Directional Derivatives

Useful Information. For a function $f(x, y, z)$ the *gradient vector* is

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\end{aligned}$$

and the *directional derivative* in the direction of a **unit vector** u is given by

$$D_{\boldsymbol{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \boldsymbol{u}$$

1. Let $f(x, y) = \sin(2x + 3y)$. Find the gradient of f and evaluate the gradient at the point $P(-6, 4)$. Find the rate of change of f at P in the direction of the vector $u = \left\langle \sqrt{3}/2, -1/2 \right\rangle$.

Solution:

$$\nabla f = \langle 2\cos(2x + 3y), 3\cos(2x + 3y) \rangle$$

So at P the gradient is

$$\nabla f(-6, 4) = \langle 2\cos(0), 3\cos(0) \rangle = \langle 2, 3 \rangle .$$

Since $\boldsymbol{u}$ is a unit vector, we have

$$D_{\boldsymbol{u}} f(-6, 4) = \langle 2, 3 \rangle \cdot \left\langle \sqrt{3}/2, -1/2 \right\rangle = \sqrt{3} - \frac{3}{2}.$$

2. Find the directional derivative of $f(x, y) = e^x \cos y$ at the point $(0, 0)$ in the direction given by the angle $\theta = \pi/4$.

Solution: The unit vector in the direction of $\theta = \pi/4$ is $\boldsymbol{u} = \langle \cos(\pi/4), \sin \pi/4 \rangle = \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle$. From $\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$,

$$D_{\boldsymbol{u}} f(0, 0) = \left\langle e^0 \cos(0), -e^0 \sin(0) \right\rangle \cdot \left\langle \sqrt{2}/2, \sqrt{2}/2 \right\rangle = \sqrt{2}/2.$$

3. Find the gradient of $f(x, y, z) = x^2yz - xyz^3$. What is the rate of change of f at the point $P(2, -1, 1)$ in the direction of the point $(2, 3, -2)$?

Solution: The gradient is

$$\nabla f = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz \right\rangle$$

The direction from P to Q is given by $\langle 2-2, 3-(-1), -2-1\rangle = \langle 0,4,-3\rangle$. A unit vector in that direction is $\langle 0, 4/5, -3/5\rangle$. So the rate of change is given by

$$\left\langle 2^2(-1)(1)-(-1)(1)^3, (2)^2(1)-(2)(1), 2^2(-1)-3(2)(-1)(1)\right\rangle\cdot\langle 0,4/5,-3/5\rangle = \langle -3,2,2\rangle\cdot\langle 0,4/5,-3/5\rangle = \frac{2}{5}.$$

4. (a) How do you find the maximum rate of change for a function f at a given point (x_0, y_0, z_0)?

   Solution: The maximum rate of change occurs in the direction ∇f and is equal to $|\nabla f|$.

   (b) Let $f(x,y) = xe^y$. Show that at the point $P(2,0)$, the direction in which f is increasing the fastest is in the direction $\langle 1,2\rangle$.

   Solution: $\nabla f = \langle e^y, xe^y\rangle$. At $2,0$ this is $\langle 1,2\rangle$.

   (c) What is the largest possible rate of change of f at the point P?

   Solution: The largest possible rate of change is $|\nabla f| = \sqrt{5}$.

5. Recall that for a diffentiable function f and any unit vector $\boldsymbol{u}$

$$D_u f = \nabla f\cdot\boldsymbol{u} = |\nabla f||\boldsymbol{u}||\cos(\theta) = |\nabla f|\cos(\theta)$$

   where θ is the angle between ∇f and $\boldsymbol{u}$.

   (a) Explain why the largest value of $D_{\boldsymbol{u}} f$ occurs when $\boldsymbol{u}$ is in the same direction as ∇f.

   Solution: Since $|\nabla f| \geq 0$ and $-1 \leq \cos(\theta) \leq 1$, the largest value occurs when $\cos(\theta) = 1$, i.e. $\theta = 0$. Which means the rate of change is maximized when the angle between $\boldsymbol{u}$ and ∇f is 0 (i.e they are the same direction).

   (b) In what direction is the rate of change of f minimized? Why?

   Solution: Similar to above the rate is minimized $\cos\theta = -1$ which occurs when $\theta = \pi$, i.e. when $\boldsymbol{u}$ points in opposite direction of ∇f.

6. Find the maximum and minimum rates of change for f at the given point and the direction in which they occur

   (a) $f(x,y) = 4y\sqrt{x}$ at $(4,1)$.

   Solution: $\nabla f = \langle 2y/\sqrt{x}, 4\sqrt{x}\rangle$, so at $(4,1)$ $\nabla f(4,1) = \langle 1,8\rangle$.
   The maximum rate of change occurs in the direction $\langle 1,8\rangle$ and has rate of change equal to $\sqrt{65}$.
   The minimum rate of change occurs in the direction $\langle -1,-8\rangle$ and has rate of change $-\sqrt{65}$.

   (b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $(3,6,-2)$.

Minimum: -1 in direction $\left\langle -\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}\right\rangle$

Solution:

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}} \right\rangle, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

and

$$\nabla f(3,6,-2) = \left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$$

Maximum: 1 in direction $\left\langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$,

7. Show that at every point on the line $y = x + 1$ the fastest rate of change for $f(x,y) = x^2 + y^2 - 2x - 4y$ is in the direction $\langle 1, 1\rangle$ and that this does not occur at any other point.

Solution: We need to find all points at which ∇f points in the direction of $\langle 1, 1\rangle$. First,

$$\nabla f = \langle 2x - 2, 2y - 4\rangle$$

Then ∇f points in the direction $\langle 1, 1\rangle$ if $2x - 2 = 2y - 4$ which is equivalent to $y = x + 1$.