In[ ]:=
```
f[x_] := x^3 - I s x^2 + x + I s
```

In[ ]:=
```
PolynomialRemainder[f[x], f'[x], x]
```

Out[ ]= $\frac{10\,i\,s}{9} + \left(\frac{2}{3} + \frac{2\,s^2}{9}\right) x$

- which is not a zero function for all s

In[ ]:=
```
r1 = PolynomialRemainder[f[x], f'[x], x];
PolynomialRemainder[f'[x], r1, x]
Solve[PolynomialRemainder[f'[x], r1, x] == 0, s]
```

Out[ ]= $1 - \frac{100\,s^2}{27\left(\frac{2}{3} + \frac{2\,s^2}{9}\right)^2} - \frac{20\,s^2}{9\left(\frac{2}{3} + \frac{2\,s^2}{9}\right)}$

Out[ ]= $\left\{\left\{s \to -\sqrt{-\frac{11}{2} + \frac{5\sqrt{5}}{2}}\right\}, \left\{s \to \sqrt{-\frac{11}{2} + \frac{5\sqrt{5}}{2}}\right\}, \left\{s \to -i\sqrt{\frac{1}{2}\left(11 + 5\sqrt{5}\right)}\right\}, \left\{s \to i\sqrt{\frac{1}{2}\left(11 + 5\sqrt{5}\right)}\right\}\right\}$

- Hence, when s is large enough, f[x] has no multiple zeros.
- From the discussion before (2.10), set $\alpha=2\pi/L$ there exists integer k>l>0, such that

In[ ]:=
```
ξ2 = ξ1 + k α;
ξ3 = ξ1 + l α;
Collect[(ξ - ξ1) (ξ - ξ2) (ξ - ξ3), ξ]
```

Out[ ]= $\xi^3 + \xi^2\left(-k\,\alpha - l\,\alpha - 3\,\xi1\right) - k\,l\,\alpha^2\,\xi1 - k\,\alpha\,\xi1^2 - l\,\alpha\,\xi1^2 - \xi1^3 + \xi\left(k\,l\,\alpha^2 + 2\,k\,\alpha\,\xi1 + 2\,l\,\alpha\,\xi1 + 3\,\xi1^2\right)$

- **(ξ - ξ1) (ξ - ξ2) (ξ - ξ3) = ξ^3 + I s ξ^2 - ξ + I s,**
  where in my draft, I use "s" denote the eigenvalue λ.
- **Now we know e1 = e2 = e3 = 0, which is the same as equation (2.11),**
  where e1, e2, e3 defined as follows

In[1]:=
```
e1 = (-k α - l α - 3 ξ1) - I s;
e2 = (k l α^2 + 2 k α ξ1 + 2 l α ξ1 + 3 ξ1^2) + 1;
e3 = -k l α^2 ξ1 - k α ξ1^2 - l α ξ1^2 - ξ1^3 - I s;
```

- Do the following calculation.

In[4]:=
```
e4 = Collect[ξ1 e2 + 3 e3, ξ1]
```

Out[4]= $-3\,i\,s + \left(1 - 2\,k\,l\,\alpha^2\right)\xi1 + \left(-k\,\alpha - l\,\alpha\right)\xi1^2$

In[5]:=
```
e5 = Collect[Simplify[(k + l) α e2 + 3 e4], ξ1]
```

Out[5]= $-9\,i\,s + k\,\alpha + l\,\alpha + k^2\,l\,\alpha^3 + k\,l^2\,\alpha^3 + \left(3 + 2\,k^2\,\alpha^2 - 2\,k\,l\,\alpha^2 + 2\,l^2\,\alpha^2\right)\xi1$

In[10]:=
```
e6 = Collect[Simplify[e5 /. Solve[e1 == 0, s][[1]]], ξ1]
```

Out[10]= $10\,k\,\alpha + 10\,l\,\alpha + k^2\,l\,\alpha^3 + k\,l^2\,\alpha^3 + \left(30 + 2\,k^2\,\alpha^2 - 2\,k\,l\,\alpha^2 + 2\,l^2\,\alpha^2\right)\xi1$

```
In[18]:= e7 = -FullSimplify[(e2 /. Solve[e6 == 0, ξ1][[1]]) * 4 (15 + k^2 α^2 - k l α^2 + l^2 α^2)^2 / 9]
```

Out[18]= $-100+20\left(k^2-k\,l+l^2\right)\alpha^2+4\left(k^2-k\,l+l^2\right)^2\alpha^4+k^2\,(k-l)^2\,l^2\,\alpha^6$

```
In[19]:= g[y_] := -100 + 20 (k^2 - k l + l^2) y + 4 (k^2 - k l + l^2)^2 y^2 + k^2 (k - l)^2 l^2 y^3
```

```
In[20]:= g'[y]
```

Out[20]= $20\left(k^2-k\,l+l^2\right)+8\left(k^2-k\,l+l^2\right)^2 y+3\,k^2\,(k-l)^2\,l^2\,y^2$

```
In[29]:= Factor[g[3 / (k^2 - k l + l^2)]]
```

Out[29]= $-\frac{(k-2\,l)^2\,(2\,k-l)^2\,(k+l)^2}{\left(k^2-k\,l+l^2\right)^3}$

```
In[31]:= Factor[g[5 (Sqrt[5] - 1) / 2 / (k^2 - k l + l^2)]]
```

Out[31]= $\frac{125\left(-2+\sqrt{5}\right)k^2\,(k-l)^2\,l^2}{\left(k^2-k\,l+l^2\right)^3}$