14.5 The Chain Rule

Useful Information.
Given a differentiable function $z = f(x, y)$ with $x = x(t)$ and $y = y(t)$ differentiable functions of t, then z is a differentiable function of t with

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

or equivalently

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$f_t = f_x x_t + f_y y_t$$

More generally, if $z = f(x, y)$ and $x = x(s, t)$,

1. Suppose $z = f(x, y)$ where f is differentiable and

$$\begin{aligned} x &= g(t) & y &= h(t) \\ g(3) &= 2 & h(3) &= 7 \\ g'(3) &= 5 & h'(3) &= -4 \\ f_x(2,7) &= 6 & f_y(2,7) &= -8 \end{aligned}$$

Find $\dfrac{\partial z}{\partial t}$ when $t = 3$.

Solution:

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= f_x g'(t) + f_y h'(t) \frac{\partial z}{\partial t}|_{t=3} \qquad = f_x(2,7)g'(3) + f_y(2,7)h'(3) \\ &= 6(5) + (-8)(-4) \\ &= 62 \end{aligned}$$

2. Find $\dfrac{\partial z}{\partial t}$ if $z = x^2 + y^2 + xy$ with $x = \sin t$ and $y = e^t$

Solution:

$$(2x + y)\cos t + (2y + x)e^t$$

3. Find $\dfrac{\partial w}{\partial t}$ if $w = xe^{y/z}$ with $x = t^2$, $y = 1 - t$, $z = 1 + 2t$.

Solution:

$$e^{y/z}\left(2t-(x/z)-(2xy/z^2)\right)$$

4. Use the chain rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if $z = e^r\cos(\theta)$ with $r = st$ and $0 = \sqrt{s^2+t^2}$

Solution:

$$\frac{\partial z}{\partial s} = e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

5. Let $z = x^4 + x^2y$, with $x = s + 2t - u$ and $y = stu^2$. Find $\dfrac{\partial z}{\partial s}$, $\dfrac{\partial z}{\partial t}$, and $\dfrac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$.

Solution: We will need

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4x^3 + 2xy\\ \frac{\partial z}{\partial y} &= x^2\\ \frac{\partial x}{\partial s} &= 1\\ \frac{\partial y}{\partial s} &= tu^2\end{aligned}$$

When $s = 4, t = 2, u = 1,$

$$x = 4 + 2(2) - 1 = 7 \quad \text{and} \quad y = (4)(2)(1)^2 = 8$$

Then

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= \left(4(7)^3 + 2(7)(8)\right)(1) + \left(7^2\right)\left((2)(1)^2\right)\\ &= 1582\end{aligned}$$

Proceeding similarly gives the following

$$\begin{aligned}\frac{\partial z}{\partial s} &= 1582\\ \frac{\partial z}{\partial t} &= 3164\\ \frac{\partial z}{\partial u} &= -700\end{aligned}$$

Useful Information. If an equation can be expressed as $F(x, y) = 0$ then

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \tag{1}$$

and if $z = f(x, y)$, then we can rewrite this in the form $F(x, y, z) = 0$ and

$$\frac{\partial z}{\partial x} = -\frac{\partial F_x}{\partial F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F_y}{\partial F_z} \tag{2}$$

6. Use Equation (1) to find $\dfrac{\partial y}{\partial x}$

(a) $y \cos x = x^2 + y^2$

Solution: Set $F(x, y) = x^2 + y^2 - y \cos x$. Then

$$F_x = 2x + y \sin x \quad \text{and} \quad F_y = 2y - \cos(x)$$

so

$$\frac{\partial y}{\partial x} = -\frac{2x + y \sin x}{2y - \cos x} = \frac{2x + y \sin x}{\cos x - 2y}$$

(b) $\tan^{-1}(x^2 y) = x + xy^2$

Use Equation (2) to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

(a) $x^2 + 2y^2 + 3z^2 = 1$

Solution: If $F(x, y, z) = x^2 + 2y^2 + 3z^2 - 1$,

$$F_x = 2xF_y \quad = 4yF_z = 6z \tag{3}$$

So

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2y}{3z}$$

(b) $e^z = xyz$

Solution: Set $F(x, y, z) = e^z - xyz$.

$$\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}, \quad \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$