MTH 234 – Chapter 15

Multiple Integrals

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Contents

2 Iterated Integrals

2.1 A Simpler Life - During Class

Objective(s):

- Learn how to to calculate double integrals much quicker.
- Define and visualize level curves of multivariable functions.

In calc 1 our saving grace for integration is the Fundamental Theorem of Calculus. It told us that integral (area under the curve) was easily computable using anti-derivatives. In calc 3 the equivalent is Fubini's Theorem

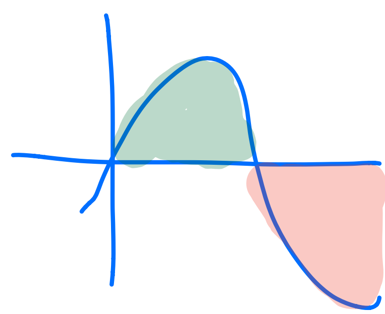
Theorem 2.1 (Fubini's Theorem)**.** If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

More generally, this is true if we assume that f is bounded on R and that f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

And just as integration and anti-derivatives became synonymous so two will double integrals and iterated integrals.

Example 2.2. Evaluate the iterated integral $\int_0^3 \left(\int_1^2 \overset{f}{\overset{\|}{x^2 y}}\,dy\right) dx$



$$= \int_0^3 \left[x^2 \frac{y^2}{2}\right]_1^2 dx$$

$$= \int_0^3 x^2 \frac{1}{2}(4-1)\,dx$$

$$= \frac{3}{2}\int_0^3 x^2\,dx = \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_0^3$$

$$= \frac{1}{2}(27-0) = \frac{27}{2}$$

Remark 2.3. The main way to make double integrals more difficult (over rectangles) is to make the integration itself is difficult (tests more that you remember calc 2. Okay for HW, not used as much for quizzes and exams).

Integration by Parts

U-Sub

Example 2.4. Evaluate $\iint_R \frac{\sqrt{y}}{x^2}\, dA$ where R is the rectangle bounded by $x = 1$, $x = 3$, $y = 0$, and $y = 1$.

$$\int_1^3 \int_0^1 \underline{y^{1/2}} \cdot \underline{x^{-2}}\, dy\, dx$$

$$= \int_1^3 \left[x^{-2}\, y^{3/2} \cdot \frac{2}{3} \right]_0^1 dx$$

$$= \int_1^3 \frac{2x^{-2}}{3} (1 - 0)\, dx$$

$$= \left[\frac{2}{3} x^{-1} (-1) \right]_1^3$$

$$= -\frac{2}{3}\left(3^{-1} - 1^{-1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{3} - 1\right) = \boxed{\frac{4}{9}}$$

Now you may have noticed in both of the previous problems that

$$\longrightarrow \left(\int_0^3 x^2\, dx\right)\left(\int_1^2 y\, dy\right) = 9(2 - 1/2) = 27/2$$

$$\longrightarrow \left(\int_1^3 \frac{1}{x^2}\, dx\right)\left(\int_0^1 \sqrt{y}\, dy\right) = (-1/3 + 1)(2/3) = 4/9$$

And this is a nice coincidence that will help us evaluate quickly but it isn't always true (See the example on the next page).

Here is a theorem for when we can use this trick.

Theorem 2.5. If $\underline{f(x,y)} = \underline{g(x)}\underline{h(y)}$ and $R = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$ then:

$$\iint_R f(x,y)\, dA = \iint_R g(x)h(y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

$$f(x,y) = x + y = x\left(1 + \frac{y}{x}\right) \quad \overset{\shortparallel}{\frown}$$

$|A| \leq K \iff -K \leq A \leq K$

$-\frac{\pi}{2} \leq x - \pi/2 \leq \pi/2$ $\quad \frac{\pi}{2} \leq y \leq 3\pi/2$

$0 \leq x \leq \pi$

Example 2.6. Evaluate $\displaystyle\iint_R x\sin(x+y)\,dA$ where $R = \{(x,y) \mid |x - \pi/2| \leq \pi/2 \text{ and } |y - \pi| \leq \pi/2\}$

d

$0 \quad \pi/2 \quad \pi \qquad \pi/2 \quad \pi \quad 3\pi/2$

Solution. Let's integrate with respect to y first.

$\int x \sin x \, dx$

$$\begin{aligned}
\iint_R x\sin(x+y) &= \int_0^{\pi}\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\sin(x+y)dy\right)dx \\
&= \int_0^{\pi}\left[-x\cos(x+y)\right]_{\pi/2}^{3\pi/2} dx \\
&= \int_0^{\pi}\left[-x\cos\left(x+\frac{3\pi}{2}\right) + x\cos\left(x+\frac{\pi}{2}\right)\right] dx
\end{aligned}$$

Using some unit circle magic we can realize that

$$\cos\left(x+\frac{3\pi}{2}\right) = \sin x$$

$$\cos\left(x+\frac{\pi}{2}\right) = -\sin x$$

So we get that

$$\begin{aligned}
\int_0^{\pi}\left[-x\cos\left(x+\frac{3\pi}{2}\right) + x\cos\left(x+\frac{\pi}{2}\right)\right] dx &= \int_0^{\pi} -x\sin x - x\sin x\, dx \\
&= -2\int_0^{\pi} x\sin x\, dx
\end{aligned}$$

This is a very traditional integration by parts problem.

$$\begin{aligned}
u &= x & dv &= \sin x\, dx \\
du &= dx & v &= -\cos x
\end{aligned}$$

$$\begin{aligned}
-2\left[\int_0^{\pi} x\sin x\, dx\right] &= -2\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\, dx\right] \\
&= -2\left[-\pi(-1) + \int_0^{\pi}\cos x\, dx\right] \\
&= -2\pi - 2\sin x\Big|_0^{\pi} \\
&= -2\pi
\end{aligned}$$

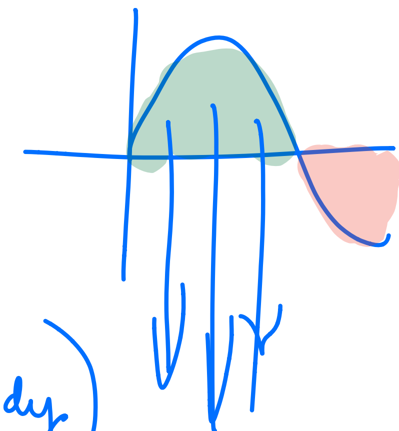
And so we have:

$$\boxed{\iint_R x\sin(x+y) = \int_0^{\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} x\sin(x+y)dydx = -2\pi}$$

Example 2.7. Find the volume of the solid enclosed by the surface $z = x\sec^2 y$ and the planes $z = 0, x = 0, x = 2, y = 0$, and $y = \pi/4$

$$V = \iint_R f(x,y)\, dA$$

$$= \int_0^2 \int_0^{\pi/4} x \cdot \sec^2 y \, dy\, dx$$



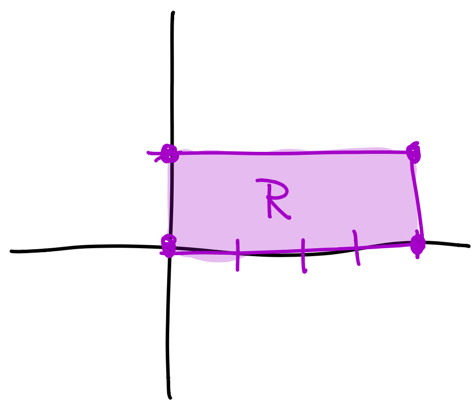
$$= \left(\int_0^2 x\, dx\right) \cdot \left(\int_0^{\pi/4} \sec^2 y\, dy\right)$$

$$= \left[\frac{x^2}{2}\right]_0^2 \cdot \left[\tan y\right]_0^{\pi/4}$$

$$= (2-0)\cdot(1-0) = 2$$

Example 2.8. Find the average value of $f(x,y) = e^y\sqrt{x+e^y}$ over the rectangle with vertices $(0,0), (4,0), (0,1)$, and $(4,1)$.

$$\frac{1}{A(R)} \int_0^4 \int_0^1 e^y \sqrt{x+e^y}\, dy\, dx$$



$$\frac{1}{4} \int_0^4 \int_0^1 e^y \sqrt{x+e^y}\, dy\, dx$$

$$\frac{1}{4} \int_0^4 \left[\frac{2}{3}(x+e^y)^{3/2}\right]_0^1 dx$$

$$\frac{1}{6} \int_0^4 (x+e)^{3/2} - (x+1)^{3/2}\, dx$$

$$\frac{1}{6}\left(\frac{2}{5}(x+e)^{5/2} - \frac{2}{5}(x+1)^{5/2}\right)_0^4$$

$$= \frac{1}{15}\left((4+e)^{5/2} - e^{5/2} - (5^{5/2} - 1)\right)$$

Bonus problem - From previous textbook

$$\iint_R \frac{x}{x^2y^2+1}\,dA \qquad R: \text{rectangle bounded by } x=0,\ \ x=1,\ \ y=0,\ \ y=1$$

Solution. I recall that integrating $\int \frac{x}{x^2+1}dx$ was easier than $\int \frac{1}{y^2+1}dy$ so let's attempt x first.

$$\iint_R \frac{x}{x^2y^2+1} = \int_0^1 \left(\int_0^1 \frac{x}{x^2y^2+1}dx\right) dy$$

Let's try a u substitution here: where

$$u = x^2y^2+1$$
$$du = 2xy^2\,dx$$

So we get:

$$\begin{aligned}\int_0^1 \left(\int_0^1 \frac{x}{x^2y^2+1}dx\right) dy &= \int_0^1 \left(\int \frac{1}{u}\frac{du}{2y^2}\right) dy \\ &= \int_0^1 \left[\frac{\ln u}{2y^2}\right] dy \\ &= \int_0^1 \left[\frac{\ln(x^2y^2+1)}{2y^2}\right]_0^1 dy\end{aligned}$$

We should be a little paranoid about u substitution in multi variables. Check that this is legit by taking the partial of $\frac{\ln(x^2y^2+1)}{2y^2}$ with respect to x to see that we get back to where we started.

$$\begin{aligned}\int_0^1 \left[\frac{\ln(x^2y^2+1)}{2y^2}\right]_0^1 dy &= \int_0^1 \frac{\ln(y^2+1)}{2y^2} - \frac{\ln(1)}{2y^2}dy \\ &= \int_0^1 \frac{\ln(y^2+1)}{2y^2}dy\end{aligned}$$

Here goes nothing. After starring at this for sometime hopefully you agree that integration by parts is once again the winning technique.

$$\begin{aligned} u &= \ln(y^2+1) & dv &= \frac{1}{2y^2}\,dy \\ du &= \frac{2y}{y^2+1}\,dy & v &= -\frac{1}{2y}\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{\ln(y^2+1)}{2y^2}dy &= -\frac{\ln(y^2+1)}{2y}\Big|_0^1 + \int_0^1 \frac{1}{y^2+1}dy \\
&= -\frac{\ln(y^2+1)}{2y}\Big|_0^1 + \arctan y\Big|_0^1 \\
&= -\frac{\ln(y^2+1)}{2y}\Big|_0^1 + \arctan(1) - \arctan(0) \\
&= -\frac{\ln(y^2+1)}{2y}\Big|_0^1 + \arctan(1) - \arctan(0) \\
&= -\frac{\ln(1^2+1)}{2} + \frac{\ln(0^2+1)}{0} + \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4} - \frac{\ln 2}{2} + \frac{\ln(1)}{0}
\end{aligned}$$

And now we have a problem. $\dfrac{\ln(1)}{0}$ is an indeterminate of the form $\dfrac{0}{0}$ so we need to evaluate this with care. Mainly we should be using:

$$\frac{\pi}{4} - \frac{\ln 2}{2} + \frac{\ln(1)}{0} = \frac{\pi}{4} - \frac{\ln 2}{2} + \lim_{y\to 0}\frac{\ln(y^2+1)}{2y}$$

Using L'Hopital's Rule we get:

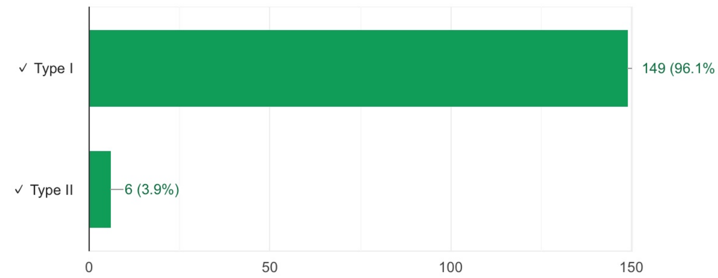
$$\begin{aligned}
\tfrac{\pi}{4} - \frac{\ln 2}{2} + \lim_{y\to 0}\frac{\ln(y^2+1)}{2y} &= \frac{\pi}{4} - \frac{\ln 2}{2} + \lim_{y\to 0}\frac{y}{y^2+1} \\
&= \frac{\pi}{4} - \frac{\ln 2}{2} + \frac{0}{0^2+1} \\
&= \frac{\pi}{4} - \frac{\ln 2}{2}
\end{aligned}$$

And so after u-substitution, integration by parts, arctan, and an indeterminate we are finally done! What a calculus workout!

$$\boxed{\iint_R \frac{x}{x^2y^2+1} = \int_0^1 \left(\int_0^1 \frac{x}{x^2y^2+1}dx\right)dy = \frac{\pi}{4} - \frac{\ln 2}{2}}$$

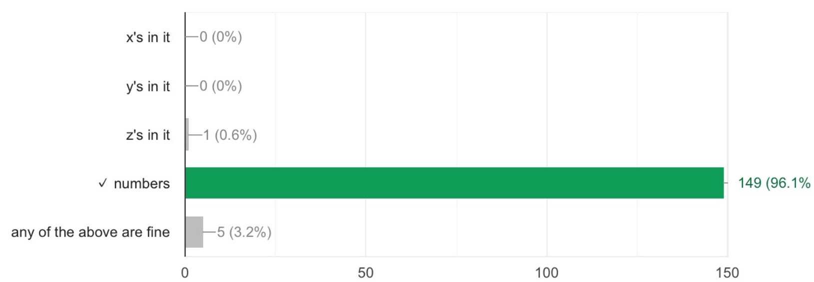
For better or worse, algebra has trained us to be more comfortable with which type of regions?

155 / 155 correct responses



In a double integral the limits of the final integral should only have...

149 / 155 correct responses



3.2 Additional Practice and Switching the Order - During Class

Objective(s):

- Get more practice integrating over general regions.
- Learn how to switch the order of double integrals and why it is useful.

Here are some additional properties that we would have suspected to be true for double integration:

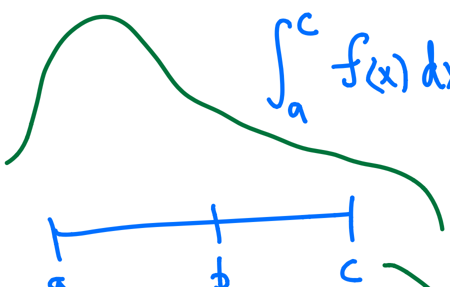
Theorem 3.4. If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region D, then the following properties hold.

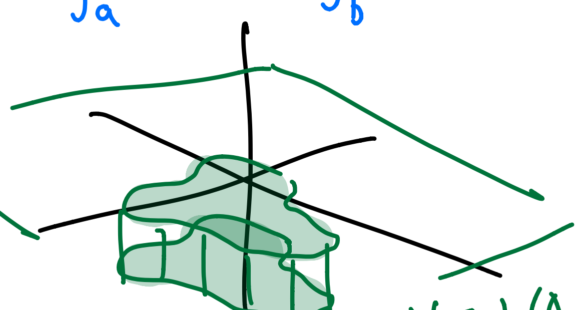
(a) If $D = D_1 \cup D_2$ then $\iint_D f(x, y)\, dA =$ $\iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$

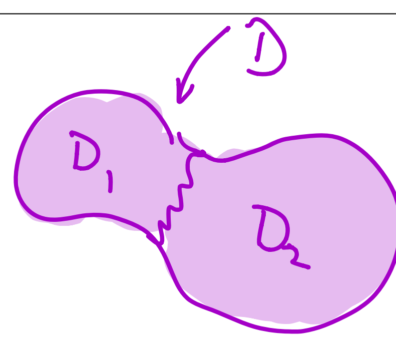
(b) $\iint_D 1\, dA =$ Area(D)

Let's draw a picture of what (a) tells us. What is this equivalent to for Calc1?

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$







V = h(Area of Base)

Example 3.5. Sketch the region of integration and write an equivalent double integral with the order of integration reversed for

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy$$

$x^2+y^2=1$

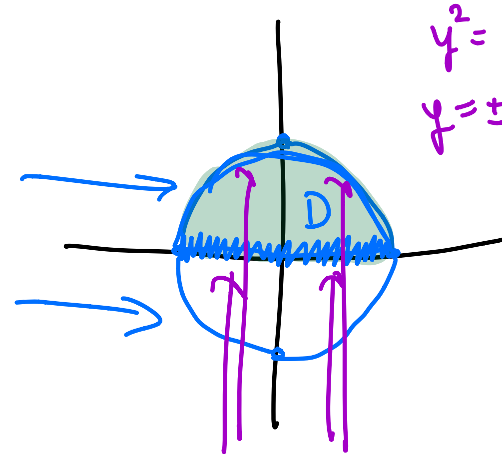
$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

$x=-\sqrt{1-y^2}$

$x^2=1-y^2$

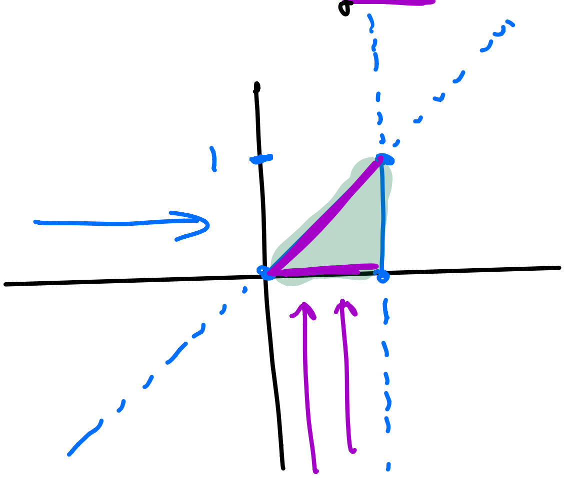
$x^2+y^2=1$



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 3y \; dy\, dx$$

Example 3.6. Evaluate $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy$

$$= \int_0^1 \int_0^x \frac{\sin x}{x} \, dy \, dx$$



$$= \int_0^1 \left[\frac{\sin x}{x} y \right]_0^x dx$$

$$= \int_0^1 \frac{\sin x}{\cancel{x}} \cdot \cancel{(x-0)} \, dx$$

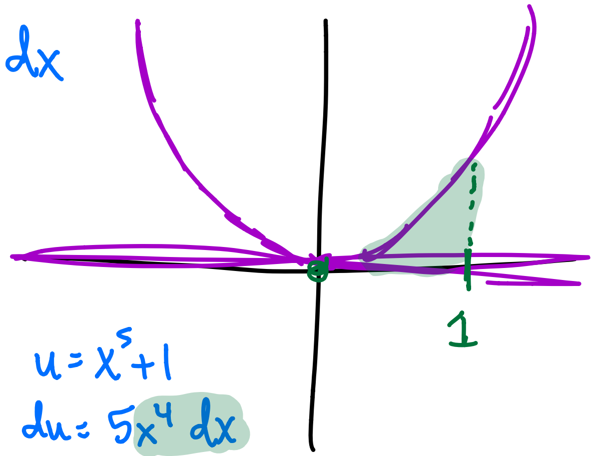
$$= \left[-\cos x \right]_0^1$$

$$= -\cos(1) + 1$$

Example 3.7. Sketch the region of integration and evaluate the double integral

$$\iint_D \frac{y}{x^5+1}\, dA, \quad D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\}$$

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx$$



$$= \int_0^1 \left[\frac{1}{x^5+1} \frac{y^2}{2}\right]_0^{x^2} dx$$

$$= \int_0^1 \frac{1}{2} \frac{x^4}{x^5+1}\, dx$$

$u = x^5+1$

$du = 5x^4\, dx$

$\frac{du}{5} = x^4\, dx$

$$= \int \frac{1}{2} \frac{1}{u} \frac{du}{5}$$

$$= \left[\frac{1}{10} \ln|x^5+1|\right]_0^1$$

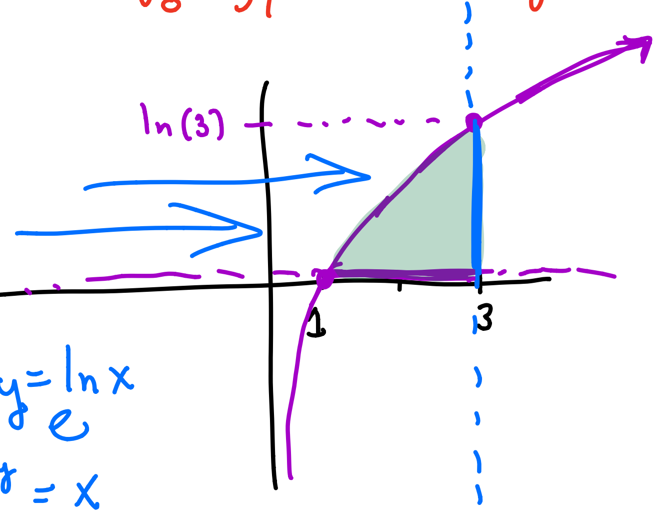
$$= \frac{1}{10}\ln|2| - \frac{1}{10}\ln|1|$$

$$= \frac{1}{10}\ln(2)$$

Example 3.8. Sketch the region of integration and change the order of integration: $\int_1^3 \int_0^{\ln x} f(x,y)\, dy\, dx$.

$$= \int_0^{\ln 3} \int_{e^y}^{3} f(x,y)\, dx\, dy$$

$$\int_0^{\ln x} \int_1^3 f(x,y)\, dx\, dy$$



$e^{y} = e^{\ln x}$

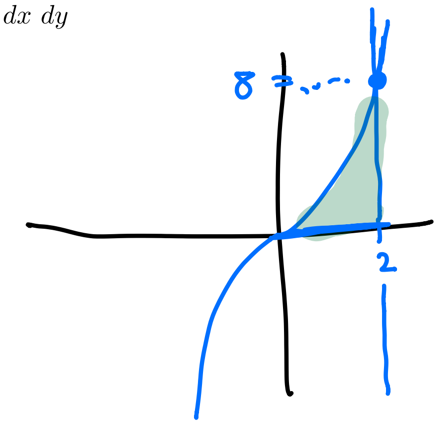
$e^y = x$

Example 3.9. Evaluate $\displaystyle\int_{0}^{8}\int_{\sqrt[3]{y}}^{2} e^{x^4}\,dx\,dy$

y= x= y= x=

$x = \sqrt[3]{y}$

$x^3 = y$



$$= \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 \left[e^{x^4} y\right]_0^{x^3} dx$$

$$= \int_0^2 e^{x^4}(x^3 - 0)\,dx$$

$$= \left[\frac{1}{4} e^{x^4}\right]_0^2$$

$$= \boxed{\frac{1}{4}\left(e^{16} - 1\right)}$$

Which of the following is equivalent to x^2y in polar form?

$r^2 \cos\theta \sin\theta$

$r^3 \cos\theta \sin\theta$

C. $r^3 \cos^2\theta \sin\theta$

$r^3 \cos\theta \sin^2\theta$

$r^3 \cos^2\theta \sin^2\theta$

$(r\cos\theta)^2 (r\sin\theta)$

$r^3 \cos^2\theta \sin\theta$

Question 2

Which of the following is equivalent to $\iint_D x^2y \, dy \, dx$ in polar form?

$\iint_D r^3 \cos^2\theta \sin\theta \; dr \; d\theta$

B. $\iint_D r^4 \cos^2\theta \sin\theta \; dr \; d\theta$

$\iint_D r^3 \cos\theta \sin^2\theta \; dr \; d\theta$

$\iint_D r^4 \cos\theta \sin^2\theta \; dr \; d\theta$

None of the above.

4.2 Polarizing Problems - During Class

Objective(s):

- Define and visualize multivariable functions.
- Define and visualize level curves of multivariable functions.

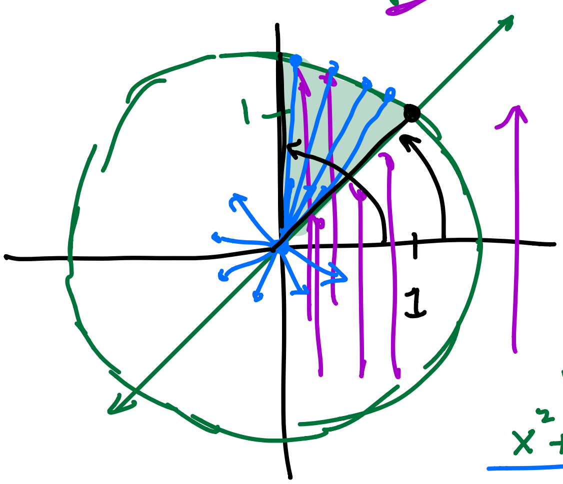
Theorem 4.3.

$$\iint_R f(x,y)\,dy\,dx = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Theorem 4.4.

$$\text{Area of } R = \iint_R 1\,dy\,dx = \iint_R r\,dr\,d\theta$$

Example 4.5. Convert $\displaystyle\int_0^1 \int_{y=x}^{y=\sqrt{2-x^2}} x + 2y\,dy\,dx$ into a polar integral and evaluate



$$\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\, dr\, d\theta$$

$y = \sqrt{2-x^2} \rightarrow y = \sqrt{2-1}$

$y^2 = 2 - x^2 \qquad = \sqrt{1}$

$x^2 + y^2 = 2 \qquad = 1$

$r^2 = 2$

$r = \sqrt{2}$

$y = x$

$\cancel{r}\sin\theta = \cancel{r}\cos\theta$

$\sin\theta = \cos\theta$

$\tan\theta = 1$

$$= \left(\int_{\pi/4}^{\pi/2} \cos\theta + 2\sin\theta\, d\theta\right) \cdot \left(\int_0^{\sqrt{2}} r^2\, dr\right)$$

$$= \Big[\sin\theta - 2\cos\theta\Big]_{\pi/4}^{\pi/2} \cdot \left[\frac{r^3}{3}\right]_0^{\sqrt{2}}$$

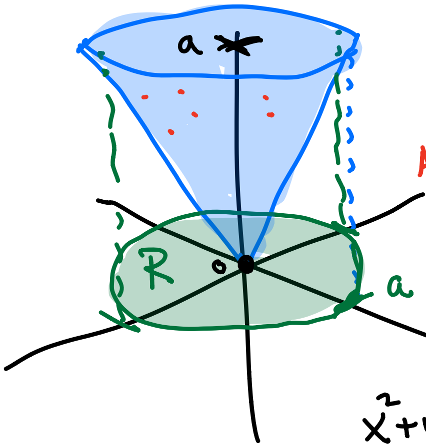
$$= \left[(1 - 0) - \left(\frac{\sqrt{2}}{2} - \sqrt{2}\right)\right]\left(\frac{\sqrt{8}}{3}\right)$$

$$= \left(1 + \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{8}}{3}\right)$$

$z^2 = x^2 + y^2$

Example 4.6. Find the average height of the (single) cone $z = \sqrt{x^2+y^2}$ above the disk $x^2+y^2 \le a^2$ in the xy-plane.

f_{ave} on height function



ANS? $\frac{a}{2}$

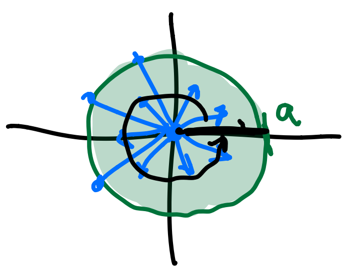
$$f_{ave} = \frac{1}{A(R)} \iint_R f \, dA$$

$$= \frac{1}{\pi a^2} \iint \sqrt{x^2+y^2} \; r \, dr \, d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi}\int_0^a r \cdot r \, dr \, d\theta$$

$x^2+y^2=a^2$

$z = \sqrt{x^2+y^2} = \sqrt{a^2} = a$



$$= \frac{1}{\pi a^2} \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{1}{\pi \cancel{a^2}} \int_0^{2\pi} \frac{a^{\cancel{3}}}{3} \, d\theta = \frac{a}{3\pi}\left[\theta\right]_0^{2\pi}$$

$$= \frac{2a\cancel{\pi}}{3\cancel{\pi}}$$

Example 4.7. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$

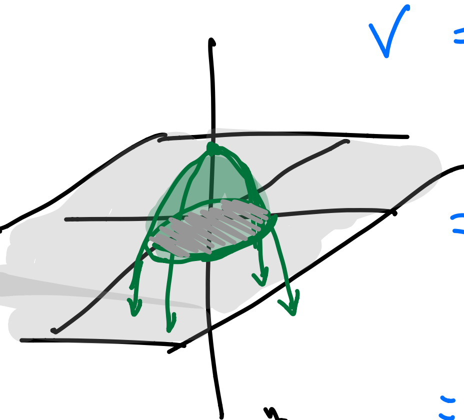
$0 = 1 - x^2 - y^2$

$x^2+y^2 = 1$

$r^2 = 1$

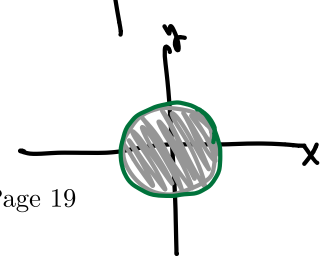
$r = 1$

$$V = \iint_R f \, dA$$



$$= \iint_R (1-x^2-y^2) \, dA$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2) r \, dr \, d\theta$$

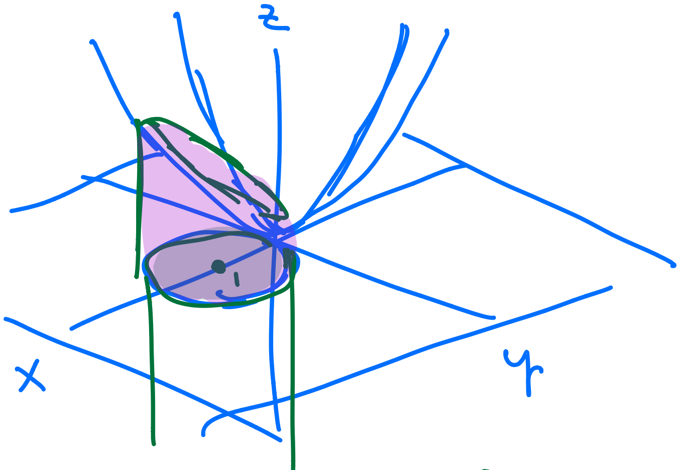


$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$$

Skip to Section 5 then come back here.

Example 4.8 (Bonus, as time permits)**.** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy-$plane, and inside the cylinder $x^2 + y^2 = 2x$.

Hint: $\cos^4 u = \left(\frac{1+\cos 2u}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2u + \frac{1}{8}(1 + \cos 4u)$



$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} x^2+y^2 \, dA$$

$$x^2 - 2x + y^2 = 0 \longrightarrow r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta = 0$$

$$r^2 - 2r\cos\theta = 0$$

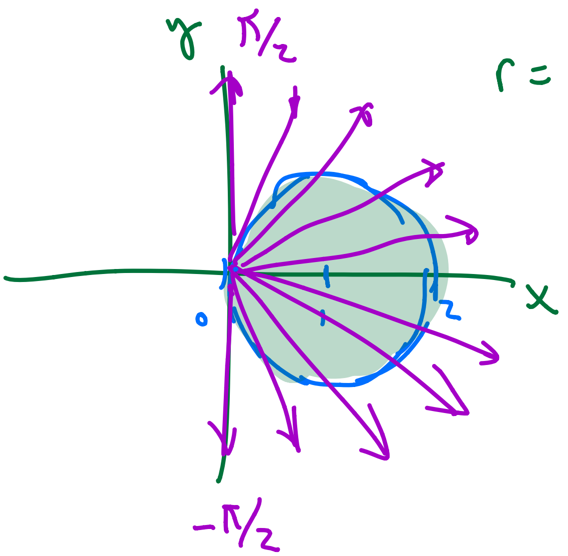
$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

$$r = 2\cos\theta$$



$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{16}{4}\cos^4\theta \, d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta \xrightarrow{\text{SAD CALCULUS 2}} 4\left[\frac{3\pi}{8}\right] = \boxed{\frac{3\pi}{2}}$$

5 Applications of Multiple Integration

5.1 Mass from Density - During Class

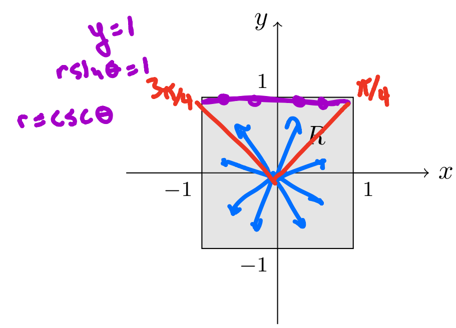
Objective(s):

- Define and visualize multivariable functions.
- Define and visualize level curves of multivariable functions.

Remark 5.1. If $\sigma(x, y)$ is a density function of a lamina (thin sheet) then the mass of the lamina can be given by

$$\text{Mass} = \iint \sigma(x,y)\, dy\, dx$$

Example 5.2. A lamina has density function $\sigma(x, y) = x^2 + y^2$ over the region R is shown below. Calculate the mass.



$$\iint f(x)g(y)\, dy\, dx = \int f(x)\,dx \cdot \int g(y)\,dy$$

$$\text{Mass} = \iint x^2 + y^2 \, dA$$

$$= \int_{-1}^{1}\int_{-1}^{1} x^2 + y^2 \, dy\, dx$$

$$= \int_{-1}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) + \left(+x^2 + \frac{1}{3}\right) dx$$

$$= \left[2 \cdot \frac{x^3}{3} + \frac{2}{3}x \right]_{-1}^{1}$$

$$= \left[\frac{2}{3} + \frac{2}{3}\right] + \left[+\frac{2}{3} + \frac{2}{3}\right]$$

$$= 8/3$$

Remark 5.3. After leaving section 4 you will have to decide whether to integrate in Cartesian or polar coordinates. A good rule of thumb is look at region instead of the function.

7.2 Setting up and Solving - During Class

Objective(s):

- Setup and evaluate triple integrals.

Now let's bring back up Fubini's Strong Theorem

Theorem 7.4 (Fubini's Theorem (Stronger Form)(Short)).

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Notice that in 1 the functions $g(x)$ depend on x because x is integrated 2nd (hasn't happened yet)

Notice that in 2 the functions $h(y)$ depend on y because y is integrated 2nd (hasn't happened yet)

Now suppose we are doing some Triple integrals:

$$\int_{C_1}^{C_2} \int_{B_1}^{B_2} \int_{A_1}^{A_2} f(x,y,z)\,dz\,dy\,dx$$

What can A_1, A_2 depend on? y & x

What can B_1, B_2 depend on? x

What can C_1, C_2 depend on?

only numbers

$$\int_{C_1}^{C_2} \int_{B_1}^{B_2} \int_{A_1}^{A_2} f(x,y,z)\,dy\,dx\,dz$$

What can A_1, A_2 depend on? x & z

What can B_1, B_2 depend on? z

What can C_1, C_2 depend on?

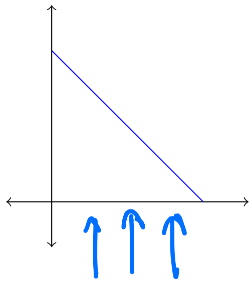
only numbers

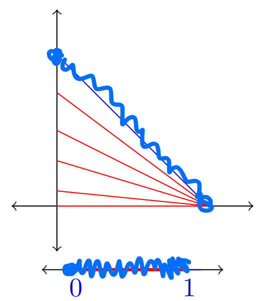
So triple integration itself is not any harder than double integration. However figuring out general regions of integration, and switching orders of integration is considerably more difficult. In addition we need to be fairly comfortable in 3D graphing.

Example 7.5. Integrate $\iint_D x\, dA$ over the region D bounded in the first quadrant by $y = 1 - x$

Solution.

$$\iint_D x\, dA = \int_a^b \int_0^{1-x} x\, dy\, dx$$





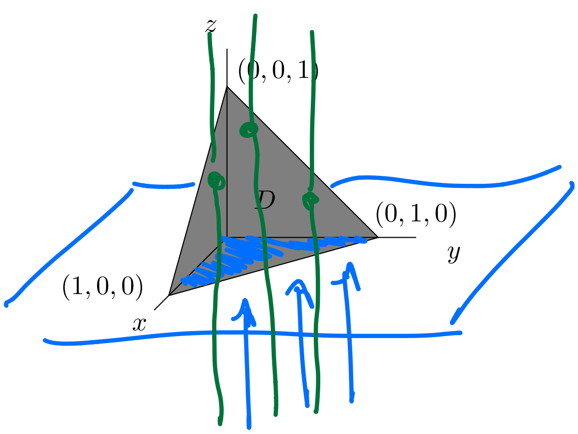
In order to find a and b "where we need to care" let's pretend that we have already integrated with respect to y so then y no longer exists the line $y = 1 - x$ get "projected" onto the x axis.

$$\int_0^1 \int_0^{1-x} x\, dy\, dx$$

This is the right idea to use for finding bounds of triple integrals:

Example 7.6. Integrate $\iiint_D x\, dV$ over the region D bounded in the first octant by the plane $x + y + z = 1$

$z = 1 - x - y$



So now we need to consider the integral

$$\iiint x\, dz\, dy\, dx$$

So let's travel along z. What is the first surface you run into?

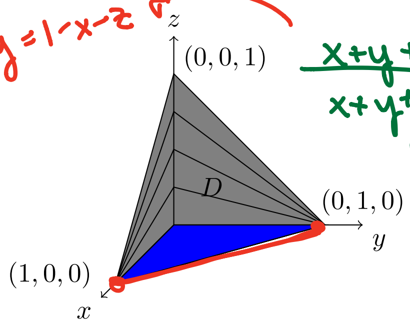
$z = 0$

What is the second surface you run into?

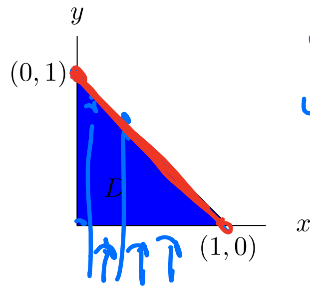
$z = 1 - x - y$

So let's consider that z is killed and so it all get's projected to the xy-plane.

$y = 1 - x - z$



$x + y + z = 1$
$x + y + 0 = 1$
$y = 1 - x$



$y = mx + b$
$y = -x + 1$

0 1

Now all that is left is x's and y's. Next we must find the y bounds of integration. And we see that we are back in 15.2 world.

It is clear we should have:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \; dz \, dy \, dx$$

$y = 0$
$y = -x + 1$

Let's integrate this quickly to get an answer (this is the boring part)

$$= \int_0^1 \int_0^{1-x} \Big[xz \Big]_0^{1-x-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x - x^2 - xy \; dy \, dx$$

$$= \int_0^1 \Big[xy - x^2 y - x\frac{y^2}{2} \Big]_0^{1-x} dx$$

$$= \int_0^1 x(1-x) - x^2(1-x) - \frac{x}{2}(1-x)^2 \; dx$$

$$= \int_0^1 x - x^2 - x^2 + x^3 - \frac{x}{2}(1 - 2x + x^2) \; dx$$

$$= \int_0^1 x - 2x^2 + x^3 - \frac{x}{2} + x^2 - \frac{x^3}{2} \; dx$$

$$= \int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2} \; dx$$

$$= \Big[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \Big]_0^1 = \boxed{\frac{1}{4} - \frac{1}{3} + \frac{1}{8}}$$

Ex

Example 7.7. Evaluate the triple integral: $\iiint_E 2y\, dV$ where $E = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le 3x, x - 8y \le z \le 4y + x\}$

$$\int_0^1 \int_0^{3x} \int_{x-8y}^{4y+x} 2y \, dz\, dy\, dx$$

$$\int_0^1 \int_0^{3x} 2y(4y+x) - 2y(x-8y)\, dy\, dx$$

$$\int_0^1 \int_0^{3x} 8y^2 + \cancel{2yx} - \cancel{2yx} + 16y^2\, dy\, dx$$

$$\int_0^1 \int_0^{3x} 24y^2\, dy\, dx$$

$$\int_0^1 8y^3 \Big]_0^{3x} dx$$

$$\int_0^1 216x^3\, dx$$

$$54x^4 \Big]_0^1$$

$$\boxed{54}$$

$27 \times 8 = 216$

$216 \div 4 = 54$

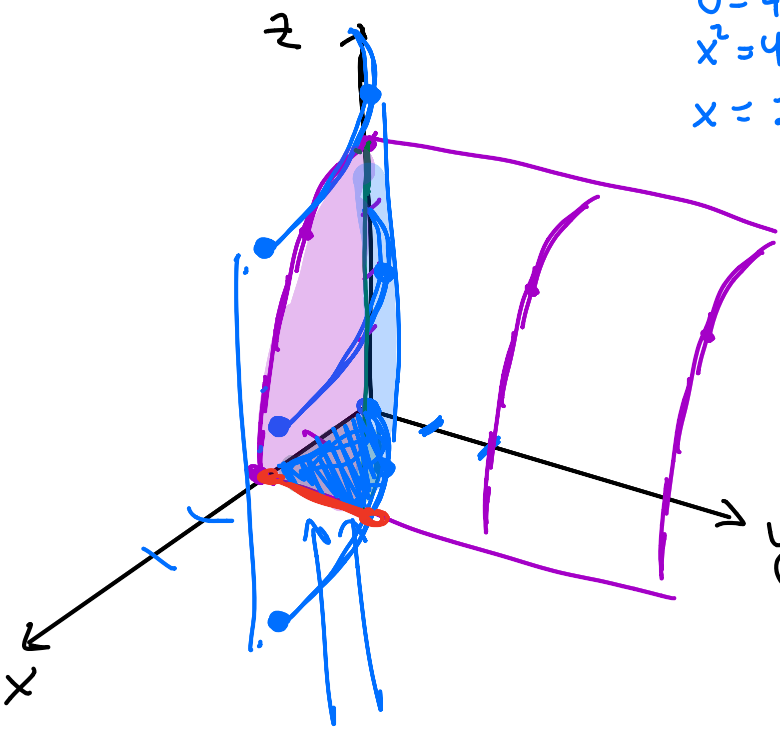
Example 7.8. A solid in the first octant is bounded by $z = 4 - x^2$ and $x = y^2$, where $x, y, \& z$ are in meters.

(a) Sketch the region and setup a triple integral representing the volume of the solid.

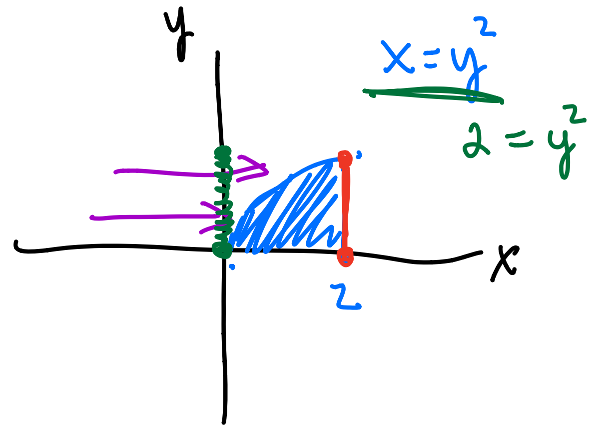
$0 = 4 - x^2$

$x^2 = 4$

$x = 2$



$$\int_0^{\sqrt{2}} \int_{y^2}^{2} \int_0^{4-x^2} 1 \; dz\,dx\,dy$$



(b) There are bees inside the solid with a bee density function given by $B(x, y, z) = xy \ \dfrac{\text{bees}}{\text{m}^3}$.

How many bees are in the container?

alternative

$$\int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} xy \; dz\,dy\,dx$$

$$= \int_0^2 \int_0^{\sqrt{x}} xy(4-x^2)\,dy\,dx = \int_0^2 (4x - x^3)\frac{y^2}{2}\Big|_0^{\sqrt{x}} dx$$

$$= \int_0^2 \frac{x}{2}(4x - x^3)\,dx = \int_0^2 2x^2 - \frac{x^4}{2}\,dx = \frac{2x^3}{3} - \frac{x^5}{10}\Big|_0^2$$

$$= \frac{16}{3} - \frac{32}{10} = \frac{160}{30} - \frac{96}{30} = \frac{64}{30} = \boxed{\frac{32}{15}} \approx \boxed{2 \text{ bees}}$$

7 (B) – Triple Integrals Day 2!

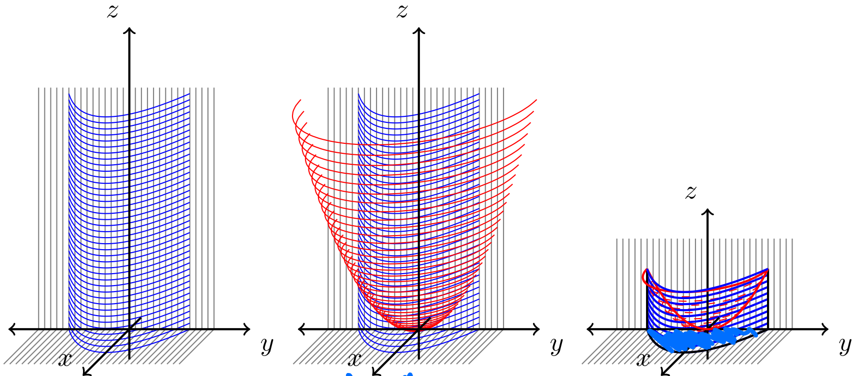
7.1 A New Day a New Order - During Class

Objective(s):

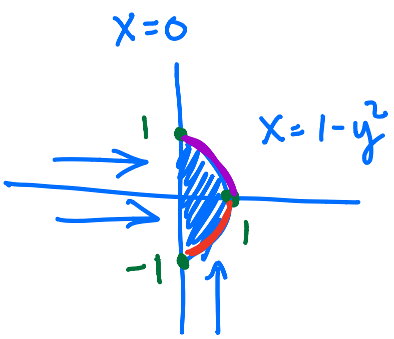
- Get more practice evaluating triple integrals.
- Switch the order of triple integrals.

Example 7.1. Find the volume of the region bounded in the back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$ and by the xy-plane on bottom.

$z=0$



$$\int_{-1}^{1}\int_{0}^{1-y^2}\int_{0}^{x^2+y^2} 1\,dz\,dx\,dy$$



$$=\int_{-1}^{1}\int_{0}^{1-y^2} x^2+y^2\,dx\,dy$$

$$=\int_{-1}^{1}\left[\frac{x^3}{3}+xy^2\right]_{0}^{1-y^2} dy$$

$$=\int_{-1}^{1}\frac{(1-y^2)^3}{3}+(1-y^2)y^2\,dy$$

$$=\int_{-1}^{1}\frac{1}{3}-\cancel{\frac{3y^2}{3}}+\cancel{\frac{3y^4}{3}}-\frac{y^6}{3}+\cancel{y^2}-\cancel{y^4}\,dy$$

$$\left[\frac{1}{3}y-\frac{1}{21}y^7\right]_{-1}^{1} = \left(\frac{1}{3}-\frac{1}{21}\right)-\left(\frac{-1}{3}+\frac{1}{21}\right)$$

$$=\frac{2}{3}-\frac{2}{21}$$

Example 7.2. Consider the integral $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx$. Rewrite the integral as an equivalent iterated integral in the order

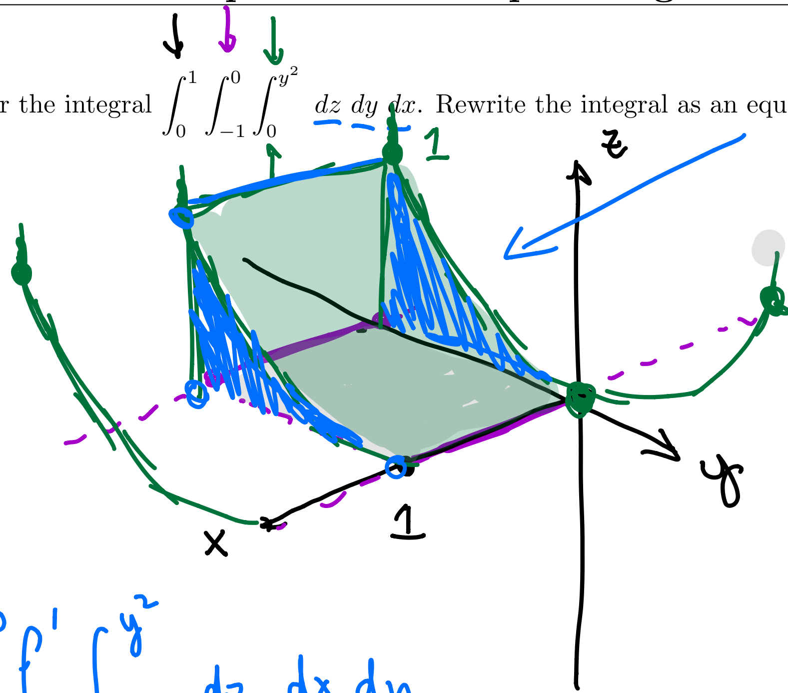
(a) $dz\, dx\, dy$

(b) $dy\, dx\, dz$

(c) $dy\, dz\, dx$

(d) $dx\, dy\, dz$

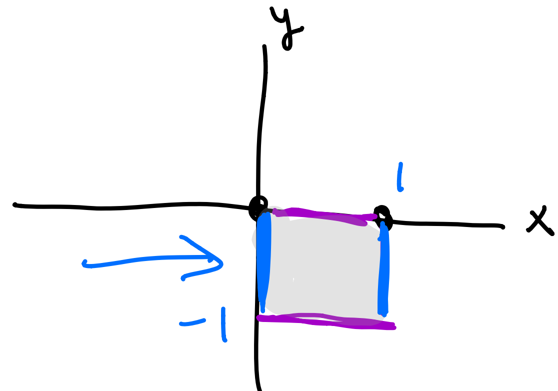
(e) $dx\, dz\, dy$



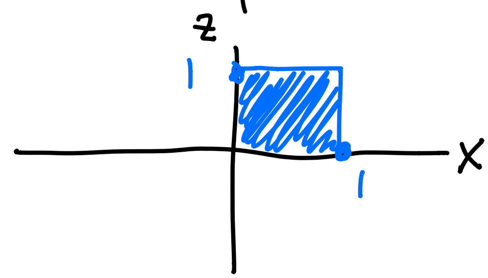
$z = y^2$

$y = -\sqrt{z}$

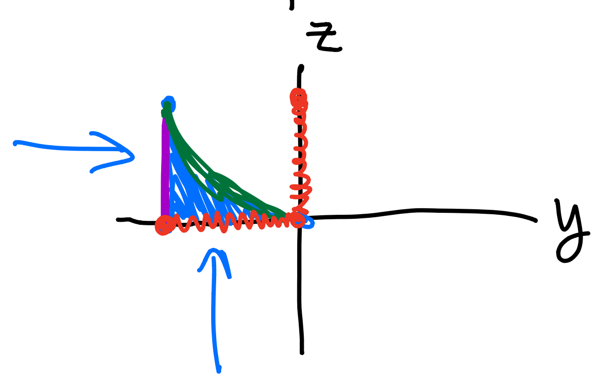
(a) $\int_{-1}^{0} \int_0^1 \int_0^{y^2} dz\, dx\, dy$



(b) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$



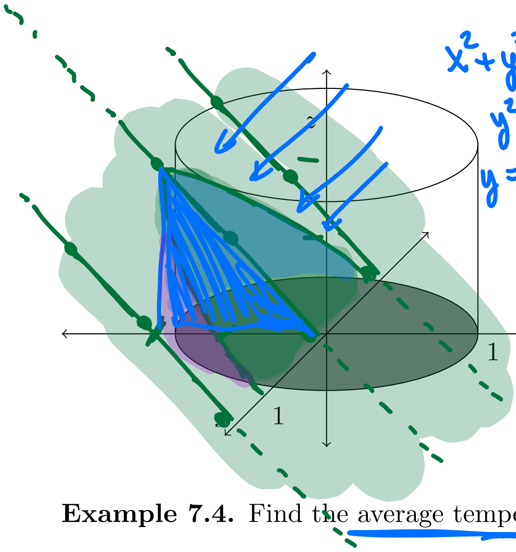
(c) $\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$



(d) $\int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\, dy\, dz$

(e) $\int_{-1}^{0} \int_0^{y^2} \int_0^1 dx\, dz\, dy$

Example 7.3. Express the volume of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ above $z = 0$ in two different ways. Choose one of the two and evaluate.



$x^2+y^2=1$

$y^2=1-x^2$

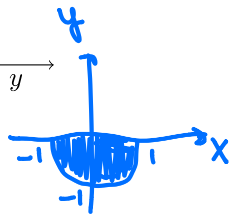
$y=-\sqrt{1-x^2}$

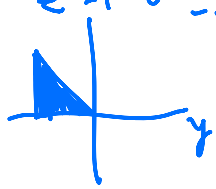
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} 1\,dz\,dy\,dx$$

$x^2=1-y^2$

$x=\pm\sqrt{1-y^2}$

$$\int_{-1}^{0}\int_{0}^{-y}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy = 2/3$$

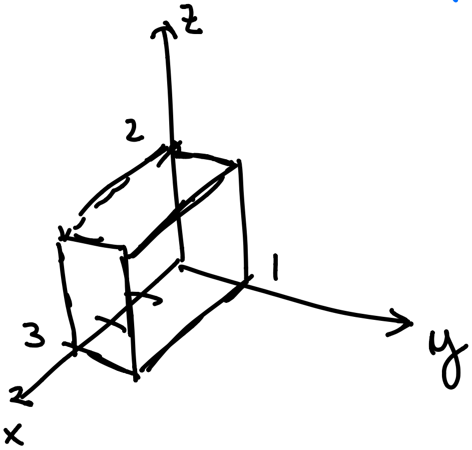




Example 7.4. Find the average temperature or the over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 3$, $y = 1$, and $z = 2$ where the temperature function is given by $T(x, y, z) = x + y - z$

$x=0$
$y=0$
$z=0$

$$\text{Avg } f = \frac{1}{V(E)}\iiint_E f\,dV$$



$$= \frac{1}{V(E)}\iiint x+y-z\;dV$$

$$= \frac{1}{6}\int_0^3\int_0^1\int_0^2 x+y-z\;dz\,dy\,dx$$

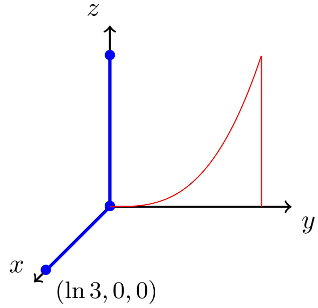
$$= \frac{1}{6}\int_0^3\int_0^1 (x+y)z-\frac{z^2}{2}\Big|_0^2\,dy\,dx$$

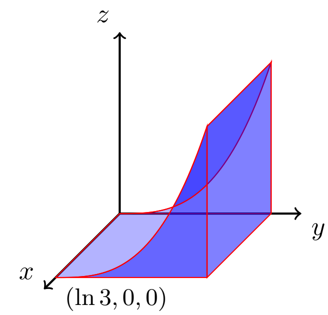
$$= \frac{1}{6}\int_0^3\int_0^1 2x+2y-2\;dy\,dx$$

$$= \frac{1}{6}\int_0^3 (2x-2)y+y^2\Big|_0^1\,dx = \frac{1}{6}\int_0^3 2x-2+1\,dx$$

$$= \frac{1}{6}\left(x^2-x\right)_0^3 = \frac{1}{6}(9-3) = \boxed{1}$$

Example 7.5. You have a strange shaped container holding bees whose volume could be represented as $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} dx\,dy\,dz$. Determine the total number of bees in your container given that the bee density function is given by $B(x,y,z) = \dfrac{\pi e^{2x}\sin\left(\pi y^2\right)}{y^2}$ (units are $\dfrac{bees}{m^3}$).



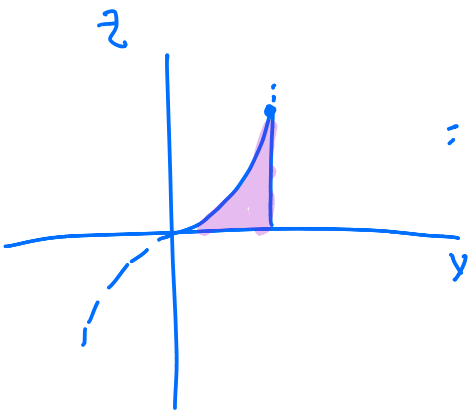


$$\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\,dy\,dz$$

$$=\int_0^1\int_{\sqrt[3]{z}}^1 \frac{\pi\sin(\pi y^2)}{y^2}\,\frac{1}{2}e^{2x}\Big|_0^{\ln 3}\,dy\,dz$$

$e^{2\ln 3} = e^{\ln 3^2} = 3^2$

$$=\int_0^1\int_{\sqrt[3]{z}}^1 \frac{\pi\sin(\pi y^2)}{y^2}\,\frac{1}{2}(9-1)\,dy\,dz$$



$$=4\pi\int_0^1\int_0^{y^3}\frac{\sin(\pi y^2)}{y^2}\,dz\,dy$$

$y=\sqrt[3]{z}$

$y^3=z$

$$=4\pi\int_0^1 y\sin(\pi y^2)\,dy$$

$$=-2\left(\cos(\pi y^2)\right)\Big|_0^1$$

$$=2+2=\boxed{4}\ \text{bees}$$

Convert the point $(x, y, z) = (\sqrt{3}, 1, 2)$ into ~~polar~~ cylindrical coordinates

A. $(r, \theta, z) = (2, \pi/6, 2)$

~~B.~~ $(r, \theta, z) = (4, \pi/6, 2)$

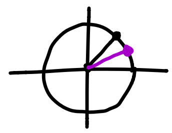
C. $(r, \theta, z) = (2, \pi/3, 2)$

~~D.~~ $(r, \theta, z) = (4, \pi/3, 2)$

E. None of the above.

$$x^2 + y^2 = r^2$$
$$3 + 1 = r^2$$
$$4 = r^2$$
$$2 = r$$

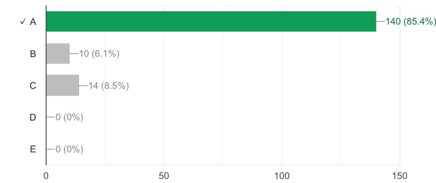
$$\frac{1}{\sqrt{3}} = \frac{y}{x} = \tan\theta$$
$$\frac{\pi}{6} = \theta$$



$$\frac{1}{\sqrt{3}}, \quad 1, \quad \sqrt{3}$$

Question 1

140 / 164 correct responses



Question 2

Evaluate $\int_0^{2\pi} \int_0^1 \int_0^1 r \, dz \, dr \, d\theta$

A. $\pi/4$

B. $\pi/2$

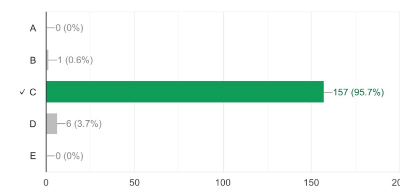
C. π

D. 2π

E. 4π

Question 2

157 / 164 correct responses



8.2 Class Time is Play Time - During Class

Objective(s):

- Experience more problems that can be solved using integration in cylindrical coordinates.
- Get some practice!

Example 8.7. Convert $\boxed{\int_0^1 \int_0^{\sqrt{1-x^2}}} \int_0^y (x^2 + y^2)\, dz\, dy\, dx$ into cylindrical coordinates and evaluate the result

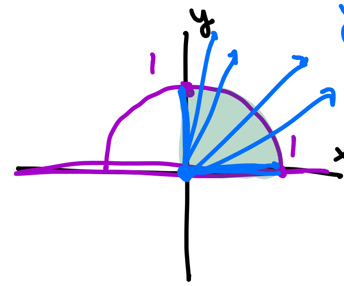
$$\int_0^{\pi/2} \int_0^1 \int_0^{r\sin\theta} r^2 \cdot r \, dz \, dr \, d\theta$$

$y = \sqrt{1-x^2}$

$y^2 + x^2 = 1$

$r^2 = 1$

$r = 1$



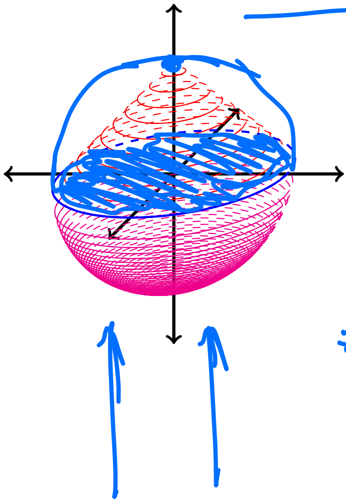
$$= \int_0^{\pi/2} \int_0^1 r^3 z \Big|_0^{r\sin\theta} dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^4 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^5}{5} \sin\theta \right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{5} \sin\theta \, d\theta = \left[\frac{1}{5} \cdot -\cos\theta \right]_0^{\pi/2} = \frac{1}{5}(1) = \frac{1}{5}$$

Example 8.8. Find the volume of the following solid:



$z = 1 - \sqrt{x^2 + y^2}$

$1 - \sqrt{r^2} = 1 - |r|$

$z = -\sqrt{1 - x^2 - y^2}$

$z = -\sqrt{1 - r^2}$

$$\iiint 1 \, dV$$

$$= \iiint r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} r \, dz \, dr \, d\theta$$

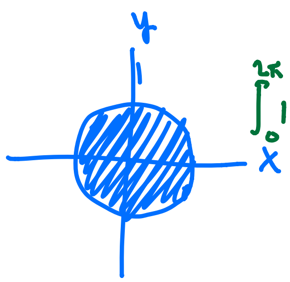
$$1 - r = z = -\sqrt{1-r^2}$$

$$\cancel{1} - 2r + r^2 = \cancel{1} - r^2$$

$$2r^2 - 2r = 0$$

$$2r(r-1) = 0$$

$$r = 0 \quad r = 1$$



$$= \int_0^{2\pi} \int_0^1 r\left(1 - r + \sqrt{1-r^2}\right) dr \, d\theta$$

$$\int_0^{2\pi} 1 \, d\theta$$

$$= 2\pi \int_0^1 r - r^2 + r\sqrt{1-r^2} \, dr$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^3}{3} + (1-r^2)^{3/2} \frac{2}{3} \cdot \frac{1}{-2} \right]_0^1$$

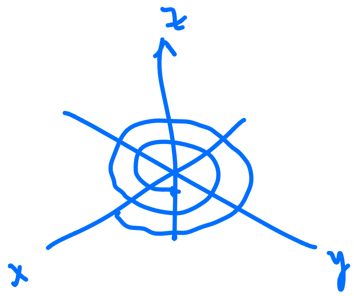
$$= 2\pi \left(\frac{1}{2} - \frac{1}{3} + 0 + (1)^{3/2} \cdot \frac{1}{3} \right)$$

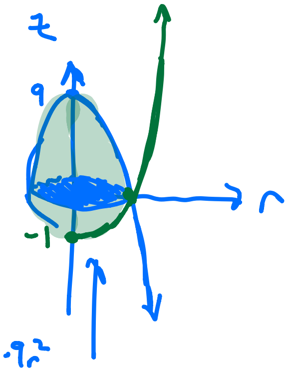
$$= \pi$$

12.6 12.6

Example 8.9. Find the volume of the solid enclosed by the curves $z = 9 - 9(x^2 + y^2)$ and $z = (x^2 + y^2)^2 - 1$

rz-half plane $z = 9 - 9r^2$ $z = r^4 - 1$





$$\int_0^{2\pi}\int_0^1\int_{r^4-1}^{9-9r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r(9 - 9r^2 - r^4 + 1)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 10r - 9r^3 - r^5\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 10r - 9r^3 - r^5\,dr\right)$$

$$= (2\pi)\left(5r^2 - \frac{9r^4}{4} - \frac{r^6}{6}\Big]_0^1\right)$$

$$= \boxed{2\pi\left(5 - \frac{9}{4} - \frac{1}{6}\right)}$$

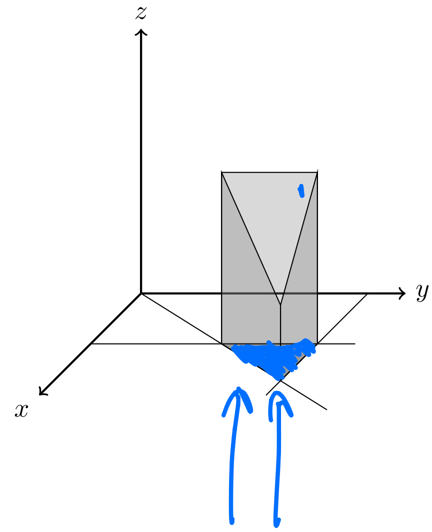
Example 8.10. (Old SS14 Exam)

Consider the region R below bounded by the surfaces:

$x = \sqrt{3}$ $\quad y = 3$

$y = x$ $\quad z = 4 - x$

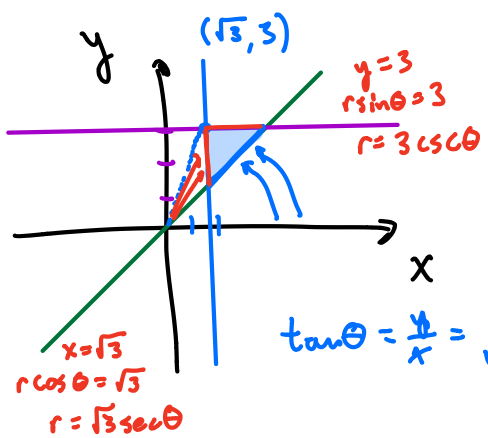
$z = 0$



(a) Determine the limits of integration for $\iiint_R f(x,y,z)\ dz\ dy\ dx$. Do not evaluate.

(b) Determine the limits of integration for $\iiint_R f(x,y,z)\ dy\ dz\ dx$. Do not evaluate.

(c) Determine the limits of integration for $\iiint_R f(r,\theta,z) r\ dz\ dr\ d\theta$. Do not evaluate.



$\tan\theta = \frac{y}{x} = \frac{3}{\sqrt{3}} = \sqrt{3}$

(c) $$\int_{\pi/4}^{\pi/3}\int_{\sqrt{3}\sec\theta}^{3\csc\theta}\int_0^{4-r\cos\theta} f(r,\theta,z)\, r\ dz\ dr\ d\theta$$

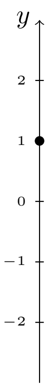
9.2 Spherical Problems - During Class

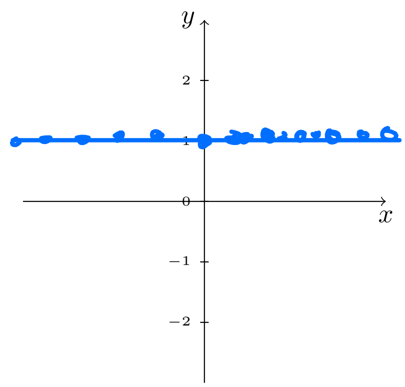
Objective(s):

- Learn a better way to sketch functions in Spherical.
- Transform Cartesian equations to spherical (and vise versa).
- Evaluate spherical integrals.

In order to help visualize 3D spherical graphs we can graph them in the rz-halfplane so long as the surfaces do not depend on θ.

Idea: Consider graphing $y = 1$ on the $y-$axis and $y = 1$ on the $xy-$plane.





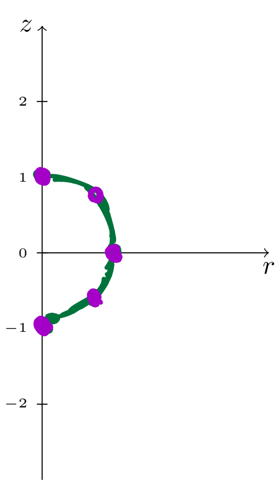
because there is no x in the equations, when we graph in the xy-plane x can “run free”.

Now lets consider graphing $\rho = 1$ is the rz-halfplane (where $r \geq 0$). Recall:

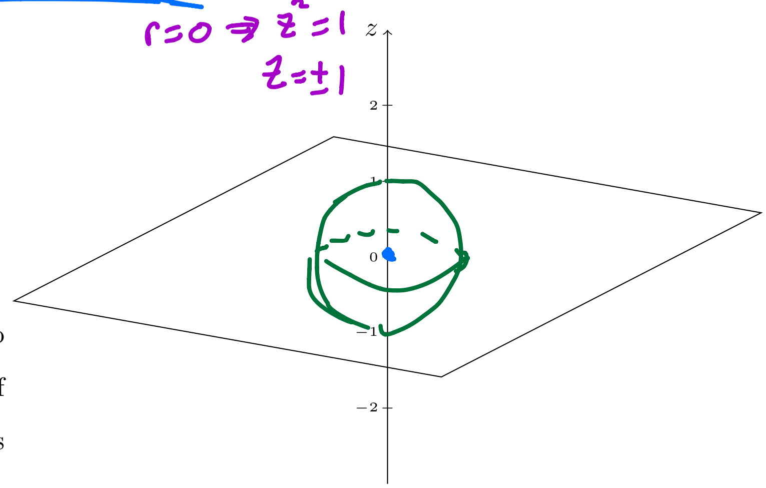
$$\rho = 1 \implies \rho^2 = 1 \implies x^2 + y^2 + z^2 = 1 \implies r^2 + z^2 = 1$$

no θs!

$r=0 \Rightarrow z^2=1$
$z=\pm 1$



And now we let θ “run free” to get our picture of the sphere. Of course θ runs around the z-axis so we get the picture



This also gives us a very nice way to “smash” with respect to θ to get a nice 2D picture. Let′s do an example to see.

cone on bottom

Example 9.3. Use spherical integration to find the volume of the solid bounded below by the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$ and above by the hemisphere $z = \sqrt{1-x^2-y^2}$

$z = \sqrt{\frac{r^2}{3}}$

$z = \frac{r}{\sqrt{3}}$

$\rho\cos\phi = \frac{\rho\sin\phi}{\sqrt{3}}$

$\sqrt{3} = \tan\phi$

$\frac{\pi}{3} = \phi$

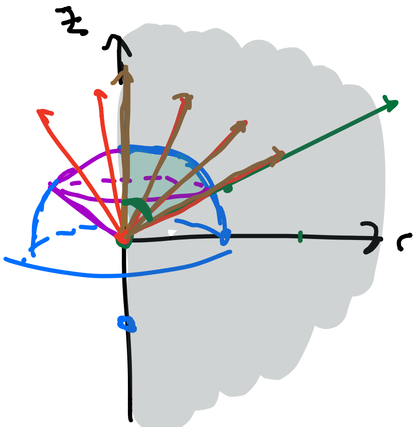
hemisphere on top

$z^2 = 1-x^2-y^2$

$x^2+y^2+z^2 = 1$

$\rho^2 = 1$

$\rho = 1$



$$\iiint 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \cdot \sin\phi \cdot 1\, d\rho\, d\phi\, d\theta$$

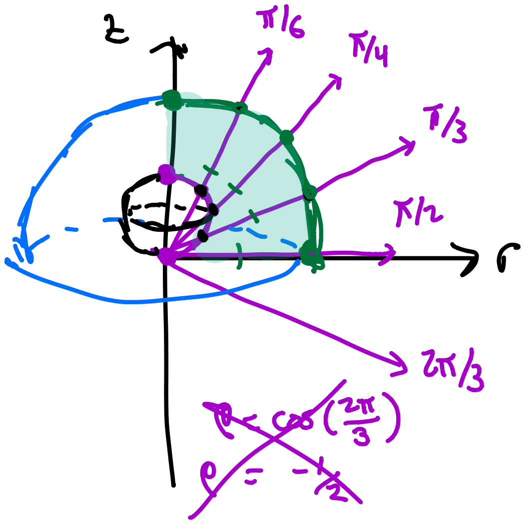
$$= \int_0^{2\pi} 1\, d\theta \cdot \int_0^{\pi/3} \sin\phi\, d\phi \cdot \int_0^1 \rho^2\, d\rho$$

$$= 2\pi \cdot \left[-\tfrac{1}{2} + 1\right] \cdot \left[\tfrac{1^3}{3}\right]$$

$$= \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \cdot \frac{1}{3} = \boxed{\pi/3}$$

Example 9.4. Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geq 0$.

Sketch a picture of the region in the $rz-$halfplane and a picture in $xyz-$space.



$\rho^2 = 4$

$x^2+y^2+z^2=4$

$r^2+z^2=4$

$$\iiint 1\, dV$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \left[\frac{\rho^3}{3}\sin\phi\right]_{\cos\phi}^{2} d\phi$$

$$= 2\pi \int_0^{\pi/2} \frac{8}{3}\sin\phi - \frac{\cos^3\phi}{3}\sin\phi \, d\phi$$

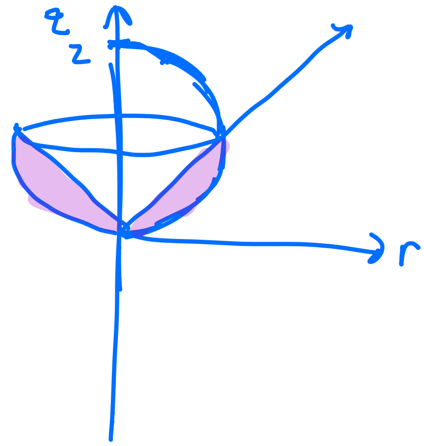
$$= 2\pi \left[\frac{-8\cos\phi}{3} + \frac{\cos^4\phi}{3}\cdot\frac{1}{4}\right]_0^{\pi/2}$$

$$= 2\pi\left(0 + \frac{8}{3}(1) + 0 - \frac{1}{12}\right)$$

$$= 2\pi\left(\frac{8}{3} - \frac{1}{12}\right)$$

Example 9.5. Find the volume of the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$

$z = r$



$z = r$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$\phi = \pi/4$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_0^{2\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_{\pi/4}^{\pi/2} 8\cos^3\phi \sin\phi \, d\phi$$

$$= \frac{4\pi}{3} \left(-\cos^4\phi\right)_{\pi/4}^{\pi/2}$$

$$= \frac{4\pi}{3}\left(0^4 + \left(\frac{\sqrt{2}}{2}\right)^4\right)$$

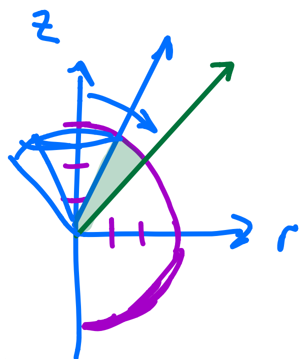
$$= \frac{4\pi}{3}\left(\frac{4}{16}\right) = \boxed{\frac{\pi}{3}}$$

Example 9.6. Find the volume between two cones $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{3\left(x^2 + y^2\right)}$ and bounded by the hemisphere $x^2 + y^2 + z^2 = 9$.

$\rho^2 = 9$

$\rho = 3$

$r^2 + z^2 = 9$



$z = \sqrt{r^2}$

$z = r$

$z = \sqrt{3}\, r$

$\cancel{\rho}\cos\phi = \sqrt{3}\,\cancel{\rho}\sin\phi$

$\frac{1}{\sqrt{3}} = \tan\phi$

$\pi/6 = \phi$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^3 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_{\pi/6}^{\pi/4} \sin\phi\, d\phi \int_0^3 \rho^2\, d\rho$$

$$= 2\pi \Big(-\cos\phi\Big)_{\pi/6}^{\pi/4} \left(\frac{\rho^3}{3}\right)_0^3$$

$$= 2\pi\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)(9-0)$$

$$= 18\pi\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right) = \boxed{9\pi\left(\sqrt{3}-\sqrt{2}\right)}$$

Example 9.7 (Old SS14 Exam Problem)**.** Consider the surface $\rho = \cos\phi$. I fill the enclosed portion of the surface with liquid X (a mysterious liquid) which has density function $\delta(x,y,z) = xyz\,\dfrac{grams}{m^3}$. Setup an integral to express the weight of liquid X. You do not need to evaluate.

Solution. This much should be relatively clear:

$$\begin{aligned}\iiint_D \delta(x,y,z)\,dV &= \int\int\int xyz(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta \\ &= \int\int\int_0^{\cos\phi} (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0\int_0^{\cos\phi} \rho^5\sin\theta\cos\theta\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

The only thing we have to be careful about is ϕ. If we do $\phi \in [0,\pi]$ we actually cover the sphere twice over (and double count everything). In order to just count it once we need $\phi \in [0,\pi/2]$. Giving us the solution:

$$\boxed{\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^5\sin\theta\cos\theta\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta}$$

or equivalently:

$$\boxed{\int_0^{\pi}\int_0^{\pi}\int_0^{\cos\phi} \rho^5\sin\theta\cos\theta\sin^3\phi\cos\phi\,d\rho\,d\phi\,d\theta}$$