# Stokes Theorem

**Theorem** (Stokes Theorem)**.** *Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then*

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

**The following is a list of the results you should get for the associated exercise**:

1. $16\pi$ two times.
2. 0
3. $81\pi/2$
4. Satisfaction and a better understanding of this section's material.
5. More of the above

1. Evaluate $\iint_S$ curl $\mathbf{F}$ $d\mathbf{S}$ where $\mathbf{F}(x, y, z) = -y\boldsymbol{i} + x\boldsymbol{j} - 2\boldsymbol{k}$ and $S$ is the cone $z^2 = x^2 + y^2$ with $0 \leq z \leq 4$. Then, verify Stokes' theorem is true for $\mathbf{F}$ and $S$ by evaluating $\int_C \mathbf{F} \cdot d\boldsymbol{r}$.

**Solution:** We first compute $\iint_S$ curl $\mathbf{F} \cdot d\mathbf{S}$.

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \langle -y, x, -2 \rangle \\ &= \langle 0, 0, 2 \rangle \end{aligned}$$

Viewing $S$ as a graph with $z = g(x, y) = \sqrt{x^2 + y^2}$ over $D = \{(x, y) | x^2 + y^2 \leq 4\}$,

$$\begin{aligned} \iint_S \text{curl } \mathbf{F} \, d\mathbf{S} &= \iint_D \left( -(-y)\frac{\partial z}{\partial x} - (x)\frac{\partial z}{\partial y} + (-2) \right) \, dA \\ &= \int_D xy/\sqrt{x^2 + y^2} - xy/\sqrt{x^2 + y^2} - 2 \, dA \\ &= -2 \int_D \, dA \end{aligned}$$

Since $D$ is a cirle of radius 4, the result is $-2\pi(4)^2 = -32\pi$. However, the above assumed upward orientation, so the actual result is

$$\boxed{\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 32\pi}$$

We now evaluate Let $C$ be the curve of intersection between the cone and the plane $z = 4$. Then $C$, oriented counterclockwise, is given by

$$\boldsymbol{r}(t) = \langle 4\cos t, 4\sin t, 4 \rangle, \quad 0 \leq t \leq 2\pi.$$

$$
\begin{aligned}
\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\boldsymbol{r} \\
&= \int_0^{2\pi} \mathbf{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\ dt \\
&= \int_0^{2\pi} \langle -4\sin t, 4\cos t, -2\rangle \cdot \langle -4\sin t, 4\cos t, 0\rangle\ dt \\
&= \int_0^{2\pi} 16\sin^2 t + 16\cos^2 t\ dt \\
&= \int_0^{2\pi} 16\ dt \\
&= 32\pi.
\end{aligned}
$$

2. Evaluate $\iint_S$ curl $\mathbf{F} \cdot d\mathbf{S}$ using Stokes' Theorem where $\mathbf{F} = x^2z^2\boldsymbol{i} + y^2z^2\boldsymbol{j} + xyz\boldsymbol{k}$ and $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies in the cylinder $x^2 + y^2 = 4$, oriented upward.

**Solution:** The boundary of $S$ is the curve $C$ defined by $x^2 + y^2 = 4$ restricted to the plane $z = 4$. $C$ can be parameterized by $\boldsymbol{r}(t) = \langle \cos t, \sin t, 4\rangle$. By Stokes' Theorem

$$
\begin{aligned}
\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\boldsymbol{r} \\
&= \int_0^{2\pi} \mathbf{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\ dt \\
&= \int_0^{2\pi} \left\langle 16\cos^2(t), 16\sin^2(t), 4\sin(t)\cos(t)\right\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle\ dt \\
&= \int_0^{2\pi} -16\sin(t)\cos^2(t) + 16\cos(t)\sin^2(t)\ dt \\
&= 0
\end{aligned}
$$

3. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\boldsymbol{r}$ where $\mathbf{F}(x, y, z) = \langle x^2z, xy^2, z^2\rangle$ and $C$ is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise when viewed from above.

**Solution:**

$$
\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \left\langle x^2z, xy^2, z^2\right\rangle \\
&= \left\langle 0 - 0, -0 + x^2, y^2 - 0\right\rangle \\
&= \left\langle 0, x^2, y^2\right\rangle
\end{aligned}
$$

Any surface bounded by $C$ will work, but we will use the surface $x + y + z = 1$ with the restriction $x^2 + y^2 \leq 9$ and viewed as a graph $z = g(x, y) = 1 - x - y$. Then

$$
\begin{aligned}
\iint_S \text{curl } \mathbf{F} \; dS &= \iint_D \left( -(0)\frac{\partial g}{\partial x} - x^2 \frac{\partial g}{\partial y} + y^2 \right) \; dA \\
&= \iint_D x^2 + y^2 \; dA \\
&= \int_0^{2\pi} \int_0^3 r^3 \; dr \; d\theta \\
&= (2\pi) \left( \frac{3^4}{4} \right) \\
&= \boxed{\frac{81}{2}\pi}
\end{aligned}
$$

4. Show that if $S$ is a sphere and $\mathbf{F}$ is a vector field that satisfies the conditions for Stokes' theorem everywhere, then
$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$
*Hint: The sphere can be thought of as two half-sphere's sharing a common boundary. Think carefully about how the boundary is oriented on each half if the sphere is oriented outward.*

**Solution:** We divide the sphere into two half-spheres $H_1$ and $H_2$, the upper and lower halves respectively. Let $C$ be the shared boundary of the two halves. Let $C_1$ and $C_2$ denote $C$ oriented as determined by the orientation of $H_1$ and $H_2$ respectively.

By stokes' theorem
$$\begin{aligned}\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \iint_{H_1} \text{curl } \mathbf{F} \cdot \ d\mathbf{S} + \iint_{H_2} \text{curl } \mathbf{F} \cdot \ d\mathbf{S} \\ &= \iint_{C_1} \mathbf{F} \cdot d\boldsymbol{r} + \iint_{C_2} \mathbf{F} \cdot d\boldsymbol{r}\end{aligned}$$

Since the sphere is oriented outwardly, $C_2 = -C_1$. Therefore

$$\begin{aligned}\iint_{C_1} \mathbf{F} \cdot d\boldsymbol{r} + \iint_{C_2} \mathbf{F} \cdot d\boldsymbol{r} &= \iint_{C_1} \mathbf{F} \cdot d\boldsymbol{r} + \iint_{-C_1} \mathbf{F} \cdot d\boldsymbol{r} \\ &= \iint_{C_1} \mathbf{F} \cdot d\boldsymbol{r} - \iint_{C_1} \mathbf{F} \cdot d\boldsymbol{r} \\ &= 0.\end{aligned}$$

5. Let $C$ be any simple closed smooth curve that lies in the plane $x+y+z=1$. Show that the line integral

$$\int_C z\ dx - 2x\ dy + 3y\ dz$$

depends only on the area of the region enclosed by $C$ and not on the shape of $C$ or its location in the plane above.

**Solution:**

$$\begin{aligned}\int_C z\ dx - 2x\ dy + 3y\ dz &= \int_C \langle z, -2x, 3y\rangle \cdot \langle dx, dy, dz\rangle \\ &= \int_C \langle z, -2x, 3y\rangle \cdot d\boldsymbol{r}.\end{aligned}$$

Let $S$ be any surface bounded by $C$. Let $D$ be the part of the plane $x+y+z=1$ bounded by $C$.

By stokes theorem,

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} = \iint_S \text{curl } \mathbf{F}\ d\mathbf{S}.$$

We have

$$\begin{aligned}\text{curl } \mathbf{F} &= \nabla \times \langle z, -2x, 3y\rangle \\ &= \langle 3-0, 1-0, -2-0\rangle \\ &= \langle 3, 1, -2\rangle .\end{aligned}$$

A normal vector to the plane $x+y+z=1$ is $\boldsymbol{n} = \langle 1,1,1\rangle$. So

$$\begin{aligned}\iint_S \text{curl } \mathbf{F}\ d\mathbf{S} &= \iint_S \text{curl } \mathbf{F}\cdot \boldsymbol{n}\ dS \\ &= \iint_S \langle 3, 1, -2\rangle \cdot \langle 1, 1, 1\rangle\ \ dS \\ &= 2 \iint_S 1\ dS.\end{aligned}$$

If $z = g(x,y) = 1-x-y$ and $f(x,y,z) = 1$,

$$\begin{aligned}2\iint_S 1\ dS &= 2\iint_S f(x,y,z)\ dS \\ &= 2\iint_D f(x,y,g(x,y))\ dA \\ &= 2\iint_D 1\ dA \\ &= 2\cdot \text{Area}(D).\end{aligned}$$

Therefore the initial line integral depends only on the area of the subsurface of $z = 1-x^2-y^2$ bounded by $C$. Futhermore, if $C_1$ and $C_2$ are different curves in the plane $x+y+z=1$ and they both bound subsurfaces with area $A$,

$$\int_{C_1} z\ dx - 2x\ dy + 3y\ dz = 2A = \int_{C_2} z\ dx - 2x\ dy + 3y\ dz.$$