1.2 Surfaces in Space – During Class

Objective(s):

- Sketch simple surfaces in space.
- Determine when a point lies on a specified surface.

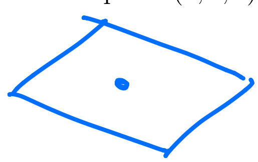
Now that we can draw points let's draw lots of them! So many that we start making some surfaces. To help us upgrade let's start by thinking about what it takes for a point to be on a surface... or a curve in $\mathbb{R}^2$.

Example 1.5.

(a) Determine if the point $(1, 4)$ is on the line $x - 4y = 1$

$1 - 4(4) = 1$
$-15 = 1$ No!

(b) Determine if the point $(1, 4, 2)$ is on the plane $x - 4y + 8z = 1$



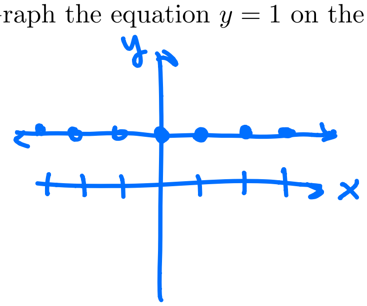
$1 - 4(4) + 8(2) = 1$
$1 - 16 + 16 = 1$
$1 = 1$ Yes!

(c) Determine if the point $(1, -3, 0)$ is on the surface $xyz + x^2 = y$

$(1)(-3)(0) + 1^2 = -3$
$1 = -3$ No!

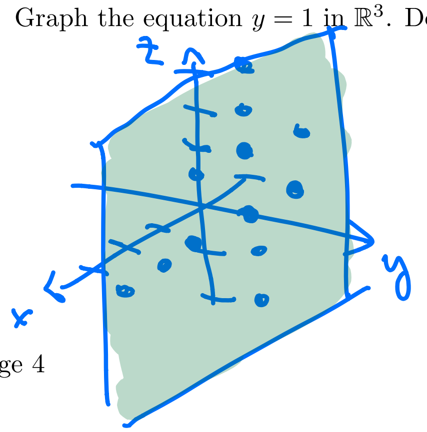
Example 1.6.

(a) Graph the equation $y = 1$ on the xy-plane. Describe it in words as best as possible.



horizontal line with y-int of $\underline{1}$

(b) Graph the equation $y = 1$ in $\mathbb{R}^3$. Describe it in words as best as possible.

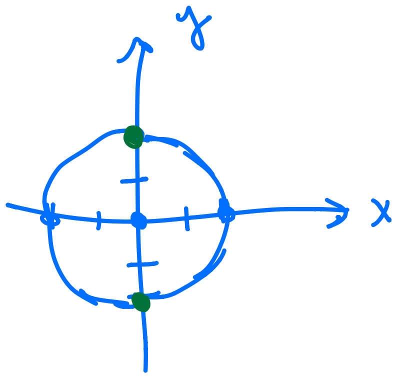


vertical plane

Example 1.7.

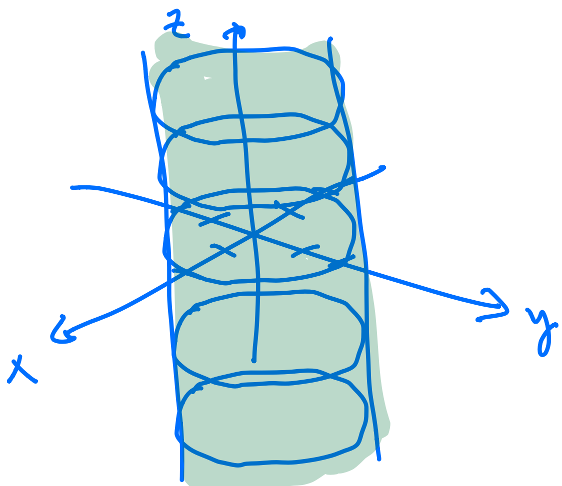
2^2
$\downarrow$

(a) Graph the equation $x^2 + y^2 = 4$ in the xy-plane. Describe it in words as best as possible.



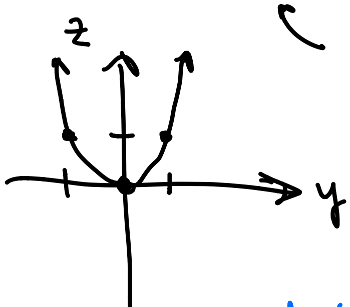
$y^2=4$

(b) Graph the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$. Describe it in words as best as possible.



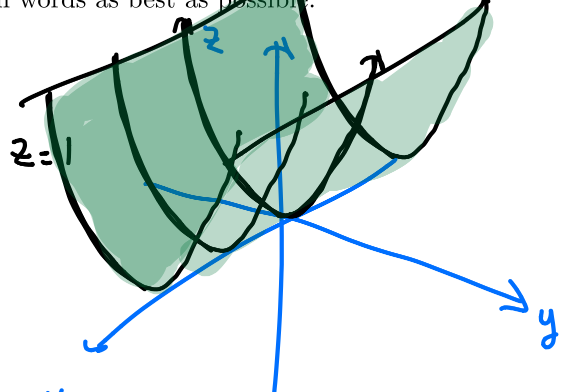
infinite cylinder

(c) Graph the equation $z = y^2$ in $\mathbb{R}^3$. Describe it in words as best as possible.



$y=0$

$\left.\begin{array}{l} y=1 \\ y=-1 \end{array}\right\} \to z=1$



parabola
shifted freely
in x direction

Equations like these that are missing a variable are quite nice and will get a special name in 12.6. They will continue to come up throughout the course

1.3 Spheres in Space – During Class

Objective(s):

- Extend our well known distance equation from 2 variables to 3 variables.
- Draw a sphere in space.
- Be able to describe a sphere given its equation.

There will be several times in this course where we can upgrade from a well known 2 dimensional equation / system by "sprinkling in some z's". This is indeed one of those times

Definition(s) 1.8. The distance $|P_1 P_2|$ between the points $P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is given by:

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Example 1.9. Calculate the distance between $(1, 2, 3)$ and $(3, -1, 0)$.

$$\sqrt{(3-1)^2+(-1-2)^2+(0-3)^2} = \sqrt{4+9+9} = \sqrt{22}$$

$$\sqrt{(1-3)^2+(2+1)^2+(3-0)^2} = \sqrt{4+9+9}$$

And with 3 dimensional distance we can define the set of all points equidistant from a center point (aka a sphere)

Definition(s) 1.10. An equation of a sphere with center $C(h, k, l)$ and radius r is:

$$(x-h)^2+(y-k)^2+(z-l)^2 = r^2$$

Example 1.11 (Now you try! WW 12.1.3)**.** Find an equation of a sphere with radius 2 centered at the point $(1, -2, 3)$.

$$(x-1)^2+(y+2)^2+(z-3)^2 = 4$$

This type of problem is relatively straight forward but we can turn it around to make a harder problem.

Example 1.12. Describe the surface $x^2 + y^2 + z^2 = 1$ in words as best as possible.

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = 1^2$$

Sphere centered at

$(0,0,0)$

with radius 1.

Example 1.13. Describe the surface $x^2 - 2x + y^2 + z^2 + 4z = 4$ in words as best as possible.

$$x^2 - x - x + 1$$

$$(x-1)(x-1)$$

$$\underbrace{x^2 - 2x + 1}_{(x-1)^2} + y^2 + \underbrace{z^2 + 4z + 4}_{(z+2)^2} = 4 + 1 + 4$$

$$(x-1)^2 + (y-0)^2 + (z-(-2))^2 = 9$$

$(1,0,-2)$ radius 3

We will always try to get as far as possible in class. Completed notes will be available on the course site for the "During Class" portions of the notes. The filled in video notes are available in the video!

2.2 Basic Vector Operations – During Class

Objective(s):

- Define vector addition and scalar multiplication and be able to visualize their actions.
- Develop some properties of vector addition and scalar multiplication.
- Get exposure to types of problems that can be asked regarding vector addition and scalar multiplication.

Now that we have these new mathematical objects we want to know how they interact with each other and real numbers (also called scalars).

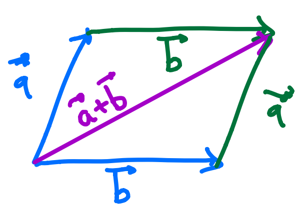
Definition(s) 2.5. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be vectors and c be a scalar. Then:

$$\mathbf{a} + \mathbf{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$$

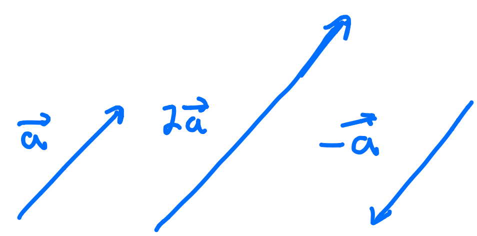
$$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$$

Similarly for 2 dimensional vectors (or really any dimension we want).

Geometrically vector addition is very elegant:



Scalar multiplication acts as expected:



Example 2.6. Consider the vectors $\mathbf{a} = \langle 1, 3 \rangle$ and $\mathbf{b} = \langle -3, 2 \rangle$.

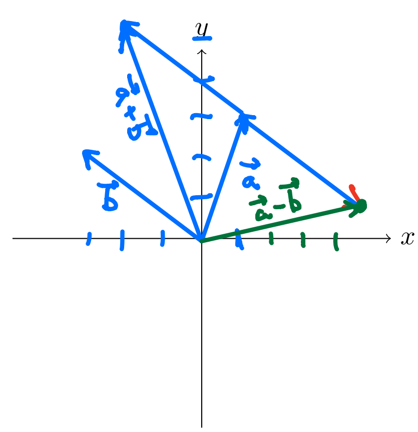
(a) Evaluate $\mathbf{a} + \mathbf{b}$

$\langle 1, 3 \rangle + \langle -3, 2 \rangle = \langle -2, 5 \rangle$

(b) Evaluate $-3\mathbf{a} + 2\mathbf{b}$

$\langle -3, -9 \rangle + \langle -6, 4 \rangle = \langle -9, -5 \rangle$

(c) Sketch $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$, and $\mathbf{a} - \mathbf{b}$ on the graph to the right



$\langle 1, 3 \rangle - \langle -3, 2 \rangle = \langle 4, 1 \rangle$

Now lets see how magnitude is influenced by scalar multiplication:

Theorem 2.7. $|c \cdot \mathbf{a}| = |c| \cdot \|\vec{a}\|$

Proof:

Mini Ex:

$|\langle 100, 300, 0\rangle|$

$= |100\langle 1,3,0\rangle|$

$= 100\,|\langle 1,3,0\rangle|$

$= 100\sqrt{1+9+0} = 100\sqrt{10}$

$\sqrt{100^2+300^2}$

$$\begin{aligned}|c \cdot \mathbf{a}| &= |\langle ca_1, ca_2, ca_3\rangle| \\ &= \sqrt{c^2a_1^2 + c^2a_2^2 + c^2a_3^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2 + a_3^2)} \\ &= |c|\sqrt{a_1^2 + a_2^2 + a_3^2} \\ &= |c| \cdot \|\mathbf{a}\|\end{aligned}$$

Definition(s) 2.8. A vector **a** is called a **unit vector** if it has length 1.

Example 2.9. Find a unit vector in the same direction as $\mathbf{a} = \langle 1, 2, 3\rangle$.

$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$

$\frac{1}{\sqrt{14}}\langle 1,2,3\rangle = \frac{\langle 1,2,3\rangle}{\sqrt{14}}$

$\frac{1}{\sqrt{14}} \cdot \sqrt{14} = 1$

Example 2.10. Find a vector of length 3 in the direction opposite of $\mathbf{a} = \langle -2, 4, 1\rangle$.

$|\vec{a}| = \sqrt{4+16+1} = \sqrt{21}$

$\frac{-3}{\sqrt{21}}\langle -2,4,1\rangle = \left\langle \frac{6}{\sqrt{21}}, \frac{-12}{\sqrt{21}}, \frac{-3}{\sqrt{21}}\right\rangle$

Theorem 2.11. Suppose that $\vec{a} \neq \vec{0}$, then: $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector in the same direction as $\vec{a}$.

Theorem 2.12 (Properties of Vector Operations)**.**

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors and c, d be scalars:

(a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(b) $\mathbf{a} + \mathbf{0} = \mathbf{a}$

(c) $0\mathbf{a} = \vec{\mathbf{0}}$

$0 \cdot \langle a_1, a_2, a_3 \rangle$
$= \langle 0 \cdot a_1, 0 \cdot a_2, 0 \cdot a_3 \rangle = \langle 0, 0, 0 \rangle$

(d) $c(d\mathbf{a}) = (cd)\mathbf{a}$

(e) $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

(f) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

(g) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

(h) $1\mathbf{a} = \mathbf{a}$

(i) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

Definition(s) 2.13. The standard base vectors are:

$\vec{i} = \langle 1, 0, 0 \rangle \quad \vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$

Note: The standard base vectors are all unit vectors.

Any vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ can be written as a linear combination of the standard unit vectors

$$\begin{aligned}
\mathbf{a} &= \langle a_1, a_2, a_3 \rangle \\
&= \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\
&= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle \\
&= a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}
\end{aligned}$$

Therefore we call a_1 the **i-component** of the vector $\mathbf{a}$, a_2 the **j-component** of the vector $\mathbf{a}$, and a_3 the **k-component** of the vector $\mathbf{a}$.

Example 2.14. Write the following vectors as a linear combination of the standard unit vectors

(a) $\mathbf{a} = \langle 5, 2, -4 \rangle$

$5\vec{i} + 2\vec{j} - 4\vec{k}$

(b) $\mathbf{b} = \langle 0, \pi, 100 \rangle$

$\pi\vec{j} + 100\vec{k} = 100\vec{k} + \pi\vec{j}$

3.2 Dot Product Properties and Applications – During Class

Objective(s):

- Sketch simple surfaces in space.
- Determine when a point lies on a specified surface.

Theorem 3.5. This is the exact same as **Theorem 3.1** but with dot product notation.

The angle between two vectors **a** and **b** is given by:

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

or equivalently,

$$\underline{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

Example 3.6. Find the angle between the following vectors.

(a) $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle -1, 3, 2\rangle$

$$\vec{a}\cdot\vec{b} = -1+0+2 = \underline{1}$$

$$|\vec{a}| = \sqrt{1+0+1} = \sqrt{2}$$

$$|\vec{b}| = \sqrt{1+9+4} = \sqrt{14}$$

$$1 = \sqrt{2}\cdot\sqrt{14}\cos\theta$$

$$\frac{1}{\sqrt{28}} = \cos\theta$$

$$\cos^{-1}\left(\frac{1}{\sqrt{28}}\right) = \theta$$

A. 0
B. $\pi/6$
C. $\pi/4$
D. $\pi/3$
E. $\pi/2$

<u>Alt:</u>

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

(b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 4\mathbf{k}$

$a = \langle 1, -1, 3\rangle \quad b = \langle 0, 2, 4\rangle$

$|a| = \sqrt{1+1+9} = \sqrt{11}$

$|b| = \sqrt{0+4+16} = \sqrt{20}$

$\vec{a}\cdot\vec{b} = 0 + (-2) + 12 = 10$

$20 \times 11 = 20 + 200 = 220$

$$\boxed{\theta = \cos^{-1}\left(\frac{10}{\sqrt{220}}\right)}$$

So now that we have this great new form of multiplication between vectors we have to wonder what properties of multiplication hold!?

Theorem 3.7 (Properties of the Dot Product)**.** Let **a, b, c** be vectors and c a scalar:

(a) $\mathbf{a} \cdot \mathbf{b} =$ $\vec{b} \cdot \vec{a}$

(b) $(c\mathbf{a}) \cdot \mathbf{b} =$ $c(\vec{a} \cdot \vec{b})$

(c) $\mathbf{a} \cdot (\mathbf{b}+\mathbf{c}) =$ $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ ←

(d) $\mathbf{0} \cdot \mathbf{a} =$ 0 ← scalar

(e) $\mathbf{a} \cdot \mathbf{a} =$ $|\vec{a}|^2$

Proof of (c)

$\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$
$\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\vec{a} \cdot (\vec{b}+\vec{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1+c_1, b_2+c_2, b_3+c_3 \rangle$$
$$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$$
$$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1 + a_2c_2 + a_3c_3$$

And now a nice physical application, **work**. Force does work. Recall that if we apply a constant force then we have the formula

$$W = f \cdot d$$

And before we had it where force and distance were just numbers so you used regular old multiplication. However we can consider **F** =Force and **D** =*Displacement* to be vectors. The the constant (or scalar) W =Work is given by the formula:

$$W = \vec{F} \cdot \vec{D}$$

Example 3.8. If $|\mathbf{F}| = 20$ Newtons, $|\mathbf{D}| = 5$ meters, and the angle between force and displacement is $\theta = 45$ degrees. What is the work was done by **F**?

$$W = \vec{F} \cdot \vec{D} = |F||D| \cos\theta$$
$$20 \cdot 5 \; \frac{\sqrt{2}}{2} = \boxed{50\sqrt{2} \; N \cdot m}$$

Example 3.9. A box is pushed with a constant force of $\mathbf{F} = \langle 1, 2, 3 \rangle$ Newtons. How much work is done in moving the box from $(1, 0, 1)$ to $(2, 1, 1)$? m

$$D = \langle 1, 1, 0 \rangle \; m$$

$$W = \langle 1,2,3 \rangle \cdot \langle 1,1,0 \rangle$$
$$= 1 + 2 + 0 = \boxed{3 \; N \cdot m}$$

3.3 Projections – During Class

Objective(s):

- Define and calculate Scalar and Vector projections.
- Define and calculate orthogonal projections.
- Be able to visualize and interpret how projections can decompose vectors.

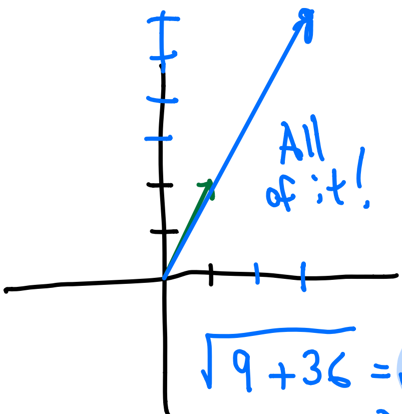
The first projection we will learn about is scalar projection. It answers how much of the vector is in the same direction? Let's do three mini examples before I give the formal definition.

Example 3.10. Consider the vector $\mathbf{a} = \langle 1, 2\rangle$. Sketch it and answer the following questions.

(a) How much of the vector

$\mathbf{b} = \langle 3, 6\rangle$

is in the same direction as $\mathbf{a}$?



All of it!

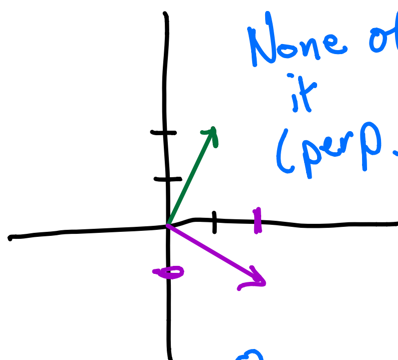
$\sqrt{9+36} = \sqrt{45}$

(its entire length)

(b) How much of the vector

$\mathbf{b} = \langle 2, -1\rangle$

is in the same direction as $\mathbf{a}$?



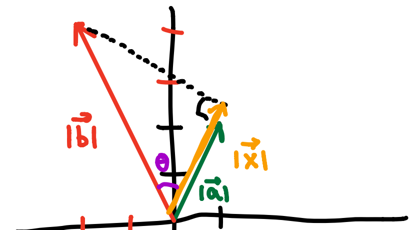
None of it (perp.)

0

(c) How much of the vector

$\mathbf{b} = \langle -2, 4\rangle$

is in the same direction as $\mathbf{a}$?



$\cos\theta = \frac{|x|}{|b|}$

$\frac{|a||b|\cos\theta}{|a|} = |x|$

$\frac{a\cdot b}{|a|} = |x|$

Definition(s) 3.11. The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ is notated by $\text{comp}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{a\cdot b}{|a|}$$

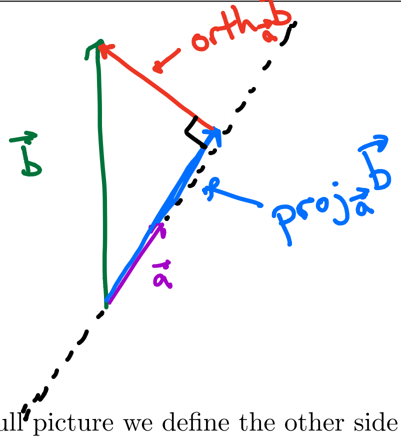
While this is a good start it is a little awkward. It seems the questions of "*How much of the vector is in the same direction?*" could be better answered with a vector. Which leads to the next projection

Definition(s) 3.12. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is notated by $\text{proj}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}$$

Which is all well and nice but what does it look like?

Picture:



$$\text{proj}_{\vec{a}}\vec{b} + \text{orth}_{\vec{a}}\vec{b} = b$$

And so to really have a full picture we define the other side of this triangle.

Definition(s) 3.13. The orthogonal projection of **b** onto **a** is notated by $\text{orth}_{\mathbf{a}}(\mathbf{b})$ and can determined by:

$$\text{orth}_{\mathbf{a}}(\mathbf{b}) = \vec{b} - \text{proj}_{\vec{a}}\vec{b} = b - \left(\frac{a \cdot b}{|a|^2}\right)\vec{a}$$

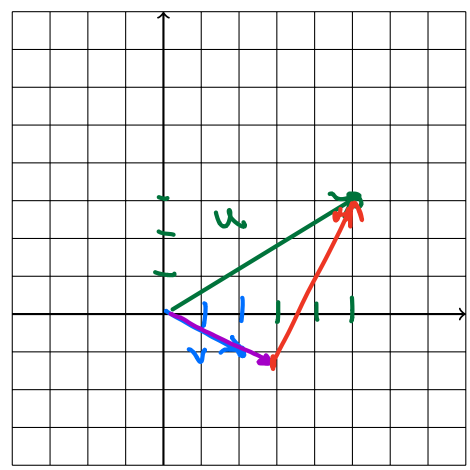
Note $\text{proj}_{\mathbf{a}}(\mathbf{b})$ is parallel to $\vec{a}$ and $\text{orth}_{\mathbf{a}}(\mathbf{b})$ is orthogonal to $\vec{a}$.

Example 3.14. Consider the vector $\mathbf{u} = \langle 5, 3\rangle$ and the vector $\mathbf{v} = \langle 2, -1\rangle$.

(a) Sketch ~~a~~ v on the axes to the right.

(b) Calculate $\text{proj}_{\mathbf{v}}(\mathbf{u})$ and sketch it on the axes as well.

(c) Calculate $\text{orth}_{\mathbf{v}}(\mathbf{u})$ and sketch it too.



(b) $$\frac{u \cdot v}{|\vec{v}|^2}\vec{v} = \frac{7}{5}\langle 2, -1\rangle = \left\langle \frac{14}{5}, \frac{-7}{5}\right\rangle$$

(c) $$\vec{u} - \left\langle \frac{14}{5}, \frac{-7}{5}\right\rangle$$

$$\left\langle \frac{25}{5}, \frac{15}{5}\right\rangle - \left\langle \frac{14}{5}, \frac{-7}{5}\right\rangle = \left\langle \frac{11}{5}, \frac{22}{5}\right\rangle \approx \langle 2.2, 4.4\rangle$$

Example 3.15 (Now you try! WW 12.3.3)**.** Consider $\mathbf{a} = -5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 8\mathbf{j} - \mathbf{k}$.

Calculate the following scalar and vector projections: $\langle -5, 5, 2\rangle$ $\langle -1, 8, -1\rangle$

(a) $\text{proj}_{\mathbf{a}}\mathbf{b}$

$$\frac{\vec{a} \cdot \vec{b}}{|a|^2}\vec{a} = \frac{5+40-2}{25+25+4}\langle -5, 5, 2\rangle = \frac{43}{54}\langle -5, 5, 2\rangle$$

(b) $\text{comp}_{\mathbf{b}}\mathbf{a}$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{5+40-2}{\sqrt{1+64+1}} = \frac{43}{\sqrt{66}}$$

4.2 Properties and Applications of the Cross Product – During Class

Objective(s):

- Determine and utilize properties of the cross product.
- Use the cross product to calculate areas of triangles and parallelograms.
- Apply the cross product to the physical application of torque.

Example 4.6. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a} = \langle 1, 1, -1 \rangle$ and $\mathbf{b} = \langle 2, 4, 6 \rangle$.

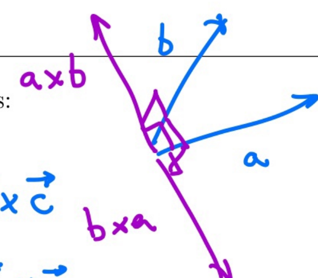
$$\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = (6+4)i - (6+2)j + (4+2)k$$

$$= \langle 10, -8, 2 \rangle$$

$$\langle 10,-8,2\rangle \cdot \langle 1,1,-1\rangle = 10-8-2 = 0$$

$$\langle 10,-8,2\rangle \cdot \langle 2,4,6\rangle = 20-32+12 = 0$$

$\langle 0,0,0 \rangle$ $\qquad$ 0



Theorem 4.7 (Properties of the Cross Product)**.** Let **a, b, c** be vectors and r, s are scalars:

(a) $\mathbf{a} \times \mathbf{b} = -(\vec{b} \times \vec{a})$

(b) $(r\mathbf{a}) \times (s\mathbf{b}) = (rs)(\vec{a} \times \vec{b})$

4 1 4

(c) $\mathbf{0} \times \mathbf{a} = \vec{0}$

(d) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(e) $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$

Example 4.8. Given that $\langle 1, 1, 0 \rangle \times \langle 3, 4, -2 \rangle = \langle -2, 2, 1 \rangle$ quickly calculate the following:

(a) $\langle 3, 4, -2 \rangle \times \langle 1, 1, 0 \rangle = -\langle -2, 2, 1\rangle = \langle 2, -2, -1 \rangle$

(b) $\langle 4, 4, 0 \rangle \times \langle 3, 4, -2 \rangle$

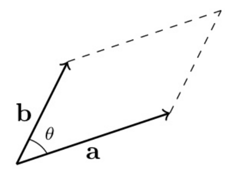
$4\langle 1,1,0\rangle \times \langle 3,4,-2\rangle = 4\langle -2,2,1\rangle$

$= \langle -8, 8, 4 \rangle$

Recall from geometry that a parallelogram with side lengths x and y with the angle θ between its sides has area: $A = xy\sin\theta$. Translated into MTH 234 notation:

Theorem 4.9. The parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$ with angle θ between them is given by:

Area of $\|$-ogram $= |a||b|\cdot\sin\theta = |\vec{a}\times\vec{b}|$



Example 4.10. Find the area of the parallelogram generated by $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 3\mathbf{k}$.

$v\times u = \langle 3,3,-1\rangle$

$u = \langle 1,-1,0\rangle$

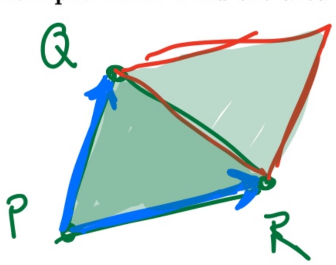
$v = \langle 0,1,3\rangle$

$$u\times v = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = (-3-0)i - (3-0)j + (1-0)k$$

$$= \langle -3,-3,1\rangle$$

$$|u\times v| = \sqrt{9+9+1} = \sqrt{19}$$

Example 4.11. Find the area of the triangle with vertices $P(1,0,1), Q(-2,1,3)$, and $R(4,2,5)$.



$\vec{PQ} = \langle -3,1,2\rangle$

$\vec{PR} = \langle 3,2,4\rangle$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = (4-4)i - (-12-6)j + (-6-3)k$$

$$= \langle 0,18,-9\rangle$$

$$|\vec{PQ}\times\vec{PR}| = \sqrt{18^2+81}$$

$$\text{Area } \Delta PQR = \frac{\sqrt{18^2+81}}{2} = \frac{9\sqrt{5}}{2}$$

Example 4.12. Find two unit vectors orthogonal to both $\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$.

length 1

$\sqrt{4}=2$

$-\sqrt{4}=-2$

$\langle 0,1,-1\rangle \quad \langle 1,1,0\rangle$

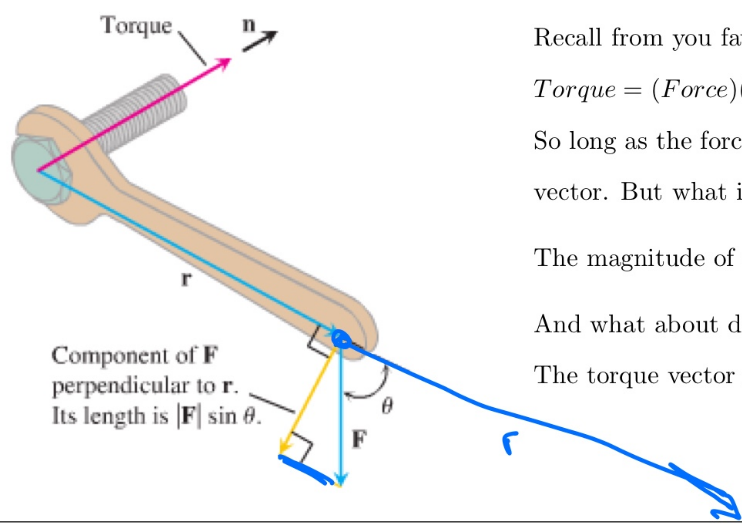
$$\begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (0+1)i-(0+1)j+(0-1)k$$

$$\boxed{(\pm)\langle 1,-1,-1\rangle}$$

$\sqrt{3}$ $\quad \sqrt{1+1+1}=\sqrt{3}$

$$\frac{\langle 1,-1,-1\rangle}{\sqrt{3}}=\frac{1}{\sqrt{3}}\langle 1,-1,-1\rangle=\left\langle \frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right\rangle$$

So our application to the real world of the day is Torque! Here is the picture



Recall from you favorite physics class that

$Torque=(Force)(Distance$ from pivot$)$.

So long as the force is being applied perpendicular to the distance vector. But what if its not?

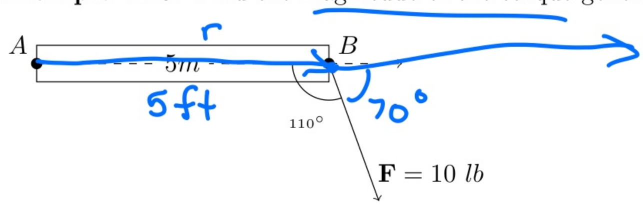
The magnitude of the torque vector is $=|r||F|\sin\theta$

And what about direction?

The torque vector is of course given by:

$$\vec{\tau}=\vec{r}\times\vec{F}$$

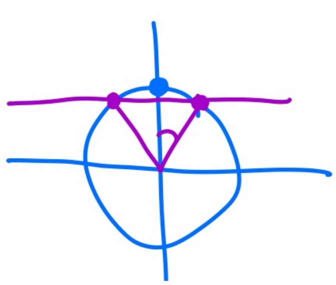
Example 4.13. Find the magnitude of the torque generated by force $\mathbf{F}$ at the pivot point A in the figure below



$$|\vec{\tau}|=|\vec{r}\times\vec{F}|=|r||F|\sin\theta$$

$$=5\,ft\cdot 10\,lb\cdot\sin(\cancel{110^\circ})$$

$$=50\sin(70^\circ)\ ft\,lb$$



5.2 Parametrizations, Line Segments, and More Examples – During Class

Objective(s):

- Determine when two parametrizations describe the same line.
- Create a way to parametrize a piece of a line.
- Gain more exposure to types of line problems that can be asked.

Example 5.5. One small annoyance with parametrizing lines is that the parametrization is not unique. Using a graphing utility show that

$$1 \qquad 5 \qquad 0$$

$$L_1 : \quad x = 1 + 2t \quad y = 5 - 2t \quad z = 6t \qquad v_1 = \langle 2, -2, 6 \rangle$$

$$L_2 : \quad x = 3 - s \quad y = 3 + s \quad z = 6 - 3s \qquad v_2 = \langle -1, 1, -3 \rangle$$

are the same line

Theorem 5.6. Two parametrizations $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ describe the same line if they are intersecting and parallel ✓.

Now let's use **Theorem 5.6** to show that L_1 and L_2 describe the same line in **Example 5.5**.

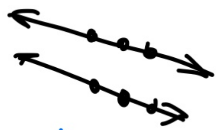
$v_1 k = v_2$

$k = -1/2$

$v_2 k = v_1$

$\langle -1, 1, -3 \rangle k = \langle 2, -2, 6 \rangle$

$k = -2$ parallel ✓



$$\begin{aligned} 1 + 2t &= 3 - s \\ 5 - 2t &= 3 + s \\ \hline 6 &= 6 \\ 6t &= 6 - 3s \end{aligned}$$

$z = 0$

$t = 0$

$s = 2$

they intersect at $(1,5,0)$ ✓

Theorem 5.7 (Equation of a line segment)**.** The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is give by the vector equation

$$\vec{r}(t) = t\vec{r}_1 + (1-t)\vec{r}_0 \qquad 0 \le t \le 1$$

Example 5.8. Find an equation for the line segment from $(1, 2, 3)$ to $(5, 2, 0)$.

$$\begin{aligned} \vec{r}(t) &= t\langle 5,2,0\rangle + (1-t)\langle 1,2,3\rangle \\ &= \langle 5t, 2t, 0\rangle + \langle 1,2,3\rangle - \langle t, 2t, 3t\rangle \\ &= \langle 4t, 0, -3t\rangle + \langle 1,2,3\rangle \\ &= \langle 4t+1, 2, -3t+3\rangle \qquad 0 \le t \le 1 \end{aligned}$$

Example 5.9

(a) Find parametric equations of the line that passes through the points $A(2,3,4)$ and $B(1,0,-1)$.

$\vec{AB} = \langle -1,-3,-5\rangle$ $\quad \vec{r}(t) = \langle 1,0,-1\rangle + t\langle -1,-3,-5\rangle$

$x = 1$

$= \langle 1-t, -3t, -1-5t\rangle$

$\uparrow \quad \uparrow \quad \uparrow$

$x \quad y \quad z$

(b) At what point does the line intersect the xy-plane.

$z = 0$

$-1-5t = 0$

$-1/5 = t$

$\left(\frac{6}{5}, \frac{3}{5}, 0\right)$

Example 5.10 (Now you try! WW 12.5.9). Find the t value for which the line $L(t) = \langle -2, 8+3t, 2+3t\rangle$ intersects the plane $x + 2y + z = 7$.

$x + 2y + z = 7$

$-2 + 2(8+3t) + (2+3t) = 7$

$9t + 16 = 7$

$\boxed{t = -1}$

Example 5.11. The lines $\mathbf{r}_1(t) = \langle 1+t, 1-t, 2t\rangle$ and $\mathbf{r}_2(s) = \langle 2-s, s, 2\rangle$ intersect at $(2,0,2)$. Determine the angle between the lines.

$v_1 = \langle 1,-1,2\rangle \qquad v_2 = \langle -1,1,0\rangle$

$v_1 \cdot v_2 = -1-1+0 = -2$

this obtuse ∡ so consider $w_2 = -v_2 = \langle 1,-1,0\rangle$ so

$v_1 \cdot w_2 = 2$

$|v_1| = \sqrt{6}$ so $\quad 2 = \sqrt{12}\cos\theta$

$|w_2| = \sqrt{2}$

$\theta = \cos^{-1}\left(\frac{2}{\sqrt{12}}\right)$

MTH 234 Chapter 12 - Vectors and Geometry of Space MSU

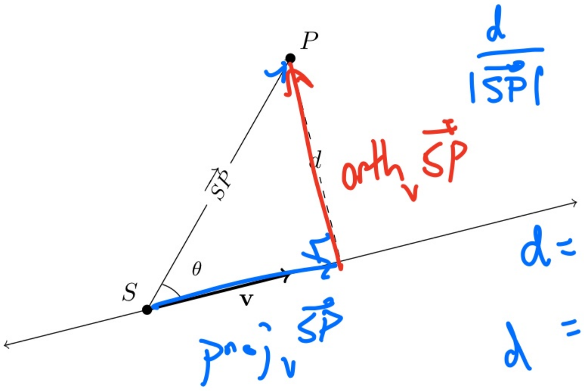
5.3 Distance from a Point to a Line – During Class

Objective(s):

- Develop a formula to determine the distance from a point to a line.
- Utilize the newly developed formula to calculate the distance from a point to a line in space!

Now we have lines, we have points, lets talk about distance!

Here is a pretty picture



$$\frac{d}{|\vec{SP}|} = \sin\theta$$

$$d = |\vec{SP}|\sin\theta$$

$$d = \frac{|\vec{v}||\vec{SP}|\sin\theta}{|\vec{v}|}$$

$$d = \frac{|\vec{v} \times \vec{SP}|}{|\vec{v}|}$$

Theorem 5.14. The distance from a Point P to a line through S parallel to $\mathbf{v}$ is given by

$$d = \frac{|\vec{v} \times \vec{SP}|}{|\vec{v}|}$$

And this is a perfectly good Theorem but that triangle looks like something we have seen before when we were talking about projections. So in fact....

Theorem 5.15. The distance from a Point P to a line through S parallel to $\mathbf{v}$ is given by

$$d = |\text{orth}_v \vec{SP}|$$

Example 5.16. Find the distance between the point $\overset{P}{(0,1,0)}$ and the line containing the points $\overset{S}{(1,1,0)}$ and $(2,-4,1)$.

(a) By using **Theorem 5.14**

$\vec{v} = \langle 1,-5,1\rangle$

$\vec{SP} = \langle -1,0,0\rangle$

$$d = \frac{|\vec{SP}\times\vec{v}|}{|\vec{v}|} = \frac{\sqrt{0+1+25}}{\sqrt{1+25+1}} = \boxed{\frac{\sqrt{26}}{\sqrt{27}}} = \boxed{\frac{\sqrt{26}\cdot\sqrt{27}}{27}}$$

$$\vec{SP}\times\vec{v} = \begin{vmatrix} i & j & k \\ -1 & 0 & 0 \\ 1 & -5 & 1 \end{vmatrix} = (0)i-(-1)j+(5)k = \langle 0,1,5\rangle$$

$\vec{v} = \langle 1,-5,1\rangle$

$\vec{SP} = \langle -1,0,0\rangle$

(b) By using **Theorem 5.15**

$$d = |\text{orth}_{\vec{v}}\vec{SP}| = |\vec{SP} - \text{proj}_{\vec{v}}\vec{SP}|$$

$$= \left|\langle -1,0,0\rangle + \frac{1}{27}\langle 1,-5,1\rangle\right|$$

$$= \left|\left\langle \frac{-26}{27}, \frac{-5}{27}, \frac{1}{27}\right\rangle\right|$$

$$= \boxed{\sqrt{\frac{26^2}{27^2}+\frac{25}{27^2}+\frac{1}{27^2}}}$$

$$= \sqrt{\frac{26^2+26}{27^2}}$$

$$= \frac{\sqrt{26^2+26}}{27} = \boxed{\frac{\sqrt{26(27)}}{27}}$$

$$\text{proj}_{\vec{v}}\vec{SP} = \frac{\vec{SP}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{-1}{27}\langle 1,-5,1\rangle$$

5.5 Double the Planes, Double the Fun – During Class

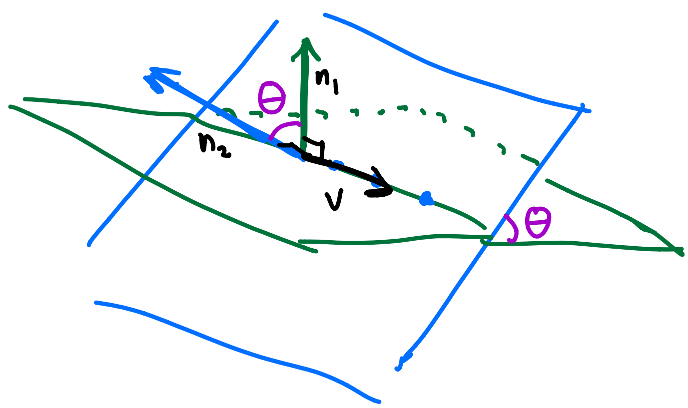
Objective(s):

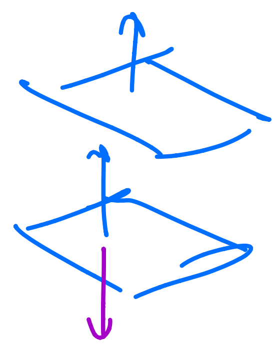
- Determine when two planes are parallel.
- If two planes intersect find the line of intersection.
- Calculate the angle of intersection between two planes.

Let's take a look at how planes interact: `https://goo.gl/hCbfzE`

Definition(s) 5.21.

(a) The angle between two planes is defined to be the acute angle between their normal vectors





(b) Two planes are parallel if their normal vectors are parallel.

(c) If two planes are not parallel then they intersect at a straight line.

Example 5.22. Show that planes $x + y = 2z + 4$ and $4z - 2x = 2y + 5$ are parallel.

~~$\langle 1,1,2\rangle$~~

$x+y-2z=4$

$\langle 1,1,-2\rangle$ or $\langle -1,-1,2\rangle$

$-2x-2y+4z=5$

$\langle -2,-2,4\rangle$

$k\langle 1,1,-2\rangle = \langle -2,-2,4\rangle$

$-2\langle 1,1,-2\rangle = \langle -2,-2,4\rangle$

so the normal vectors are parallel
therefore the planes are parallel

Example 5.23. Find the line of intersection for the planes $x + y + z = 1$ and $x + 2y + 2z = 1$.

https://goo.gl/hCbfzE

$\vec{r}(t) = \vec{r}_0 + \vec{v}\,t$

$z = 0$

$\rightarrow x + y = 1$

$\rightarrow x + 2y = 1$

$\cancel{x} + y = \cancel{x} + 2y$

$0 = y$

$1 = x$

$\langle 1, 0, 0 \rangle$

$n_1 = \langle 1, 1, 1 \rangle$

$n_2 = \langle 1, 2, 2 \rangle$

$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (2-2)i - (2-1)j + (2-1)k$

$= \langle 0, -1, 1 \rangle$

$\vec{r}(t) = \langle 1, 0, 0 \rangle + \langle 0, -1, 1 \rangle t$

$= \langle 1, -t, t \rangle$

Theorem 5.24. If P_1 and P_2 are non-parallel planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ then their line of intersection has direction vector:

$$\mathbf{v} = n_1 \times n_2$$

Example 5.25. Find the angle between planes $3x - 6y - 2z = 15$ and $2x - 2z = 5 - y$

$\langle 3, -6, -2 \rangle$ $\cancel{\langle 2, -2 \rangle}$ $\langle 2, 1, -2 \rangle$

$\langle -3, 6, 2 \rangle$

$\sqrt{9 + 36 + 4} = \sqrt{49} = 7$

$\sqrt{4 + 1 + 4} = \sqrt{9} = 3$

$6 - 6 + 4 = \dfrac{-4}{\sqrt{441}}$

$\begin{array}{r} 36 \\ +4 \\ \hline 40 \end{array}$

$\begin{array}{r} 49 \\ \times 9 \\ \hline 441 \end{array}$

$\cos^{-1}\left(\dfrac{-4}{\sqrt{441}}\right) = \theta$

$\cos^{-1}\left(\dfrac{-4}{21}\right) = \theta$

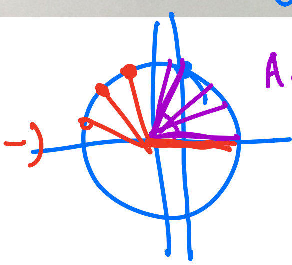
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Acute?

$\cos\theta = \dfrac{4}{21}$

Obtuse

$\cos\theta$ is $(-)$



Acute

$\cos\theta$ is $(+)$

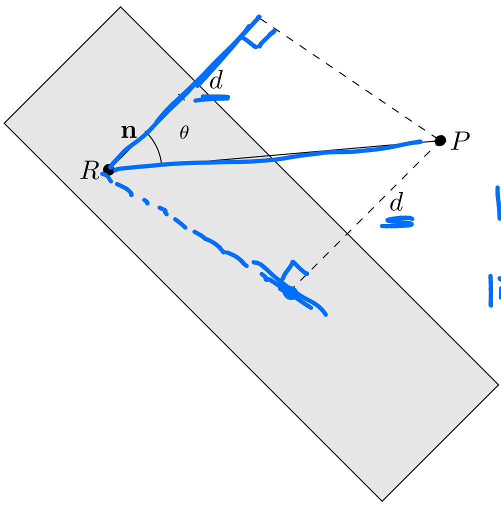
get rid of $(-)$ in dot product for acute angle

5.6 Distance from a Point to a Plane – During Class

Objective(s):

- Develop a formula to determine the distance from a point to a plane.
- Utilize the newly developed formula to calculate the distance from a point to a plane in space!
- Use the same formula to determine distance between a line and plane and between two planes.

Now we should develop a theorem for the distance between a point and a plane



$\cos\theta = \dfrac{d}{|\vec{RP}|}$

$|\vec{RP}|\cos\theta = d$

$\dfrac{|\vec{n}||\vec{RP}|\cos\theta}{|\vec{n}|} = d$

$\dfrac{|\vec{n}\cdot\vec{RP}|}{|\vec{n}|} = d$

Theorem 5.26. The distance from a Point P to a plane containing R with $\mathbf{n}$ normal to the plane is given by

$$d = \frac{|\vec{n}\cdot\vec{RP}|}{|\vec{n}|} = |\text{comp}_{\vec{n}}\vec{RP}|$$

Example 5.27. Find the distance between the point $P(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$

$\vec{n} = \langle 3, 2, 6\rangle$

$R = (1, 1, 0)$ $\qquad R = (\frac{5}{3}, 0, 0)$

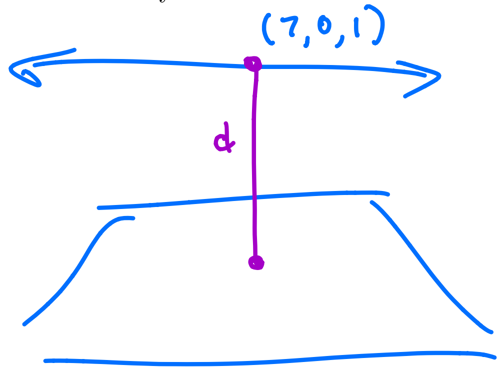
$P = (1, -2, 4)$

$\vec{RP} = \langle 0, -3, 4\rangle$

$$d = \frac{|\vec{n}\cdot\vec{RP}|}{|\vec{n}|} = \frac{|-6+24|}{\sqrt{9+4+36}} = \frac{18}{\sqrt{49}} = \frac{18}{7}$$

Example 5.28. Consider the line $\mathbf{r}(t) = \langle \underbrace{7+4t}_{x}, \underbrace{2t}_{y}, \underbrace{1-t}_{z} \rangle$ and the plane $x - y + 2z = 5$. If they intersect, find the point of intersection. If they don't intersect find the distance between them.

$$7+4t-2t+2(1-t)=5$$
$$\vdots$$
$$9=5$$

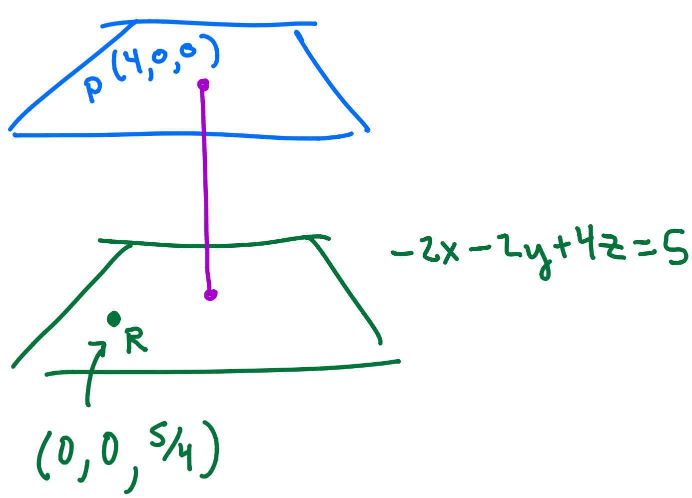


$$P=(7,0,1)$$
$$R=(5,0,0)$$
$$\vec{RP}=\langle 2,0,1\rangle$$

$$\vec{n}=\langle 1,-1,2\rangle$$

$$d=\frac{|2+0+2|}{\sqrt{1+1+4}}=\boxed{\frac{4}{\sqrt{6}}}$$

Example 5.29. Consider the planes $x + y = 2z + 4$ and $4z - 2x = 2y + 5$. If they intersect, find the line of intersection. If they don't intersect find the distance between them.



$$\vec{n}=\langle -2,-2,4\rangle$$
$$\vec{RP}=\langle 4,0,-5/4\rangle$$

$$d=\frac{|-8+0-5|}{\sqrt{4+4+16}}=\boxed{\frac{13}{\sqrt{24}}}=\boxed{\frac{13}{2\sqrt{6}}}$$

6.2 Battling Quadric Surfaces – During Class

Objective(s):

- Identify and Sketch Quadric Surfaces.
- Do lots of problems.

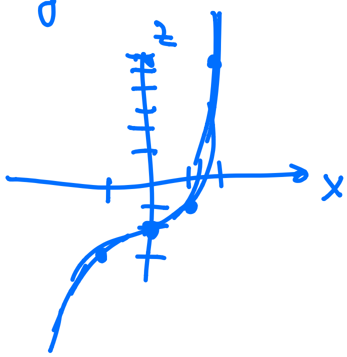
Example 6.4. For all of the questions complete the following.

(i) Classify the curve as a: Cylinder, Ellipsoid, Elliptical Paraboloid, Elliptical Cone, Hyperboloid of one sheet, Hyperboloid of two sheets, or a Hyperbolic Paraboloid.

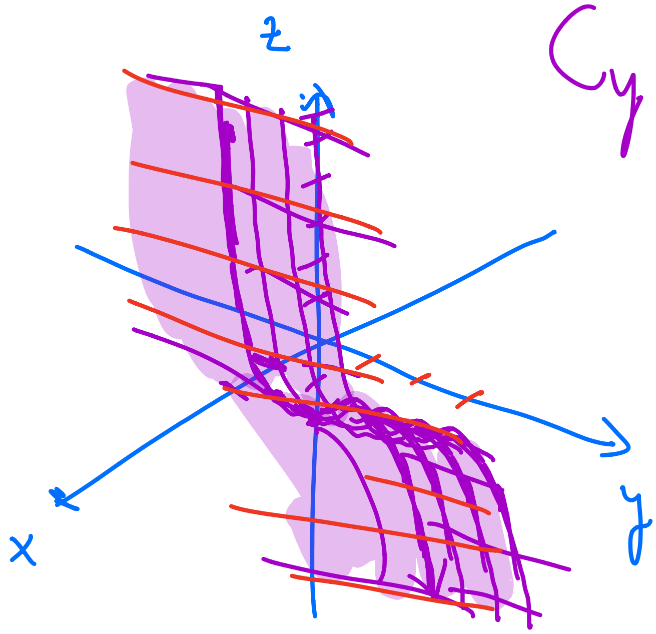
(ii) Draw a sketch of each curve.

(a) $z = x^3 - 2$

$y=0$



$y=1,\ y=2,\ y=3$



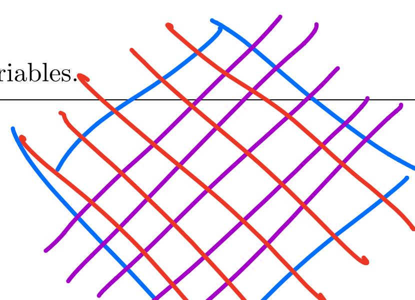
Cylinder

Theorem 6.5. If a surface is missing a variable then it is a cylinder.

Remark 6.6 (All thumbs are fingers but not all fingers are thumbs).

Theorem 6.5 does NOT say that all cylinders are missing variables.

$2x+3y+4z=1$



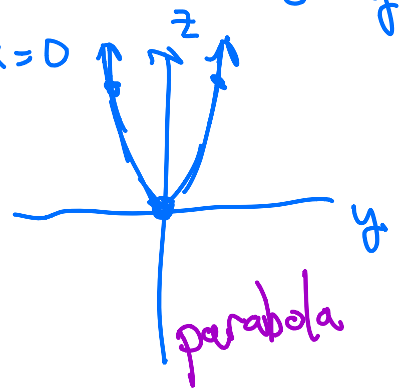
Example 6.7. For all of the questions complete the following.

(i) Classify the curve as a: Cylinder, Ellipsoid, Elliptical Paraboloid, Elliptical Cone, Hyperboloid of one sheet, Hyperboloid of two sheets, or a Hyperbolic Paraboloid.

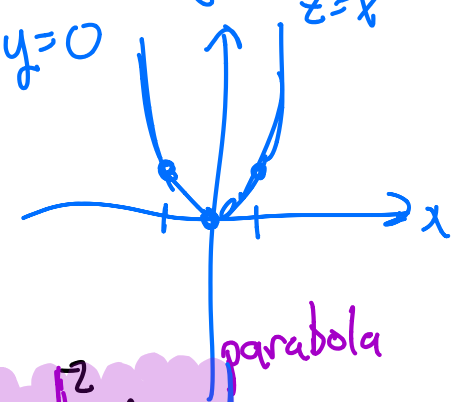
(ii) Draw a sketch of each curve.

(a) $z = x^2 + 9y^2$

$x = 0$ $\quad z = 9y^2$

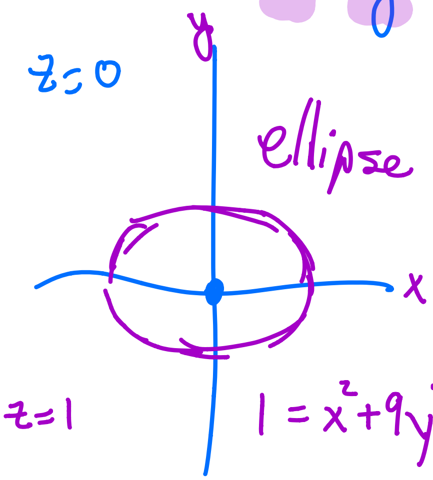


$y = 0$ $\quad z = x^2$



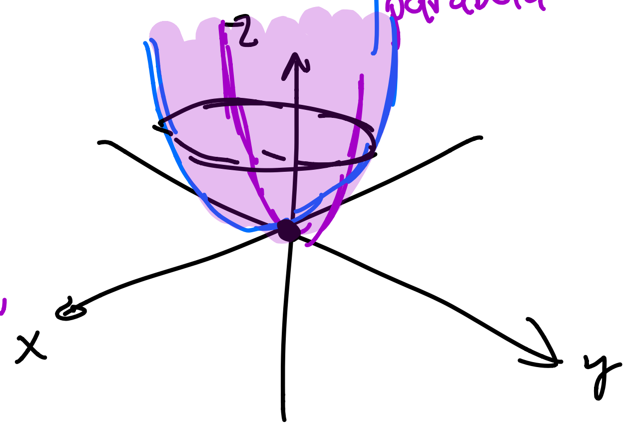
$z = 0$ $\quad 0 = x^2 + 9y^2$

ellipse



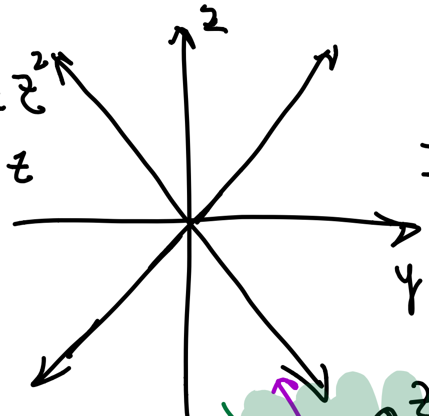
$z = 1$ $\quad 1 = x^2 + 9y^2$

Elliptical Paraboliod

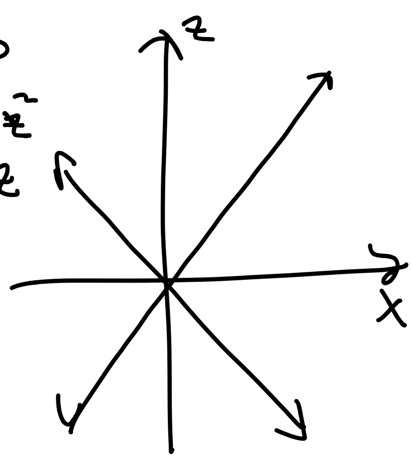


(b) $x^2 + y^2 = z^2$

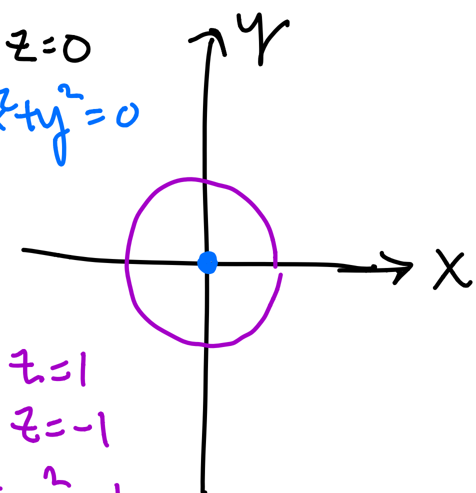
$x = 0$
$y^2 = z^2$
$\pm y = z$



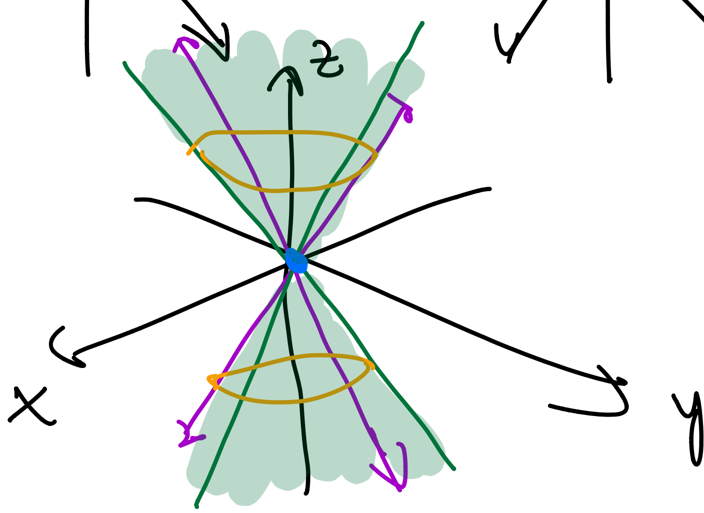
$y = 0$
$x^2 = z^2$
$\pm x = z$



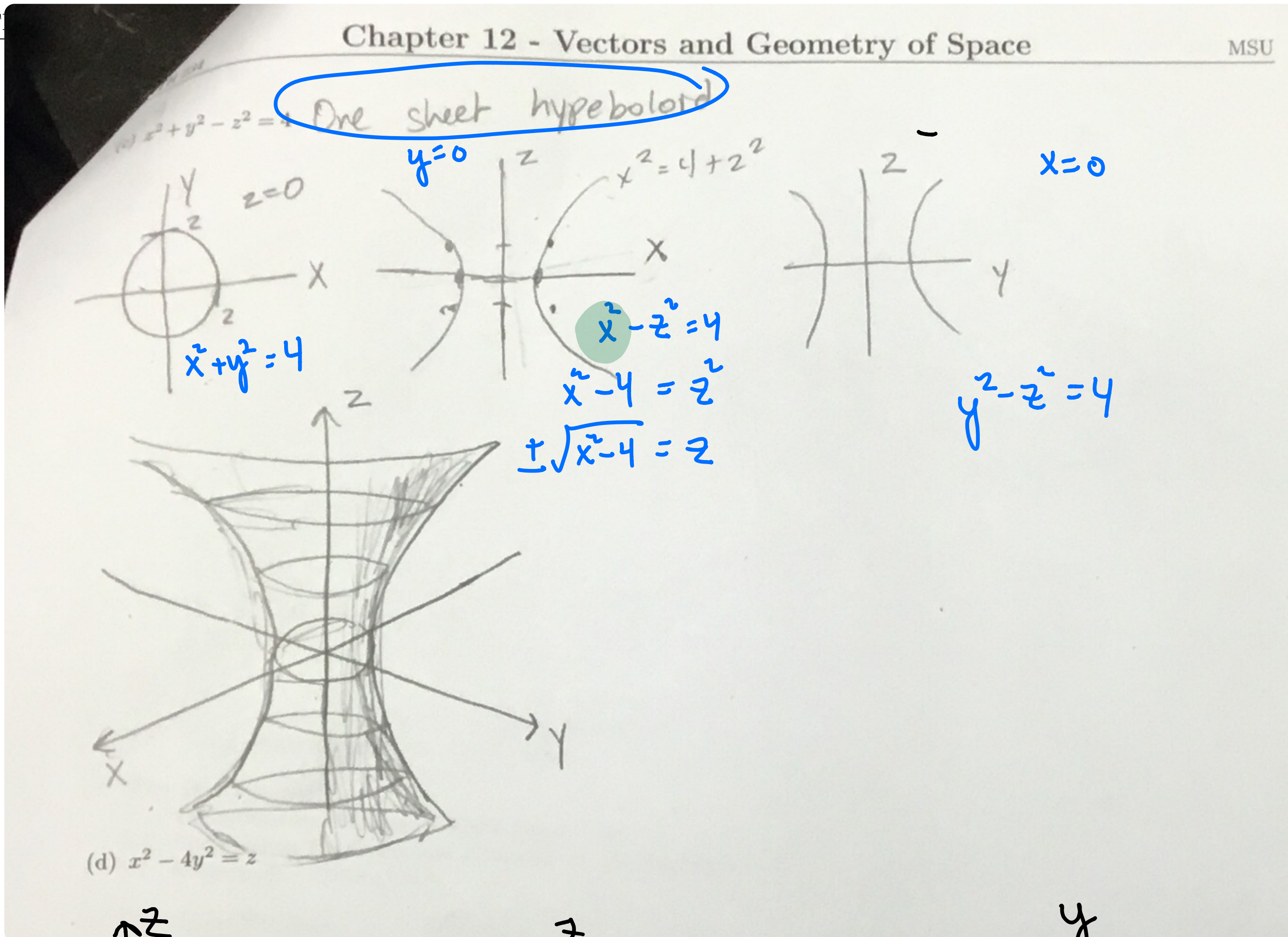
$z = 0$
$x^2 + y^2 = 0$



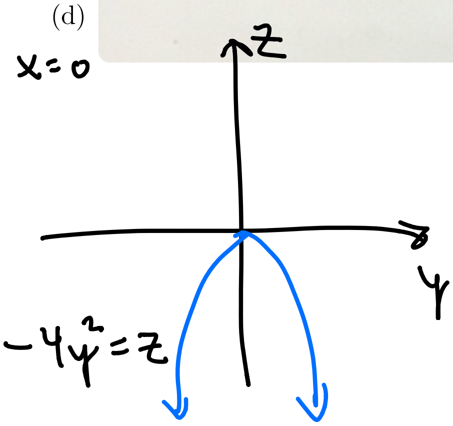
$z = 1$
$z = -1$
$x^2 + y^2 = 1$

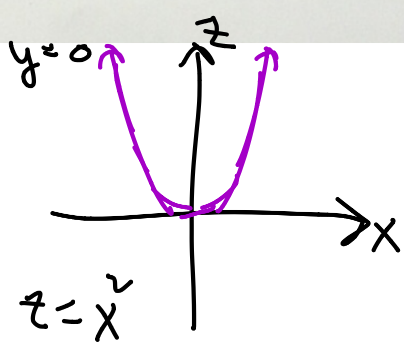


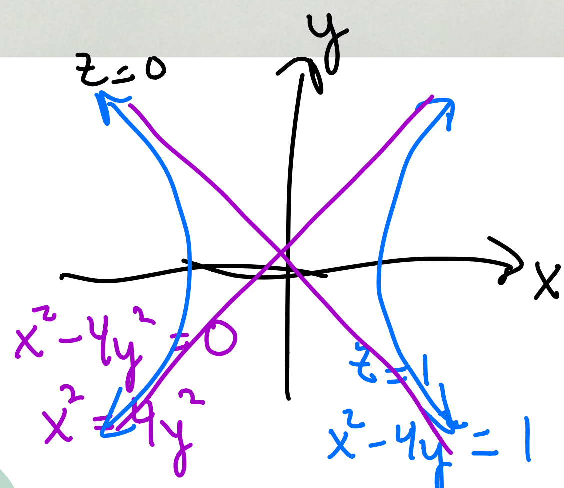
(c)



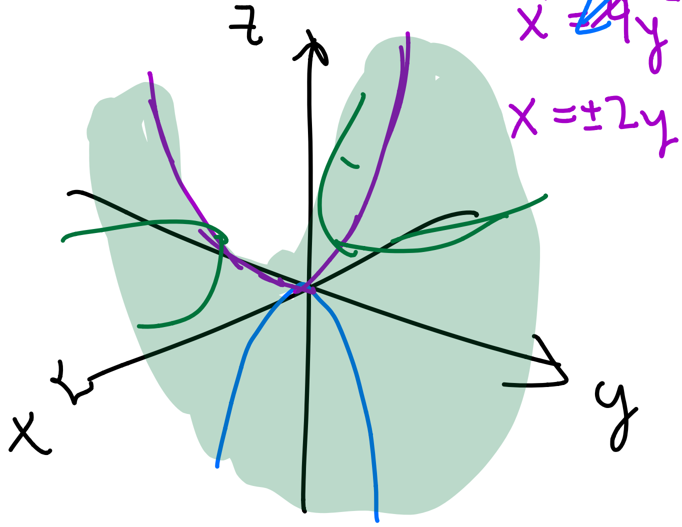
(d)







Hyperbolic Paraboloid



(e) $x^2 + 2z^2 - 6x - y + 10 = 0$

$$x^2 - 6x + 9 - y + 2z^2 + 10 = 9$$

$$(x-3)^2 - y + 2z^2 + 1 = 0$$

Elliptical Paraboloid

$x = 3$ $\quad -y + 2z^2 + 1 = 0$

$2z^2 + 1 = y$ parabola

$y = 0$ $\quad (x-3)^2 + 2z^2 = -1$

$y = 2$ $\quad (x-3)^2 + 2z^2 = 1$ ellipse

$z = 0$

$$(x-3)^2 - y + 1 = 0$$

$$(x-3)^2 + 1 = y$$ parabola