Surface Integrals

1. **Parametric Surfaces:**

   For a surface S given by $\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ The integral of $f(x,y,z)$ over S is given by

$$\boxed{\iint_S f(x,y,z)\; dS = \iint_D f(\boldsymbol{r}(u,v))\; |\boldsymbol{r}_u \times \boldsymbol{r}_v|\; dA}$$

2. **Graphs:**

   For S given by $z = g(x,y)$ with $(x,y) \in D$,

$$\boxed{\iint_S f(x,y,z)\; dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA}$$

3. **Vector Fields:**

   If $\boldsymbol{F}$ is a continuous vector field on an oriented surface S with unit normal vector $\boldsymbol{n}$ then

$$\begin{aligned} \iint_S \boldsymbol{F}\cdot\; d\boldsymbol{S} &= \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\; dS \\ &= \iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_u \times \boldsymbol{r}_v)\; dA \quad \text{(If } S \text{ is given by } r(u,v)\text{)} \end{aligned}$$

4. **Here are the answers you should get:**

   (a) $(364/3)\sqrt{2}\pi$

   (b) $(2/3)\left(2^{3/2} - 1\right)$

   (c) ≈ 5.84728

   (d) 4

   (e) $-(4/3)\pi$

1. Evaluate $\iint_S x^2z^2\, dS$ where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Solution: Since z is always positive between 1 and 3, we may rewrite the surface as

$$z = \sqrt{x^2 + y^2}.$$

Note that

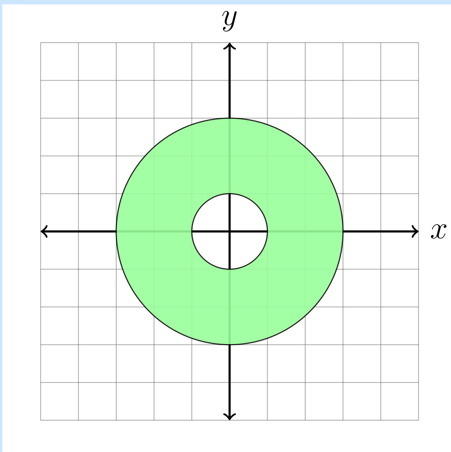
$$\begin{aligned}\sqrt{\frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2 + 1} &= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \\ &= \sqrt{2}\end{aligned}$$

So our integral becomes

$$\iint_D x^2 \left(\sqrt{x^2+y^2}\right)^2 \sqrt{2}\, dA = \sqrt{2} \iint_D x^4 + x^2y^2\, dA$$

What is D?

When $z = 1$ we have $x^2 + y^2 = 1$ and when $z = 3$ $x^2 + y^2 = 9$. So D is the area in the xy-plane between the circles of radius 1 and 3 both centered at the origin. Shown in green below.



This is clearly easier if we use polar coordinates, i.e. $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\sqrt{2}\iint_D x^4 + x^2y^2\, dA &= \sqrt{2}\int_0^{2\pi}\int_1^3 \left(r^4\cos^4\theta + r^4\cos^2\theta\sin^2\theta\right) r\, dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_1^3 r^5\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)\, dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi}\cos^2\theta\int_1^3 r^5\, dr\, d\theta \\ &= \boxed{\left(\frac{364}{3}\right)\sqrt{2}\pi}\end{aligned}$$

2. Evaluate $\iint_S y\, dS$ where S is the helicoid with vector equation

$$\boldsymbol{r}(u,v) = \langle u\cos v, u\sin v, v\rangle\,,\ 0 \le u \le 1,\ 0 \le v \le \pi.$$

Solution: Note that

$$\begin{aligned}
\boldsymbol{r}_u \times \boldsymbol{r}_v &= \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle \\
&= \langle \sin v, -\cos v, u\rangle \\
|\boldsymbol{r}_u \times \boldsymbol{r}_v| &= \sqrt{\sin^2 v + \cos^2 v + u^2} \\
&= \sqrt{1+u^2}
\end{aligned}$$

So the integral becomes

$$\begin{aligned}
\iint_D (u\sin v)\sqrt{1+u^2}\, dA &= \int_0^1 \int_0^\pi u\sqrt{1+u^2}\sin v\ dv\ du \\
&= \int_0^1 u\sqrt{1+u^2}\left(\int_0^\pi \sin v\ dv\right)\ du \\
&= 2\int_0^1 u\sqrt{1+u^2}\ du \\
&= \frac{2}{3}\left(1+u^2\right)^{3/2}\Big|_0^1 \\
&= \boxed{\frac{2}{3}\left(2^{3/2}-1\right)}.
\end{aligned}$$

3. Find the mass of the surface S, given by $x = 3z^2 - 4y$, $0 \le y \le 1$, $0 \le z \le 1$, if its density function is $\rho(x,y,z) = 2z$.

Solution:

Since the surface is of the form $x = g(y,z)$ we use

$$\begin{aligned}
\iint_S \rho\left(g(y,z), y, z\right)\ \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 1}\ dS &= \int_0^1 \int_0^1 2z\sqrt{17+36z^2}\ dz\ dy \\
&= \left(\frac{2}{3}\right)\left(\frac{1}{36}\right)\left((17+36)^{3/2} - 17^{3/2}\right) \\
&\approx \boxed{5.84728}
\end{aligned}$$

4. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ as given

(a) $\mathbf{F}(x, y, z) = ze^{xy}\boldsymbol{i} - 3ze^{xy}\boldsymbol{j} + xy\boldsymbol{k}$
S is the parallelogram with parametric equations

$$x = u + v, \quad y = u - v \quad , z = 1 + 2u + v$$

with

$$0 \leq u \leq 2, \ 0 \leq v \leq 1$$

oriented upwards.

Solution: A normal vector is

$$\begin{aligned} \boldsymbol{r}_u \times \boldsymbol{r}_v &= \langle 1, 1, 2\rangle \times \langle 1, -1, 1\rangle \\ &= \langle 3, 1, -2\rangle \end{aligned}$$

This is pointing downwards, so we use $\boldsymbol{n} = \langle -3, -1, 2\rangle = \boldsymbol{r}_v \times \boldsymbol{r}_u$.

$$\begin{aligned} \mathbf{F}(r(u,v)) \cdot (\boldsymbol{r}_v \times \boldsymbol{r}_u) &= \left\langle (1 + 2u + v)e^{u^2 - v^2}, -3(1 + 2u + v)e^{u^2 - v^2}, u^2 - v^2 \right\rangle \cdot \langle -3, -1, 2\rangle \\ &= 2(u^2 - v^2) \end{aligned}$$

and

$$\begin{aligned} \int_0^2 \int_0^1 \mathbf{F}(r(u,v)) \cdot (\boldsymbol{r}_v \times \boldsymbol{r}_u) \ dv \ du &= \int_0^2 \int_0^1 2(u^2 - v^2) \ dv \ du \\ &= \boxed{4}. \end{aligned}$$

(b) $\mathbf{F}(x, y, z) = \langle x, -z, y\rangle$
S is the part of sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation towards the origin.

Solution: In the first octant we can describe this surface as

$$z = g(x, y) = \sqrt{4 - x^2 - y^2}$$

with

$$\frac{\partial g}{\partial x} = -\frac{x}{\sqrt{4 - x^2 - y^2}} \quad \text{and} \quad \frac{\partial g}{\partial y} = -\frac{y}{\sqrt{4 - x^2 - y^2}}$$

Then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right)\ dA \\
&= \iint_D -\frac{x^2}{\sqrt{4-x^2-y^2}} - (-\sqrt{4-x^2-y^2}) \left(-\frac{y}{\sqrt{4-x^2-y^2}} \right) + y\ dA \\
&= \iint_D -\frac{x^2}{\sqrt{4-x^2-y^2}}\ dA \\
&= \int_0^{\pi/2} \int_0^2 -\frac{r^2\cos^2\theta}{\sqrt{4-r^2}} r\ dr\ d\theta \\
&= \left(\int_0^{\pi/2} \cos^2\theta\ d\theta \right) \left(\int_0^2 \frac{r^3}{\sqrt{4-r^2}}\ dr \right) \\
&= \left(\frac{\pi}{4} \right) \left(\frac{16}{3} \right) \\
&= \frac{4}{3}\pi.
\end{aligned}$$

Since we were meant to integrate assuming a downward orientation, the answer is

$$\boxed{-\frac{4}{3}\pi}$$