15.10

Useful Information.

- The **Jacobian** for a tranformation T given by $x = x(u, v)$ and $y = y(u, v)$ is

$$\frac{(x,y)}{(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

-
$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{(u,v)}{(u,v)} \right| \, du\, dv$$

1. Show that the Jacobian of the transformation $x = 5u - v, y = u + 3v$ is 16.

Solution:

$$\begin{aligned} \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} &= (5)\,(3) - (-1)\,(1) \\ &= 16. \end{aligned}$$

2. Show that the Jacobian of the transformation $x = e^{-r} \sin\theta$, $y = e^{r} \cos\theta$ is $\sin^2\theta - \cos^2\theta$.

Solution:

$$\begin{aligned} \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \cdot \frac{\partial y}{\partial r} &= \left(-e^{-r} \sin\theta\right) \left(-e^{r} \sin\theta\right) - \left(e^{-r} \cos\theta\right) \left(e^{r} \cos\theta\right) \\ &= \sin^2\theta - \cos^2\theta. \end{aligned}$$

3. Evaluate $\iint_R (x-3y)dA$ where R is the triangular region with vertices $(0,0)$,$(2,1)$, and $(1,2)$ using the transformation $x = 2u+v$, $y = u+2v$. Make sure you evaluate it in such a way that the answer you get is -3.

Solution: If T is the transformation $x = 2u+v$ and $y = u+2v$ we will first find the image of the triangle R. We can find the inverse transformation by combining the above equations,

$$x - 2y = 2u + v - 2(u+2v) = -3v \quad \Rightarrow \quad v = \frac{1}{3}(2y - x)$$

and

$$2x - y = 2(2u+v) - (u+2v) = 3u \quad \Rightarrow \quad u = \frac{1}{3}(2x - y)$$

R is bounded by the lines

$$\begin{aligned} L1: &\quad y = \frac{1}{2}x &&\quad \text{with } 0 \le x \le 2 \\ L2: &\quad y = 2x &&\quad \text{with } 0 \le x \le 1 \\ L3: &\quad y = 3 - x &&\quad \text{with } 1 \le x \le 2 \end{aligned}$$

Along $L1$,

$$y = \frac{1}{2}x \quad \Rightarrow \quad u + 2v = u + \frac{1}{2}v \quad \Rightarrow \quad v = 0$$

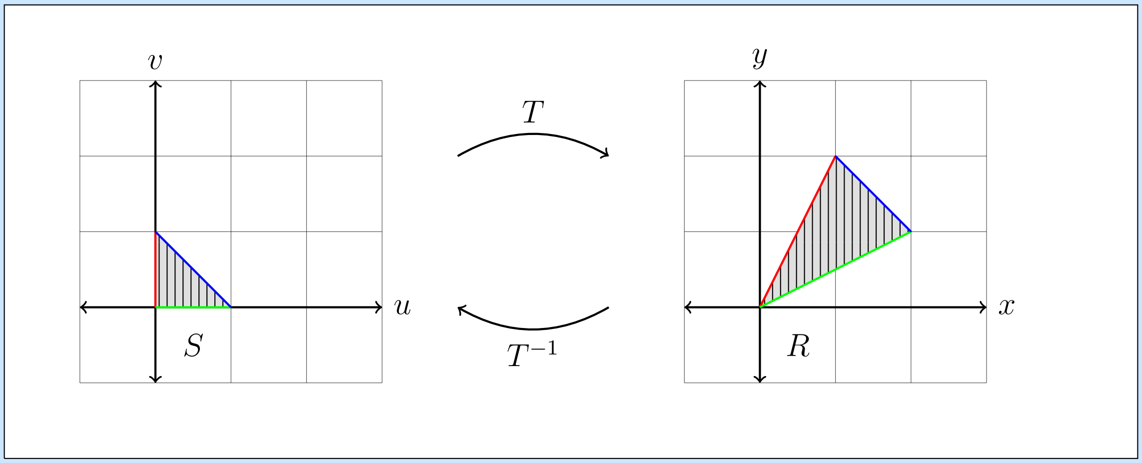
$L2$,

$$y = 2x \quad \Rightarrow \quad u + 2v = 2(2u+v) \quad \Rightarrow \quad u = 0$$

$L3$,

$$y = 3 - x \quad \Rightarrow \quad u + 2v = 3 - (2u+v) \quad \Rightarrow \quad v = 1 - u$$

As seen in the image below, for $0 \le u \le 1$, $0 \le v \le 1 - u$.



From

$$\begin{aligned}\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} &= (2)\,(2)-(1)(1)\\ &= 3\end{aligned}$$

we compute

$$\begin{aligned}\iint_R (x-3y)dA &= \iint_S (2u+v-3(u+2v))(3)dA\\ &= 3\int_0^1\int_0^{1-u} -u-5v\ dv\ du\\ &= 3\int_0^1 -uv-\frac{5}{2}v^2|_{v=0}^{v=1-u}\ du\\ &= 3\int_0^1 u-u^2-\frac{5}{2}(1-u)^2\ du\\ &= 3\frac{1}{2}u^2-\frac{1}{3}u^3+\frac{5}{6}(1-u)^3|_{u=0}^{u=1}\\ &= 3\frac{1}{2}-\frac{1}{3}+0-\left(0-0+\frac{5}{6}\right)\\ &= 3(-1)=-3.\end{aligned}$$

4. Use the transformation $x=2u$, $y=3v$ to evaluate $\iint_R x^2 dA$ where R is the region bounded by the ellipse $9x^2+4y^2=36$

Solution: Under this substitution,

$$9x^2+4y^2=36 \quad\Rightarrow\quad 9(2u)^2+4(3v)^2=36 \quad\Rightarrow\quad u^2+v^2=1.$$

The integral becomes

$$\iint_R x^2 dA =_S 4u^2\ dA$$

where S is the circle of radius 1 centered at $(0,0)$ in the uv-plane.

The rest of the integral can be done similiarly to above, but will be easier after we cover integration with polar coordinates.