1.2 Domains and Ranges of Multivariable Functions – During Class

Objective(s):

- Describe the domain of a two dimensional function.
- Find the range of a two dimensional function.

Recall from algebra that the *domain* of a function f is the allowable set of inputs. The *range* of a function is the set of possible outputs.

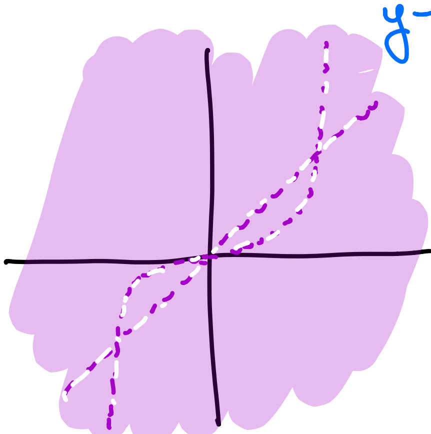
Remark 1.7. For a two variable function $z = f(x, y)$ the domain is a set of (x,y) points
These can be expressed in set notation or via a sketch in the xy-plane.

Remark 1.8. For a two variable function $z = f(x, y)$ the range is a set of z values.
These can be most easily be expressed using interval notation.

Example 1.9. Consider the function $f(x, y) = \dfrac{(x-1)(y+2)}{(y-x)(y-x^3)}$. Sketch the domain of the following functions and write it in set notation:

$y - x \neq 0$, $y - x^3 \neq 0$

$y \neq x$ $y \neq x^3$



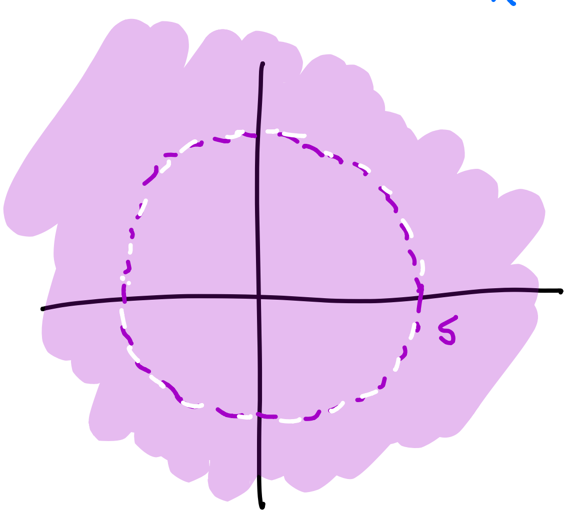
$D = \{(x,y) \in \mathbb{R}^2 \mid y \neq x \text{ and } y \neq x^3\}$

in reals such that

Example 1.10. Consider the function $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$. Sketch the domain of the following functions and write it in set notation:

$x^2 + y^2 - 25 \neq 0$

$x^2 + y^2 \neq 25$



$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 25\}$

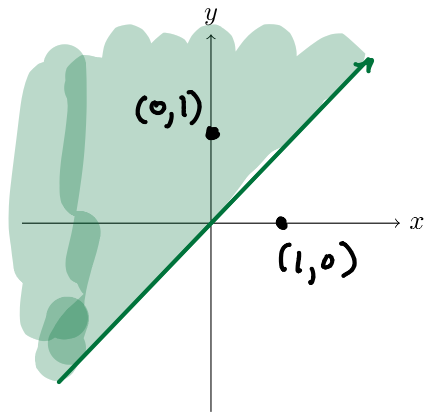
Example 1.11. Consider the function $f(x,y) = \sqrt{y-x}$. Find the following

(a) the domain of f. (written in set notation and sketched)

$y - x \geq 0$

$y \geq x$

$y = x$



$D = \{(x,y) \in \mathbb{R}^2 \mid y \geq x\}$

(b) the range of f.

$\sqrt{\star}$ $[0, \infty)$

$y - x$

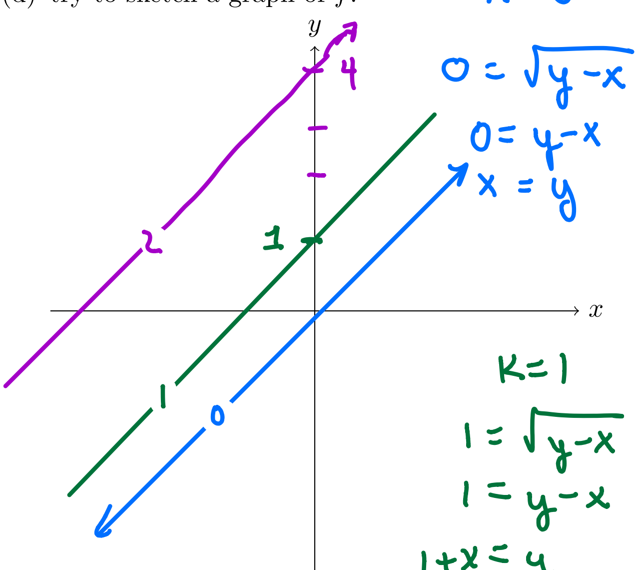
no additional restrictions

① look at outer most func. and determine its range.

② look inside this func. and see if ~~the~~ there are additional restrictions.

(c) sketch some level curves of f.

(d) try to sketch a graph of f.



$k = 0$

$0 = \sqrt{y-x}$

$0 = y - x$

$x = y$

$k = 1$

$1 = \sqrt{y-x}$

$1 = y - x$

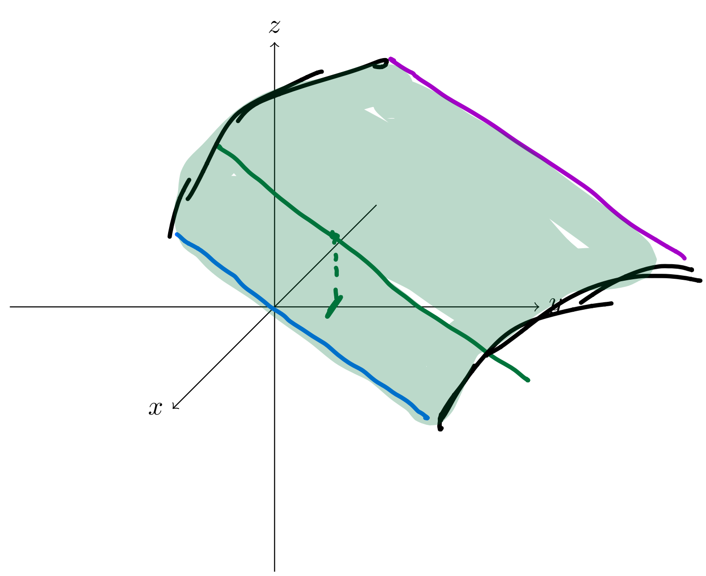
$1 + x = y$

$k = 2$

$2 = \sqrt{y-x}$

$4 = y - x$

$4 + x = y$



Example 1.12. Consider the function $f(x, y) = \ln(xy + x - y - 1)$. Sketch the domain of the following functions and write it in set notation:

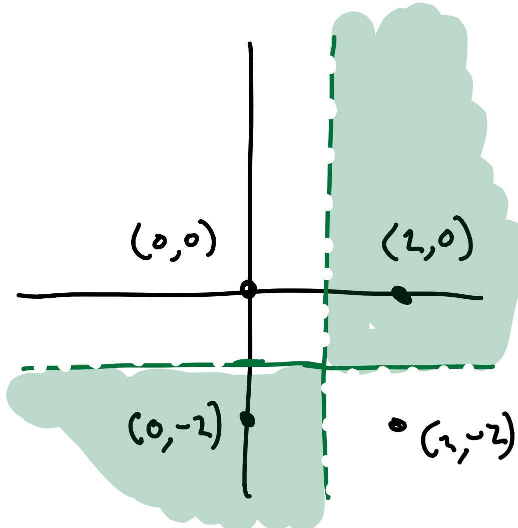
$xy + x - y - 1 > 0$

$xy + x - y - 1 = 0$

$x(y+1) - 1(y+1) = 0$

$(x-1)(y+1) = 0$

$x = 1 \quad y = -1$



$D = \{(x,y) \in \mathbb{R}^2 \mid (x > 1 \text{ and } y > -1) \text{ or } (x < 1 \text{ and } y < -1)\}$

Example 1.13. Find and sketch the domain of the function $f(x, y, z) = \ln(16 - x^2 - y^2 - 4z^2)$. What is the range of f? Describe the level surfaces.

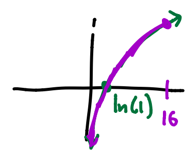
Alternate problem

$f(x,y) = \ln(16 - x^2 - y^2)$

Range?

① $\ln(\star)$

$(-\infty, \infty)$



② $16 - x^2 - y^2$

$(4,0) \rightarrow 16 - 16 - 0$

$(3.999$

$(-\infty, \ln(16)]$

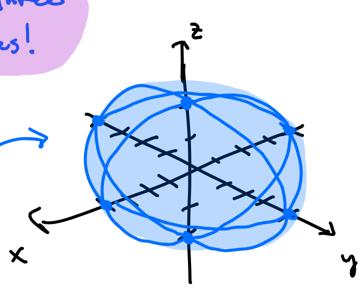
$16 - x^2 - y^2 - 4z^2 > 0$

$16 > x^2 + y^2 + 4z^2$

$1 > \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2$

New since three variables!

inside the ellipsoid



$D = \{(x,y,z) \in \mathbb{R}^3 \mid 16 > x^2 + y^2 + 4z^2\}$

Range is $(-\infty, \ln(16)]$

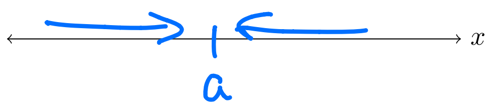
2.2 More Complicated Limits – During Class

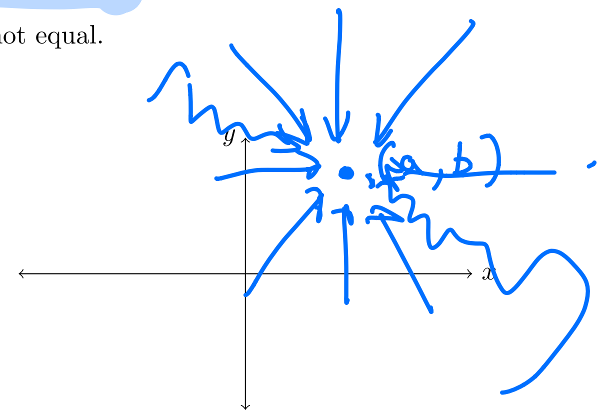
Objective(s):

- Understand and apply the two path test for showing limits to not exist.
- Use the Squeeze Theorem to show limits do exist.

Technique 2 for multivariable limits To prove a limit doesn't exist take limits from different paths and see that they are not equal.

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$



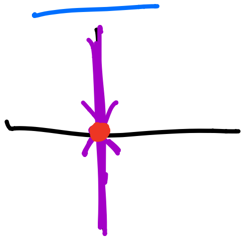


Example 2.3. Find the limit of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ if it exists, or show that the limit does not exist.

Tech 1 $\frac{0}{0+0}$ $= \frown$

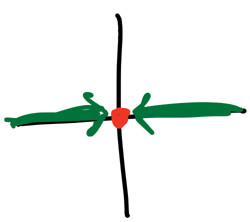
Tech 2

Path 1 $x=0,\ y\to 0$



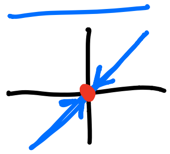
$$\lim_{\substack{x=0\\ y\to 0}} \frac{0(y)}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Path 2 $y=0,\ x\to 0$



$$\lim_{\substack{y=0\\ x\to 0}} \frac{x(0)}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Path 3 $y=x,\ x\to 0$



$$\lim_{\substack{y=x\\ x\to 0}} \frac{x(x)}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Since the limit gives different values along different paths we have

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} \quad \text{DNE.}$$

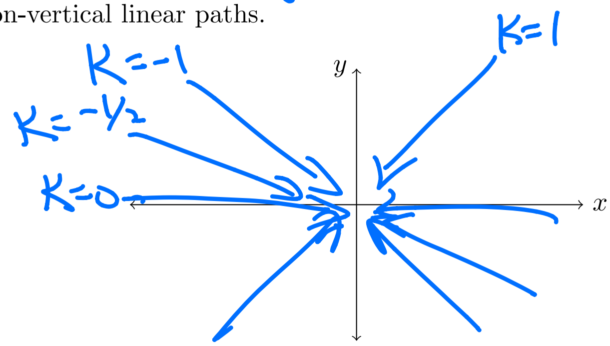
Note: Please include the words:

Since we have obtained different limits from different paths, the limit DNE.

Technique 2+ for multivariable limits To prove a limit doesn′t exist take limits from two different paths and see that they are not equal.

But why stop at two. If we consider the paths $y = Kx$ for arbitrary value K then we can effectively observe the limit from all non-vertical linear paths.

↑
slope



Example 2.4. Find the limit of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ if it exists, or show that the limit does not exist.

$$\rightarrow \lim_{\substack{y=Kx\\ x\to 0}} \frac{x(Kx)}{x^2+(Kx)^2} = \lim_{x\to 0} \frac{K\cancel{x^2}}{\cancel{x^2}+K^2\cancel{x^2}} = \lim_{x\to 0} \frac{K}{1+K^2}$$
$$= \frac{K}{1+K^2}$$

$K=0 \rightarrow L = \frac{0}{1} = 0$

$K=1 \rightarrow L = \frac{1}{2}$

since the limits depends on K the limit does not exist.

Warning: 2+ is great for $(x,y)\to(0,0)$

Note: Please include the words:

Since we have obtained different limits from different paths, the limit DNE.

Warning: Just because the limit agrees on all non-vertical linear paths does not mean that the limit exists. This technique only proves that the limit DNE!!!

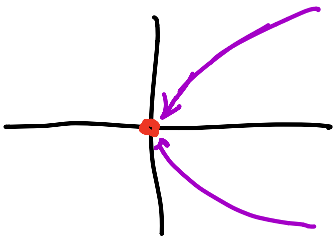
Example 2.5. Find the limit of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy^2}{x^2+y^4}$ if it exists, or show that the limit does not exist.

Path 1: $y=kx,\ x\to 0$

$$\lim_{\substack{y=kx\\ x\to 0}} \frac{x(kx)^2}{x^2+(kx)^4} = \lim_{x\to 0} \frac{k^2x^{\cancel{3}\,1}}{\cancel{x^2}+k^4x^{\cancel{4}\,2}} = \lim_{x\to 0}\frac{k^2x}{1+k^4x^2}$$

$$= \frac{0}{1+0} = \frac{0}{1} = 0$$

Path 2: $x=y^2,\ y\to 0$



$$\lim_{\substack{x=y^2\\ y\to 0}} \frac{(y^2)y^2}{(y^2)^2+y^4} = \lim_{y\to 0}\frac{y^4}{2y^4} = \frac{1}{2}$$

2 paths w/ diff lims

DNE

Technique for choosing paths: Try to have x's and y's with same exponents

Example 2.6. Find the limit of $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy^2}{x^2+y^2}$ if it exists, or show that the limit does not exist.

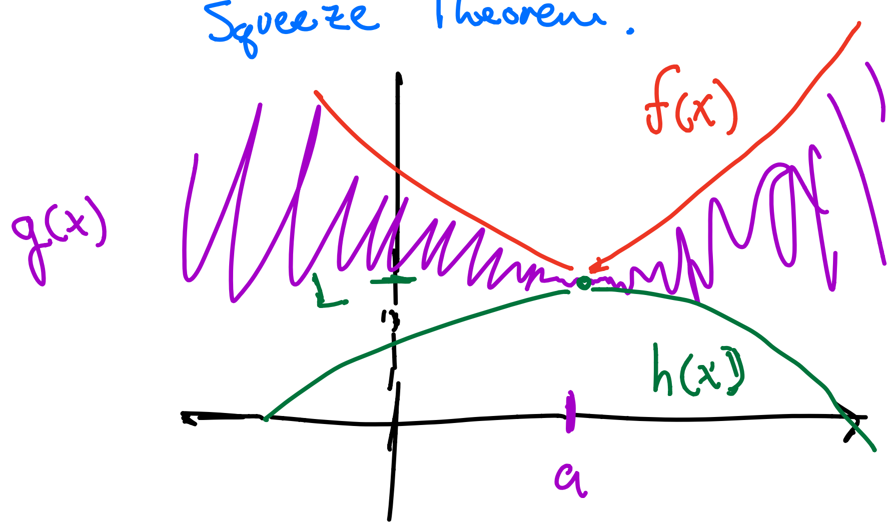
Tech 1 ☹

Tech 2 2path test. Tech 2+ ☹

All yield 0.

Tech 3 try to show exists

Squeeze Theorem.

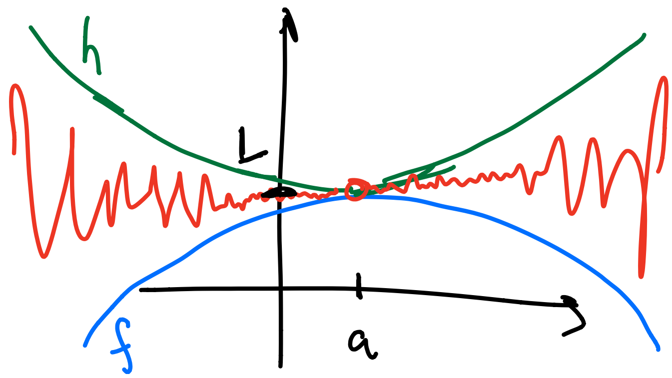


Technique 3 for multivariable limits If 1. and 2. don't work try Squeeze theorem to show a limit exists.

Note: a useful property to know that aid this technique is that:

$(\star)$ $f(x,y) \to 0$ if and only if $|f(x,y)| \to 0$

Try again! Find the limit of $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^2}$ if it exists, or show that the limit does not exist.



$$? \leq \frac{xy^2}{x^2+y^2} \leq ?$$

$$0 \leq \frac{|x|\,y^2}{x^2+y^2} \leq |x|\,1$$

as $x \to 0$, $y \to 0$: $\quad 0 \to 0, \quad \frac{|x|\,y^2}{x^2+y^2} \to 0, \quad |x|\,1 \to 0$

$$\frac{y^2}{x^2+y^2} \leq 1$$

$0 \leq x^2$

$y^2 \leq x^2+y^2$

$\frac{y^2}{x^2+y^2} \leq 1$

So by the squeeze $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^2} = 0$

Definition(s) 2.7.

A function f of two variables is called continuous at (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Note: There are also limits for functions with 3 variables. Treat them similarly.

3.2 Examples and Applications for Partial Derivatives – During Class

Objective(s):

- Get more practice calculating partial derivatives.
- Interpret what partial derivatives are telling us.
- Go over applications of partial derivatives.

Let's do a little warm up to get the juices flowing.

Example 3.5. Find all the first and second partial derivatives of $v = \frac{xy}{x-y}$

$$\frac{2}{x} = 2x^{-1}$$

$$V_x = \frac{y(x-y) - xy(1)}{(x-y)^2} = \frac{\cancel{xy} - y^2 - \cancel{xy}}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$x^2(x-y)^{-2} = V_y = \frac{x^2}{(x-y)^2} = \frac{x(x-y) - (xy)(-1)}{(x-y)^2}$$

$$V_{xx} = \frac{-y^2(-2)}{(x-y)^3} = \frac{2y^2}{(x-y)^3}$$

$$V_{yy} = x^2(-2)(x-y)^{-3}(-1)$$
$$= \frac{2x^2}{(x-y)^3}$$

$$V_{xy} = V_{yx} = \frac{-2y(x-y)^{\cancel{4}} - (-y^2)2(x-y)(-1)}{(x-y)^{\cancel{4}\,3}} = \frac{-2yx + \cancel{2y^2} - \cancel{2y^2}}{(x-y)^3} = \frac{-2yx}{(x-y)^3}$$

Example 3.6. Find f_{xyz} for the function $f = xyz + \underbrace{(x^2+y^2)\frac{\sin^{-1}(x\sqrt{y})}{\tan(x)}}$

$$f_{xyz} = f_{zxy}$$

$$f_z = xy + 0$$

$$f_{zx} = y$$

$$f_{zxy} = 1$$

Great so these are fun to calculate but what are they good for?

Interpretation:

1. Consider fixing $y = b$ and considering the partial derivative $f_x(x, b)$. This is equivalent to the slope of the curve of intersection between the function $f(x, y)$ and the plane $y = b$.
2. Another way to think about it is: If you took one step in the positive x direction from (a, b) then f would change approximately $f_x(a, b)$.
3. Likewise instead we could fix $x = a$ and consider the partial derivative $f_y(a, y)$. This is equivalent to the slope of the curve of intersection between the function $f(x, y)$ and the plane $x = a$.
4. Another way to think about it is: If you took one step in the positive y direction from (a, b) then f would change approximately $f_y(a, b)$.
5. **Monroe demo** >> Show a fx tangent line at point / Show a fy tangent line at point

Example 3.7. Suppose you are surrounded by bees given by the bee density function

$B(1,1) = 100 - 1^2 + 1^2 + 3(1) = 103$ bees/unit^2

$$B(x, y) = 100 - x^2 + y^2 + 3y \text{ bees/unit}^2$$

You are currently standing at (1,1). Which of the four directions would be best to run in $\{\mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}\}$?

$B_x = -2x$

$B_x(1,1) = -2$ (~101 bees if $\vec{i}$)
(~105 bees if $-\vec{i}$)

$B_y = 2y + 3$

$B_y(1,1) = 5$ (~108 bees if $\vec{j}$)
(~98 bees if $-\vec{j}$)

In addition, as always we can consider partial derivatives with more variables ($f(x, y, z)$) and we can consider higher order partial derivatives (3rd order, 4th order, etc.). However we usually stop at second order partial derivatives because of Clairaut′s Thoerem and because many famous Partial Differential Equations use second order partial derivatives

Definition(s) 3.8.

(a) The Laplace Equation is given by

$$f_{xx} + f_{yy} = 0$$

Solutions to this equation are called harmonic functions; they play a role in problems of heat conduction, fluid flow, and electric potential.

(b) The Wave equation is given by

$$f_{tt} = a^2 f_{xx}$$

for some a.

which describes the motion of a waveform (ocean, sound, light, string).

Example 3.9. Does the function $f(x, t) = \sin(x + 2t)$ satisfy the wave equation?

$$f_t = \cos(x+2t)\cdot(2)$$

$$f_{tt} = -\sin(x+2t)\cdot 2\cdot 2 = -4\sin(x+2t)$$

$$f_x = \cos(x+2t)\cdot 1$$

$$f_{xx} = -\sin(x+2t)$$

Yes!

$a = 2$

$a = -2$

Example 3.10. Verify that $f(x, y) = e^x \sin y$ is a harmonic function.

yes!

3.3 A Taste of Things to Come – During Class

Objective(s):

- Determine if there is a function with a particular set of partial derivatives.

While it is not obvious right now that this is a game worth playing, in Chapter 16 we will want to be able to find a function with a particular set of partial derivatives (if it exists). Here I would like to outline the technique for 2 variables

Example 3.11. Suppose that $f_x = 2x + 6y^2 + 1$, $f_y = 12xy + 2y$

(a) Find a function $f(x, y)$ such that $f_x = 2x + 6y^2 + 1$

$$\int 2x + 6y^2 + 1 \, dx$$

$$= x^2 + 6y^2 x + x + C(y)$$

(b) Find a function $f(x, y)$ such that $f_y = 12xy + 2y$

$$\int 12xy + 2y \, dy$$

$$= 6xy^2 + y^2 + C(x)$$

(c) Is there a function that satisfies both (a) and (b)? $6xy^2 + y^2 + x^2 + x + C$

Example 3.12. You are told that there is a function f whose partial derivatives are $f_x(x, y) = x + 4y$ and $f_y(x, y) = 3x - y$. Should you believe it?

No!

$f_{xy} = 4$

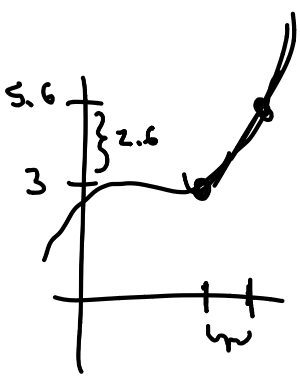
$f_{yx} = 3$

4.2 Practice, Alternatives, and Upgrades – During Class

Objective(s):

- Practice finding and using linearizations.
- Define and use differentials.
- Recognize how to do this all again for 3 variables.

Example 4.6. Find the linear approximation of $f(x,y) = \dfrac{2x+3}{4y+1}$ at $(0,0)$. Use it to approximate $f(0.1,-0.2)$.



$f(0,0) = \frac{3}{1} = 3$

$f(x_0,y_0)$

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z - 3 = f_x(0,0)(x-0) + f_y(0,0)(y-0)$

$z - 3 = 2x - 12y \Rightarrow z = 3 + 2x - 12y$

$\Rightarrow L(x,y) = 3 + 2x - 12y$

$f_x = \dfrac{2}{4y+1} \rightarrow f_x(0,0) = 2$

$f_y = (2x+3)(-1)(4y+1)^{-2}(4) \rightarrow f_y(0,0) = -3(4) = -12$

$f(x,y) \approx L(x,y)$ if (x,y) is close to (x_0,y_0)

$f(.1,-.2) \approx L(.1,-.2) = 3 + 2(.1) - 12(-.2) = 3.2 + 2.4 = 5.6$

Definition(s) 4.7. Sometimes its valuable to look at the change in a function so we define differentials.
Let dx and dy by independent variables (think a $dx =$"small change in x" and $dy =$"small change in y"). Then the **differential** dz, also called the total differential is given by:

$$dz = f_x(x,y)\cdot dx + f_y(x,y)\cdot dy$$

Note: the similarity to $z - z_0 = f_x(x,y)(x-x_0) + f_y(x,y)(y-y_0)$

Note2: again this is a good way to approximate the $\Delta z = z - z_0$ the change in z. That is $\Delta z \approx dz$

Example 4.8. Consider the function $z = x^2 + 3xy - y^2$.

(a) Find the differential dz

$f_x = 2x + 3y$
$f_y = 3x - 2y$

$dz = (2x+3y)\,dx + (3x-2y)\,dy$

$x = 2$ $\quad dx = .05$ $\qquad y = 3$ $\quad dy = -.04$

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

$z(2,3) = 4 + 18 - 9 = 13$

$z(2.05, 2.96) = 4.2025 + 18.204 - 8.7616 = 13.6449$

$\Delta z = 0.6449$

$dz = (4+9)(.05) + (6-6)(-.04)$

$= 13(.05) = .65$

This is a nice example because you see that $\Delta z \approx dz$ but dz is easier to compute!

x y z

Example 4.9. The dimensions of a box are measure to be 10cm, 5cm, and 8cm. If each measurement is correct to within 1/5 cm (.2cm) approximate the largest possible error when the volume of the box is calculated from these measurements.

$dx = dy = dz$

$V = 10 \cdot 5 \cdot 8 = 400 \text{ cm}^3$

$dV = (yz) \cdot dx + (xz) dy + (xy) dz$

$V = x \cdot y \cdot z$

$V_x = yz$

$V_y = xz$

$V_z = xy$

$= (40)\left(\frac{1}{5}\right) + 80\left(\frac{1}{5}\right) + (50)\left(\frac{1}{5}\right)$

$= 8 + 16 + 10 = 34 \text{ cm}^3$

Finally all these things can be done with more variables. In particular you will want to know

Definition(s) 4.10.

(a) The **linear approximation** of $f(x, y, z)$ at (a, b, c) and is given by:

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

(b) If $w = f(x, y, z)$ then the **differential** dw, is given by:

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + +f_z(x, y, z)dz$$

Example 4.11. Consider the surface $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1$

~~$(\frac{1}{3}, 2, 2)$~~

(a) Find the equation of the tangent plane to this surface at the point $\left(\frac{1}{3}, 2, 2\right)$.

$$z = f(x,y)$$
$$z^2 = 9\left(1 - x^2 - \frac{y^2}{9}\right)$$
$$z = 3\sqrt{1 - x^2 - y^2/9}$$

$$z - 2 = f_x \cdot \left(x - \tfrac{1}{3}\right) + f_y \cdot (y - 2)$$
$$z - 2 = -\tfrac{3}{2}\left(x - \tfrac{1}{3}\right) - 1\,(y - 2)$$

$$f_x = \frac{3}{2}\left(1 - x^2 - \frac{y^2}{9}\right)^{-1/2}(-2x) \qquad f_y = \frac{\cancel{3}}{\cancel{2}}\left(1 - x^2 - \frac{y^2}{9}\right)^{-1/2}\left(-\frac{\cancel{2}}{\cancel{9}\,3}y\right)$$

$$f_x\left(\tfrac{1}{3}, 2\right) = \frac{\cancel{3}}{\cancel{2}}\left(1 - \frac{1}{9} - \frac{4}{9}\right)^{-1/2}\left(-\frac{\cancel{2}}{\cancel{3}}\right) = -\left(\frac{4}{9}\right)^{-1/2} = -\frac{3}{2}$$

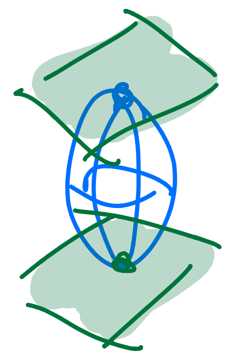
$$f_y\left(\tfrac{1}{3}, 2\right) = -\left(\frac{4}{9}\right)^{-1/2}\left(\frac{2}{3}\right) = -\frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} = -1$$

(b) Find a point at which the tangent plane to this surface is horizontal. Are there any other such points?

$f_x = 0$ AND $f_y = 0$

$$f_x = \frac{1}{2}(9 - 9x^2 - y^2)^{-1/2}(-18x)$$
$$0 = \frac{-9x}{\sqrt{9 - 9x^2 - y^2}}$$
$$0 = -9x$$
$$0 = x$$

$$f_y = \cancel{\tfrac{1}{2}}(9 - 9x^2 - y^2)^{-1/2}(-\cancel{2}y)$$
$$0 = y$$
$$(0, 0, 3),\ (0, 0, -3)$$



(c) Find a point at which the tangent plane to this surface is vertical. Are there any other such points?

f_x OR f_y is undefined

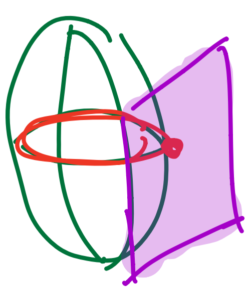
$(1, 0, 0)$

$(-1, 0, 0)$

$(0, 3, 0)$

$(0, -3, 0)$

infinitely many!



$$\frac{-9x}{\sqrt{9 - 9x^2 - y^2}} \to DNE \qquad \frac{-y}{\sqrt{9 - 9x^2 - y^2}} \to DNE$$

divide by 0

$$9 - 9x^2 - y^2 = 0 \qquad 9 - 9x^2 - y^2 = 0$$
$$9 = 9x^2 + y^2 \qquad 9 = 9x^2 + y^2$$
$$1 = x^2 + \frac{y^2}{9} + 0^2 \qquad 1 = x^2 + \frac{y^2}{9} + 0^2$$

5.2 Upgrades to the Multivariable Chain Rule – During Class

Objective(s):

- Define the chain rule in a variety of situations.
- Practice!

Theorem 5.5.

(a) Suppose that $w = f(x, y, z)$ is a differentiable function of x and y, where $x = x(t)$, $y = y(t)$, and $z = z(t)$ are all differentiable functions of t. Then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$
$$= f_x \cdot x_t + f_y \cdot y_t + f_z \cdot z_t$$

(b) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = x(s, t)$ and $y = y(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of s and t and

$$\frac{\partial z}{\partial t} = f_x \cdot x_t + f_y \cdot y_t$$

and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

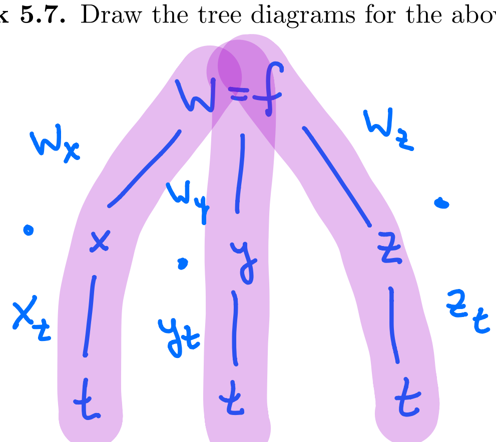
Definition(s) 5.6. In the above case (5.3.2)

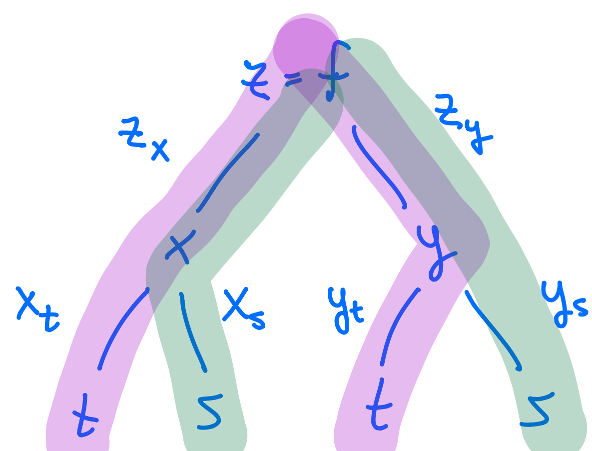
(a) z is a dependent variable.

(b) s and t are independent variables.

(c) x and y are intermediate variables.

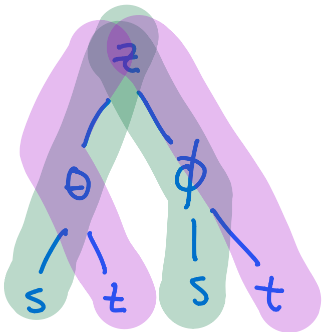
Remark 5.7. Draw the tree diagrams for the above situations.





Example 5.8. Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ where

$$z = \sin\theta\cos\phi \qquad \theta = st^2 \qquad \phi = s^2 t$$



$z_\theta = \cos\theta\cos\phi$

$z_\phi = \sin\theta(-\sin\phi)$

$\theta_s = t^2 \qquad \theta_t = 2st$

$\phi_s = 2st \qquad \phi_t = s^2$

$$\frac{\partial z}{\partial t} = \cos\theta\cos\phi\cdot 2st - \sin\theta\sin\phi\cdot s^2$$

$$= \cos(st^2)\cos(s^2t)\cdot 2st - \sin(st^2)\sin(s^2t)\cdot s^2$$

$$\frac{\partial z}{\partial s} = \cos(st^2)\cos(s^2t)\cdot t^2 - \sin(st^2)\sin(s^2t)\cdot 2st$$

Example 5.9. Suppose $z = f(x, y)$ where f is differentiable, and

$$x = x(t) \qquad y = y(t)$$
$$x(3) = 2 \qquad y(3) = 7$$
$$x'(3) = 5 \qquad y'(3) = -4$$
$$f_x(2,7) = 6 \qquad f_y(2,7) = -8$$

Find $\frac{dz}{dt}$ when $t = 3$.

$$\frac{dz}{dt} = f_x\cdot x_t + f_y\cdot y_t$$

$$= f_x(2,7)(5) + f_y(2,7)(-4)$$

$$= 6\cdot 5 + (-8)(-4)$$

$$= 30 + 32 = 62$$

5.3 Implicit Differentiation, A Better Way – During Class

Objective(s):

- Find a better way to do implicit differentiation.

Finally we learn the best way to do implicit differentiation (Calc I upgrade)

Example 5.10. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$ using the old calc 1 way.

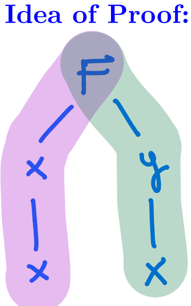
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6y + 6x \cdot \frac{dy}{dx}$$
$$3x^2 - 6y = 6x \cdot \frac{dy}{dx} - 3y^2 \frac{dy}{dx}$$
$$3x^2 - 6y = (6x - 3y^2) \cdot \frac{dy}{dx}$$
$$\frac{3x^2 - 6y}{6x - 3y^2} = \frac{dy}{dx}$$

Theorem 5.11. Suppose that an equation of the form: $F(x, y) = c$ defines y implicitly as a differentiable function of x where $c \in \mathbb{R}$ is any constant. Then we have

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

Provided that $F_y \neq 0$.

Idea of Proof:



$$F(x, y) = c$$
$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0 \qquad \text{(Use chain rule (since } y \text{ is not indepent from } x.\text{)}$$
$$F_y \frac{dy}{dx} = -F_x \qquad \text{(Subtraction and } \frac{dx}{dx} = 1\text{)}$$
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

This is just the idea of the proof because there is something much more subtle going on with the statement "$F(x, y) = c$ defines y implicitly as a differentiable function of x" which requires the **Implicit Function Theorem** which you can learn all about in MTH421.

Example 5.12. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$ using the new calc 3 way!

$$F(x,y) = c$$
$$x^3 + y^3 - 6xy = 0$$
$$F_x = 3x^2 - 6y$$
$$F_y = 3y^2 - 6x$$

$$\frac{dy}{dx} = \frac{-(3x^2 - 6y)}{3y^2 - 6x} = \frac{3x^2 - 6y}{6x - 3y^2}$$

Similarly we can upgrade this theorem to include more variables if we want:

Theorem 5.13. Suppose that an equation of the form: $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y where $c \in \mathbb{R}$ is any constant. Then we have

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Provided that $F_z \neq 0$.

Example 5.14. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $yz + x \ln y = z^2$.

$F = yz + x\ln y - z^2 + 5$

$F_x = \ln y$

$F_y = z + \frac{x}{y}$

$F_z = y - 2z$

$$\frac{\partial z}{\partial x} = \frac{-(\ln y)}{(y-2z)}$$

$$\frac{\partial z}{\partial y} = \frac{-\left(z + \frac{x}{y}\right)}{(y-2z)}$$

Example 5.15. Find $\frac{dy}{dx}$ where $e^y \sin x = x + xy$

$\underbrace{e^y \sin x - x - xy}_{F} = 0$

$F_x = e^y \cos x - 1 - y$

$F_y = e^y \sin x - x$

$$\frac{dy}{dx} = \frac{-(e^y \cos x - 1 - y)}{e^y \sin x - x}$$

6.2 More Things the Directional Derivative is Good For – During Class

Objective(s):

- Minimizing the directional derivative.
- No changes to the directional derivative.
- Practice more problems.

$= x(x^2+y^2)^{-1}$

Example 6.8. Find the directional derivative of the function $f(x,y) = \dfrac{x}{x^2+y^2}$, at $(1,2)$ in the direction of $\mathbf{v} = \langle 3,5 \rangle$.

$$f_x = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$f_y = -x(x^2+y^2)^{-2} \cdot (2y)$$

$$\nabla f(1,2) = \left\langle \frac{5-2}{25}, -1(5)^{-2}(4) \right\rangle$$

$$= \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle$$

$$D_v f = \left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle \cdot \langle 3,5 \rangle \frac{1}{\sqrt{9+25}} = \frac{9-20}{25} \cdot \frac{1}{\sqrt{34}} = \frac{-11}{25\sqrt{34}}$$

Example 6.9. Find the maximum rate of change of $f(x,y,z) = \sin(xy) + z$ at the point $(1,0,1)$ and the direction in which it occurs.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

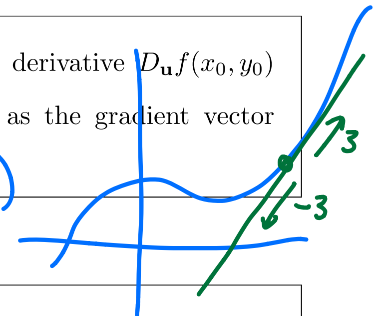
$$= \langle \cos(xy) \cdot y, \cos(xy) \cdot x, 1 \rangle$$

$$\nabla f(1,0,1) = \langle 1 \cdot 0, 1 \cdot 1, 1 \rangle$$

$$= \langle 0,1,1 \rangle$$

$$|\nabla f(1,0,1)| = \sqrt{0+1+1} = \sqrt{2} \leftarrow \text{max rate of change}$$

$$\langle 0,1,1 \rangle \frac{1}{\sqrt{2}} \leftarrow \text{direction of max rate of change.}$$



Theorem 6.10. Suppose f is differentiable function. The minimum value of the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ is $-|\nabla f(x_0,y_0)|$ and it occurs when $\mathbf{u}$ is in the opposite direction as the gradient vector $\nabla f(x_0, y_0)$.

$-\nabla f(x_0,y_0)$

Finally it also helps determine where the function does not change

Theorem 6.11. Suppose f is differentiable function. The function sees no change (think level curve) when the directional derivative is 0. That is when $\mathbf{u}$ is perpendicular to gradient vector $\nabla f(x_0, y_0)$.

$\nabla f(x_0,y_0)\cdot\vec{u} = 0$

Example 6.12. Suppose $f(x, y) = \dfrac{y^2}{x}$, $P(1, 2)$, and $\mathbf{u} = \langle 2, \sqrt{5}\rangle$. Find the following

(a) The gradient of f.

$$\nabla f = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle$$

(b) $\nabla f(P)$.

$$\nabla f(P) = \left\langle \frac{-4}{1}, \frac{4}{1} \right\rangle = \langle -4, 4\rangle$$

(c) Find the rate of change of f at P in the direction of $\mathbf{u}$.

$$\langle -4, 4\rangle \cdot \langle 2, \sqrt{5}\rangle \frac{1}{\sqrt{4+5}} = \left(-8+\sqrt{5}\cdot 4\right)\frac{1}{3}$$

(d) Find a direction $\mathbf{v}$ in which f neither increases nor decreases.

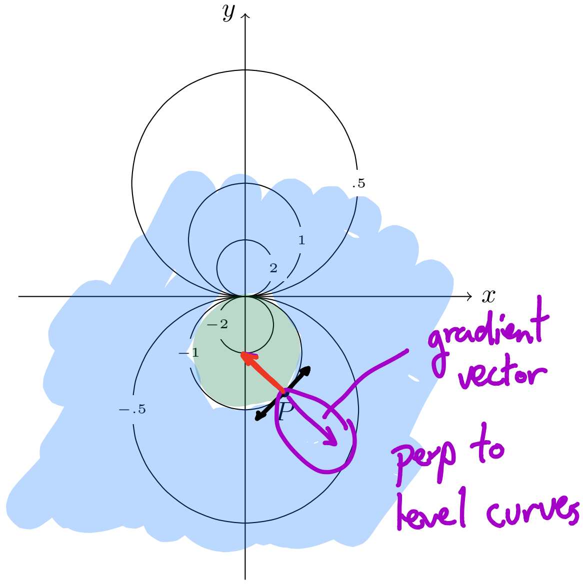
$$\langle -4, 4\rangle \cdot \langle 1, 1\rangle = 0$$

$$\langle 1, 1\rangle$$

Of course we can upgrade all this to three variables as well.

Example 6.13. Suppose f has the contour plot shown below.

Draw the direction in which the function increases the fastest at P



$f(x,y) = K$

$f(x,y,z) = K$

$x^2 + y^2 + z^2 = 1$

Definition(s) 6.14.

- $\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

Theorem 6.15.

- $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$
- The maximum value of the directional derivative $D_{\mathbf{u}} f(x_0, y_0, z_0)$ is $|\nabla f(x_0, y_0, z_0)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x_0, y_0, z_0)$.

Theorem 6.16. Suppose $f(x, y, z)$ is differentiable function. The function sees no change (think level surface) when the directional derivative is 0. That is when $\mathbf{u}$ is perpendicular to gradient vector $\nabla f(x_0, y_0, z_0)$.

Another way to word that is:

$\nabla f(x_0, y_0, z_0)$ is perpendicular to the the level surface $f(x,y,z) = f(x_0,y_0,z_0)$ $= c$! Which means that $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent plane of the level surface through (x_0, y_0, z_0)!

Example 6.17. Find a vector perpendicular to $xy + xz + yz = 3$ at $(1, 1, 1)$.

$f = c$

$\nabla f = \langle y+z, x+z, x+y \rangle$

$\nabla f(1,1,1) = \langle 2,2,2 \rangle$

6.3 Tangent Planes and Normal Lines... UPGRADE! – During Class

Objective(s):

- Give a better way to find tangent planes.
- Calculate normal lines to an implicit surface.

Recall from last time that:

Theorem 6.18. $\nabla f(x_0, y_0, z_0)$ is perpendicular to the the level surface $f(x, y, z) = f(x_0, y_0, z_0)$! Which means that $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent plane of the level surface through (x_0, y_0, z_0)!

Therefore

Theorem 6.19. If $F(x, y, z)$ is differentiable and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$ then the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is given by: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$f_x(P)(x-x_0)+f_y(P)(y-y_0)+f_z(P)(z-z_0)=0$$

Example 6.20. Find the equation of the tangent plane to the surface $x^2 + \frac{y^2}{9} + \frac{z^2}{9} = 1$ at the point $\left(\frac{1}{3}, 2, 2\right)$

$f = K$

$$\frac{2}{3}\left(x-\frac{1}{3}\right)+\frac{4}{9}(y-2)+\frac{4}{9}(z-2)=0$$

$$f_x = 2x \Rightarrow f_x(P) = \frac{2}{3}$$

$$f_y = \frac{2y}{9} \Rightarrow f_y(P) = \frac{4}{9}$$

$$f_z = \frac{2z}{9} = f_z(P) = \frac{4}{9}$$

Theorem 6.21. If $F(x,y,z)$ is differentiable and $\nabla F(x_0,y_0,z_0) \neq \mathbf{0}$ then the normal line to the level surface $F(x,y,z)=k$ at $P(x_0,y_0,z_0)$ is given by:

$$\mathbf{r}(t) = \overrightarrow{\nabla F(P)} \cdot t + \langle x_0, y_0, z_0 \rangle$$

Example 6.22. Find the tangent plane and normal line of $z = x^2 - y^2$ through the point $(4,3,7)$

$$0 = \underbrace{x^2 - y^2 - z}_{f}$$

$$\nabla f = \langle 2x, -2y, -1 \rangle$$

$$\nabla f(4,3,7) = \langle 8, -6, -1 \rangle$$

Tangent Plane

$$8(x-4) - 6(y-3) - 1(z-7) = 0$$

Normal Line

$$\langle 8, -6, -1 \rangle t + \langle 4, 3, 7 \rangle$$

7.2 Locating Local Mins and Maxes – During Class

Objective(s):

- Upgrade the second derivative test to be able to solve for local minimums and maximums.
- Calculate normal lines to an implicit surface.

Theorem 7.4 (2 Variable - Second Derivative Test)**.** Suppose that $f, f_x, f_y, f_{xx}, f_{xy}$, and f_{yy} are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Let:

$$D = D(a, b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Calc 1 world

1. If $D(a, b) > 0$ and $f_{xx} < 0$ at (a, b) then f has a local maximum at (a, b).
2. If $D(a, b) > 0$ and $f_{xx} > 0$ at (a, b) then f has a local minimum at (a, b).

Calc 3 world

3. $D(a, b) < 0$ at (a, b) then f has a saddle point at (a, b).
4. $D(a, b) = 0$ at (a, b) then it's inconclusive.

Let's use it! I claimed in 12.6 that $z = y^2 - x^2$ has a saddle point. Show it!

Example 7.5. Find and classify all critical points of $f(x, y) = y^2 - x^2$

$f_x = -2x \Rightarrow 0 = -2x$ and $x = 0$

$f_y = 2y \Rightarrow 0 = 2y$ and $y = 0$

so $(0,0)$ is only crit pt.

$f_{xx} = -2$

$f_{xy} = 0$

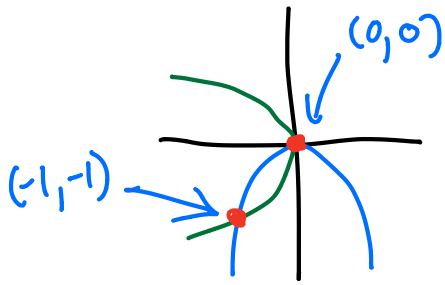
$f_{yy} = 2$

$D(0,0) = -2(2) - 0^2$

$= -4 < 0$

so $(0,0)$ is a saddle point

Example 7.6. Find and identify all the local maxima, minima, and saddle points for the function $f(x, y) = x^3 + 3xy + y^3$.



$f_x = 3x^2 + 3y$

$f_y = 3x + 3y^2$

$0 = 3x^2 + 3y$ & $0 = 3x + 3y^2$

$-x^2 = y$ & $-y^2 = x$

$-(-x^2)^2 = x$

$-x^4 = x$

$-x - x^4 = 0$

$\underbrace{-x}(1 + x^3) = 0$

$x = 0$ $\quad 1 + x^3 = 0$

$x^3 = -1$

$x = \sqrt[3]{-1} = -1$

$f_{xx} = 6x$

$f_{xy} = 3$ $\qquad D = 36xy - 9$

$f_{yy} = 6y$

$D(0,0) = -9 < 0$

so $(0,0)$ is a saddle pt.

$D(-1,-1) = 36 - 9 = 27 > 0$

$f_{xx} = 6(-1) = -6 < 0$

so $(-1,-1)$ is a local max.

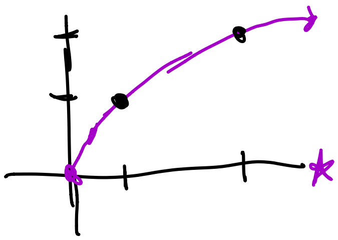
Example 7.7. Use 14.7 method to find the shortest distance from the point $(1, 0, 2)$ to the plane $x + 2y + z = 4$.

$z = 4 - x - 2y$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (4-x-2y-2)^2}$$

$\sqrt{☆}$



$$d^2 = f = (x-1)^2 + (y-0)^2 + (4-x-2y-2)^2$$

$$f_x = 2(x-1) + 2(2-x-2y)^1(-1)$$

$$f_y = 2y + 2(2-x-2y)^1(-2)$$

$$0 = 2x - 2 - 4 + 2x + 4y \rightarrow 6 = 4x + 4y$$

$$0 = 2y - 8 + 4x + 8y \rightarrow - \; 8 = 4x + 10y$$

$$-2 = -6y$$

$$1/3 = y$$

$$\frac{18}{3} = 4x + \frac{4}{3}$$

$$\frac{14}{3} = 4x$$

$$\frac{7}{6} = x$$

So $\left(\frac{7}{6}, \frac{1}{3}\right)$ is a critical pt.

$f_{xx} = 4$

$f_{yy} = 10$

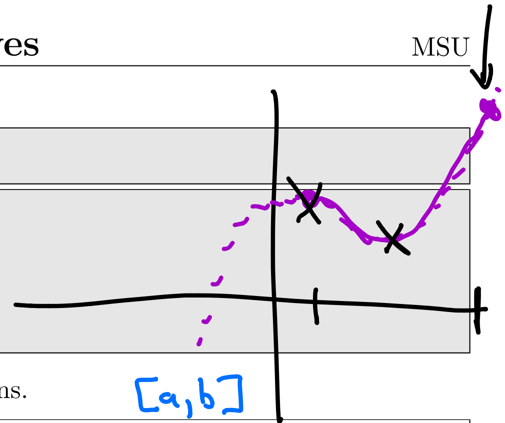
$f_{xy} = 4$

$D\left(\frac{7}{6}, \frac{1}{3}\right) = 40 - 16 > 0$

$f_{xx} > 0$

so $\left(\frac{7}{6}, \frac{1}{3}\right)$ is a min

$$d = \sqrt{\left(\frac{7}{6} - 1\right)^2 + \left(\frac{1}{3}\right)^2 + \left(2 - \frac{7}{6} - 2\cdot\frac{1}{3}\right)^2}$$



7.3 Day 2... Absolutely! – During Class

Objective(s):

- Learn a technique for finding absolute minimums and maximums.
- Apply it to a variety of problems.

Now we will learn how to find absolute maximums and minimums in certain situations.

Theorem 7.14 (Extreme Value Theorem)**.** If a real-valued function f is continuous on the closed and bounded set D in $\mathbb{R}^2$, then f must attain a maximum and a minimum, each at least once.

This was very similar to the Calc 1 version. Now we need to recall what closed and bounded means in $\mathbb{R}^2$.

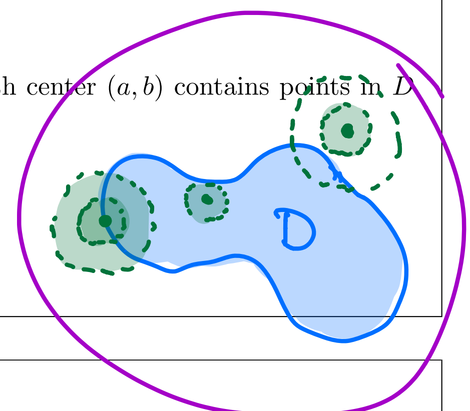
Definition(s) 7.15.

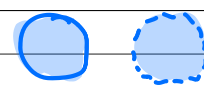
(a) A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.

(b) A closed set in $\mathbb{R}^2$ is one that contains all its boundary points.

(no dashed lines)

(c) A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.

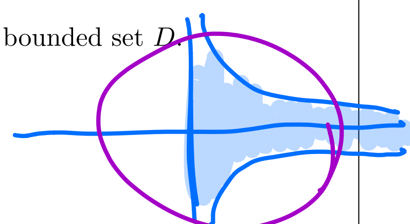




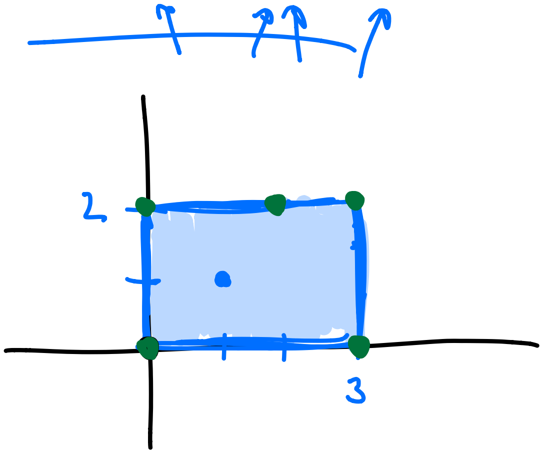
Theorem 7.16 (Strategy for Finding Abs Max/Min)**.**

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D.

1. Find the values of f at the critical points in D.
2. Find the critical values of f on the boundary of D.
3. The largest of the values from step 1 & 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.



Example 7.17. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.



① $f = x^2 - 2xy + 2y$

$f_x = 2x - 2y$

$f_y = -2x + 2$

$-2x+2=0$ and $2x-2y=0$

$x=1$ and $y=1$ $(1,1)$

② Bottom $y=0$, $x \in [0,3]$

$f = x^2$

$f' = 2x \Rightarrow 2x = 0$

$x = 0$

$(0,0)$ & $(3,0)$

Top $y=2$ $x \in [0,3]$

$f = x^2 - 4x + 4$

$f' = 2x - 4 \Rightarrow 2x-4=0$

$x = 2$

$(2,2)$, $(0,2)$, $(3,2)$

Left: $x=0$, $y \in [0,2]$

$f = 2y$

$f' = 2 \Rightarrow 2=0$ ✗

$(0,0)$, $(0,2)$

Right: $x=3$, $y \in [0,2]$

$f = 9-4y$

$f' = -4 \Rightarrow -4=0$ ✗

$(3,0)$ & $(3,2)$

③ $f(0,0) = 0 \leftarrow$ abs min

$f(3,0) = 9 \leftarrow$ abs max

$f(1,1) = 1$

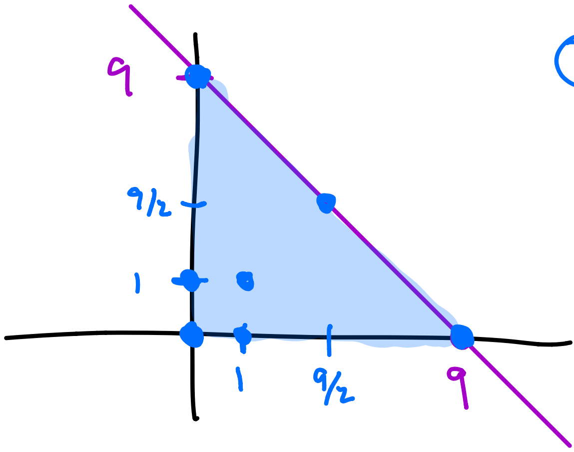
$f(2,2) = 0 \leftarrow$ abs min

$f(0,2) = 4$

$f(3,2) = 1$

$2+2+0-1-0$

Example 7.18. Find the absolute maximum and minimum values of the function $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by and including the line $y = 9 - x$.



① $f_x = 2-2x$ $\quad 2-2x=0$ and $2-2y=0$

$f_y = 2-2y$ $\quad x=1$ and $y=1$

② Bottom $y=0$, $x\in[0,9]$

$f = 2+2x-x^2$

$f' = 2-2x \Rightarrow 2-2x=0$

$x=1$

$(1,0), (0,0), (9,0)$

Left: $x=0$, $y\in[0,9]$

$f = 2+2y-y^2$

$f' = 2-2y \Rightarrow 2-2y=0$

$y=1$

$(0,1), (0,0), (0,9)$

Hypotenuse $y=9-x$ $\quad x\in[0,9]$

$f = 2+2x+2(9-x)-x^2-(9-x)^2$

$f' = 2-2-2x+2(9-x)'(+1)$

$= -2x+18-2x$

$= 18-4x = 0$

$18 = 4x$

$\frac{9}{2} = x \quad y = 9-\frac{9}{2} = \frac{9}{2}$

$(\frac{9}{2},\frac{9}{2}), (0,9), (9,0)$

③ $f(0,0) = 2$

$f(1,0) = 3$

$f(0,1) = 3$

$f(1,1) = 4 \leftarrow$ abs max

$f(9,0) = -61 \leftarrow$ abs min

$f(0,9) = -61 \leftarrow$

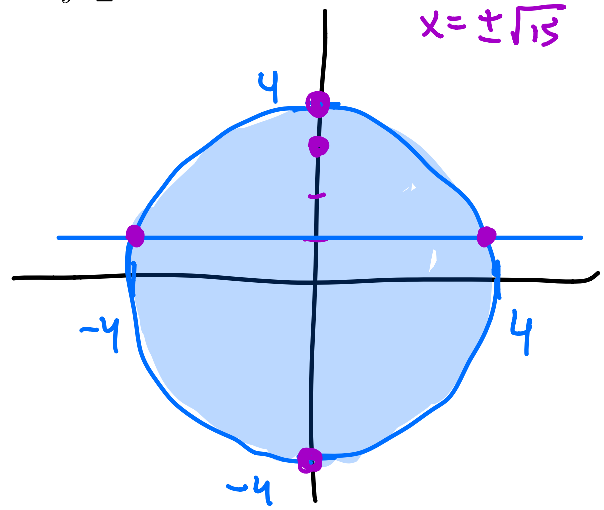
$f(\frac{9}{2},\frac{9}{2}) = \frac{-41}{2}$

$x^2+y^2=16$

$x^2+1=16$

$x=\pm\sqrt{15}$

Example 7.19. Consider $f(x, y) = 2x^2 - y^2 + 6y$ on the disk D given by $x^2 + y^2 \leq 16$.



(a) Find all critical points of f.

$f_x = 4x$

$f_y = -2y+6$

$4x=0$ and $-2y+6=0$

$x=0 \qquad y=3$

(b) Find the functions values of f on the boundary of D.

$\cancel{f=2x^2-(16-x^2)+6(\quad)} \qquad \cancel{y^2=16-x^2}$

$f=2(16-y^2)-y^2+6y \qquad x^2=16-y^2$

$= 32-2y^2-y^2+6y = -3y^2+6y+32$

(c) Find the absolute minimum and absolute maximum of $f(x, y) = 2x^2 - y^2 + 6y$ on the disk given by $x^2 + y^2 \leq 16$.

$f' = -6y+6 \qquad y \in [-4, 4]$

$-6y+6=0$

$y=1$

$(\sqrt{15},1), (-\sqrt{15},1), (0,-4), (0,4)$

3

$f(0,3) = 9$

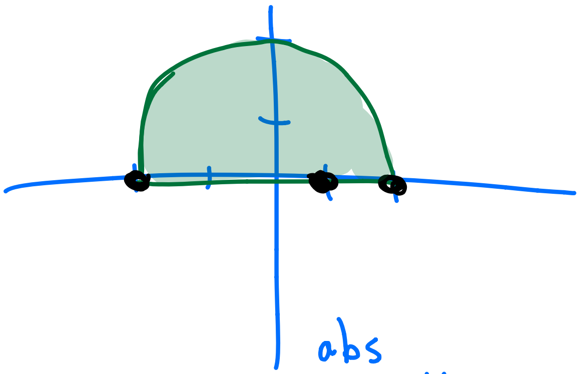
$f(0,4) = 8$

$f(0,-4) = -40$ abs min

$f(-\sqrt{15},1) = 35$ abs max

$f(\sqrt{15},1) = 35$

Example 7.20. Find the absolute maximum and minimum values of the function $f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}$ on the closed half-disk $R = \{(x, y) \mid x^2 + y^2 \leq 4 \text{ with } y \geq 0\}$.



1

$f_x = \frac{1}{2}(x^2+y^2-2x+2)^{-1/2}(2x-2)$

$f_y = \frac{1}{2}(x^2+y^2-2x+2)^{-1/2}(2y)$

$2x-2=0 \rightarrow x=1 \qquad 2y=0 \rightarrow y=0$

crit pt @ $(1,0)$

2

$y=0, \quad x \in [-2,2]$

$f(x,0) = \sqrt{x^2-2x+2}$

$f'(x) = \frac{1}{2}(x^2-2x+2)^{-1/2}(2x-2)$

$\rightarrow 2x-2=0 \rightarrow x=1$

$\rightarrow (1,0)$

$(-2,0)$

$(2,0)$

$y^2 = 4-x^2, \quad x \in [-2,2]$

$f(x) = \sqrt{x^2+(4-x^2)-2x+2} = \sqrt{6-2x}$

$f' = \frac{1}{2}(6-2x)^{-1/2}(-2)$

$\rightarrow \quad (-2,0)$

$(2,0)$

3

abs max

$f(-2,0) = \sqrt{4+4+2} = \sqrt{10}$

$f(2,0) = \sqrt{4-4+2} = \sqrt{2}$

$f(1,0) = \sqrt{1-2+2} = 1$

abs min