The Banach-Tarski Paradox and Amenability of Groups

Aahana Jain

TIFR Visiting Students Research Program, 2025

The Banach-Tarski paradox is a mathematical theorem that says that given a solid ball, it is possible to decompose it into disjoint pieces and reassemble those pieces (via rotations and translations) to form two exact copies of the original ball.

In this report, we will sketch the outline of the proof. We are interested in the action of free groups and first introduce the following definitions:

Definition 1. Let G be a group acting on a set X. For $g \in G$ and $A \subset X$, we define

$$gA = \{g \cdot a : a \in A\}.$$

Two sets $A, B \subset X$ are said to be **congruent** ($A \sim B$) if there exists $g \in G$ such that $gA = B$. They are said to be **equidecomposable** ($A \sim_2 B$) if there exists $k \geq 1$ and subsets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ such that A is the disjoint union of $A_1, \ldots, A_k$, B is the disjoint union of $B_1, \ldots, B_k$, and A_i is congruent to B_i for all i.

It is easy two see that congruence and equidecomposability are equivalence relations. We introduce the notion of paradoxical sets:

Definition 2. Let G be a group acting on a set X. We say a subset $E \subset X$ is *G-paradoxical* if it is equidecomposable with the union of two disjoint copies of itself.

Remark: Let G be a group and $H \leq G$ be its subgroup. If $A, B \subseteq X$ are equidecomposable with respect to the action of H, then $A \sim_2 B$ with respect to the action of G since we can take the same decomposition as was taken for the action of H. Thus, if X is H-paradoxical, it is also G-paradoxical.

Now, we can formula the Banach-Tarski paradox as follows:

Theorem 1. *The solid ball in $\mathbb{R}^3$ is G-paradoxical, where G denotes the group of all homeomorphisms of $\mathbb{R}^3$ of the form*

$$x \mapsto Ax + b, \quad A \in \mathrm{SO}(3) \text{ and } b \in \mathbb{R}^3$$

The free group on two letters F_2 plays an important role in proving this theorem; we use the following intermediate lemmas:

Lemma 1. *F_2 is F_2-paradoxical.*

Lemma 2. *If the group F_2 (the free group on two generators) acts freely on a set X, then the set X is paradoxical with respect to that action.*

Lemma 3. *The group $SO(3)$ contains an isomorphic copy of F_2.*

Lemma 4. *For any countable subset $D \subset \mathbb{S}^2$, the sets $\mathbb{S}^2 \setminus D$ and $\mathbb{S}^2$ are equidecomposable under the action of* $\mathrm{SO}(3)$.

Lemma 5. *Let G be as in theorem 1. The closed unit ball $B := \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ and the punctured closed unit ball $B \setminus \{0\}$ are equidecomposable with respect to the action of G.*

We now shift our focus to amenable groups, which are interesting since they are groups such that paradoxical decompositions *do not* exist. The key point is this: suppose there exists a finitely additive left-invariant measure on $\mathcal{P}(G)$. We can then use this to find a finitely additive G-invariant measure on $\mathcal{P}(X)$.

Definition 3. For a set S, let $\mathbb{R}^S$ denote the set of all functions from S to $\mathbb{R}$ equipped with the product topology. An **affine set** is a compact convex subset of $\mathbb{R}^S$ for some set S.

Definition 4. Let G be a discrete group. An action of G on an affine set Δ is called **affine** if the maps $x \mapsto g \cdot x$ are continuous for all $g \in G$ and

$$g \cdot (\alpha x + \beta y) = \alpha(g \cdot x) + \beta(g \cdot y)$$

for all $x, y \in \Delta$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Definition 5. A discrete group G is called **amenable** if every affine action of G has a fixed point.

Definition 6. For a set S, let $\mathbb{R}^S$ denote the set of all functions from S to $\mathbb{R}$, equipped with the product topology. Let X be a set and let $\mathcal{P}(X)$ be the collection of all subsets of X. A **finitely additive measure** on X is a map

$$\mu : \mathcal{P}(X) \to [0, \infty)$$

such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$.

We can now use the following result to show that the Banach-Tarski paradox fails in $\mathbb{R}^2$:

Lemma 6. *Let $\mathcal{S}$ denote the collection of all bounded subsets of $\mathbb{R}^2$ and let Δ denote the collection of all $\mu \in \mathbb{R}^{\mathcal{S}}$ such that*

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \textit{whenever } A \cap B = \emptyset,$$

and

$$\mu((a, b] \times (c, d]) = (b - a)(d - c)$$

for all rectangles. Then Δ is a non-empty affine subset of $\mathbb{R}^{\mathcal{S}}$.

Thus, the existence of a non-empty affine set Δ of finitely additive measures, combined with the amenability of the isometry group of $\mathbb{R}^2$, allows the construction of an isometry-invariant finitely additive measure ν on all bounded subsets of $\mathbb{R}^2$. This measure assigns positive values to sets, such as disks, and is preserved under isometries. Atempting to create a Banach-Tarski paradoxical decomposition then leads to a contradiction, since the measure of the original set cannot be doubled. Therefore, the closed unit disk in $\mathbb{R}^2$ cannot be decomposed into finitely many pieces and reassembled via isometries into two copies of itself.