Dot Product Properties and Applications Learning Objectives

- Sketch simple surfaces in space
- Determine when a point lies on a specified surface.

Dot Product examples

Theorem. *The angle between two vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is given by*

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$

or equivalently

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$$

Theorem (Properties of the dot product)**.** *Let* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *be vectors and let* c *be a scalar:*

(a) $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$

(b) $(c\boldsymbol{a})\cdot\boldsymbol{b} = c(\boldsymbol{a}\cdot\boldsymbol{b})$

(c) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$

(d) $0\boldsymbol{a} = 0$

(e) $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

1. Find the angle between the following vectors

   (a) $\langle 4, 1, 1/4\rangle$, $\langle 6, -3, -8\rangle$

   Solution:

   $$\begin{aligned} |\langle 4, 1, 1/4\rangle &= \sqrt{16+1+1/16} = \sqrt{273/16} = \sqrt{273}/4 \\ |\langle 6, -3, -8\rangle| &= \sqrt{36+9+64} = \sqrt{109} \\ \langle 4, 1, 1/4\rangle\cdot\langle 6, -3, -8\rangle &= 19 \end{aligned}$$

   So the angle is

   $$\cos^{-1}\left(\frac{19}{\sqrt{109}\sqrt{273/4}}\right) \approx 1.1146$$

(b) $\boldsymbol{i}+\boldsymbol{j}$, $\boldsymbol{k}$

$$\cos^{-1}\left(\frac{\langle 1,1,0\rangle\cdot\langle 0,0,1\rangle}{|\langle 1,1,0\rangle||\langle 0,0,1\rangle|}\right)=\cos^{-1}(0)=\frac{\pi}{2}$$

(c) $\langle p,-p,2p\rangle$, $\langle 2q,q,-q\rangle$ where p and q are any two non-zero real numbers.

Solution:

$$\begin{aligned}|\langle p,-p,2p\rangle| &= \sqrt{p^2+p^2+4p^2}=\sqrt{6p^2}=\sqrt{6}|p|\\ |\langle 2q,q,-q\rangle| &= \sqrt{4q^2+q^2+q^2}=\sqrt{6q^2}=\sqrt{6}|q|\\ \langle p,-p,2p\rangle\cdot\langle 2q,q,-q\rangle &= 2pq-pq-2pq=-pq\end{aligned}$$

So

$$\theta=\cos^{-1}\left(\frac{-pq}{6|pq|}\right)$$

2. Let $\boldsymbol{a}=\langle -2,2,1\rangle$, $\boldsymbol{b}=\langle 1,2,0\rangle$ and $\boldsymbol{c}=\langle 0,-1,-1\rangle$.

(a) Find a vector $\boldsymbol{v}$ so that $\boldsymbol{b}\neq\boldsymbol{v}$ but $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{v}$

Solution: Many vectors work. Since

$$\langle -2,2,1\rangle\cdot\langle 1,2,0\rangle=2,$$

if $\boldsymbol{v}=\langle x,y,z\rangle$

$$\boldsymbol{a}\cdot\boldsymbol{v}=-2x+2y+z$$

Any values for x,y,z that make $-2x+2y+z=2$ give a vector that works. E.g. $\langle -1,0,0\rangle$.

(b) Verify part (c) above by computing $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ and $\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$ and showing they are the same.

Solution:

$$\begin{aligned}\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) &= \langle -2,2,1\rangle\cdot(\langle 1,2,0\rangle+\langle 0,-1,-1\rangle)\\ &= \langle -2,2,1\rangle\cdot\langle 1,1,-1\rangle\\ &= -2+2-1=-1.\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c} &= \langle -2,2,1\rangle\cdot\langle 1,2,0\rangle+\langle -2,2,1\rangle\cdot\langle 0,-1,-1\rangle\\ &= (-2+4+0)+(0-2-1)\\ &= 2-3=-1\end{aligned}$$

Projections

Definition. *The **scalar projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is*

$$\text{comp}_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

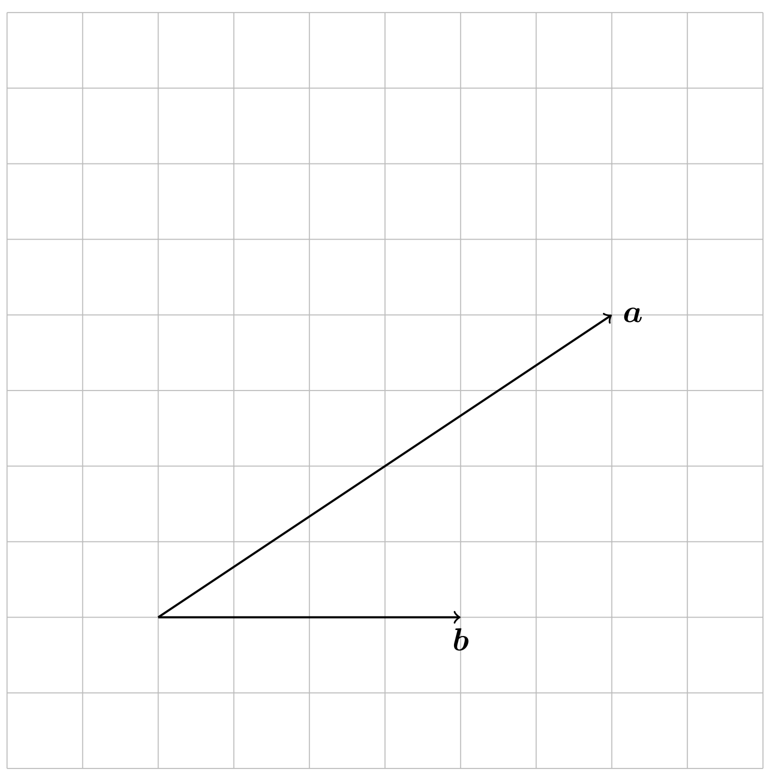
*The **vector projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is*

$$\text{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right) = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$$

*The **orthogonal projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is*

$$\text{orth}_{\boldsymbol{a}}(\boldsymbol{b}) = \boldsymbol{b} - \text{proj}_{\boldsymbol{a}}(\boldsymbol{b})$$

Let $\boldsymbol{a} = \langle 6, 4\rangle$ and $\boldsymbol{b} = \langle 4, 0\rangle$, shown below

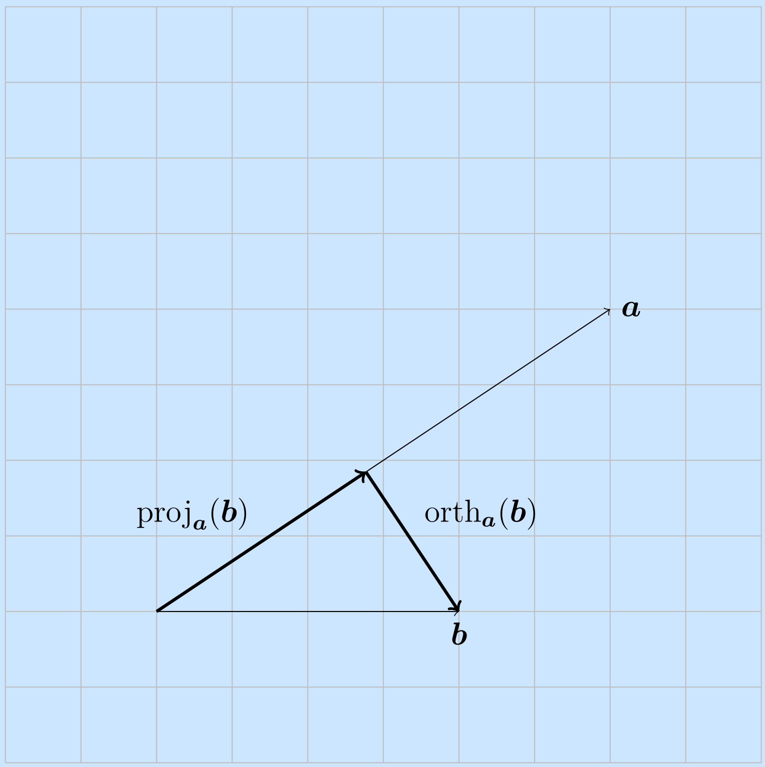


1. Compute the following values and sketch them alongside the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ above

   (a) $\text{proj}_{\boldsymbol{a}}(\boldsymbol{b})$ and $\text{orth}_{\boldsymbol{a}}(\boldsymbol{b})$

Solution:

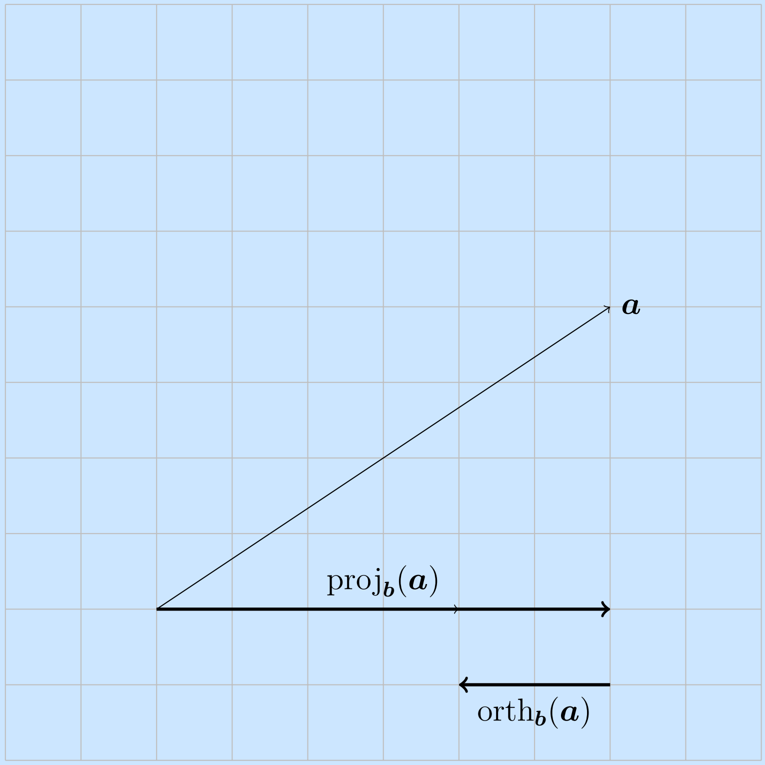
$$\begin{aligned}
\operatorname{proj}_{\boldsymbol{a}}(\boldsymbol{b}) &= \left(\frac{\langle 6,4\rangle \cdot \langle 4,0\rangle}{\langle 6,4\rangle \cdot \langle 6,4\rangle}\right)\langle 6,4\rangle \\
&= \frac{24+0}{36+16}\langle 6,4\rangle \\
&= \frac{6}{13}\langle 6,4\rangle = \left\langle \frac{36}{13}, \frac{24}{13}\right\rangle \\
\operatorname{orth}_{\boldsymbol{a}}(\boldsymbol{b}) &= \boldsymbol{b} - \operatorname{proj}_{\boldsymbol{a}}(\boldsymbol{b}) \\
&= \langle 4,0\rangle - \left\langle \frac{36}{13}, \frac{24}{13}\right\rangle \\
&= \left\langle \frac{16}{13}, -\frac{24}{13}\right\rangle
\end{aligned}$$



(b) $\operatorname{proj}_{\boldsymbol{b}}(\boldsymbol{a})$ and $\operatorname{orth}_{\boldsymbol{b}}(\boldsymbol{a})$

Solution:

$$\operatorname{proj}_{\boldsymbol{b}}(\boldsymbol{a}) = \langle 6,0\rangle \ \operatorname{orth}_{\boldsymbol{b}}(\boldsymbol{a}) \qquad = \langle -2,0\rangle$$



2. What vector is equal to $\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) + \mathrm{orth}_{\boldsymbol{a}}(\boldsymbol{b})$?

Solution: $\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b}) + \mathrm{orth}_{\boldsymbol{a}}(\boldsymbol{b})$ is always equal to $\boldsymbol{b}$.

3. Let $\boldsymbol{a} = -5\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 8\boldsymbol{j} - \boldsymbol{k}$

   (a) Compute $\mathrm{proj}_{\mathrm{a}}(\mathrm{b})$

Solution: Using $\boldsymbol{a} = \langle -5, 5, 2\rangle$ and $\boldsymbol{b} = \langle -1, 8, -1\rangle$

$$\mathrm{proj}_{\mathrm{a}}(\mathrm{b}) = \left\{-\frac{215}{54}, \frac{215}{54}, \frac{43}{27}\right\} \approx \{-3.98148, 3.98148, 1.59259\}$$

   (b) Compute $\mathrm{orth}_{\mathrm{a}}(\mathrm{b})$

Solution:

$$\mathrm{orth}_{\mathrm{a}}(\mathrm{b}) = \left\{\frac{161}{54}, \frac{217}{54}, -\frac{70}{27}\right\} \approx \langle 2.98148, 4.01852, -2.59259\rangle$$