Directional Derivatives

Useful Information. For a function $f(x, y, z)$ the *gradient vector* is

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\end{aligned}$$

and the *directional derivative* in the direction of a **unit vector** u is given by

$$D_{\boldsymbol{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \boldsymbol{u}$$

1. Let $f(x, y) = \sin(2x + 3y)$. Find the gradient of f and evaluate the gradient at the point $P(-6, 4)$. Find the rate of change of f at P in the direction of the vector $u = \left\langle \sqrt{3}/2, -1/2 \right\rangle$.

2. Find the directional derivative of $f(x, y) = e^x \cos y$ at the point $(0, 0)$ in the direction given by the angle $\theta = \pi/4$.

3. Find the gradient of $f(x, y, z) = x^2yz - xyz^3$. What is the rate of change of f at the point $P(2, -1, 1)$ in the direction of the point $(2, 3, -2)$?

4. (a) How do you find the maximum rate of change for a function f at a given point (x_0, y_0, z_0)?

   (b) Let $f(x, y) = xe^y$. Show that at the point $P(2, 0)$, the direction in which f is increasing the fastest is in the direction $\langle 1, 2\rangle$.

   (c) What is the largest possible rate of change of f at the point P?

5. Recall that for a diffentiable function f and any unit vector $\boldsymbol{u}$

$$D_u f = \nabla f \cdot \boldsymbol{u} = |\nabla f||\boldsymbol{u}|| \cos(\theta) = |\nabla f| \cos(\theta)$$

where θ is the angle between ∇f and $\boldsymbol{u}$.

   (a) Explain why the largest value of $D_{\boldsymbol{u}} f$ occurs when $\boldsymbol{u}$ is in the same direction as ∇f.

   (b) In what direction is the rate of change of f minimized? Why?

6. Find the maximum and minimum rates of change for f at the given point and the direction in which they occur

   (a) $f(x, y) = 4y\sqrt{x}$ at $(4, 1)$.

   (b) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 6, -2)$.

7. Show that at every point on the line $y = x + 1$ the fastest rate of change for $f(x, y) = x^2 + y^2 - 2x - 4y$ is in the direction $\langle 1, 1 \rangle$ and that this does not occur at any other point.