13.2 Derivatives and Integrals of Vector Functions

1. For each of the following

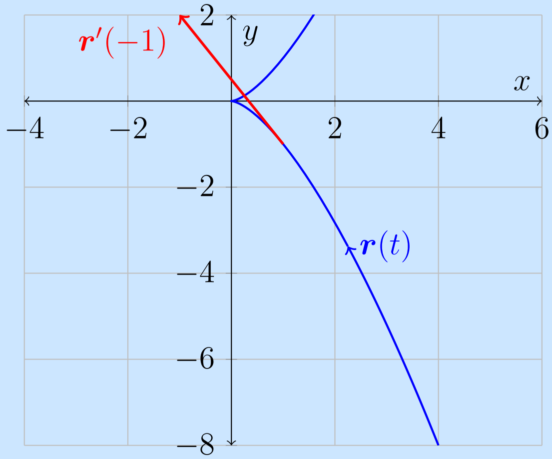
   (i) Sketch the following curves in the xy-plane

   (ii) Find $\boldsymbol{r}'(t)$

   (iii) Sketch the tangent vector $\boldsymbol{r}'(t)$ for the given value of t

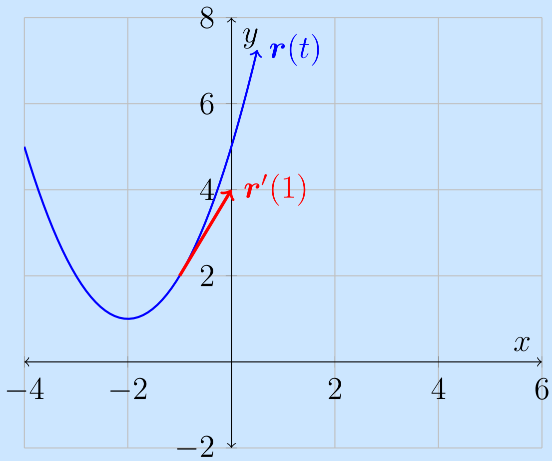
   (a) $\boldsymbol{r}(t) = \langle t^2, t^3 \rangle$ with $t = -1$

   Solution: At $t = -1$ the curve $\boldsymbol{r}(t)$ passes through $\boldsymbol{r}(-1) = (1, -1)$. Since $\boldsymbol{r}'(t) = \langle 2t, 3t^2 \rangle$, the tangent vector at that point is given by $\boldsymbol{r}'(-1) = \langle -2, 3 \rangle$.



   (b) $\boldsymbol{r}(t) = \langle t - 2, t^2 + 1 \rangle$ with $t = 1$

   Solution: At $t = 1$ the curve passes through $\boldsymbol{r}(1) = \langle -1, 2 \rangle$. Also $\boldsymbol{r}'(t) = \langle 1, 2t \rangle$ and $\boldsymbol{r}'(1) = \langle 1, 2 \rangle$.



2. Find $\boldsymbol{r}'(t)$ given $\boldsymbol{r}(t) = \langle t\sin(t), t^2, \cos(2t)\rangle$

Solution:

$$\begin{aligned}\boldsymbol{r}'(t) &= \left\langle \frac{d}{dt}(t\sin(t)), \frac{d}{dt}(t^2), \frac{d}{dt}(\cos(2t))\right\rangle \\ &= \langle (1)\sin(t) + t\cos(t), 2t, -2\sin(2t)\rangle \\ \boldsymbol{r}'(t) &= \langle \sin(t) + t\cos(t), 2t, -2\sin(2t)\rangle\end{aligned}$$

3. Find parametric equations for the tangent line to the curve $\boldsymbol{v}(t) = \left\langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t\right\rangle$ at the point $(3, 0, 2)$

Solution: To find the value of t that makes $\boldsymbol{v}(t) = (3, 0, 2)$ set the components equal:

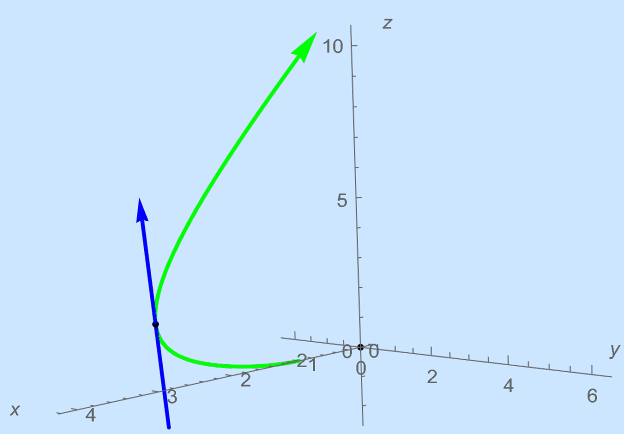
$$1 + 2\sqrt{t} = 3 \quad \Rightarrow \quad t = 1$$

From $\boldsymbol{v}'(t) = \left\langle \frac{1}{\sqrt{t}}, 3t^2 - 1, 3t^2 + 1\right\rangle$ and $\boldsymbol{v}'(1) = \langle 1, 2, 4\rangle$, the tangent line is represented by

$$\begin{aligned}\boldsymbol{u}(t) &= \langle 3, 0, 2\rangle + t\,\langle 1, 2, 4\rangle \\ &= \langle 3 + t, 2t, 2 + 4t\rangle\,.\end{aligned}$$

In particular

$$\begin{aligned}x(t) &= 3 + t \\ y(t) &= 2t \\ z(t) &= 2 + 4t\end{aligned}$$



In the above $\boldsymbol{v}(t)$ is shown in green and the the line we found is shown in blue.

4. Evaluate the given integrals

(a)

$$\int_0^2 \left(t\boldsymbol{i} - t^2\boldsymbol{j} + 3t^5\boldsymbol{k}\right)\ dt$$

Solution:

$$\begin{aligned}
\int_0^2 \left(t\boldsymbol{i} - t^2\boldsymbol{j} + 3t^5\boldsymbol{k}\right)\ dt &= \left(\int_0^2 t\ dt\right)\boldsymbol{i} - \left(\int_0^2 t^2\ dt\right)\boldsymbol{j} + \left(\int_0^2 3t^5\ dt\right)\boldsymbol{k} \\
&= \left(\frac{1}{2}t^2\bigg|_0^2\right)\boldsymbol{i} - \left(\frac{1}{3}t^3\bigg|_0^2\right)\boldsymbol{j} + \left(\frac{1}{2}t^6\bigg|_0^2\right)\boldsymbol{k} \\
&= 2\boldsymbol{i} - \frac{8}{3}\boldsymbol{j} + 32\boldsymbol{k}
\end{aligned}$$

(b)

$$\int_0^1 \left(\frac{4}{1+t^2}\boldsymbol{j} + \frac{2t}{1+t^2}\boldsymbol{k}\right)\ dt$$

Solution:

$$\begin{aligned}
\int_0^1 \left(\frac{4}{1+t^2}\boldsymbol{j} + \frac{2t}{1+t^2}\boldsymbol{k}\right)\ dt &= \left(4\tan^{-1}(t)\bigg|_0^1\right)\boldsymbol{j} + \left(\ln(1+t^2)\bigg|_0^1\right)\boldsymbol{k} \\
&= 4\cdot\frac{\pi}{4}\boldsymbol{j} + \ln(2)\boldsymbol{k} \\
&= \pi\boldsymbol{j} + \ln(2)\boldsymbol{k}
\end{aligned}$$

13.3 Arc Length and Curvature

$$\begin{aligned}
\text{Unit tangent vector} \quad & \mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \\
\text{Unit normal vector} \quad & \mathbf{N}(t) = \frac{\mathbf{T}(t)}{|\mathbf{T}'(t)|} \\
\text{Binormal vector} \quad & \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \\
\text{Curvature} \quad & \kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}
\end{aligned}$$

5. Find the unit tangent vector $\boldsymbol{T}(\pi/4)$ to the curve $\left\langle \sin^2(t), \cos^2(t), \tan^2(t) \right\rangle$.

Solution:

$$\begin{aligned}
\boldsymbol{r}'(t) &= \left\langle 2\sin(t)\cos(t), -2\cos(t)\sin(t), 2\tan(t)\sec^2(t) \right\rangle \\
\boldsymbol{r}'(\pi/4) &= \langle 1, -1, 4 \rangle \\
|\boldsymbol{r}'(\pi/4)| &= \sqrt{1+1+16} \\
\mathbf{T}(\pi/4) &= \left\langle \frac{1}{\sqrt{18}}, -\frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}} \right\rangle
\end{aligned}$$

6. Suppose you are at the point $(0,0,0)$ and move 2 units in the positive direction along the curve $\boldsymbol{r}(t) = \langle t^2, t^3, t^4 \rangle$. What point did you end up at? (Your answer may be messy, if so do not try to simplify.)

Solution:

7. Find the length of the curve $\boldsymbol{r}(t) = \boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$ with $0 \le t \le 1$. Then find the unit tangent vector $\boldsymbol{T}(t)$ and unit normal vector $\boldsymbol{N}(t)$.

Solution: