Problem 1

Let G be a group acting on a set X. For $g \in G$ and $A \subset X$, we define

$$gA = \{g \cdot a : a \in A\}.$$

Two sets $A, B \subset X$ are said to be **congruent** ($A \sim B$) if there exists $g \in G$ such that $gA = B$. They are said to be **equidecomposable** ($A \sim_2 B$) if there exists $k \geq 1$ and subsets $A_1, \dots, A_k$ and $B_1, \dots, B_k$ such that:

- A is the disjoint union of $A_1, \dots, A_k$,
- B is the disjoint union of $B_1, \dots, B_k$, and
- A_i is congruent to B_i for all i.

1.1

Show that congruence is an equivalence relation.

Solution: We have to show the following properties:

1. $\sim$ *is reflexive*: Let e be the identity element of G. Then it follows from the fact that $eA = A$.
2. $\sim$ *is symmetric*: If given $A \sim B$, i.e. $gA = B$ for some $g \in G$, then $A = g^{-1}B \;\Rightarrow\; B \sim A$.
3. $\sim$ *is transitive*: Given $A \sim B$ and $B \sim C$, i.e. $gA = B$ and $hB = C$ for some $g, h \in G$. Then $hB = h(gB) = (hg)A = C$, where $hg \in G$. Thus, $A \sim C$.

1.2

Show that equidecomposability is an equivalence relation.

Solution: We have to show the following properties:

1. $\sim_2$ *is reflexive*: Suppose $A = \sqcup_{i=1}^{k} A_i$. By reflexivity of $\sim$, $A_i \sim A_i \;\forall\; 1 \leq i \leq k \;\Rightarrow A \sim_2 A$.
2. $\sim_2$ *is symmetric*: Suppose $A \sim_2 B$, i.e. $A = \sqcup_{i=1}^{k} A_i$ and $B = \sqcup_{i=1}^{k} B_i$. Then $A_i \sim B_i \quad \forall i \;\Rightarrow$ $B_i \sim A_i \;\forall i \quad \Rightarrow B \sim_2 A$.
3. $\sim_2$ *is transitive*: Suppose $A \sim_2 B$ and $B \sim_2 C$. Then $A = \sqcup_{i=1}^{k} A_i$, $B = \sqcup_{i=1}^{k} B_i$, and $C = \sqcup_{i=1}^{k} C_i$. By transitivity of $\sim$, $A_i \sim C_i \;\; \forall i \quad \Rightarrow A \sim_2 C$.

1.3

Suppose $x \in X$ and there exists $g \in G$ such that $g^k \cdot x \neq x$ for all $k \geq 1$. Show that the set $X \setminus \{x\}$ and X are equidecomposable.

Solution: Let $B = \{g^k x \mid k \geq 0\}$ and $B' = X \setminus B$. Clearly, $X = B \sqcup B'$.

Note that $gB = \{g^k x \mid k \geq 1\} = B \setminus \{x\} \Rightarrow gB = B \setminus \{x\} \sim B$, and $B' \sim B'$. Since $X \setminus \{x\} = gB \sqcup B'$, $X \setminus \{x\}$ and X are equidecomposable.

1.4

Let F_2 denote the free group generated by the set $\{a, b\}$. Let A denote the set of all elements that start with a, and let B denote the set of all elements that start with a^{-1}. Show that $A \cup B$ is equidecomposable with F_2.

Solution: Every element of F_2 can be written uniquely as a reduced word in the generators a, a^{-1}, b, b^{-1} such that no letter is immediately followed by its inverse. Clearly $A \cap B = \emptyset$.

We partition F_2 as:

$$F_2 = \{e\} \sqcup A \sqcup B \sqcup C,$$

where:

- $\{e\}$ is the identity element,
- A is the set of elements starting with a,
- B is the set of elements starting with a^{-1},
- C is the set of elements starting with b or b^{-1}.

Observe that for $x \in A$,

$$a^{-1} \cdot x = a^{-1} \cdot a \cdot y \text{ for some } y \in F_2 \text{ such that } y \text{ does not start with } a, \text{ i.e., } y \in F_2 \setminus B$$

This implies that $a^{-1} \cdot A = F_2 \setminus B$, i.e., $A \sim F_2 \setminus B$. By reflexivity of $\sim$, we have that $B \sim B$. Since $F_2 = (F_2 \setminus B) \sqcup B$, F_2 is equidecomposable with $A \sqcup B$.

1.5

Show that the group F_2 is equidecomposable with the disjoint union of two copies of F_2.

Solution: Partition C as $C = D \sqcup E$, where D is the set of elements starting with b and E is the set of elements starting with b^{-1}. Then, from 1.d., we have that $F_2 \ \sim_2 \ A \sqcup B \ \sim_2 D \sqcup E$. Since $F_2 = A \sqcup aB = D \sqcup bD$, we can write the disjoint union as

$$F_2 \sqcup F_2 = (A \sqcup aB) \sqcup (D \sqcup bE)$$

Since $A \sqcup B \ \sim_2 \ A \sqcup aB \ \sim_2 \ D \sqcup bE \ \sim_2 \ F_2$, it follows that $F_2 \sim_2 F_2 \sqcup F_2$.

Problem Set 2

2.1

Let $\mathbb{S}^1$ be the circle group acting on itself by translations. Show that for any countable set $D \subset \mathbb{S}^1$, the sets $\mathbb{S}^1 \setminus D$ and $\mathbb{S}^1$ are equidecomposable.

Solution: Let $D = \{d_1, d_2, d_3, ...\}$. Each $d_k = e^{i\theta_k}$ for some $\theta_k \in [0, 2\pi)$. Since the action by translation (or left-multiplication) by any element of $\mathbb{S}^1$ induces a bijection:

$$f_\theta : \mathbb{S}^1 \to \mathbb{S}^1$$

Consider the following set to collect all possible rotations taking d_k to d_j for some $d_k, d_j \in D$:

$$E = \left\{\theta \in [0, 2\pi) : \exists d \in D \; \exists k \in \mathbb{N}^+ \mid \quad f_\theta^k(d) \in D\right\}$$

E is countable since it is a countable union of countable sets. Thus, $\exists \phi \in [0, 2\pi)$ such that $f_\phi^k(d) \notin D \; \forall d \in D \; \forall k \in \mathbb{N}^+$ (equivalently, $\phi \notin E$).

Claim: $\{f_\phi^k[D] : k \in \mathbb{N}\}$ consists of pairwise disjoint sets. Assume that there exists $x \in f_\phi^k[D] \cap f_\phi^\ell[D]$ for some natural numbers $k \neq \ell$. Without loss of generality, assume $k < \ell$. Then we would have

$$f_\phi^{-k}(x) \in D \cap f_\phi^{\ell-k}[D],$$

but this is not possible by choice of ϕ. Hence, the sets $D, f_\phi[D], f_\phi^2[D], \ldots$ are pairwise disjoint.

Let

$$A = \bigsqcup_{k=0}^{\infty} f_\phi^k(D), \quad \text{and} \quad B = \mathbb{S}^1 \setminus A.$$

Then we have that

$$A \sqcup B = \mathbb{S}^1, \quad \text{and} \quad f_\phi(A) \sqcup B = \mathbb{S}^1 \setminus D.$$

Therefore, $\mathbb{S}^1 \setminus D$ and $\mathbb{S}^1$ are equidecomposable.

2.2

Show that the $\mathbb{S}^1$-action on $\mathbb{S}^1 \times (0, 1)$ has the same property, where the action on the second component is trivial.

Solution: Let $D = \{(d_i, e_i) \mid i \in \mathbb{N}\}$ and $D_1 = \{d_i \mid i \in \mathbb{N}\}$.

We can choose $A \subseteq \mathbb{S}^1$ as in problem 1, so we have

$$A \sqcup B = \mathbb{S}^1, \quad \text{and} \quad f_\phi(A) \sqcup B = \mathbb{S}^1 \setminus D_1$$

Let $A' = \bigsqcup_{k=0}^{\infty} f_\phi^k(D)$ and B' be its complement. Then $B' = \{(d, e) \mid d \notin D_2 \text{ and } e \notin D_2\}$.

Note that $\bigsqcup_{k=1}^{\infty} f_\phi^k(D) = \{(d, e) \mid d \notin D_2 \text{ and } e \in D_2\}$. Thus, we have

$$f_\phi(A') \sqcup B' = \mathbb{S}^1 \times (0,1) \setminus D$$

and the required result follows.

2.3

Show that the SO(3) action on the 2-sphere $\mathbb{S}^2$ has the same property; that is, for any countable subset $D \subset \mathbb{S}^2$, the sets $\mathbb{S}^2 \setminus D$ and $\mathbb{S}^2$ are equidecomposable under the SO(3) action.

Solution: SO(3) acts on $\mathbb{R}^2$ by rotating a point about the origin. This is equivalent to rotating a point about a line through the origin.

Now, choose a line l that does not intersect D. As in problem 1, the set W of rotations r corresponding to a rotation about l by some angle θ such that for $d \in D$, $r_\theta^n(d) \in D$ is countable. Thus, we can find an angle ψ such that $r_\psi^n(d) \notin D$ $\forall d$ in D, i.e. $r_\psi^n(d) \cap D = \phi \forall\ n \geq 1$.

Let

$$A = \bigsqcup_{k=0}^{\infty} r_\psi^k(D), \quad \text{and} \quad B = \mathbb{S}^1 \setminus A.$$

Then we have that

$$A \sqcup B = \mathbb{S}^2, \quad \text{and} \quad r_\psi(A) \sqcup B = \mathbb{S}^2 \setminus D.$$

Thus, the sets $\mathbb{S}^2 \setminus D$ and $\mathbb{S}^2$ are equidecomposable under the SO(3) action.

Remark: Problem 1 and Problem 3 in this section essentially follow from Problem 1.3.

2.4

Let G be a group of homeomorphisms of $\mathbb{R}^3$ that contains SO(3) and all translations. Show that the closed unit ball $B := \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ and the punctured closed unit ball $B \setminus \{0\}$ are equidecomposable with respect to the action of G.

Solution: We use problem 1.3. again. Note that every element of SO(3) fixes the origin while every translation t is such that $t^n \cdot 0 \neq\ 0\ \forall n \geq 1$. Then, from problem 1.3., $B \sim_2 B \setminus 0$.

Problem Set 3

Let G be a group acting on a set X. We call a subset $E \subset X$ *paradoxical* if it is equidecomposable with the union of two disjoint copies of itself.

Remark (ii): Let G be a group and $H \leq G$ be its subgroup. If $A, B \subseteq X$ are equidecomposable with respect to the action of H, then $A \sim_2 B$ with respect to the action of G since we can take the same decomposition as was taken for the action of H. Thus, if X is H-paradoxical, it is also G-paradoxical.

3.1

Show that if the group F_2 (the free group on two generators) acts freely on a set X, then the set X is paradoxical with respect to that action.

Solution: Let M be a set of representatives for the F_2-orbits of X. For $c \in F_2$, define

$$X_c := \{zm \mid z \in W(c),\ m \in M\}.$$

Then the sets X_a, $X_{a^{-1}}$, X_b, and $X_{b^{-1}}$ are disjoint such that

$$X = X_a \sqcup aX_{a^{-1}} = X_b \sqcup bX_{b^{-1}}$$

Thus $X \sim X_a \sqcup aX_{a^{-1}} \sim X_b \sqcup bX_{b^{-1}} \ \Rightarrow\ X \sim_2 X_a \sqcup aX_{a^{-1}} \sqcup X_a \sqcup aX_{a^{-1}} = X \sqcup X$

Thus, X is paradoxical with respect to the action of F_2.

3.2

Show that any non-trivial element $A \in \mathrm{SO}(3)$ has at most two fixed points in S^2.

Solution: The fixed point set of $A \in \mathrm{SO}(3)$ is $X_A = \{x \in S^2 \mid A \cdot x = x\}$. We already know that this action of A on a point x can be seen as the rotation of x about some line l passing through the origin, so the set of points fixed by A in $\mathbb{R}^3$ corresponds exactly to the line l, which intersects S^2 in exactly two points that are anti-podal.

Hence, any non-trivial element $A \in \mathrm{SO}(3)$ has exactly two fixed points in S^2.

3.3

Assume that $\mathrm{SO}(3)$ contains a copy of F_2. Show that there exists a countable set $D \subset S^2$ such that $S^2 \setminus D$ is paradoxical with respect to the action of $\mathrm{SO}(3)$ on S^2.

Solution: Let F be the isomorphic copy of F_2 contained in $\mathrm{SO}(3)$ and $D_1 = \bigcup_{A \in F} X_A$ be the set of points that are fixed by elements of F. Let $D = \bigcup_{A \in F} A \cdot D_1$. Then from problem 3.1, since F_2 embeds in $\mathrm{SO}(3)$, it acts freely on $S^2 \setminus D$, i.e. $S^2 \setminus D$ is paradoxical with respect to the action of F_2 and by extension, is paradoxical with respect to the action of $\mathrm{SO}(3)$.

Note: D is countable since F is countable and each element has exactly 2 fixed points.

3.4

Show that S^2 is a paradoxical set with respect to the SO(3)-action on S^2.

Solution: This follows from the transitive nature of equidecomposability:

$$S^2 \;\sim_2\; S^2 \setminus D \;\sim_2\; S^2 \setminus D \sqcup S^2 \setminus D \;\sim_2\; S^2 \sqcup S^2$$

Thus, S^2 is SO(3)-paradoxical.

3.5

Show that the closed unit ball minus the origin is a paradoxical subset of $\mathbb{R}^3$ with respect to the SO(3)-action on $\mathbb{R}^3$.

Solution: Since S^2 is $\mathrm{SO}_3(\mathbb{R})$-paradoxical, there exists a partition

$$\{A_1, \dots, A_n, B_1, \dots, B_m\}$$

of S^2 and rotations $g_1, \dots, g_n, h_1, \dots, h_m \in \mathrm{SO}_3(\mathbb{R})$ such that

$$S^2 = \bigsqcup_{i=1}^{n} g_i[A_i] = \bigsqcup_{j=1}^{m} h_j[B_j].$$

Let B denote the unit ball in $\mathbb{R}^3$ and let 0 denote the origin. Then, we have

$$B \setminus \{0\} = \bigcup_{0<a\leq 1} \{(ax, ay, az) : x^2 + y^2 + z^2 = 1\}$$

Let

$$C_i = \bigcup_{0<a\leq 1} \{(ax, ay, az) : (x, y, z) \in A_i\}$$

$$D_i = \bigcup_{0<a\leq 1} \{(ax, ay, az) : (x, y, z) \in B_j\}$$

Then we can rewrite $B \setminus \{0\}$ as

$$B \setminus \{0\} = \bigsqcup_{i=1}^{n} g_i \cdot C_i = \bigsqcup_{j=1}^{m} h_j \cdot D_i$$

Thus, $B \setminus \{0\}$ is $\mathrm{SO}_3(\mathbb{R})$-paradoxical.

3.6

Let G denote the group of all homeomorphisms of $\mathbb{R}^3$ of the form

$$x \mapsto Ax + b,$$

where $A \in \mathrm{SO}(3)$ and $b \in \mathbb{R}^3$. Show that the closed unit ball is a paradoxical set with respect to the G-action on $\mathbb{R}^3$.

Solution: From 3.4, it suffices to show that $B \setminus \{0\} \sim_2 B$. From 1.3, we only need an element $g \in G$ such that $g^k \cdot 0 \neq 0 \ \forall k \geq 1$ to show this. Any translation $b \in \mathbb{R}^3$ will satisfy the required property.

1 Problem Set 4

4.1

Let R be a field and let $M(3, R)$ be the set of all 3×3 matrices with entries in R. For $m = 1, 2, 3$, let $Q_m \subset M(3, R)$ denote the set of all A with the property that $A_{ij} = 0$ if and only if either $i = m$ or $j = m$.

If $k \geq 1$, $x_1, \ldots, x_k \in Q_1$ and $y_1, \ldots, y_k \in Q_3$, then show that both

$$x_1 y_1 \cdots x_k y_k \quad \text{and} \quad y_1 x_1 \cdots y_k x_k$$

are non-zero.

Solution: We will prove this by induction. Note that in Q_1, matrices have zeros in the first row and first column, with non-zero entries elsewhere:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

In Q_3, matrices have zeros in the third row and third column, with non-zero entries elsewhere:

$$\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Base Case ($k = 1$):

Let $x_1 \in Q_1$, $y_1 \in Q_3$. Compute $x_1 y_1$:

$$x_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}, \quad y_1 = \begin{bmatrix} p & q & 0 \\ r & s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then,

$$x_1 y_1 = \begin{bmatrix} 0 & 0 & 0 \\ ar & as & 0 \\ cr & cs & 0 \end{bmatrix}$$

Since a, c, r, s are non-zero elements of a field R, the product $x_1 y_1 \neq 0$.

Similarly,

$$y_1 x_1 = \begin{bmatrix} p & q & 0 \\ r & s & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} = \begin{bmatrix} 0 & qa & qb \\ 0 & sa & sb \\ 0 & 0 & 0 \end{bmatrix}$$

which is also non-zero.

Inductive Step:

Assume $x_1y_1\cdots x_ky_k \neq 0$ for some $k \geq 1$. Let $z_i = x_iy_i$. Let $P_1 \subset M(3,R)$ denote the set of all A with the property that $A_{ij} = 0$ if and only if either $i = 1$ or $j = 3$.

Note that $\forall i,\ z_i \in P_1$. By associativity of matrix multiplication, $x_1y_1\cdots x_ky_k = z_1\cdots z_k = M_k \in P_1$. Thus, if M_k is non-zero and since we know z_{k+1} is non-zero, $M_kz_{k+1} \neq 0$. Thus,

$$x_1y_1\cdots x_ky_k \neq 0 \ \ \forall k \geq 1$$

Similarly, $y_1x_1\cdots y_kx_k \neq 0 \ \ \forall k \geq 1$.

4.2.

Let $A \in M(3,R)$ be defined by

$$e_1^TA = (3,4,0), \quad e_2^TA = (-4,3,0), \quad e_3^TA = (0,0,5),$$

and let $B \in M(3,R)$ be defined by

$$B_{ij} = A_{i-1 \bmod 3,\, j-1 \bmod 3}.$$

Show that $A/5$ and $B/5$ are elements of SO(3).

Solution: We are given:

$$e_1^TA = (3,4,0), \quad e_2^TA = (-4,3,0), \quad e_3^TA = (0,0,5)$$

Hence,

$$A = \begin{bmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \frac{A}{5} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to see that $\frac{A}{5}$ is orthogonal with determinant 1.

Define B by

$$B_{ij} = A_{i-1 \bmod 3,\, j-1 \bmod 3}$$

Then,

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{bmatrix}, \quad \frac{B}{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$\frac{B}{5}$ is also orthogonal with determinant 1. Hence, both $\frac{A}{5}, \frac{B}{5} \in \mathrm{SO}(3)$.

4.3.

Let $R = \mathbb{Z}/5\mathbb{Z}$. Show that for any integer k, A^k is an element of Q_3 and B^k is an element of Q_1 when viewed as elements of $M(3,R)$.

Solution: After reduction modulo 5,

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, $A \in Q_3$ and $A_{ij} = 0$ for $i = 3$ or $j = 3$. To show if and only if,

$$A^2 = \begin{bmatrix} 13 & 24 & 0 \\ 6 & 13 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A \in Q_3$$

Thus, $A^k = A \in Q_3 \ \forall k \geq 1$.

Similarly,

$$B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 13 & 24 \\ 0 & 6 & 13 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \in Q_1$$

Thus, $B^k = B \in Q_1 \ \forall k \geq 1$.

4.4.

Show that the subgroup of SO(3) generated by $A/5$ and $B/5$ is isomorphic to the free group F_2.

Solution: For simplicity, rewrite $\dfrac{A}{5}$ as a and $\dfrac{B}{5}$ as b. Let $F_2 = \langle p, q \rangle$ and $\phi : \langle A, B \rangle \to F_2$ be the map defined by

$$\phi(p) = a \quad \text{and} \quad \phi(q) = b$$

Now, we want to show that ϕ is an isomorphism. It suffices to show that a and b have no non-trivial relations, i.e. show the following equivalent statements to be true:

Let e be the identity element of SO(3). For any $k \geq 1$ and natural numbers $n_1, m_1, ..., n_k, m_k$

1. $a^{n_1} b^{m_1} ... a^{n_k} b^{m_k} \neq e$
2. $b^{n_1} a^{m_1} ... a^{n_k} b^{m_k} \neq e$
3. $a^{n_1} b^{m_1} ... a^{n_k} \neq e$
4. $b^{n_1} a^{m_1} ... b^{n_k} \neq e$

$(1) \Rightarrow (2):\ a^{n_1} b^{m_1} ... a^{n_k} b^{m_k} \neq e \ \Rightarrow\ a^{n_1} b^{m_1} ... a^{n_k} \neq b^{-m_k} \ \Rightarrow\ b^{m_k} a^{n_1} b^{m_1} ... a^{n_k} \neq e$

Similarly, we can show $(2) \Rightarrow (1)$.

$(3) \Rightarrow (1):\ a^{n_1} b^{m_1} ... a^{n_k} \neq e \ \Rightarrow\ a^{n_1 - n_k} b^{m_1} ... a^{n_{k-1}} b^{n_{k-1}} \neq e$

$(4) \Rightarrow (2):\ b^{n_1} a^{m_1} ... b^{n_k} \neq e \ \Rightarrow\ b^{n_1 - n_k} a^{m_1} ... b^{n_{k-1}} a^{n_{k-1}} \neq e$

Now, it remains to show (1) $\ \forall k \geq 1$ and any natural numbers $n_1, m_1, ... n_k, m_k$. We will prove by reduction.

Suppose there exists $k \geq 1$ and natural numbers $n_1, m_1, ...n_k, m_k$ such that $a^{n_1}b^{m_1}...a^{n_k}b^{m_k} = e$. This implies that in $M(3, \mathbb{Z}/5\mathbb{Z})$, $A^{n_1}B^{m_1}...A^{n_k}B^{m_k} = 0$. But this contradicts 4.1. since each $A^{n_i} \in Q_1$ and each $B^{m_i} \in Q_3$. Thus, (1) holds and $\langle a, b \rangle \cong F_2$.

Problem Set 5

For a set S, let $\mathbb{R}^S$ denote the set of all functions from S to $\mathbb{R}$ equipped with the product topology. An **affine set** is a compact convex subset of $\mathbb{R}^S$ for some set S.

Let G be a discrete group. An action of G on an affine set Δ is called **affine** if the maps $x \mapsto g \cdot x$ are continuous for all $g \in G$ and

$$g \cdot (\alpha x + \beta y) = \alpha(g \cdot x) + \beta(g \cdot y)$$

for all $x, y \in \Delta$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A discrete group G is called **amenable** if every affine action of G has a fixed point.

5.1

Show that finite groups are amenable.

Solution: Let G be a finite group acting affinely on a compact convex set Δ. For any $x \in \Delta$, define:

$$x_0 = \frac{1}{|G|} \sum_{g \in G} g \cdot x.$$

Since Δ is convex and compact, $x_0 \in \Delta$. Moreover, for any $h \in G$,

$$h \cdot x_0 = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot x) = \frac{1}{|G|} \sum_{g \in G} (hg) \cdot x = x_0.$$

Thus, x_0 is fixed by the action of G, so G is amenable.

5.2

Show that $\mathbb{Z}$ is amenable.

Solution: Let $\mathbb{Z}$ act affinely on a compact convex set Δ. Let the generator $1 \in \mathbb{Z}$ act via a continuous affine map $T \colon \Delta \to \Delta$. Then the action is given by $n \cdot x = T^n(x)$.

Define the Cesàro sums:

$$x_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x),$$

which lie in the compact set Δ. By compactness, the sequence (x_n) has a convergent subsequence whose limit $x_\infty \in \Delta$ satisfies $T(x_\infty) = x_\infty$, using continuity and affinity of T. Hence, $\mathbb{Z}$ is amenable.

5.3

Show that if G is an increasing union of amenable subgroups, then G is amenable.

Solution: Let $G = \bigcup_{n=1}^{\infty} G_n$, where $G_1 \subset G_2 \subset \cdots$, and each G_n is amenable.

Let G act affinely on a compact convex set Δ. Since each G_n is amenable, there exists a fixed point $x_n \in \Delta$ for the action of G_n.

Since the sequence (x_n) lies in a compact set Δ, it has an accumulation point $x \in \Delta$. For any $g \in G$, there exists N such that $g \in G_N$, and for all $n \geq N$, $g \cdot x_n = x_n \ \Rightarrow \ g \cdot x = x$. Thus x is a fixed point for G, so G is amenable.

5.4

If G has a normal subgroup N such that both N and G/N are amenable, then show that G is amenable.

Solution: Suppose $N \trianglelefteq G$ and both N and G/N are amenable.

Let G act affinely on a compact convex set Δ. Since N is amenable, the fixed point set $\Delta^N = \{x \in \Delta : n \cdot x = x \text{ for all } n \in N\}$ is non-empty, convex, closed, and compact.

The quotient G/N acts on Δ^N, and this action is affine and continuous. Since G/N is amenable, it has a fixed point in Δ^N, which is then fixed by all of G. Hence, G is amenable.

5.5

Show that abelian groups are amenable.

Solution: We already showed that $\mathbb{Z}$ is amenable. Any finitely generated abelian group is of the form $\mathbb{Z}^n \oplus T$, where T is a finite abelian group. Finite groups and $\mathbb{Z}^n$ are amenable.

Any abelian group is a union of its finitely generated subgroups, each of which is amenable. By problem 3, increasing unions of amenable subgroups are amenable. Hence, all abelian groups are amenable.

5.6

Show that solvable groups are amenable.

Solution: A group G is solvable if there exists a finite derived series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$$

with $G_{i+1} \trianglelefteq G_i$ and G_i/G_{i+1} abelian.

We proceed by induction on the length of the derived series. The base case $G_n = \{e\}$ is trivially amenable. Assume G_{i+1} is amenable. Then G_i/G_{i+1} is abelian and hence amenable. By problem 4, G_i is amenable.

Thus, G is amenable.

2 Problem Set 6

For a set S, let $\mathbb{R}^S$ denote the set of all functions from S to $\mathbb{R}$, equipped with the product topology. Let X be a set and let $\mathcal{P}(X)$ be the collection of all subsets of X. A **finitely additive measure** on X is a map

$$\mu : \mathcal{P}(X) \to [0, \infty)$$

such that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$.

6.1

Let X be any set and $c \in \mathbb{R}$. Show that the collection of finitely additive measures on X satisfying $\mu(X) = c$ is a non-empty affine subset of $\mathbb{R}^{\mathcal{P}(X)}$.

Solution: Define $S = \{\mu : P(X) \to [0, \infty) \mid \mu \text{ is finitely additive}, \mu(\emptyset) = 0\}$ and $T = \{\mu \in S \mid \mu(X) = c\}$. An affine subset of $\mathbb{R}^{P(X)}$ is of the form μ_0

If $c < 0$, then $T = \emptyset$, as $\mu(X) \geq 0$. Assume $c \geq 0$. Fix $x_0 \in X$ and define:

$$\mu(A) = \begin{cases} c & \text{if } x_0 \in A, \\ 0 & \text{if } x_0 \notin A. \end{cases}$$

This satisfies:

- $\mu(A) \geq 0$, since $c \geq 0$.
- $\mu(\emptyset) = 0$, as $x_0 \notin \emptyset$.
- For disjoint A, B:
  - If $x_0 \in A$, then $x_0 \notin B$, so $x_0 \in A \cup B$, and $\mu(A \cup B) = c = c + 0 = \mu(A) + \mu(B)$.
  - If $x_0 \in B$, similarly.
  - If $x_0 \notin A, B$, then $x_0 \notin A \cup B$, so $\mu(A \cup B) = 0 = 0 + 0$.
- $\mu(X) = c$, as $x_0 \in X$.

Thus, $\mu \in T$, so $T \neq \emptyset$ for $c \geq 0$.

Let $V = \{\nu \in \mathbb{R}^{P(X)} \mid \nu \text{ is finitely additive}, \nu(\emptyset) = 0, \nu(X) = 0\}$. Then V is a vector subspace, as it is closed under addition and scalar multiplication. Fix $\mu_0 \in T$. For $\mu \in T$, we have $\mu - \mu_0 \in V$, since $(\mu - \mu_0)(X) = c - c = 0$. For $\nu \in V$, if $\mu_0 + \nu \geq 0$, then $\mu_0 + \nu \in T$, as $(\mu_0 + \nu)(X) = c$. Thus, $T = \mu_0 + V$, and T is affine. If $c < 0$, $T = \emptyset$, which is trivially affine.

6.2

Show that the SO(2)-action on S^1 is not paradoxical.

Solution: The group $\mathrm{SO}(2) \cong S^1$ acts on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by rotation: $e^{i\theta} \cdot z = e^{i\theta} z$. An action is paradoxical if there exist disjoint $A, B \subseteq S^1$, and $g_1, \ldots, g_n, h_1, \ldots, h_m \in \mathrm{SO}(2)$, such that $S^1 = \bigcup_i g_i A$ and $S^1 = \bigcup_j h_j B$.

Suppose such a decomposition exists. Let μ be the Lebesgue measure on S^1 with $\mu(S^1) = 1$, invariant under rotations. Then:

$$\mu(S^1) \leq \sum_i \mu(g_i A) = n\mu(A), \quad \mu(S^1) \leq m\mu(B).$$

Since $A \cap B = \emptyset$, $\mu(A) + \mu(B) \leq 1$. Thus, $\mu(A) \leq \frac{1}{n}$, $\mu(B) \leq \frac{1}{m}$. Since $\mu(A) + \mu(B) \leq \frac{1}{n} + \frac{1}{m} \leq 1$, a paradoxical decomposition requires $n = m = 1$, implying $S^1 = A = B$, contradicting $A \cap B = \emptyset$.

6.3

Let $X = (a, b] \times (c, d] \subset \mathbb{R}^2$ be any rectangle. Show that the collection of finitely additive measures on X satisfying $\mu(R) = \text{Area}(R)$ for all rectangles $R \subset X$ is a non-empty affine subset of $\mathbb{R}^{\mathcal{P}(X)}$.

Solution: A rectangle $R = (a_1, b_1] \times (c_1, d_1] \subset X$ has $\text{Area}(R) = (b_1 - a_1)(d_1 - c_1)$. Let $S = \{\mu : P(X) \to [0, \infty) \mid \mu \text{ is finitely additive}, \mu(\emptyset) = 0\}$, and $T = \{\mu \in S \mid \mu(R) = \text{Area}(R) \text{ for all } R\}$.

Lebesgue measure μ_0 on X, extended to $P(X)$ via the axiom of choice, satisfies $\mu_0(R) = (b_1 - a_1)(d_1 - c_1)$. Thus, $\mu_0 \in T$, so $T \neq \emptyset$.

Let $V = \{\nu \in \mathbb{R}^{P(X)} \mid \nu \text{ is finitely additive}, \nu(\emptyset) = 0, \nu(R) = 0 \text{ for all } R\}$. Then V is a subspace. Fix $\mu_0 \in T$. For $\mu \in T$, $\mu - \mu_0 \in V$, as $(\mu - \mu_0)(R) = \text{Area}(R) - \text{Area}(R) = 0$. For $\nu \in V$, $\mu_0 + \nu \in T$ if $\mu_0 + \nu \geq 0$, satisfying $\mu_0 + \nu(R) = \text{Area}(R)$. Thus, $T = \mu_0 + V$ is affine.

6.4

Let $\mathcal{S}$ denote the collection of all bounded subsets of $\mathbb{R}^2$ and let Δ denote the collection of all $\mu \in \mathbb{R}^{\mathcal{S}}$ such that

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{whenever } A \cap B = \emptyset,$$

and

$$\mu((a, b] \times (c, d]) = (b - a)(d - c)$$

for all rectangles. Show that Δ is a non-empty affine subset of $\mathbb{R}^{\mathcal{S}}$.

Solution: Extending the Lebesgue measure μ_0 on bounded measurable sets to S via the axiom of choice satisfies $\mu_0((a, b] \times (c, d]) = (b - a)(d - c)$. Thus, $\mu_0 \in \Delta$, so $\Delta \neq \emptyset$.

Let $V = \{\nu \in \mathbb{R}^S \mid \nu \text{ is finitely additive}, \nu(\emptyset) = 0, \nu((a, b] \times (c, d]) = 0 \text{ for all rectangles}\}$. Then V is a subspace. Fix $\mu_0 \in \Delta$. For $\mu \in \Delta$, $\mu - \mu_0 \in V$, as $(\mu - \mu_0)((a, b] \times (c, d]) = 0$. For $\nu \in V$, $\mu_0 + \nu \in \Delta$ if $\mu_0 + \nu \geq 0$, satisfying the rectangle condition. Thus, $\Delta = \mu_0 + V$ is affine.

6.5

Show that the Banach–Tarski paradox fails in $\mathbb{R}$.

Solution: One can define a nonnegative finitely additive set function $m(P)$, for all subsets P of the circle, that is invariant under rotation.

This implies in particular that it is impossible to decompose S^1 paradoxically into a disjoint union of finitely many pieces $A_1, \ldots, A_n$ in such a way that S^1 can be written as a disjoint union of rotated versions of $A_1, \ldots, A_k$ as well as $A_{k+1}, \ldots, A_n$, i.e., $r_1 A_1 \cup \cdots \cup r_k A_k = S^1$ and $r_{k+1} A_{k+1} \cup \cdots \cup r_n A_n = S^1$, where $r_1, \ldots, r_n$ are some rotations.

Then, we have

$$1 = m(S^1) = m(A_1 \cup \cdots \cup A_n) = m(A_1) + \cdots + m(A_n)$$

as well as

$$\begin{aligned} 1 &= m(S^1) = m(r_1 A_1 \cup \cdots \cup r_k A_k) = m(A_1) + \cdots + m(A_k) \\ &= m(S^1) = m(r_{k+1} A_{k+1} \cup \cdots \cup r_n A_n) = m(A_{k+1}) + \cdots + m(A_n) \end{aligned}$$

by finite additivity and invariance of m under rotations. This implies

$$1 = m(A_1) + \cdots + m(A_n) = [m(A_1) + \cdots + m(A_k)] + [m(A_{k+1}) + \cdots + m(A_n)] = 2$$

which is a contradiction.