Surface Integrals

1. **Parametric Surfaces:**

   For a surface S given by $\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D$ The integral of $f(x,y,z)$ over S is given by

$$\boxed{\iint_S f(x,y,z)\ dS = \iint_D f(\boldsymbol{r}r(u,v))\ |\boldsymbol{r}_u \times \boldsymbol{r}_v|\ dA}$$

2. $\boldsymbol{z = g(x,y)}$

   For S given by $z = g(x,y)$ with $(x,y) \in D$,

$$\boxed{\iint_S f(x,y,z)\ dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA}$$

3. **Vector Fields**

   If $\boldsymbol{F}$ is a continuous vector field on an oriented surface S with unit normal vector $\boldsymbol{n}$ then

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot\ d\boldsymbol{S} &= \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\ dS \\ &= \iint_D \boldsymbol{F}\cdot(\boldsymbol{r}_u \times \boldsymbol{r}_v)\ dA \quad \text{(If } S \text{ is given by } r(u,v))\end{aligned}$$

1. Evaluate $\iint_S x^2z^2\ dS$ where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Solution: Since z is always positive between 1 and 3, we may rewrite the surface as

$$z = \sqrt{x^2+y^2}.$$

Note that

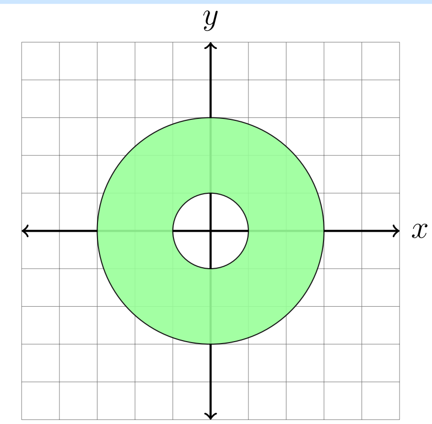
$$\begin{aligned}\sqrt{\frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2 + 1} &= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \\ &= \sqrt{2}\end{aligned}$$

So our integral becomes

$$\iint_D x^2\left(\sqrt{x^2+y^2}\right)^2\ \sqrt{2}\ dA = \iint_D$$

What is D?

When $z = 1$ we have $x^2 + y^2 = 1$ and when $z = 3$ $x^2 + y^2 = 9$. So D is the area in the xy-plane between the circles of radius 1 and 3 both centered at the origin. Shown in green below.


y
x

2. Evaluate $\iint_S yz\, dS$ where S is the surface with parametric equations

$$x = u^2, \quad y = u \sin v, \quad z = u \cos v$$

$$0 \leq u \leq 1, \quad 0 \leq v \leq \pi/2$$

Solution:

3. Find the mass of the surface S, given by $x = 3z^2 - 4y$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, if its density function is $\rho(x, y, z) = 2z$.

Solution:

Since the surface is of the form $x = g(y, z)$ we use

$$\begin{aligned}\iint_S \rho\left(g(y,z), y, z\right) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 1}\; dS &= \int_0^1 \int_0^1 2z\sqrt{17 + 36z^2}\; dz\; dy \\ &= \left(\frac{2}{3}\right)\left(\frac{1}{36}\right)\left((17+36)^{3/2} - 17^{3/2}\right) \\ &\approx 5.84728\end{aligned}$$

4. Evaluate $\iint_S (-xy\boldsymbol{i} - yz\boldsymbol{j} + zx\boldsymbol{k}) \cdot \, d\boldsymbol{S}$ where S is the part of the parabaloid $z = 1 - x^2 - y^2$ that lies above the square $0 \le x \le 2$, $0 \le y \le 4$, and has upward orientation.

Solution:

5. Evaluate $\iint_S \boldsymbol{F} \cdot \, d\boldsymbol{S}$ where

$$\boldsymbol{F}(x, y, z) = xze^y\boldsymbol{i} + xze^y\boldsymbol{j} + z\boldsymbol{k},$$

and S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation.

Solution: