The Divergence Theorem

Theorem (The Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives in an open region that contains E. Then*

$$\iint_S \mathbf{F} \cdot d\boldsymbol{S} = \iiint_E \text{div } \mathbf{F} \; dV$$

1. Verify the divergence theorem is true for the vector field $\mathbf{F}F(x, y, z) = \langle z, y, x \rangle$ on E where E is the solid ball $x^2 + y^2 + z^2 \leq 16$ with boundary sphere S defined by $x^2 + y^2 + z^2 = 16$. In other words, evaluate both

$$\iint_S \mathbf{F} \cdot d\boldsymbol{S}$$

and

$$\iiint_E \text{div } \mathbf{F} \; dV$$

and show they are equal.

Solution: Parameterize S by

$$\boldsymbol{r}(\phi, \theta) = \langle 4 \sin\phi \cos\theta, 4 \sin\phi \sin\theta, 4 \cos\phi \rangle .$$

Then

$$\mathbf{F}(\boldsymbol{r}(\phi, \theta)) = \langle 4 \cos\phi, 4 \sin\phi \sin\theta, 4 \sin\phi \cos\theta \rangle$$

and

$$\mathbf{F}(\boldsymbol{r}(\phi, \theta)) \cdot (r_\phi \times r_\theta) = 64 \left(2 \cos\phi \sin^2\phi \cos\theta + \sin^3\phi \sin^2\theta \right)$$

$$\begin{aligned} \iiint_E \text{div } \mathbf{F} \; dV &= \iint_E 1 \; dV \\ &= \frac{64}{3}\pi. \end{aligned}$$

2. Use the divergence theorem (assume all conditions are satisfied) to prove the identity

$$\iint_S \text{curl } \mathbf{F} \cdot \ d\text{S} = 0.$$

Solution: Follows immedietly from the fact that div curl $\mathbf{F} = 0$.

3. Use the divergence theorem to calculate the flux of $\mathbf{F}(x, y, z) = xye^z\boldsymbol{i} + x^2z^3\boldsymbol{j} - ye^z\boldsymbol{k}$ across the surface of the box bounded by the coordinate planes and the planes $x = 3, y = 2, z = 1$.

Solution:

$$\text{div } \boldsymbol{F} = ye^z + 0 - ye^z = 0 \Longrightarrow \iiint_E \div \ \mathbf{F} \ dV = 0.$$

4. Let
$$\mathbf{F}(x,y,z) = \left\langle z^2x, \frac{1}{3}y^3 + \tan z,\ (x^2z + y^2) \right\rangle$$
and let S be the top half of the sphere $x^2 + y^2 + z^2 = 1,\ \ z \geq 0$.

Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot d\boldsymbol{S}$.

Hint: S is not a closed surface, but S_0 equals S along with the disk D_0, $x^2 + y^2 \leq 1$ is. Use the divegence theorem with S_0 and use $S = S_0 - D_0$.

Solution:
$$\text{div } \mathbf{F} = z^2 + y^2 + x^2$$
S_0 is $\{(\rho, \phi, \theta) \mid 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi\}$.
$$\begin{aligned}\iiint_E \text{div } \mathbf{F}\ dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta \\ &= 2\pi \left(\int_0^{\pi/2} \sin\phi\ d\phi \right) \left(0^1 \rho^4\ d\rho \right) \\ &= \end{aligned}$$