16.6

More parameterization of surfaces

1. Let $\boldsymbol{r}(u,v) = \langle u+v, u-v, u^2-v^2\rangle$

   (a) Evaulate $\boldsymbol{r}(2,-1)$ and $\boldsymbol{r}(-1,2)$.

   Solution:

   $$\begin{aligned}\boldsymbol{r}(2,-1) &= (2-1, 2-(-1), 2^2-(-1)^2) \\ &= (1,3,3) \\ \boldsymbol{r}(-1,2) &= (-1+2, -1-2, (-1)^2-2^2) \\ &= (1,-3,-3)\end{aligned}$$

   (b) Find u, v so that $\boldsymbol{r}(u,v) = (3,-1,-3)$.

   Solution: If $u+v=3$ then $v = 3-u$. Then

   $$u - v = -1 \quad\Longleftrightarrow\quad u-(3-u) = -1 \quad\Longleftrightarrow\quad 2u = 2 \quad\Longleftrightarrow\quad u = 1$$

   Which implies $v = 3-1 = 2$. To verify $u=1, v=2$ is correct,

   $$\boldsymbol{r}(1,2) = (1+2, 1-2, (1)^2-(2)^2) = (3,-1,-3).$$

   (c) Show that $(0,0,1)$ is not a point on the surface.

   Solution: Since $u-v=0$, $u=v$. Then $u+v=0$ gives $u+u=0$ which means $u=0=v$. But $\boldsymbol{r}(0,0) = (0,0,0) \neq (0,0,1)$.

   (d) Recall that a surface $\boldsymbol{r}(u,v)$ is *smooth* if $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are both continuous and $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is never 0 for (u,v) in the interior of the domain (this means that at any point on the surface, the normal vector to the tangent plane is not $\mathbf{0}$). Show that $\boldsymbol{r}(u,v)$ is continuous.

Solution: We first compute $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, $\boldsymbol{r}_u \times \boldsymbol{r}_v$, and $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$.

$$\begin{aligned}
\boldsymbol{r}_u(u,v) &= \langle 1+0, 1-0, 2u-0\rangle \\
&= \langle 1, 1, 2u\rangle \\
\boldsymbol{r}_v(u,v) &= \langle 0+1, 0-1, 0-2v\rangle \\
&= \langle 1, -1, -2v\rangle . \\
\boldsymbol{r}_u \times \boldsymbol{r}_v &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 2u \\ 1 & -1 & -2v \end{vmatrix} \\
&= \langle 2u-2v, 2u+2v, -2\rangle \\
|\boldsymbol{r}_u \times \boldsymbol{r}_v| &= \sqrt{(2u-2v)^2 + (2u+2v)^2 + (-2)^2} \\
&= \sqrt{4u^2 - 4uv + 4v^2 + 4u^2 + 4uv + 4v^2 + 4} \\
&= \sqrt{8u^2 + 8v^2 + 4}
\end{aligned}$$

It is clear that $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are continuous everywhere (constant functions and polynomials). The only way $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{8u^2+8v^2+4} = 0$ is if $8u^2+8v^2+4=0$. Since $8u^2+8v^2+4 \geq 4$ for any (u,v) we know $|\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is never 0.

This means the surface is continuous

(e) Find an equation of the tangent plane to the parametric surface $\boldsymbol{r}(u,v)$ at the point $u=1$ and $v=-1$.

Solution: A point on the tangent plane is given by $\boldsymbol{r}(1,-1) = (0,2,0)$. Using $\boldsymbol{r}_u \times \boldsymbol{r}_v$ found above,

$$\begin{aligned}
\boldsymbol{r}_u \times \boldsymbol{r}_v(1,-1) &= \langle 2(1) - 2(-1), 2(1) + 2(-1), -2\rangle \\
&= \langle 4, 0, -2\rangle
\end{aligned}$$

Therefore an equation for the tangent plane is

$$\begin{aligned}
\langle 4, 0, -2\rangle \cdot \langle x-0, y-2, z-0\rangle &= 0 \\
4(x-0) + 0(y-2) - 2(z-0) &= 0 \\
4x - 2z &= 0
\end{aligned}$$