Properties and Applications of the Cross Product

Remark. *For convenience, if* $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ *and* $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$,

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - b_2 a_3)\boldsymbol{i} - (a_1 b_3 - b_1 a_3)\boldsymbol{j} + (a_2 b_3 - b_2 a_3)\boldsymbol{k}$$

Theorem. *Let* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *be vectors and let* r, s *be scalars.*

a. $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

b. $(r\boldsymbol{a}) \times (s\boldsymbol{b}) = (rs)(\boldsymbol{a} \times \boldsymbol{b})$

c. $\boldsymbol{0} \times \boldsymbol{a} = \boldsymbol{0}$

d. $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

e. $(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{a}$

1. Let $\boldsymbol{a} = \langle 1, 1, -1 \rangle$ and $\boldsymbol{b} = \langle -2, -4, -6 \rangle$.

   (a) Compute $\boldsymbol{a} \times \boldsymbol{b}$.

   Solution:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ -2 & -4 & -6 \end{vmatrix} \\ &= (-6 - 4)\boldsymbol{i} - (-6 - 2)\boldsymbol{j} + (-4 - (-2))\boldsymbol{k} \\ &= -10\boldsymbol{i} + 8\boldsymbol{j} - 2\boldsymbol{k} \\ &= \langle -10, 8, -2 \rangle \end{aligned}$$

   Use your result and the properties above to compute the following:

   (b) $\boldsymbol{b} \times \boldsymbol{a}$

   Solution:

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b} = \langle 10, -8, 2 \rangle$$

   (c) $\langle 4, 4, -4 \rangle \times \langle 2, 4, 6 \rangle$ (hint: try to express each vector as a scalar times $\boldsymbol{a}$ or $\boldsymbol{b}$)

Solution:
$$\langle 4,4,-4\rangle \times \langle 2,4,6\rangle = (4\boldsymbol{a}) \times (-\boldsymbol{b}) = -4(\boldsymbol{a} \times \boldsymbol{b}) = \langle 40,-32,8\rangle$$

(d) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{b})$

Solution:
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{b}) = \boldsymbol{a} \times (2\boldsymbol{b}) = 2(\boldsymbol{a} \times \boldsymbol{b}) = \langle -20,16,-4\rangle$$

(e) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) + (\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a}$ where $\boldsymbol{c}$ is any vector.

Solution: Since $(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} = -\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})$, the answer is $\mathbf{0}$.

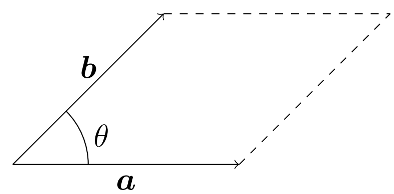
Remark. *Recall that a parallelogram with side lengths x and y has area*

$$A(x, y) = xy\sin\theta$$

If we represent the sides of a parallelogram as vectors, we can rewrite this as the following

Theorem. *The area of a parallelogram formed by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if given by the magnitude of their crossproduct, i.e.*

$$A = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta = |\boldsymbol{a}\times\boldsymbol{b}|$$



2. Find the area of the paralellogram with sides represented by

$$\boldsymbol{a} = \langle 2, -2, 0\rangle \quad \text{and} \quad \boldsymbol{b} = \langle 0, 3, 2\rangle$$

Solution: First we will find the crossproduct

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -2 & 0 \\ 0 & 3 & 2 \end{vmatrix} = \langle -4, -4, 6\rangle$$

Then the area is

$$|\,\langle -4, -4, 6\rangle\,| = \sqrt{16+16+36} = 2\sqrt{17}\text{ units}^2$$

3. Find the area of the triangle with vertices

$$P(0, 0, -3), \quad Q(4, 2, 0), \quad \text{and} \quad R(3, 3, 1).$$

Solution: We first find vectors PQ and PR.

$$PQ = \langle 4, 2, 0\rangle - \langle 0, 0, -3\rangle = \langle 4, 2, 3\rangle$$
$$PR = \langle 3, 3, 1\rangle - \langle 0, 0, -3\rangle = \langle 3, 3, 4\rangle$$

Then the area of the **paralellogram** with sides PQ and PR is

$$|PQ\times PR| = |\,\langle -1, -7, 6\rangle\,| = \sqrt{86}\text{ units}^2.$$

And the area of the triangle is

$$\frac{1}{2}(\text{area of the paralellogram}) = \frac{1}{2}\sqrt{86}.$$

Remark. *Recall that $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$*

4. Find two unit vectors orthogonal to $\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} + \boldsymbol{j}$.

Solution: We will first find the crossproduct

$$\langle 0, 1, -1\rangle \times \langle 1, 1, 0\rangle = \langle 1, -1, -1\rangle$$

This is a vector orthogonal to both of the given vectors. Then a unit vector is

$$\frac{1}{|\langle 1, -1, -1\rangle}\langle 1, -1, -1\rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle.$$

Multiplying this result by -1 gives a vector in the opposite direction with the same magnitude, so a second distinct unit vector is

$$\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$$

5. Let $\boldsymbol{a} = \langle -2, 10, 0\rangle$ and $\boldsymbol{b} = \langle 7, 7, 7\rangle$. Compute $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{i} + \boldsymbol{k})$.

Solution:

$$\langle -2, 10, 0\rangle \times \langle 7, 7, 7\rangle = \langle 70, 14, -84\rangle.$$

Then

$$(\langle -2, 10, 0\rangle \times \langle 7, 7, 7\rangle) \cdot \langle 1, 0, 1\rangle = 70 - 84 = -14.$$

6. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be vectors. Determine if the following are true or false. If false, give an example showing why the statement is false or explain your thinking.

   (a) $2(\boldsymbol{a} \times \boldsymbol{b}) = (2\boldsymbol{a}) \times (2\boldsymbol{b})$

   A. True

   B. False

   Solution: See property (b) of the Theorem on page 1.

   (b) $(2\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (2\boldsymbol{b})$

   A. True

   B. False

   (c) $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{b} \times \boldsymbol{a}|$

   A. True

   B. False

   (d) $|(-2\boldsymbol{a}) \times (\pi \boldsymbol{b})| = -2\pi|\boldsymbol{a} \times \boldsymbol{b}|$

A. True

B. False

Solution:

$$|(-2\boldsymbol{a}) \times (\pi \boldsymbol{b})| = |-2\pi(\boldsymbol{a} \times \boldsymbol{b})| = |-2\pi||\boldsymbol{a} \times \boldsymbol{b}| = 2\pi|\boldsymbol{a} \times \boldsymbol{b}|$$

7. What would you expect $\text{proj}_{\boldsymbol{a}}(\boldsymbol{a} \times \boldsymbol{b})$ to be? Draw a picture explaining your thinking.

Solution: Since $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$, the projection should be a single point, or the 0-vector. See the image below