16.6 More Parametric Surfaces

1. Let $\boldsymbol{r}(u, v) = \langle u+v, u-v, u^2 - v^2\rangle$

(a) Evaulate $\boldsymbol{r}(2, -1)$ and $\boldsymbol{r}(-1, 2)$.

Solution:

$$\begin{aligned}
\boldsymbol{r}(2,-1) &= (2-1, 2-(-1), 2^2-(-1)^2) \\
&= (1, 3, 3) \\
\boldsymbol{r}(-1,2) &= (-1+2, -1-2, (-1)^2-2^2) \\
&= (1, -3, -3)
\end{aligned}$$

(b) Find u, v so that $\boldsymbol{r}(u, v) = (3, -1, -3)$.

Solution: If $u + v = 3$ then $v = 3 - u$. Then

$$u - v = -1 \quad\iff\quad u - (3-u) = -1 \quad\iff\quad 2u = 2 \quad\iff\quad u = 1$$

Which implies $v = 3 - 1 = 2$. To verify $u = 1, v = 2$ is correct,

$$\boldsymbol{r}(1,2) = (1+2, 1-2, (1)^2-(2)^2) = (3, -1, -3).$$

(c) Show that $(0, 0, 1)$ is not a point on the surface.

Solution: Since $u - v = 0$, $u = v$. Then $u + v = 0$ gives $u + u = 0$ which means $u = 0 = v$. But $\boldsymbol{r}(0, 0) = (0, 0, 0) \neq (0, 0, 1)$.

(d) Recall that a surface $\boldsymbol{r}(u, v)$ is *smooth* if $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are both continuous and $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is never 0 for (u, v) in the interior of the domain (this means that at any point on the surface, the normal vector to the tangent plane is not $\mathbf{0}$). Show that $\boldsymbol{r}(u, v)$ is continuous.

Solution: We first compute $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, $\boldsymbol{r}_u \times \boldsymbol{r}_v$, and $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$.

$$\begin{aligned}
\boldsymbol{r}_u(u,v) &= \langle 1+0, 1-0, 2u-0\rangle \\
&= \langle 1, 1, 2u\rangle \\
\boldsymbol{r}_v(u,v) &= \langle 0+1, 0-1, 0-2v\rangle \\
&= \langle 1, -1, -2v\rangle\,. \\
\boldsymbol{r}_u \times \boldsymbol{r}_v &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 2u \\ 1 & -1 & -2v \end{vmatrix} \\
&= \langle 2u-2v, 2u+2v, -2\rangle \\
|\boldsymbol{r}_u \times \boldsymbol{r}_v| &= \sqrt{(2u-2v)^2 + (2u+2v)^2 + (-2)^2} \\
&= \sqrt{4u^2 - 4uv + 4v^2 + 4u^2 + 4uv + 4v^2 + 4} \\
&= \sqrt{8u^2 + 8v^2 + 4}
\end{aligned}$$

It is clear that $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are continuous everywhere (constant functions and polynomials). The only way $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{8u^2 + 8v^2 + 4} = 0$ is if $8u^2 + 8v^2 + 4 = 0$. Since $8u^2 + 8v^2 + 4 \geq 4$ for any (u, v) we know $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is never 0.

This means the surface is continuous

(e) Find an equation of the tangent plane to the parametric surface $\boldsymbol{r}(u, v)$ at the point $u = 1$ and $v = -1$.

Solution: A point on the tangent plane is given by $\boldsymbol{r}(1, -1) = (0, 2, 0)$. Using $\boldsymbol{r}_u \times \boldsymbol{r}_v$ found above,

$$\begin{aligned} \boldsymbol{r}_u \times \boldsymbol{r}_v(1, -1) &= \langle 2(1) - 2(-1), 2(1) + 2(-1), -2 \rangle \\ &= \langle 4, 0, -2 \rangle \end{aligned}$$

Therefore an equation for the tangent plane is

$$\begin{aligned} \langle 4, 0, -2 \rangle \cdot \langle x - 0, y - 2, z - 0 \rangle &= 0 \\ 4(x - 0) + 0(y - 2) - 2(z - 0) &= 0 \\ 4x - 2z &= 0 \end{aligned}$$

2. Find two parametric representations for the part of the plane $z = 3x + 5y$ that lies inside the cylinder $x^2 + y^2 = 4$ by completing the following

   (a)
   $$\boldsymbol{r}_1(u, v) = \langle u, \underline{\qquad\qquad}, \underline{\qquad\qquad} \rangle \qquad \text{with} \quad u^2 + v^2 \leq \underline{\qquad\qquad}$$

   Solution:

   (b)
   $$\boldsymbol{r}_2(s, t) = \langle s\cos(t), s\sin(t), \underline{\qquad\qquad} \rangle \quad \text{with} \quad s \in [\underline{\quad\;}, \underline{\quad\;}], \quad t \in [\underline{\quad\;}, \underline{\quad\;}]$$

   Solution:

3. Let S be the surface consisting of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.

   (a) Find a parametric representatino $\boldsymbol{r}(u, v)$ of S by completing the following
   $$\boldsymbol{r}(u, v) = \langle 4\cos(u)\sin(v), \underline{\quad\;}, \underline{\quad\;} \rangle$$
   $$u \in [0, 2\pi] \quad \text{and} \quad t \in [0, 2\pi]$$

   Solution:

   (b) Find a parametric representation of the boundary curve of S where $z = 2$
   $$\boldsymbol{r}_1(t) = \langle \underline{\quad\;}, \underline{\quad\;}, 2 \rangle, \quad t \in [0, 2\pi]$$

   Solution:

   (c) Find a parametric representation of the boundary curve of S where $z = -2$
   $$\boldsymbol{r}_2(t) = \langle \underline{\quad\;}, \underline{\quad\;}, -2 \rangle, \quad t \in [0, 2\pi]$$

   Solution: