13.1 Vector Functions and Curves

Parameterizing curves in $\mathbb{R}^2$

1. Sketch $\boldsymbol{r}(t) = \langle t, t^2 \rangle$ in $\mathbb{R}^2$ and express the curve as a function of x and y.

Solution: Let $x = t$. Then

$$t + y^2 = 4 \quad \iff \quad y^2 = 4 - t \quad \iff \quad y = \pm\sqrt{4-t} \ldots$$

2. Sketch $\boldsymbol{r}(t) = \langle 4, \cos(t), \sin(t) \rangle$ in $\mathbb{R}^3$ and express the curve as a function of x, y, and z.

Solution: Let $x = t$. Then

$$t + y^2 = 4 \quad \iff \quad y^2 = 4 - t \quad \iff \quad y = \pm\sqrt{4-t} \ldots$$

Definition. A vector function $\boldsymbol{r}(t)$ is *continuous* at $t = a$ if

$$\lim_{t \to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$$

Note that by the above definition to check $\boldsymbol{r}(t)$ is continuous at $t = a$ you need to verify that $\boldsymbol{r}(a)$ and $\lim_{t \to a} \boldsymbol{r}(t)$ are both defined and equal to each other.

3. Show that $\boldsymbol{r}(t) = \langle \sin(\pi t) + 1, t^2 + t, 1 \rangle$ is continuous at $t = -1$.

Solution: Let $x = t$. Then

$$t + y^2 = 4 \quad \iff \quad y^2 = 4 - t \quad \iff \quad y = \pm\sqrt{4-t} \ldots$$

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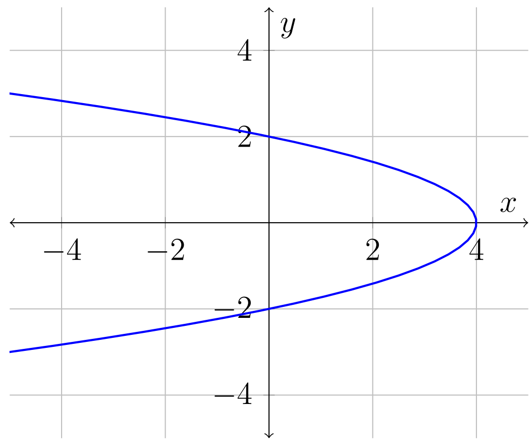
4. Find a vector function $\boldsymbol{r}(t)$, $t \in \mathbb{R}$ that represents the curve $x + y^2 = 4$ shown below.



(a) Try parametezing the curve by setting $x = t$ and explain what makes that approach difficult (there is not one correct answer)..

Solution: Let $x = t$. Then

$$t + y^2 = 4 \quad \Longleftrightarrow \quad y^2 = 4 - t \quad \Longleftrightarrow \quad y = \pm\sqrt{4-t}\ldots$$

(b) Now try parameterizing by setting $y = t$.

Solution: Let $y = t$. Then

$$x + t^2 = 4 \quad \Longleftrightarrow \quad x = 4 - t^2$$

and we have the vector equation

$$\boldsymbol{r}(t) = \langle 4 - t^2, t \rangle$$

(c) Verify the vector function you found satisfies $x + y^2 = 4$ for any arbitrary choice of t.

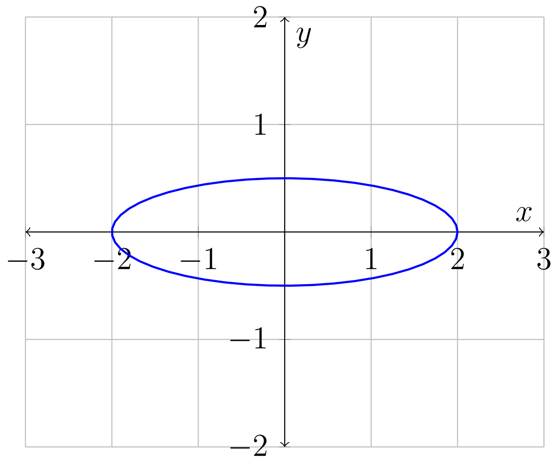
Solution:

$$\begin{aligned} x + y^2 &= 4 \\ (4 - t^2) + (t^2) &= 4 \\ 4 &= 4 \end{aligned}$$

(d) What would you do differently for the curve $x^2 + y = 4$?

Solution: Almost nothing. The roles of x and y are exchanged in the parameterization, i.e. set $x = t$ etc.

5. Find a vector function $\boldsymbol{r}(t)$ that represents the curve $x^2 + 16y^2 = 4$



(a) Try parameterizing by setting $x = t$ and then try setting $y = t$. What makes these so difficult to work with?

Solution: If we set $x = t$, then $t^2 + 16y^2 = 4$ gives $y = \pm\frac{\sqrt{4-t^2}}{4}$ which is not ideal. Setting $y = t$ is similar.

(b) Rewrite the above equation so that it has the form

$$(f(x))^2 + (g(y))^2 = 1$$

Solution: Divide by 4:

$$\frac{x^2}{4} + 4y^2 = 1 \quad \Longleftrightarrow \quad \left(\frac{x}{2}\right)^2 + (2y)^2 = 1$$

(c) Find $\boldsymbol{r}(t)$ by setting $f(x) = \cos t$ and $g(y) = \sin t$. Be sure to include the domain for t.

Solution: From

$$\frac{x}{2} = \cos(t) \quad \text{and} \quad 2y = \sin(t)$$

we get the parameterization

$$\boldsymbol{r}(t) = \left\langle 2\cos(t), \frac{1}{2}\sin(t) \right\rangle, \quad 0 \le t \le 2$$

(d) What do you need to change to represent the portion of $x^2 + 36y^2 = 4$ where $y \ge 0$? What about $x > 0$?

Solution:

Parameterizing curves in $\mathbb{R}^3$

6. Find a vector function that represents the curve of intersection between the two surfaces

$$y = 4z^2 + x^2 \quad \text{and} \quad x = z^2$$

Hint: Try $x = t$, $y = t$, and $z = t$ and see which gives you something you can work with.

Solution: