Surfaces Learning Objectives

- Sketch simple surfaces in $3D$.
- Determine when a point lines on a specified surface.

Surfaces Examples

1.

   (a) Determine if the point $(1, 4)$ is on the line $x - 4y = 1$.

   Solution: False; $1 - 4(4) \neq 1$

   (b) Determine if the point $(1, 4, 2)$ is on the plane $x - 4y + 8z = 1$.

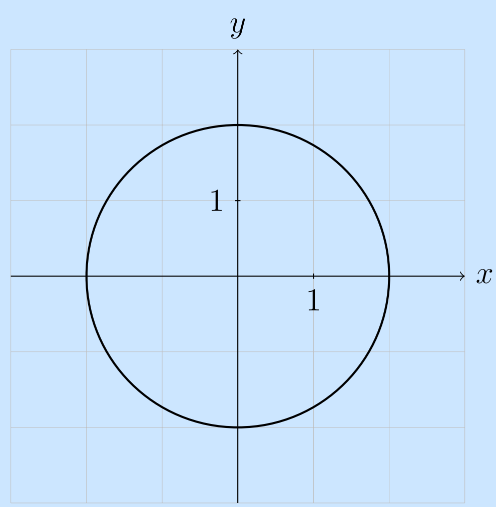
   Solution: True; $1 - 4(4) + 8(2) = 1$

   (c) Determine if the point $(1, -3, 0)$ is on the surface $xyz + x^2 = y$.

   Solution: False

2. (a) Graph the equation $x^2 + y^2 = 4$ in the xy-plane. Describe it in words.

   Solution:

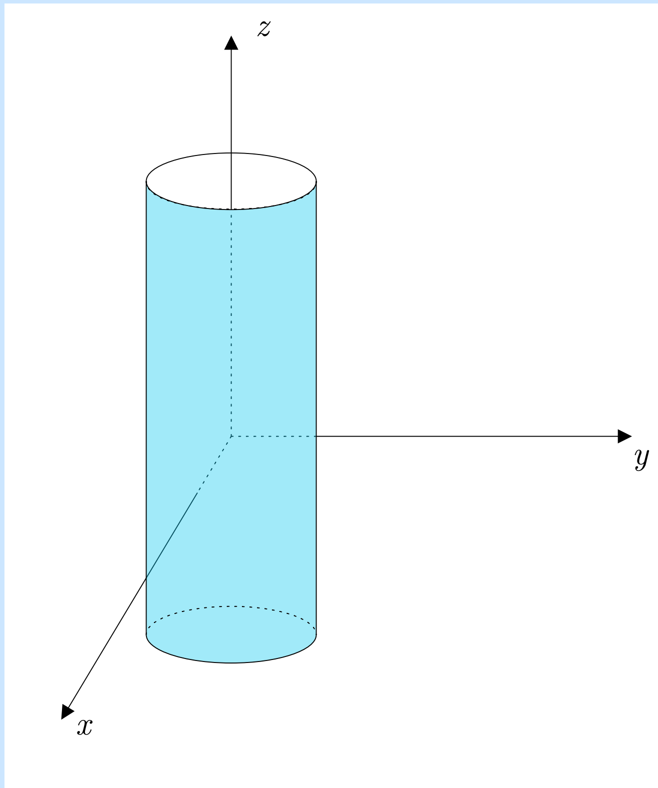
   

   A circle of radius 2 centered at the origin

   (b) Graph the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$ (i.e. in space). Describe it in words.
   Hint: Which values of z satisfy this equation?

Solution: In $\mathbb{R}^2$ the equation $x^2 + y^2 = 4$ is a circle of radius 2 in the xy-plane, in other words all points (x, y) that are 2 units away from the origin (the point $(0, 0)$). For instance, the point $(\sqrt{2}, \sqrt{2})$.

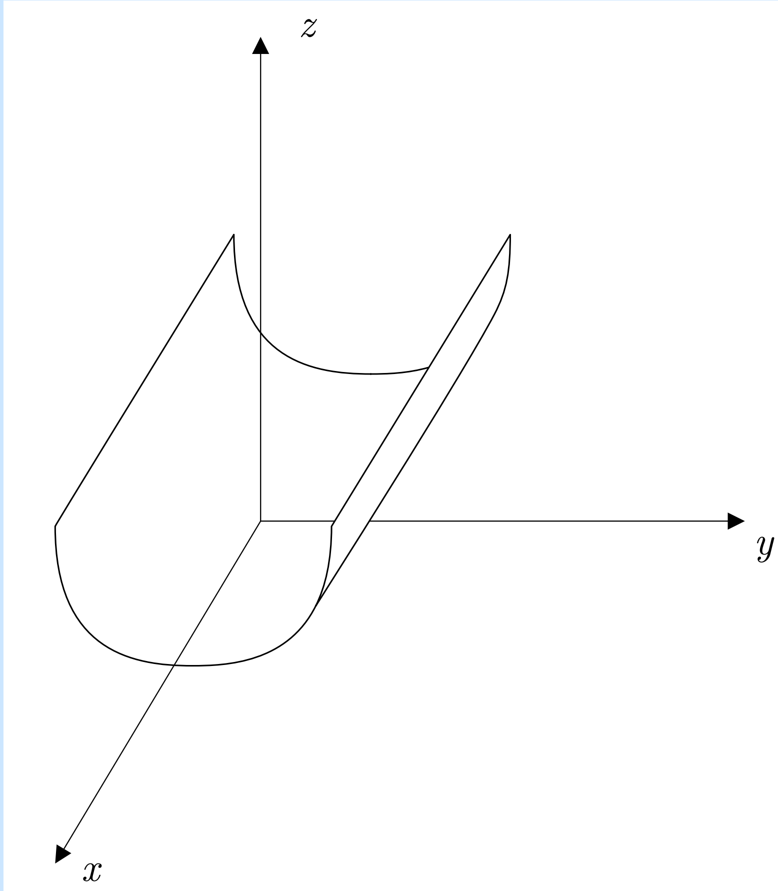
In $\mathbb{R}^3$, for any value of z the point $(\sqrt{2}, \sqrt{2}, 0)$ also satisfies the equation $x^2 + y^2 = 4$. Thinking of it like this, you can imagine a circle of radius 2 at every value of z.



The cylinder above should extend infinitely in both z-axis directions.

(c) Graph the equation $z = y^2$ in $\mathbb{R}^3$. Describe it in words.

Solution:



The equation represents a parabola in the yz-plane extended infinitely along the x-axis.

Spheres Leaning Objectives

- Extend the usual distance equation to three variables.
- Draw spheres.
- Describe a sphere given its equation

Spheres Examples

Definition. *The distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, written $|P_1P_2|$, is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1. Calculate the distance between $(1, 2, 3)$ and $(3, -1, 0)$.

Solution:

$$d\left((1,2,3),(3,-1,0)\right) = \sqrt{(1-3)^2+(2-(-1))^2+(3-0)^2}$$
$$= \sqrt{4+9+9} = \sqrt{22}$$

Definition. *The set of all points in $\mathbb{R}^3$ equidistant from a center point is called a sphere.*
An equation of a sphere with center $C(h,k,l)$ and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2 = r^2.$$

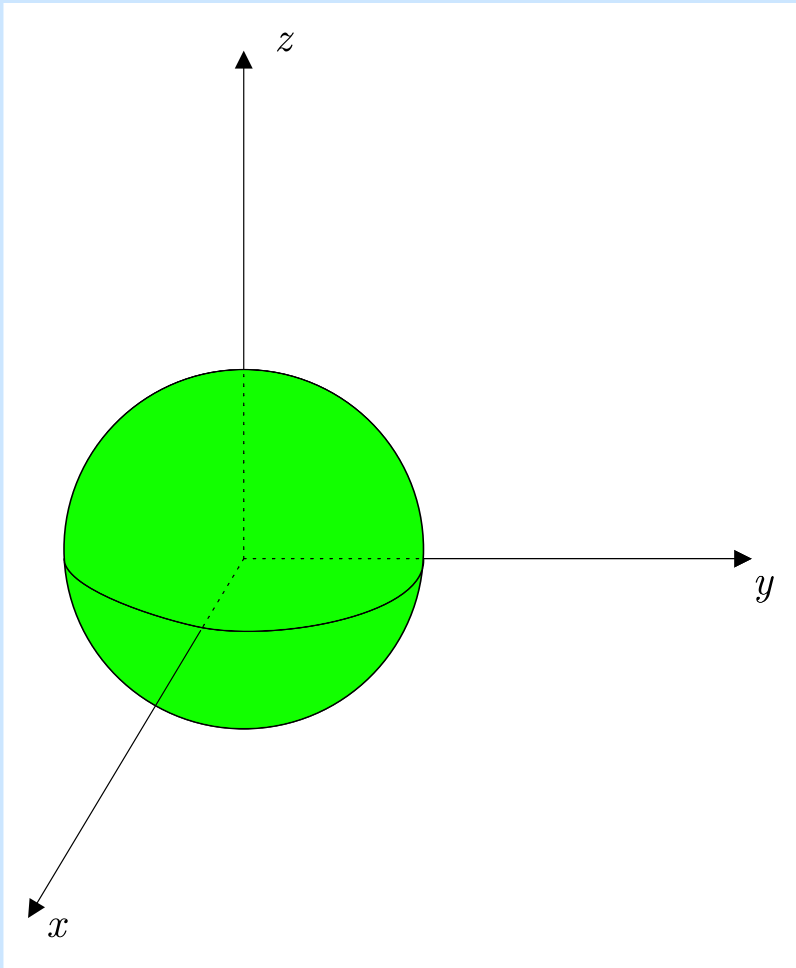
2. Find an equation of the sphere centered at $(1,-2,3)$ with radius 2.

Solution: Using the above with $h=1$, $k=-2$, $l=3$, and $r=2$ gives us

$$(x-1)^2+(y+2)^2+(z-3)^2 = 4$$

3. Describe in words and draw the surface $x^2+y^2+z^2=1$.

Solution: A sphere centered at the origin (the point $(0,0,0)$) with radius 1.



4. Describe the surface $x^2 - 2x + y^2 + z^2 + 4z = 4$ in words.

Solution: We need to complete the square to rewrite this in the form above.

$$\begin{aligned} x^2 - 2x \qquad\qquad + y^2 + z^2 + 4z \qquad\qquad &= 4 \\ + \left(-\frac{2}{2}\right)^2 \qquad\qquad\qquad + \left(\frac{4}{2}\right)^2 \quad &+ \left(-\frac{2}{2}\right)^2 + \left(\frac{4}{2}\right)^2 \\ x^2 - 2x + 1 + y^2 + z^2 + 4z + 4 &= 4 + 1 + 4 \\ (x-1)^2 + y^2 + (z+2)^2 &= 9 \end{aligned}$$

This means the surface is a sphere centered at $(1, 0, -2)$ with radius 3.