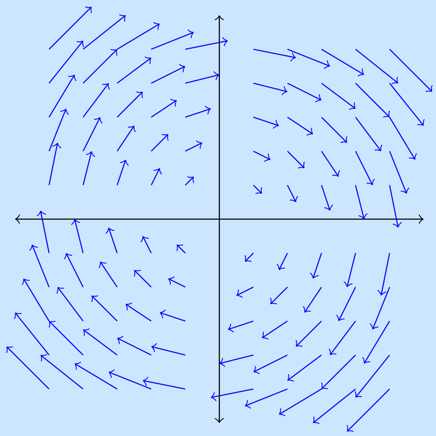
## Vector Fields

1. Sketch the vector field $\boldsymbol{F}(x,y) = \langle y, x \rangle$.



**Solution:**

2. Match the vector fields $\boldsymbol{F}$ with their plots

   (a) $\boldsymbol{F}(x,y) = \langle x, -y \rangle$

   **Solution:** A

   (b) $\boldsymbol{F}(x,y) = \langle -x, y \rangle$

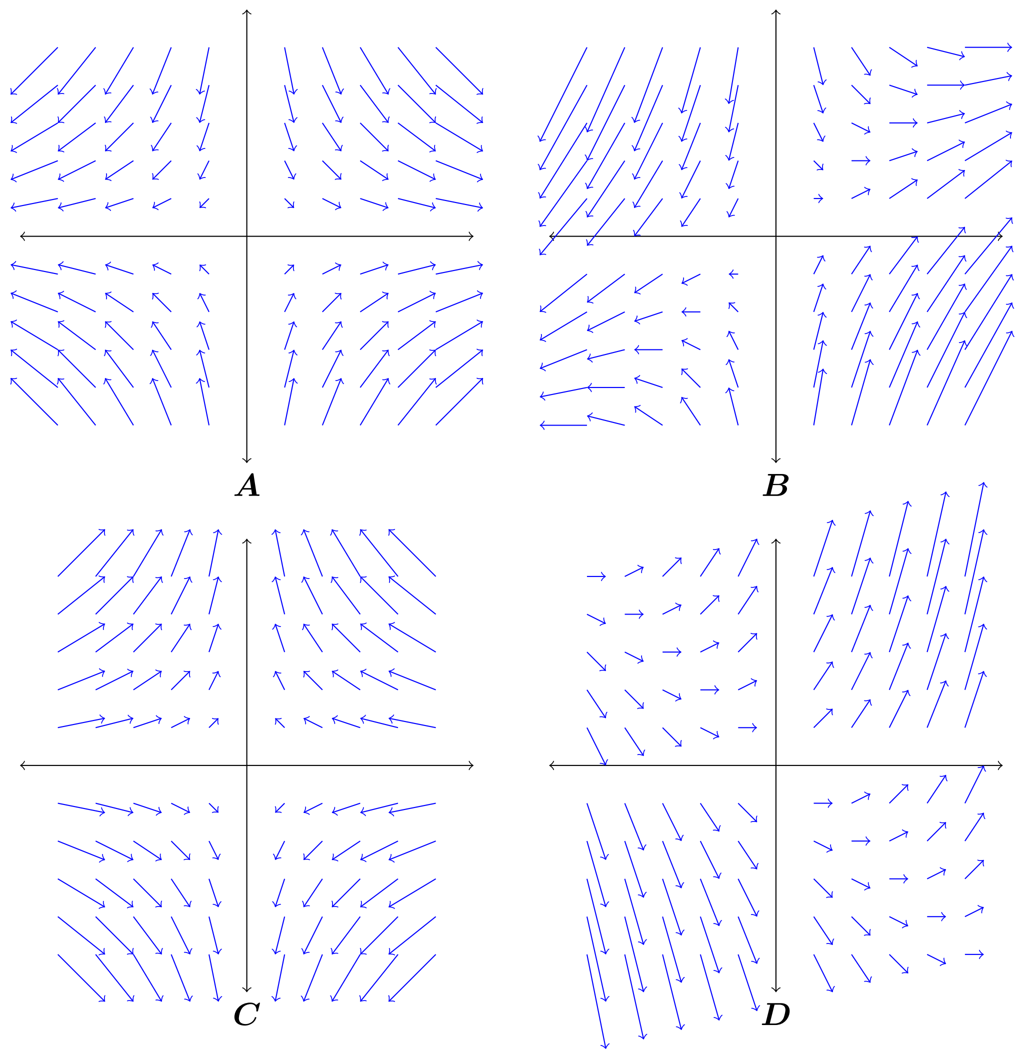
   **Solution:** C

   (c) $\boldsymbol{F}(x,y) = \langle x, x-y \rangle$

   **Solution:** B

   (d) $\boldsymbol{F}(x,y) = \langle 1, x+y \rangle$

   **Solution:** D



## Line Integrals

3. Evaluate $\int_C y^3\ ds$ where $C$ is the curve $x = t^3, y = t$ with $0 \le t \le 2$.

**Solution:**

$$\begin{aligned}\int_0^2 (y(t))^3 \sqrt{x'(t)^2 + y'(t)^2}dt &= \int_0^2 t^3 \sqrt{(3t^2)^2 + 1^2}dt \\ &= \int_0^2 t^3 \sqrt{9t^4+1}dt \\ &= \frac{1}{54}(9t^4+1)^{3/2}|_0^2 \qquad = \frac{1}{54}(145)^{3/2} - 1)\end{aligned}$$

4. Evaluate

$$\int_C (x+2y)\ dx + x^2\ dy,$$

where $C$ is the curve consisting of line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$.

**Solution:** First we compute the line integral along $C_1$ from $0,0$ to $(2,1)$ parameterized by $x = 2t, y = t$ with $0 \le t \le 1$.

$$\begin{aligned}\int_0^1 (2t+2t)2 + (2t)^2\ dt &= 4t^2 + \frac{4}{3}t^3|_0^1 \\ &= 4 + 4/3.\end{aligned}$$

Now along $C_2$ from $(2,1)$ to $(3,0)$ parameterized by $x(t) = 2+t$, $y(t) = 1-t$ with $0 \le t \le 1$.

$$\begin{aligned}\int_0^1 (2+t+2(1-t)) + (2+t)^2(-1)\ dt &= \int_0^1 4 - t - (t^2+4t+4)dt \\ &= \int_0^1 -t^2 - 5tdt \\ &= -\frac{1}{3}t^3 - \frac{5}{2}t^2|_0^1 \\ &= -\frac{1}{3} - \frac{5}{2}\end{aligned}$$

So in total,

$$\int_C (x+2y)\ dx + x^2\ dy = 4 + 4/3 - 1/3 - 5/2 = 5/2.$$

5. Find the work done by the force field $F(x,y,z) = \langle x - y^2, y - z^2, z - x^2\rangle$ on a particle that moves along the line segment from $(0,0,1)$ to $(2,1,0)$.

**Solution:** If $\boldsymbol{r}(t) = \langle 2t, t, 1-t \rangle$, $0 \leq t \leq 1$, then

$$\begin{aligned}\int_0^1 F(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt &= \int_0^1 \left\langle 2t - t^2, t - (1-t)^2, 1 - t - (2t)^2 \right\rangle \cdot \langle 2, 1, -1 \rangle \, dt \\ &= \int_0^1 t^2 + 8t - 2dt \\ &= \frac{1}{3}t^3 + 4t^2 - 2t|_0^1 \\ &= \frac{7}{3}.\end{aligned}$$

6. Find the work done by the force field $F(x,y) = x\boldsymbol{i} + (y+2)\boldsymbol{j}$ in moving an object along an arch of the cyloid $r(t) = (t - \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j}$, $0 \leq t \leq 2\pi$.

**Solution:**

$$\begin{aligned}\int_0^{2\pi} \langle t - \sin t, 1 - \cos t + 2 \rangle \cdot \langle 1 - \cos t, \sin t \rangle \ dt &= \int_0^{2\pi} (t - \sin t)(1 - \cos t) + (1 - \cos t + 2)(\sin t) \ dt \\ &= \int_0^{2\pi} t - t\cos(t) + 2\sin(t) \ dt \\ &= \frac{1}{2}t^2 - (\cos(t) + t\sin(t)) - 2\cos(t)|_0^{2\pi} \\ &= \frac{1}{2}(2\pi)^2 - (1+0) - 2(1) - (0 - (1+0) - 2) \\ &= 2\pi^2\end{aligned}$$