In[ ]:=
```
f[x_] := x^3 - I s x^2 + x + I s
```

In[ ]:=
```
PolynomialRemainder[f[x], f'[x], x]
```

Out[ ]= $\frac{10\, i\, s}{9} + \left(\frac{2}{3} + \frac{2 s^2}{9}\right) x$

- which is not a zero function for all s

In[ ]:=
```
r1 = PolynomialRemainder[f[x], f'[x], x];
PolynomialRemainder[f'[x], r1, x]
Solve[PolynomialRemainder[f'[x], r1, x] == 0, s]
```

Out[ ]= $1 - \frac{100 s^2}{27\left(\frac{2}{3} + \frac{2 s^2}{9}\right)^2} - \frac{20 s^2}{9\left(\frac{2}{3} + \frac{2 s^2}{9}\right)}$

Out[ ]= $\left\{\left\{s \to -\sqrt{-\frac{11}{2} + \frac{5\sqrt{5}}{2}}\right\}, \left\{s \to \sqrt{-\frac{11}{2} + \frac{5\sqrt{5}}{2}}\right\}, \left\{s \to -i\sqrt{\frac{1}{2}\left(11 + 5\sqrt{5}\right)}\right\}, \left\{s \to i\sqrt{\frac{1}{2}\left(11 + 5\sqrt{5}\right)}\right\}\right\}$

- Hence, when s is large enough, f[x] has no multiple zeros.
- From the discussion before (2.10), set $\alpha=2\pi/L$ there exists integer k>l>0, such that

In[6]:=
```
ξ2 = ξ1 + k α;
ξ3 = ξ1 + l α;
Collect[(ξ - ξ1) (ξ - ξ2) (ξ - ξ3), ξ]
```

Out[8]= $\xi^3 + \xi^2\left(-k\,\alpha - l\,\alpha - 3\,\xi 1\right) - k\, l\, \alpha^2\, \xi 1 - k\,\alpha\,\xi 1^2 - l\,\alpha\,\xi 1^2 - \xi 1^3 + \xi\left(k\, l\,\alpha^2 + 2\, k\,\alpha\,\xi 1 + 2\, l\,\alpha\,\xi 1 + 3\,\xi 1^2\right)$

- **$(\xi-\xi 1)(\xi-\xi 2)(\xi-\xi 3) = \xi$^3 + I s ξ^2 − ξ + I s, where in my draft, I use "s" denote the eigenvalue λ. We aim to eliminate s and $\xi 1$.**
  - Now we know e1 = e2 = e3 = 0, which is the same as equation (), where e1, e2, e3 defined as follows

In[12]:=
```
e1 = (-k α - l α - 3 ξ1) - I s;
e2 = (k l α^2 + 2 k α ξ1 + 2 l α ξ1 + 3 ξ1^2) + 1;
e3 = -k l α^2 ξ1 - k α ξ1^2 - l α ξ1^2 - ξ1^3 - I s;
```

- Eliminate ξ1^3 term.

In[17]:=
```
e4 = Collect[ξ1 e2 + 3 e3, ξ1]
```

Out[17]= $-3\, i\, s + \left(1 - 2\, k\, l\,\alpha^2\right)\xi 1 + \left(-k\,\alpha - l\,\alpha\right)\xi 1^2$

- Notice that e1=e2=e3=0 is equivalent to e1=e2=e4=0, since (e4-ξ1e2)/3=e3.
- Eliminate ξ1^2 term in e3.

In[23]:=
```
e5 = Collect[Simplify[(k + l) α e2 + 3 e4], ξ1]
```

Out[23]= $-9\, i\, s + k\,\alpha + l\,\alpha + k^2\, l\,\alpha^3 + k\, l^2\,\alpha^3 + \left(3 + 2\, k^2\,\alpha^2 - 2\, k\, l\,\alpha^2 + 2\, l^2\,\alpha^2\right)\xi 1$

- Notice that e1=e2=e4=0 is equivalent to e1=e2=e5, since (e5-(k+l)αe2)/3=e4.

- Now we solve s from e1, and then plug-in e5 to obtain e6.

```
In[31]:= e6 = Collect[Simplify[e5 /. Solve[e1 == 0, s][[1]]], ξ1]
```

Out[31]= $10\,k\,\alpha + 10\,l\,\alpha + k^2\,l\,\alpha^3 + k\,l^2\,\alpha^3 + \left(30 + 2\,k^2\,\alpha^2 - 2\,k\,l\,\alpha^2 + 2\,l^2\,\alpha^2\right)\,\xi 1$

- Now we have e2=e6=0 is equivalent to e1=e2=e5=0.

```
In[37]:= e7 = e2 /. Solve[e6 == 0, ξ1][[1]]
```

Out[37]=
$$1 + k\,l\,\alpha^2 + \frac{k\,\alpha\,\left(-10\,k\,\alpha - 10\,l\,\alpha - k^2\,l\,\alpha^3 - k\,l^2\,\alpha^3\right)}{15 + k^2\,\alpha^2 - k\,l\,\alpha^2 + l^2\,\alpha^2} + \frac{l\,\alpha\,\left(-10\,k\,\alpha - 10\,l\,\alpha - k^2\,l\,\alpha^3 - k\,l^2\,\alpha^3\right)}{15 + k^2\,\alpha^2 - k\,l\,\alpha^2 + l^2\,\alpha^2} + \frac{3\,\left(-10\,k\,\alpha - 10\,l\,\alpha - k^2\,l\,\alpha^3 - k\,l^2\,\alpha^3\right)^2}{4\,\left(15 + k^2\,\alpha^2 - k\,l\,\alpha^2 + l^2\,\alpha^2\right)^2}$$

```
In[49]:= e8 = FullSimplify[-e7 * 4 (15 + k^2 α^2 - k l α^2 + l^2 α^2)^2 / 9]
```

Out[49]= $-100 + 20\,\left(k^2 - k\,l + l^2\right)\,\alpha^2 + 4\,\left(k^2 - k\,l + l^2\right)^2\,\alpha^4 + k^2\,(k - l)^2\,l^2\,\alpha^6$

- If we let k=k'+l, now we have as k' not equal to l, since k>l, i.e. k' and l are positive integers such that k' no equal to l.

```
In[52]:= e9 = Collect[FullSimplify[e8 /. {k → k' + l}], α]
```

Out[52]= $-100 + l^2\,\alpha^6\,(k')^2\,(l + k')^2 + 20\,\alpha^2\,\left(l^2 + k'\,(l + k')\right) + 4\,\alpha^4\,\left(l^2 + k'\,(l + k')\right)^2$

- Set y=α^2>0, we aim to prove for all positive integers k>l>0, we have only one positive solution for equation g(y)=0, i.e. e8=0, where g(y) is defined as follows.

```
In[55]:= g[y_] := -100 + 20 (k^2 - k l + l^2) y + 4 (k^2 - k l + l^2)^2 y^2 + k^2 (k - l)^2 l^2 y^3
```

```
In[63]:= g'[y]
```

Out[63]= $20\,\left(k^2 - k\,l + l^2\right) + 8\,\left(k^2 - k\,l + l^2\right)^2\,y + 3\,k^2\,(k - l)^2\,l^2\,y^2$

- Since g'[y]>0,for all y>0, g[0]=-100, and g[+∞]=+∞, hence g[y] has only one positive solution, for each k>l>0.