# That time I took Modular Forms as an overload course

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## 1 Introduction

Title says it all. Oh wait, no. These are my notes for my course on Modular Forms so that I do not go mad trying to keep up. This section will be updated as I go on.

(This document is incomplete. An asterisk* by the side of a statement means it requires verification or more explanation.)

## 2 Some info-dumping to motivate the course

Anyone interested in Modular forms would be aware that these are very useful functions with some nice properties that, and they naturally appear in number-theoretic problems. To understand what this means, one will need some knowledge of complex analysis and group theory.

The theory of modular forms is still under development, but for now, we are interested in looking at certain analytic (holomorphic) functions in the upper half-plane, and their behaviour under the action of the special linear group $SL_2(\mathbf{R})$ and its discrete subgroups. We are particularly interested in the modular group, $SL_2(\mathbf{Z})$.

### 2.1 What are these Modular Forms?

Consider the Riemann sphere, i.e. the one-point compactification of the complex plane, i.e. $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. Recall the following theorem from complex analysis:

**Theorem:** Every automorphism $f$ of $\widehat{\mathbb{C}}$ is of the form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

We wish to find all *analytic* automorphisms $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

This can be viewed as the bijective maps given by the group $SL_2(\mathbb{R})$, i.e. for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$, define:

$$t_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ by } t_A(z) = Az = \frac{az+b}{cz+d} \text{ and } t_A(\infty) = \begin{cases} \frac{a}{c} \text{ if } c \neq 0 \\ \infty \text{ if } c = \lambda \end{cases}$$

This action by fractional linear transformations is crucial to the study of modular forms. Now, we turn our focus to $\Gamma = SL_2(\mathbb{Z})$ and its action on the upper half plane $\mathbb{H}$:

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H} \cup \mathbb{Q}$ by

$$A\tau = \frac{a\tau + b}{c\tau + d} \quad \text{for } \tau \in \mathbb{H},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

And we have $Ai\infty$,

$$Ai\infty = \frac{a}{c} = r.$$

Thus, define

$$A^{-1}r = \infty.$$

With this, we define a **modular form of weight** $k$ to be an analytic function $f : \mathbb{H} \to \mathbb{C}$ such that its value at different orbit points under the action of $SL_2(\mathbb{Z}) \setminus \mathbb{H} \cup \mathbb{Q}$ is given by:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau).$$

We call this the *Petersson slash action* and denote it by

$$f|_k A = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$$

The space of all modular forms $M_k$ is a linear space over $\mathbf{C}$.

Note that $(f|_k A)|_k B = f|_k(AB)$ for $A, B \in SL_2(\mathbb{R})$, giving an action of $SL_2(\mathbb{R})$ on functions from $\mathbb{H} \to \mathbb{C}$.

Note that $f$ has a Fourier expansion:

$$f(\tau) = \sum_{n \geq 0} a_f(n) e^{2\pi i n \tau}$$

This is because

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) \Rightarrow f(z+1) = f(z).$$

Thus, f is analytic.

## 2.2 Factor of Automorphy

We now fix an invariant subspace for its actions. Poincaré proved that this action holds for $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ and $\mathbb{R} \cup \{\infty\}$ are the only fixed subspaces.

We introduce certain classes of analytic functions on $\mathbb{H}$ that are determined by a fixed automorphic factor. Let $J(A, z)$ be one such analytic function, knows as a **factor of automorphy**. Consider the quotient:

$$\frac{f(Az)}{f(z)}$$

Then we have the following properties:

$$J(AB, z) = \frac{f(AB(z))}{f(z)} = J(A, Bz)J(B, z).$$

$$j(A, \tau)^k = (c\tau + d)^k.$$

$$j(AB, \tau)^k = j(A, B\tau)^k j(B, \tau)^k.$$

This is a neat representation of modular forms (and automorphic forms in general) as a quotient of two functions whose zeroes are invariant under the action of the modular group*.

## 2.3 The Valence Formula

Let us recall another result from complex analysis:

$$\frac{1}{2\pi i} \oint_{\partial(\Gamma \setminus \mathbb{H})} \frac{f'(\tau)}{f(\tau)} d\tau = \text{number of zeros} - \text{number of poles in } \Gamma \setminus \mathbb{H}.$$

We know $\partial(\Gamma \setminus \mathbb{H})$ explicitly. We cut this by a horizontal line at maximum height, and this line becomes the boundary of a neighborhood of $i\infty$.

The Riemann mapping theorem gives us the integral over this line:

$$\text{from } \frac{1}{2} im \text{ into } -\frac{1}{2} im \Rightarrow \text{Order of zero/pole at } \infty \text{ with no sign.}$$

Similarly, we get the integral values at elliptic points $\rho, i, -\bar{\rho}$.

$$N(f) = -N_{i\infty}(f) + \frac{1}{3}N_\rho(f) - \frac{1}{2}N_i(f) + \frac{k}{12}.$$

$$\Rightarrow \frac{k}{12} = N_\infty + \frac{N_p}{3} + \frac{N_i}{2} + N.$$

## 2.4 Eisenstein series and bases for $M_k$

We will later study Eisenstein series explicitly, which are concrete examples of modular forms. In this section, $E_k$ denotes the Eisenstein series of weight $k$:

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n\geq 1} \sigma_{k-1}(n) e^{2\pi i n z},$$

where

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha, \quad \zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \quad (Res > 1)$$

The division of

$$\Delta = \frac{1}{12^3}(E_4^3 - E_6^2)$$

produces an injective map that is surjective from $S_k$ to $M_{k-12}$ by the valence formula:

$$\Delta \neq 0 \text{ if } \tau \in \mathbb{H}.$$

$$\dim S_k = \dim M_{k-12} = 1 + \dim S_{k-12}.$$

where

$$\dim S_k = \begin{cases} \frac{k}{12}, & k \not\equiv 2 \pmod{12} \\ \frac{k}{12} - 1, & k \equiv 2 \pmod{12}. \end{cases}$$

$M_k$ has 3 types of bases:

1. $E_4, E_6$ basis $\{a_4 + b_6 \mid k, a, b \geq 0 \text{ integers}\}$.
2. $\sum_i r E_{k-12i}$ with $0 \leq i \leq l, \quad l = \dim S_k$.
3. $e^x = q^x + O(q^x + 1), \quad 0 \leq x \leq k$.

And that concludes the info-dumping section (for now). This is more or less what will be covered in these notes (along with some Hecke theory).

# 3 Fundamental domain for $SL_2(\mathbb{Z})$ action on $\mathbb{H}$

Let $D_\Gamma$ be a fundamental domain, $\Gamma = SL_2(\mathbb{Z})$. The fundamental domain is such that no two actions of $\Gamma$ by fractional linear transformations are equivalent on this domain. That is,

1. If $z \in \mathbb{H}$, $\exists \tau \in D_\Gamma$ such that $z = \frac{a\tau+b}{c\tau+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
2. $D \subseteq \mathbb{C}$ open and if $\tau_1, \tau_2 \in D_\Gamma$ with $\tau_2 = \frac{\alpha\tau_1+\beta}{\gamma\tau_1+\delta}$ for some $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $\tau_2 = \tau_1$.

**Theorem 1:** $D_\Gamma = \{x + iy \mid y > 0, -\frac{1}{2} < x < \frac{1}{2}, x^2 + y^2 > 1\}$

**Proof:** For condition (1), we need to look at a single point in each orbit. If $z \in \mathbb{H}$,

$$z = \frac{a\tau + b}{c\tau + d}, \quad \Im(z) = \Im\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\Im(\tau)}{|c\tau + d|^2}$$

Let $\inf |c\tau + d| \to$ lattice generated by 1 and $\tau$. Let $z$ be one such whose $\Im(z)$ attains maximum.

$$\gcd(c, d) = 1, \quad (c, d) \in \mathbb{Z}^2$$

Let $z \in \mathbb{H}$. Its orbit is given by:

$$O_z = \left\{\frac{az + b}{cz + d} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}\right\}.$$

We seek an element $\tau \in O_z \cap D$ maximising $\mathrm{Im}(\tau)$, which occurs when $|cz + d|^2$ is minimized.

**Claim:** $\inf_{(c,d)=1} |cz + d|^2 > 0$

Let $z = x + iy \in \mathbb{H} \Rightarrow y > 0$.

- If $c = 0$, $d = \pm 1 \Rightarrow \inf |cz + d|^2 = 1$.
- If $c \neq 0$, then
$$|c(x + iy) + d|^2 = (cx + d)^2 + c^2y^2 > c^2y^2 \geq y^2 > 0.$$

Fix one such maximal imaginary part element $\tau$. Since $\Im(\tau) = \Im(\tau + 1)$, we further let our orbit element $\tau$ satisfy

$$-\frac{1}{2} < \Re(\tau) < \frac{1}{2}, \quad \Im(\tau) > \Im(z).$$

Thus, we define the set:

$$\left\{\tau \in \mathbb{H} \mid -\frac{1}{2} < \Re(\tau) < \frac{1}{2}, \quad |c\tau + d|^2 > 1, \quad c \neq 0, \quad (c, d) = 1\right\} = D.$$

$\forall z = \dfrac{a\tau + b}{c\tau + d}$, i.e. $\Im(\tau) > \dfrac{\Im(\tau)}{|c\tau + d|^2} \Rightarrow |c\tau + d|^2 > 1$

Let $\tau \in D$. Then,

$$|c\tau + d|^2 > 1 \quad \forall d, c \neq 0, \quad \gcd(c, d) = 1, \quad -\frac{1}{2} < \Re(\tau) < \frac{1}{2}.$$

Put $c = 1$, $d = 0$, then $|\tau|^2 > 1$ and $-\frac{1}{2} < \Re(\tau) < \frac{1}{2} \Rightarrow \tau \in D_\Gamma$.

Suppose $\tau \in D_\Gamma$, then we have $|\tau|^2 > 1$ and $-\frac{1}{2} < \Re(\tau) < \frac{1}{2}$.

For any $c \neq 0$, $d \in \mathbb{Z}$, $(c, d) = 1$,

$$|cz + d|^2 = c^2(x^2 + y^2) + 2cdx + d^2 > c^2 \cdot 1 - 2|c||d| + d^2 = (|c| - |d|)^2 + |c||d| \geq 1.$$

Thus,

$$D_\Gamma \subseteq D.$$