12.5 Part 1: Lines

A note about notation:

If $P(x_0, y_0, z_0)$ is a point then $\boldsymbol{p}$ will denote the vector $\boldsymbol{p} = \langle x_0, y_0, z_0 \rangle$.

Useful Information.

- The **line through a point** P **parallel to** $\boldsymbol{v}$ is given by

$$\boldsymbol{r}(t) = \boldsymbol{p} + t\boldsymbol{v}, \quad t \in \mathbb{R} \tag{1}$$

- The **line segment** beginning at a point R_0 and ending at the point R_1 is given by

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 \quad t \in \mathbb{R} \tag{2}$$

$$\text{or equivalently} \quad \boldsymbol{r}(t) = \boldsymbol{r}_0 + t(\boldsymbol{r}_1 - \boldsymbol{r}_0) \quad t \in \mathbb{R} \tag{3}$$

- A **parametric equation** for a line is of the form

$$\boldsymbol{r}t = \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

and if $a, b, c \neq 0$ a **symmetric equation**[a] is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[a]This comes from $\langle x, y, z \rangle = \langle at + x_0, bt + y_0, ct + z_0 \rangle$ and solving each component for t, e.g. $x = at + x_0, \ldots$

1. Find an equation for the line segment $\boldsymbol{r}(t)$ with $\boldsymbol{r}(0) = (3, -7, 0)$ and $\boldsymbol{r}(1) = (18, -7, \pi)$, $0 \leq t \leq 1$.

Solution:

$$\begin{aligned} \boldsymbol{r}(t) &= \langle 3, -7, 0 \rangle + t(\langle 18, -7, \pi \rangle - \langle 3, -7, 0 \rangle) \\ &= \langle 3, -7, 0 \rangle + t(\langle 15, 0, \pi \rangle) \\ &= \langle 3 + 15t, -7, \pi t \rangle \end{aligned}$$

2. Let $\boldsymbol{f}(t) = \langle 1 + 2t, -10t, 2200111 - 7t \rangle$. Let L be a line parallel to $\boldsymbol{f}(t)$ that passes through the point $(0, 0, 0)$.

(a) Find a parametric equation for L.

Solution: From

$$\begin{aligned} \boldsymbol{f}(t) &= \langle 1 + 2t, -10t, 2200111 - 7t \rangle \\ &= \langle 1, 0, 2200111 \rangle + t \langle 2, -10, -7 \rangle \end{aligned}$$

We see that L is a line through $(0,0,0)$ and parallel to the vector $\langle 2,-10,-7\rangle$ and is given by

$$\begin{aligned}\boldsymbol{r}(t) &= \langle 0,0,0\rangle + t\,\langle 2,-10,-7\rangle \\ &= \langle 2t,-10t,-7t\rangle\,.\end{aligned}$$

(b) Use the result above to find a symmetric equation for L.

Solution:

$$\begin{aligned} x = 2t &\iff \frac{x}{2} = t \\ y = -10t &\iff \frac{-y}{10} = t \\ z = -7t &\iff \frac{-z}{7} = t \end{aligned}$$

Which gives

$$\frac{x}{2} = \frac{y}{-10} = \frac{z}{-7}$$

3. Suppose L is represented by the equation $\boldsymbol{r}_1(t) = \langle 2t, 1-t, 2\pi + \pi t\rangle$. Determine which of the following points belong to L

(a) $P(-4,3,0)$

Solution: If P is in L then when $x=-4$ we have

$$-4 = 2t \quad \iff \quad t = -2.$$

When $t=-2$ the line L passes through the point

$$(2(-2), 1-(-2), 2\pi + (-2)\pi) = (-4,3,0).$$

Which means the line L passes through P.

(b) $Q(1, 1/2, 5\pi)$

Solution: Q is not a point on L

(c) $R(20,-9,12\pi)$

Solution: The line L passes through R when $t=10$.

4. Let L denote the line with symmetric equation

$$\frac{x-1}{2} = y = \frac{z+1}{3}$$

(a) Find a parametric equation representing L.

Solution: We have

$$t = \frac{x-1}{2} \iff x = 2t+1$$
$$t = y$$
$$t = \frac{z+1}{3} \iff z = 3t-1$$

which gives the parametric equation

$$\boldsymbol{r}(t) = \langle 2t+1, t, 3t-1 \rangle$$

(b) Determine if the line that passes through the points $(1, -5, 5)$ and $(-1, 0, 2)$ intersects L. If not, determine if it is parallel or skew to L.

Solution: The line through $(1, -5, 5)$ and $(-1, 0, 2)$ is given by

$$\begin{aligned} \boldsymbol{s}(t) &= \langle -1, 0, 2 \rangle + t * (\langle 1, -5, 5 \rangle - \langle -1, 0, 2 \rangle) \\ &= \langle -1, 0, 2 \rangle + t \langle 2, -5, 3 \rangle \\ &= \langle 2t-1, -5t, 3t+2 \rangle . \end{aligned}$$

The lines do not intersect (assume $2t+1 = 2t-1$ and it follows). From

$$\begin{aligned} \boldsymbol{r}(t) &= \langle 1, 0, -1 \rangle + t \langle 2, 1, 3 \rangle \\ \boldsymbol{s}(t) &= \langle -1, 0, 2 \rangle + t \langle 2, -5, 3 \rangle \end{aligned}$$

We see the lines are not parallel since

$$\frac{2}{2} \neq \frac{1}{-5} \neq \frac{3}{3}$$

which means the lines are skew.

12.5 Part 2: Planes

Useful Information. A plane with normal vector $\boldsymbol{n} = \langle a, b, c\rangle$ containing the point $P(x_0, y_0, z_0)$ is represented by the equation

$$\begin{aligned} 0 &= \boldsymbol{n} \cdot (\langle x, y, z\rangle - \boldsymbol{p}) \\ &= \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle \\ &= ax + by + cz - (ax_0 + by_0 + cz_0) \end{aligned}$$

or

$$ax + by + cz = d \quad \text{where} \quad d = ax_0 + by_0 + cz_0$$

5. Find a unit normal vector for the plane that contains the line $\boldsymbol{r}(t) = \langle t, -t, 2t\rangle$ and the point $P(0, 0, 1)$.

Solution: Since P is not on $\boldsymbol{r}(t)$ we can use it to create another line not parallel to $\boldsymbol{r}(t)$ and contained in the plane.

When $t = 0$, $\boldsymbol{r}(0) = \langle 0, 0, 0\rangle$, so the plane contains a line segment connecting $(0, 0, 0)$ to $(0, 0, 1)$ which is parallel to the vector $\langle 0, 0, 1\rangle$.

Then a normal vector is

$$\begin{aligned} \langle 0, 0, 1\rangle \times \langle 1, -1, 2\rangle &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} \\ &= \langle 0 - (-1), -(0 - 1), 0 - 0\rangle \\ &= \langle 1, 1, 0\rangle . \end{aligned}$$

From

$$||\,\langle 1, 1, 0\rangle\,|| = \sqrt{2},$$

a unit normal vector to the plane is

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle \quad (\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle \ \text{ is also correct}).$$

6. Find the equation of a plane that contains the triangle with vertices $P(0, 0, 0)$, $Q(5, 2, 0)$, and $R(0, 1, 1)$.

Solution:

Use vectors PQ and PR to find a normal vector to the plane $PQ \times PR = \langle 2, -5, 5\rangle$. So the plane is given by

$$0 = \langle 2, -5, 5\rangle \cdot (\langle x, y, z\rangle - \langle 0, 0, 0\rangle) \qquad \Longleftrightarrow \qquad 0 = 2x - 5y + 5z.$$

7. Find an equation for the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

Solution:

The vector $\eta = \langle 5, -1, -1 \rangle$ is normal to $5x - y - z = 6$ so an equation for a parallel plane through $(1, -1, -1)$ is

$$0 = \langle 5, -1, -1 \rangle \cdot (\langle x - 1, y + 1, z + 1 \rangle \quad \Longleftrightarrow \quad 5x - y - z = 7$$

8. At what point does the line through the points $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$.

Solution: The line is represented by the equation

$$\boldsymbol{r}(t) = \langle 1, 0, 1 \rangle + t \langle 4 - 1, -2 - 0, 2 - 1 \rangle \quad \Longleftrightarrow \quad \boldsymbol{r}(t) = \langle 3t + 1, -2t, t + 1 \rangle$$

We want to find the value of t for which $\boldsymbol{r}(t)$ gives a point tthat satisfies $x + y + z = 6$.

This implies that

$$(3t + 1) + (-2t) + (t + 1) = 6 \quad \Rightarrow \quad t = 2$$

Which means the line intersects the plane at the point

$$\boldsymbol{r}(2) = (7, -4, 3).$$

9. The planes $x + y + z = 1$ and $x - 2y + 3z = 1$ intersect in a line.

   (a) Find the angle between the two planes (hint: think about normal vectors)

   Solution: The given planes have normal vectors $\eta_1 = \langle 1, 1, 1\rangle$ and $\eta_2 = \langle 1, -2, 3\rangle$. The angle between these two vectors will give the angle between the planes:

   $$\cos\theta = \frac{\langle 1,1,1\rangle \cdot \langle 1,-2,3\rangle}{|\langle 1,1,1\rangle||\langle 1,-2,3\rangle|} = \frac{2}{\sqrt{42}}$$

   so the angle is

   $$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 1.257 \text{ radians}$$

   (b) Find the parametric equation representing the line determined by the intersection of the two planes (hint: think about normal vectors again)

   Solution: We can find a vector parallel to the line of intersection by computing $\eta_1 \times \eta_2$ (as they were defined above).

   $$\begin{aligned} \boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\ &= \langle 5, -2, -3\rangle \end{aligned}$$

   A point that lies on both planes is $(1, 0, 0)$ so the line of intersection can be represented by

   $$\boldsymbol{r}(t) = \langle 1,0,0\rangle + t\langle 5,-2,-3\rangle = \langle 5t+1, -2t, -3t\rangle$$

10. Let $\mathcal{P}$ be the plane represented by

    $$x + 2y - z = 4$$

    Find the equation for the plane that intersects $\mathcal{P}$ and is parallel to a normal vector for $\mathcal{P}$.

    Solution: Please ignore this question. I did not type it up correctly so it doesn't really make sense.

11. The goal of this exercise is to find the distance from the point $P(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$.

(a) Find a point Q on the plane $3x + 2y + 6z = 5$ (the choice of point does not matter, you just need to choose one).

Solution: The easiest way to make 5 out of 3,2, and 6 is $3(1) + 2(1) + 6(0)$, so I will choose $Q = (1, 1, 0)$.

(b) Find the vector PQ using the points above.

Solution:

$$\begin{aligned} PQ &= \langle 1, 1, 0\rangle - \langle 1, -2, 4\rangle \\ &= \langle 0, 3, -4\rangle \end{aligned}$$

Remark. Unless you were very lucky, the vector PQ is not the shortest vector connecting P to the plane. To find the shortest distance possible, we project PQ onto the normal vector for the plane.

(c) Find the normal vector $\boldsymbol{\eta}$ for the plane.

Solution:

$$\boldsymbol{\eta} = \langle 3, 2, 6\rangle$$

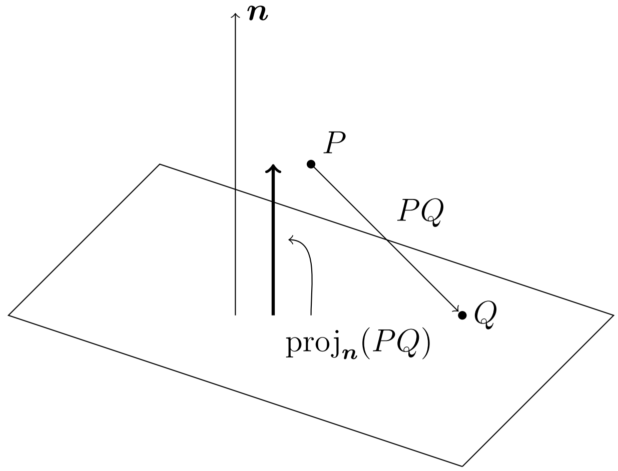
(d) Find $||\text{proj}_{\boldsymbol{n}}(PQ)||$. This is the distance between P and the plane. See the image below.

Solution: First we find

$$\begin{aligned} \text{proj}_{\boldsymbol{n}}(PQ) &= \left(\frac{\eta \cdot PQ}{\eta \cdot \eta}\right)\eta \\ &= \left(\frac{3(0) + 2(3) + 6(-4)}{3^2 + 2^2 + 6^2}\right)\langle 3, 2, 6\rangle \\ &= \left(\frac{-18}{49}\right)\langle 3, 2, 6\rangle \end{aligned}$$

So the distance is given by

$$\begin{aligned} |\text{proj}_{\boldsymbol{n}}(PQ)| &= \left|\frac{18}{49}\right| |\langle 3, 2, 6\rangle| \\ &= \frac{18}{49}\sqrt{9 + 4 + 36} = \frac{18}{7} \quad \text{units} \end{aligned}$$



$\boldsymbol{n}$
P
PQ
Q
$\text{proj}_{\boldsymbol{n}}(PQ)$