15.8 Triple Integrals in Cylindrical Coordinates

1. Sketch or describe the surface given (all coordinates are cylindrical):

   (a) $r = 5$

   Solution: A cylinder of radius 5, i.e. a circle of radius 5 in the xy-plane extended infinitely in both directions of the z-axis.

   (b) $\theta = 0$

   Solution: The xz-plane.

   (c) $\theta = \pi/4$

   Solution: A plane parallel to the z-axis and the line $y = x$.

   (d) $z = 4 - r^2$

   Solution: When $\theta = 0$ this is the line $z = 4 - x^2$ in the xz-plane. Since θ can vary, the surface is the surface obtained by rotating this line around as seen below

   (e) $0 \leq r \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$, $0 \leq z \leq 1$.

   Solution: The right half of a circle of radius 2 that is 1 unit thick.

2. Convert the following (given in rectangular coordinates) to cylindrical coordinates:

   (a) The point $(-1, 1, 1)$

   Solution:

$$
\begin{aligned}
r^2 &= (-1)^2 + 1^2 \\
r &= \sqrt{2} \\
\tan\theta &= \frac{1}{-1} \\
\theta &= \tan^{-1}(-1) \\
\theta &= -\pi/4
\end{aligned}
$$

$$\boxed{(\sqrt{2}, -\pi/4, 1)}$$

   (b) The point $(2, -\pi/2, 1)$

   Solution:

$$r = \sqrt{4 + \pi^2/4}$$

$$\theta = \tan^{-1}\left(-\frac{\pi}{4}\right)$$

$$\boxed{\left(sqrt4 + \pi^2/4, \tan^{-1}\left(-\frac{\pi}{4}\right), 1\right)}$$

(c) $z = x^2 - y^2$

Solution:

$$\begin{aligned} z &= (r\cos\theta)^2 - (r\sin\theta)^2 \\ &= r^2\cos^2\theta - r^2\sin^2\theta \end{aligned}$$

$$\boxed{z = r^2\left(\cos^2\theta - \sin^2\theta\right)}$$

(d) $x^2 - x + y^2 + z^2 = 1$

Solution: $z^2 = 1 + r\cos\theta - r^2$

3. Sketch the surface whose volume is given by the integral

$$\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^{r^2} r\ dz\ dr\ d\theta.$$

What is the volume of this surface?

Solution: The volume is given by

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_0^2\int_0^{r^2} r\ dz\ dr\ d\theta &= \int_{-\pi/2}^{\pi/2}\int_0^2 r^3\ dr\ d\theta \\ &= \int_{-\pi/2}^{\pi/2} 4\ d\theta \\ &= 4\pi\end{aligned}$$

4. Use cylindrical coordinates to evaluate the following:
   (a) $\iiint_E (x+y+z)\ dV$ where E is the solid in the first octant that lies under the paraboloid $z = 4-x^2-y^2$.

   Solution:

   (b) $\iiint_E \sqrt{x^2+y^2}\ dV$ where E is the region that lies inside the cylinder $x^2+y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

   Solution:

5. Evaluate by changing to cylindrical coordinates:

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\ dz\ dx\ dy.$$