R401: Statistical and Mathematical Foundations

Lecture 18: Deterministic Optimal Control in Continuous Time: The Infinite Horizon Case

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Lecture Contents

Introduction

The rationale behind infinite horizons

- We now start studying an important class of optimal control problems for which there is no finite terminal time T. Thus, the objective functional will look like

$$\int_0^\infty F(x(t), u(t), t)\, dt$$

or a version thereof.

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- **Why do we need the infinite planning horizon?**
- After all, people are mortal and we'll stop planning one day...

The rationale behind infinite horizons

- There are two (related) economic reasons why an infinite-horizon formulation might be appropriate:
  1. Entities such as households and firms may exist indefinitely despite turnover in their composition (i.e. family members dying or moving, employees changing jobs etc.).
  2. Often there is uncertainty about the end of the planning horizon. This can be conveniently modelled as an infinite horizon, especially when it is reasonable to assume that the true, finite horizon is sufficiently distant.
- A technical complication with finite planning horizons arises when state variables represent economically valuable resources (wealth, capital). In these common cases we need to either:
  - exhaust the respective resource fully as required by optimality if there is no scrap value, which is often implausible, or
  - specify an appropriate scrap value term in the objective function, which may be difficult.

The rationale behind infinite horizons

- Apart from matters of interpretation, an infinite-horizon formulation eliminates some mathematical difficulties (one generally obtains simpler and cleaner expressions).
- However, this is not costless, as certain other complications arise.
- More specifically, we need to modify appropriately our definition of optimality to capture situations that arise in the case of an infinite horizon.

The basic problem

The problem we shall be studying is the following:

$$
\begin{aligned}
&\max_{u(t)\in\Omega(t)} \int_0^\infty F(x(t),u(t),t)\,dt \\
&\text{s.t.} \\
&\dot{x}(t) = f(x(t),u(t),t), \quad x(0) = x_0.
\end{aligned}
\tag{1}
$$

Sometimes problem (1) is replaced by a simplified version that reflects the structure of typical economic problems:

$$
\begin{aligned}
&\max_{u(t)\in\Omega(t)} \int_0^\infty e^{-\rho t}\phi(x(t),u(t))\,dt, \quad \rho > 0, \\
&\text{s.t.} \\
&\dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0.
\end{aligned}
\tag{2}
$$

Specificities of the infinite-horizon setting

- One obvious requirement in order to have a well-defined problem is for the improper integral forming the objective functional to converge for every admissible state-control pair.
- This leads to a relatively simple case, obtainable for instance for problem (2) when
$$|\phi(x,u)| \leq M, \ \forall (x,u).$$
- Unfortunately many problems do not admit such a clean characterization, necessitating a generalization of optimality criteria.

Specificities of the infinite-horizon setting

- It is possible to have situations in which the objective functional does not converge, yet for some state-control pair its value is greater than the value for every alternative state-control pair for every finite time horizon.
- In such cases it is natural to think of the "dominant" state-control pair as optimal, even though the verification is more complex than comparing two numbers.
- Other possibilities for defining optimality exist as well.

Optimality criteria in infinite-horizon problems

Among popular optimality criteria when the integral in (1) or (2) does not converge, the most permissive definition is the following:

Definition 1 (Piecewise (PW) optimality)

An admissible pair $(x^*(t), u^*(t))$ is *piecewise optimal* if for every $T \geq 0$, the restriction of $(x^*(t), u^*(t))$ to $[0, T]$ is optimal for the corresponding fixed horizon problem with terminal condition $x(T) = x^*(T)$ and $\int_0^T F(x(t), u(t), t)\, dt$ as the objective functional.

Optimality criteria in infinite-horizon problems

In contrast, one of the stricter definitions is that of overtaking optimality. We need to define the quantity

$$D(t) := \int_0^t F(x^*(\tau), u^*(\tau), \tau)\, d\tau - \int_0^t F(x(\tau), u(\tau), \tau)\, d\tau.$$

Definition 2 (Overtaking (OT) optimality)

An admissible pair $(x^*(t), u^*(t))$ is *overtaking optimal* if there exists a number t' such that $D(t) \geq 0$, $\forall t \geq t'$.

It can be shown that overtaking optimality implies piecewise optimality.

There are definitions of optimality that take an intermediate position between PW optimality and OT optimality in terms of strictness.

Pitfalls of applying the transversality condition in infinite-horizon problems

- In the finite-horizon setting the transversality conditions impose constraints on the terminal values of the adjoint variables $\lambda(T)$.
- In the absence of a salvage value term they take the form $\lambda(T) = 0$.
- One is tempted to conjecture that the natural extension of this requirement to the infinite-horizon setting would be

$$\lim_{T \to \infty} \lambda(T) = 0.$$

- Unfortunately there are counterexamples showing that in general this is not the case and additional modifications are required to obtain NCs or SCs for optimality.

Optimality conditions in the infinite-horizon setting

Necessary conditions for PW optimality

The following result by H. Halkin provides NCs at the expense of a slight complication of the formulation of the Hamiltonian.

Fact 1 (Halkin)

Let $(x^*(t), u^*(t))$ be a PW-optimal state-control pair for the problem (1) and define the Hamiltonian as

$$H = \lambda_0 F(x,u,t) + \sum_{i=1}^{n} \lambda_i f_i(x,u,t). \tag{3}$$

Then there exist a constant λ_0 and (piecewise) continuously differentiable functions $\lambda_i(t),\ i = 1,\ldots,n$, such that for all t

1. $(\lambda_0, \lambda(t)) \neq (0,\ldots,0)$,
2. $H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u(t), \lambda(t), t),\ \forall u(t) \in \Omega(t)$,
3. $\dot{\lambda}(t) = -H'_x(x^*(t), u^*(t), \lambda(t), t)$.

Moreover, either $\lambda_0 = 0$ or $\lambda_0 = 1$.

A comment on necessary conditions

- Halkin's necessary conditions encompass a broad range of candidates and may create difficulties in filtering potential solutions.
- There exist other versions of necessary conditions which one may try to apply (see SS, Sec. 3.9).
- However, they tend to be rather cumbersome to apply.
- As a result, using appropriate sufficient conditions for a well-structured problem may be your best bet.

Sufficient conditions for OT optimality

Fact 1 is relatively easy to check but often provides too many candidates. Using sufficiency results is a partial solution to this.

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Fact 2 (Mangasarian-type sufficiency)

Consider the problem (1), assuming that the set Ω is convex and F, f are continuously differentiable with respect to u. Suppose (x^*, u^*) is an admissible pair and there exists a continuous function $\lambda(t)$ such that, using the Hamiltonian (3), for $\lambda_0 = 1$ and $t \geq 0$ we have

1. $u^*(t)$ maximizes $H(x^*(t), u(t), \lambda(t), t)$ for $u(t) \in \Omega$,
2. $\dot{\lambda}(t) = -H'_x(x^*(t), u^*(t), \lambda(t), t)$,
3. $H(x, u, \lambda(t), t)$ is concave in (x, u), $\forall t \geq 0$.

Then the following transversality condition guarantees that (x^*, u^*) is overtaking optimal:
For all admissible $x(t)$ there exists a number t' such that

$$\lambda(t) \cdot (x(t) - x^*(t)) \geq 0, \quad t \geq t'.$$

Comments on the SCs for OT optimality

- Fact 2 provides sufficiency conditions for a fairly general problem and a strict definition of optimality.
- One can resort to a more permissive definition of optimality (e.g. the so-called *catching up optimality*, see SS, Ch. 3, or SHSS, Sec. 10.3) at the expense of additional mathematical complications.
- The price we pay in any case is that the transversality condition is rather cumbersome to check, as it requires us to compare a candidate trajectory x^* to all feasible alternatives x.
- This is realistic only in special cases.
- As a result, a typical approach in economic applications is to resort to conditions that are easier to check but are applicable to less general problems.

A sufficiency result in a less general setting

- Suppose we have a simplified version of problem (2) with only **one** state variable $x(t)$ and **one** control variable $u(t)$ taking values in a constant region Ω:

$$
\begin{aligned}
&\max_{u(t)} \int_0^\infty e^{-\rho t}\phi(x(t),u(t))\,dt, \quad \rho > 0,\\
&\text{s.t.}\\
&\dot{x}(t) = f(x(t),u(t)), \quad x(0) = x_0, \quad u(t) \in \Omega.
\end{aligned}
\tag{4}
$$

- Suppose also that the integral in (4) converges for all admissible (x,u).
- We therefore can employ the conventional definition of optimality.
- Also, because of the specific discounting structure, we can work in a current-value setting.

A sufficiency result in a less general setting

- Construct the current-value Hamiltonian

$$H^c(x, u, \lambda) = \lambda_0 \phi(x, u) + \lambda f(x, u).$$

Fact 3

Let an admissible pair $(x^*(t), u^*(t))$ for problem (4) satisfy the following conditions for some $\lambda(t)$, $\forall t \geq 0$, with $\lambda_0 = 1$:

1. $u^*(t)$ maximizes $H^c(x^*, u, \lambda)$ w.r.t. $u \in \Omega$,
2. $\dot{\lambda}(t) - \rho\lambda(t) = -\frac{\partial}{\partial x} H^c(x^*(t), u^*(t), \lambda(t))$,
3. $H^c(x, u, \lambda(t))$ is concave w.r.t. (x, u),
4. $\lim\limits_{t \to \infty} \lambda(t) e^{-\rho t}(x(t) - x^*(t)) \geq 0$ for all admissible $x(t)$.

Then the pair $(x^*(t), u^*(t))$ is optimal.

Example
State-space analysis

Readings

Main references:

Seierstad and Sydsæter [SS]. *Optimal control theory with economic applications*. Chapter 3.
Sydsæter et al. [SHSS] *Further mathematics for economic analysis*. Chapter 9.

Additional readings:
Sethi and Thompson [ST]. *Optimal control theory: applications to management science and economics*. Chapter 3.