

Properties and Applications of the Cross Product

Remark. For convenience, if $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= (a_2b_3 - b_2a_3)\mathbf{i} - (a_1b_3 - b_1a_3)\mathbf{j} + (a_2b_3 - b_2a_3)\mathbf{k}\end{aligned}$$

Theorem. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors and let r, s be scalars.

a. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

d. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

b. $(r\mathbf{a}) \times (s\mathbf{b}) = (rs)(\mathbf{a} \times \mathbf{b})$

c. $\mathbf{0} \times \mathbf{a} = \mathbf{0}$

e. $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$

1. Let $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle -2, -4, -6 \rangle$.

(a) Compute $\mathbf{a} \times \mathbf{b}$.

Solution:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -2 & -4 & -6 \end{vmatrix} \\ &= (-6 - 4)\mathbf{i} - (-6 - 2)\mathbf{j} + (-4 - (-2))\mathbf{k} \\ &= -10\mathbf{i} + 8\mathbf{j} - 2\mathbf{k} \\ &= \langle -10, 8, -2 \rangle\end{aligned}$$

Use your result and the properties above to compute the following:

(b) $\mathbf{b} \times \mathbf{a}$

Solution:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = \langle 10, -8, 2 \rangle$$

(c) $\langle 4, 4, -4 \rangle \times \langle 2, 4, 6 \rangle$ (hint: try to express each vector as a scalar times \mathbf{a} or \mathbf{b})

Solution:

$$\langle 4, 4, -4 \rangle \times \langle 2, 4, 6 \rangle = (4\mathbf{a}) \times (-\mathbf{b}) = -4(\mathbf{a} \times \mathbf{b}) = \langle 40, -32, 8 \rangle$$

(d) $\mathbf{a} \times (\mathbf{b} + \mathbf{b})$ **Solution:**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{b}) = \mathbf{a} \times (2\mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) = \langle -20, 16, -4 \rangle$$

(e) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + (\mathbf{b} + \mathbf{c}) \times \mathbf{a}$ where \mathbf{c} is any vector.**Solution:** Since $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = -\mathbf{a} \times (\mathbf{b} + \mathbf{c})$, the answer is $\mathbf{0}$.

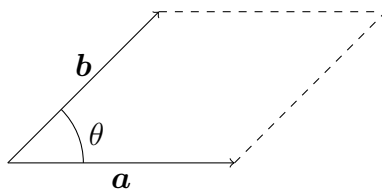
Remark. Recall that a parallelogram with side lengths x and y has area

$$A(x, y) = xy \sin \theta$$

If we represent the sides of a parallelogram as vectors, we can rewrite this as the following

Theorem. The area of a parallelogram formed by vectors \mathbf{a} and \mathbf{b} is given by the magnitude of their crossproduct, i.e.

$$A = |\mathbf{a}||\mathbf{b}| \sin \theta = |\mathbf{a} \times \mathbf{b}|$$



2. Find the area of the parallelogram with sides represented by

$$\mathbf{a} = \langle 2, -2, 0 \rangle \quad \text{and} \quad \mathbf{b} = \langle 0, 3, 2 \rangle$$

Solution: First we will find the crossproduct

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 0 \\ 0 & 3 & 2 \end{vmatrix} = \langle -4, -4, 6 \rangle$$

Then the area is

$$|\langle -4, -4, 6 \rangle| = \sqrt{16 + 16 + 36} = 2\sqrt{17} \text{ units}^2$$

3. Find the area of the triangle with vertices

$$P(0, 0, -3), \quad Q(4, 2, 0), \quad \text{and} \quad R(3, 3, 1).$$

Solution: We first find vectors PQ and PR .

$$PQ = \langle 4, 2, 0 \rangle - \langle 0, 0, -3 \rangle = \langle 4, 2, 3 \rangle$$

$$PR = \langle 3, 3, 1 \rangle - \langle 0, 0, -3 \rangle = \langle 3, 3, 4 \rangle$$

Then the area of the **parallelogram** with sides PQ and PR is

$$|PQ \times PR| = |\langle -1, -7, 6 \rangle| = \sqrt{86} \text{ units}^2.$$

And the area of the triangle is

$$\frac{1}{2}(\text{area of the parallelogram}) = \frac{1}{2}\sqrt{86}.$$

Remark. Recall that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b}

4. Find two unit vectors orthogonal to $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

Solution: We will first find the crossproduct

$$\langle 0, 1, -1 \rangle \times \langle 1, 1, 0 \rangle = \langle 1, -1, -1 \rangle$$

This is a vector orthogonal to both of the given vectors. Then a unit vector is

$$\frac{1}{|\langle 1, -1, -1 \rangle|} \langle 1, -1, -1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle.$$

Multiplying this result by -1 gives a vector in the opposite direction with the same magnitude, so a second distinct unit vector is

$$\left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

5. Let $\mathbf{a} = \langle -2, 10, 0 \rangle$ and $\mathbf{b} = \langle 7, 7, 7 \rangle$. Compute $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{i} + \mathbf{k})$.

Solution:

$$\langle -2, 10, 0 \rangle \times \langle 7, 7, 7 \rangle = \langle 70, 14, -84 \rangle.$$

Then

$$(\langle -2, 10, 0 \rangle \times \langle 7, 7, 7 \rangle) \cdot \langle 1, 0, 1 \rangle = 70 - 84 = -14.$$

6. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors. Determine if the following are true or false. If false, give an example showing why the statement is false or explain your thinking.

(a) $2(\mathbf{a} \times \mathbf{b}) = (2\mathbf{a}) \times (2\mathbf{b})$

A. True

B. False

Solution: See property (b) of the Theorem on page 1.

(b) $(2\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (2\mathbf{b})$

A. True

B. False

(c) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{a}|$

A. True

B. False

(d) $|(-2\mathbf{a}) \times (\pi\mathbf{b})| = -2\pi|\mathbf{a} \times \mathbf{b}|$

A. True

B. False

Solution:

$$|(-2\mathbf{a}) \times (\pi\mathbf{b})| = |-2\pi(\mathbf{a} \times \mathbf{b})| = |-2\pi||\mathbf{a} \times \mathbf{b}| = 2\pi|\mathbf{a} \times \mathbf{b}|$$

7. What would you expect $\text{proj}_{\mathbf{a}}(\mathbf{a} \times \mathbf{b})$ to be? Draw a picture explaining your thinking.

Solution: Since $\mathbf{a} \times \mathbf{b}$ is orthogonal to \mathbf{a} , the projection should be a single point, or the 0-vector. See the image below

