) for which

$$J(x_0, \mathbf{u}) = \sum_{t=0}^{\infty} \beta^t F(x_t, u_t) \tag{5}$$

attains a maximum

$$v(x_0) = \max_{\mathbf{u}} J(x_0, \mathbf{u}). \tag{6}$$

Formulation of the problem with explicit controls

The number $\beta \in (0, 1)$ is the *discount factor* in the model.

Denote by $\mathrm{FC}(x_0)$ the set of all feasible control sequences $\{u_t\}_{t=0}^{\infty}$ for initial $x_0 \in X$, i.e. x_{t+1} satisfies (4) for $u_t \in \Omega(x_t)$, $t = 0, 1, 2, \ldots,$ and a given x_0.

We denote the optimal sequence of pairs of state variables and controls for problem (4)-(6) by $\{x_{t+1}^*, u_t^*\},\ t = 0, 1, 2, \ldots$, i.e. $\{u_t^*\} \in \mathrm{FC}(x_0)$, and

$$v(x_0) = J(x_0, \mathbf{u}^*),\ \mathbf{u}^* = \{u_t^*\}.$$

Necessary conditions for optimality

A variation of the familiar approach for problems of this type is the following:

Algorithm

1. Construct the Lagrangian

$$\mathcal{L}(x_1, x_2, \ldots, u_0, u_1, \ldots) = \sum_{t=0}^{\infty} \beta^t \left[F(x_t, u_t) + \lambda_t' [f(x_t, u_t) - x_{t+1}] \right],$$

where $\lambda_t = (\lambda_t^1, \ldots, \lambda_t^n)', t = 0, 1, 2, \ldots,$ are the Lagrange multipliers and the prime ($'$) denotes transposition as usual:

$$\lambda_t' [f(x_t, u_t) - x_{t+1}] = \sum_{i=1}^{n} \lambda_t^i [f^i(x_t, u_t) - x_{t+1}^i].$$

Necessary conditions for optimality

Algorithm (cont.)

2. Differentiate $\mathcal{L}$ w.r.t. x_t and u_t, set the resulting expressions equal to zero and obtain first-order necessary conditions for optimality:

$$\begin{aligned} \beta\left[F_{x_t^k}(x_t,u_t)+\sum_{i=1}^{n}\lambda_t^i f^i_{x_t^k}(x_t,u_t)\right] &= \lambda_{t-1}^k,\ k=1,\ldots,n, \\ F_{u_t^j}(x_t,u_t)+\sum_{i=1}^{n}\lambda_t^i f^i_{u_t^j}(x_t,u_t) &= 0,\ j=1,\ldots,m. \end{aligned} \tag{7}$$

3. Equations (4) and (7) are solved as a system and we obtain a candidate solution $\{u_t\}_{t=0}^{\infty}$ or, more precisely, a sequence $\{x_{t+1},u_t\}_{t=0}^{\infty}$.

Note: It is common to find a stationary point of the system (4) and (7), and work with a linearised version of the system around that point.

Necessary conditions for optimality

In matrix notation the above takes the form:

$$\mathcal{L}_x = \beta^t \nabla_x F(x_t, u_t) + \beta^t \nabla'_x f(x_t, u_t)\lambda_t - \beta^{t-1}\lambda_{t-1} = 0 \Rightarrow$$
$$\beta(\nabla_x F(x_t, u_t) + \nabla'_x f(x_t, u_t)\lambda_t) = \lambda_{t-1}. \tag{8}$$

$$\mathcal{L}_u = \beta^t \nabla_u F(x_t, u_t) + \beta^t \nabla'_u f(x_t, u_t)\lambda_t = 0 \Rightarrow$$
$$\nabla_u F(x_t, u_t) + \nabla'_u f(x_t, u_t)\lambda_t = 0 \tag{9}$$

Necessary conditions for optimality

Why is this algorithm valid?

The value function for problem (4)-(6) satisfies a version of the Bellman equation:

$$v(x) = \max_{u \in \Omega(x)} \left[F(x, u) + \beta v(f(x, u)) \right]. \tag{10}$$

Let the maximum in (10) be attained on the interior of the set $F(x)$. Denote this point by $u = \nu(x)$ and assume that all objects used below are differentiable.

Necessary conditions for optimality

We have

$$v(x) = F(x, \nu(x)) + \beta v(f(x, \nu(x))). \tag{11}$$

Also the extremum condition is

$$\nabla_u F(x, \nu(x)) + \beta \nabla'_u f(x, \nu(x)) \nabla v(f(x, \nu(x))) = 0. \tag{12}$$

Differentiating (11) w.r.t. x, we obtain

$$\begin{aligned}
\nabla v(x) =& \nabla_x F(x, \nu(x)) + \nabla' \nu(x) \nabla_u F(x, \nu(x)) + \\
& \beta \left[\nabla'_x f(x, \nu(x)) + \nabla' \nu(x) \nabla'_u f(x, \nu(x)) \right] \nabla v(f(x, \nu(x))) \\
=& \nabla_x F(x, \nu(x)) + \beta \nabla'_x f(x, \nu(x)) \nabla v(f(x, \nu(x))) + \\
& \underbrace{\nabla' \nu(x) \nabla_u F(x, \nu(x)) + \beta \nabla' \nu(x) \nabla'_u f(x, \nu(x)) \nabla v(f(x, \nu(x)))}_{=0 \text{ in view of (12)}}.
\end{aligned}$$

Necessary conditions for optimality

We thus get

$$\nabla v(x) = \nabla_x F(x, \nu(x)) + \beta \nabla'_x f(x, \nu(x)) \nabla v(f(x, \nu(x))). \tag{13}$$

For $x = x_t^*$ and $u_t^* = \nu(x_t^*)$, equations (12) and (13) take the form

$$\nabla_u F(x_t^*, u_t^*) + \beta \nabla'_u f(x_t^*, u_t^*) \nabla v(x_{t+1}^*) = 0, \tag{14}$$

$$\nabla v(x_t^*) = \nabla_x F(x_t^*, u_t^*) + \beta \nabla'_x f(x_t^*, u_t^*) \nabla v(x_{t+1}^*). \tag{15}$$

Set $\lambda_t := \beta \nabla v(x_{t+1}^*)$ in (14) and (15), to obtain precisely (8) and (9).

Example: NCs for a problem with explicit controls

Compute the NCs for the problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_{t+1} = (1-\delta)k_t + y_t - c_t, \quad k_0 > 0 \text{ – given}$$

$$y_t = Ak_t^{\alpha}, \quad 0 < \underline{\epsilon} \le c_t \le \overline{\epsilon}, \quad \alpha, \beta \in (0,1), \quad A > 0$$

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$$y_t = Ak_t^{\alpha}, \quad 0 < \underline{\epsilon} \leq c_t \leq \overline{\epsilon}, \quad \alpha, \beta \in (0,1), \quad A > 0$$

The Lagrangian for the problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + \lambda_t((1-\delta)k_t + Ak_t^{\alpha} - c_t - k_{t+1})\right].$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \beta^t \lambda_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{c_t}.$$

Example: NCs for a problem with explicit controls

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial k_t} =&\beta^t \lambda_t (1-\delta+\alpha A k_t^{\alpha-1}) - \beta^{t-1}\lambda_{t-1} = 0 \quad \Rightarrow \\ &\lambda_{t-1} = \beta \lambda_t (1-\delta+\alpha A k_t^{\alpha-1}).\end{aligned}$$

Substituting the $1/c_t$ for λ_t, we obtain

$$c_t = \beta c_{t-1}(1-\delta+\alpha A k_t^{\alpha-1}).$$

This is a version of a *consumption Euler equation* showing how consumption changes between two periods.

Sufficient conditions

Fact 1

Let $\{\lambda_t\}$ and $\{x^*_{t+1}, u^*_t\}, t = 0, 1, 2, \ldots,$ be obtained by using (4) and (7). If

1. The functions $F(x, u)$ and $f(x, u)$ are concave in (x, u),
2. The Lagrange multipliers $\lambda^1_t, \ldots, \lambda^n_t,\ t = 0, 1, 2, \ldots$ are nonnegative,
3. The state space X is a subset of $\mathbb{R}^n_+$ and the following transversality condition is valid

$$\lim_{T\to\infty} \beta^T \lambda'_T x^*_{T+1} = 0,$$

then the sequence $\{x^*_{t+1}, u^*_t\}$ (for a given x_0) is optimal for problem (4)-(6).

Sufficient conditions

We shall verify the validity of Fact 1.

Recall that (7) in matrix terms is given by (8) and (9).
Consider

$$\mathcal{L}_T(x_t, u_t) = \sum_{t=0}^{T} \beta^t \left\{ F(x_t, u_t) + \lambda_t'[f(x_t, u_t) - x_{t+1}] \right\}.$$

We have

$$\begin{aligned} D := \mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) = \sum_{t=0}^{T} \beta^t \lambda_t'(x_{t+1}^* - x_{t+1}) + \\ \sum_{t=0}^{T} \beta^t [F(x_t, u_t) + \lambda_t' f(x_t, u_t) - F(x_t^*, u_t^*) - \lambda_t' f(x_t^*, u_t^*)]. \end{aligned} \tag{16}$$

Sufficient conditions

Then, in view of concavity, we get

$$\mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) \ \ (= D) \ \ \leq \sum_{t=0}^{T} \beta^t \lambda_t'(x_{t+1}^* - x_{t+1}) +$$

$$\sum_{t=0}^{T} \beta^t [\nabla_x' F(x_t^*, u_t^*)(x_t - x_t^*) + \nabla_u' F(x_t^*, u_t^*)(u_t - u_t^*) +$$

$$\lambda_t'[\nabla_x f(x_t^*, u_t^*)(x_t - x_t^*) + \nabla_u f(x_t^*, u_t^*)(u_t - u_t^*)]] =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t'(x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T} \beta^t \underbrace{[\nabla_x' F(x_t^*, u_t^*) + \lambda_t' \nabla_x f(x_t^*, u_t^*)]}_{= \frac{\lambda_{t-1}'}{\beta} \text{ in view of (8)}} (x_t - x_t^*)$$

$$+ \sum_{t=0}^{T} \beta^t \underbrace{[\nabla_u' F(x_t^*, u_t^*) + \lambda_t' \nabla_u f(x_t^*, u_t^*)]}_{= \mathbf{0}' \text{ in view of (9)}} (u_t - u_t^*).$$

Sufficient conditions

We therefore have

$$D \leq \sum_{t=0}^{T} \beta^t \lambda_t' (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T} \beta^t \frac{\lambda_{t-1}'}{\beta} \underbrace{(x_t - x_t^*)}_{\text{N.B.: } x_0 = x_0^*} =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t' (x_{t+1}^* - x_{t+1}) + \sum_{t=1}^{T} \beta^{t-1} \frac{\lambda_{t-1}'}{\beta} (x_t - x_t^*) =$$

$$\sum_{t=0}^{T} \beta^t \lambda_t' (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda_t'}{\beta} (x_{t+1} - x_{t+1}^*) =$$

$$\underbrace{\sum_{t=0}^{T-1} \beta^t \lambda_t' (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda_t'}{\beta} (x_{t+1} - x_{t+1}^*)}_{=0} +$$

$$\beta^T \lambda_T' (x_{T+1}^* - x_{T+1}) \underbrace{\leq}_{\text{since } \lambda_t, x_t \geq \mathbf{0}} \beta^T \lambda_T' x_{T+1}^*.$$

Sufficient conditions

In view of the transversality condition, we have:

$$D \leq \beta^T \lambda_T' x_{T+1}^* \underset{T \to \infty}{\longrightarrow} 0,$$

i.e.

$$\mathcal{L}_T(x_t^*, u_t^*) - \mathcal{L}_T(x_t, u_t) \geq 0,$$

which proves the optimality of the sequence $\{x_{t+1}^*, u_t^*\}$.

Readings

Additional readings:

Stokey, Lucas and Prescott. 1989. *Recursive methods in economic dynamics*. Chapters 2 and 4.

Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 12.