On definitions of the rank for interval matrices: A letter to the editor

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Abstract

We discuss various definitions of the rank of interval matrices adopted in the literature and show drawbacks of some popular variants.

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In the paper [1] recently published by O. Mullier, E. Goubault, M. Kieffer, and S. Putot, one of the key concepts is that of the rank of interval matrices. The authors give the following definitions:

Definition 4.2 (regular interval square matrix) Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval matrix. $\boldsymbol{A}$ is regular if and only if, for all matrices $A \in \boldsymbol{A}$, A is not singular.

Definition 4.3 (rank of an interval matrix) Let $\boldsymbol{A} \in \mathbb{IR}^{n\times m}$ be an interval matrix. $\boldsymbol{A}$ is of constant rank r if and only if the largest regular interval square sub-matrix $\boldsymbol{A}_0$ of $\boldsymbol{A}$, is of dimension r.

Definition 4.2 is quite traditional, and we are not going to call it into question.

The object that should be paid close attention is Definition 4.3 that the authors attribute to the earlier paper [2]. Although the definition seems quite natural, it runs contrary to the other presently existing definitions as well as to the general principle according to which we extend usual point concepts to interval objects. We believe that O. Mullier, E. Goubault, M. Kieffer, and S. Putot should alter their terminology at this point.

Let us remind yet another classical definitions. A point $n \times m$-matrix is called *full-rank matrix*, if its rank is equal to the minimal of the numbers m and n (it cannot be greater). An interval matrix is called *full-rank matrix*, if it contains only full-rank point matrices (see e.g. [3, 4]). Otherwise, we shall speak that the matrix, either point or interval, has *incomplete rank*.

Any interval full-rank $m \times n$-matrix has thus the rank $\min\{m, n\}$ according to the above definition from [3, 4]. But Definition 4.3 may be violated for such matrices, so that their rank, according to Definition 4.3, is evidently less than $\min\{m, n\}$. As a specific example, we consider the matrix proposed by Irene Sharaya and previously used in the paper [4]:

$$\boldsymbol{A} = \begin{pmatrix} 1 & [0,1] \\ -1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}. \tag{1}$$

It demonstrates that, for interval matrices, traditional ways of thinking based on the classical linear algebra and its intuition may fail. The matrix (1) has full rank 2, which one can make sure of after using e. g. the criteria described in [4]. At the same time, the matrix (1) does not contain non-singular interval 2×2-submatrices. This can be revealed by exhaustive search of all 2×2-submatrices: there are only three of them, namely

$$\begin{pmatrix} 1 & [0,1] \\ -1 & [0,1] \end{pmatrix}, \qquad \begin{pmatrix} 1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & [0,1] \\ [-1,1] & 1 \end{pmatrix}.$$

These matrices contain singular point matrices

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

respectively. Let us recall that the full-rank matrix in a usual noninterval case has, by definition, a square nonsingular matrix whose order is equal to the matrix rank.

The authors themselves seem to realize possible confusion with Definition 4.3, they write in [1]: "Definition 4.3 means that for all $A \in \boldsymbol{A} \in \mathbb{IR}^{n\times m}$, the rank of A is larger than or equal to r". This is really so, as the above example shows. However, just formulating a warning like the above is not sufficient. To cope with the problem, we need to revise the terminology more deeply.

My suggestion is to give up Definition 4.3, since it is misleading and conflicting with the previous stable definitions. Instead, it makes sense to speak e.g. of *strong rank* of interval matrices defining it through the interval submatrices similar to what was done in the former Definition 4.3.

References

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