15.10

Useful Information.

- The **Jacobian** for a tranformation T given by $x = x(u, v)$ and $y = y(u, v)$ is

$$\frac{(x, y)}{(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

- $$\iint_R f(x, y)\ dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{(u, v)}{(u, v)} \right|\ du\ dv$$

1. Let S be the square $\{(u, v)\ :\ 0 \leq u \leq 1, 0 \leq v \leq 1\}$ and let T be the transformation

$$x = v,\ \ y = u(1 + v^2)$$

(a) Let L_1,L_2,L_3, and L_4 denote the left, bottom, right, and top sides of S respectively. L_1 is the line $u = 0$ and $0 \leq v \leq 1$. So on L_1, $x = v$ and $y = 0$ with

$$L_1 : \quad y = 0, \quad \text{with} 0 \leq x \leq 1$$

Express the image of L_2, L_3, and L_4 similarly.

Solution: Along $L2$ $v = 0$ and $0 \leq u \leq 1$, which gives

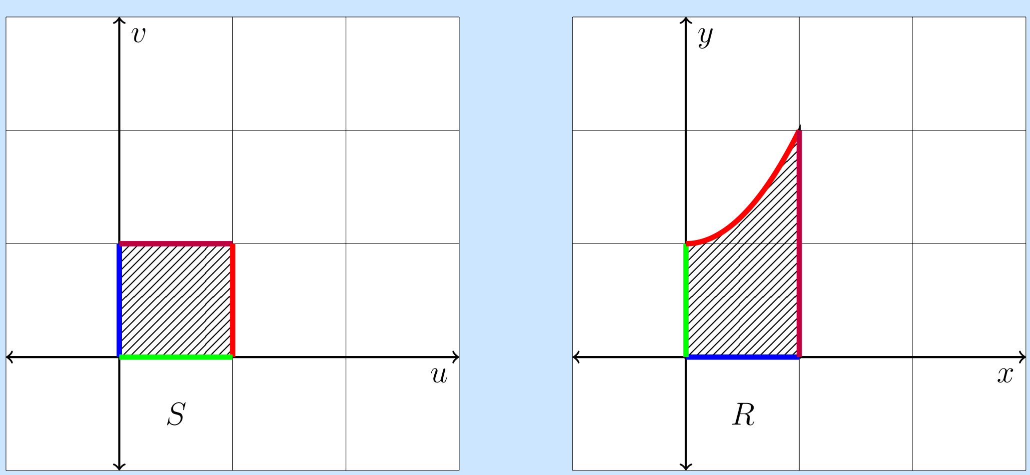
$$x = 0, \quad 0 \leq y \leq 1$$

Similiarly

$$\begin{aligned} L3 : &\quad y = 1 + x^2 \quad 0 \leq x \leq 1 \\ L4 : &\quad x = 1 \qquad\quad\ 0 \leq y \leq 2 \end{aligned}$$

(b) Sketch the image of S under the transformation given.

Solution:

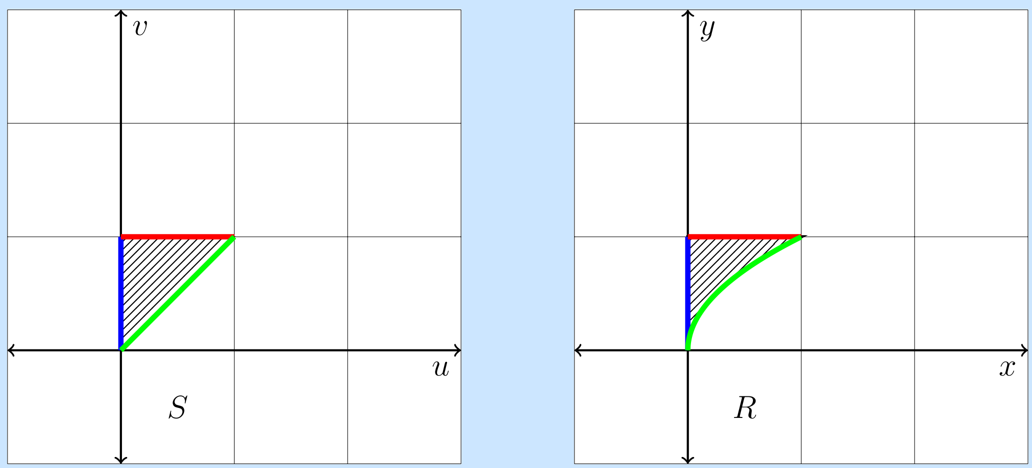


2. Repeat the above instructions for S the triangular region of the uv-plane with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$ with the transformation

$$x = u^2, y = v.$$

Solution: The lines bounding S are

$$\begin{aligned}
L1: &\quad u = 0 &\implies&\quad x = 0, \quad 0 \leq y \leq 1\\
L2: &\quad v = 1 &\implies&\quad y = 1, \quad 0 \leq x \leq 1\\
L3: &\quad v = u &\implies&\quad x = y^2, \quad 0 \leq y \leq 1
\end{aligned}$$



3. Let R be the parallelogram with vertices $(0,0),(4,3),(2,4),(-2,1)$. Let S be the square $[0,1]\times[0,1]$. Find a transformation that maps S onto R.

   Suggestion: Experiment and try stuff. What points in S are sent to the corners of R?

   Solution:
   $$x = 4u - 2v, \qquad y = 3u + v$$

4. Evaluate the integral
   $$\iint_R e^{(x+y)/(x-y)}\, dA$$
   where R is the trapezoidal region with vertices $(1,0),(2,0),(0,-2),(0,-1)$.

   (a) Since this is not easy as written, we want to do a change of variables. Based on the given function, we will try
   $$u = x+y, \quad v = x-y.$$
   Then we want to use the transformation T given by $x = \frac{1}{2}(u+v)$ and $y =$?

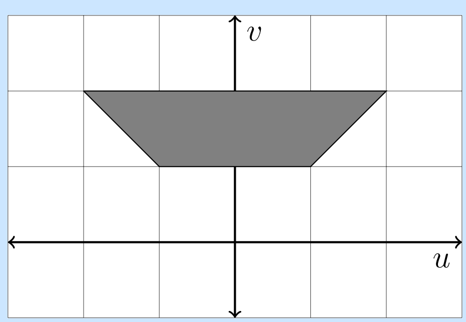
   Solution: Solve for y by using the above and $u - v = (x+y) - (x-y) = 2y$ to get
   $$y = \frac{1}{2}(u-v).$$

   (b) Setting $u = x+y$ and $v = x-y$, what is the image of the trapezoidal region given?

   Solution:

(x,y)	(u,v)
$(1,0)$	$(1,1)$
$(2,0)$	$(2,2)$
$(0,-2)$	$(-2,2)$
$(0,-1)$	$(-1,1)$

   With



   (c) Evaluate the integral.

Solution:

$$\begin{aligned}
\iint_R e^{(x+y)/(x-y)}\ dA &= \iint_S e^{u/v}\frac{\partial(x,y)}{\partial(u,v)}\ dA \\
&= \iint_S e^{u/v}\left(\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right)\ dA \\
&= \frac{1}{2}\int_1^2\int_{-v}^{v} e^{u/v}\ du\ dv \\
&= \frac{1}{2}\int_1^2 \left(v(e^{v/v}-e^{-v/v})\right)\ dv \\
&= \frac{e-e^{-1}}{2}\int_1^2 v\ dv \\
&= \frac{3}{4}\left(e-e^{-1}\right).
\end{aligned}$$

5. Evaluate $\iint_R xy\, dA$ where R is the region in the first quadrant bounded by

$$y = x, \quad y = 3x, \quad xy = 1, \quad xy = 3.$$

using the transformation $x = u/v, y = v$.

(a) Complete the following, determining the image of each line or curve:

- $$y = x \quad \mapsto \quad v^2 = u \ \text{ or } \ v = \pm\sqrt{u}$$
- $$y = 3x \quad \mapsto$$

Solution:

$$v = 3u/v \quad \Rightarrow \quad \frac{1}{3}v^2 = u \ \text{ or } \ v = \pm\sqrt{3u}$$

- $$xy = 1 \quad \mapsto \quad u = 1$$
- $$xy = 3 \quad \mapsto$$

Solution:

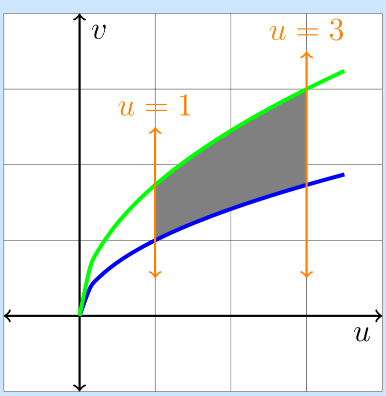
$$\frac{u}{v} \cdot v = 3 \quad \Rightarrow \quad u = 3$$

(b) Rewrite the original double integral using the given transformation

$$\int_a^b \int_c^d f(u, v)\, dv\, du$$

What are the values of a, b, c, d and $f(u, v)$?

Solution:
After the transformation we are integrating over the region bounded by the lines above, as seen below:



Since

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = \left(\frac{1}{v}\right)(1) - (0) = \frac{1}{v},$$

the integral is

$$\begin{aligned}\iint_R xy\ dA &= \iint_S \left(\frac{u}{v}\right)(v)\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\ dA\\ &= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\ dv\ du\end{aligned}$$

(c) Evaluate the integral.

Solution: From above,

$$\begin{aligned}\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\ dv\ du &= \int_1^3 u\left(\ln(v)\big|_{\sqrt{u}}^{\sqrt{3u}}\right)\ du\\ &= \int_1^3 u\left(\ln(\sqrt{3u}) - \ln(\sqrt{u})\right)\ du\\ &= \int_1^3 u\left(\frac{1}{2}\ln(3) + \frac{1}{2}\ln(u) - \frac{1}{2}\ln(u)\right)\ du\\ &= \frac{\ln(3)}{2}\left(\int_1^3 u\ du\right)\\ &= \frac{\ln(3)}{2}\left(\frac{9}{2} - \frac{1}{2}\right)\\ &= 2\ln(3).\end{aligned}$$