

R401: Statistical and Mathematical Foundations

Lecture 19: Infinite-Horizon Deterministic Optimal Control in Discrete Time

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Introduction

Switching to discrete time

- You are already familiar with a number of dynamic optimization problems in which time is continuous.
- In many applications, however, it is natural to work in discrete time.
- This provides a bridge to validating/calibrating models with data or estimating them, among others.
- We therefore need to develop the counterpart of the continuous-time optimal control framework for the case of discrete time.

Switching to discrete time

- Some of the details of such a transition are easily predictable:
 - Differential equations will be replaced by difference equations.
 - The objective functional will involve a series instead of an integral.
- Other details need to be specified further. In particular, there exist two broad classes of dynamic optimization problems in discrete time:
 - Problems in variational form
 - Problems with explicit controls
- Both classes can be used to address a wide variety of problems.
- However, problems with explicit controls are a bit more transparent in terms of their structure.

Specific assumptions

- We will work in an infinite-horizon setup. Finite-horizon formulations for discrete-time problems exist but are less common in economic applications.
- We sacrifice some generality from the outset by assuming a specific structure of the problems:
 - Special (time) discounting in the objective functionals.
 - Autonomous difference equations describing the evolution of the system that is being modelled.

Problems in variational form

Formulation

A dynamic optimization problem in variational form is defined as follows:

$$\begin{aligned}
 & \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
 & \text{s.t. } x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \dots, \\
 & \quad x_0 \in X - \text{given}
 \end{aligned} \tag{1}$$

The problem is characterized by the following:

- We choose directly the sequence $\{x_t\}_{t=1}^{\infty}$. For any element x_t we have $x_t \in X$, where X is the set of states.
- At any point in time x_t defines a set $\Gamma(x_t)$ of admissible values for x in the following period.
- The number β is called the *discount factor* and $\beta \in (0, 1)$.
- We shall assume differentiability of the function F as needed.
- We write \max everywhere with some sacrifice of mathematical precision.

Additional information on problems in variational form

The Bellman equation

- A problem of the form given in (1) has an associated equation of the form

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}, \quad \forall x \in X. \quad (2)$$

- The equation is called the *Bellman equation*.
- The Bellman equation is a *functional* equation: it involves finding an unknown function v .
- The Bellman equation may not have a solution or it may have multiple solutions.
- Notice that solving the Bellman equation (however it may be done) involves finding the maximizing value $y^* \in \Gamma(x)$.

Additional information on problems in variational form

The value function

- Denote the set of all feasible sequences $\{x_t\}_{t=0}^{\infty}$ starting from x_0 by $\Pi(x_0)$.
- Define the function $v^*(x_0)$ as

$$v^*(x_0) := \max_{\{x_t\} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}).$$

- This function is known as the *value function*.
- It can be shown that the value function is (one) solution to the Bellman equation.
- Moreover, v^* is the only solution to the Bellman equation that satisfies the boundedness condition

$$\lim_{t \rightarrow \infty} \beta^t v^*(x_t) = 0 \text{ for all } (x_0, x_1, \dots) \in \Pi(x_0) \text{ and all } x_0 \in X.$$

Additional information on problems in variational form

- It can also be shown that an optimal sequence $\{x_t\}$ for problem (1) satisfies the relations

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \quad t = 0, 1, 2, \dots$$

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- The above statements are only indicative. The precise formulations require specific assumptions on the mathematical structure of the problem.**

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- The above statements are only indicative. The precise formulations require specific assumptions on the mathematical structure of the problem.**
- The Bellman equation approach is quite general. However, it often leads to situations which are analytically intractable and computationally demanding.
- For this reason it is typical to resort to more restrictive but tractable approaches.

The Euler equations

- One approach that is essentially the discrete-time counterpart of the Euler equation from the calculus of variations can be used to provide necessary conditions for optimality.
- Because of this similarity, the resulting necessary conditions are also called *Euler equations*.
- They take the form

$$\nabla_y F(x_t, x_{t+1}) + \beta \nabla_x F(x_{t+1}, x_{t+2}) = 0. \quad (3)$$

- Notice that they lead to a second-order difference equation (or, more precisely, a system of second-order difference equations), just like the Euler equation for the continuous-time variational problem produced a second-order ODE.

Sufficiency

- The Euler equations can be supplemented with appropriate transversality conditions to obtain sufficient conditions for optimality.
- The precise formulation requires technical concepts and assumptions that are beyond this course (see Stokey and Lucas, Ch. 4).
- The main assumptions are that X is a convex subset of \mathbb{R}_+^n , $\Gamma(x)$ is nonempty and compact, F is bounded, concave, differentiable and strictly increasing in x_t . There are additional assumptions and qualifications.
- The essence of the sufficiency result is that, under the required assumptions, a feasible sequence $\{x_t^*\}_{t=0}^\infty$ satisfying the Euler equations (3) and the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t \nabla_x F(x_t^*, x_{t+1}^*) \cdot x_t^* = 0$$

is optimal for problem (1).

Example: NCs for a problem in variational form

Compute the NCs for the problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t, \quad k_0 > 0 - \text{given}$$

$$y_t = Ak_t^\alpha, \quad 0 < \underline{\epsilon} \leq c_t \leq \bar{\epsilon}, \quad \alpha, \beta \in (0, 1), \quad A > 0$$

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To write the NCs we transform the problem in variational form:

$$\sum_{t=0}^{\infty} \beta^t \underbrace{\ln((1 - \delta)k_t + Ak_t^\alpha - k_{t+1})}_{=F(x_t, x_{t+1})}.$$

Example: NCs for a problem in variational form

Applying the Euler equation, we get:

$$\frac{-1}{(1-\delta)k_t + Ak_t^\alpha - k_{t+1}} + \beta \frac{(1-\delta) + \alpha Ak_{t+1}^{\alpha-1}}{(1-\delta)k_{t+1} + Ak_{t+1}^\alpha - k_{t+2}} = 0.$$

The last result has the form

$$G(x_{t+2}, x_{t+1}, x_t) = 0,$$

i.e. a nonlinear second-order difference equation.

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Question: Can you compute the solution of the equation for $\beta = 0.95$, $A = 1$, $\delta = 0.05$, $\alpha = 0.5$ and $k_0 = 2$?

Example: NCs for a problem in variational form

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Question: Can you compute the solution of the equation for $\beta = 0.95$, $A = 1$, $\delta = 0.05$, $\alpha = 0.5$ and $k_0 = 2$? If not, what else do you need?

Problems with explicit controls

Formulation of the problem with explicit controls

It is also possible to introduce controls explicitly, as a direct counterpart of the continuous-time formulation. This is done as follows.

Let $X \subset \mathbb{R}^n$ be the *state space* for a model, where the state variables are $x = (x^1, \dots, x^n)$.

We assume that $\forall x \in X, \exists \Omega(x) \subset \mathbb{R}^m, \Omega(x) \neq \emptyset$. The elements of $u = (u^1, \dots, u^m)$ are our *controls*.

The (instantaneous) objective function is $F(x, u) \quad x \in X, u \in \Omega(x)$

The state equations are

$$x_{t+1} = f(x_t, u_t), \quad x_0 - \text{given} \quad (4)$$

where $f(x, u)$ is a vector function taking values in X , for $x \in X, u \in \Omega(x)$.

Formulation of the problem with explicit controls

We need to find a sequence of admissible controls $\mathbf{u} = \{u_t\}, t = 0, 1, 2, \dots$, which determine a sequence of state variables $\{x_{t+1}\}, t = 0, 1, 2, \dots$ via (4) for which

$$J(x_0, \mathbf{u}) = \sum_{t=0}^{\infty} \beta^t F(x_t, u_t) \quad (5)$$

attains a maximum

$$v(x_0) = \max_{\mathbf{u}} J(x_0, \mathbf{u}). \quad (6)$$

Formulation of the problem with explicit controls

The number $\beta \in (0, 1)$ is the *discount factor* in the model.

Denote by $\text{FC}(x_0)$ the set of all feasible control sequences $\{u_t\}_{t=0}^{\infty}$ for initial $x_0 \in X$, i.e. x_{t+1} satisfies (4) for $u_t \in \Omega(x_t)$, $t = 0, 1, 2, \dots$, and a given x_0 .

We denote the optimal sequence of pairs of state variables and controls for problem (4)-(6) by $\{x_{t+1}^*, u_t^*\}$, $t = 0, 1, 2, \dots$, i.e. $\{u_t^*\} \in \text{FC}(x_0)$, and

$$v(x_0) = J(x_0, \mathbf{u}^*), \quad \mathbf{u}^* = \{u_t^*\}.$$

Necessary conditions for optimality

A variation of the familiar approach for problems of this type is the following:

Algorithm

- 1 Construct the Lagrangian

$$\mathcal{L}(x_1, x_2, \dots, u_0, u_1, \dots) = \sum_{t=0}^{\infty} \beta^t [F(x_t, u_t) + \lambda'_t \cdot [f(x_t, u_t) - x_{t+1}]] ,$$

where $\lambda_t = (\lambda_t^1, \dots, \lambda_t^n)$, $t = 0, 1, 2, \dots$, are the Lagrange multipliers and the dot (\cdot) denotes a matrix product or a scalar product:

$$\lambda'_t \cdot [f(x_t, u_t) - x_{t+1}] = \sum_{i=1}^n \lambda_t^i [f^i(x_t, u_t) - x_{t+1}^i].$$

Necessary conditions for optimality

Algorithm (cont.)

- ② Differentiate \mathcal{L} w.r.t. x_t and u_t , set the resulting expressions equal to zero and obtain first-order necessary conditions for optimality:

$$\beta \left[F_{x_t^k}(x_t, u_t) + \sum_{i=1}^n \lambda_t^i f_{x_t^k}^i(x_t, u_t) \right] = \lambda_{t-1}^k, \quad k = 1, \dots, n, \quad (7)$$

$$F_{u_t^j}(x_t, u_t) + \sum_{i=1}^n \lambda_t^i f_{u_t^j}^i(x_t, u_t) = 0, \quad j = 1, \dots, m.$$

- ③ Equations (4) and (7) are solved as a system and we obtain a candidate solution $\{u_t\}_{t=0}^{\infty}$ or, more precisely, a sequence $\{x_{t+1}, u_t\}_{t=0}^{\infty}$.

Note: It is common to find a stationary point of the system (4) and (7), and work with a linearised version of the system around that point.

Necessary conditions for optimality

In matrix notation the above takes the form:

$$\begin{aligned}\mathcal{L}_x &= \beta^t F_x(x_t, u_t) + \beta^t f'_x(x_t, u_t) \cdot \lambda_t - \beta^{t-1} \lambda_{t-1} = 0 \Rightarrow \\ &\beta(F_x(x_t, u_t) + f'_x(x_t, u_t) \cdot \lambda_t) = \lambda_{t-1}.\end{aligned}\tag{8}$$

$$\begin{aligned}\mathcal{L}_u &= \beta^t F_u(x_t, u_t) + \beta^t f'_u(x_t, u_t) \cdot \lambda_t = 0 \Rightarrow \\ &F_u(x_t, u_t) + f'_u(x_t, u_t) \cdot \lambda_t = 0\end{aligned}\tag{9}$$

Necessary conditions for optimality

Why is this algorithm valid?

The value function for problem (4)-(6) satisfies a version of the Bellman equation:

$$v(x) = \max_{u \in \Omega(x)} [F(x, u) + \beta v(f(x, u))]. \quad (10)$$

Let the maximum in (10) be attained on the interior of the set $F(x)$. Denote this point by $u = v(x)$ and assume that all objects used below are differentiable.

Necessary conditions for optimality

We have

$$v(x) = F(x, v(x)) + \beta v(f(x, v(x))). \quad (11)$$

Also the extremum condition is

$$F_u(x, v(x)) + \beta f'_u(x, v(x)) \cdot v_x(f(x, v(x))) = 0. \quad (12)$$

Differentiating (11) w.r.t. x , we obtain

$$\begin{aligned} v_x(x) &= F_x(x, v(x)) + v'_x(x) \cdot F_u(x, v(x)) + \\ &\quad \beta [f'_x(x, v(x)) + v'_x(x) \cdot f'_u(x, v(x))] \cdot v_x(f(x, v(x))) \\ &= F_x(x, v(x)) + \beta f'_x(x, v(x)) \cdot v_x(f(x, v(x))) + \\ &\quad \underbrace{v'_x(x) \cdot F_u(x, v(x)) + \beta v'_x(x) \cdot f'_u(x, v(x)) \cdot v_x(f(x, v(x)))}_{=0 \text{ in view of (12)}}. \end{aligned}$$

Necessary conditions for optimality

We thus get

$$v_x(x) = F_x(x, v(x)) + \beta f'_x(x, v(x)) \cdot v_x(f(x, v(x))). \quad (13)$$

For $x = x_t^*$ and $u_t^* = v(x_t^*)$, equations (12) and (13) take the form

$$F_u(x_t^*, u_t^*) + \beta f'_u(x_t^*, u_t^*) \cdot v_x(x_{t+1}^*) = 0, \quad (14)$$

$$v_x(x_t^*) = F_x(x_t^*, u_t^*) + \beta f'_x(x_t^*, u_t^*) \cdot v_x(x_{t+1}^*). \quad (15)$$

Set $\lambda_t := \beta v_x(x_{t+1}^*)$ in (14) and (15), to obtain precisely (8) and (9).

Example: NCs for a problem with explicit controls

Compute the NCs for the problem:

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t, \quad k_0 > 0 - \text{given}$$

$$y_t = Ak_t^\alpha, \quad 0 < \underline{\epsilon} \leq c_t \leq \bar{\epsilon}, \quad \alpha, \beta \in (0, 1), \quad A > 0$$

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The Lagrangian for the problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [\ln c_t + \lambda_t ((1 - \delta)k_t + Ak_t^\alpha - c_t - k_{t+1})].$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \frac{1}{c_t} - \beta^t \lambda_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{c_t}.$$

Example: NCs for a problem with explicit controls

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \lambda_t (1 - \delta + \alpha A k_t^{\alpha-1}) - \beta^{t-1} \lambda_{t-1} = 0 \quad \Rightarrow$$

$$\lambda_{t-1} = \beta \lambda_t (1 - \delta + \alpha A k_t^{\alpha-1}).$$

Substituting the $1/c_t$ for λ_t , we obtain

$$c_t = \beta c_{t-1} (1 - \delta + \alpha A k_t^{\alpha-1}).$$

This is a version of a *consumption Euler equation* showing how consumption changes between two periods.

Example: NCs for a problem with explicit controls

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Homework: Use the state equation for the model to eliminate c_t from the above consumption Euler equation and obtain a difference equation involving only k . How does this result differ from the result for the same example in variational form?

Sufficient conditions

Fact 1

Let $\{\lambda_t\}$ and $\{x_{t+1}^*, u_t^*\}$, $t = 0, 1, 2, \dots$, be obtained by using (4) and (7). If

- ① The functions $F(x, u)$ and $f(x, u)$ are concave in (x, u) ,
- ② The Lagrange multipliers $\lambda_t^1, \dots, \lambda_t^n$, $t = 0, 1, 2, \dots$ are nonnegative,
- ③ The state space X is a subset of \mathbb{R}_+^n and the following transversality condition is valid

$$\lim_{T \rightarrow \infty} \beta^T \lambda'_T \cdot x_{T+1}^* = 0,$$

then the sequence $\{x_{t+1}^*, u_t^*\}$ (for a given x_0) is optimal for problem (4)-(6).

Sufficient conditions

We shall verify the validity of Fact 1.

Recall that (7) in matrix terms is given by (8) and (9).
Consider

$$\mathcal{L}_T(x_t, u_t) = \sum_{t=0}^T \beta^t \{F(x_t, u_t) + \lambda'_t \cdot [f(x_t, u_t) - x_{t+1}]\}.$$

We have

$$\begin{aligned} D := \mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) &= \sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \\ &\sum_{t=0}^T \beta^t [F(x_t, u_t) + \lambda'_t \cdot f(x_t, u_t) - F(x_t^*, u_t^*) - \lambda'_t \cdot f(x_t^*, u_t^*)]. \end{aligned} \tag{16}$$

Sufficient conditions

Then, in view of concavity, we get

$$\begin{aligned}
 \mathcal{L}_T(x_t, u_t) - \mathcal{L}_T(x_t^*, u_t^*) \quad (= D) &\leq \sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \\
 &\sum_{t=0}^T \beta^t [f_x^{0'}(x_t^*, u_t^*) \cdot (x_t - x_t^*) + f_u^{0'}(x_t^*, u_t^*) \cdot (u_t - u_t^*) + \\
 &\lambda'_t \cdot [f_x(x_t^*, u_t^*) \cdot (x_t - x_t^*) + f_u(x_t^*, u_t^*) \cdot (u_t - u_t^*)]] = \\
 &\sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^T \beta^t \underbrace{[f_x^{0'}(x_t^*, u_t^*) + \lambda'_t \cdot f_x(x_t^*, u_t^*)]}_{=\frac{\lambda'_{t-1}}{\beta} \text{ in view of (8)}} \cdot (x_t - x_t^*) \\
 &+ \sum_{t=0}^T \beta^t \underbrace{[f_u^{0'}(x_t^*, u_t^*) + \lambda'_t \cdot f_u(x_t^*, u_t^*)]}_{=0' \text{ in view of (9)}} \cdot (u_t - u_t^*).
 \end{aligned}$$

Sufficient conditions

i.e.

$$\begin{aligned}
 D &\leq \sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^T \beta^t \frac{\lambda'_{t-1}}{\beta} \cdot \underbrace{(x_t - x_t^*)}_{\text{N.B.: } x_0 = x_0^*} = \\
 &\sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=1}^T \beta^{t-1} \frac{\lambda'_{t-1}}{\beta} \cdot (x_t - x_t^*) = \\
 &\sum_{t=0}^T \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda'_t}{\beta} \cdot (x_{t+1} - x_{t+1}^*) = \\
 &\underbrace{\sum_{t=0}^{T-1} \beta^t \lambda'_t \cdot (x_{t+1}^* - x_{t+1}) + \sum_{t=0}^{T-1} \beta^t \frac{\lambda'_t}{\beta} \cdot (x_{t+1} - x_{t+1}^*)}_{=0} + \\
 &\beta^T \lambda'_T \cdot (x_{T+1}^* - x_{T+1}) \underbrace{\leq}_{\text{since } \lambda_t, x_t \geq 0} \beta^T \lambda'_T \cdot x_{T+1}^*.
 \end{aligned}$$

Sufficient conditions

In view of the transversality condition, we have:

$$D \leq \beta^T \lambda'_T \cdot x_{T+1}^* \xrightarrow{T \rightarrow \infty} 0,$$

i.e.

$$\mathcal{L}_T(x_t^*, u_t^*) - \mathcal{L}_T(x_t, u_t) \geq 0,$$

which proves the optimality of the sequence $\{x_{t+1}^*, u_t^*\}$.

Homework

Derive the NCs for the problem

$$\sum_{t=0}^{\infty} \beta^t \ln(x_t - u_t)$$

s.t.

$$x_{t+1} = u_t, \quad x_0 > 0, \quad u_t > 0.$$

Express the result as a system of difference equations for x and λ .

Readings

Additional readings:

Stokey, Lucas and Prescott. 1989. *Recursive methods in economic dynamics*. Chapters 2 and 4.

Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 12.