# Some model-theoretic methods in algebra

Aahana Jain

IISER Bhopal

## Why would one be interested in model-theoretic algebra?

In its early years, model theory was traditionaly related to universal algebra. Later on, with M. Morley and S. Shehlah's vast work, it became more of a "classification" or "stability" theory.

Model-theoretic algebra has been used to prove far more advanced results well outside of the purely model-theoretic realm. Model theory also aims to identify unifying ideas, and could hence develop ideas that can be used to prove theorems in various mathematical theories.

## Some model theoretic notions

A model-theoretic structure contains interpretations of certain relational, functional, and constant symbols; each relational or functional symbol has a fixed arity. The collection $K$ of these symbols is called the *signature* of the structure.

A *theory* $T$ is a set of axioms that is used to define a class of structures.

The **first-order theory of fields** (denoted $T_{\text{fields}}$) is the collection of all first-order statements in a formal language of fields (with constants $0$ and $1$, operations $+$ and $\times$, and equality) that are true in all fields, which includes axioms expressing the basic field properties.

A theory is

- **Complete** if any two models of the theory are *elementarily equivalent*, i.e. they satisfy the same first-order statements.
- **Stable** if it satisfies certain combinatorial restrictions on its complexity.
- **Decidable** if there is an algorithm that can determine whether any given sentence is a theorem of the theory. For fields, decidability means that we can algorithmically verify whether statements about the field hold in the theory.
- **Quantifier-free** if it does not use any existential or universal quantifiers (e.g., $\forall$ or $\exists$). In field theory, quantifier-free formulas describe properties directly in terms of field elements without referring to all elements or subsets.
- **Model complete** if every embedding between models of the theory is an *elementary embedding* (preserving truth of all first-order formulas).

**Definitions** Let $M = (M, \dots)$ be an $\mathcal{L}$-structure. If $X \subseteq M^n$, then $X$ is *definable* if and only if there is an $L$-formula $\varphi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $b \in M^m$ such that

$$X = \{a \in M^n : M \models \varphi(a, b)\}.$$

## A first order-theory of Algebraically Closed Fields

Let ACF be the theory of algebraically closed fields together with the axiom

$$\forall \alpha_0 \dots \forall \alpha_{n-1} \exists z \left( x^n + \sum_{i=0}^{n-1} \alpha_i x^i = 0 \right)$$

for each $n$. ACF is not a complete theory since it does not decide the characteristic of the field. For each $n$, let $\varphi_n$ be the formula

$$\forall x\, (x + \cdots + x = 0) \quad \text{(with } n \text{ repetitions of } x\text{)}.$$

For $p$ prime, let ACF$_p$ be the theory ACF $+ \varphi_p$, and let ACF$_0$ = ACF $\cup \{\neg\varphi_n : n = 1, 2, \dots\}$.

For a cardinal $\kappa$, a theory is *$\kappa$-categorical* if there is, up to isomorphism, a unique model of cardinality $\kappa$.

**Proposition.** Let $p$ be a prime or zero, and let $\kappa$ be an uncountable cardinal. The theory ACF$_p$ is $\kappa$-categorical, complete, and decidable.

**Theorem.** The theorem of algebraically closed fields has quantifier elimination. As a result, *ACF* is model-complete. Additionally, $ACF_p$ is complete, where $p = 0$ or is a prime.

## Sketch of a model-theoretic proof of Hilbert's Nullstellensatz

**Hilbert's Nullstellensatz:** For $S \subseteq K[\bar{x}]$, let $V(S) = \{\bar{a} \in K^n : f(\bar{a}) = 0 \text{ for all } f \in S\}$. Then if $I$ and $J$ are radical ideals in $K[\bar{x}]$ with $I \subset J$, then $V(J) \subset V(I)$.

Chapter 7 of [4] provides a detailed proof. For an overview, once we have:

Let $K$ be an algebraically closed field, and let $m$ be a maximal ideal in $k[x_1, \dots, x_n]$. Let $L$ be the algebraic closure of $K[x_1, \dots, x_n]/m$.

We can show that there exists $a_1, \dots, a_n \in k$, which is a common root of some set of generators for $m$. This assertion is a first-order sentence using the language of fields and symbols in $K$. Thus, model completeness of ACF$_p$ implies this sentence is true in $k$ if and only if it is true in some (and hence every) algebraically closed extension field of $K$. But we know that this sentence is true in the algebraically closed extension $L$ by construction, and so we are done.

## Model-theory of Galois Theory

In the model-theoretic sense, Galois theory can be thought of as the classification of definably closed subsets of a model-theoretic algebraic closure according to the structure of a profinite automorphism group. (Poizat, 1983)

For instance, the fundamental theorem of Galois theory can be translated to model theory by working in a monster model $\mathbb{M} \models T$ for a complete first-order theory $T$ which eliminates imaginaries. Here,

1. A *monster model* $\mathbb{M}$ is a very large, highly saturated and homogeneous model of a complete first-order theory $T$.
2. $T$ has elimination of imaginaries if for every definable set
$$X = \{m \in \mathbb{M} \mid \varphi(m, b)\},$$
there exists a formula $\psi(x, y)$ and a tuple $c$ with the same sort as $y$ such that $c$ uniquely satisfies
$$X = \{m \in \mathbb{M} \mid \psi(m, c)\}.$$
3. Let $A$ be a subset of a monster model $\mathbb{M} \models T$. The *definable closure* $\mathrm{dcl}(A)$ of $A$ is the set of all tuples $b \in \mathbb{M}$ such that there exists a formula $\varphi(x, y)$ and a tuple $a$ from $A$ such that $b$ is the unique solution to $\varphi(a, y)$, i.e.,
$$\varphi(\mathbb{M}, b) = \{b\}.$$

**A model-theoretic Galois correspondence:** Let $K$ be a definably closed parameter set. Let $A$ be a normal extension of $K$ generated by the finite algebraic tuple $\gamma$. Then there is an order-reversing bijective correspondence between the subgroups of $\mathrm{Aut}(A/K)$ and the definably closed intermediate extensions of $A/K$. The correspondence is given by maps $\mathrm{Fix}$ sending a subgroup to its fixed points and $\mathrm{Stab}$ sending an intermediate definably closed extension to its stabilizer subgroup.

## Two interesting developments in model-theoretic algebra

**Differential Galois Theory** is the theory of solutions of differential equations over a differential base field.

In [7], [6], [5], model-theoretic methods are used to develop differential Galois theory. For instance, in suitable model-theoretic contexts, automorphism groups have the structure of definable groups.

[7] developed a theory of generalised strongly normal extensions of differential fields (which is a generalisation of Kolchin's theory), and showed that any finite-dimensional differential algebraic group is the Galois group of some generalised strongly normal extension $K$ of some $F$. [5] investigates inverse problems for generalised strongly normal extensions.

**Hrushovski's proof of the Mordell-Lang conjecture:**

The Mordell-Lang conjecture describes the properties of the intersection of a subvariety $X$ of a semiabelian variety $A$, both defined by polynomial equations over a function field $K$, with special subgroups $\Gamma$ of $A$.

Let **DCF** stand for the theory of differentially closed fields of characteristic 0. To prove the Mordell-Lang conjecture in characteristic $p = 0$. [1] uses DCF as it is a complete $\omega$-stable theory and hence has saturated models of any cardinality, and admits quantifier and imaginary elimination.

Hrushovski also utilised concepts from *stable group theory* and introduced the notion of *Zariski geometries*, which generalises classical algebraic geometry into the model-theoretic realm. Objects of finite Morley rank in model theory have properties analogous to those of algebraic varieties, thus translating geometric notions into logical ones.

The notes [3] describes the proof in great detail by presenting the required preliminaries.

## References

[1] Ehud Hrushovski.
The mordell-lang conjecture for function fields.
*J. AMS*, 9, 1996.

[2] E. R. Kolchin.
Galois theory of differential fields.
*American Journal of Mathematics*, 1953.

[3] Robert Lakatos.
Model theory and the moredll-lang conjecture.

[4] David Marker.
*An invitation to mathematical logic*.
2024.

[5] David Marker and Anand Pillay.
Differential galois theory iii: Some inverse problems.
*Illinois Journal of Mathematics*, 41, 1997.

[6] Anand Pillay.
Differential galois theory ii.
*Annals of Pure and Applied Logic*, 88, 1997.

[7] Anand Pillay.
Differential galois theory i.
*Illinois Journal of Mathematics*, 42, 1998.