# 12.5 Part 1: Lines

A note about notation:

If $P(x_0, y_0, z_0)$ is a point then $\boldsymbol{p}$ will denote the vector $\boldsymbol{p} = \langle x_0, y_0, z_0 \rangle$.

**Useful Information.**

- The **line through a point $P$ parallel to $\boldsymbol{v}$** is given by

$$\boldsymbol{r}(t) = \boldsymbol{p} + t\boldsymbol{v}, \quad t \in \mathbb{R} \tag{1}$$

- The **line segment** beginning at a point $R_0$ and ending at the point $R_1$ is given by

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 \quad t \in \mathbb{R} \tag{2}$$

$$\text{or equivalently} \quad \boldsymbol{r}(t) = \boldsymbol{r}_0 + t(\boldsymbol{r}_1 - \boldsymbol{r}_0) \quad t \in \mathbb{R} \tag{3}$$

- A **parametric equation** for a line is of the form

$$\boldsymbol{r}t = \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

and if $a, b, c \neq 0$ a **symmetric equation**[a] is given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

[a] This comes from $\langle x, y, z \rangle = \langle at + x_0, bt + y_0, ct + z_0 \rangle$ and solving each component for $t$, e.g. $x = at + x_0, \ldots$

1. Find an equation for the line segment $\boldsymbol{r}(t)$ with $\boldsymbol{r}(0) = (3, -7, 0)$ and $\boldsymbol{r}(1) = (18, -7, \pi)$, $0 \leq t \leq 1$.

**Solution:**

$$\begin{aligned}\boldsymbol{r}(t) &= \langle 3, -7, 0 \rangle + t(\langle 18, -7, \pi \rangle - \langle 3, -7, 0 \rangle) \\ &= \langle 3, -7, 0 \rangle + t(\langle 15, 0, \pi \rangle) \\ &= \langle 3 + 15t, -7, \pi t \rangle\end{aligned}$$

2. Let $\boldsymbol{f}(t) = \langle 1 + 2t, -10t, 2200111 - 7t \rangle$. Let $L$ be a line parallel to $\boldsymbol{f}(t)$ that passes through the point $(0, 0, 0)$.

   (a) Find a parametric equation for $L$.

**Solution:** From

$$\begin{aligned}\boldsymbol{f}(t) &= \langle 1 + 2t, -10t, 2200111 - 7t \rangle \\ &= \langle 1, 0, 2200111 \rangle + t \langle 2, -10, -7 \rangle\end{aligned}$$

We see that $L$ is a line through $(0,0,0)$ and parallel to the vector $\langle 2,-10,-7\rangle$ and is given by

$$\begin{aligned}\boldsymbol{r}(t) &= \langle 0,0,0\rangle + t\,\langle 2,-10,-7\rangle \\ &= \langle 2t,-10t,-7t\rangle\,.\end{aligned}$$

(b) Use the result above to find a symmetric equation for $L$.

**Solution:**

$$\begin{aligned} x = 2t &\iff \frac{x}{2} = t \\ y = -10t &\iff \frac{-y}{10} = t \\ z = -7t &\iff \frac{-z}{7} = t \end{aligned}$$

Which gives

$$\frac{x}{2} = \frac{y}{-10} = \frac{z}{-7}$$

3. Suppose $L$ is represented by the equation $\boldsymbol{r}_1(t) = \langle 2t, 1-t, 2\pi + \pi t\rangle$. Determine which of the following points belong to $L$

(a) $P(-4,3,0)$

**Solution:** If $P$ is in $L$ then when $x=-4$ we have

$$-4 = 2t \qquad \iff \qquad t = -2.$$

When $t=-2$ the line $L$ passes through the point

$$(2(-2), 1-(-2), 2\pi + (-2)\pi) = (-4,3,0).$$

Which means the line $L$ passes through $P$.

(b) $Q(1, 1/2, 5\pi)$

**Solution:** $Q$ is not a point on $L$

(c) $R(20,-9,12\pi)$

**Solution:** The line $L$ passes through $R$ when $t=10$.

4. Let $L$ denote the line with symmetric equation

$$\frac{x-1}{2} = y = \frac{z+1}{3}$$

(a) Find a parametric equation representing $L$.

**Solution:** We have

$$\begin{aligned} t = \frac{x-1}{2} &\iff x = 2t+1 \\ t &= y \\ t = \frac{z+1}{3} &\iff z = 3t-1 \end{aligned}$$

which gives the parametric equation

$$\boldsymbol{r}(t) = \langle 2t+1, t, 3t-1 \rangle$$

(b) Determine if the line that passes through the points $(1, -5, 5)$ and $(-1, 0, 2)$ intersects $L$. If not, determine if it is parallel or skew to $L$.

**Solution:** The line through $(1, -5, 5)$ and $(-1, 0, 2)$ is given by

$$\begin{aligned} \boldsymbol{s}(t) &= \langle -1, 0, 2 \rangle + t * (\langle 1, -5, 5 \rangle - \langle -1, 0, 2 \rangle) \\ &= \langle -1, 0, 2 \rangle + t \langle 2, -5, 3 \rangle \\ &= \langle 2t-1, -5t, 3t+2 \rangle . \end{aligned}$$

The lines do not intersect (assume $2t+1 = 2t-1$ and it follows). From

$$\begin{aligned} \boldsymbol{r}(t) &= \langle \phantom{-}1, 0, -1 \rangle + t \langle 2, \phantom{-}1, 3 \rangle \\ \boldsymbol{s}(t) &= \langle -1, 0, \phantom{-}2 \rangle + t \langle 2, -5, 3 \rangle \end{aligned}$$

We see the lines are not parallel since

$$\frac{2}{2} \neq \frac{1}{-5} \neq \frac{3}{3}$$

which means the lines are skew.