14.4 Group Work

14.4 - Tangent Planes and Linear Approximations

1. Find the equation to the tangent plane of $z = x^2 + xy + 3y^2$ at the point $1, 1, 5$.

Solution:

$$\frac{\partial z}{\partial x}(x, y) = 2x + y$$
$$\frac{\partial z}{\partial x}(1, 1) = 3$$
$$\frac{\partial z}{\partial y}(x, y) = x + 6y$$
$$\frac{\partial z}{\partial y}(1, 1) = 7$$

Which gives the equation

$$z - 5 = 3(x - 1) + 7(y - 1)$$

or equivalently

$$z = 3x + 7y - 5$$

2. Find the linearization to $f(x, y) = x^3y^4$ at the point $(1, 1)$ and use it to approximate the value of $(1.01)^3(.9)^4$.

Solution:

$$\frac{\partial f}{\partial x}(x, y) = 3x^2y^4$$
$$\frac{\partial f}{\partial x}(1, 1) = 3$$
$$\frac{\partial f}{\partial y}(x, y) = 4x^3y^3$$
$$\frac{\partial f}{\partial xy}(1, 1) = 4$$

So the linearization is given by

$$\begin{aligned} L(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\ &= 1 + 3(x - 1) + 4(y - 1) \\ &= 3x + 4y - 6 \end{aligned}$$

Then we can approximate

$$\begin{aligned}(1.01)^3(0.9)^4 &\approxeq L(1.01, 0.9)\\ &= 3(1.01) + 4(.9) - 6\\ &= 3.03 + 3.6 - 6\\ &= 0.63\end{aligned}$$

If you use an expensive calculator to estimate the value you get $(1.01)^3(0.9)^4 \approxeq 0.6300000000000008$.

3. Find the linearization to
$$f(x,y) = \sqrt{x+y} + (y)^4$$
at the point $(3,1)$ and use this to estimate the value of $\sqrt{4.25} + (.75)^4$.

Solution:

$$\begin{aligned}f(1,1) &= \sqrt{3+1} + (1)^4 = 2 + 1 = 3\\ f_x(x,y) &= \frac{1}{2\sqrt{x+y}} \cdot \frac{\partial f}{\partial x}(x+y) = \frac{1}{2\sqrt{x+y}}\\ f_x(3,1) &= \frac{1}{2\sqrt{3+1}} = \frac{1}{4} = 0.25\\ f_y(x,y) &= \frac{1}{2\sqrt{x+y}} + 4y^3\\ f_y(3,1) &= \frac{1}{2\sqrt{3+1}} = \frac{1}{4} + 4 = 4.25\end{aligned}$$

$$\begin{aligned}L(x,y) &= f(1,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)\\ &= 3 + (0.25)(x-3) + (4.25)(y-1)\end{aligned}$$

So an estimate is given when $y = .75$ which required $x + .75 = 4.25 \Rightarrow x = 3.5$ and

$$\begin{aligned}\sqrt{4.25} + (0.75)^4 &\approx 3 + (.25)(3.5-3) + (4.25)(0.75-1)\\ &= 3 + (0.25)(.5) + (4.25)(-.25)\\ &= 3 + 0.125 - 1.0625 = 2.0625\end{aligned}$$

The estimate given by a calculator is 2.378. The estimate is not as good as the one above because we increased the distance from the estimated value and the point used in our linearization.

4. If $f(x,y)$ is differentiable with $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$, estimate the value of $f(2.2, 4.9)$.

Solution:

$$\begin{aligned} f(2.2, 4.9) &\approx f(2,5) + f_x(2,5)(2.2-2) + f_y(2,5)(4.9-5) \\ &= 6 + 1(0.2) + (-1)(-0.1) \\ &= 6.3 \end{aligned}$$

5.