FRAÏSSÉ LIMITS AND THE CLASS OF P-GROUPS

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Abstract. This survey paper presents the prerequisites to understand Fraïssé's theorem, and an overview of the amalgamation property in different classes of groups, with an emphasis on the class of p-groups.

1. Introduction

When studying a certain class of structures, an intriguing problem is to find a "universal" object within which member of that class "embeds" into, with these embeddings preserving the common substructures. Fraïssé's limit does exactly that for a certain class of structures, using first order logic. Roland Fraïssé described this in [5], and since then, the Fraïssé limit of various structures have been investigated. This paper is primarily concerned with the amalgamation property - the most interesting property of such classes - in various types of groups.

We will outline the required model theory to describe Fraïssé's construction and look at some examples to understand the universality and homogeneity of the structures. We begin with defining a model and spend some time on *homogeneous structures*. [14] discusses properties of homogeneous structures and its connections to other topics of mathematics.

We then focus on the amalgamation property in the language of groups with the goal of investigating whether or not, fixing p, the class of finite p-groups form a Fraïssé class. According to a theorem of Otto Schreier [21], every amalgam of two groups A, B can be embedded in the free product of A and B with amalgamated subgroup H, and we will describe this product later. [19] talks about this generalised amalgamated free product, including examples, the conditions of existence, and various properties. [17] also talks about expands on this with a focus on the permutational product, which will be essential for understanding the amalgamation property in finite groups. [9] writes about the amalgams of p-groups and some of their properties. [2] obtains a necessary and sufficient condition for the free product of two residually finite p-groups with finite amalgamated subgroup to be a residually finite p-group, which is a generalisation of Higman's theorem.

This discourse on universal properties brings us to the *pushout* (or fibre-product) from category theory, and we will present a short note on the same. Chapter 2 of [16] provides a nice overview of the link between this idea and the Seifert-van Kampen Theorem from algebraic topology, which gives the structure of the fundamental group of a topological space X in terms of the fundamental groups of two open, path-connected subspaces that cover X. The pushout here actually corresponds to the free product with amalgamation of the fundamental groups of the two subspaces covering X. This idea raises the question of extending Fraïssé's theorem to second-order logic and examining whether or not it can be applied to structures in a language of topological spaces.

2. Some Model Theory

We begin by outlining the required concepts from model theory. In this article, we will only focus on the basic and relevant information. Readers looking for a detailed description of Fraïssé limits and a wide arrayof examples of the same should refer to [1].

Definition 2.1. A *structure or model* $\mathcal{M}$ is a set with given specified functions M and relations (subsets of M^n) for various n, and some differentiated elements called "constants".

Definition 2.2. A language $\mathcal{L}$ is *relational* if it has no function or constant symbols.

Definition 2.3. A structure $\mathcal{M}$ is *homogeneous* if:

(1) $\mathcal{M}$ is countable.
(2) Whenever U and V are finite substructures of $\mathcal{M}$, there exists an isomorphism $f : U \to V$ that extends to an automorphism of $\mathcal{M}$.

Definition 2.4. [3] Let L be a language. An L-structure is a pair $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$, where

- A is a non-empty set, the universe of $\mathcal{A}$,
- $Z^{\mathcal{A}} \in A$ if Z is a constant,
- $Z^{\mathcal{A}} : A^n \to A$ if Z is an n-ary function symbol, and
- $Z^{\mathcal{A}} \subseteq A^n$ if Z is an n-ary relation symbol.

Definition 2.5. The *age* of $\mathcal{C}$ is the class $\mathcal{K}$ of all finitely generated structures that can be embedded in $\mathcal{C}$. We take these structures to be, up to isomorphism, the finitely generated substructures of $\mathcal{C}$.

If $\mathcal{K}$ is an age of a structure, then $\mathcal{K}$ is non-empty and has the following two properties:

(1) $\mathcal{K}$ is closed under substructures **(Hereditary property "HP")**.
(2) Whenever $A, B \in \mathcal{K}$, there exists $D \in \mathcal{K}$ such that $A \subseteq D$ and $B \subseteq D$ **(Joint Embedding Property "JEP")**.

The above two properties of the age are an if and only if condition, which will be important to prove Fraïsse's theorem.

Theorem 2.6. *Suppose $\mathcal{L}$ is a signature and $\mathcal{K}$ is a non-empty finite or countable set of finitely generated $\mathcal{L}$-structures which has the Hereditary Property and the Joint Embedding Property. Then $\mathcal{K}$ is the age of some finite or countable structure.*

Proof. See the proof of theorem 7.1.1 of [10].

3. Fraïssé's construction

As the name suggests, there is a notion of the structrues in a class "tending to" their Fraïssé limit. The key underlying property is the amalgamation property, and it is often the only property that is tricky to verify or disprove.

Definition 3.1. [3] The class $\mathcal{K}$ satisfies the *(amalgamation property "AP")* whenever given $A, B_1, B_2 \in \mathcal{K}$ and embeddings $f_i : A \to B_i$ (for $i = 1, 2$), there exists $D \in \mathcal{K}$ and embeddings $g_i : B_i \to D$ (for $i = 1, 2$) such that $(g_1 \circ f_1)(a) = (g_2 \circ f_2)(a)$ for all $a \in A$.

If D and the embeddings g_i can be chosen such that:

$$g_1(B_1) \cap g_2(B_2) = (g_1 \circ f_1)(a) = (g_2 \circ f_2)(a)$$

then $\mathcal{C}$ has *strong amalgamation.*

Note: The empty set is not considered as a structure in model theory.

Definition 3.2. A class $\mathcal{K}$ of finite $\mathcal{L}$-structures is an *amalgamation class* if it contains at most countably many non-isomorphic structures, has the amalgamation property, and is closed under isomorphisms and taking induced substructures.

Some graph-theoretic examples and non-examples of amalgamation classes:

(1) The class of all finite forests (simple, acyclic graphs) is *not* an amalgamation class.
(2) The class of all finite tournaments (directed graph without self-loops with exactly one edge in one direction between any two vertices) *is* an amalgamation class.
(3) [3] Let D be the tournament obtained from the directed cycle C_3 by adding a new vertex u, and adding the edges (u, v) for each vertex v of C_3. Let D' be the tournament obtained by flipping the orientation of each edge in D. The class of all finite tournaments that embeds neither D nor D' *is* an amalgamation class.

Theorem 3.3 (Fraïssé)**.** [1] *Let $\mathcal{L}$ be a countable language and $\mathcal{K}$ a class of finitely generated $\mathcal{L}$-structures. Suppose $\mathcal{L}$ is closed under isomorphism, and there are countably many isomorphism types of said structures. Then, there exists a (up to isomorphism) unique countable $\mathcal{L}$-structure $\mathcal{C}$ such that Age($\mathcal{C}$) $= \mathcal{K}$ if and only if $\mathcal{K}$ satisfies **HP**, **JEP**, and **AP**. In this case, $\mathcal{C}$ is called the Fraïssé limit of $\mathcal{K}$.*

Proof. See the proof of theorem 7.1.2 of [10].

Some examples of Fraïssé limits:

(1) The Fraïssé limit of the class of all finite linear orderings is equivalent to the structure $\langle \mathbb{Q}, < \rangle$ up to isomorphism.
(2) The Fraïssé-limit of the class of all **finite graphs** is the countable random graph, or the *Rado graph*.
(3) The Fraïssé limit of the class of all finite abelian groups is the group $A = \bigoplus_{i<\omega} \mathbb{Q}/\mathbb{Z}$ (see the proof of proposition 6 in [12]).

4. The Random Graph and Some of its Properties

Here, we will take a detour from the main topic to understand this classic example of a Fraïssé limit, as it will provide an intuitive understanding of how such a homogeneous and "universal" object behaves.

Definition 4.1. Let R denote the countable random graph *(Rado Graph)*.

- R is a countably infinite graph that can be constructed (with probability 1) by choosing independently at random for each pair of its vertices whether to connect the vertices by an edge (i.e. connect each pair of vertices by an edge independently with probability 1/2).
- **(Number theoretic definition:)** Let $\left(\frac{p}{q}\right)$ denote the Legendre symbol. Take the base set $S = \{p \mid \text{prime } p, p \equiv 1 \bmod 4\}$. Join $p, q \in S$ by an edge when $\left(\frac{p}{q}\right) = 1 \iff \left(\frac{q}{p}\right) = 1$.

Definition 4.2. A graph is *homogeneous* if any isomorphism between finite induced subgraphs extends to an automorphism of the graph.

The finite and countably infinite homogeneous graphs have been classified [6], of which the Rado graph is one.

Definition 4.3 (The Extension Property)**.** Let Γ denote an infinite graph and denote its vertex set (indexed by the natural numbers). Γ has the *extension property* if for any two disjoint finite subsets $A, B \subset N$, there is a vertex $v \in N - A - B$ such that v is connected to all vertices in A and no vertices in B.

Lemma 4.4. *If Γ_1 and Γ_2 are two infinite graphs, both having the extension property, then $\Gamma_1 \cong \Gamma_2$.*

Remark 4.5. As R has the extension property, any two Rado graphs are isomorphic. Hence, there is only one Rado graph (unique up to isomorphism).

5. The Class of Finite Groups

Before investigating the class of p-groups, we will first look at another standard example of the Fraïssé limit: *Hall's Universal Group.*

Groups are some of the most fundamental structures of algebra. To apply Fraïssé's theorem here, we first need to see how a group is defined in model theory:

Definition 5.1. [1] Let 1 be a constant, and $\cdot$ a binary operator. We denote the language of groups by $\mathcal{L}_{\text{Group}} = \{1, \cdot\}$. The theory of groups consists of the following axioms:

- $\forall a \; \forall \; b \; \forall c, \; (a \cdot (b \cdot c)) = ((a \cdot b) \cdot c)$ (associativity)
- $\forall a, \; 1 \cdot a = a \cdot 1 = a$ (identity element)
- $\forall a, \; \exists b \; (a \cdot b = 1)$ (inverse)

Let $\mathcal{K}$ be the class of all finite groups. $\mathcal{K}$ is clearly closed under isomorphism and taking induced substructures. It follows the **JEP** as two finite groups can be embedded in their direct product. We now only have to prove that $\mathcal{K}$ follows the amalgamtion property to see that it has a Fraïssé limit.

What construction could work as the amalgam of two finite groups? One may think of the free product with amalgamation, but it clearly will not work for two groups having trivial intersection, as in that case, the free product with amalgamation will simply be the free product of the two groups, which is an infinite group.

To understand this better, we will now describe the free product with amalgamation.

5.1. Free Product with Amalgamation.

Definition 5.2 (The free product of groups)**.** Given two groups G_1 and G_2, the free product G can be thought of as all possible words that can be made freely using "letters" (elements) from G_1 abd G_2 (and hence contains the two groups as subgroups), and is the "universal" group having the individual group properties. For finite groups, mathematically:

Let $\langle A_1 | R_1 \rangle$ and $\langle A_2 | R_2 \rangle$ be the group presentations of G_1 and G_2 respectively, where A_i is the set of generators of G_i and R_i is the set of defining relations for $i = 1, 2$. Then,

$$G_1 * G_2 = \langle A_1, A_2 \mid R_1, R_2 \rangle$$

Note that the free product is finite (except when one of the groups is trivial).

Free product with amalgamation:

As noted before, the amalgam of two groups can be embedded in its free product with amalgamated subgroup. We will only look at the definition of this product, as that is enough to understand why it doesn't work for the class of finite groups. [18] talks about the construction and applications of the amalgamated free product in detail.

We are looking for a group $G_1 *_H G_2$ satisfying the diagram below:

$$\begin{tikzcd}
& H \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
G_1 \arrow[dr] & & G_2 \arrow[dl] \\
& G_1 *_H G_2 \arrow[d] & \\
& X &
\end{tikzcd}$$

with $G_1 *_H G_2$ having the *universal property*, i.e. if any other group X satisfies the diagram in place of $G_1 *_H G_2$, then there is a homomorphism from $G_1 *_H G_2$ to X.

Suppose G_1 and G_2 are given as above, along with the monomorphisms:

$$\varphi : H \to G_1 \quad \text{and} \quad \psi : H \to G_2,$$

where, without loss of generality, H can be taken to be a common subgroup of G_1 and G_2 (as $\varphi(H) \leq G_1$ and $\psi(H) \leq G_2$).

Start with the free product $G_1 * G_2$ and adjoin as relations

$$\varphi(h)\psi(h)^{-1} = 1 \text{ (identity of } G_1 * G_2) \tag{5.1}$$

for every $h \in H$. This is equivalent to taking the *smallest normal subgroup* N of $G_1 * G_2$ containing all elements on the left-hand side of equation (5.1), as considered in $G_1 * G_2$ by the inclusion mappings of G_1 and G_2 into $G_1 * G_2$. Thus, we adjoin the relations of N with the relations of $G_1 * G_2$ to obtain the free product with amalgamation:

$$(G_1 * G_2)/N \quad = \langle A_1, A_2 \mid R_1, R_2, \{\varphi(h)\psi(h)^{-1} = 1 | h \in H\}\rangle$$

Example 5.3. Let $G_1 = (\mathbb{Z}_6, +) = \langle a \mid a^6 = 1\rangle$ and $G_2 = (\mathbb{Z}_4, +) = \langle b \mid b^4 = 1\rangle$. Clearly, $H = G_1 \cap G_2 = (\mathbb{Z})_2, +) = \{1, a^3\} = \{1, b^2\}$. Take the monomorphisms as the inclusion maps

$$i_1 : H \hookrightarrow G_1 \text{ and } i_2 : H \hookrightarrow G_2$$

Now, we want to adjoin as relations $i_1(h)i_2(h)^{-1} = 1$, which gives the relation $a^3 = b^2\}$. Now, the free product with amalgamation is the following group:

$$\langle a, b \mid a^6 = 1, b^4 = 1, a^3 = b^2\rangle$$

6. The Permutational Product

A construction of an amalgam of two finite groups is the permutational product, and it will be described in this section.

Definition 6.1. Let G be a finite group, $H \leq G$ a subgroup. We define the *left transversal* S of H as the fixed set of left coset representatives (representatives of the aforementioned equivalence classes). Given an element $a \in G$ and its unique product decomposition $a = sh$ in terms of $s \in S$, $h \in H$, we denote $s = a^\sigma$, $h = a^{-\sigma+1}$.

We use these transversals to construct the following group:

Definition 6.2. Let $A, B, H \in \mathcal{K}$ such that $H = A \cap B$ is a common subgroup of A and B. Let K be the Cartesian product $K := A \times B \times H$. There exist maps ρ and ρ' from A and B respectively to a permutation group $\rho(A)$ and $\rho'(B)$ of K such that the maps $\rho(h_0)$ and $\rho'(h_0)$ coincide for any element $h_0 \in H$. The permutational product is a subgroup of the group of permutations on K, generated by the images $\rho(A)$ and $\rho'(B)$.

Theorem 6.3. *The class $\mathcal{K}$ of all finite groups satisfies the amalgamation property.*

Proof. See the proof of Theorem 4.13 in [1].

The Fraïssé limit of the class of finite groups is known to be *Hall's universal group*:

Definition 6.4. Let $\mathcal{K}$ be as above and U be a countable, locally finite group. We say U is Hall's universal group if the following hold:

- Every finite group $G \in \mathcal{K}$ admits an embedding $G \subseteq U$, i.e. G is a subgroup of U.
- Given $G_0, G_1 \in \mathcal{K}$, any embeddings $f_0 : G_0 \to U, f_1 : G_1 \to U$ are conjugate by some inner automorphism of U.

Philip Hall constructed this Universal Group before Fraïsse's theorem was developed. It is very easy to see that U is the age of $\mathcal{K}$. The challenging part would be proving its existence, which was done in [7]. However, Fraïssé's theorem guarantees its existence as the "limit" of $\mathcal{K}$.

Example 6.5 (Demonstration of the Permutational Product)**.** To understand how exactly the permutational product works, here is an example. Note that this product depends on our choice of left transversals.

Consider the Dihedral group $D_8 = \langle x, y \mid x^4 = 1, y^2 = 1\rangle$ and the cyclic group $\mathbb{Z}_8 = \langle r \mid r^8 = 1\rangle$. $D_8 \cap \mathbb{Z}_8 = \mathbb{Z}_4 = \langle r^2 \mid r^8 = 1\rangle = \langle x \mid x^4 = 1\rangle$.

We will build the permutational product of D_8 and $\mathbb{Z}_8$ using the condition in equation 2.1. For that, we have to determine $\rho(\mathbb{Z}_8)$ and $\rho'(D_8)$.

Let the left transversal of $\mathbb{Z}_4$ in $\mathbb{Z}_8$ be $S = \{1, r\}$ and the left transversal of $\mathbb{Z}_4$ in D_8 be $T = \{1, y\}$.

Computing $\rho(\mathbb{Z}_8)$: Consider an element $a = r^n \in \mathbb{Z}_8, n \in \{2, 4, 6, 8\}$.

- For $(1, t, r^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho_a(1, t, r^m) = ((r^{n+m})^\sigma, t, (r^{n+m})^{-\sigma+1}) = \begin{cases} (1, t, (r^{n+m})) & \text{m is even} \\ (r, t, (r^{n+m-1})) & \text{m is odd} \end{cases} \tag{6.1}$$

- For $(r, t, r^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho_a(r, t, r^m) = ((r^{n+m+1})^\sigma, t, (r^{n+m+1})^{-\sigma+1}) = \begin{cases} (r, t, (r^{n+m})) & \text{m is even} \\ (1, t, (r^{n+m+1})) & \text{m is odd} \end{cases} \tag{6.2}$$

Now, consider an element $a = r^n \in \mathbb{Z}_8, n \in \{1, 3, 5, 7\}$.

- For $(1, t, r^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho_a(1, t, r^m) = ((r^{n+m})^\sigma, t, (r^{n+m})^{-\sigma+1}) = \begin{cases} (r, t, (r^{n+m-1})) & \text{m is even} \\ (1, t, (r^{n+m})) & \text{m is odd} \end{cases} \tag{6.3}$$

- For $(r, t, r^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho_a(r, t, r^m) = ((r^{n+m+1})^\sigma, t, (r^{n+m+1})^{-\sigma+1}) = \begin{cases} (1, t, (r^{n+m+1})) & \text{m is even} \\ (r, t, (r^{n+m})) & \text{m is odd} \end{cases} \tag{6.4}$$

$\rho'(D_8)$**:** Consider an element $b = y^l x^n \in D_8,\ l \in 0, 1$.

- For $(s, 1, x^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho'_b(s, 1, x^m) = (s, (y^l x^{n+m})^\tau, (y^l x^{n+m})^{-\tau+1}) = \begin{cases} (s, 1, x^{n+m}) & l = 0 \\ (s, y, x^{n+m}) & l = 1 \end{cases} \tag{6.5}$$

- For $(y, t, x^m) \in \mathbb{Z}_8 \times D_8 \times \mathbb{Z}_4$

$$\rho_a(s, y, x^m) = (s, (yx^m y^l x^n)^\tau, (yx^m y^l x^n)^{-\tau+1}) = \begin{cases} (s, y, (x^{n+m})) & l = 0 \\ (s, 1, (x^{n-m})) & l = 1 \end{cases} \tag{6.6}$$

Clearly, for $h_0 \in \mathbb{Z}_4$, ρ_h and ρ'_h coincide:

$$\rho_{h_0}(s, t, h) = (s, t, hh_0) = \rho'_{h_0}(s, t, h)$$

as required.

7. Some Group Theory

Before investigating the amalgamation property for p-groups, we need some group-theoretic definitions.

Definition 7.1. If G is a group, then the *centre* of G is

$$Z(G) = \{a \in G \mid ab = ba \text{ for all } b \in G\}$$

Note that $Z(G) \trianglelefteq G$.

Definition 7.2. The *upper central series* of a group G is a sequence of normal subgroups of G:

$$\{e\} = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \dots$$

where

$$Z_0(G) = \{e\},$$
$$Z_1(G) = Z(G),$$

and

$$Z_{i+1} = \{x \in G \mid \forall y \in G : [x, y] \in Z_i\}$$

Definition 7.3. A group G is *nilpotent* if $Z_i(G) = G$ in the above series for some i. If G is a nilpotent group, then the *nilpotency class* of G is the smallest $n \geq 0$ such that $Z_n(G) = G$.

Definition 7.4. Let $\mathcal{P}$ be a property of a group (e.g., finite, nilpotent, free). A group is *residually* $\mathcal{P}$ if for all $g \in G \setminus \{1\}$, there exists a normal subgroup $N \triangleleft G$ such that

$$g \notin N \quad \text{and} \quad G/N \in \mathcal{P}.$$

Note: A free product of two finite p-groups amalgamating a cyclic subgroup is residually p-finite [9].

Definition 7.5. A **chief series** of a group G is a finite collection of normal subgroups $N_i \trianglelefteq G$,

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \dots \trianglelefteq N_n = G,$$

where each N_i is a normal subgroup of G.

8. The Class of p-groups

Fixing p, is the class of p-groups a Fraisse class? The answer is *no*, and a counter-example to the amalgamtion property will be presented at the end of this section. However, the class of **abelian p-groups** *do* form a Fraïssé class, and their Fraïssé limit is the direct sum of countably many finite Prufer groups.

Before that, we will first highlight some interesting notions that are known regarding the amalgams of p-groups. [9] describes these amalgams, of which the following theorem highlights some of its properties:

Theorem 8.1. *Let $A \cup B$ be an amalgam of finite p-groups A, B with $A \cap B = U$. Let F be the free product of A and B with the amalgamation U. Then the following four statements are equivalent:*

(i) *F is residually nilpotent.*
(ii) *F is residually a finite p-group.*
(iii) *$A \cup B$ is embeddable in a finite p-group.*
(iv) *There exists a chief series (A_i) of A and (B_i) of B such that $U \cap (A_i) = U \cap (B_i)$.*

Corollary 8.2. *If U is cyclic, statements (i) to (iii) hold. For in this case U has only one chief series, so (iv) must be true.*

Next, if U is a subgroup of a group G, we denote by $\mathrm{Aut}_G(U)$ the image of the normalizer of U in G in its natural homomorphism into the automorphism group of U.

Corollary 8.3. *If U is normal both in A and in B, then (i) to (iii) hold if and only if $Aut_A(U)$ and $Aut_B(U)$ generate a p-group.*

Example 8.4 (The class of 2-groups does not have the amalgamation property.)**.** Recall that the exponent of a group G is the smallest natural number n such that $g^n = 1$ for all $g \in G$.

Counter-example: Let $k \in \mathbb{N},\ 2|\ k$ but $k \neq 2^m$. Consider $G = (\mathbb{Z}/2\mathbb{Z})^k$, where $(\mathbb{Z}/2\mathbb{Z}) = \{[0], [1]\} = \{0, 1\}$. Let G_1 be the semidirect product $G \rtimes_\psi \mathbb{Z}/2\mathbb{Z}$ with ψ such that

$$\psi : \mathbb{Z}/2\mathbb{Z} \to \ \mathrm{Aut}(G),\ \psi(0) = \psi_0 \text{ and } \psi(1) = \psi_1$$

where $\psi_0 =$ identity automorphism, and $\psi_1(g_1, g_2, g_3, g_4, ..., g_{k-1}, g_k) = (g_2, g_1, g_4, g_3, ..., g_k, g_{k-1})$, where

$$g_i \in G \text{ with 1 in the ith place of the } k\text{-tuple and 0 in the other positions.}$$

Then, G_1 is of exponent 4. For instance, the element

$$((1, 0, 1, 0, ..., 1, 0), 1)$$

is of order 4.

Let G_2 be the semidirect product $G \rtimes_{\psi'} \mathbb{Z}/2\mathbb{Z}$ with ψ' such that

$$\psi' : \mathbb{Z}/2\mathbb{Z} \to \ \mathrm{Aut}(G),\ \psi'([0]) = \psi_0 \text{ and } \psi'([1]) = \psi'_1$$

where $\psi'_1(g_1, g_2, g_3, g_4, ..., g_{k-1}, g_k) = (g_1, g_3, g_2, g_5, ..., g_{k-2}, g_k)$.

Then, G_2 is also of exponent 4.

Note that $G_1 \cap G_2 = G$. (By construction, G is a subgroup of both G_1 and G_2 of index 2, and $G_1 \neq G_2$).

Now, assume an amalgam G_0 of G_1 and G_2 exists. Then, there exist monomorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ as shown below

$$\begin{tikzcd} & G_1 \arrow[dr, "\beta_1"] & \\ G \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & G_0 \\ & G_2 \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

such that $\beta_1 \circ \alpha_1(g) = \beta_2 \circ \alpha_2 \ \forall g \in G$

Let $h \in G_1, G_2, h = ((0, ..., 0), 1)$. Then, G_0 has an element of order k, which implies that G_0 of order k. The key point is the action of $\beta_1(h)\beta_2(h)$ on elements in the amalgam. $\beta_1(h)$ permutes elements as $\sigma(12)(34)...(k-1k)$ and β_2 as $\sigma' = (23)(45)...(k-2k-1)$. The action of $\beta_1(h)\beta_2(h)$ corresponds to the permutation $(12)(34)...(k-1k)(23)(45)...(k-2\ k-1)$, which (on reducing it to the standard representation) is a k-cycle. Corollary 8.3 then tells us that the group generated by $\mathrm{Aut}_{G_1}(G)$ and $\mathrm{Aut}_{G_2}(G)$ will not be a p-group as it will contain the aforementioned k-cycle. Hence, the class of 2-groups do not satisfy the amalgamation property and do not form a Fraïssé class.

The above example can be modified for other primes p. Hence, **the class of p-groups do not form an amalgamation class.** Having said that, the following notion can realise the property in a "restriction" of the class:

Definition 8.5. [15] A group $H \in \mathcal{K}$ is called an *amalgamation base* for $\mathcal{K}$ if all amalgams with constituents $G_1, G_2, \in \mathcal{K}$ and common subgroup H can be realised in $\mathcal{K}$.

For example, the amalgamation bases in the class of finite p-groups are the cyclic p-groups.

8.1. **Further ideas.** What other classes of groups could satisfy the amalgamation property? For further reading, [17] summarises some conditions on the existence of an amalgam of two groups satisfying a certain property. [11] defines some modified notions of the Fraïssé class and describes classes of groups with those properties. [4] talks about the Fraïssé limit of 2-nilpotent groups, and some other model theoretic properties of nilpotent groups. [22] presents some neat ideas on *universal abelian groups*,

9. A Category-Theoretic Analogue

The Fraïssé limit can be thought of as a special case of the notion of a *direct limit* from category theory, and the amalgamation property is analogous to the *pushout*. As this article is not about category theory, we will only highlight some interesting bits of information to outline a parallel picture. For a better understanding of category theory, [20] is a good source.

The first idea we are concerned with is that of a *colimit*. We will avoid the definition as that requires some category theory. Intuitively, colimit constructions build new objects by "gluing together" existing ones along specified morphisms. They actually generalise the notion of a direct sum.

Example 9.1 ([20])**.** The colimit of the sequence of inclusions

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \cdots \hookrightarrow \mathbb{Z}/p^n \hookrightarrow \cdots$$

is the Prüfer p-group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, which is an abelian group with the following group presentation:

$$\mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z} \cong \langle g_1, g_2, g_3, \cdots \mid pg_1 = 0, pg_2 = g_1, pg_3 = g_2, \ldots \rangle.$$

Recall that the the Prüfer group is the Fraïssé limit of the class of finite abelian groups!

Definition 9.2 (Pushout)**.** The pushout of the morphisms f and g consists of an object P and two morphisms $i_1 : X \to P$ and $i_2 : Y \to P$ such that the following diagram commutes:

$$\begin{tikzcd} P & Y \arrow[l, "i_2"'] \\ X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

and such that (P, i_1, i_2) is *universal* with respect to this diagram, i.e, for any other such triple (Q, j_1, j_2) satsifying the above commuting diagram, there must exist a unique $u : P \to Q$ also making the following diagram commute (this is the *universal property*):

$$\begin{tikzcd} & Q & \\ X \arrow[ur, "j_1"] \arrow[r, "i_1"] & P \arrow[u, dashed, "u"] & Y \arrow[ul, "j_2"'] \arrow[l, "i_2"'] \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Hence, the pushout, if it exists, *is unique up to isomorphism.*

Remark 9.3. In the category of groups, the pushout gives the amalgamated free product.

9.1. **The Van-Kampen theorem.** The correlation between the pushout and the free product with amalgamation is clearly seen in the Van-Kampen theorem. We will write the theorem in the usual algebraic-topologic manner [8] and from a category-theoretic perspective:

Theorem 9.4 ([8])**.** *Given a pointed space (X, x_0), if X is the union of path-connected open sets A_α with $x_0 \in A_\alpha$ $\forall\alpha$, and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism*

$$\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective. If, in addition, each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism

$$\pi_1(X) \cong \frac{*_\alpha \pi_1(A_\alpha)}{N}$$

The above can be formulated to look at X as a union of two subspaces. Following is the above theorem (for X as the union of two subspaces) in terms of the ***pushout****:*

Let X be as above with $\{A_\alpha\} = U - 1, U_2$. Suppose $U_1 \cap U_2$ is path-connected and nonempty, and let x_0 be a point in $U_1 \cap U_2$ that will be used as the basepoint of all fundamental groups. The inclusion maps of U_1 and U_2 into X induce group homomorphisms

$$j_1 : \pi_1(U_1, x_0) \to \pi_1(X, x_0) \quad \textit{and} \quad j_2 : \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Then X is path-connected and j_1 and j_2 form a commutative pushout diagram:

$$\begin{tikzcd}
 & \pi_1(U_1) \arrow[d, dashed] \arrow[dr, "j_1"] & \\
\pi_1(U_1 \cap U_2) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2) \arrow[r, dashed, "k"] & \pi_1(X) \\
 & \pi_1(U_2) \arrow[u, dashed] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

*The natural morphism k is an isomorphism. That is, the fundamental group of X is the **free product** of the fundamental groups of U_1 and U_2 with the amalgamated subgroup $\pi_1(U_1 \cap U_2, x_0)$.*

In algebraic topology, an important notion of a universal object is the *universal covering space*. We will not go into the details as that will require a course on algebraic topology, but it is essentially a solution to the universal property, as noted in the proof of 13.2.4 in [23]. A natural question that arises is whether or not a universal covering space can be realised as the Fraïssé limit of a certain class of topological spaces. However, topology is non-first orderisable; There are various constructions of topology (for some examples, see the introduction of [13]), but they all lead to a class of structures larger than the class of topological spaces. Thus, we will have to formulate an extended notion of Fraïssé's constructions to second-order logic to answer such a question.

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