16.6 More Parametric Surfaces

1. Let $\boldsymbol{r}(u,v) = \langle u+v, u-v, u^2-v^2\rangle$

   (a) Evaulate $\boldsymbol{r}(2,-1)$ and $\boldsymbol{r}(-1,2)$.

   Solution:

$$\begin{aligned}\boldsymbol{r}(2,-1) &= (2-1, 2-(-1), 2^2-(-1)^2)\\ &= (1,3,3)\\ \boldsymbol{r}(-1,2) &= (-1+2,-1-2,(-1)^2-2^2)\\ &= (1,-3,-3)\end{aligned}$$

   (b) Find u, v so that $\boldsymbol{r}(u,v) = (3,-1,-3)$.

   Solution: If $u+v=3$ then $v=3-u$. Then

$$u-v=-1 \quad\Longleftrightarrow\quad u-(3-u)=-1 \quad\Longleftrightarrow\quad 2u=2 \quad\Longleftrightarrow\quad u=1$$

   Which implies $v=3-1=2$. To verify $u=1, v=2$ is correct,

$$\boldsymbol{r}(1,2) = (1+2, 1-2, (1)^2-(2)^2) = (3,-1,-3).$$

   (c) Show that $(0,0,1)$ is not a point on the surface.

   Solution: Since $u-v=0$, $u=v$. Then $u+v=0$ gives $u+u=0$ which means $u=0=v$. But $\boldsymbol{r}(0,0)=(0,0,0)\neq(0,0,1)$.

   (d) Recall that a surface $\boldsymbol{r}(u,v)$ is *smooth* if $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are both continuous and $|\boldsymbol{r}_u\times\boldsymbol{r}_v|$ is never 0 for (u,v) in the interior of the domain (this means that at any point on the surface, the normal vector to the tangent plane is not $\mathbf{0}$). Show that $\boldsymbol{r}(u,v)$ is continuous.

   Solution: We first compute $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, $\boldsymbol{r}_u\times\boldsymbol{r}_v$, and $|\boldsymbol{r}_u\times\boldsymbol{r}_v|$.

$$\begin{aligned}\boldsymbol{r}_u(u,v) &= \langle 1+0, 1-0, 2u-0\rangle\\ &= \langle 1,1,2u\rangle\\ \boldsymbol{r}_v(u,v) &= \langle 0+1, 0-1, 0-2v\rangle\\ &= \langle 1,-1,-2v\rangle\,.\\ \boldsymbol{r}_u\times\boldsymbol{r}_v &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 1 & 2u\\ 1 & -1 & -2v\end{vmatrix}\\ &= \langle 2u-2v, 2u+2v, -2\rangle\\ |\boldsymbol{r}_u\times\boldsymbol{r}_v| &= \sqrt{(2u-2v)^2+(2u+2v)^2+(-2)^2}\\ &= \sqrt{4u^2-4uv+4v^2+4u^2+4uv+4v^2+4}\\ &= \sqrt{8u^2+8v^2+4}\end{aligned}$$

It is clear that $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are continuous everywhere (constant functions and polynomials). The only way $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{8u^2 + 8v^2 + 4} = 0$ is if $8u^2 + 8v^2 + 4 = 0$. Since $8u^2 + 8v^2 + 4 \geq 4$ for any (u, v) we know $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is never 0.

This means the surface is continuous

(e) Find an equation of the tangent plane to the parametric surface $\boldsymbol{r}(u, v)$ at the point $u = 1$ and $v = -1$.

Solution: A point on the tangent plane is given by $\boldsymbol{r}(1, -1) = (0, 2, 0)$. Using $\boldsymbol{r}_u \times \boldsymbol{r}_v$ found above,

$$\begin{aligned} \boldsymbol{r}_u \times \boldsymbol{r}_v(1, -1) &= \langle 2(1) - 2(-1), 2(1) + 2(-1), -2\rangle \\ &= \langle 4, 0, -2\rangle \end{aligned}$$

Therefore an equation for the tangent plane is

$$\begin{aligned} \langle 4, 0, -2\rangle \cdot \langle x - 0, y - 2, z - 0\rangle &= 0 \\ 4(x - 0) + 0(y - 2) - 2(z - 0) &= 0 \\ 4x - 2z &= 0 \end{aligned}$$

2. Find two parametric representations for the part of the plane $z = 3x + 5y$ that lies inside the cylinder $x^2 + y^2 = 4$ by completing the following

(a)
$$\boldsymbol{r}_1(u, v) = \langle u, \underline{\hspace{2cm}}, \underline{\hspace{2cm}}\rangle \qquad \text{with} \quad u^2 + v^2 \leq \underline{\hspace{2cm}}$$

Solution:
$$\boldsymbol{r}_1(u, v) = \langle u, v, 3u + 5v\rangle\,, \quad u^2 + v^2 \leq 4$$

(b)
$$\boldsymbol{r}_2(s, t) = \langle s\cos(t), s\sin(t), \underline{\hspace{2cm}}\rangle \quad \text{with} \quad s \in [\underline{\hspace{1cm}}, \underline{\hspace{1cm}}], \quad t \in [\underline{\hspace{1cm}}, \underline{\hspace{1cm}}]$$

Solution:
$$\boldsymbol{r}_2(s, t) = \langle s\cos(t), s\sin(t), 3s\cos(t) + 5s\sin(t)\rangle\,, \quad s \in [0, 2], \quad t \in [0, 2\pi]$$

3. Let S be the surface consisting of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.

(a) Find a parametric representation $\boldsymbol{r}(u, v)$ of S by completing the following

$$\boldsymbol{r}(u, v) = \langle 4\cos(u)\sin(v), \underline{\hspace{1cm}}, \underline{\hspace{1cm}}\rangle$$

$$u \in [0, 2\pi] \quad \text{and} \quad v \in [\_\_\_\_, \_\_\_\_]$$

Solution:

$$\boldsymbol{r}(u, v) = \langle 4\cos(u)\sin(v), 4\sin(u)\sin(v), 4\cos(v)\rangle$$

$$u \in [0, 2\pi] \quad \text{and} \quad v \in [\pi/3, 2\pi/3]$$

(b) Find a parametric representation of the boundary curve of S where $z = 2$

$$\boldsymbol{r}_1(t) = \langle \_\_\_\_, \_\_\_\_, 2\rangle, \quad t \in [0, 2\pi]$$

Solution: If $z = 2$, our equation becomes $x^2 + y^2 = 12$, i.e. a circle with radius $2\sqrt{3}$.

$$\boldsymbol{r}_1(t) = \left\langle 2\sqrt{3}\cos(t), 2\sqrt{3}\sin(t), 2\right\rangle, \quad t \in [0, 2\pi]$$

(c) Find a parametric representation of the boundary curve of S where $z = -2$

$$\boldsymbol{r}_2(t) = \langle \_\_\_\_, \_\_\_\_, -2\rangle, \quad t \in [0, 2\pi]$$

Solution:

$$\boldsymbol{r}_2(t) = \left\langle 2\sqrt{3}\cos(t), 2\sqrt{3}\sin(t), -2\right\rangle, \quad t \in [0, 2\pi]$$