Stokes Theorem

Theorem (Stokes Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

The following is a list of the results you should get for the associated exercise:

1. 16π two times.
2. 0
3. $81\pi/2$
4. Satisfaction and a better understanding of this section's material.
5. More of the above

Exercises

:

1. Evaluate $\iint_S$ curl $\mathbf{F}\ d\mathbf{S}$ where $\mathbf{F}(x, y, z) = -y\boldsymbol{i} + x\boldsymbol{j} - 2\boldsymbol{k}$ and S is the cone $z^2 = x^2 + y^2$ with $0 \leq z \leq 4$. Then, verify Stokes' theorem is true for $\mathbf{F}$ and S by evaluating $\int_C \mathbf{F} \cdot d\boldsymbol{r}$.

2. Evaluate $\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ using Stokes' Theorem where $\mathbf{F} = x^2z^2\boldsymbol{i} + y^2z^2\boldsymbol{j} + xyz\boldsymbol{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 4$, oriented upward.

3. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ where $\mathbf{F}(x, y, z) = \langle x^2z, xy^2, z^2 \rangle$ and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise when viewed from above.

4. Show that if S is a sphere and $\mathbf{F}$ is a vector field that satisfies the conditions for Stokes' theorem everywhere, then

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

Hint: The sphere can be thought of as two half-sphere's sharing a common boundary. Think carefully about how the boundary is oriented on each half if the sphere is oriented outward.

5. Let C be any simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\ dx - 2x\ dy + 3y\ dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane above.