1.2 Ideas, Sketches, and Gradients - During Class

Objective(s):

- Recognize natural phenomenon that can be expressed using vector fields
- Calculate gradient fields
- Verify a nice relation between gradient fields and level curves
- Match vector field equations and pictures.
- Start looking ahead to 16.2

Here are some pretty pictures from the book.

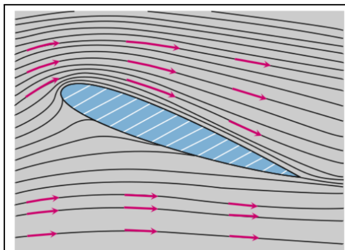

FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

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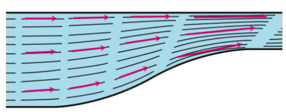


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

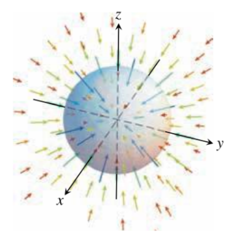
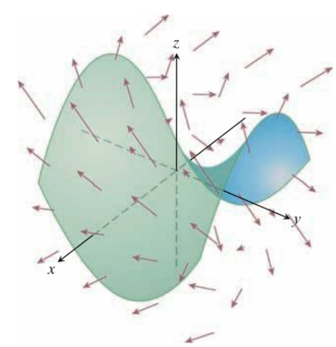


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.



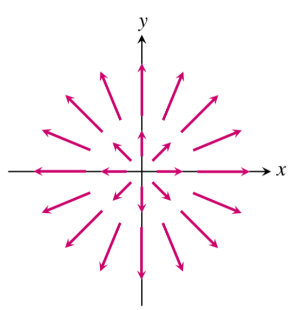


FIGURE 16.11 The radial field

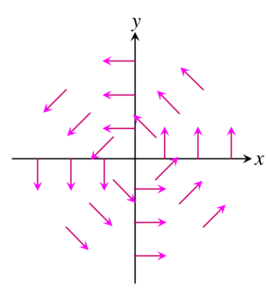


FIGURE 16.12 A “spin” field of

Vector fields can represent many different things. The main applications we will focus on are:

1. Force (gravity, electromagnetism, etc)
2. Velocity

We have technically seen vector fields before even though we never used it’s full potential. Any guesses?

The gradient

$f = x^2 + y^2$ $\nabla f = \langle 2x, 2y \rangle$

Definition(s) 1.3.

(a) A gradient field is vector field found by taking the gradient of a function.

Ex: $f = xy \Rightarrow \nabla f = \langle y, x \rangle$

(b) A vector field **F** is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $\vec{F} = \nabla f$. In this situation f is called a potential function for **F**.

Ex: $\vec{F} = \langle y, x \rangle$ since $f = xy + 3$ and $\nabla f = \langle y, x \rangle$

Example 1.4. Which of the vector field describes the plot to the right?

A. $\langle x, x - y \rangle$

B. $\langle y, x - y \rangle$

C. $\langle x, x + y \rangle$

D. $\langle y, x + y \rangle$

E. None of the above

@ $x = 1$, $y = 1$

A. $\langle 1, 0 \rangle$

B. $\langle 1, 0 \rangle$

C. $\langle 1, 2 \rangle$

D. $\langle 1, 2 \rangle$

@ $x = -1$, $y = -2$

C. $\langle -1, -3 \rangle$

D. $\langle -2, -3 \rangle$

@ $x = -3$, $y = 1$

C. $\langle -3, -2 \rangle$

~~D.~~ $\langle 1, -2 \rangle$

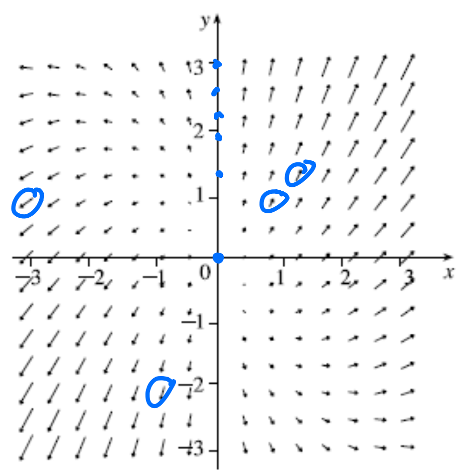


Figure may be scaled down

Example 1.5. Find the gradient vector field of $f(x, y) = 2xy + 3x - e^{-xy}$

$$\vec{F} = \nabla f = \langle 2y + 3 - e^{-xy}(-y), 2x - e^{-xy} \cdot (-x) \rangle$$

$$= \langle 2y + 3 + ye^{-xy}, 2x + xe^{-xy} \rangle$$

Example 1.6. For each of the following functions, draw level curves $f(x, y) = k$ for the indicated values of k. Then compute the gradient vector field, and sketch it at one or two points on each level curve.

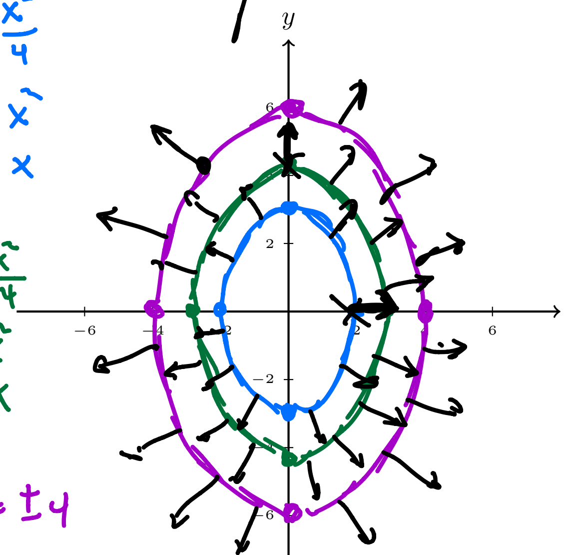
(a) $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$; $k = 1, 2, 4$

$\nabla f = \langle \frac{x}{2}, \frac{2y}{9} \rangle$

$1 = \frac{x^2}{4} + \frac{y^2}{9}$ $\quad x=0 \Rightarrow 1 = \frac{y^2}{9}$, $9 = y^2$, $\pm 3 = y$ $\quad y = 0 \Rightarrow 1 = \frac{x^2}{4}$, $4 = x^2$, $\pm 2 = x$

$2 = \frac{x^2}{4} + \frac{y^2}{9}$ $\quad x=0 \Rightarrow 2 = \frac{y^2}{9}$, $18 = y^2$, $\pm\sqrt{18} = y$ $\quad y=0 \Rightarrow 2 = \frac{x^2}{4}$, $8 = x^2$, $\pm\sqrt{8} = x$

$4 = \frac{x^2}{4} + \frac{y^2}{9}$ $\quad x=0$ $\quad 36 = y^2$, $\pm 6 = y$ $\quad y=0$ $\quad x = \pm 4$



$\nabla f(2,0) = \langle 1, 0 \rangle$

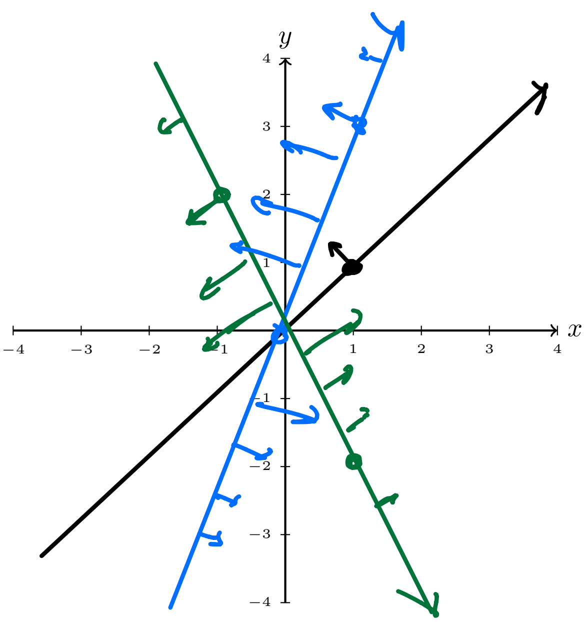
$\nabla f(0,\sqrt{18}) = \langle 0, \frac{2\sqrt{18}}{9} \rangle$

(b) $f(x, y) = \frac{y}{x + y}$, $x \neq -y$, $k = 1/2, 3/4, 2$

$\frac{1}{2} = \frac{y}{x+y} \Rightarrow x+y = 2y \Rightarrow y = x$

$\frac{3}{4} = \frac{y}{x+y} \Rightarrow 3x+3y = 4y \Rightarrow y = 3x$

$2 = \frac{y}{x+y} \Rightarrow 2x+2y = y \Rightarrow y = -2x$



$\nabla f = \langle \frac{-y}{(x+y)^2}, \frac{(1)(x+y) - y(1)}{(x+y)^2} \rangle$

$= \langle \frac{-y}{(x+y)^2}, \frac{x}{(x+y)^2} \rangle$

$\nabla f(1,1) = \langle \frac{-1}{4}, \frac{1}{4} \rangle$

Looking ahead it will be important to be able to answer the following questions.

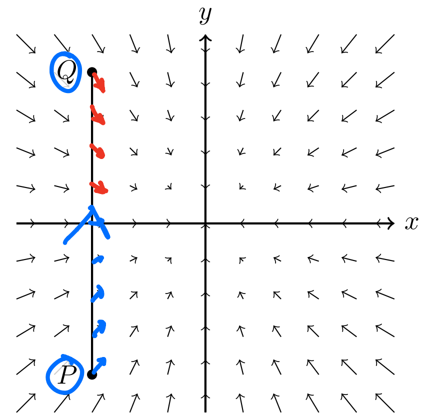
Example 1.7. Consider the vector field to the right. Overall would the vector field help

A. Push particles from P to Q along the curve C.

B. Push particles from Q to P along the curve C.

C. Neither.

$$\int_C \vec{F} \cdot d\vec{r} = 0$$



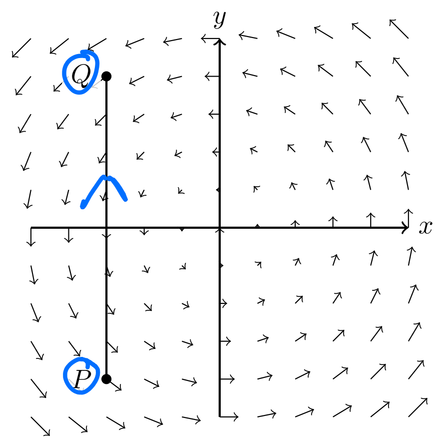
Example 1.8. Consider the vector field to the right. Overall would the vector field help

A. Push particles from P to Q along the curve C.

B. Push particles from Q to P along the curve C.

C. Neither.

$$\int_C \vec{F} \cdot d\vec{r} < 0$$



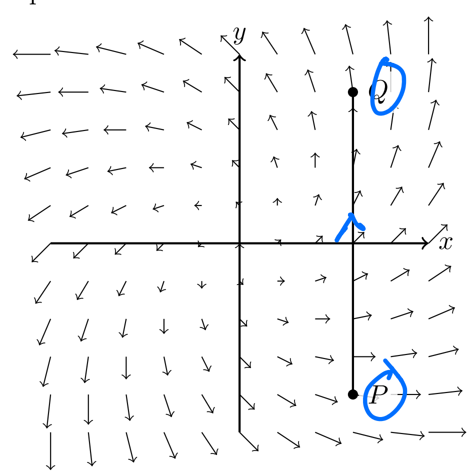
Example 1.9. Consider the vector field to the right. Overall would the vector field help

A. Push particles from P to Q along the curve C.

B. Push particles from Q to P along the curve C.

C. Neither.

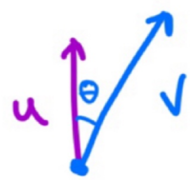
$$\int_C \vec{F} \cdot d\vec{r} > 0$$



Warm up:

In each case determine if $u \cdot v$ is positive / negative / zero.

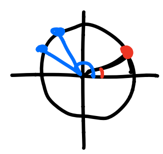
(a)



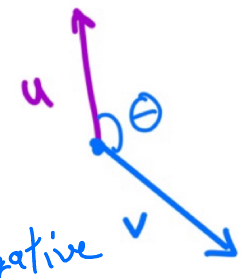
$u \cdot v = |u||v| \cos\theta$

$(+)(+)(+)$

Positive!

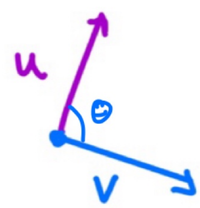


(c)



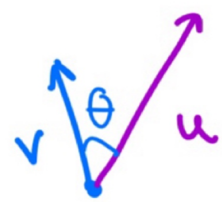
Negative

(b)



$0 = u \cdot v$

(d)



Positive!

Q1

Which of the following is equivalent to arc length of a curve C?

A. $\int_C 1 \; ds$

$\int_C 1 \; dt$

$\int_C t \; ds$

$\int_C s \; dt$

$\int_C 0 \; ds$

Q2

$x \quad y$

Evaluate $\int_C xy \; ds$ where C is parametrized by $\mathbf{r}(t) = \langle t, t \rangle, \; t \in [0, 1]$.

1/3

1/2

C. $\sqrt{2}/3$

$\sqrt{2}/2$

$\sqrt{2}$

$$\int f \, ds = \int_0^1 t \cdot t \; \sqrt{2} \; dt$$

$ds = |r'(t)| \, dt$

$r'(t) = \langle 1, 1 \rangle$

$|r'(t)| = \sqrt{1+1} = \sqrt{2}$

$$= \left[\sqrt{2} \; \frac{t^3}{3} \right]_0^1 = \frac{\sqrt{2}}{3}$$

2.2 Line Integrals with Scalar Functions and Vector Fields - During Class

Objective(s):

- Find more uses for line integrals with scalar functions
- Introduce and compute line integrals with vector fields.
- Using pictures reasonably approximate the sign of the answer.

How many bricks it takes to build the Great wall of China - Or area under space curves

Example 2.6. A portion of the great wall of china can be parametrized by $\mathbf{r}(t) = \langle 2\sin t, 2\cos t\rangle \quad t \in [0, \pi]$ where the height is given by $H(x, y) = xy^2$ *meters*. Each brick has a cross-sectional area of 100 cm^2. How many bricks are needed to build this portion of the great wall of china?

$x \quad y$

$r'(t) = \langle 2\cos t, -2\sin t\rangle$

$|r'(t)| = \sqrt{4\cos^2 t + 4\sin^2 t} = \sqrt{4(1)} = 2$

$\int H \, m \; ds \; m$

$$\int xy^2 \, ds = \int_0^{\pi} 2\sin t\,(2\cos t)^2 \cdot 2 \cdot dt$$

$$= 16\int_0^{\pi} \sin t\cos^2 t \, dt$$

$$= 16\left[-\frac{\cos^3 t}{3}\right]_0^{\pi}$$

$$= \frac{16}{3} + \frac{16}{3} = \frac{32}{3}\cancel{m^2} \cdot \frac{100\,\cancel{cm}}{1\,\cancel{m}} \cdot \frac{100\,\cancel{cm}}{1\,\cancel{m}} \cdot \frac{1 \text{ brick}}{\cancel{100\,cm^2}}$$

$$= \frac{3200}{3} \text{ bricks} \approx 1067 \text{ bricks}$$

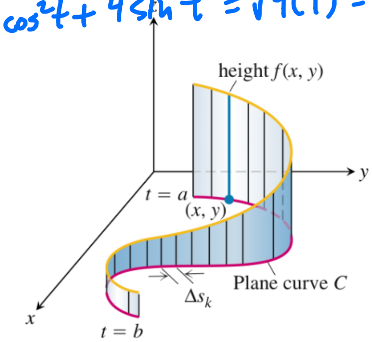


FIGURE 16.5 The line integral $\int_C f\,ds$ gives the area of the portion of the cylindrical surface or "wall" beneath $z = f(x, y) \geq 0$.

How heavy stuff is (given a density function)

Example 2.7. Suppose I have a nice spring that seems to follow the curve $\mathbf{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$ with $t \in [0, 8\pi]$ which happens to have a density function of $\delta(x, y, z) = 10 - x - y \; g/cm$. How heavy is the spring?

$x \quad y \quad z$

$\int 10 - x - y \, ds$

$$= 5\int_0^{8\pi} 10 - 3\sin t - 3\cos t \; dt$$

$r'(t) = \langle 3\cos t, -3\sin t, 4\rangle$

$\sqrt{9\cos^2 t + 9\sin^2 t + 16}$

$\sqrt{9 + 16} = 5$

$$= 5\left[10t + 3\cos t - 3\sin t\right]_0^{8\pi}$$

$$= 5\left(80\pi + \cancel{3} - 0 - (0 + \cancel{3} - 0)\right)$$

$$= 400\pi \; g$$

Now let's consider a special case where $f(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \overbrace{\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}^{T(t)}$. This gives us a way to do line integrals over vector fields!

Definition(s) 2.8. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along** C is

$$\int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] \overbrace{|\mathbf{r}'(t)|\, dt}^{ds} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C \vec{F}(r(t)) \cdot r'(t)\, dt \quad \bigstar$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ then

$$= \int_C P\, dx + Q\, dy$$

If $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then

$$= \int_C P\, dx + Q\, dy + R\, dz$$

So besides a great way to torture math students what is this used for? work!

Theorem 2.9. Take $\mathbf{F}$ to be a force field then the work done by the field over the curve C is given by

$$W = \int_C F \cdot T\, ds$$

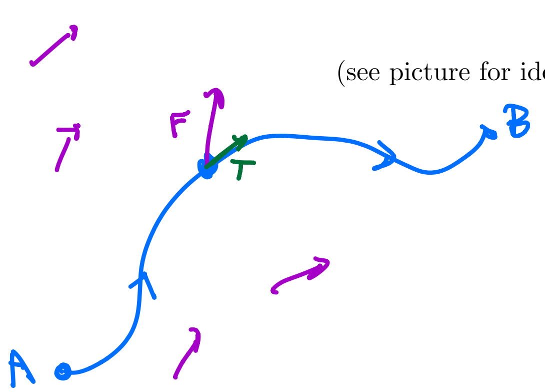
Idea of proof: Recall

$$W = \vec{F} \cdot \vec{D} \qquad \text{(where } \mathbf{D} \text{ is displacement vector)}$$

So if we consider the work that the force field is doing at each point on the curve we have:

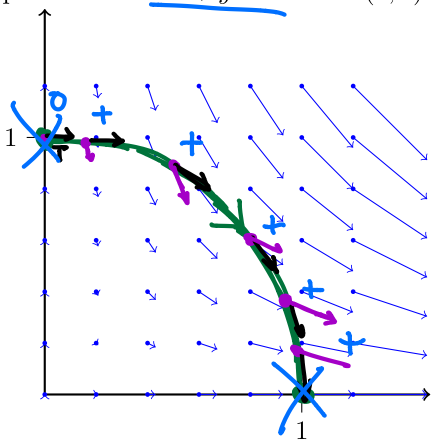
$$W_{\text{at a point}} = \vec{F} \cdot \vec{T} \qquad \text{(see picture for idea)}$$

Summing these all up over the curve we get



$$W_{\text{over the curve}} = \int_C \vec{F} \cdot \vec{T}\, ds$$

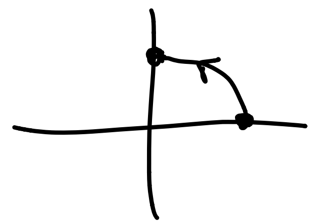
Example 2.10. A picture of the force field $\mathbf{F}(x, y)$ is given below. Determine if the work in moving a particle along the quarter circle $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$ is positive or negative using the picture.



$$\int F \cdot T \, ds$$

$$= \int (+) \, ds$$

$$= (+)$$



Example 2.10 (again)**.** Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, -xy \rangle$ in moving a particle along the quarter circle $x^2 + y^2 = 1$ from $(0, 1)$ to $(1, 0)$.

$$W = \int_C F \cdot T \, ds$$

$$= \int_a^b F(r(t)) \cdot r'(t) \, dt$$

~~$r(t) = \langle \cos t, \sin t \rangle$~~

$t \in [0, \frac{\pi}{2}]$

$r(t) = \langle \sin t, \cos t \rangle$

$r(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle$

$t \in [\frac{\pi}{2}, 0]$

$$= \int_{\pi/2}^{0} \langle x^2, -xy \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{\pi/2}^{0} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_{\pi/2}^{0} -\sin \cos^2 t - \cos^2 t \sin t \, dt$$

$$= -\int_{\pi/2}^{0} 2 \sin t \cos^2 t \, dt = \int_0^{\pi/2} 2 \sin t \cos^2 t \, dt$$

$$= 2\left[-\frac{\cos^3 t}{3}\right]_0^{\pi/2} = 0 + \frac{2}{3} = \frac{2}{3}$$

$t \in [0,1]$

$r(t) = \langle 2+t, 0+4t, 5t \rangle$

Example 2.11.

P Q R

(a) Evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

$r'(t) = \langle 1, 4, 5 \rangle$

$r(t) = \langle 3, 4, -5t \rangle$

$r'(t) = \langle 0, 0, -5 \rangle$

$$= \int_C \langle y, z, x \rangle \cdot dr$$

$$= \int_{C_1} \langle y, z, x \rangle \cdot dr + \int_{C_2} \langle y, z, x \rangle \cdot dr$$

$$= \int_0^1 \langle 4t, 5t, 2+t \rangle \cdot \langle 1, 4, 5 \rangle\, dt + \int_0^1 \langle 4, 5-5t, 3 \rangle \cdot \langle 0, 0, -5 \rangle\, dt$$

$$= \int_0^1 4t + 20t + 10 + 5t\, dt + \int_0^1 -15\, dt$$

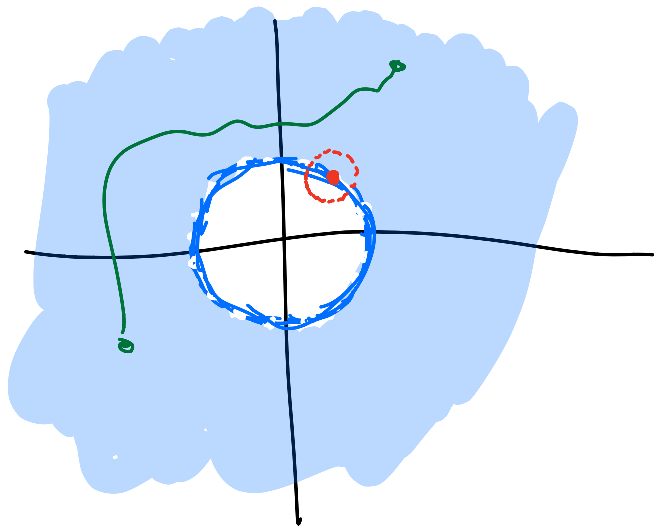
$$= \frac{29}{2}t^2 - 5t \Big|_0^1 = \frac{29}{2} - 5 = \boxed{19/2}$$

(b) What is the calculation in (a) telling you (in terms of Work)?

overall the vector field is helping push the particle.

$x^2+y^2=1$

$x^2+y^2>1$



3.2 The FUNdemental Theorem and Quick Evaluations - During Class

Objective(s):

- State and use the Fundamental Theorem of line integrals
- Understand what it takes to use the Fundamental Theorem

Remark 3.6. Moreover please notice that

$$f(x,y) = x^2 + y^2$$

$$f(1,1) - f(0,0) = (1^2+1^2) - (0^2+0^2) = 2$$

Thats to say that in our case

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

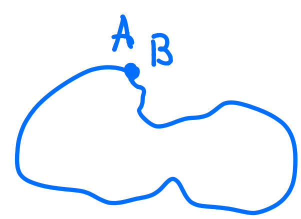
Theorem 3.7 (Fundamental Theorem of Line Integrals)**.**

Let C be a smooth curve joining the point A to the point B in the plane or space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on the domain D containing C. Then:

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

equivalently:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



Theorem 3.8. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = \underline{0}$ for every closed path C in D.

So here is where we sit:

1. We like conservative vector fields because they are path independent.
2. We like them even more because if we can find their potential function then line integrals are extremely easy to calculate.

Here are the natural questions we need to ask:

1. Given a vector field is there a way to tell if it is conservative?
2. Okay we know we have a conservative vector field... How can we find the potential function?

and very importantly we have:

$F = \langle P, Q \rangle$

Theorem 3.9 (Component Test)**.** Let $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$Q_x = P_y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Then $\mathbf{F}$ is conservative.

Finally! it's example time!!!!

Example 3.10. Consider the vector field $\mathbf{F} = \langle \underbrace{e^x \cos y + y}_{f_x}, \underbrace{x - e^x \sin y + 3}_{f_y} \rangle$. ($P$, Q)

(a) Show that $\mathbf{F}$ is conservative over its natural domain

(b) Find a potential function for $\mathbf{F}$.

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parametrized by $\mathbf{r}(t) = \langle t, \sin t \rangle$ and $t \in [0, 1]$

$r(0) = \langle 0, 0 \rangle$

$r(1) = \langle 1, \sin 1 \rangle$

(a) $Q_x = 1 - e^x \sin y \qquad P_y = -e^x \sin y + 1$

since $Q_x = P_y$ by the component test F is conservative.

$$f = e^x \cos y + xy + 3y + K$$

(b) $f_x = e^x \cos y + y$

$f = \int f_x \, dx = e^x \cos y + xy + C(y)$

$f_y = e^x(-\sin y) + x + C'(y)$

$\cancel{x} - e^x \cancel{\sin y} + 3 = e^x \cancel{(-\sin y)} + \cancel{x} + C'(y)$

$3 = C'(y)$

$3y + K = C(y)$

(c) $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$

$= f(1, \sin 1) - f(0,0)$

$= e^1 \cos(\sin 1) + \sin 1 + 3 \sin 1 - (1 + 0 + 0)$

$= e \cdot \cos(\sin 1) + 4 \sin(1) - 1$

Remark 3.11 (Technique for finding potential function). $F = \langle P, Q \rangle$

1. Integrate $\int P\,dx$ to get $f + c(y)$.
2. Try to solve for $c(y)$
   (a) Differentiate $f + c(y)$ with respect to y and set it equal to Q.
   (b) Solve for $c'(y)$.
   (c) Integrate $c'(y)$ with respect to y to get $c(y)$.

Example 3.12. Consider $\mathbf{F} = \left\langle \overset{P}{\frac{-y}{x^2+y^2}}, \overset{Q}{\frac{x}{x^2+y^2}} \right\rangle$

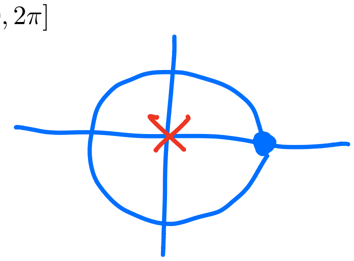
(a) Show that for $\mathbf{F}$ we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\frac{\partial Q}{\partial x} = \frac{-x(2x) + (x^2+y^2)}{(x^2+y^2)^2} = \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{+y(2y) - (x^2+y^2)}{(x^2+y^2)^2} = \frac{+2y^2 - x^2 - y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ where C is a loop parametrized by $r(t) = \langle \cos t, \sin t \rangle, \quad t \in [0, 2\pi]$



Solution.

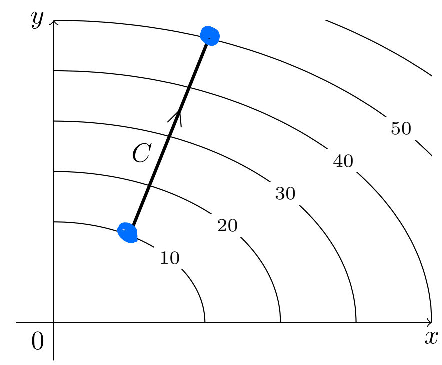
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$
$$= \int_0^{2\pi} \langle \frac{-\sin t}{1}, \frac{\cos t}{1} \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$
$$= \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

(c) Does this contradict **Theorem 3.8?**

No

F is not continuous on simply connected D.

Example 3.13. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



$$= f(B) - f(A)$$
$$= 50 - 10 = 40$$

$P \quad Q$

Example 3.14. Consider the vector field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$

(a) Determine that $\mathbf{F}$ is conservative using the component test.

$$Q_x = 2x$$
$$P_y = 2x$$

So $\vec{F}$ is conservative via the component test.

(b) Find a function f such that $\mathbf{F} = \nabla f$

$$\int 3 + 2xy \, dx = 3x + x^2 y + C(y)$$
$$f_y = x^2 + C'(y)$$
$$\cancel{x^2} - 3y^2 = \cancel{x^2} + C'(y)$$
$$-y^3 = C(y)$$

$$f = 3x + x^2 y - y^3$$

(c) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$, $0 \le t \le \pi$.

$$= f(B) - f(A)$$

$$r(0) = \langle 1 \cdot 0, 1 \cdot 1 \rangle = \langle 0, 1 \rangle = A$$
$$r(\pi) = \langle 0, -e^{\pi} \rangle = B$$

$$= -(-e^{\pi})^3 - (-1)$$
$$= \boxed{e^{3\pi} + 1}$$

Note: We currently do not have the correct theory in place to show that a vector field of 3 variables is conservative. However if we assume it is conservative then we can find potential functions. Here is an example from the book (on page 1104) of how to do this for 3 variables. Please read this through if you have trouble with your WeBWorK homework assignment.

V **EXAMPLE 5** If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + e^{3z})\,\mathbf{j} + 3ye^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

SOLUTION If there is such a function f, then

$$f_x(x, y, z) = y^2 \tag{11}$$

$$f_y(x, y, z) = 2xy + e^{3z} \tag{12}$$

$$f_z(x, y, z) = 3ye^{3z} \tag{13}$$

Integrating $\boxed{11}$ with respect to x, we get

$$f(x, y, z) = xy^2 + g(y, z) \tag{14}$$

where $g(y, z)$ is a constant with respect to x. Then differentiating $\boxed{14}$ with respect to y, we have

$$f_y(x, y, z) = 2xy + g_y(y, z)$$

and comparison with $\boxed{12}$ gives

$$g_y(y, z) = e^{3z}$$

Thus $g(y, z) = ye^{3z} + h(z)$ and we rewrite $\boxed{14}$ as

$$f(x, y, z) = xy^2 + ye^{3z} + h(z)$$

Finally, differentiating with respect to z and comparing with $\boxed{13}$, we obtain $h'(z) = 0$ and therefore $h(z) = K$, a constant. The desired function is

$$f(x, y, z) = xy^2 + ye^{3z} + K$$

It is easily verified that $\nabla f = \mathbf{F}$.

4.2 Practice Turning Line Integrals into Double Integrals - During Class

Objective(s):

- Use Green's Theorem
- Use Green's Theorem even more!

Let's verify Green's Theorem with a relatively easy example.

Example 4.3. Calculate $\int_C \langle y, -x\rangle \cdot d\mathbf{r}$ where C is the unit circle oriented counter-clockwise.

(a) By evaluating the line integral

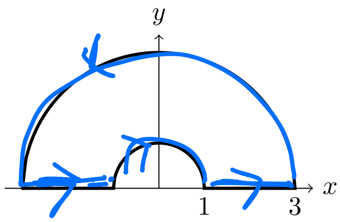
(b) By evaluating the double integral from Green's Theorem.

$$\iint_D -1-1\, dA = -2\iint_D 1\, dA$$

$$= -2(\pi 1^2) = \boxed{-2\pi}$$

Here both ways were relatively easy. Now let's witness the power of Green's Theorem

Example 4.4. Find the work done by $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ once counterclockwise around the curve given by the picture:



Solution. Let's pretend we forgot Green's Theorem on the exam.

To parametrize this curve correctly I need to break it into 4 pieces

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{BC} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{TC} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{LL} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{RL} \mathbf{F}\cdot\mathbf{T}\,ds$$

Parametrizing the four pieces we see that (in a counterclockwise direction)

$BC:$	$\mathbf{r}(t) = \langle \cos t, \sin t \rangle$	$t \in [\pi, 0]$	$\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$
$TC:$	$\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$	$t \in [0, \pi]$	$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t \rangle$
$LL:$	$\mathbf{r}(t) = \langle t, 0 \rangle$	$t \in [1, 3]$	$\mathbf{r}'(t) = \langle 1, 0 \rangle$
$RL:$	$\mathbf{r}(t) = \langle t, 0 \rangle$	$t \in [-3, -1]$	$\mathbf{r}'(t) = \langle 1, 0 \rangle$

Let's calculate these individual integrals

$$\begin{aligned}\int_{BC} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_\pi^0 \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\;dt \\ &= \int_\pi^0 \langle 4(\cos t) - 2(\sin t), 2(\cos t) - 4(\sin t) \rangle \cdot \langle -\sin t, \cos t \rangle\;dt \\ &= \int_\pi^0 -4\sin t\cos t + 2\sin^2 t + 2\cos^2 t - 4\sin t\cos t\;dt \\ &= \int_\pi^0 -8\sin t\cos t + 2\;dt \\ &= 2(0 - \pi) = -2\pi\end{aligned}$$

$$\begin{aligned}\int_{TC} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^\pi \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\;dt \\ &= \int_0^\pi \langle 4(3\cos t) - 2(3\sin t), 2(3\cos t) - 4(3\sin t) \rangle \cdot \langle -3\sin t, 3\cos t \rangle\;dt \\ &= \int_0^\pi -36\sin t\cos t + 18\sin^2 t + 18\cos^2 t - 36\sin t\cos t\;dt \\ &= \int_0^\pi -72\sin t\cos t + 18\;dt \\ &= 18(\pi - 0) = 18\pi\end{aligned}$$

$$\begin{aligned}\int_{LL} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{-3}^{-1} \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\;dt \\ &= \int_{-3}^{-1} \langle 4(t) - 2(0), 2(t) - 4(0) \rangle \cdot \langle 1, 0 \rangle\;dt \\ &= \int_{-3}^{-1} 4t\;dt \\ &= \left[2t^2\right]_{-3}^{-1} = 2(1 - 9) = -16\end{aligned}$$

$$\begin{aligned}\int_{RL} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_1^3 \langle 4x - 2y, 2x - 4y \rangle \cdot \mathbf{r}'(t)\;dt \\ &= \int_1^3 \langle 4(t) - 2(0), 2(t) - 4(0) \rangle \cdot \langle 1, 0 \rangle\;dt \\ &= \int_1^3 4t\;dt \\ &= \left[2t^2\right]_1^3 = 2(9 - 1) = 16\end{aligned}$$

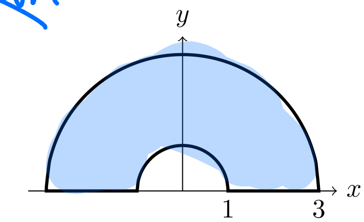
Giving us our final answer of

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = -2\pi + 18\pi - 16 + 16 = 16\pi$$

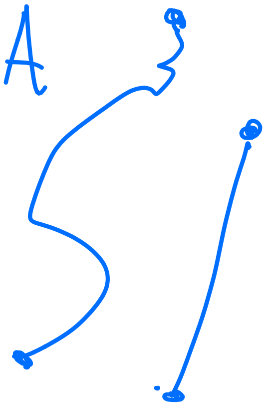
Now let's imagine you remember Green's Theorem.

$\cancel{\iint_D 4x\, dA}$

Example 4.4. Find the work done by $\mathbf{F} = \langle 4x - 2y, 2x - 4y \rangle$ once counterclockwise around the curve given by the picture:



$$\int_C F \cdot T\,ds = \iint_D 2+2\, dA = 4 \iint_D 1\, dA$$



$$= 4 \int_0^{\pi} \int_1^3 r\, dr\, d\theta$$

$$= 4\pi \left(\frac{r^2}{2} \right)_1^3 = 4\pi \left(\frac{9}{2} - \frac{1}{2} \right)$$

$$= 16\pi$$

Notation 4.5.

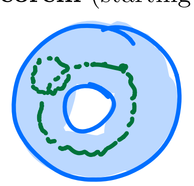
(a) The notation $\oint_C P\,dx + Q\,dy$

Is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

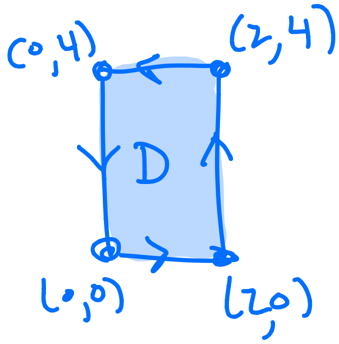
(b) Another notation for the positively oriented boundary curve of a region D is ∂D.

Fun Reads

There is additional material in 16.4 that is covered in the book that MSU will not currently be testing on. Those wishing to gain a greater understanding of the power of Green's Theorem may wish to read the section on finding area using line integrals (top of page 1111) and the section on **Extended Versions of Green's Theorem** (starting on page 1111).



Example 4.6. Use Green's Theorem to evaluate the line integral $\oint_C \underbrace{4\cos(-y)}_{P}\, dx + \underbrace{4x^2\sin(-y)}_{Q}\, dy$.
Where C is the rectangle with vertices $(0,0), (2,0), (0,4)$, and $(2,4)$.



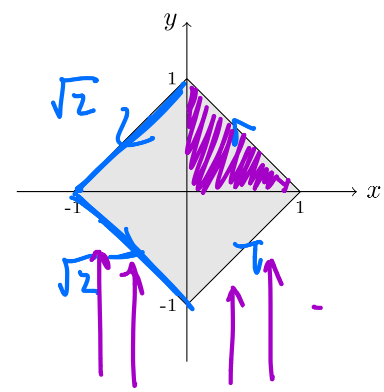
$$\iint_D 8x\sin(-y) - 4\sin(-y)\, dA$$

$$= \int_0^2\int_0^4 (8x-4)\sin(-y)\, dy\, dx$$

$$= \left[4x^2-4x\right]_0^2 \cdot \left[+\cos(-y)\right]_0^4$$

$$= (16-8)\cdot(\cos(-4)-1) = 8(\cos(-4)-1)$$

Example 4.7. Calculate $\oint_C \underbrace{(x^4+2y)}_{P}dx + \underbrace{(5x+\sin y)}_{Q}dy$ where C is the boundary of region shown to the right:



$$= \iint_D 5-2\, dA = 3\iint_D \overset{x}{1}\, dA$$

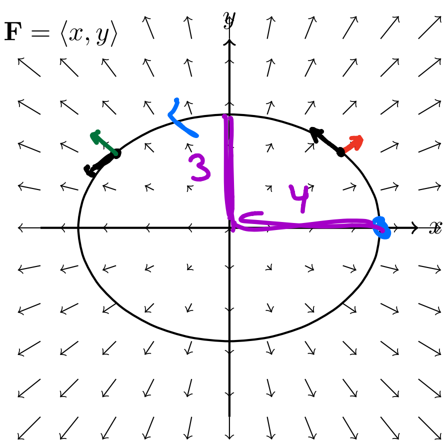
$$= 3\left(\sqrt{2}\right)^2$$

$$= 6$$

Alt:

$$3\int_{-1}^{1}\int_{-1}^{1} 1\, dy\, dx$$

Example 4.8. Consider the curve $C: \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ to the right.



(a) Use only the picture to hypothesize an answer to $\oint_C \langle x, y\rangle \cdot \mathbf{T}\, ds$. $= 0$

neither helping nor hurting

(b) Confirm your suspicions by evaluating $\oint_C \langle x, y\rangle \cdot \mathbf{T}\, ds$.

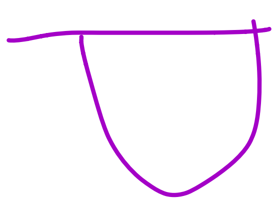
$$\iint 0 - 0 \, dA = 0$$

Area of ellipse

$$A = \pi \cdot a \cdot b$$

where a is the x-radius
b is the y-radius

$$\iint_D 5 \, dA = 5 \cdot \pi (3)(4) = 60\pi$$



$$\iint_D x \, dA$$

5.2 More Curl and More Divergence - During Class

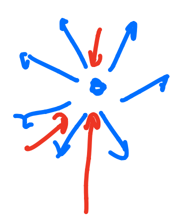
Objective(s):

- Get more practice finding potential functions
- Learn a carnival trick
- Define a few more things

Great but what are these things?

Remark 5.7. Divergence is a number that counts how much is coming in or out of a point.

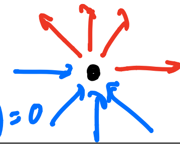
div $F(P) > 0$



(a) Positive means more is going out than in.

(b) Negative means more is going in than out.

(c) 0 means Same in & out.



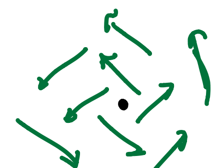
div $F(P) = 0$

Remark 5.8. Curl is a collection of numbers that counts how much is rotating around a point. ⟨ , , ⟩

(a) Positive means rotation in counter-clockwise direction.

(b) Negative means rotation in clockwise direction.

(c) 0 means no rotation.



curl F > 0

piece

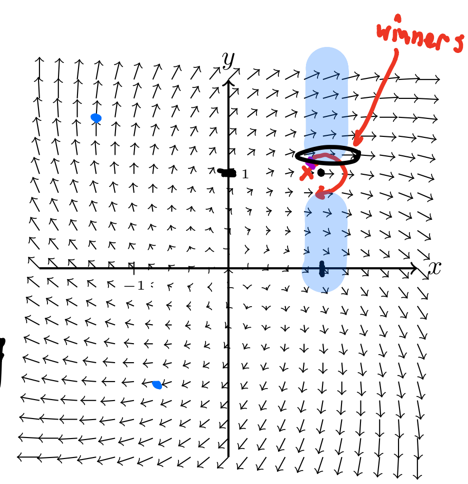
Example 5.9. Consider the picture of $\mathbf{F} = \langle x+y, y-x \rangle$ to the right. Use the picture to determine if the following values should be positive, negative, or 0. Then use the equation of the vector field to verify your guess.

(a) div **F** at $(1,1)$.

div F = 1 + 1 = 2

more is coming out than in

div F (P) > 0



(b) curl **F** at $(1,1)$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & y-x & 0 \end{vmatrix}$$

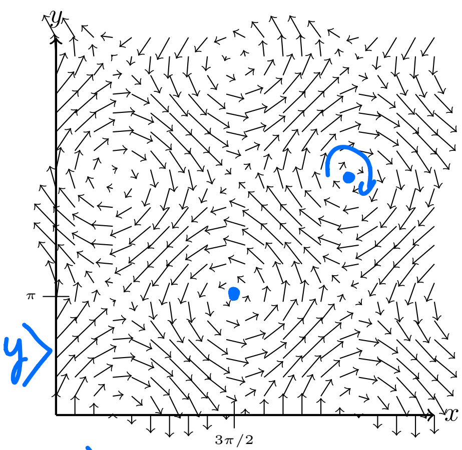
$= (0-0)i$
$-(0-0)j$
$+(-1-1)k$
$= \langle 0, 0, -2 \rangle$

xy-plane rotation

longer vectors are on top of the point trying to spin in clockwise direction

curl F (P) < 0

Example 5.10. Consider the picture of $\mathbf{F} = \langle \sin(y), \cos(x) \rangle$ to the right. Use the picture to determine if the following values should be positive, negative, or 0. Then use the equation of the vector field to verify your guess.



(a) $\nabla \cdot \mathbf{F}$ at $(3\pi/2, \pi)$.

div F = 0

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle \sin(y), \cos(x) \rangle = \frac{\partial}{\partial x}(\sin y) + \frac{\partial}{\partial y}(\cos(x)) = 0 + 0 = 0$$

(b) $\nabla \times \mathbf{F}$ at $(3\pi/2, \pi)$.

Curl F = $\langle 0, 0, + \rangle$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & \cos x & 0 \end{vmatrix} = \begin{matrix} (0)i \\ -(0)j \\ +(-\sin x - \cos y)k \end{matrix} = \langle 0, 0, -\sin x - \cos y \rangle$$

at $\left(\frac{3\pi}{2}, \pi\right)$ is $\langle 0, 0, 1+1 \rangle$

Definition(s) 5.11.

(a) Curl helps to measure rotations about a point. Because of this if curl $\mathbf{F} = 0$ at a point P then the fluid is free from rotations at P and $\mathbf{F}$ is called irotational at P.

(b) Div (Divergence) represents the net rate of change (with respect to time) of a mass of fluid (or gas) flowing from the point (x, y, z). If div $\mathbf{F} = 0$ then there is no net change and $\mathbf{F}$ is said to be incompessible.

Example 5.12. Let $\mathbf{F} = -2x\mathbf{i} - 3y\mathbf{j} + 5z\mathbf{k}$. Is $\mathbf{F}$ irrotational/incompressible/both/neither?

$$\text{div } F = -2 - 3 + 5 = 0$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x & -3y & 5z \end{vmatrix} = \begin{matrix} (0-0)i \\ -(0-0)j \\ +(0-0)k \end{matrix} = \langle 0, 0, 0 \rangle$$

Theorem 5.13. If $\mathbf{F} = \langle P, Q, R \rangle$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div}(\text{curl } F) = 0$$

Note: Proof of this is in the book on page 1119. It is very boring and not at all enlightening. It works because of Clairaut's Theorem.

Example 5.14. Your friend Eugene comes up to you and is like "*Whoa you have to check out my awesome vector field* $\boldsymbol{F}$*. You know what? I bet you can't even figure out what it is. The only thing I'll tell you is that curl* $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$." Shut Eugene up by finding his vector field if it exists or prove that Eugene is a liar.

$$\text{div}(\langle xz, xyz, -y^2 \rangle)$$

$$= z + xz + 0$$

Not alway 0.

Eugene is a liar.

Example 5.15. Consider the vector field $\mathbf{F} = \langle yz + \overset{3}{\alpha} y + 1, xz + 3x + \overset{-1}{\beta} z + 2, xy - y\rangle$.

(a) Find values for constants α and β so that $\mathbf{F}$ is conservative.

$\alpha = 3$
$\beta = -1$

$\operatorname{curl} F = \vec{0}$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+\alpha y+1 & xz+3x+\beta z+2 & xy-y \end{vmatrix} = \langle 0,0,0\rangle$$

$$(x-1-(x+\beta))\,i - (y-(y))\,j + (z+3-(z+\alpha))\,k = \langle -1-\beta, 0, 3-\alpha\rangle = \langle 0,0,0\rangle$$

(b) Using your answers from (a) evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is your favorite curve from $\underset{A}{(1,0,0)}$ to $\underset{B}{(1,2,1)}$.

$$= \int_a^b F(r(t))\cdot r'(t)\,dt = \underline{f(B)} - \underline{f(A)}$$

$$\int yz+3y+1\,dx = xyz+3xy+x+C(y,z)$$

$$\cancel{xz}+\cancel{3x}-z+2 = \cancel{xz}+\cancel{3x}+C_y(y,z)$$

$$-yz+2y+c(z) = C(y,z)$$

$$f = xyz+3xy+x-yz+2y+\cancel{c(z)}\ K$$

$$f_z = xy-y+c'(z)$$

$$\cancel{xy}-\cancel{y} = \cancel{xy}-\cancel{y}+c'(z)$$

$$0 = c'(z)$$

$$K = c(z)$$

$$f(1,2,1) = 2+6+1-2+4 = 11$$

$$f(1,0,0) = 1$$

$$11-1 = 10$$

5.3 Alternate Forms of Green's Theorem – During Class

Objective(s):

- View Green's Theorem in a few different lights

Two ideas here that will be used later. Both have to do with downgrading curl and divergence to 2 dimensions for a minute. That is take $\mathbf{F} = \langle P, Q, 0 \rangle$

Theorem 5.16 (Green's Theorem)**.** Bunch of conditions up here.

$\langle \sim, \sim, Q_x - P_y \rangle$ $\qquad \langle 0, 0, 1 \rangle$

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

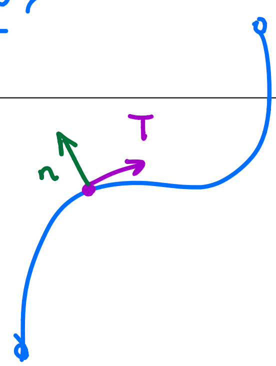
$$= \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA$$

And while this may be annoying to write right now, it is the first good step in expanding Green's Theorem to 3 dimensions and discovering Stokes' Theorem (16.8).

The second idea is instead of choosing to do the line integrals $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ instead evaluating $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ where $\mathbf{n}$ is a outward pointing unit normal vector to C.

Theorem 5.17. An outward pointing unit normal vector to a curve C parametrized counterclockwise by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is given by:

$$\vec{T} = \frac{\langle x'(t), y'(t) \rangle}{|r'(t)|} \qquad \vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{|r'(t)|}$$



Theorem 5.18 (Green's Theorem Alternate)**.** Bunch of conditions up here.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

$$= \iint_D \operatorname{div} \mathbf{F}\, dA$$

A proof of this can be found on page 1120. This will help us expand into 3 dimensions for Divergence Theorem (16.9).

6.3 More Parametric Surfaces - During Class

Objective(s):

- Understand the subtleties of parameterizing surfaces
- Practice parameterizing more surfaces!

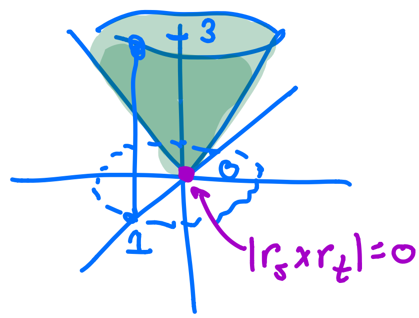
$z = 3r = 3s$

Example 6.6. Find a parametrization $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ of the following: $z = 3\sqrt{x^2+y^2}$ $\quad z \in [0,3]$

$r(s,t) = \langle s\cdot\cos t,\ s\cdot\sin t,\ 3s\rangle$

$t \in [0, 2\pi]$

$s \in [0, 1]$



$\rho^2 = 9 \Rightarrow \rho = 3$

Example 6.7. Find a parametrization $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ of the following: $x^2+y^2+z^2 = 9$

$\langle \rho\sin\phi\cos\theta,\quad ,\ \rho\cos\phi$

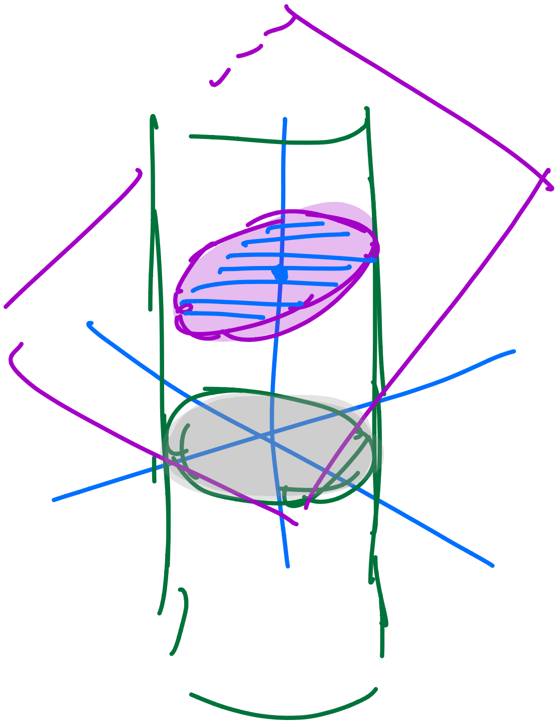
$\langle 3\cdot\sin s\cdot\cos t,\ 3\sin s\sin t,\ 3\cos s\rangle$

$t \in [0, 2\pi]$

$s \in [0, \pi]$

Remark 6.8. If you are doing things right then often <u>your limits</u> for your independent variables s, t (or u, v) should not depend on one another.

Example 6.9. Parametrize the portion of the tilted plane $x - y + 3z = 5$ that lies inside of the cylinder $x^2 + y^2 = 4$.



$z = \frac{5-x+y}{3}$

$\vec{r}(s,t) = \langle s\cdot\cos t, s\cdot\sin t, \frac{5-s\cos t+s\sin t}{3}$

$s \in [0, 2]$

$t \in [0, 2\pi]$

Remark 6.10. Compare this problem to the 6.5 video problem. These two problems together should teach us that it isn't just about the surface, it's also the area we the surface is over.

Definition(s) 6.11. A parametrized surface $\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$ is smooth if r_s & r_t are continuous and $|r_s \times r_t|$ is never zero on the interior of the parameter domain.

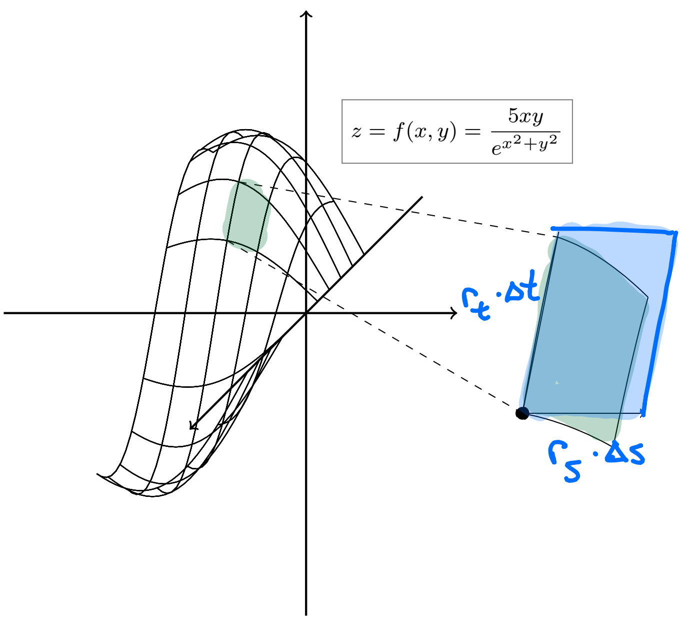
Note: our parametrization of the cone IS smooth.

6.4 Surface Area of Parametric Surfaces - During Class

Objective(s):

- Learn how to calculate surface area over parametric surfaces
- Practice calculating more surface area!

Now our goal is to find and area equation for smooth parametrized surfaces. The idea is as follows



If we chop our surface into lots of small rectangles that have side lengths $\underline{\Delta s}$ and $\underline{\Delta t}$ (where s, t parametrize the surface) then the area of the small surface piece is about the area of the parallelogram made by the vectors $\underline{\vec{r}_s \cdot \Delta s}$ and $\underline{\vec{r}_t \cdot \Delta t}$

Then we can sum all these areas together to get the summation:

$\sum$ Areas of //-ograms

$\sum |\vec{r}_s \cdot \Delta s \times r_t \cdot \Delta t|$

$\sum |r_s \times r_t| \Delta s \, \Delta t$

Since $|\mathbf{r}_s \times \mathbf{r}_t|$ is continuous we know that

$$\text{Surface Area } = \sum_n |\mathbf{r}_s \times \mathbf{r}_t| \triangle t \triangle s = \iint_S |r_s \times r_t| \, dt \, ds$$

Theorem 6.12. The **area** of a smooth surface $\mathbf{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ with $s \in [a,b]$ and $t \in [c,d]$ is:

$$\text{Area } = \int_a^b \int_c^d |r_s \times r_t| \, dt \, ds$$

Or the book writes:

Theorem 6.12. The **area** of a smooth surface $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ with $a \le u \le b$ and $c \le v \le d$ is:

$$\text{Area} \;=\; \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

Remark 6.13. Note we can use this theorem along with the parametrization $\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle$ to prove the 15.6 formula for surface area.

There is still much to cover so lets talk about the following worked out example:

Example 6.14. Find the area of the portion of the tilted plane $x - y + 3z = 5$ that lies inside of the cylinder $x^2 + y^2 = 4$.

Solution. From Example 5.2 we have that the tilted plane is parametrized by $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, \frac{5 + s\sin t - s\cos t}{3}\rangle$ where $s \in [0,2]$ and $t \in [0, 2\pi]$

So we first need to calculate out:

$$\begin{aligned}
\mathbf{r}_s &= \langle \cos t, \sin t, \frac{\sin t - \cos t}{3}\rangle \\
\mathbf{r}_t &= \langle -s\sin t, s\cos t, \frac{s\cos t + s\sin t}{3}\rangle
\end{aligned}$$

Now we can calculate $|\mathbf{r}_s \times \mathbf{r}_t|$

$$\begin{aligned}
|\mathbf{r}_s \times \mathbf{r}_t| &= \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & \frac{\sin t - \cos t}{3} \\ -s\sin t & s\cos t & \frac{s\cos t + s\sin t}{3} \end{matrix} \right\| \\
&= \left| (\sin t \frac{s\cos t + s\sin t}{3} - s\cos t \frac{\sin t - \cos t}{3})\mathbf{i} - (\cos t \frac{s\cos t + s\sin t}{3} + s\sin t \frac{\sin t - \cos t}{3})\mathbf{j} + (s\cos^2 t + s\sin^2 t)\mathbf{k} \right| \\
&= \left| (\frac{s\sin\cos t + s\sin^2 t}{3} + \frac{-s\sin t\cos t + s\cos^2 t}{3})\mathbf{i} - (\frac{s\cos^2 t + s\sin t\cos t}{3} + \frac{s\sin^2 t - s\sin t\cos t}{3})\mathbf{j} + (s)\mathbf{k} \right| \\
&= \left| (\frac{s\sin^2 t}{3} + \frac{s\cos^2 t}{3})\mathbf{i} - (\frac{s\cos^2 t}{3} + \frac{s\sin^2 t}{3})\mathbf{j} + (s)\mathbf{k} \right| \\
&= \left| (\frac{s}{3})\mathbf{i} - (\frac{s}{3})\mathbf{j} + (s)\mathbf{k} \right| \\
&= \sqrt{\left(\tfrac{s}{3}\right)^2 + \left(\tfrac{-s}{3}\right)^2 + s^2} \\
&= \sqrt{\tfrac{s^2}{9} + \tfrac{s^2}{9} + \tfrac{9s^2}{9}} = \sqrt{\tfrac{11s^2}{9}} = \frac{s}{3}\sqrt{11}
\end{aligned}$$

Integrating finally we get:

$$\begin{aligned}
\int_0^{2\pi}\int_0^2 |\mathbf{r}_s\times\mathbf{r}_t|\ ds\ dt &= \int_0^{2\pi}\int_0^2 \frac{s}{3}\sqrt{11}\ ds\ dt \\
&= \left[\int_0^{2\pi} 1\ dt\right]\left[\int_0^2 \frac{s}{3}\sqrt{11}\ ds\right] \\
&= [2\pi]\left[\frac{s^2}{6}\sqrt{11}\right]_0^2 \\
&= 2\pi\frac{4}{6}\sqrt{11} \\
&= \boxed{\frac{4\pi}{3}\sqrt{11}}
\end{aligned}$$

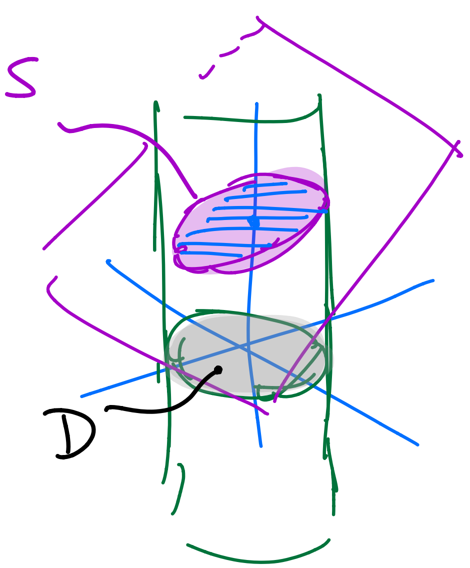
Intuitively a slanted circle like this should have more area than a non-slanted circle in the cylinder so we could check:

$$\underline{\pi(2^2)} = 4\pi < \underline{\frac{4\pi}{3}\sqrt{11}}$$

Since $\sqrt{11} > 3$. So our answer seems reasonable!

Remember: unless the problem specifies you can use 15.6 to help your evaluation. $\; // \; \frac{5}{3} - \frac{1}{3}x + \frac{1}{3}y$

Example 6.15. Find the area of the portion of the tilted plane $z = \dfrac{5-x+y}{3}$ (Look familiar?... $x - y + 3z = 5$) that lies inside of the cylinder $x^2 + y^2 = 4$



$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\ dA$$

$$= \iint_D \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 1}\ dA$$

$$= \iint_D \sqrt{\frac{11}{9}}\ dA = \iint_D \frac{\sqrt{11}}{3}\ dA$$

$$= \int_0^{2\pi}\int_0^2 \frac{\sqrt{11}}{3}\, r\ dr\, d\theta$$

$$= 2\pi\,\frac{\sqrt{11}}{3}\,\frac{4}{2} = 4\pi\frac{\sqrt{11}}{3}$$

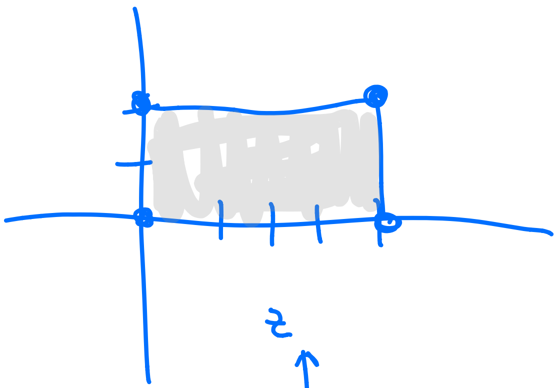
Example 6.16. Find the area of the part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices $(0, 0), (4, 0), (0, 2)$, and $(4, 2)$.

$z^2 = 9 - y^2$

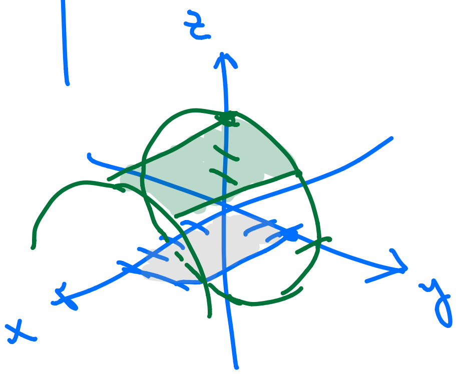
$z = \sqrt{9-y^2}$

$f_x = 0$

$f_y = \cancel{\frac{1}{2}}(9-y^2)^{-1/2}(-\cancel{2}y)$



$$\int_0^4\int_0^2 \sqrt{0^2 + \frac{y^2}{9-y^2} + 1}\, dy\, dx$$



$$4\int_0^2 \sqrt{\frac{\cancel{y^2}}{9-y^2} + \frac{9-\cancel{y^2}}{9-y^2}}\, dy = 4\int_0^2 \sqrt{\frac{9}{9-y^2}}\, dy$$

$$= 4\int_0^2 \frac{3}{\sqrt{9-y^2}}\, dy = 12\left[\sin^{-1}\left(\frac{y}{3}\right)\right]_0^2 = 12\sin^{-1}\left(\frac{2}{3}\right)$$

Example 6.17. Express that area of the surface $z = e^{-x^2-y^2}$ that lies above the disk $x^2 + y^2 \leq 4$ in terms of a single integral. Do not evaluate.

$f_x = e^{-x^2-y^2}(-2x)$

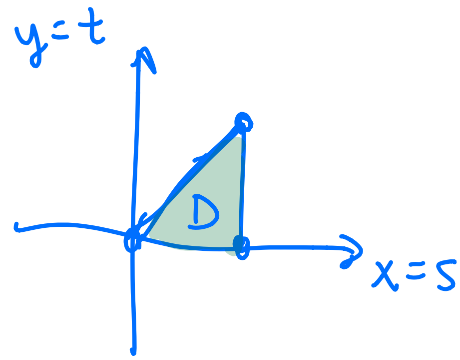
$f_y = e^{-x^2-y^2}(-2y)$

$$A(S) = \iint_D \sqrt{4x^2\left(e^{-x^2-y^2}\right)^2 + 4y^2\left(e^{-x^2-y^2}\right)^2 + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2e^{-2r^2} + 1}\; r\, dr\, d\theta$$

$$= 2\pi\int_0^2 r\sqrt{4r^2e^{-2r^2} + 1}\, dr$$

Example 6.18. Consider the surface S, given by $z = xy^2$, over the triangle with vertices $(0,0),(1,0)$, and $(1,1)$ in the $xy-$plane.



(a) Give a parametrization $\mathbf{r}(s,t)$ of the surface S.

$$r(s,t) = \langle s, t, st^2 \rangle$$

$$t \in [0, s]$$
$$s \in [0, 1]$$

(b) Use your parametrization in (a) to express the surface area as a double integral. Simplify as much as possible without evaluating the integrals.

$$r_s = \langle 1, 0, t^2 \rangle$$

$$r_t = \langle 0, 1, 2st \rangle$$

$$r_s \times r_t = \begin{vmatrix} i & j & k \\ 1 & 0 & t^2 \\ 0 & 1 & 2st \end{vmatrix} = (-t^2)i - (2st)j + (1)k$$

$$= \langle -t^2, -2st, 1 \rangle$$

$$|r_s \times r_t| = \sqrt{t^4 + 4s^2t^2 + 1}$$

$$A(S) = \int_0^1 \int_0^s \sqrt{t^4 + 4s^2t^2 + 1} \, dt \, ds$$

Here are some additional videos that may come in useful (they are hyper links so just click on them on your computer).

Video 1: https://www.khanacademy.org/math/calculus/multivariable-calculus/surface_parametrization/v/introduction-to-parametrizing-a-surface-with-two-parameters

Video 2: https://www.khanacademy.org/math/calculus/multivariable-calculus/surface_parametrization/v/determining-a-position-vector-valued-function-for-a-parametrization-of-two-parameters

7.2 Surface Integrals with Scalar Functions - During Class

Objective(s):

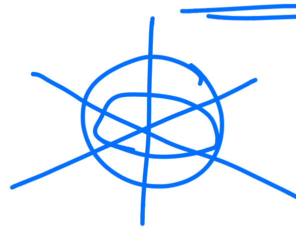
- More surface integrals with scalar functions.

$\rho = 1$

$\rho^2 = 1$

Example 7.5. Compute the surface integral $\iint_S z^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

(a) by expressing the surface parametrically.



$x = \rho \sin\phi \cos\theta$

$\quad 1 \sin s \cos t$

$y = 1 \sin s \sin t$

$z = 1 \cos s$

$t \in [0, 2\pi]$

$s \in [0, \pi]$

$r(s,t) = \langle \sin s \cos t, \sin s \sin t, \cos s \rangle$

$$\iint (\cos s)^2 \, |r_s \times r_t| \, ds\, dt$$

$$\int_0^{2\pi}\int_0^{\pi} (\cos s)^2 \sin s \, ds\, dt$$

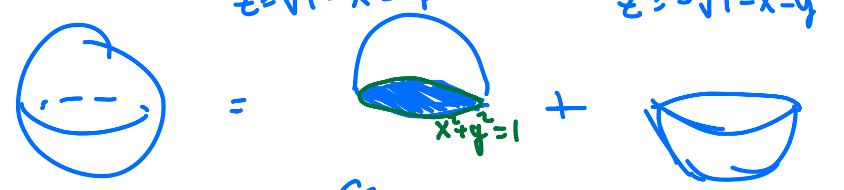
$$= 2\pi \left(-\frac{\cos^3 s}{3} \right)_0^{\pi} = \frac{-2\pi}{3}(-1-1)$$

$$= \frac{4\pi}{3}$$

$x^2+y^2+z^2=1$

(b) by carefully ripping the surface into two surfaces that can be expresses explicitly.

$z=\sqrt{1-x^2-y^2}$ $\qquad$ $z=-\sqrt{1-x^2-y^2}$



$$\iint_S z^2\,dS \qquad \iint_{S_1} z^2\,dS_1 + \iint_{S_2} z^2\,dS_2$$

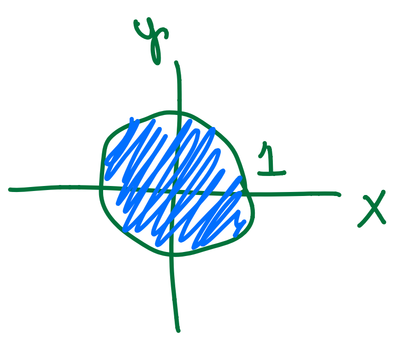
On $z=\sqrt{1-x^2-y^2}$

$$\iint_{S_1} z^2\,dS_1 = \iint (1-x^2-y^2)\sqrt{g_x^2+g_y^2+1}\;dy\,dx$$

$$g_x = \cancel{\tfrac{1}{2}}(1-x^2-y^2)^{-1/2}(-\cancel{2}x) \qquad g_y = \cancel{\tfrac{1}{2}}(1-x^2-y^2)^{-1/2}(-\cancel{2}y)$$

$$g_x^2 = \frac{x^2}{1-x^2-y^2} \qquad g_y^2 = \frac{y^2}{1-x^2-y^2}$$

$$= \iint (1-x^2-y^2)\sqrt{\frac{\cancel{x^2}}{1-x^2-y^2}+\frac{\cancel{y^2}}{1-x^2-y^2}+\frac{1-\cancel{x^2}-\cancel{y^2}}{1-x^2-y^2}}\;dy\,dx$$



$$= \iint \cancel{\sqrt{(1-x^2-y^2)}}\,\frac{1}{\cancel{\sqrt{1-x^2-y^2}}}\;dy\,dx$$

$$= \iint \sqrt{1-x^2-y^2}\;dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\;r\,dr\,d\theta$$

$$= 2\pi\left[(1-r^2)^{3/2}\,\frac{\cancel{2}}{3}\cdot\frac{1}{-\cancel{2}}\right]_0^1$$

$$= \frac{2\pi}{-3}(0-1) = \frac{2\pi}{3}$$

Same for S_2

So answer is $\frac{2\pi}{3}+\frac{2\pi}{3} = \frac{4\pi}{3}$

Example 7.6. Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $x \in [0,1]$, $y \in [0,2]$.

$g_x^2 + g_y^2 + 1$

$g_x = 1 \quad g_x^2 = 1$

$g_y = 2y \quad g_y^2 = 4y^2$

$$\int_0^1\int_0^2 y \sqrt{1 + 4y^2 + 1}\, dy\, dx$$

$$\int_0^1\int_0^2 y \sqrt{2 + 4y^2}\, dy\, dx$$

$$= 1 \cdot \int_0^2 y\sqrt{2+4y^2}\, dy$$

$u = 2 + 4y^2$

$du = 8y\, dy$

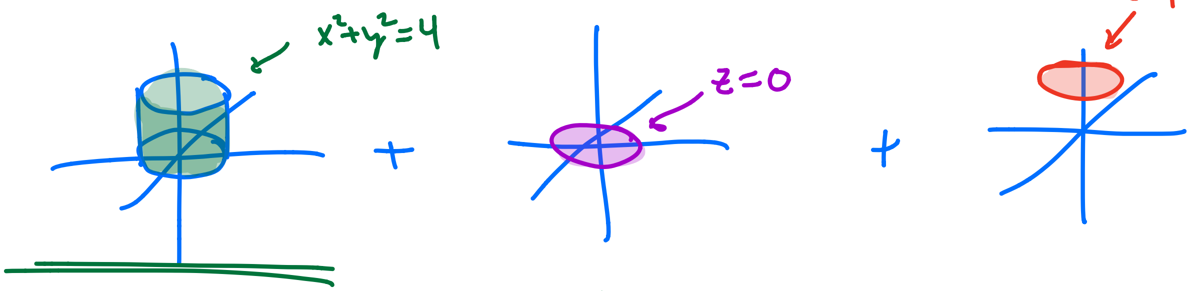
$\frac{du}{8} = y\, dy$

$$= 1 \cdot \int_2^{18} \sqrt{u}\, \frac{du}{8}$$

$$= \frac{1}{8}\left(\frac{2}{3}u\right)^{3/2}\Bigg|_2^{18}$$

$$= \frac{1}{12}\left(18^{3/2} - 2^{3/2}\right)$$

Example 7.7. Evaluate: $\displaystyle\iint_S (x^2+y^2+z^2)\,dS$ where S is the part of the cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=1$, together with its top and bottom disks.



$r(s,t) = \langle 2\cos t, 2\sin t, s\rangle$ $\qquad r_s = \langle 0,0,1\rangle$

$s\in[0,1],\ t\in[0,2\pi]$ $\qquad r_t = \langle -2\sin t, 2\cos t, 0\rangle$

$$r_s\times r_t = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -2\sin t & 2\cos t & 0 \end{vmatrix} = \begin{matrix}(-2\cos t)i \\ -(2\sin t)j \\ +(0)k\end{matrix}$$

$$|r_s\times r_t| = \sqrt{4\cos^2 t + 4\sin^2 t + 0} = 2$$

$$\iint x^2+y^2+z^2\,dS$$

$$\iint 4\cos^2 t + 4\sin^2 t + s^2\,dS$$

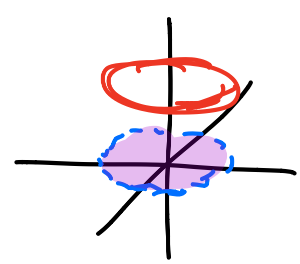
$$\int_0^{2\pi}\int_0^1 4+s^2\,|r_s\times r_t|\,ds\,dt$$

$$= \int_0^{2\pi}\int_0^1 (4+s^2)(2)\,ds\,dt$$

$$2\pi\left[8s+\frac{2}{3}s^3\right]_0^1 = 2\pi\left(8+\frac{2}{3}\right) = \boxed{\frac{52\pi}{3}}$$

$g(x,y) = z = 1$ $\qquad g_x = 0$, $\ g_y = 0$ $\qquad dS = \sqrt{0^2+0^2+1}\,dy\,dx$

$$\iint_S x^2+y^2+1\cdot\sqrt{1}\,dy\,dx$$



$$\iint_D x^2+y^2+1\,dy\,dx$$

$$\int_0^{2\pi}\int_0^2 (r^2+1)\,r\,dr\,d\theta$$

$$2\pi\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^2 = 2\pi\left[4+2\right] = \boxed{12\pi}$$

$\boxed{8\pi}$

$$\frac{52\pi}{3} + 12\pi + 8\pi = \boxed{\frac{112\pi}{3}}$$

7.4 Surface Integrals with Vector Fields - During Class

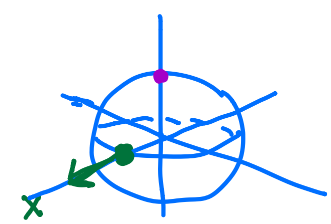
Objective(s):

- Calculate surface integrals of vector fields.
  (a) Given an explicit surface, $z = g(x, y)$.
  (b) Given a parametric surface, $\mathbf{r}(s, t)$.
- Recognize the physical interpretation of the above calculations.

Example 7.12. Find the (outward) flux of vector field $\mathbf{F} = \langle z, y, x\rangle$ across the sphere parametrized by $\mathbf{r}(s,t) = \langle \sin s\cos t, \sin s\sin t, \cos s\rangle$ with $s \in [0,\pi]$, $t \in [0, 2\pi]$. **Hints:** Recall $\mathbf{r}_s \times \mathbf{r}_t = \langle \sin^2 s\cos t, \sin^2 s\sin t, \sin s\cos s\rangle$, $\int_0^\pi \sin^3 u\, du = 4/3$, and $\int_0^{2\pi} \sin^2 u\, du = \pi$.

$s=0$, $t=0$: $\langle 0, 0, 0\rangle$!!

$t=0$, $s=\frac{\pi}{2}$: $\langle 1, 0, 0\rangle$



$t=0$, $s=0$

$t=0$ $s=\pi/2$

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \iint \langle z, y, x\rangle \cdot \pm\frac{(r_s\times r_t)}{\cancel{|r_s\times r_t|}}\, \cancel{|r_s\times r_t|}\, ds\, dt$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle \cos s, \sin s\sin t, \sin s\cos t\rangle \cdot + \langle \sin^2 s\cos t, \sin^2 s\sin t, \sin s\cos s\rangle\, ds\, dt$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos s\sin^2 s\cos t + \sin^3 s\sin^2 t + \sin^2 s\cos s\cos t\, ds\, dt$$

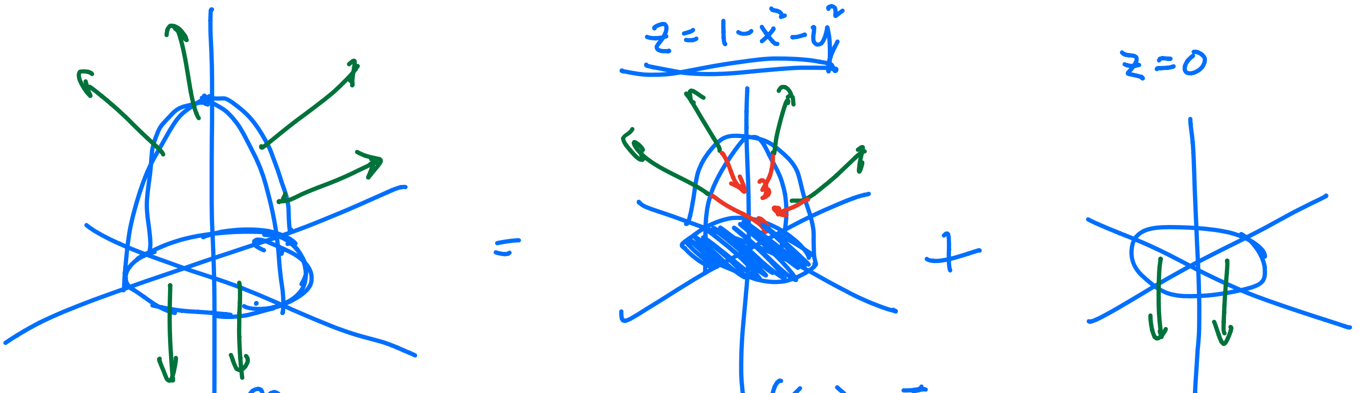
$$= \int_0^{2\pi}\int_0^{\pi} 2\cos s\sin^2 s\cos t + \sin^3 s\sin^2 t\, ds\, dt$$

$$= \int_0^{2\pi} \cancel{2\frac{\sin^3 s}{3}\Big|_0^{\pi}}^{\,0} \cos t + \frac{4}{3}\sin^2 t\, dt = \int_0^{2\pi} \frac{4}{3}\sin^2 t\, dt$$

$$= \frac{4}{3}\pi$$

Remark 7.13. Unless otherwise specified assume that your closed surfaces are always <u>positively oriented</u>.

Example 7.14. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$

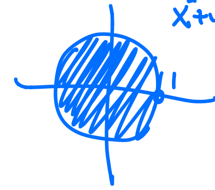


$$\iint_S F\cdot d\vec{S} = \underbrace{\iint_{S_1} \vec{F}\cdot d\vec{S}_1}_{ⓐ} + \underbrace{\iint_{S_2} \vec{F}\cdot d\vec{S}_2}_{ⓑ}$$

ⓑ $$\iint_{S_2} \langle y, x, z\rangle \cdot \pm \frac{\langle g_x, g_y, -1\rangle}{\cancel{\sqrt{g_x^2+g_y^2+1}}} \cancel{\sqrt{g_x^2+g_y^2+1}}\, dy\, dx$$

$0 = z = 1-x^2-y^2$

$x^2+y^2=1$



$$\iint_{S_2} \langle y, x, 0\rangle \cdot \oplus \langle 0,0,-1\rangle\, dy\, dx = \iint 0\, dy\, dx = 0$$

ⓐ $$\iint_{S_1} \langle y, x, 1-x^2-y^2\rangle \cdot \oplus \langle +2x, +2y, +1\rangle\, dy\, dx$$

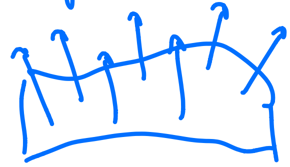
$$\iint 2xy + 2xy + 1 - x^2 - y^2\, dy\, dx = \int_0^{2\pi}\int_0^1 (4r^2\cos\theta\sin\theta + 1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \underbrace{\cancel{4r^3\cos\theta\sin\theta}}_{0}\, dr\, d\theta + \int_0^{2\pi}\int_0^1 r - r^3\, dr\, d\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}$$

Remark 7.15. In the flux definition:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \underline{\mathbf{F} \cdot \mathbf{n}}\, dS$$

We can interpret this calculation as <u>Summing</u> up movement of particles induced by the <u>vector field</u> across the surface S (Hence the <u>$\vec{n}$</u>).



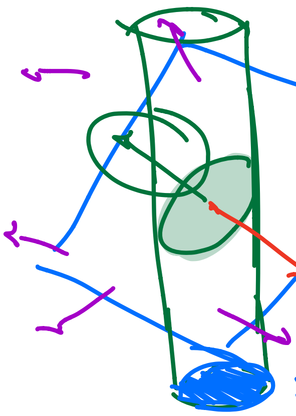
surface

Example 7.16. Find the upward flux of $\mathbf{F} = \langle x, y, z\rangle$ across the portion of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 1$.

$z = 1 - x - y$

bounds

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$



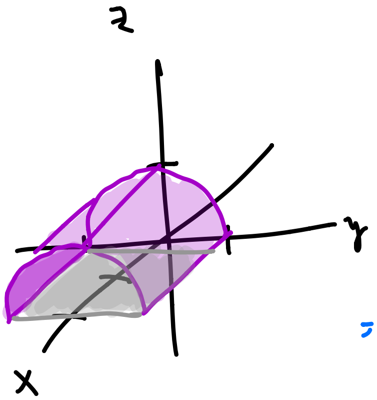
$$= \iint \langle x, y, z\rangle \cdot \langle +1, +1, +1\rangle\, dy\, dx$$

$$= \iint \cancel{x} + \cancel{y} + \overset{1-\cancel{x}-\cancel{y}}{\cancel{z}}\, dy\, dx = \iint 1\, dA$$

$$= \pi(1)^2 = \boxed{\pi}$$

Example 7.17. Evaluate the surface integral: $\iint_S \langle y^3, 4x, z^2\rangle \cdot d\mathbf{S}$
where S is given by cylinder $z^2 + y^2 = 1$ above the xy-plane with positive orientation and $0 \le x \le 2$.

$z \ge 0$ upward



$$z^2 = 1 - y^2$$
$$z = \sqrt{1-y^2}$$

$$= \iint \langle y^3, 4x, z^2\rangle \cdot \pm\left\langle 0, \tfrac{1}{2}(1-y^2)^{-1/2}(-2y), -1\right\rangle \, dy\,dx$$

$$= \iint \langle y^3, 4x, 1-y^2\rangle \cdot \left\langle 0, \frac{y}{\sqrt{1-y^2}}, 1\right\rangle \, dy\,dx$$

$$= \int_0^2 \int_{-1}^1 \frac{4xy}{\sqrt{1-y^2}} + (1-y^2)\, dy\,dx$$

$$= \int_0^2 \left[4x\left(-\sqrt{1-y^2}\right) + y - \tfrac{1}{3}y^3\right]_{-1}^{1} dx$$

$$= \int_0^2 0 + (1+1) - \tfrac{1}{3}(1+1)\, dx$$

$$= \int_0^2 2 - \tfrac{2}{3}\, dx = \tfrac{4}{3}x \Big|_0^2 = \boxed{\tfrac{8}{3}}$$

Question 1

Consider the surface S given by $x = 0$ with $y^2 + z^2 \leq 1$ oriented in the $\mathbf{i}$ direction.

Evaluate $\iint_S \text{curl}(\langle z, x, y \rangle) \cdot d\mathbf{S}$

A. π

B. 2π

C. 3π

D. -3π

E. 0

$$\text{curl}(\langle z,x,y\rangle) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix}$$

$$= (1-0)i - (0-1)j + (1-0)k$$

$$= \langle 1,1,1\rangle$$

$$\iint_S \langle 1,1,1\rangle \cdot n \, dS$$

$$= \iint_S \langle 1,1,1\rangle \cdot \langle 1,0,0\rangle \, dS$$

$$= \iint_S 1 \, dS \leftarrow \text{surface area}$$

$$= \pi(1)^2 = \pi$$

- A
- B
- C
- D
- E

8.2 Stokes' the Two Way Street - During Class

Objective(s):

- Practice turning surface integrals into line integrals
- Practice turning line integrals into surface integrals

$\cdot n \, dS$

Example 8.5. Use Stokes' Theorem to compute the integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy-$plane.

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$r'(t)\, dt$

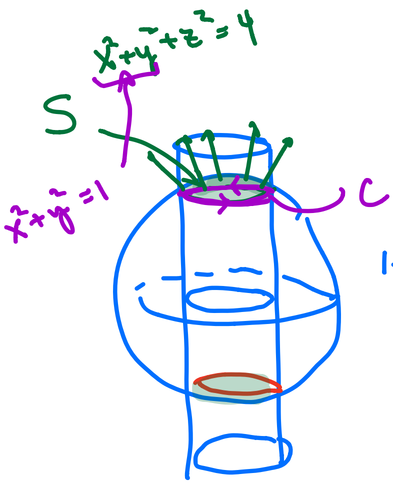
boundary curve

$$= \int_C \langle xz, yz, xy \rangle \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle \cos t \cdot \sqrt{3}, \sin t \cdot \sqrt{3}, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin t \cos t \sqrt{3} + \cos t \sin t \sqrt{3} + 0 \, dt$$

$$= \int_0^{2\pi} 0 \, dt = \boxed{0}.$$



$1 + z^2 = 4$

$z^2 = 3$

$z = \pm\sqrt{3}$

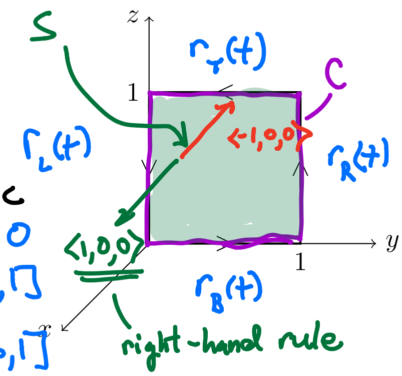
$r(t) = \langle 1\cos t, \sin t, \sqrt{3} \rangle$

$r'(t) = \langle -\sin t, \cos t, 0 \rangle$

Remark 8.6. Sometimes the difficulty in these problems identifying the boundary curve of the surface and making sure your parametrization orients the curve correctly.

Example 8.7. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle y, z, x\rangle$ and C is unit square shown below

$$= \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$



$F = c$

$x = 0$

$y \in [0,1]$

$z \in [0,1]$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= (-1)i - (1)j + (-1)k$$

$$= \langle -1,-1,-1\rangle$$

$$x^2+y^2+z^2 = 9$$

$$F = c$$

$$\vec{n} = \nabla F = \langle 2x, 2y, 2z\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \langle -1,-1,-1\rangle \cdot d\vec{S}$$

$$= \iint_S \langle -1,-1,-1\rangle \cdot \pm\vec{n}\ dS$$

$$= \iint_S \langle -1,-1,-1\rangle \cdot \langle 1,0,0\rangle\ dS$$

$$= \iint_S -1\ dS = -\iint_S 1\ dS = -(1)(1)$$

$$= \boxed{-1}$$

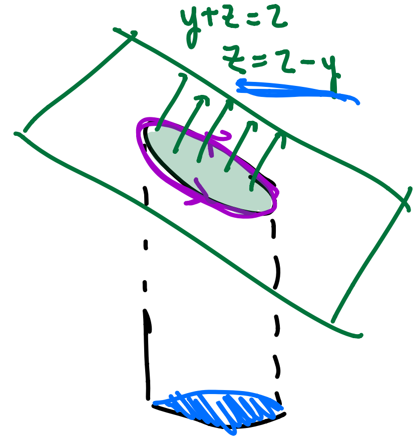
Example 8.8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Note: C is to be oriented counterclockwise when viewed from above.)

$\langle 0,1,1\rangle$

$F = c$

$y+z=2$

$z = 2-y$



$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= (0-0)i - (0-0)j + (1+2y)k$$

$$= \langle 0, 0, 1+2y \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \langle 0,0,1+2y \rangle \cdot d\vec{S}$$

$$= \iint \langle 0,0,1+2y \rangle \cdot -\langle \overset{g_x}{0}, \overset{g_y}{-1}, -1 \rangle \, dy\,dx$$

$$= \iint 1+2y \, dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r \, dr\,d\theta$$

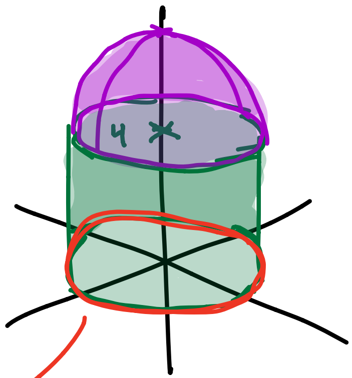
$$= \int_0^{2\pi} \frac{r^2}{2} + \frac{2}{3}r^3\sin\theta \Big|_0^1 d\theta = \frac{1}{2}(2\pi) = \pi$$

Remark 8.9. If we wanted to evaluate the line integral in **Ex 8.7** we would end up integrating: $\int_0^{2\pi} (\sin^3 t + \cos^2 t - 4\cos t + 4\sin t\cos t - \sin^2 t\cos t)dt.$

Remark 8.10. The surface in **Ex 8.7** is not unique. However it is clearly the correct choice.

Example 8.11. Let S be the surface formed by capping the piece of the cylinder $x^2 + y^2 = 2$, $0 \le z \le 4$ with the top half of the sphere $x^2 + y^2 + (z-4)^2 = 2$.

(a) Draw a rough sketch of S.



(b) What is $C = \partial S$? Parametrize C so that it has a positive orientation with respect to the outward normal.

$r = \langle \cos t, \sin t, 0 \rangle \quad t \in [0, 2\pi]$

(c) Evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle zx + z^2y + x, z^3yx + y, z^4x^2 \rangle$.

$$= \int_0^{2\pi} \langle 0 + 0 + \cos t, 0 + \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin t \cos t + \sin t \cos t + 0 \, dt$$

$$= \int_0^{2\pi} 0 \, dt = \boxed{0}$$

8.4 More Examples More Power - During Class

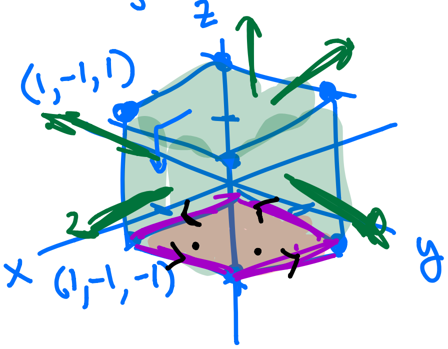
Objective(s):

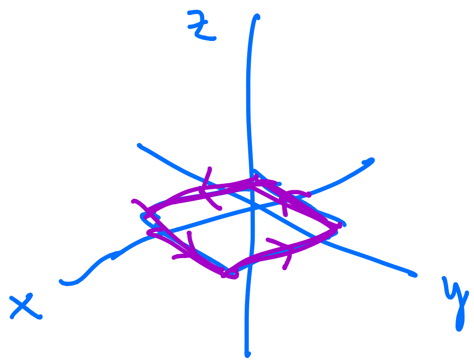
- Cement your knowledge of how to use Stokes' Theorem.

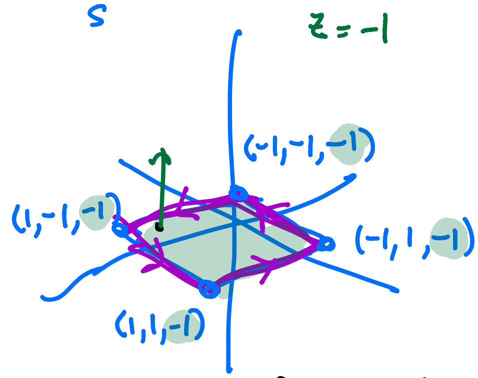
Example 8.10. Evaluate $\iint_S \overbrace{(\nabla \times \mathbf{F})}^{\text{curl F}} \cdot d\mathbf{S}$ where $\mathbf{F} = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ where S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

Note: *WW # 3 has you do this by applying Stokes' Theorem once. Here we will be extra clever and apply it twice!*

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} \quad = \quad \int_C \vec{F}\cdot d\vec{r} \quad = \quad \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$







5 surface ints — 4 line ints — 1 surface int

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & xy & x^2yz \end{vmatrix}$$

$$= (x^2z - 0)i - (2xyz - xy)j + (y - xz)k$$

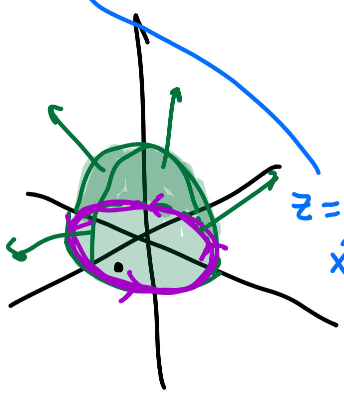
$$= \langle x^2z,\ xy - 2xyz,\ y - xz\rangle$$

$$\iint_S \langle x^2z, xy-2xyz, y-xz\rangle \cdot \langle 0, 0, -1\rangle \, dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} y - xz \; dy\,dx = \int_{-1}^{1}\int_{-1}^{1} y + x \; dy\,dx$$

$$= \boxed{0}$$

Example 8.11 (on SS01 Final Exam)**.** Use Stokes' Theorem to evaluate $\iint_S \overbrace{\nabla \times (y\mathbf{i})}^{\text{curl } F} \cdot d\mathbf{S}$ where S is the hemisphere: $x^2 + y^2 + z^2 = 1, z \geq 0$.



$z = 0$

$x^2 + y^2 = 1$

$x = 1\cos t$

$y = 1\sin t$

$r(t) = \langle \cos t, \sin t, 0 \rangle$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$\frac{dr}{dt} \cdot dt$

$r'(t)\, dt$

$$= \int_C \langle y, 0, 0 \rangle \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle \sin t, 0, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt$$

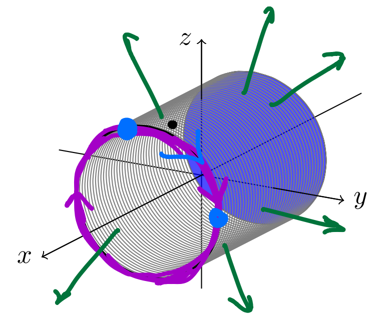
$$= \int_0^{2\pi} -\frac{1}{2}(1 - \cos(2t))\, dt$$

$$= \int_0^{2\pi} -\frac{1}{2} + \frac{1}{2}\cos(2t)\, dt$$

$$= \left[-\frac{1}{2}t + \frac{1}{4}\sin(2t) \right]_0^{2\pi}$$

$$= -\frac{1}{2}(2\pi) = -\pi$$

Example 8.12 (on E4 - FS14). Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = xyz\mathbf{i} + 2xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the cylinder $y^2+z^2=1$, $x \in [-1,1]$ along with the disk $y^2+z^2 \le 1, x=-1$, oriented outward, shown to the right.



(a) Identify and parametrize the boundary curve of S with the correct orientation.

$x = 1$

$y^2+z^2=1$

$r(t) = \langle 1, \sin t, \cos t \rangle$

$t \in [0, 2\pi]$

$t=0$

$\langle 1,0,1 \rangle$

$t = \pi/2$

$\langle 1,1,0 \rangle$

(b) Write $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ as an equivalent line integral and then evaluate.

$= \int_C F \cdot dr$

$= \int_0^{2\pi} \langle xyz, 2xy, x^2yz \rangle \cdot \langle 0, \cos t, -\sin t \rangle dt$

$= \int_0^{2\pi} 2\sin t\cos t - \sin^2 t\cos t \; dt$

$= 0$

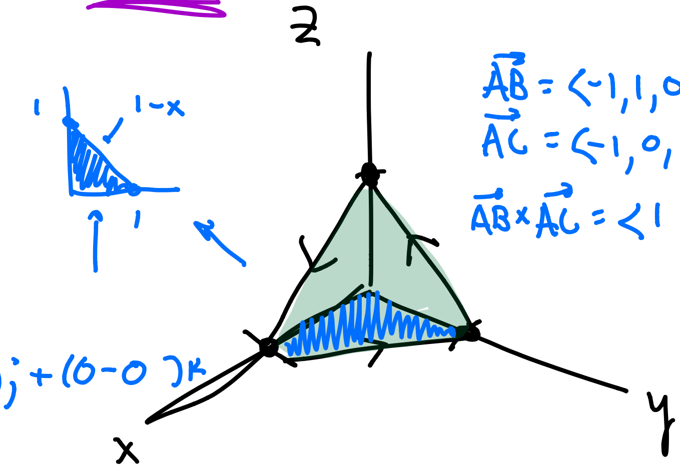
$x+y+z=1$

Example 8.13. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ with counter-clockwise rotation $=$

A B C

Hint: $\int_0^1 [1-u-u(1-u)-(1-u^2)/2]\,du = 1/6$

$$\iint_S \text{curl}\,F\cdot d\vec{S}$$



$\vec{AB} = \langle -1,1,0\rangle$

$\vec{AC} = \langle -1,0,1\rangle$

$\vec{AB}\times\vec{AC} = \langle 1,1,1\rangle$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & y^2 & x \end{vmatrix} = (0-0)i - (1-2z)j + (0-0)k$$

$$= \langle 0, 2z-1, 0\rangle$$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$1(x-0)+1(y-1)+1(z-0)=0$

$x+y-1+z=0$

$z = 1-x-y$

$$\iint \langle 0,2z-1,0\rangle \cdot \pm\langle g_x, g_y, -1\rangle\, dA$$

$$= \iint \langle 0,2z-1,0\rangle \cdot \langle +1,+1,+1\rangle\, dA$$

$$= \iint 2z-1\, dA$$

$$= \int_0^1\int_0^{1-x} 1-2x-2y\; dy\,dx$$

Hint!

$$= \int_0^1 1-x-2x(1-x)-(1-x)^2\,dx = 1/6$$

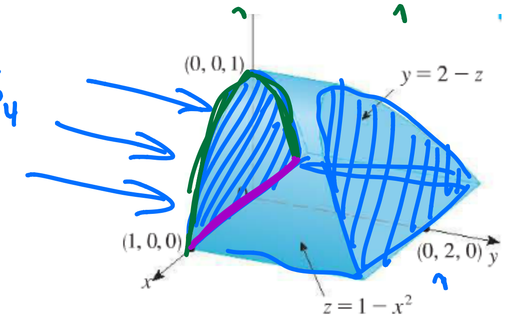
9.2 Examples Galore - During Class

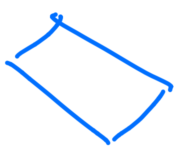
Objective(s):

- Apply the divergence theorem to more problems.

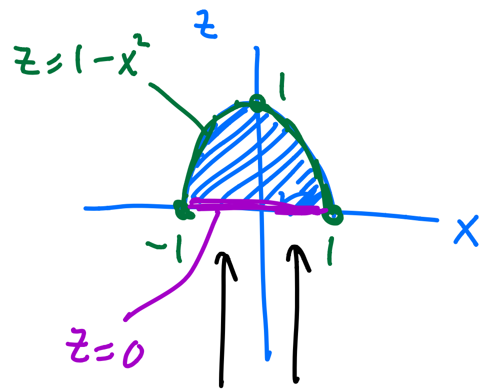
Example 9.3. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy)\rangle$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

$$\iint_{S_1} F\cdot dS_1 + \iint_{S_2} F\cdot dS_2 + \cdots \iint_{S_4} F\cdot dS_4$$





$$= \iiint_E \text{div } F \, dV$$



$$= \iiint y + 2y + 0 \, dV$$

$$= 3 \iiint y \, dV$$

$$= 3 \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} y \, dy\, dz\, dx$$

$$= 3 \int_{-1}^{1}\int_{0}^{1-x^2} \frac{y^2}{2}\Big|_{0}^{2-z} dz\, dx$$

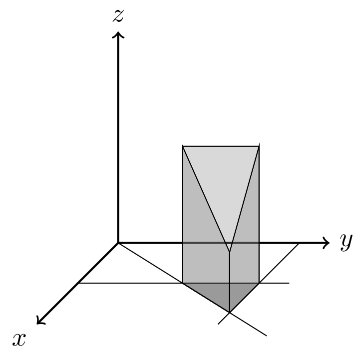
$$= 3 \int_{-1}^{1}\int_{0}^{1-x^2} \frac{(2-z)^2}{2} dz\, dx$$

$$= \vdots$$

$$= \boxed{184/35}$$

Example 9.4 (SS14 Exam 4 Question)**.** Consider the surfaces S from Exam 3 shown below:

$$\begin{aligned} x &= \sqrt{3} & y &= 3 \\ y &= x & z &= 4 - x \\ z &= 0 \end{aligned}$$



Calculate the flux of $\mathbf{F} = (3x + \tan y)\mathbf{i} + (y - \ln(z+1))\mathbf{j} + (3xy - 2z)\mathbf{k}$ outward through S. (**Hint:** the volume enclosed by S is $24 - 13\sqrt{3}$)

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint_E 3 + 1 - 2 \, dV$$

$$= \iiint_E 2 \, dV = 2 \iiint_E 1 \, dV$$

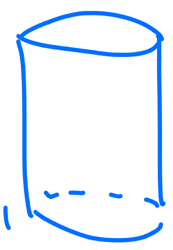
$$= \boxed{2(24 - 13\sqrt{3})}$$

Example 9.5. Consider $\mathbf{F} = \left\langle \frac{xy^2}{2}, \frac{y^3}{6}, zx^2 \right\rangle$ over the surface S, where S is the cylinder $x^2 + y^2 = 1$ capped by the planes $z = \pm 1$.

(a) Is the net flux of $\mathbf{F}$ from the surface positive or negative?

capped → include planes.

cut → don't include planes.



$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint \underbrace{\frac{y^2}{2} + \frac{y^2}{2} + x^2}\, dV$$

positive because adding up positive numbers.

(b) What is the value of the flux across S?

$$\iiint x^2+y^2 \cdot dV = \iiint r^3 \cdot dz\,dr\cdot d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\, dz\,dr\cdot d\theta = \int_0^{2\pi}\int_0^1 2r^3 \cdot dr \cdot d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{2}\Big]_0^1 d\theta = 2\pi\left(\frac{1}{2}\right) = \boxed{\pi}$$

Example 9.6. Challenging problem Evaluate $\iint_{S=\partial R} (x + y^2 + 2z)\, dS$, where R is the solid sphere $x^2 + y^2 + z^2 \le 4$ using the divergence theorem.

$x^2+y^2+z^2=4$

$g=c$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div} F\, dV$$

$\vec{n} = \nabla g = \langle 2x, 2y, 2z\rangle$

$$\frac{\vec{n}}{|\vec{n}|} = \frac{\langle 2x, 2y, 2z\rangle}{\sqrt{4x^2+4y^2+4z^2}} = \frac{\langle 2x,2y,2z\rangle}{2\sqrt{x^2+y^2+z^2}} = \left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\rangle$$

$x^2+y^2+z^2 \to 4$

$$\vec{F}\cdot\vec{n} = x+y^2+2z$$

$$\vec{F}\cdot\left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\rangle = x+y^2+2z$$

$$\underbrace{\langle 2, 2y, 4\rangle}_{\vec{F}}\cdot\left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right\rangle = x+y^2+2z$$

$$= \iiint_E 0+2+0\, dV$$

$$= 2\iiint 1\, dV = 2\cdot\frac{4}{3}\pi\, 2^3$$

$$= \boxed{\frac{64}{3}\pi}$$

We have already shown that $\iint_S Rn_3\, dS = \iiint_E R_z\, dV$. Similar proofs can be used to show

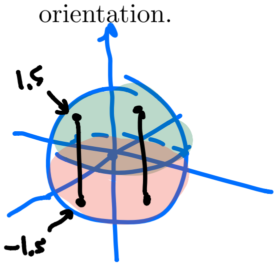
____________________ and ____________________.

9.4 The Last Time I Swear - During Class

Objective(s):

- Try a few more Divergence Theorem Problems

Example 9.1. Evaluate the surface integral $\iint_S \langle xz, -2y, 3x\rangle \cdot d\mathbf{S}$ where S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation.



$$= \iiint_E z - 2 + 0\, dV$$

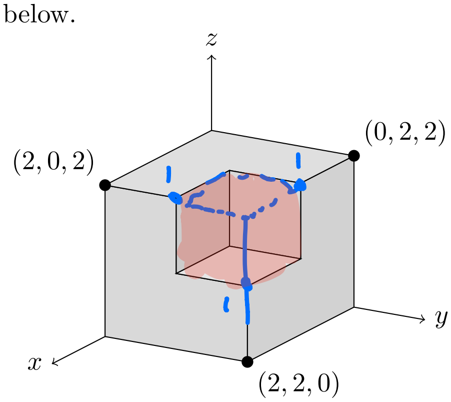
$$= \iiint z\, dV - \iiint 2\, dV$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta - 2\,\frac{4}{3}\pi\, 2^3$$

$$\left[\frac{\sin^2\phi}{2}\right]_0^{\pi} = 0$$

$$\int_0^{2\pi} 1\, d\theta \cdot \underbrace{\int_0^{\pi} \cos\phi \sin\phi\, d\phi}_{0} \cdot \int_0^{2} \rho^3\, d\rho - \frac{64\pi}{3} = \boxed{\frac{-64\pi}{3}}$$

Example 9.2. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the outwardly oriented surface shown in the figure below.



A. 0

B. 3

C. 7

D. 14

E. 21

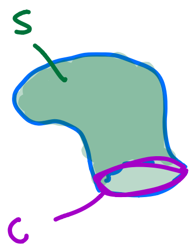
$$\iiint_E 1 + 1 + 1\, dV$$

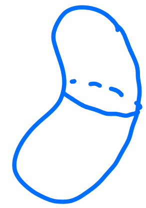
$$= 3 \iiint_E 1\, dV$$

$$3(8 - 1) = 21$$

Example 9.3. Prove that $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = 0$ assuming S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

$$\iint \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$





$$= \int F\cdot dr$$

$$= 0$$

$$\int_3^3 x^2\,dx = 0$$

$$\int_a^a f(x)\,dx = 0$$

$$\iint \text{curl}\,F\cdot dS = \iiint_E \text{div}\,(\text{curl}\,F)\,dV$$

$$= \iiint_E 0\,dV$$

$$= 0$$

Example 9.4. Use the Divergence Theorem to evaluate $\iint_S (2x + 2y + z^2)\; dS$ where S is the sphere $x^2 + y^2 + z^2 = 1$.

$$=$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint \text{div}\,F\,dV$$

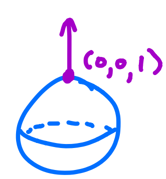
$$\iiint 1\,dV$$

$$= \frac{4}{3}\pi$$

$G(x,y,z) = c$

$\nabla G = \langle 2x, 2y, 2z\rangle$

$\langle 0, 0, 2\rangle$



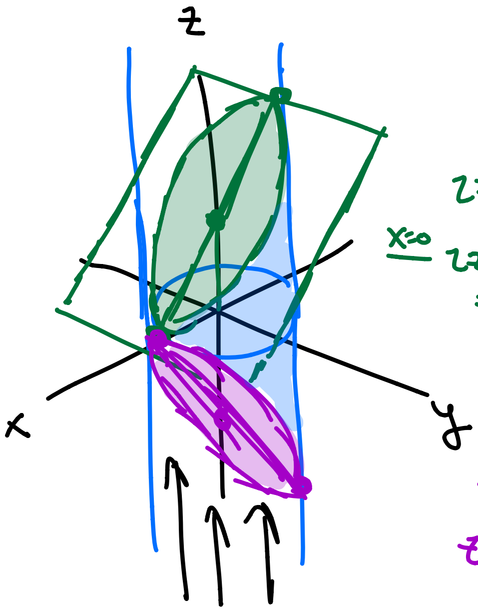
$$|\nabla G| = \sqrt{4x^2+4y^2+4z^2}$$
$$= 2\sqrt{x^2+y^2+z^2}$$
$$= 2$$

$n = \langle x, y, z\rangle$

$$\iint_S \langle 2, 2, z\rangle\cdot\langle x, y, z\rangle\,dS$$

$$\iint_S 2x+2y+z^2\;dS$$

Example 9.5. Compute $\displaystyle\iint_{S=\partial R} \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = (x-z)\mathbf{i} + (y-x)\mathbf{j} + (z-y)\mathbf{k}$ and S is the cylinder $x^2+y^2=1$ capped by the planes $2z = 1-x$ and $2z = x-1$.



$$\iiint_R 1+1+1\, dV = \iiint_R 3\, dV$$

$$\int_0^{2\pi}\int_0^1\int_{\frac{r\cos\theta-1}{2}}^{\frac{1-r\cos\theta}{2}} 3\, r\, dz\, dr\, d\theta$$

$2z = 1-x$

$\underline{x=0}$ $2z=1$, $z=\frac{1}{2}$ $\underline{x=1}$ $2z=0$, $z=0$

$\underline{x=-1}$ $2z=2$, $z=1$

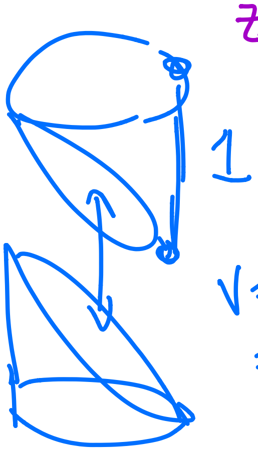
$x=0$ $z=-\frac{1}{2}$ $x=1$ $z=0$ $x=-1$ $z=-1$

$z = \frac{x-1}{2}$ $z = \frac{1-x}{2}$

$$= \int_0^{2\pi}\int_0^1 3r\left(\frac{1-r\cos\theta}{2} - \frac{r\cos\theta-1}{2}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r(1-r\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r - 3r^2\cos\theta\, dr\, d\theta$$

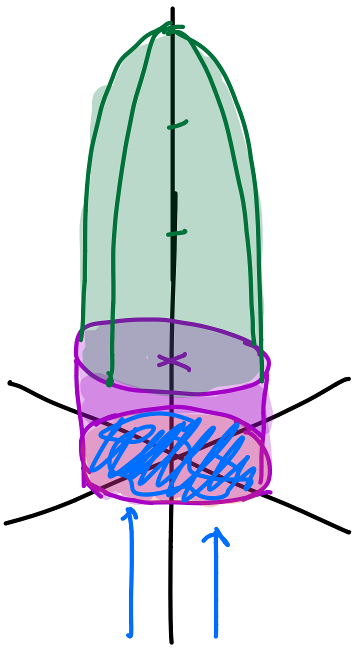


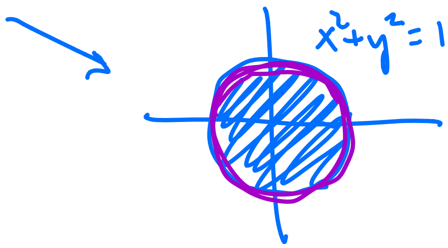
$V = 1(\pi 1^2)$
$= \pi$

$$= \int_0^{2\pi}\left[\frac{3}{2} - \cos\theta\right] d\theta$$

$$= \left[\frac{3}{2}\theta - \sin\theta\right]_0^{2\pi} = \frac{3}{2}(2\pi) = 3\pi$$

Example 9.6. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j} + z\mathbf{k}$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.





$$\iiint_E y - y + 1 \, dV = \iiint_E 1 \, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 r(4-3r^2) \, dr$$

$$= 2\pi \left[2r^2 - \frac{3}{4} r^4 \right]_0^1$$

$$= 2\pi \left(2 - \frac{3}{4} \right)$$

$$= \boxed{\frac{5\pi}{2}}$$