Dot Product Properties and Applications Learning Objectives

- Sketch simple surfaces in space
- Determine when a point lies on a specified surface.

Dot Product examples

Theorem. *The angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by*

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right)$$

or equivalently

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$$

Theorem (Properties of the dot product)**.** *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be vectors and let c be a scalar:*

(a) $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$

(b) $(c\boldsymbol{a})\cdot\boldsymbol{b} = c(\boldsymbol{a}\cdot\boldsymbol{b})$

(c) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$

(d) $0\boldsymbol{a} = 0$

(e) $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

1. Find the angle between the following vectors

   (a) $\langle 4, 1, 1/4\rangle$, $\langle 6, -3, -8\rangle$

   Solution:

   $$\begin{aligned} |\langle 4,1,1/4\rangle &= \sqrt{16+1+1/16} = \sqrt{273/16} = \sqrt{273}/4 \\ |\langle 6,-3,-8\rangle| &= \sqrt{36+9+64} = \sqrt{109} \\ \langle 4,1,1/4\rangle\cdot\langle 6,-3,-8\rangle &= 19 \end{aligned}$$

   So the angle is

   $$\cos^{-1}\left(\frac{19}{\sqrt{109}\sqrt{273/4}}\right) \approx 1.1146$$

(b) $\boldsymbol{i}+\boldsymbol{j}$, $\boldsymbol{k}$

$$\cos^{-1}\left(\frac{\langle 1,1,0\rangle\cdot\langle 0,0,1\rangle}{|\,\langle 1,1,0\rangle\,||\,\langle 0,0,1\rangle\,|}\right)=\cos^{-1}(0)=\frac{\pi}{2}$$

(c) $\langle p,-p,2p\rangle$, $\langle 2q,q,-q\rangle$ where p and q are any two non-zero real numbers.

Solution:

$$\begin{aligned}|\,\langle p,-p,2p\rangle\,| &= \sqrt{p^2+p^2+4p^2}=\sqrt{6p^2}=\sqrt{6}|p|\\ |\,\langle 2q,q,-q\rangle\,| &= \sqrt{4q^2+q^2+q^2}=\sqrt{6q^2}=\sqrt{6}|q|\\ \langle p,-p,2p\rangle\cdot\langle 2q,q,-q\rangle &= 2pq-pq-2pq=-pq\end{aligned}$$

So

$$\theta=\cos^{-1}\left(\frac{-pq}{6|pq|}\right)$$

2. Let $\boldsymbol{a}=\langle -2,2,1\rangle$, $\boldsymbol{b}=\langle 1,2,0\rangle$ and $\boldsymbol{c}=\langle 0,-1,-1\rangle$.

(a) Find a vector $\boldsymbol{v}$ so that $\boldsymbol{b}\neq\boldsymbol{v}$ but $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{v}$

Solution: Many vectors work. Since

$$\langle -2,2,1\rangle\cdot\langle 1,2,0\rangle=2$$

, if $\boldsymbol{v}=\langle x,y,z\rangle$

$$\boldsymbol{a}\cdot\boldsymbol{v}=-2x+2y+z$$

Any values for x,y,z that make $-2x+2y+z=2$ give a vector that works. E.g. $\langle -1,0,0\rangle$.

(b) Verify part (c) above by computing $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ and $\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$ and showing they are the same.

Projections

Definition. *The* ***scalar projection*** *of* $\boldsymbol{b}$ *onto* $\boldsymbol{a}$ *is*

$$\text{comp}_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

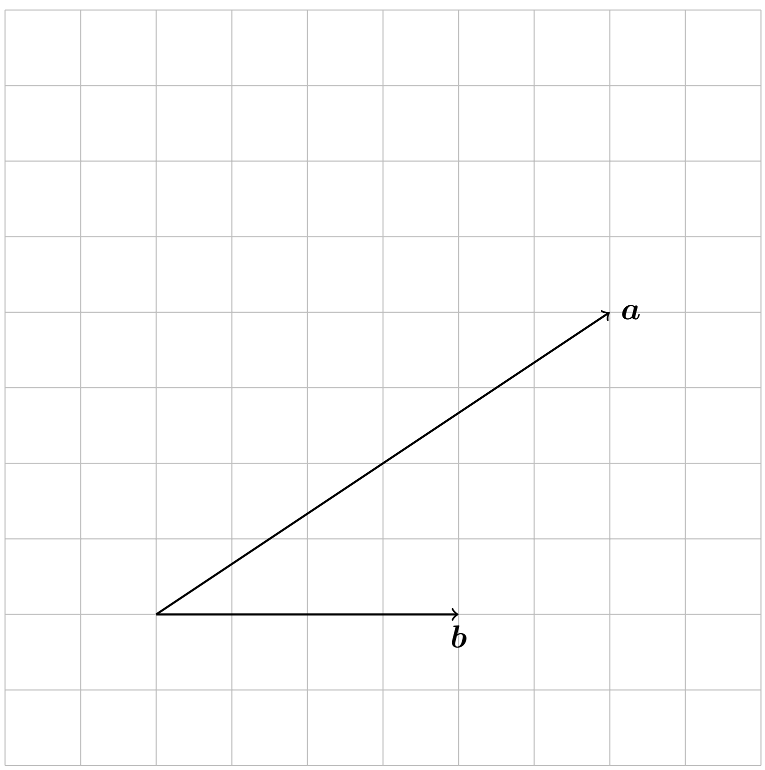
The ***vector projection*** *of* $\boldsymbol{b}$ *onto* $\boldsymbol{a}$ *is*

$$\text{proj}_{\boldsymbol{a}}(\boldsymbol{b}) = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}\right)\left(\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\right) = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\right)\boldsymbol{a}$$

The ***orthogonal projection*** *of* $\boldsymbol{b}$ *onto* $\boldsymbol{a}$ *is*

$$\text{orth}_{\boldsymbol{a}}(\boldsymbol{b}) = \boldsymbol{b} - \text{proj}_{\boldsymbol{a}}(\boldsymbol{b})$$

Let $\boldsymbol{a} = \langle 6, 4\rangle$ and $\boldsymbol{b} = \langle (4,0)\rangle$, shown below



1. Compute the following values and sketch them alongside the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ above
   (a) $\text{proj}_{\boldsymbol{a}}(\boldsymbol{b})$ and $\text{orth}_{\boldsymbol{a}}(\boldsymbol{b})$
   (b) $\text{proj}_{\boldsymbol{b}}(\boldsymbol{a})$ and $\text{orth}_{\boldsymbol{b}}(\boldsymbol{a})$
2. What vector is equal to $\text{proj}_{\boldsymbol{a}}(\boldsymbol{b}) + \text{orth}_{\boldsymbol{a}}(\boldsymbol{b})$?

3. Let $\boldsymbol{a} = -5\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + 8\boldsymbol{j} - \boldsymbol{k}$

   (a) Compute $\mathrm{proj}_{\mathrm{a}}(\mathrm{b})$

   (b) Compute $\mathrm{orth}_{\mathrm{a}}(\mathrm{b})$