

16.6 More Parametric Surfaces

1. Let $\mathbf{r}(u, v) = \langle u + v, u - v, u^2 - v^2 \rangle$

(a) Evaluate $\mathbf{r}(2, -1)$ and $\mathbf{r}(-1, 2)$.

Solution:

$$\begin{aligned}\mathbf{r}(2, -1) &= (2 - 1, 2 - (-1), 2^2 - (-1)^2) \\ &= (1, 3, 3) \\ \mathbf{r}(-1, 2) &= (-1 + 2, -1 - 2, (-1)^2 - 2^2) \\ &= (1, -3, -3)\end{aligned}$$

(b) Find u, v so that $\mathbf{r}(u, v) = (3, -1, -3)$.

Solution: If $u + v = 3$ then $v = 3 - u$. Then

$$u - v = -1 \quad \Longleftrightarrow \quad u - (3 - u) = -1 \quad \Longleftrightarrow \quad 2u = 2 \quad \Longleftrightarrow \quad u = 1$$

Which implies $v = 3 - 1 = 2$. To verify $u = 1, v = 2$ is correct,

$$\mathbf{r}(1, 2) = (1 + 2, 1 - 2, (1)^2 - (2)^2) = (3, -1, -3).$$

(c) Show that $(0, 0, 1)$ is not a point on the surface.

Solution: Since $u - v = 0$, $u = v$. Then $u + v = 0$ gives $u + u = 0$ which means $u = 0 = v$. But $\mathbf{r}(0, 0) = (0, 0, 0) \neq (0, 0, 1)$.

(d) Recall that a surface $\mathbf{r}(u, v)$ is *smooth* if \mathbf{r}_u and \mathbf{r}_v are both continuous and $|\mathbf{r}_u \times \mathbf{r}_v|$ is never 0 for (u, v) in the interior of the domain (this means that at any point on the surface, the normal vector to the tangent plane is not $\mathbf{0}$). Show that $\mathbf{r}(u, v)$ is continuous.

Solution: We first compute \mathbf{r}_u , \mathbf{r}_v , $\mathbf{r}_u \times \mathbf{r}_v$, and $|\mathbf{r}_u \times \mathbf{r}_v|$.

$$\begin{aligned}\mathbf{r}_u(u, v) &= \langle 1 + 0, 1 - 0, 2u - 0 \rangle \\ &= \langle 1, 1, 2u \rangle \\ \mathbf{r}_v(u, v) &= \langle 0 + 1, 0 - 1, 0 - 2v \rangle \\ &= \langle 1, -1, -2v \rangle. \\ \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2u \\ 1 & -1 & -2v \end{vmatrix} \\ &= \langle 2u - 2v, 2u + 2v, -2 \rangle \\ |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{(2u - 2v)^2 + (2u + 2v)^2 + (-2)^2} \\ &= \sqrt{4u^2 - 4uv + 4v^2 + 4u^2 + 4uv + 4v^2 + 4} \\ &= \sqrt{8u^2 + 8v^2 + 4}\end{aligned}$$

It is clear that \mathbf{r}_u and \mathbf{r}_v are continuous everywhere (constant functions and polynomials). The only way $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{8u^2 + 8v^2 + 4} = 0$ is if $8u^2 + 8v^2 + 4 = 0$. Since $8u^2 + 8v^2 + 4 \geq 4$ for any (u, v) we know $|\mathbf{r}_u \times \mathbf{r}_v|$ is never 0.

This means the surface is continuous

- (e) Find an equation of the tangent plane to the parametric surface $\mathbf{r}(u, v)$ at the point $u = 1$ and $v = -1$.

Solution: A point on the tangent plane is given by $\mathbf{r}(1, -1) = (0, 2, 0)$. Using $\mathbf{r}_u \times \mathbf{r}_v$ found above,

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v(1, -1) &= \langle 2(1) - 2(-1), 2(1) + 2(-1), -2 \rangle \\ &= \langle 4, 0, -2 \rangle\end{aligned}$$

Therefore an equation for the tangent plane is

$$\begin{aligned}\langle 4, 0, -2 \rangle \cdot \langle x - 0, y - 2, z - 0 \rangle &= 0 \\ 4(x - 0) + 0(y - 2) - 2(z - 0) &= 0 \\ 4x - 2z &= 0\end{aligned}$$

2. Find two parametric representations for the part of the plane $z = 3x + 5y$ that lies inside the cylinder $x^2 + y^2 = 4$ by completing the following

(a)

$$\mathbf{r}_1(u, v) = \langle u, _, _ \rangle \quad \text{with} \quad u^2 + v^2 \leq _$$

Solution:

$$\mathbf{r}_1(u, v) = \langle u, v, 3u + 5v \rangle, \quad u^2 + v^2 \leq 4$$

(b)

$$\mathbf{r}_2(s, t) = \langle s \cos(t), s \sin(t), _ \rangle \quad \text{with} \quad s \in [_, _], \quad t \in [_, _]$$

Solution:

$$\mathbf{r}_2(s, t) = \langle s \cos(t), s \sin(t), 3s \cos(t) + 5s \sin(t) \rangle, \quad s \in [0, 2], \quad t \in [0, 2\pi]$$

3. Let S be the surface consisting of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.

- (a) Find a parametric representation $\mathbf{r}(u, v)$ of S by completing the following

$$\mathbf{r}(u, v) = \langle 4 \cos(u) \sin(v), _, _ \rangle$$

$$u \in [0, 2\pi] \quad \text{and} \quad v \in [_, _]$$

Solution:

$$\begin{aligned}\mathbf{r}(u, v) &= \langle 4 \cos(u) \sin(v), 4 \sin(u) \sin(v), 4 \cos(v) \rangle \\ u &\in [0, 2\pi] \quad \text{and} \quad v \in [\pi/3, 2\pi/3]\end{aligned}$$

- (b) Find a parametric representation of the boundary curve of S where $z = 2$

$$\mathbf{r}_1(t) = \langle _, _, 2 \rangle, \quad t \in [0, 2\pi]$$

Solution: If $z = 2$, our equation becomes $x^2 + y^2 = 12$, i.e. a circle with radius $2\sqrt{3}$.

$$\mathbf{r}_1(t) = \langle 2\sqrt{3} \cos(t), 2\sqrt{3} \sin(t), 2 \rangle, \quad t \in [0, 2\pi]$$

- (c) Find a parametric representation of the boundary curve of S where $z = -2$

$$\mathbf{r}_2(t) = \langle _, _, -2 \rangle, \quad t \in [0, 2\pi]$$

Solution:

$$\mathbf{r}_2(t) = \langle 2\sqrt{3} \cos(t), 2\sqrt{3} \sin(t), -2 \rangle, \quad t \in [0, 2\pi]$$