12.5 Part 1: Lines

A note about notation:

If $P(x_0, y_0, z_0)$ is a point then $\boldsymbol{p}$ will denote the vector $\boldsymbol{p} = \langle x_0, y_0, z_0 \rangle$.

Useful Information.

- The **line through a point** P **parallel to** $\boldsymbol{v}$ is given by

$$\boldsymbol{r}(t) = \boldsymbol{p} + t\boldsymbol{v}, \quad t \in \mathbb{R} \tag{1}$$

- The **line segment** beginning at a point R_0 and ending at the point R_1 is given by

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 \quad t \in \mathbb{R} \tag{2}$$

$$\text{or equivalently} \quad \boldsymbol{r}(t) = \boldsymbol{r}_0 + t(\boldsymbol{r}_1 - \boldsymbol{r}_0) \quad t \in \mathbb{R} \tag{3}$$

- A **parametric equation** for a line is of the form

$$\boldsymbol{r}t = \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

and if $a, b, c \neq 0$ a **symmetric equation**[a] is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[a] This comes from $\langle x, y, z \rangle = \langle at + x_0, bt + y_0, ct + z_0 \rangle$ and solving each component for t, e.g. $x = at + x_0, \ldots$

1. Find an equation for the line segment $\boldsymbol{r}(t)$ with $\boldsymbol{r}(0) = (3, -7, 0)$ and $\boldsymbol{r}(1) = (18, -7, \pi)$, $0 \leq t \leq 1$.

2. Let $\boldsymbol{f}(t) = \langle 1 + 2t, -10t, 2200111 - 7t \rangle$. Let L be a line parallel to $\boldsymbol{f}(t)$ that passes through the point $(0, 0, 0)$.

   (a) Find a parametric equation for L.

   (b) Use the result above to find a symmetric equation for L.

3. Suppose L is represented by the equation $\boldsymbol{r}_1(t) = \langle 2t, 1-t, 2\pi + \pi t\rangle$. Determine which of the following points belong to L

   (a) $P(-4, 3, 0)$

   (b) $Q(1, 1/2, 5\pi)$

   (c) $Q(20, -9, 12\pi)$

4. Let L denote the line with symmetric equation

$$\frac{x-1}{2} = y = \frac{z+1}{3}$$

   (a) Find a parametric equation representing L.

   (b) Determine if the line that passes through the points $(1, -5, 5)$ and $(-1, 0, 2)$ intersects L. If not, determine if it is parallel or skew to L.

12.5 Part 2: Planes

Useful Information. A plane with normal vector $\boldsymbol{n} = \langle a, b, c \rangle$ containing the point $P(x_0, y_0, z_0)$ is represented by the equation

$$\begin{aligned} 0 &= \boldsymbol{n} \cdot (\langle x, y, z \rangle - \boldsymbol{p}) \\ &= \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= ax + by + cz - (ax_0 + by_0 + cz_0) \end{aligned}$$

or

$$ax + by + cz = d \quad \text{where} \quad d = ax_0 + by_0 + cz_0$$

5. Find a unit normal vector for the plane that contains the line $\boldsymbol{r}(t) = \langle t, -t, 2t \rangle$ and the point $P(0, 0, 1)$.

6. Find the equation of a plane that contains the triangle with vertices $P(0, 0, 0)$, $Q(5, 2, 0)$, and $R(0, 1, 1)$.

7. Find an equation for the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

8. At what point does the line through the points $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$.

9. The planes $x + y + z = 1$ and $x - 2y + 3z = 1$ intersect in a line.
   (a) Find the angle between the two planes (hint: think about normal vectors)

   (b) Find the parametric equation representing the line determined by the intersection of the two planes (hint: think about normal vectors again)

10. Let $\mathcal{P}$ be the plane represented by

$$x + 2y - z = 4$$

Find the equation for the plane that intersects $\mathcal{P}$ and is parallel to a normal vector for $\mathcal{P}$.

11. The goal of this exercise is to find the distance from the point $P(1, -2, 4)$ to the plane $3x + 2y + 6z = 5$.

    (a) Find a point Q on the plane $3x + 2y + 6z = 5$ (the choice of point does not matter, you just need to choose one).

    (b) Find the vector PQ using the points above.

    > *Remark.* Unless you were very lucky, the vector PQ is not the shortest vector connecting P to the plane. To find the shortest distance possible, we project PQ onto the normal vector for the plane.

    (c) Find the normal vector $\boldsymbol{n}$ for the plane.

    (d) Find $||\text{proj}_{\boldsymbol{n}}(PQ)||$. This is the distance between P and the plane. See the image below.