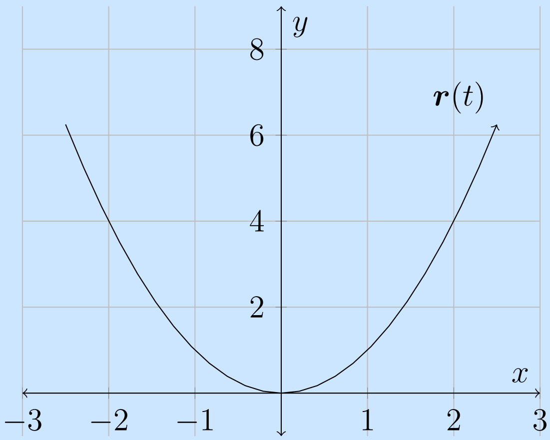
13.1 Vector Functions and Curves

Parameterizing curves in $\mathbb{R}^2$

1. Sketch $\boldsymbol{r}(t) = \langle t, t^2 \rangle$ in $\mathbb{R}^2$ and express the curve as a function of x and y.

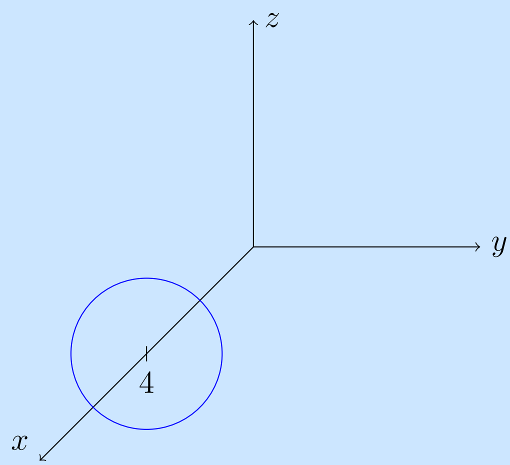
Solution: If $x = t$ then $y = t^2 = x^2$. The curve is described by $y = x^2$.



2. Sketch $\boldsymbol{r}(t) = \langle 4, \cos(t), \sin(t) \rangle$ in $\mathbb{R}^3$ and express the curve as a function of x, y, and z.

Solution: Setting $y = \cos(t)$ and $z = \sin(t)$ corresponds to a circle of radius 1 in the yz-plane. In $\mathbb{R}^3$ this is a cylinder extending infinitely parallel to the x-axis. By placing the restriction $x = 4$ we get the previously mentioned circle shifted 4 units in the positive x direction.

In particular, $x = 4$ and $y^2 + z^2 = 1$.



Definition. A vector function $\boldsymbol{r}(t)$ is *continuous* at $t = a$ if

$$\lim_{t\to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$$

Note that by the above definition to check $\boldsymbol{r}(t)$ is continuous at $t = a$ you need to verify that $\boldsymbol{r}(a)$ and $\lim_{t\to a} \boldsymbol{r}(t)$ are both defined and equal to each other.

3. Show that $\boldsymbol{r}(t) = \langle \sin(\pi t) + 1, t^2 + t, 1\rangle$ is continuous at $t = -1$.

Solution: $\boldsymbol{r}(t)$ is clearly defined when $t = -1$ with $\lim_{t->-1} \boldsymbol{r}(t) = (1, 2, 1) = \boldsymbol{r}(-1)$

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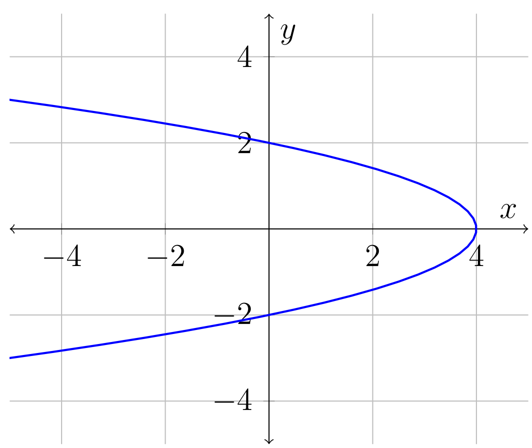
4. Find a vector function $\boldsymbol{r}(t)$, $t \in \mathbb{R}$ that represents the curve $x + y^2 = 4$ shown below.



(a) Try parametezing the curve by setting $x = t$ and explain what makes that approach difficult (there is not one correct answer)..

Solution: Let $x = t$. Then

$$t + y^2 = 4 \iff y^2 = 4 - t \iff y = \pm\sqrt{4 - t}\ldots$$

(b) Now try parameterizing by setting $y = t$.

Solution: Let $y = t$. Then

$$x + t^2 = 4 \iff x = 4 - t^2$$

and we have the vector equation

$$\boldsymbol{r}(t) = \left\langle 4 - t^2, t\right\rangle$$

(c) Verify the vector function you found satisfies $x + y^2 = 4$ for any arbitrary choice of t.

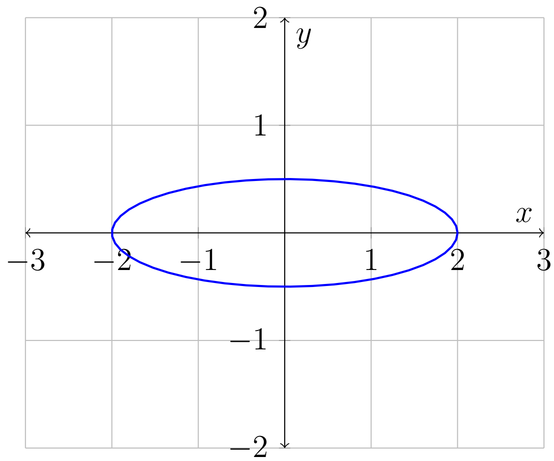
Solution:

$$\begin{aligned} x + y^2 &= 4 \\ (4 - t^2) + (t^2) &= 4 \\ 4 &= 4 \end{aligned}$$

(d) What would you do differently for the curve $x^2 + y = 4$?

Solution: Almost nothing. The roles of x and y are exchanged in the parameterization, i.e. set $x = t$ etc.

5. Find a vector function $\boldsymbol{r}(t)$ that represents the curve $x^2 + 16y^2 = 4$



(a) Try parameterizing by setting $x = t$ and then try setting $y = t$. What makes these so difficult to work with?

Solution: If we set $x = t$, then $t^2 + 16y^2 = 4$ gives $y = \pm\frac{\sqrt{4-t^2}}{4}$ which is not ideal. Setting $y = t$ is similar.

(b) Rewrite the above equation so that it has the form

$$(f(x))^2 + (g(y))^2 = 1$$

Solution: Divide by 4:

$$\frac{x^2}{4} + 4y^2 = 1 \quad \Longleftrightarrow \quad \left(\frac{x}{2}\right)^2 + (2y)^2 = 1$$

(c) Find $\boldsymbol{r}(t)$ by setting $f(x) = \cos t$ and $g(y) = \sin t$. Be sure to include the domain for t.

Solution: From

$$\frac{x}{2} = \cos(t) \quad \text{and} \quad 2y = \sin(t)$$

we get the parameterization

$$\boldsymbol{r}(t) = \left\langle 2\cos(t), \frac{1}{2}\sin(t) \right\rangle, \quad 0 \le t \le 2\pi$$

(d) What do you need to change to represent the portion of $x^2 + 36y^2 = 4$ where $y \ge 0$? What about $x > 0$?

Solution: There are multiple correct ways to adjust this. One example: since $y \ge 0$ corresponds to the upper half of the ellipse, we can simply restrict the domain of t to $0 \le t \le \pi$. For $x > 0$, we can use $-\pi/2 < t < \pi/2$

Parameterizing curves in $\mathbb{R}^3$

6. Find a vector function that represents the curve of intersection between the two surfaces

$$y = 4z^2 + x^2 \quad \text{and} \quad x = z^2$$

Hint: Try $x = t$, $y = t$, and $z = t$ and see which gives you something you can work with.

Solution: We need to find a way to describe all points (x, y, z) that satisfy both $y = 4z^2 + x^2$ and $x = z^2$. The second equation is a little simpler so we will start there

$$\begin{aligned} x = t: &\quad \Rightarrow \ z(t) = \pm\sqrt{x}\dots \\ z = t: &\quad \Rightarrow \ x(t) = t^2 \end{aligned}$$

Note that if you try $y = t$ the equation becomes too complicated.

Using the above we can also find an equation for $y(t) = 4t^2 + t^4$.

So the intersection is described by

$$x(t) = t^2, \qquad y(t) = 4t^2 + t^4, \qquad z(t) = t$$

You can verify this by plugging in $x(t), y(t), z(t)$ into the equations above.