15.7 Triple Integrals

1. Evaluate the given triple integrals

   (a) $\int_0^3 \int_0^x \int_{x-y}^{x+y} y \, dz \, dy \, dx$

   Solution:

$$\begin{aligned}
\int_0^3 \int_0^x \int_{x-y}^{x+y} y \, dz \, dy \, dx &= \int_0^3 \int_0^x yz|_{x-y}^{x+y} \, dy \, dx \\
&= \int_0^3 \int_0^x \left(2y^2\right) dy \, dx \\
&= \int_0^3 \frac{2}{3}x^3 \, dx \\
&= \frac{1}{6}x^4|_0^3 \\
&= \frac{3^4}{6} = \frac{81}{6} = \frac{27}{2}
\end{aligned}$$

   (b) $\iiint_T x^2 \, dV$ where T is the tetrahedron with vertices

$$(0,0,0), (1,0,0), (0,1,0), \text{ and } (0,0,1).$$

   Solution: A normal vector to the plane through the three non-zero points is

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

   So the plane has equation $x + y + z = 1$ or $z = 1 - x - y$.

$$\begin{aligned}
\iiint_T x^2 \, dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 \, dz \, dy \, dx \\
&= \int_0^1 \int_0^{1-x} x^2 z|_0^{1-x-y} \, dy \, dx \\
&= \int_0^1 x^2 \int_0^{1-x} 1 - x - y \, dy \, dx \\
&= \int_0^1 x^2 \left(y - xy - \frac{1}{2}y^2|_{y=0}^{y=1-x}\right) dx \\
&= \int_0^1 x^2 \left(1 - x - x(1-x) - \frac{1}{2}(1-x)^2\right) dx \\
&= \frac{1}{2}\int_0^1 x^2 - 2x^3 + x^4 \, dx \\
&= \frac{1}{2}\left(\frac{1}{30}\right) = \frac{1}{60}.
\end{aligned}$$

(c) $\iiint_E 6xy\ dV$ where E is the region under the plane $z = 1 + x + y$ and above the part of the xy-plane bounded by curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

Solution:

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\ dz\ dy\ dx &= \int_0^1 \int_0^{\sqrt{x}} 6xy + 6x^2y + 6xy^2\ dy\ dx \\
&= \int_0^1 \left(3xy^2 + 3x^2y^2 + 2xy^3 |_0^{\sqrt{x}} \right) dx \\
&= \int_0^1 3x^2 + 3x^3 + 2x^{5/2}\ dx \\
&= 1 + \frac{3}{4} + \frac{4}{7} = \frac{65}{28}
\end{aligned}$$

2. Use a triple integral to find the volume of the given solid

(a) The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

Solution:

$$\int_0^1 \int_{x^2}^1 \int_0^{1-y} dz\ dy\ dx = \frac{8}{15}$$

(b) The tetrahedron enclosed by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $2x + y + z = 4$.

Solution:

$$\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\ dy\ dx = \frac{16}{3}$$