



Universiteit Utrecht



DIRECTIONAL-FIELD PROCESSING

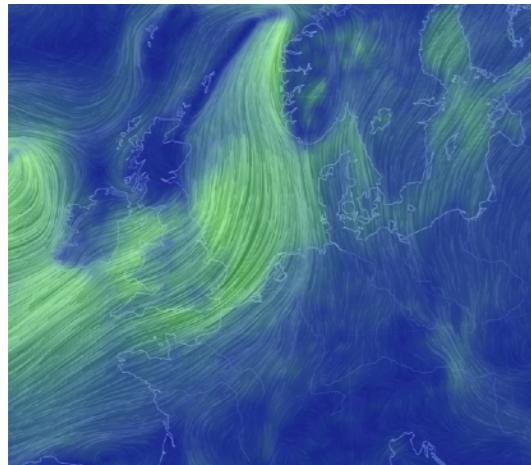
Amir Vaxman, Utrecht University

Developed with Marcel Campen, Olga Diamanti, David Bommes, Mirela Ben-Chen, Klaus Hildebrandt, and Daniele Panozzo.

github.com/avaxman/DirectionalFieldSynthesis

DIRECTIONAL FIELDS

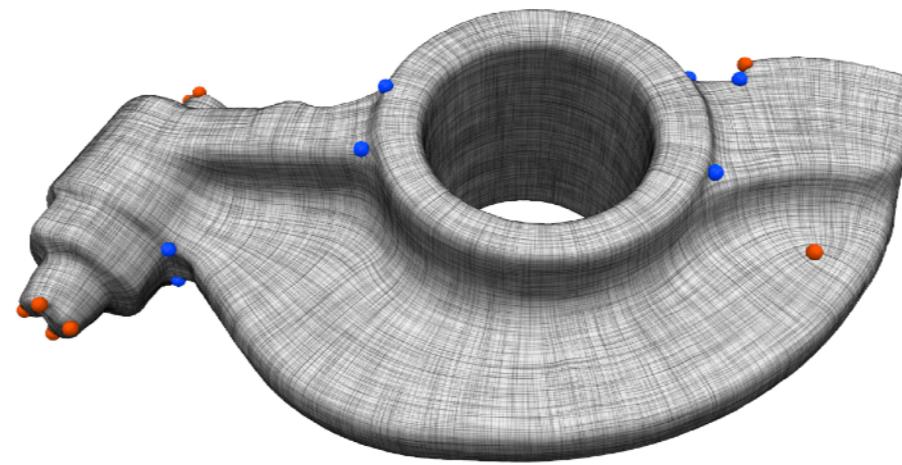
N -Directional field: a collection of vectors $\{u_1(p), u_2(p), \dots, u_N(p)\}$ associated with each point $p \in \Omega$ in a domain Ω .



<https://earth.nullschool.net/>

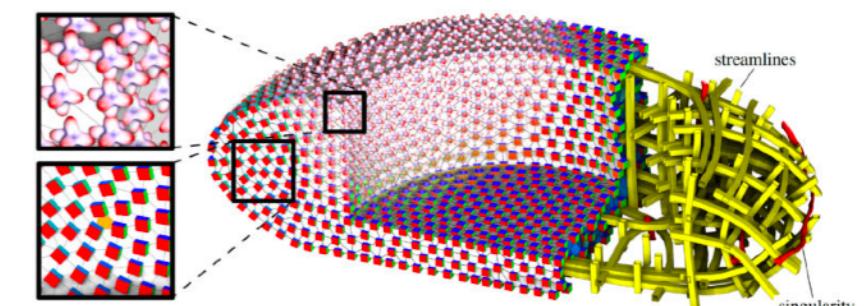
Domain: 2D* image

Field: single (1-)vector field.



Domain: 2-manifold in 3D

Field: **tangent** 4-field (“cross-field”)



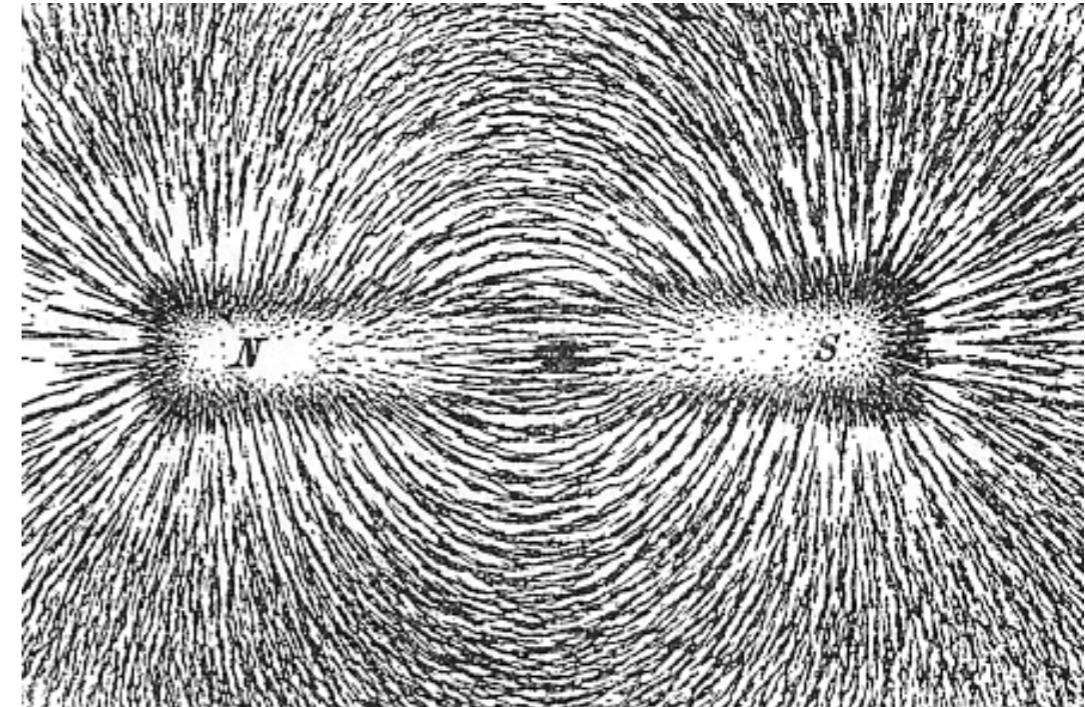
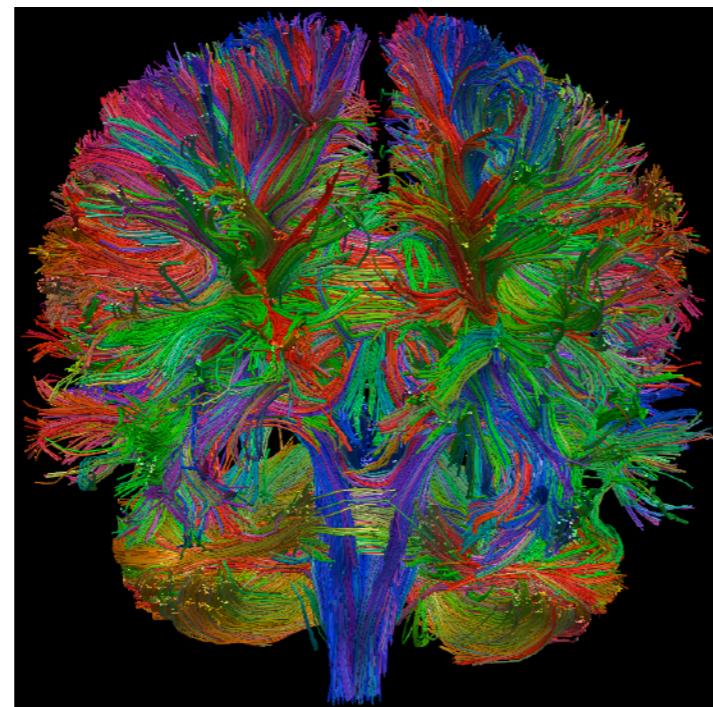
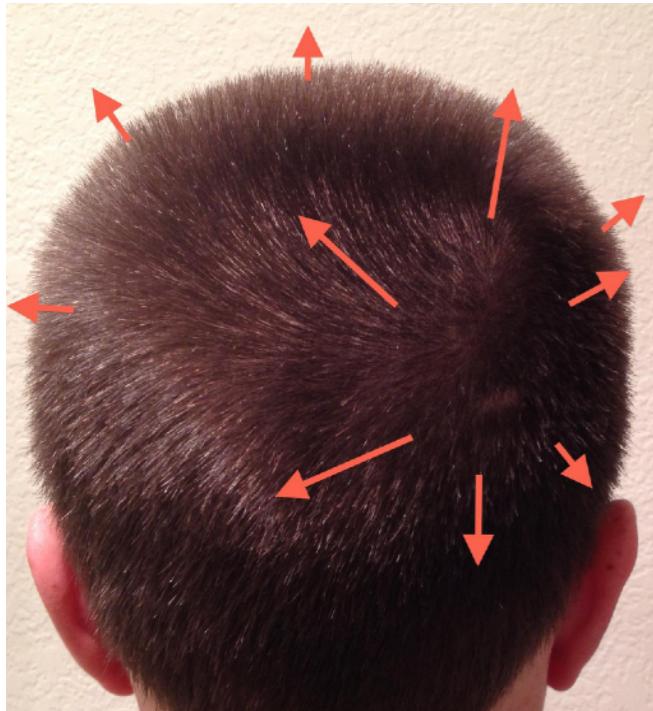
[Sokolov *et al.* 2016]

Domain: 3D bounded volume

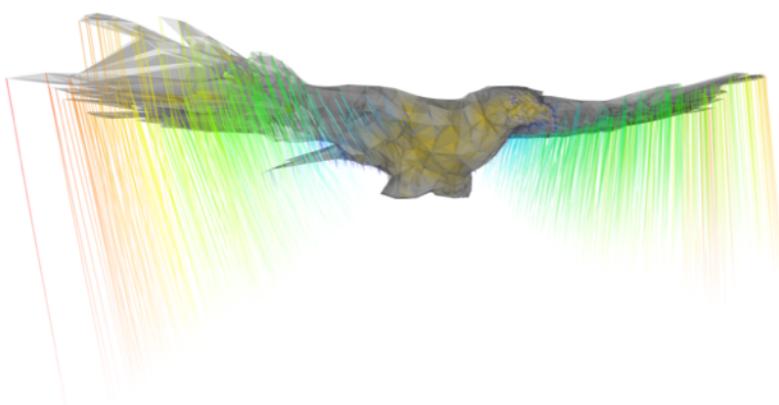
Field: 3D orthonormal frame field (“octahedral”)

**spherical*

DIRECTIONAL FIELDS IN NATURE AND DESIGN



[Alexander *et al.* 2007]



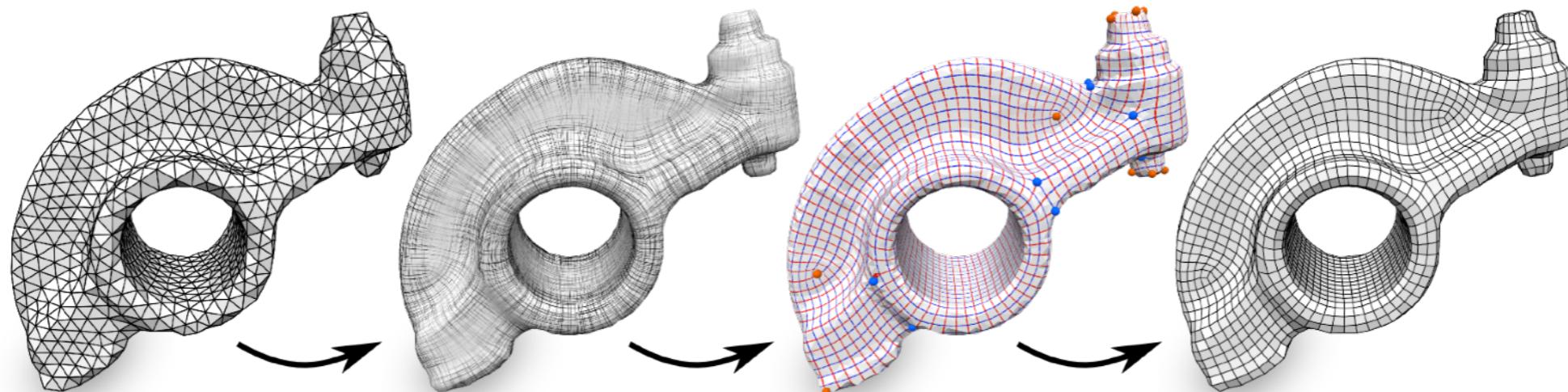
[Martinez Esturo *et al.* 2014]



[Ferreira *et al.* 2013]

APPLICATIONS

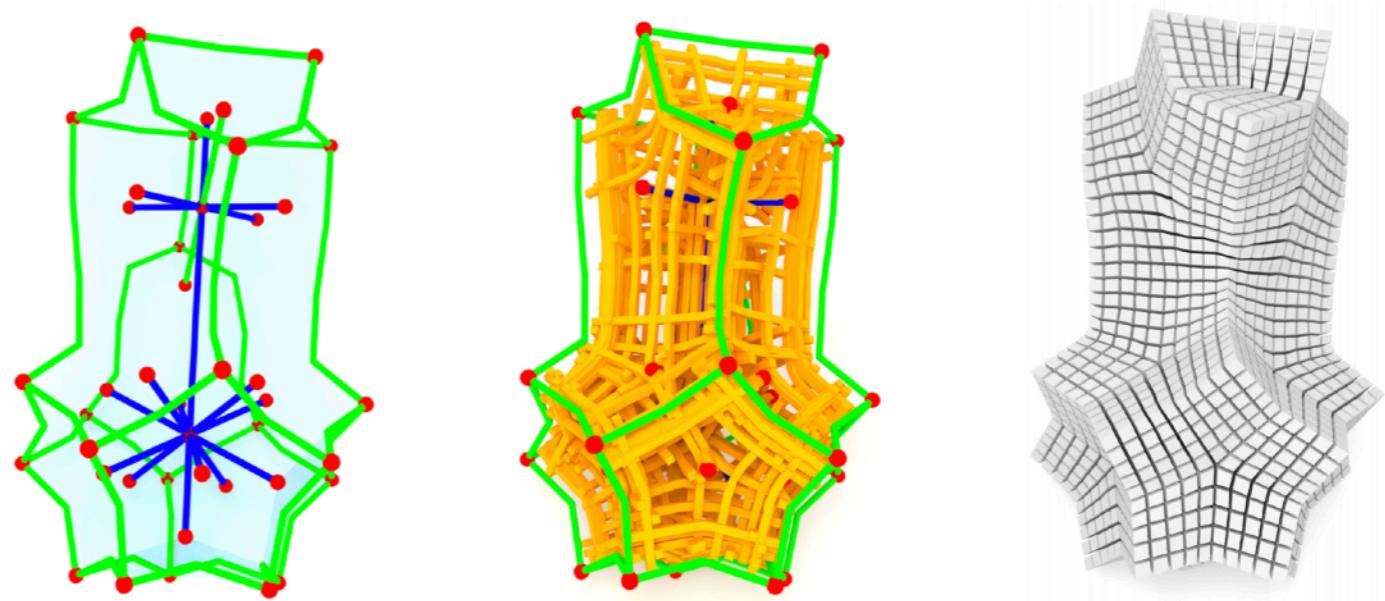
- Meshing



$N=4$ [Ebke et al. 2014]



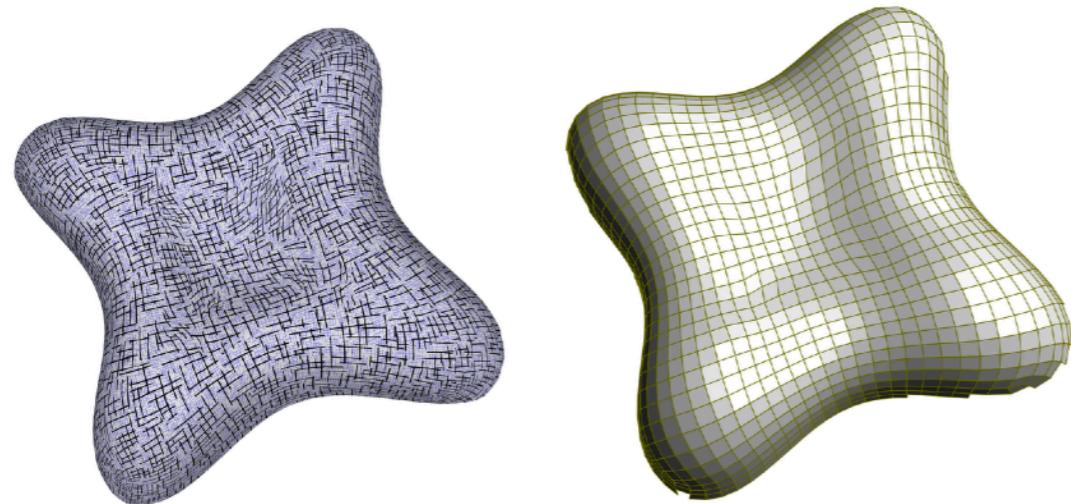
$N=3,6$ [Nieser et al. 2011]



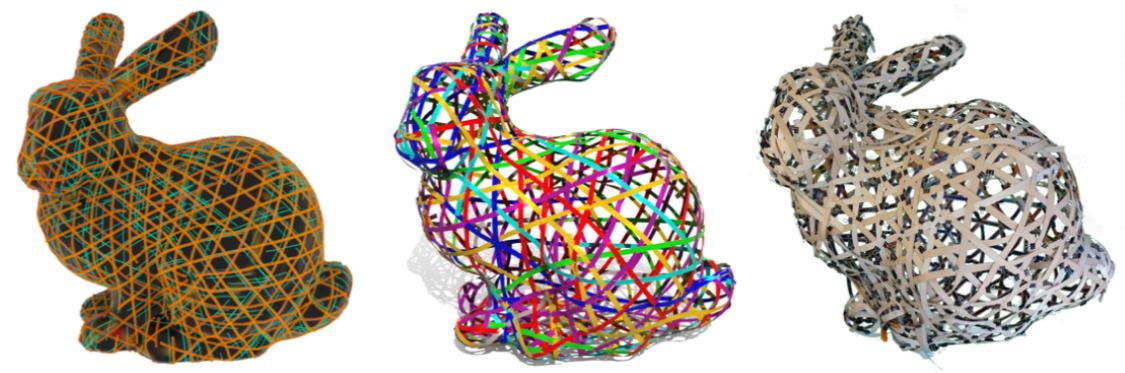
Octahedral [Liu et al. 2018]

APPLICATIONS

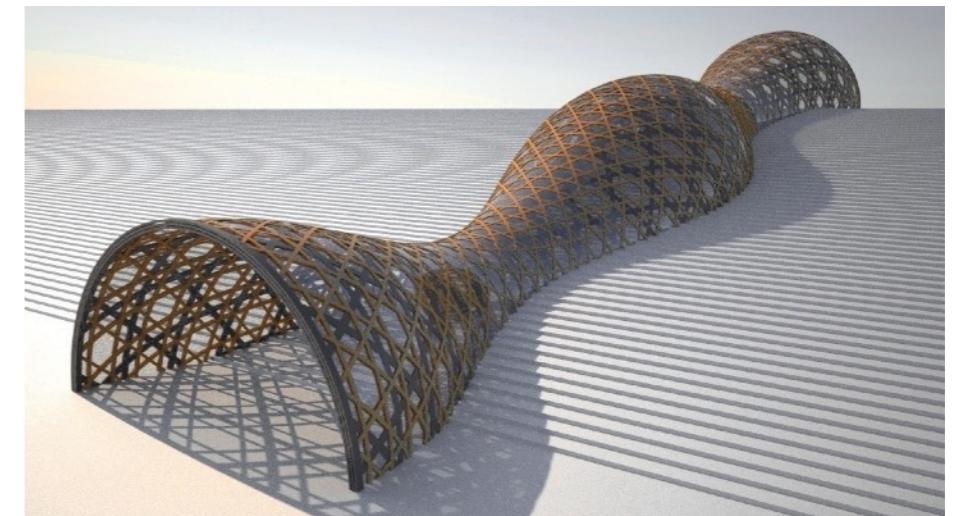
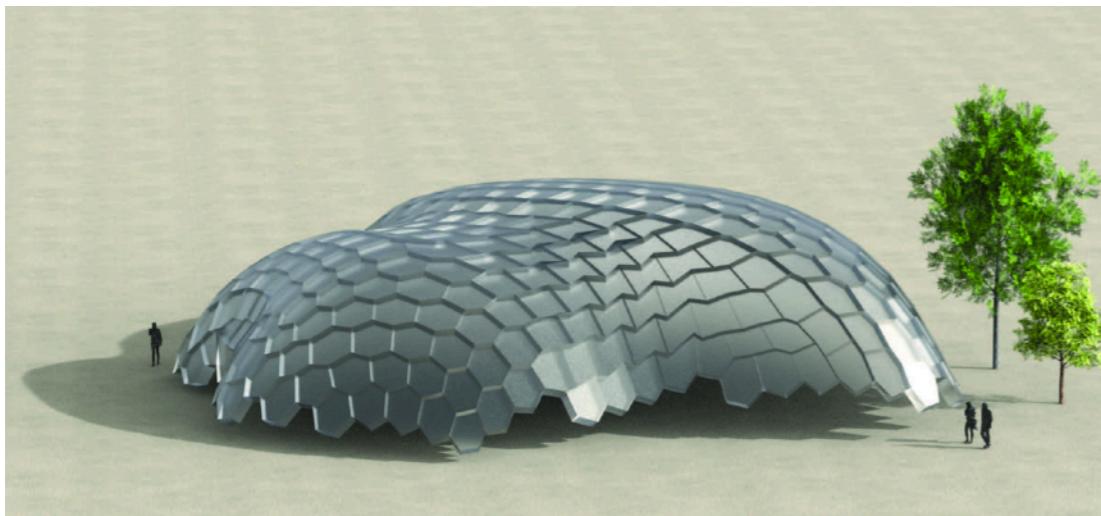
- Meshing with a special function



Conjugacy->planarity [Liu *et al.* 2011, Diamanti *et al.* 2014,2015, Pluta *et al.* 2021]

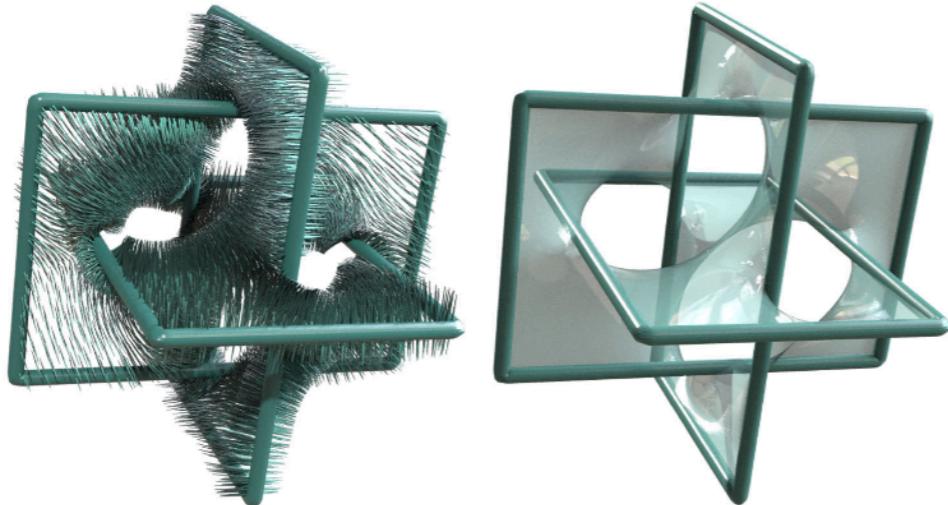


Geodesic [Pottmann et al. 2010, Vekhter *et al.* 2019]

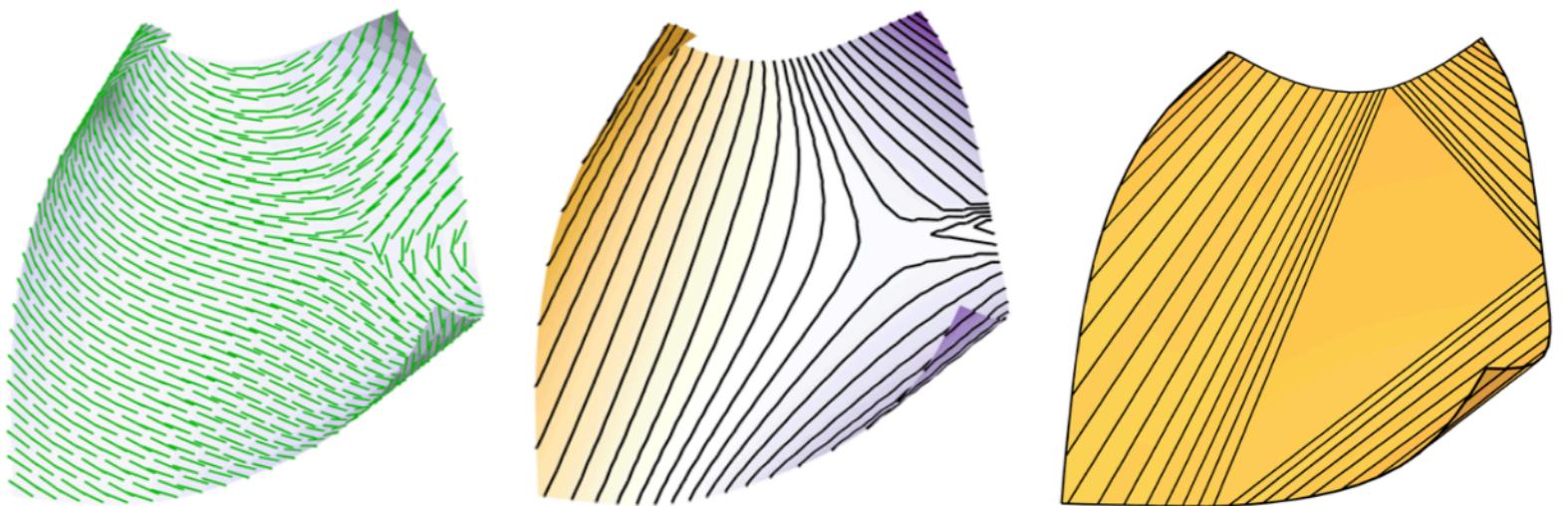


APPLICATIONS

- Meshing with a special function



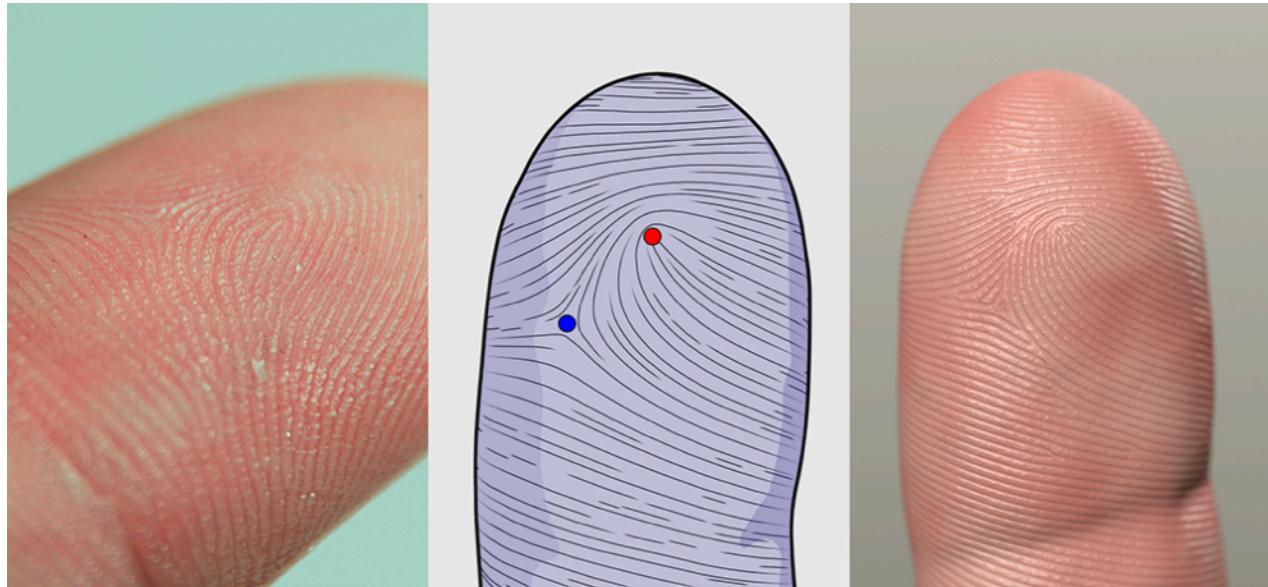
Minimal surfaces [Wang and Chern 2021]



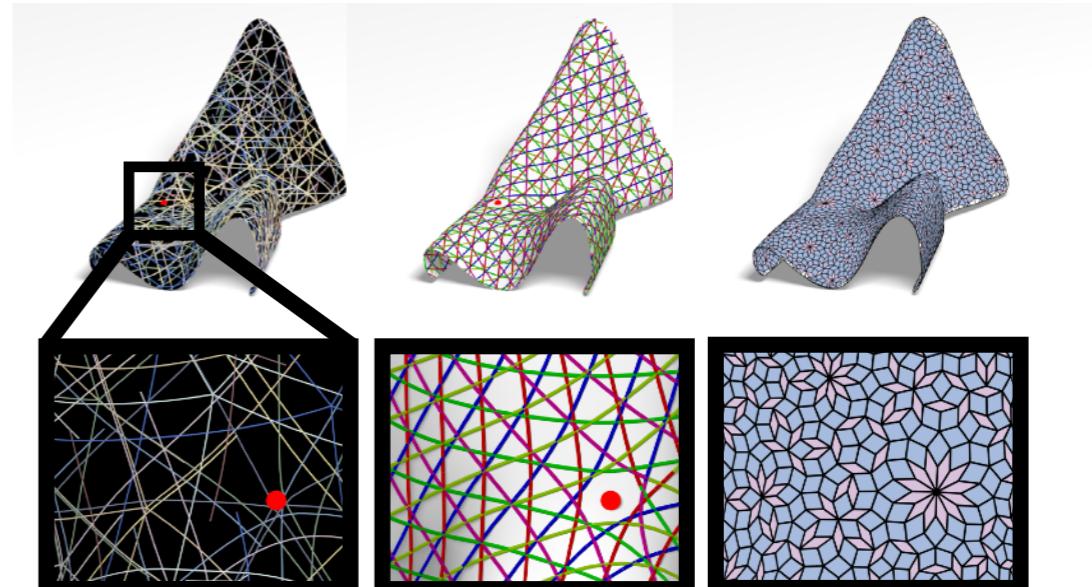
Developability [Verhoeven *et al.* 2021]

APPLICATIONS

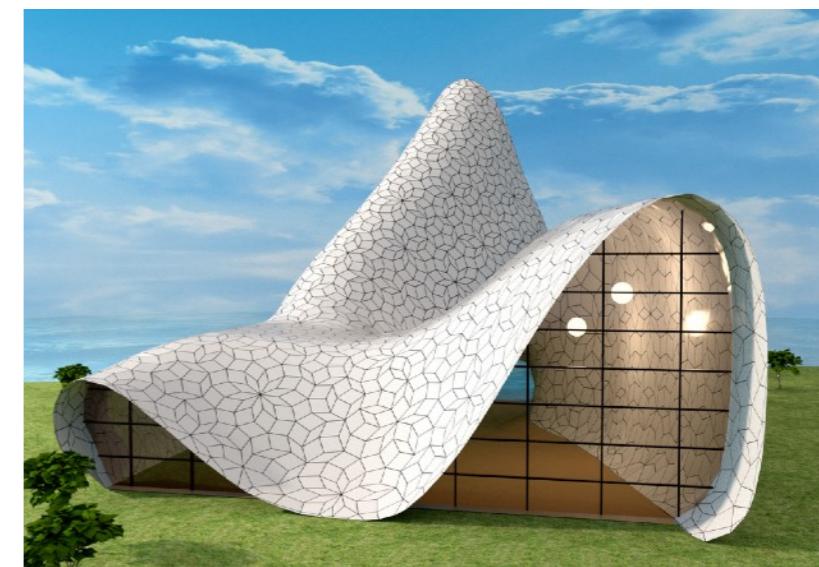
- Alternate symmetries



N=2 Stripe patterns [Knöppel *et al.* 2015]

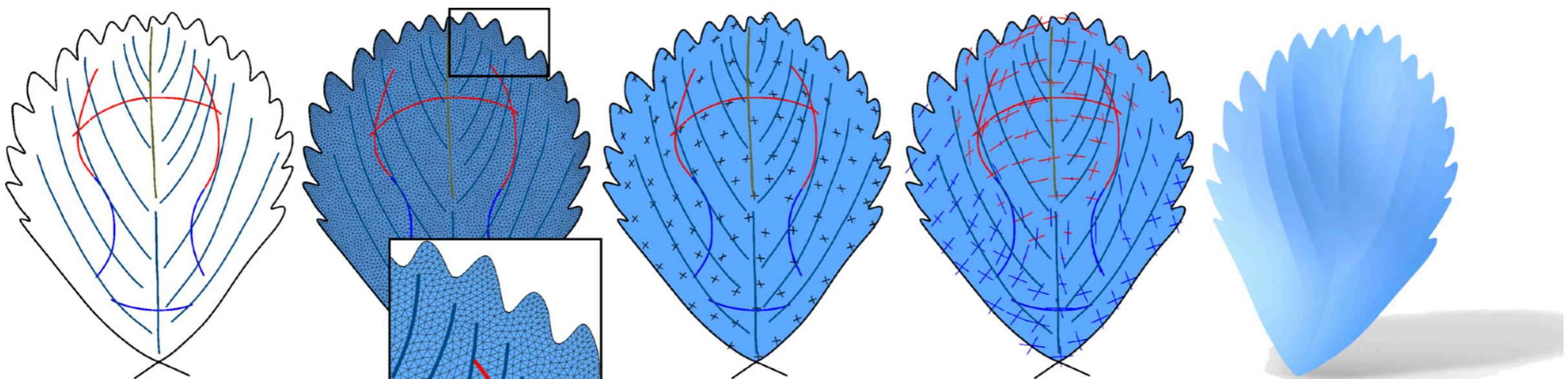


N=5,>7 [Meekes *et al.* 2021]

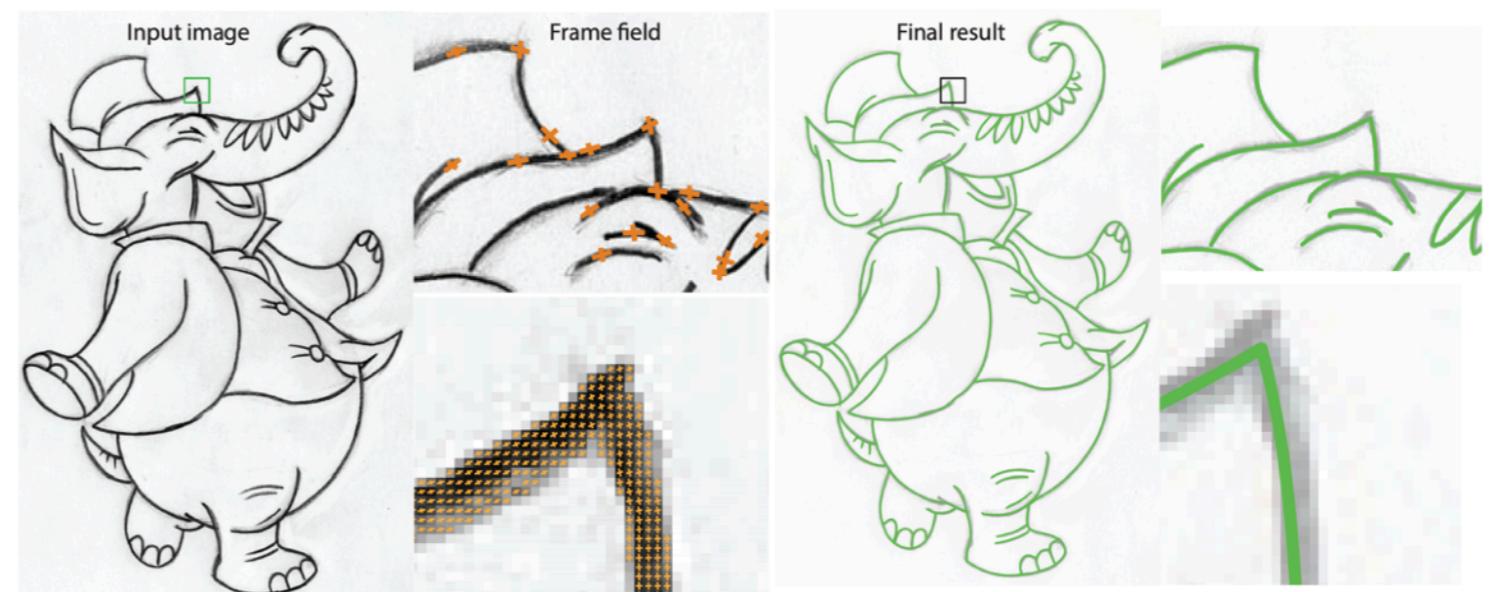


APPLICATIONS

- Shape from illustration and sketching



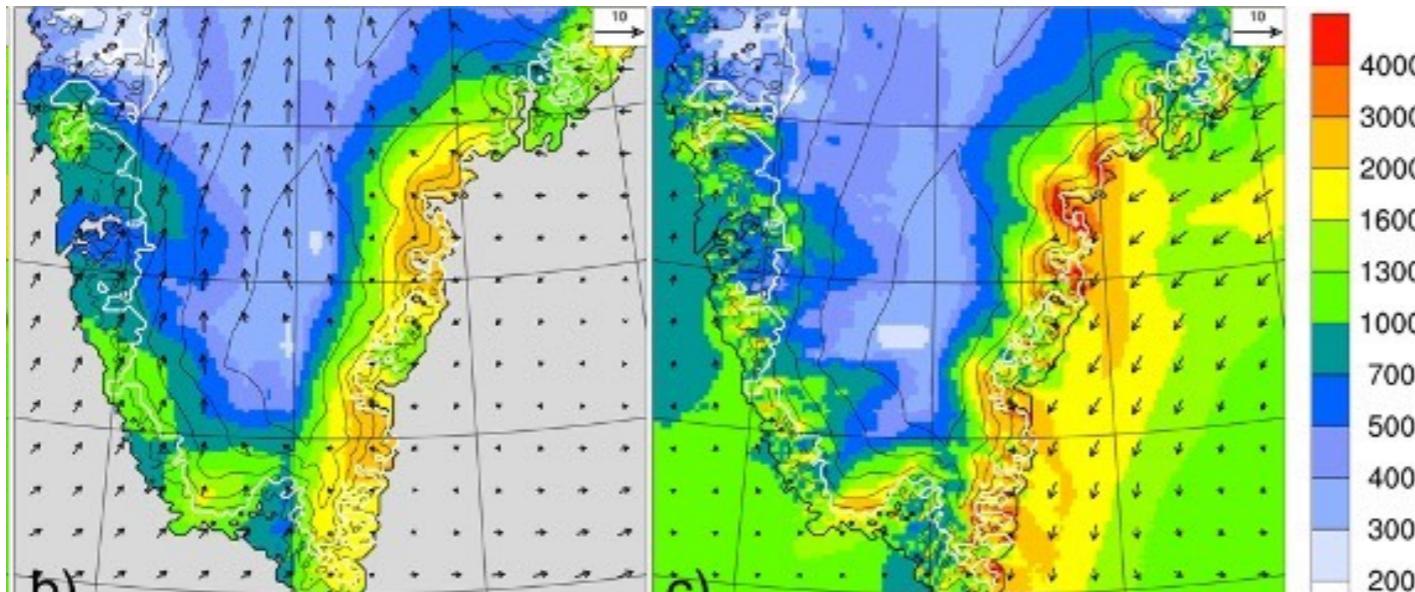
[Li *et al.* 2017]



[Bessmeltsev & Solomon 2000]

APPLICATIONS

- Simulation



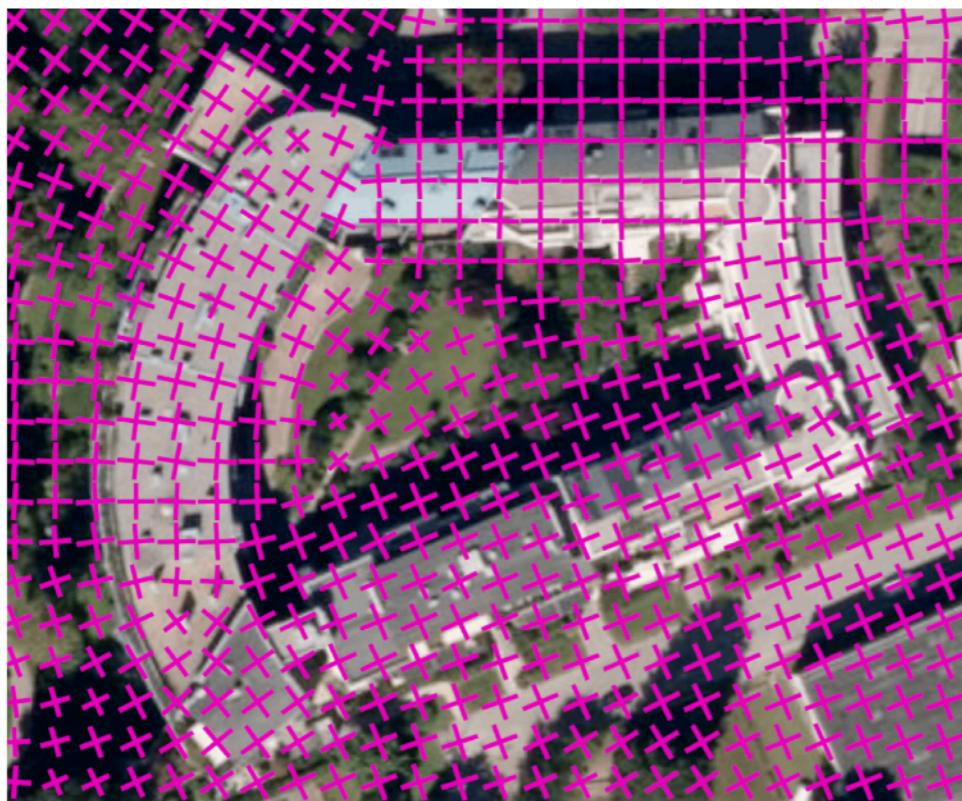
Climate [Noel *et al.* 2016]



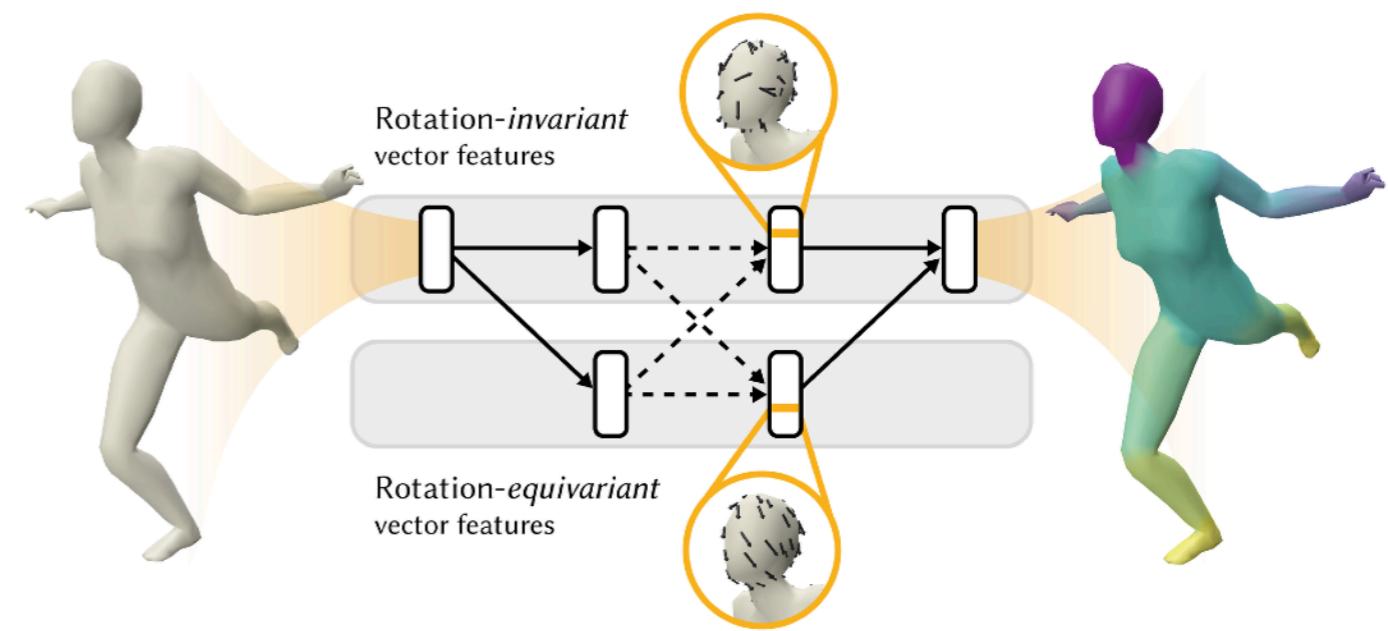
Fluid [Chern *et al.* 2017]

APPLICATIONS

- Learning



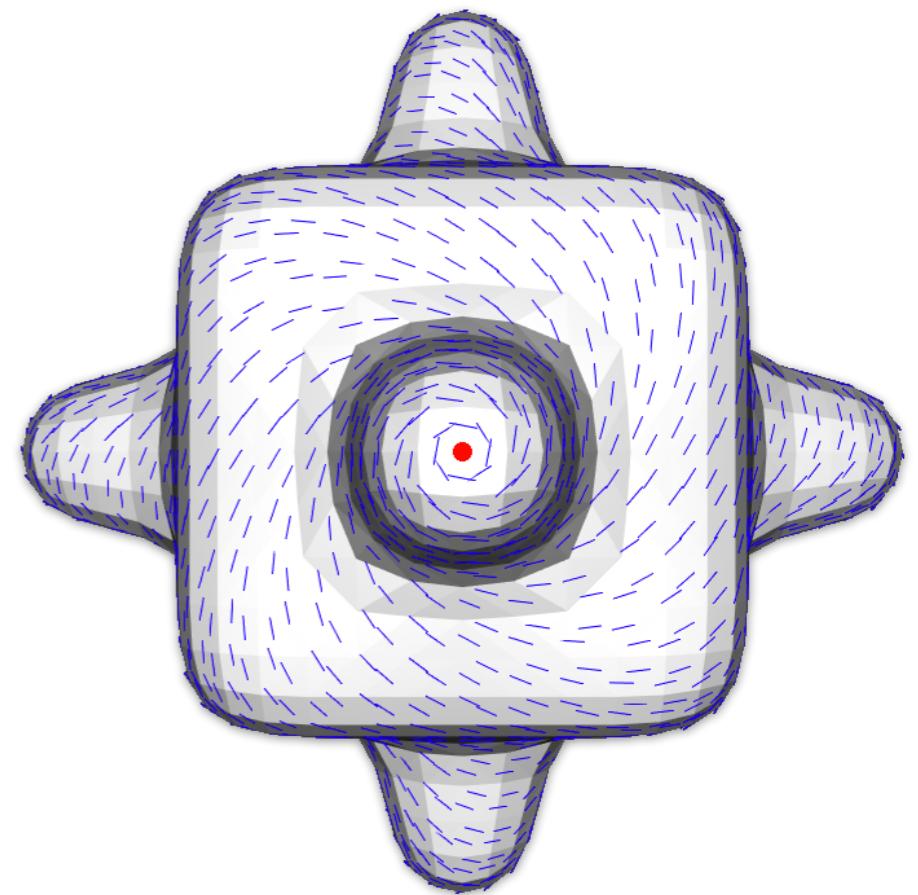
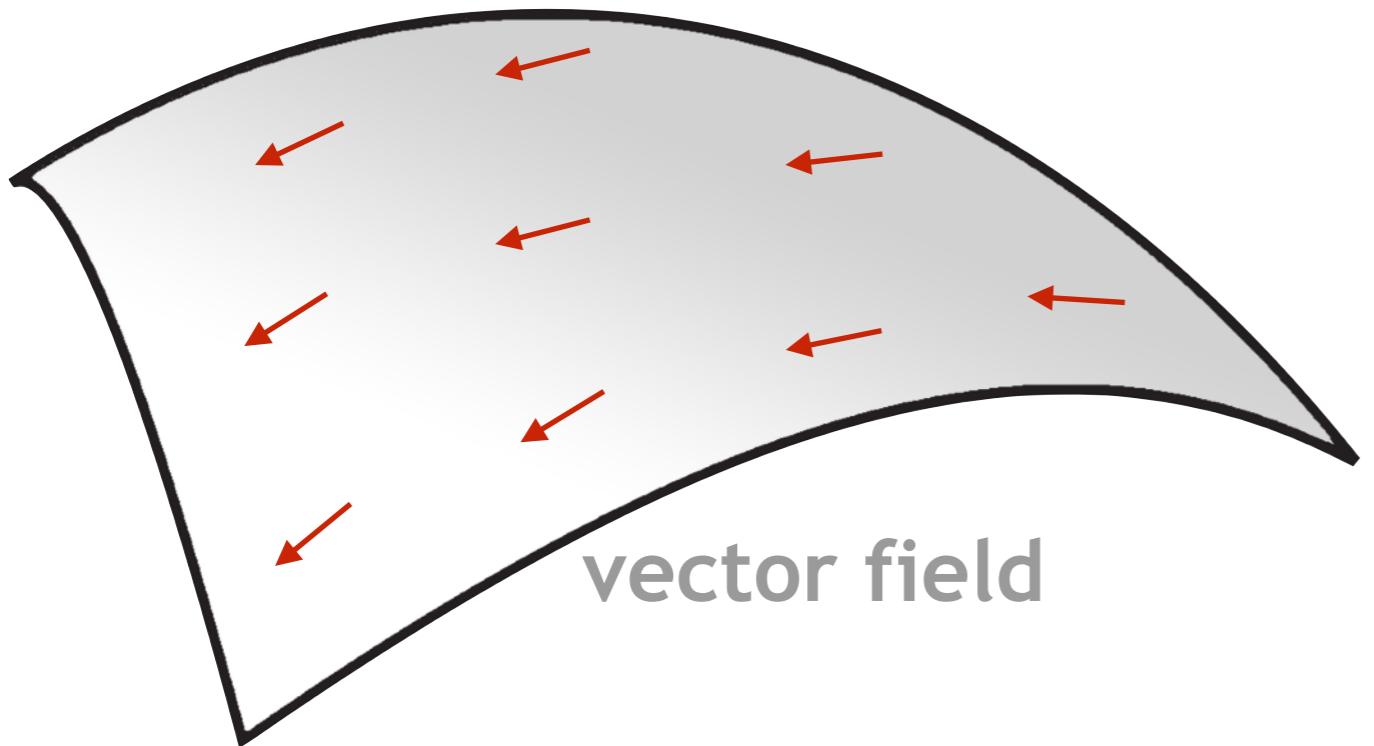
Polygonal building extraction [Girard *et al.* 2021]



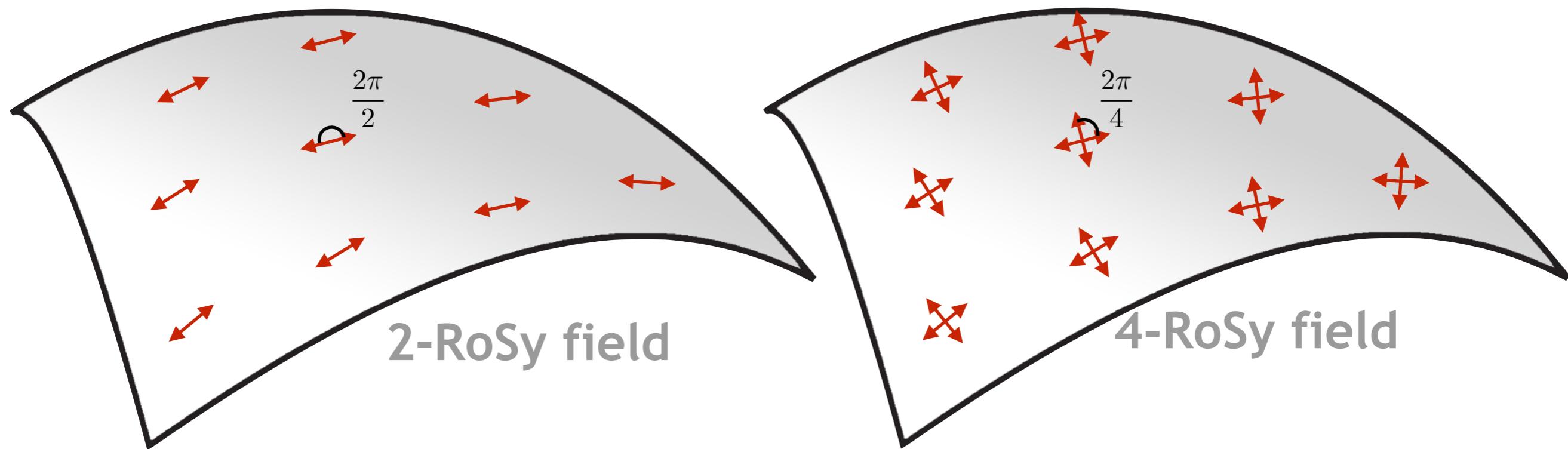
Learning surface features [Wiersma *et al.* 2020]

IN THE CONTINUUM

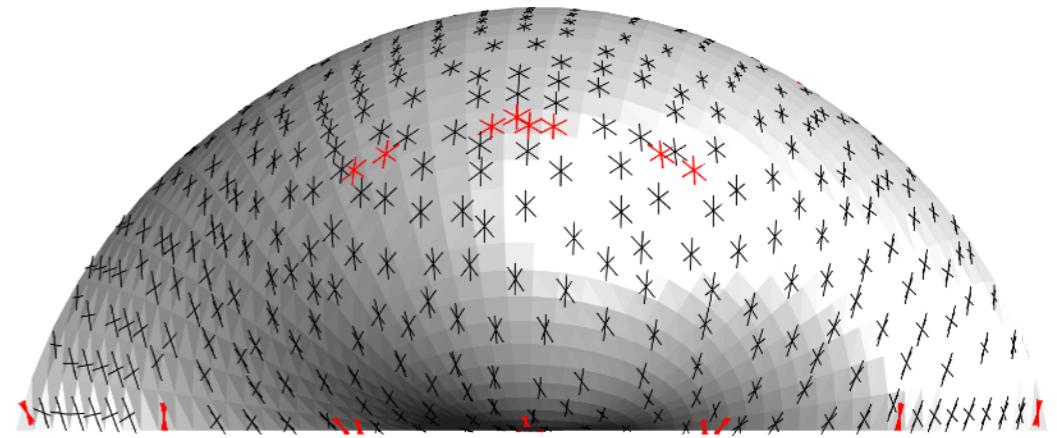
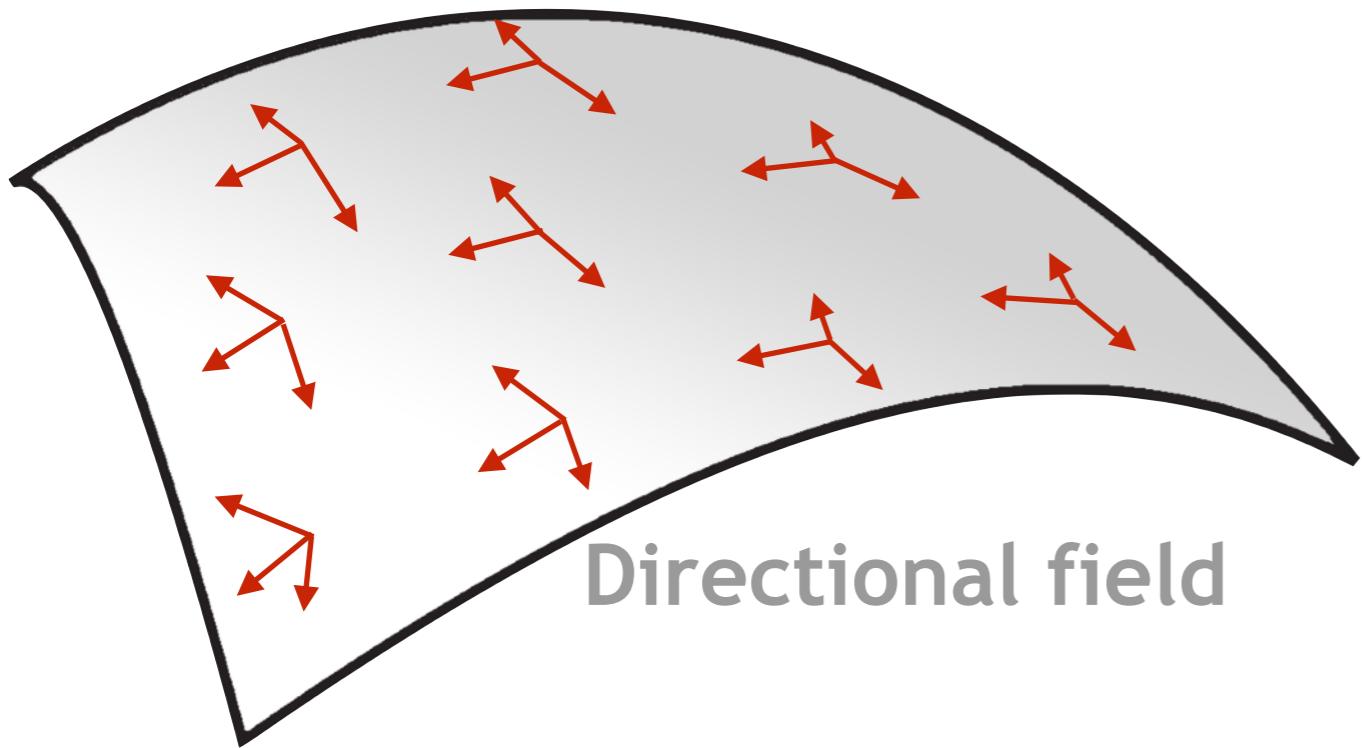
FIELDS ON SURFACES



ROTATIONALLY-SYMMETRIC FIELDS

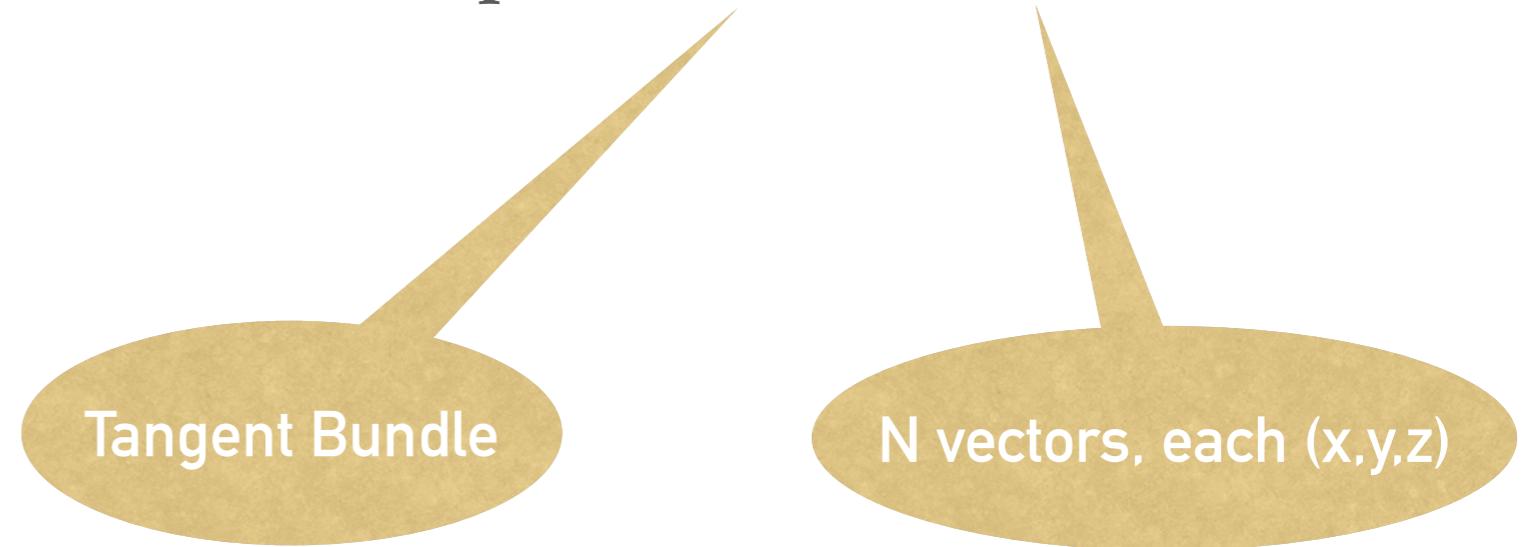


GENERAL, NON-SYMMETRIC FIELDS



FORMALITIES

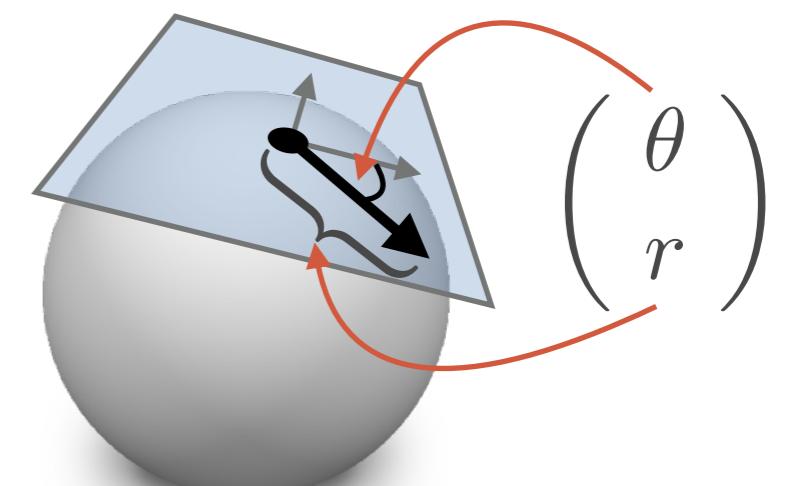
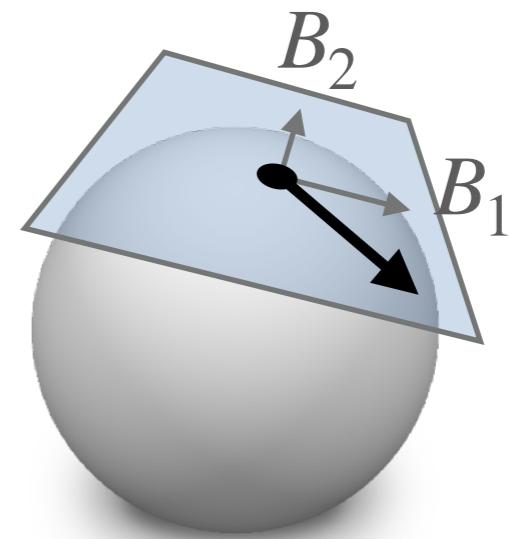
- A tangent N-directional field is a map $V: T\Omega \rightarrow \mathbb{R}^{N \times 3}$



- Tangent $\Rightarrow N \times 2$ d.o.f. in each point, where
$$V(p) = \{v_i(p) \mid \forall i, v_i \perp N(p)\}$$

LOCAL COORDINATES

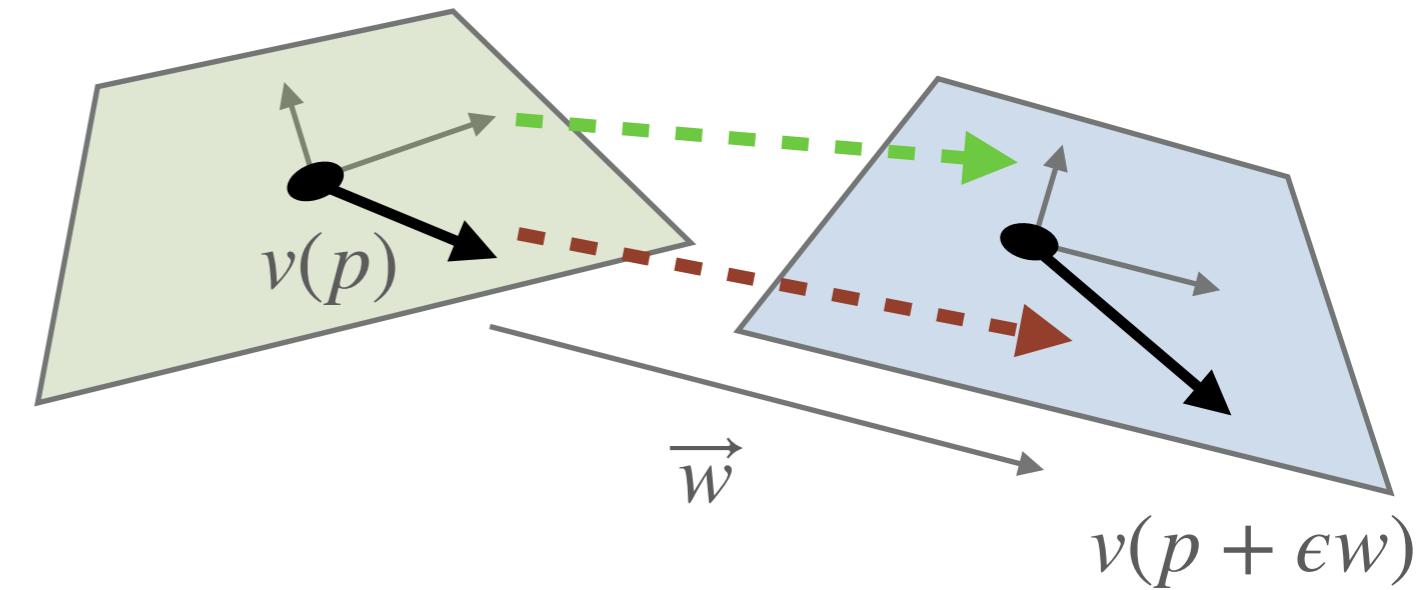
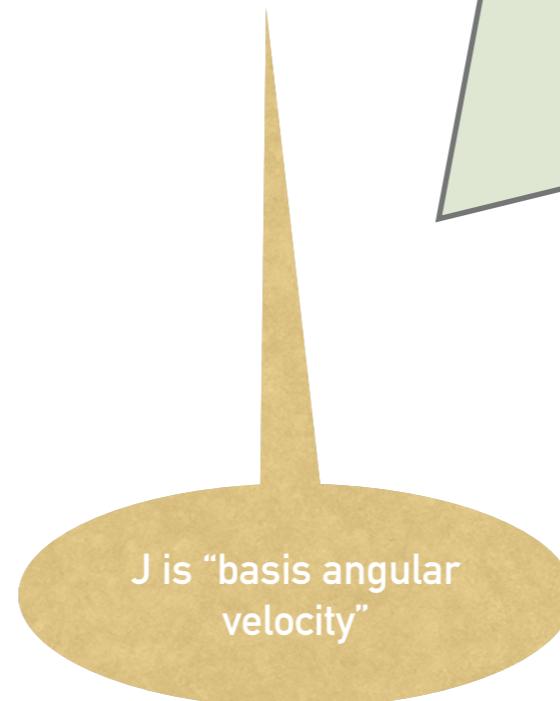
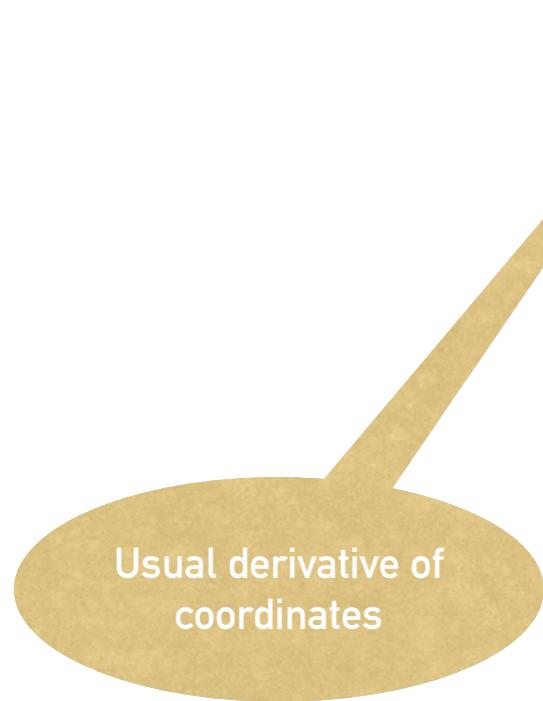
- Admit basis vectors (fields on their own!): $B_1(p), B_2(p)$.
- Then field can be represented in its true d.o.f. in *Cartesian coordinates* $v_i = (x, y)$.
- Alternatives:
 - complex representation: $z = x + iy$
 - polar representation



CONNECTION AND PARALLEL TRANSPORT

- How to differentiate/compare a vector in two points on a surface?
 - Take into account the change of basis
- Covariant derivative (assuming affine metric):

$$\nabla_w(v) = \langle \nabla v, w \rangle + Jv$$



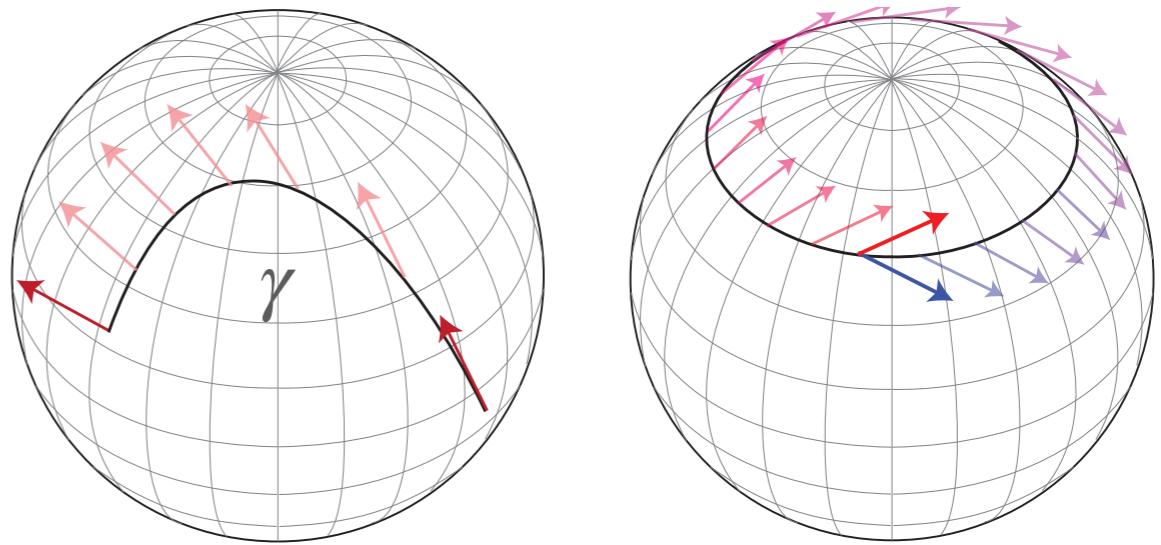
Usual derivative of
coordinates

J is “basis angular
velocity”

CONNECTION AND PARALLEL TRANSPORT

- Parallel transport: zero cov. derivative along a path:

$$\nabla_{\dot{\gamma}} v = 0$$

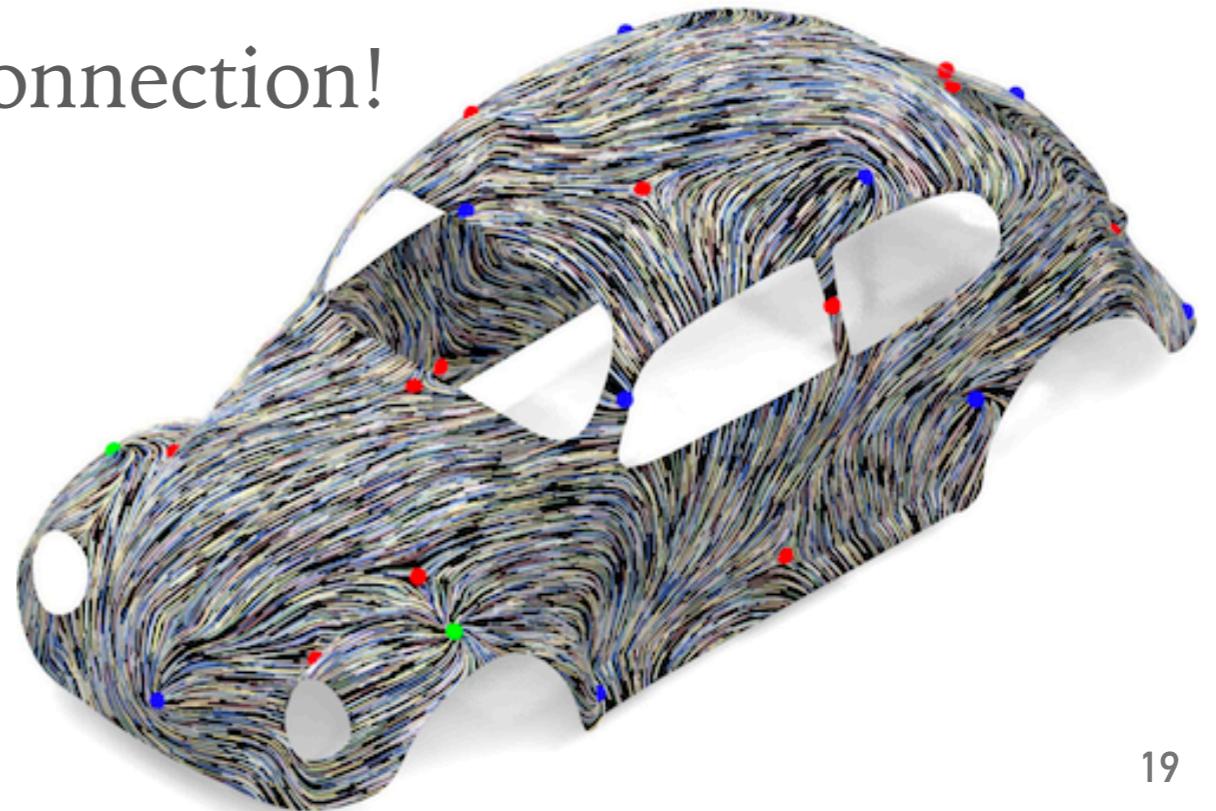


- Holonomy: transport is non-trivial with Gaussian curvature.
- If path is simply connected, then:

$$d\theta = \iint_{S(\gamma)} K dS \pmod{2\pi}$$

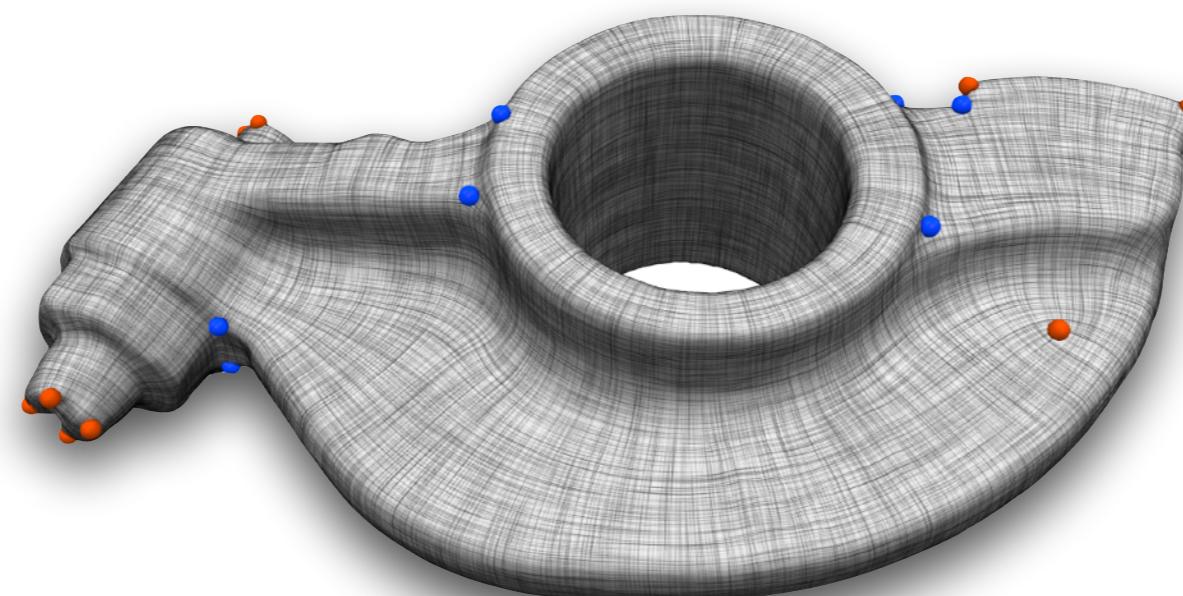
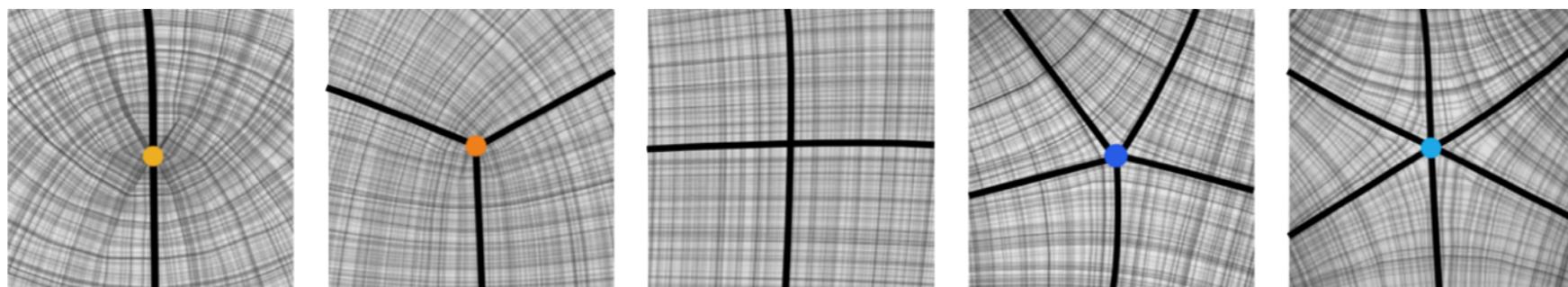
SINGULARITIES

- Index I_p of field $v(p)$: #rotations of field around an infinitesimal cycle
 - Always integer!
- Does not distinguish between source, sink, or vortex.
- Poincare-Hopf Theorem: $\sum I_p = 2\pi\chi$
- Vector field as a flat* (trivial) connection!
 - *with cone points.



N-FIELD SINGULARITIES

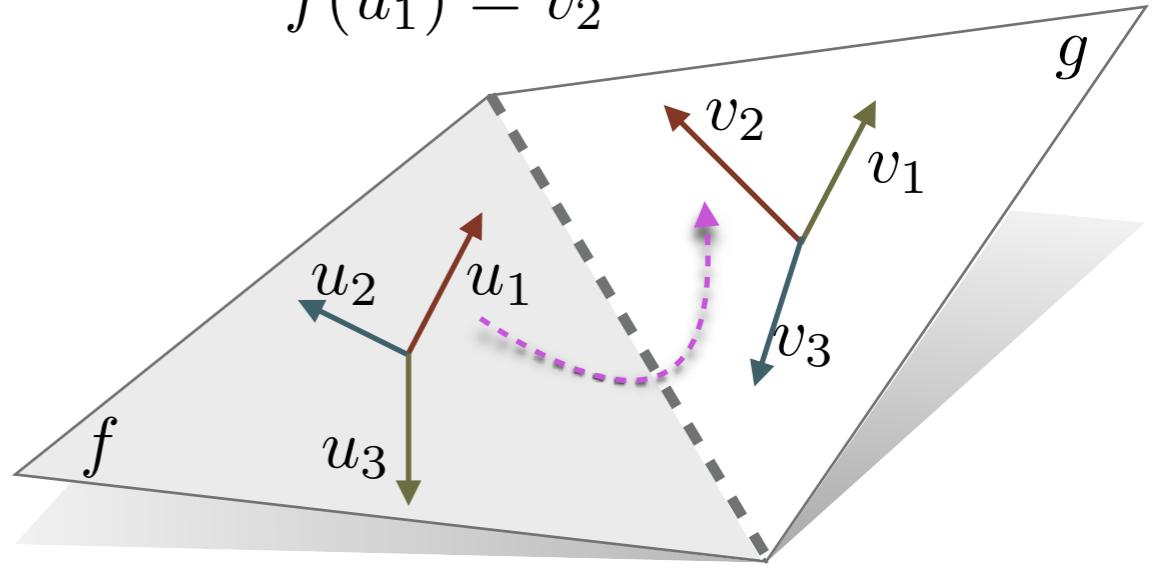
- The same definition, but index can be in multiples of $\frac{1}{N}$.
- Interpretation: field “returns to itself” but with a different *matching* of vectors.



COVECTORS (1-FORMS)

- A linear functional $\eta : T\Omega \rightarrow \mathbb{R}$.
 - Taking vectors into scalar.
 - In coordinates: $\eta_x dx + \eta_y dy$
- Given a metric, we can dualize to the vector (η_x, η_y) so that the map is implemented as:
$$\eta(v \in T_p\Omega) = \eta_x v_x + \eta_y v_y$$
- Differential: $df(v) = \langle \nabla f, v \rangle$
 - Identifying a field with an operator on functions!

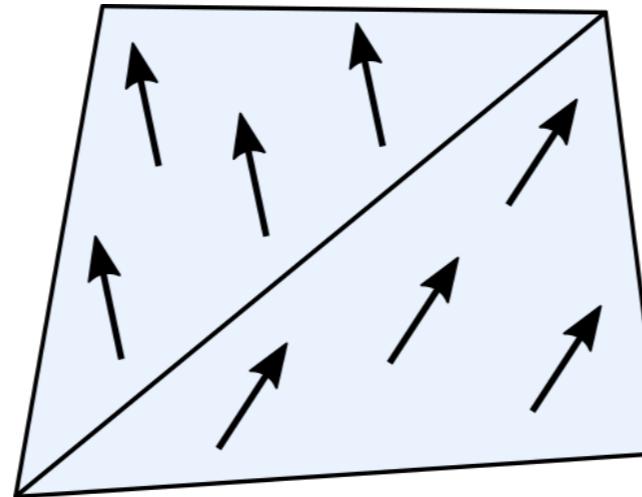
$$f(u_1) = v_2$$



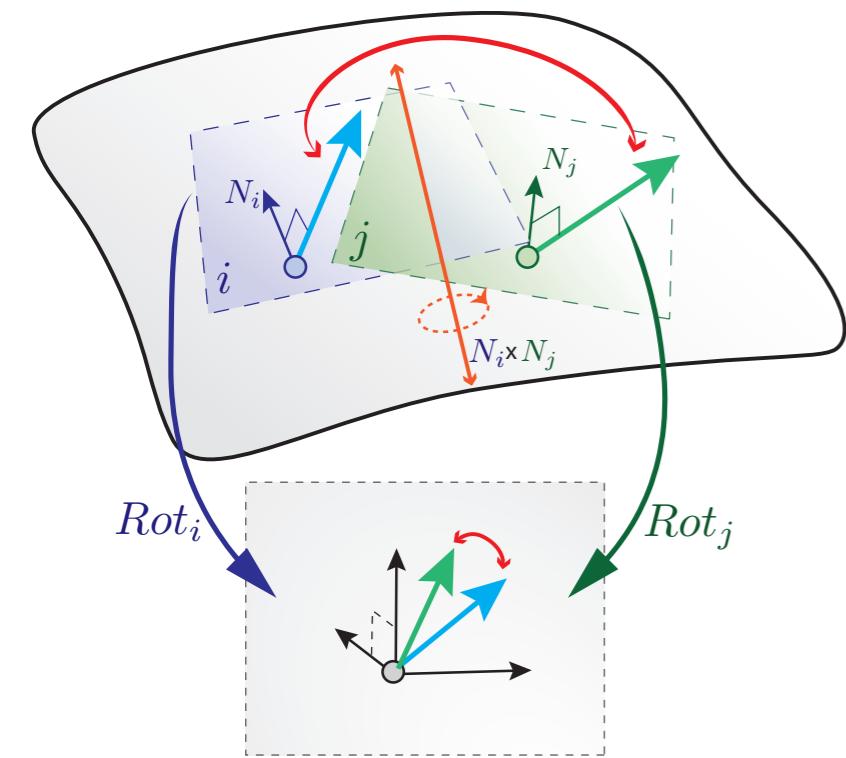
DISCRETIZATION

CHALLENGES IN THE DISCRETE SETTING

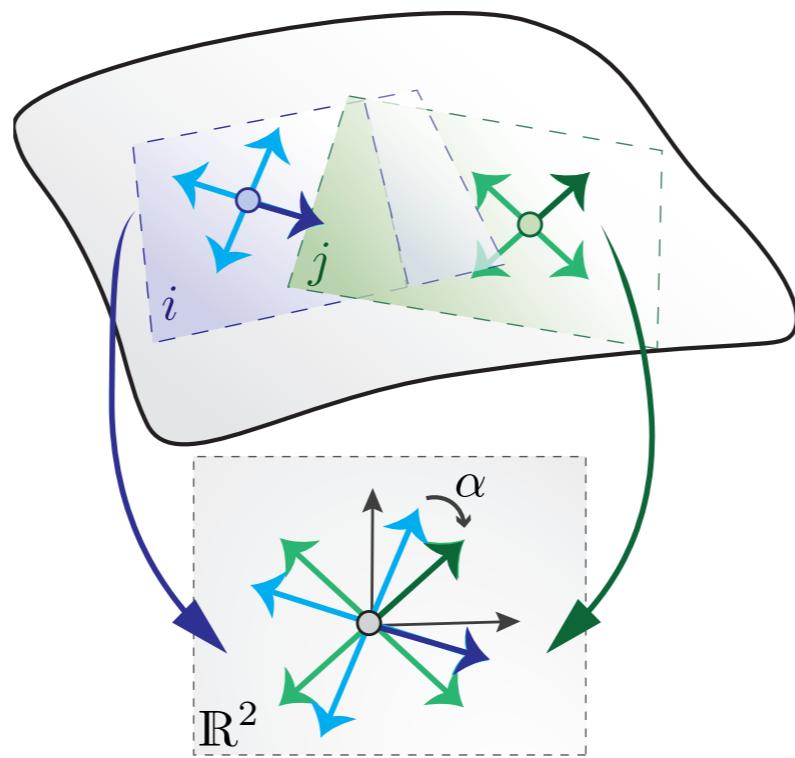
Discontinuity



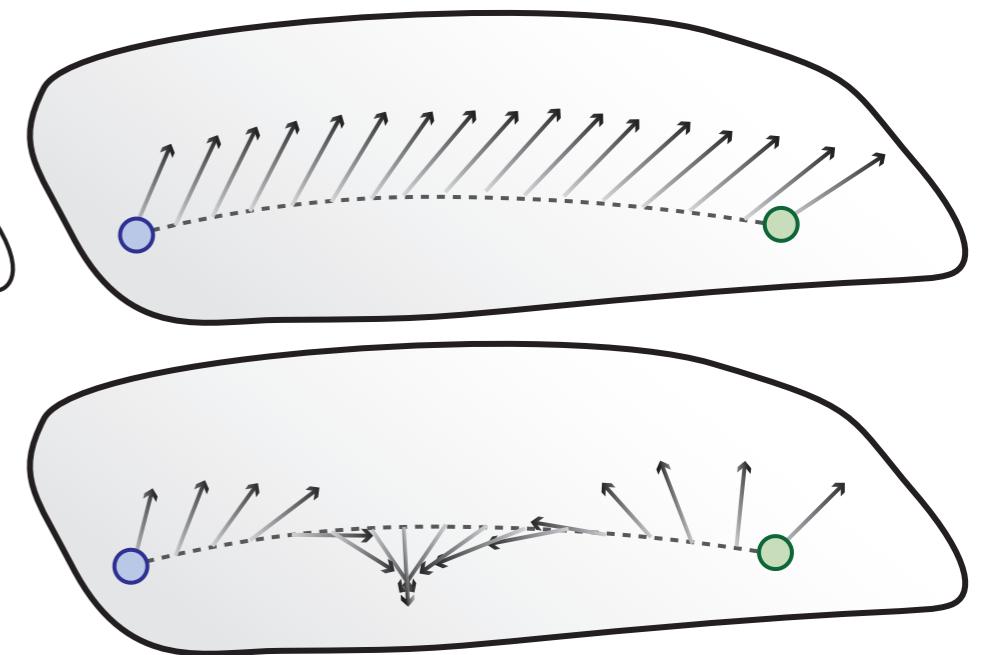
Connection



Matching

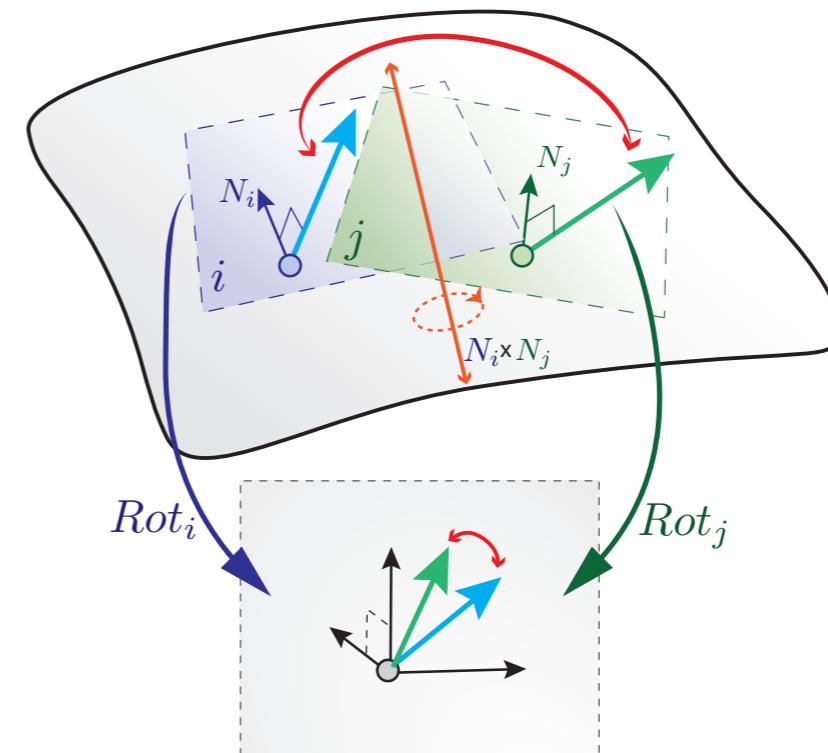


Interpolation



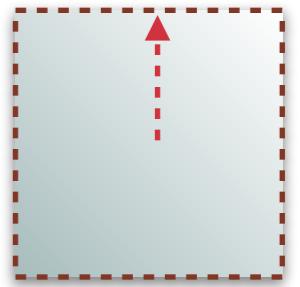
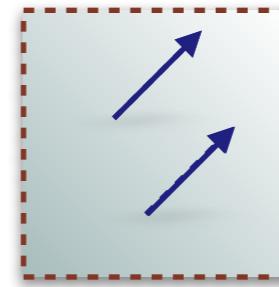
DISCRETE CONNECTION

- Bijective linear map between adjacent tangent spaces.
- Popular choice: flattening + single axis system.
- Alternatively: align the representation of a mutual vector.



DISCRETE TOPOLOGY: ROTATION

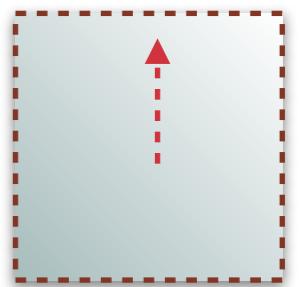
- What happens in between?
- Valid **rotation** choices:



$$\delta_{ij} = \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

Principal Rotation *Period Jump*

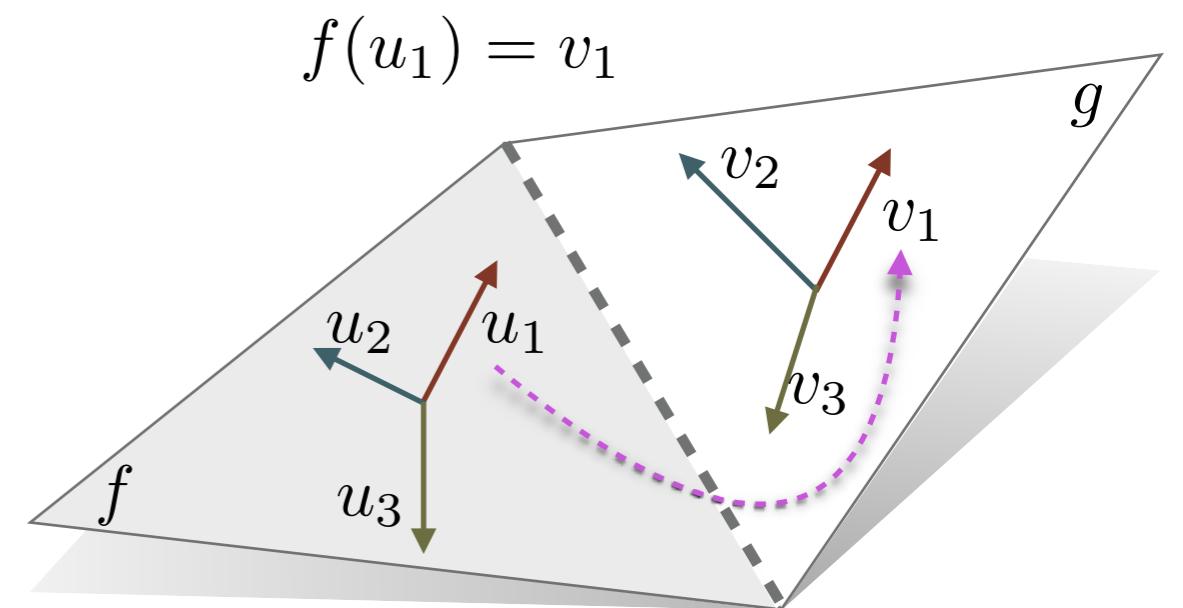
[Li *et al.* 2006]



- Implicit/Cartesian: can only assume principal.
- Explicit/polar/angle-based: period given.

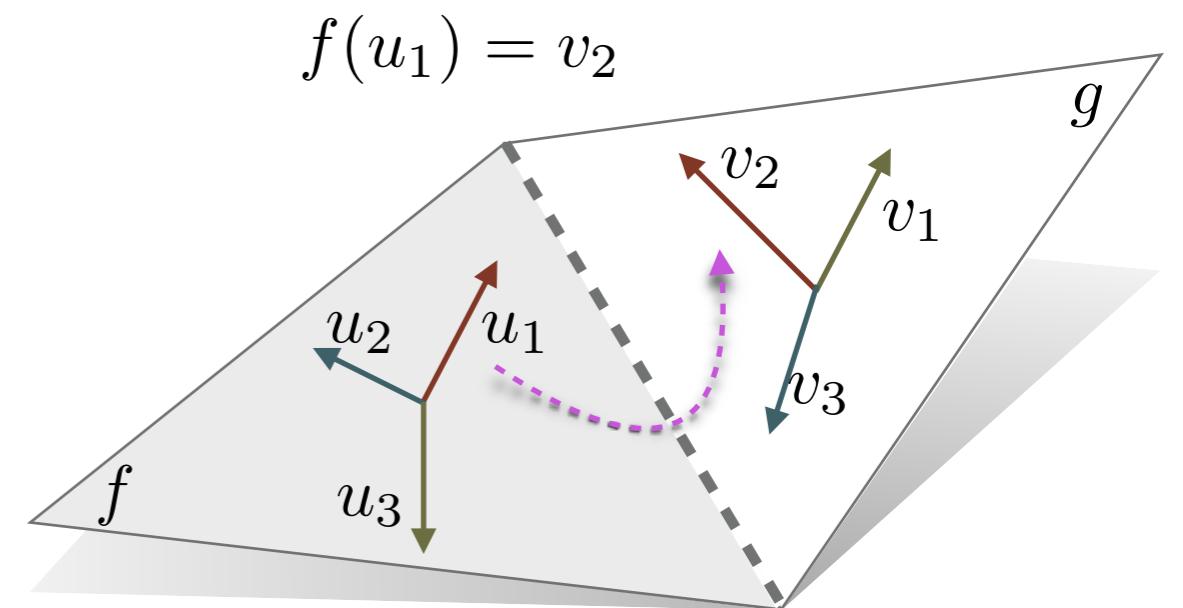
MATCHING

- Which direction to which other?
- Reduction: order-preserving. [Diamanti *et al.* 2014]
- N -directional: N possible choices.
 - How best to choose?



MATCHING

- Which direction to which other?
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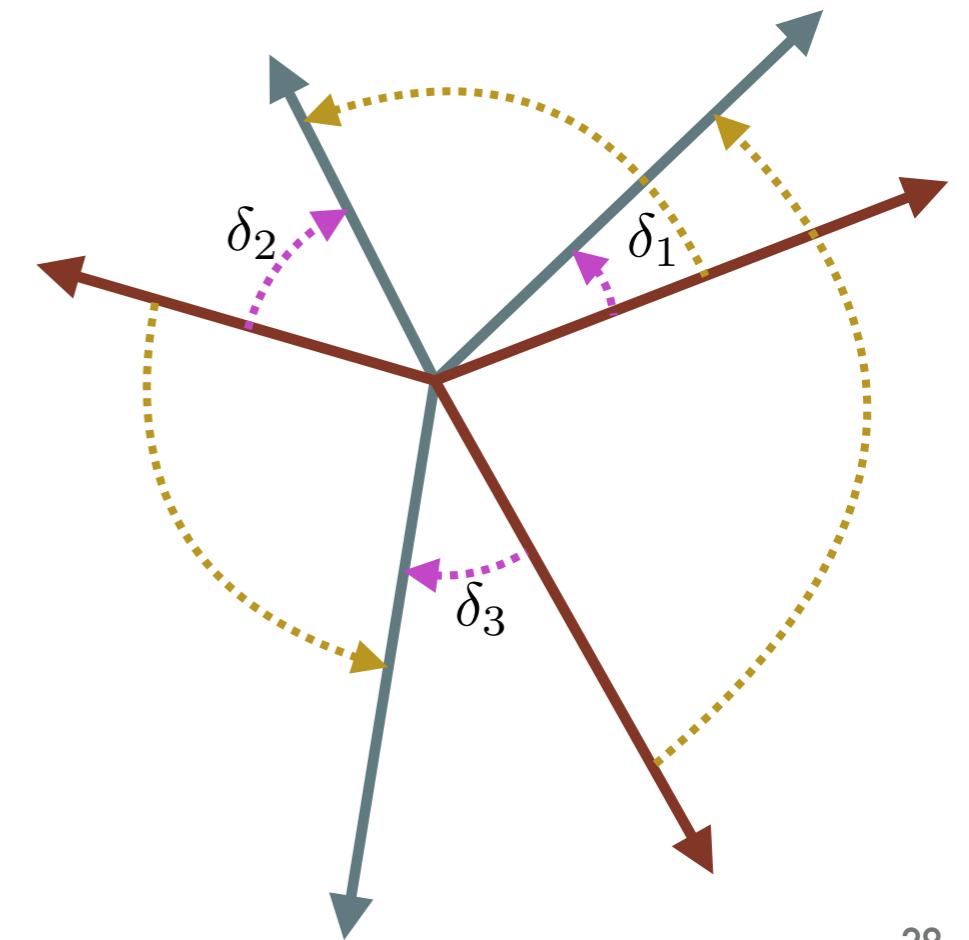


EFFORT

- Sum of matching rotations.
 - **Intuition:** generalize “closest angle” to “minimum effort”.
- All order-preserving matchings differ by $2\pi k, k \in \mathbb{Z}$
- Principal matching: the matching with $Y = [-\pi, \pi]$

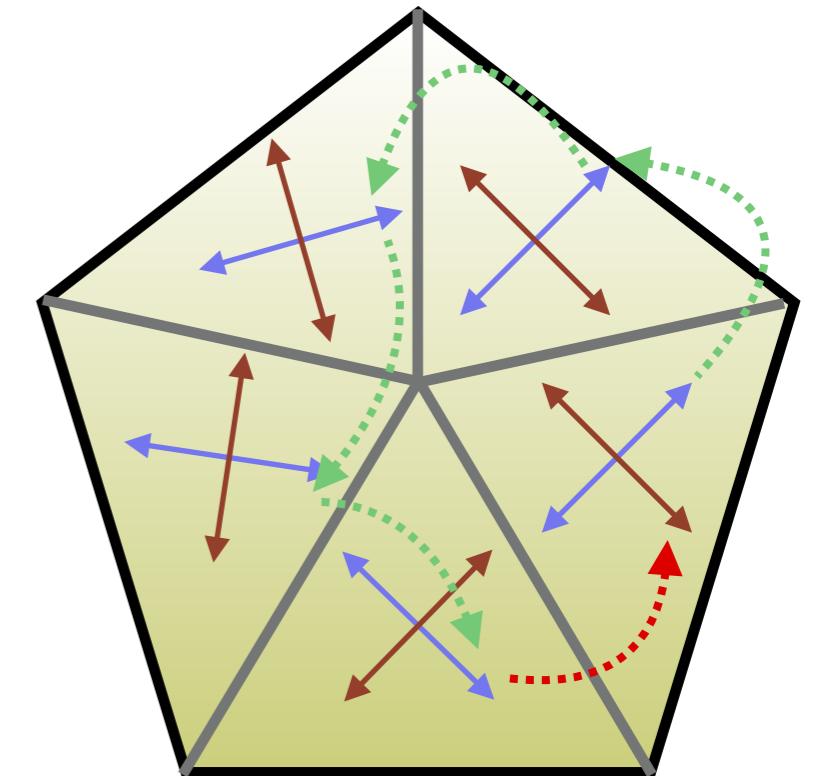
$$Y_1 = \delta_1 + \delta_2 + \delta_3$$

$$Y_2 = \delta_1 + \delta_2 + \delta_3 + 2\pi$$



SINGULARITIES

- Around a matching cycle, a directional set returns to itself.
- Up to a different matching!
- Directional field as **trivial connections**. [Crane *et al.* 2010]
 - Induced Curvature: $\frac{2\pi}{k}$
- Regular cycles: index 0
- Sum of indices: $2 - 2g$

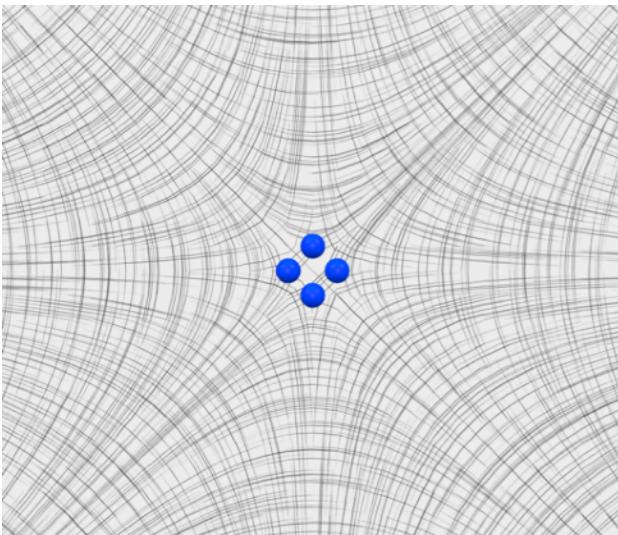


Index of singularity: $\frac{1}{k}$

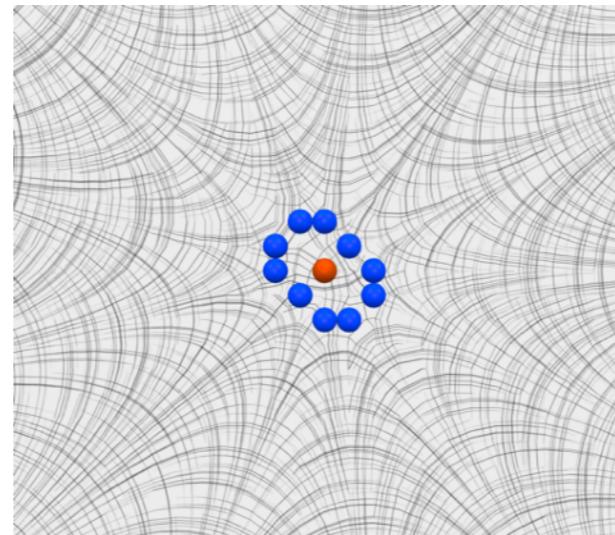
SAMPLING PROBLEM

- **Implicit field:** principal matching assumed.
- Low valence cycles: limited rotation sums.
- Higher order singularities cannot be represented!
- In practice: promoting low-degree singularity cycles (“**singularity party**”).

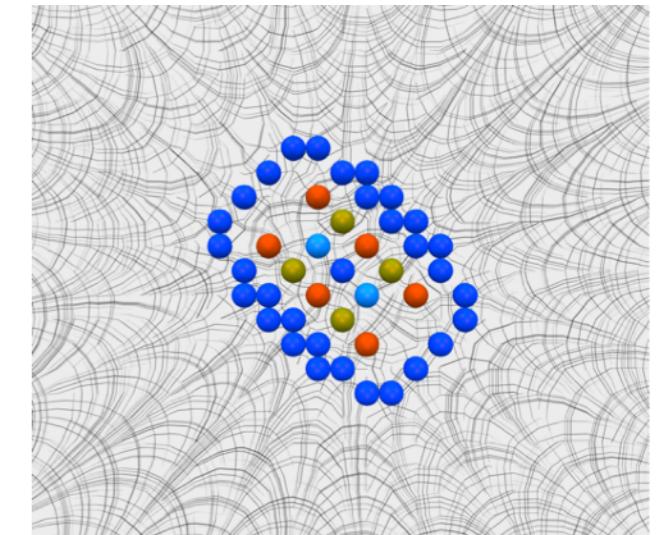
$$-\frac{4}{4}$$



$$-\frac{9}{4}$$

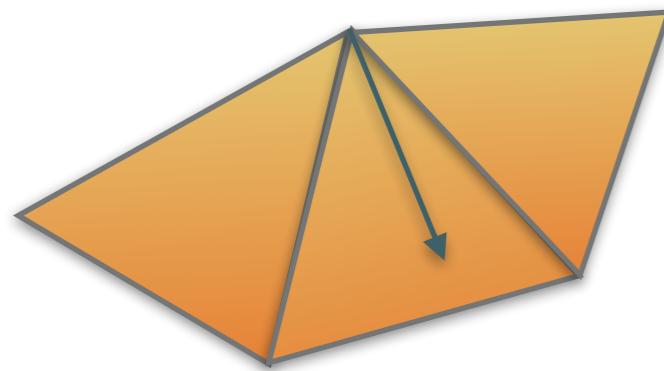


$$-\frac{21}{4}$$

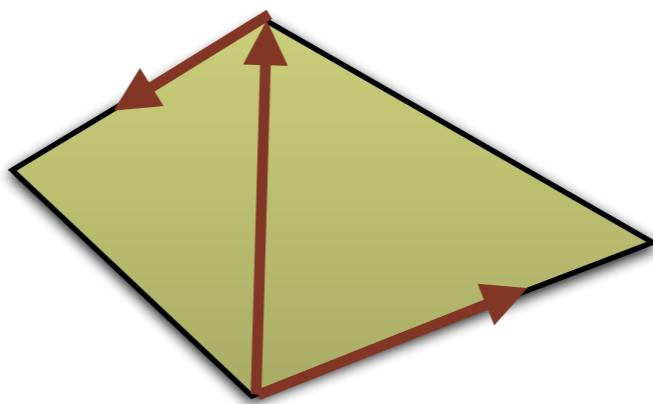


TANGENT SPACES

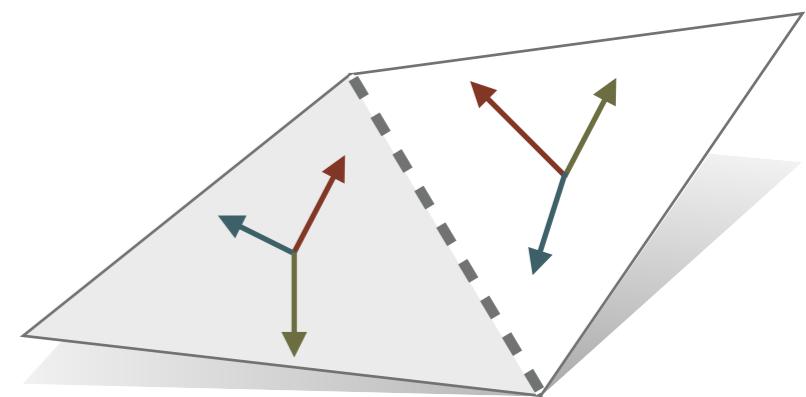
Vertex based



Edge based



Face based



[Polthier and Schmies 98]

[Zhang *et al.* 2006]

[Knöppel *et al.* 2013]

[Desbrun *et al.* 2005]

[Fisher *et al.* 2007]

[Ben-Chen *et al.* 2010]

[Bommes *et al.* 2009]

[Crane *et al.* 2010]

[Diamanti *et al.* 2014]

FACE-BASED FIELDS

- Natural normal => natural discrete tangent spaces.
- Usually piecewise constant.
- Connection: representation of mutual edge.
- Discontinuous at edges and vertices
- Structure-preserving:

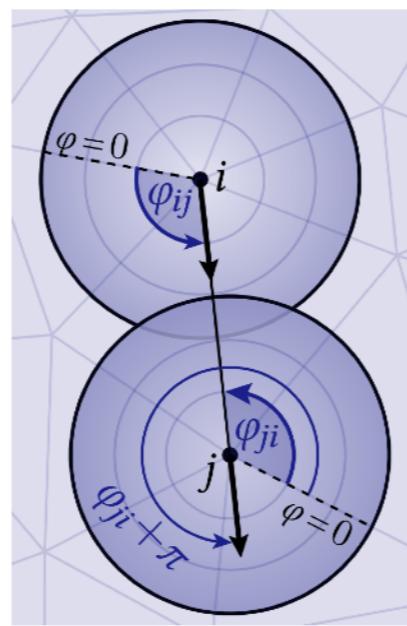
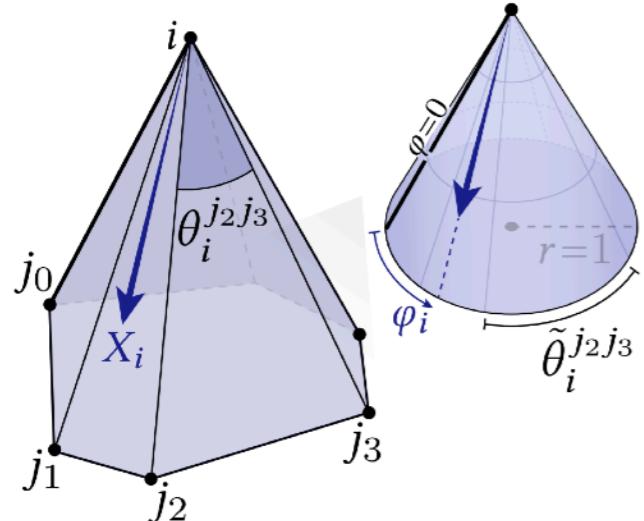
$$v = \nabla f_V + J \nabla g_E + H$$


2 |F| |V| - 1 |E| - 1 2g

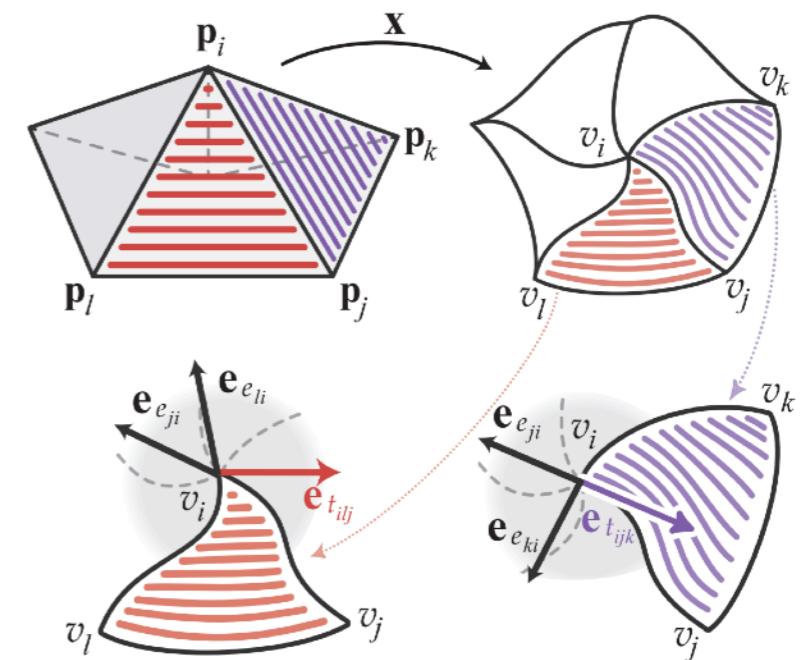


VERTEX-BASED FIELDS

- Common flavor: parameterize vertex tangent space
 - Effectively “pushing” curvature inside faces.
- Connection: through the representation of a mutual edge or by using PL charts.
- Not structure-preserving, but allows for 2nd-order point wise operators.



[Sharp *et al.* 2019]

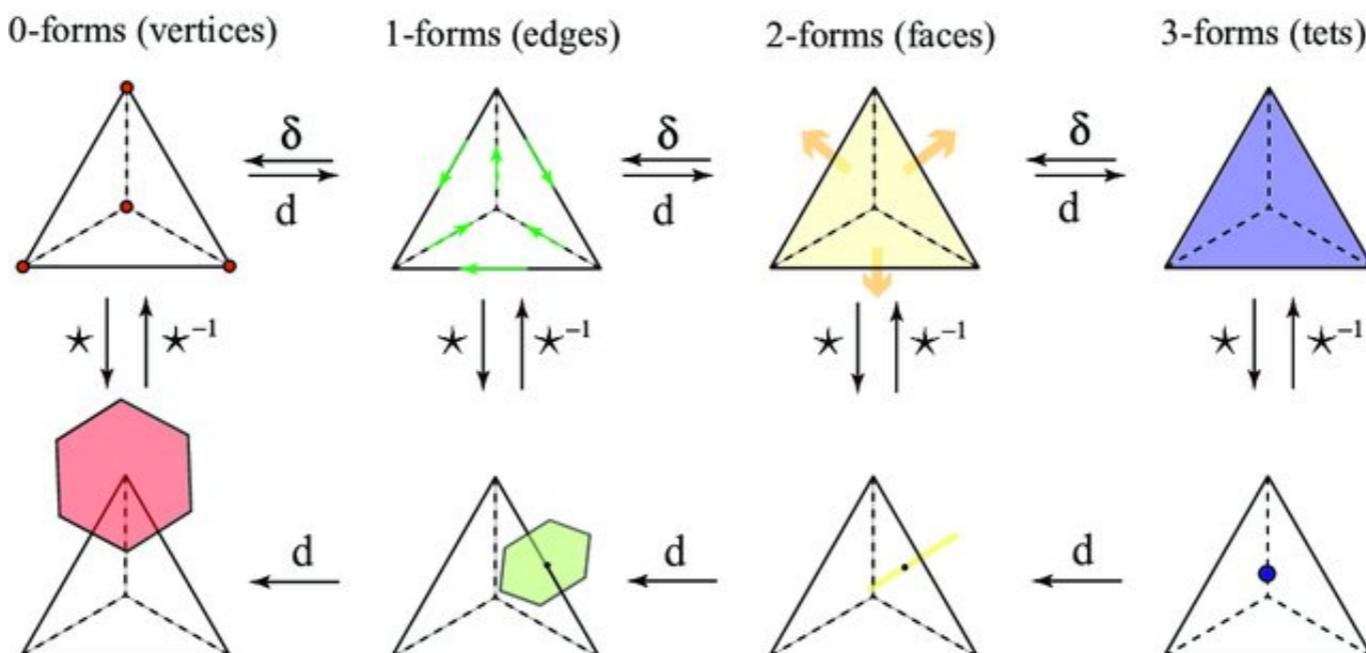


[Liu *et al.* 2016]

EDGE-BASED FIELDS - DIFFERENTIAL 1-FORMS

- Differential forms:

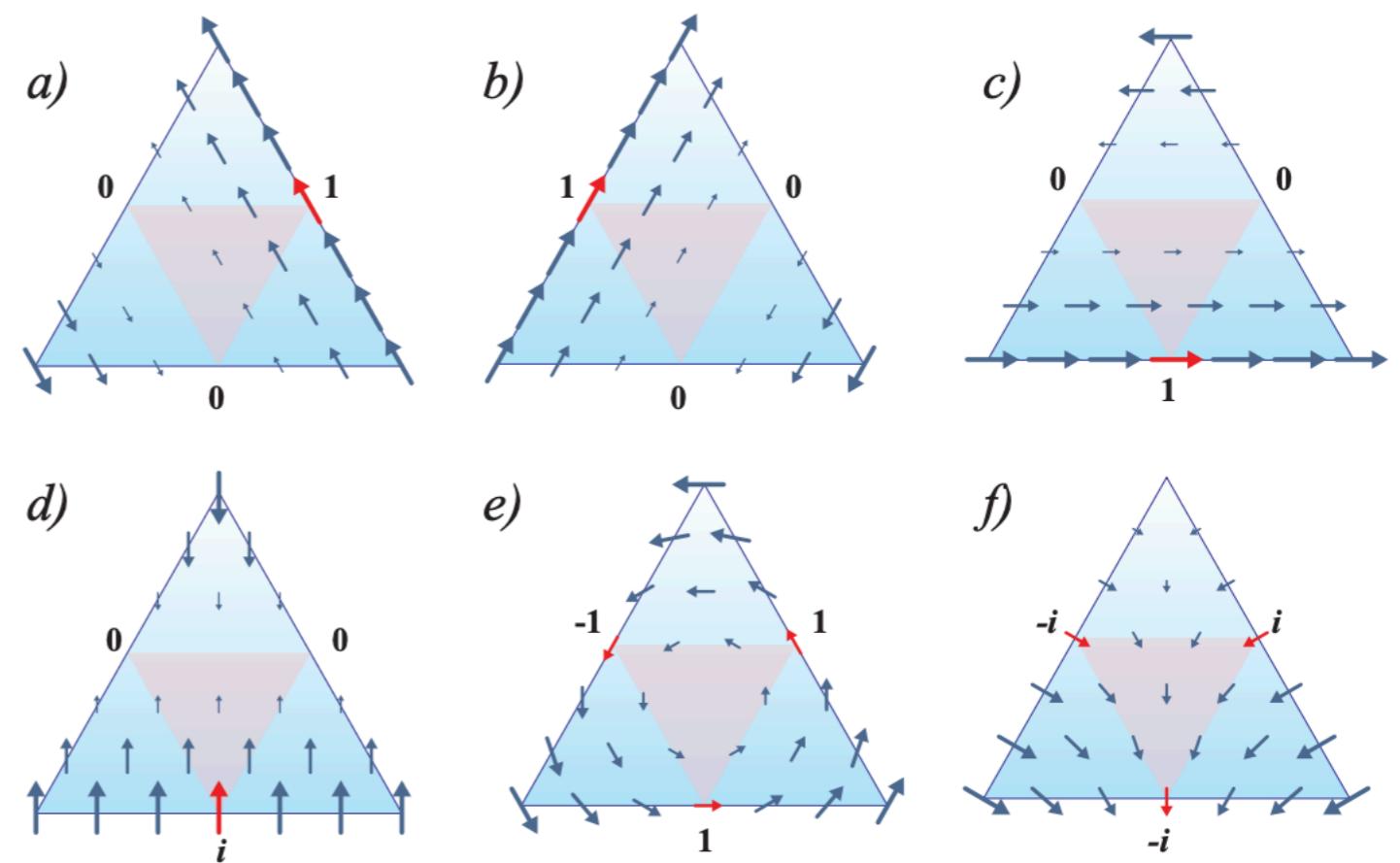
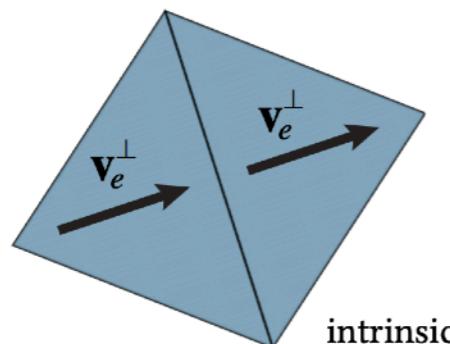
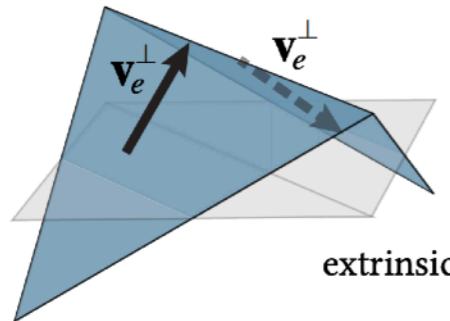
- Encoding integration of field over edges: $\int_e \langle v, e \rangle dt$
- A single number per oriented edge!
- Structure preserving: $d^2 = 0, \delta^2 = 0$.



[Desbrun *et al.* 2005]

EDGE-BASED FIELDS - NON-CONFORMING SPACES

- Field is defined per-edge and linear inside faces.
- Field is encoded in parallel and orthogonal components to the mutual edge.



[Stein *et al.* 2020]

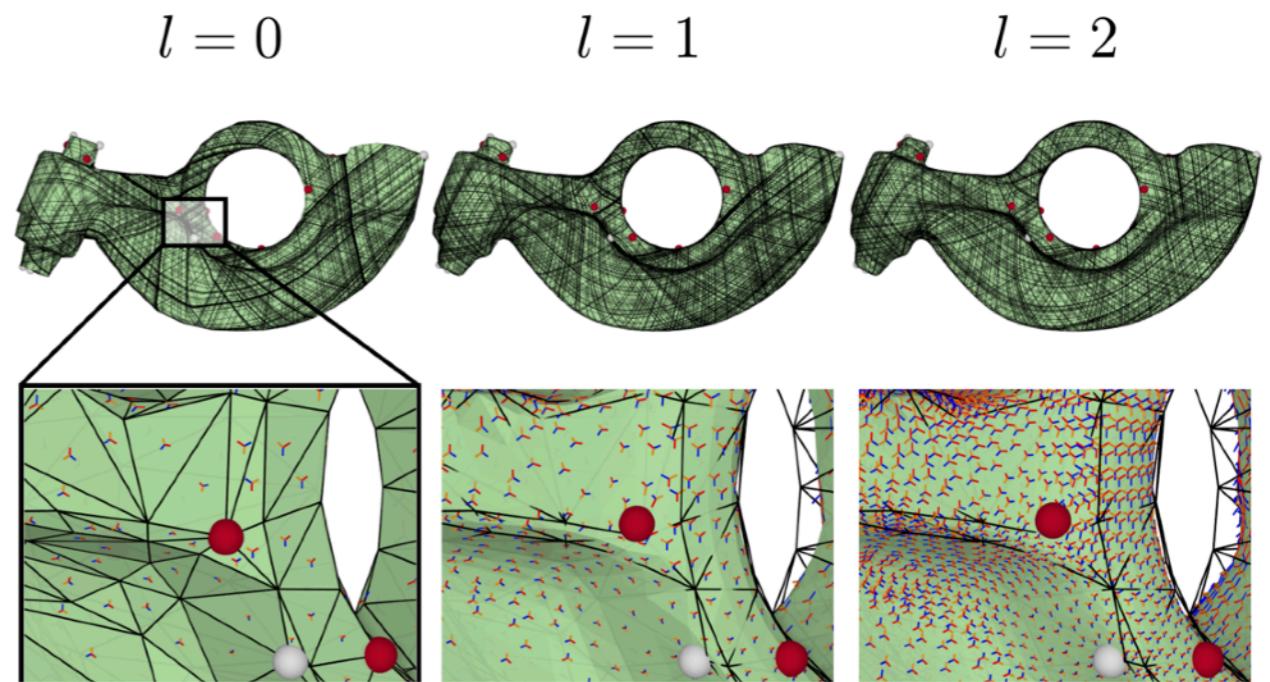
[Djerbetian and Ben-Chen 2016]

SUBDIVISION FIELDS

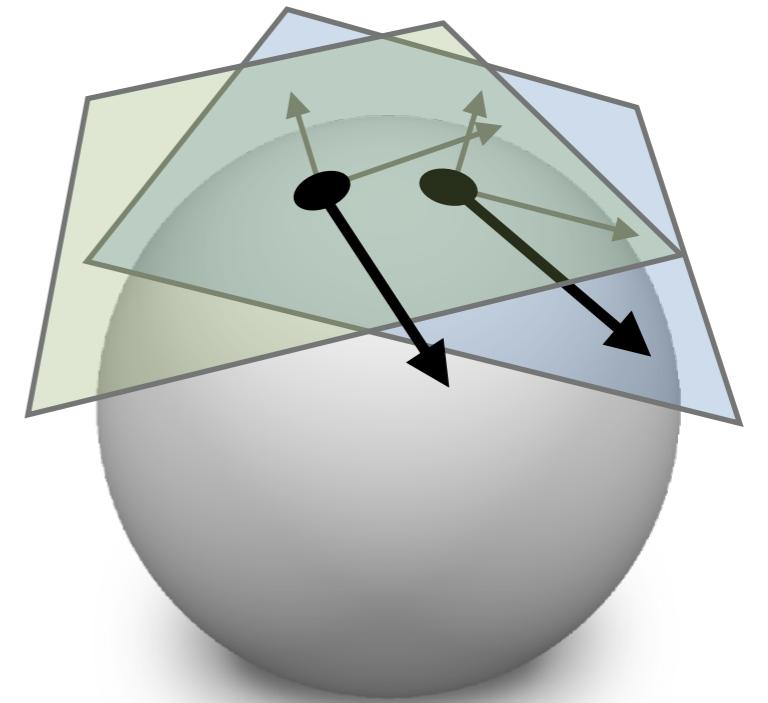
- Higher-order continuity in less d.o.f.
- Overcoming the sampling issue.
- Structure Preserving
 - Gradient coarse fields subdivide into gradient fine fields
 - The coarse curl subdivides into fine curl.



Differential forms [De Goes *et al.* 2016]



Face-based fields [Custers and Vaxman 2020]



REPRESENTATION

REPRESENTATION

- 1 directional 

REPRESENTATION

- 1 directional 
- Cartesian

REPRESENTATION

- 1 directional 

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

REPRESENTATION

- 1 directional



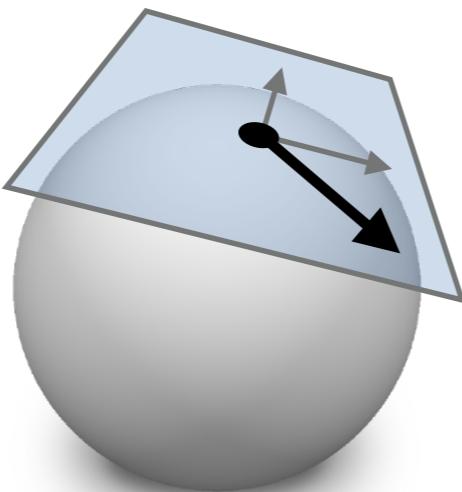
- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

REPRESENTATION

- 1 directional 
- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

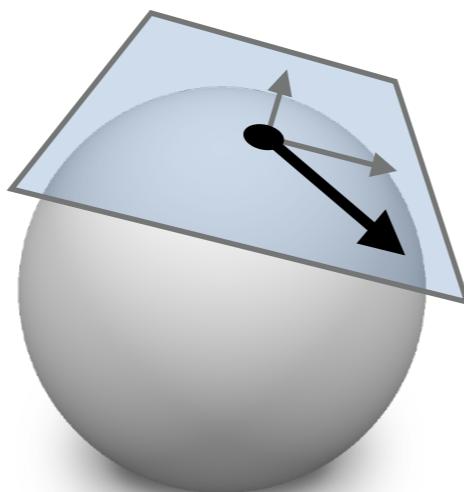


REPRESENTATION

- 1 directional 

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$$\begin{pmatrix} b_{1,p} & b_{2,p} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

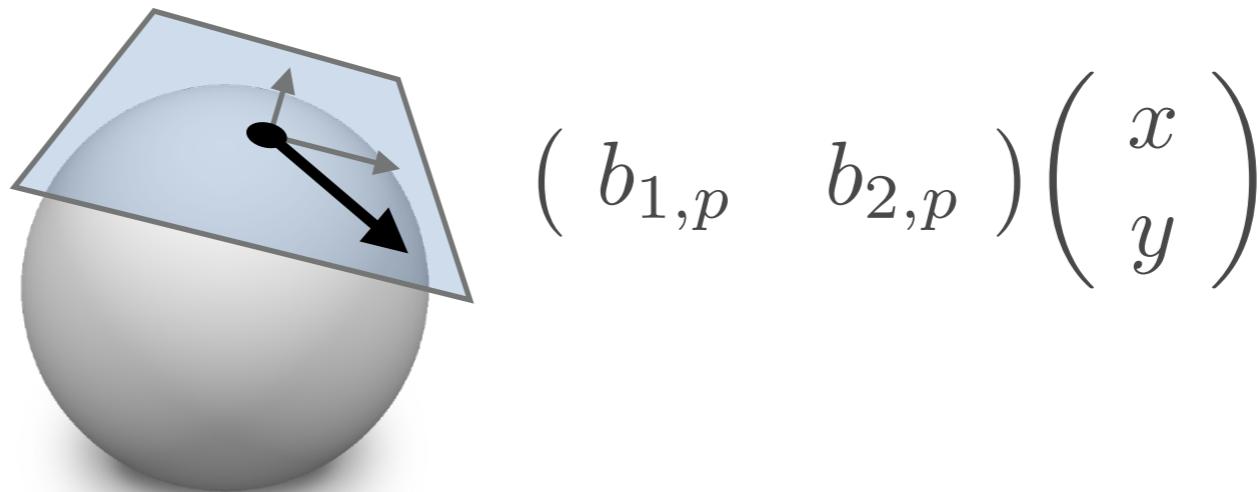
REPRESENTATION

- 1 directional

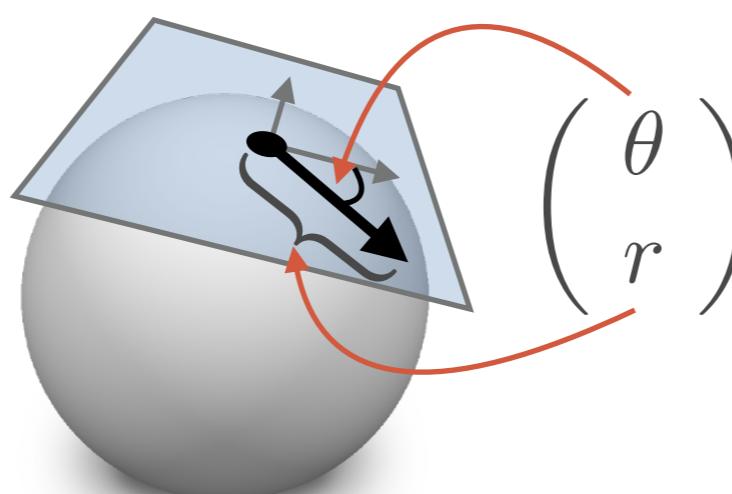


- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



- Polar



REPRESENTATION

- 1 directional



direction (unit length) field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$(\theta)$$

REPRESENTATION

- 1 directional



direction (unit length) field

- Cartesian

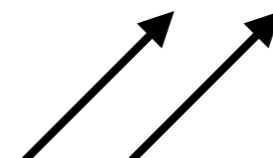
$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1 \quad \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} - \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} = 0$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$(\theta)$$



$$45^\circ - 405^\circ \neq 0$$

REPRESENTATION

- 1 directional



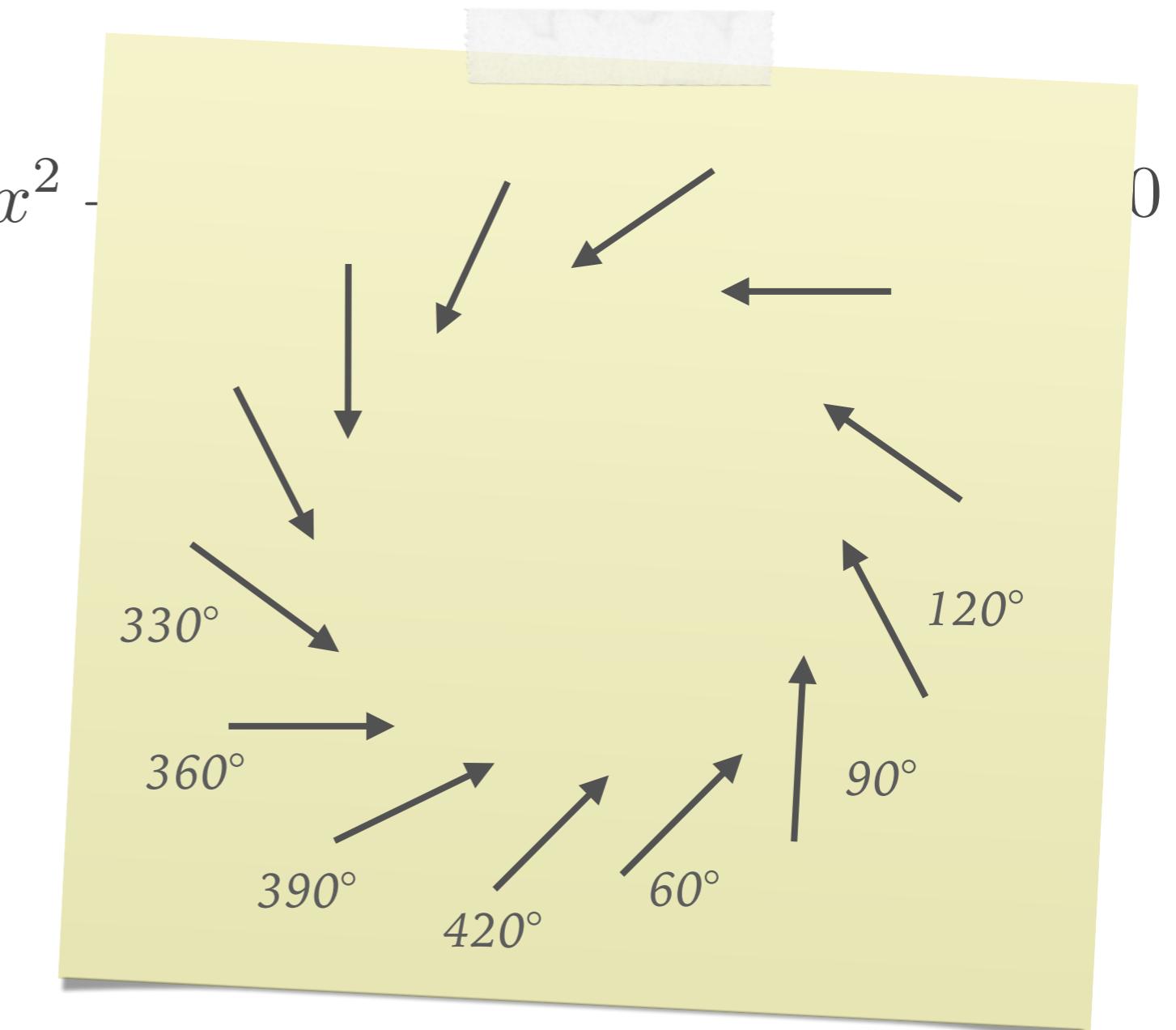
direction field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$



REPRESENTATION

- 1 directional



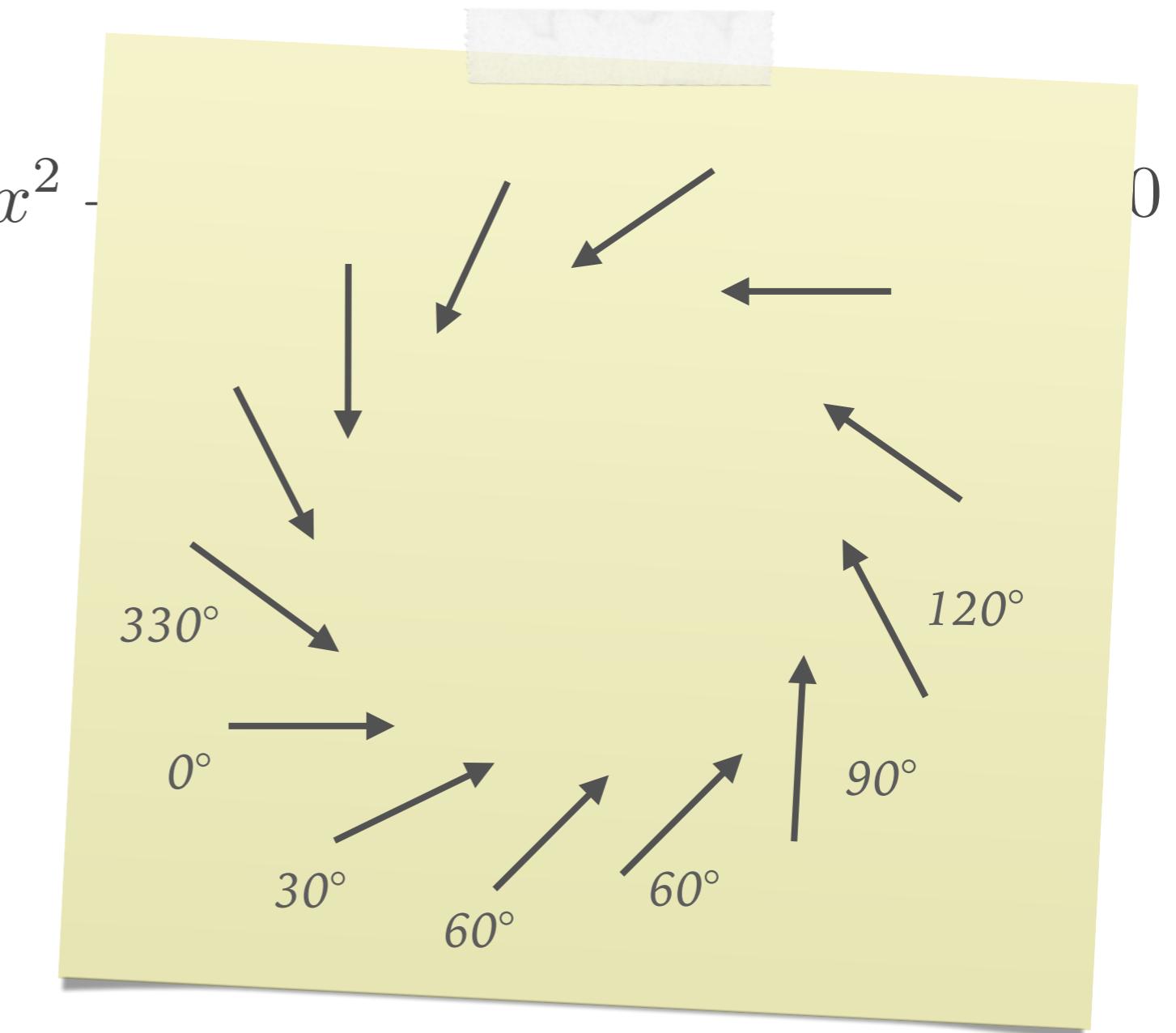
direction field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$



REPRESENTATION

- 1 directional



direction field

2π -invariance

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

built-in

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$(\theta)$$

$$\theta_i - \theta_j \bmod 2\pi$$

REPRESENTATION

- 1 directional



direction field

2π -invariance

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

built-in

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$(\theta)$$

$$\min_{k \in \mathbb{Z}} \theta_i - \theta_j + k2\pi$$

REPRESENTATION

- 1 directional



direction field

2π -invariance

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

built-in

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$\begin{pmatrix} \theta \end{pmatrix}$$

$$\theta_i - \theta_j + k2\pi$$

k const.

explicit choice of period
 \Rightarrow control over topology

REPRESENTATION

- 1 directional



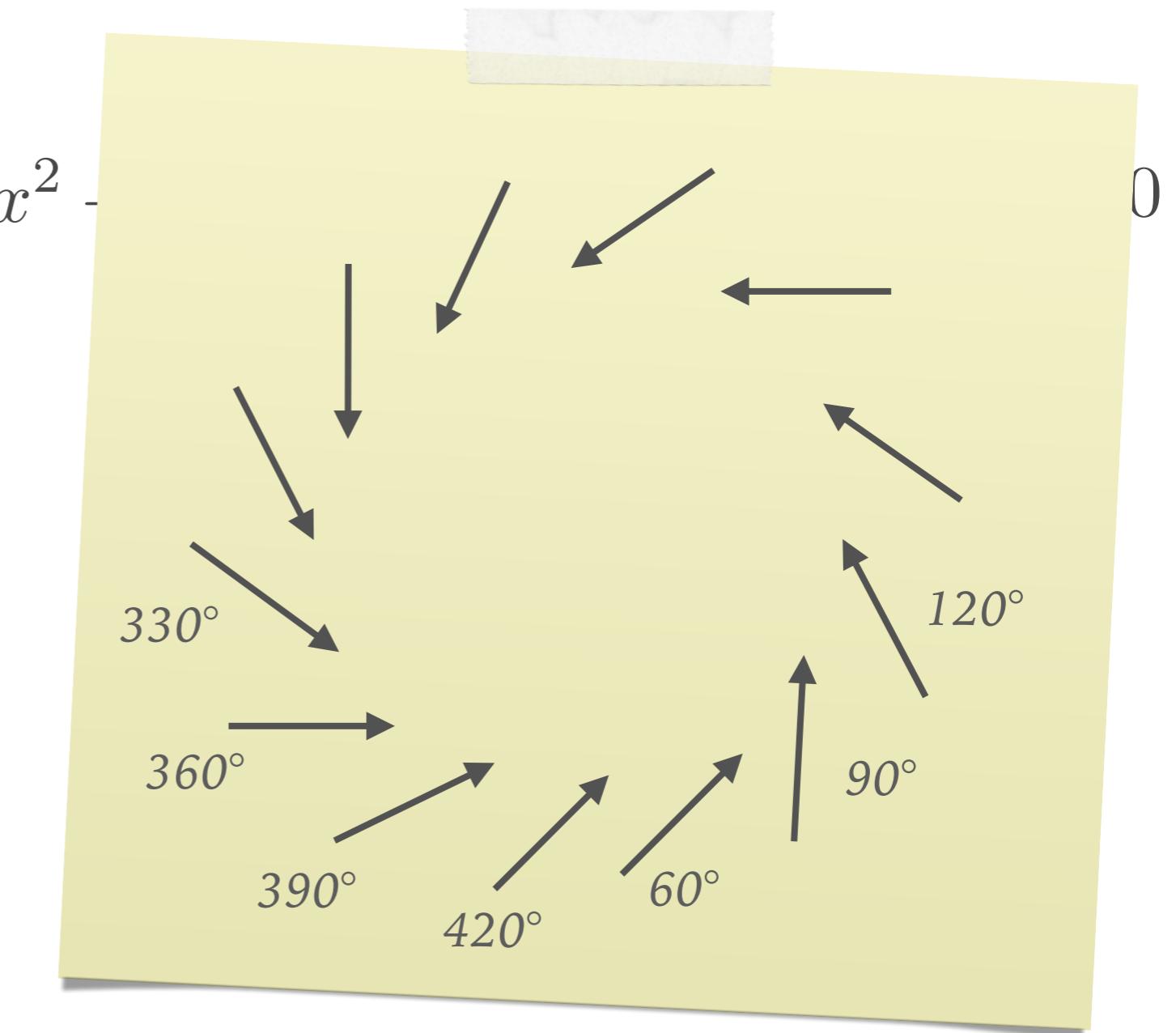
direction field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$



REPRESENTATION

- 1 directional



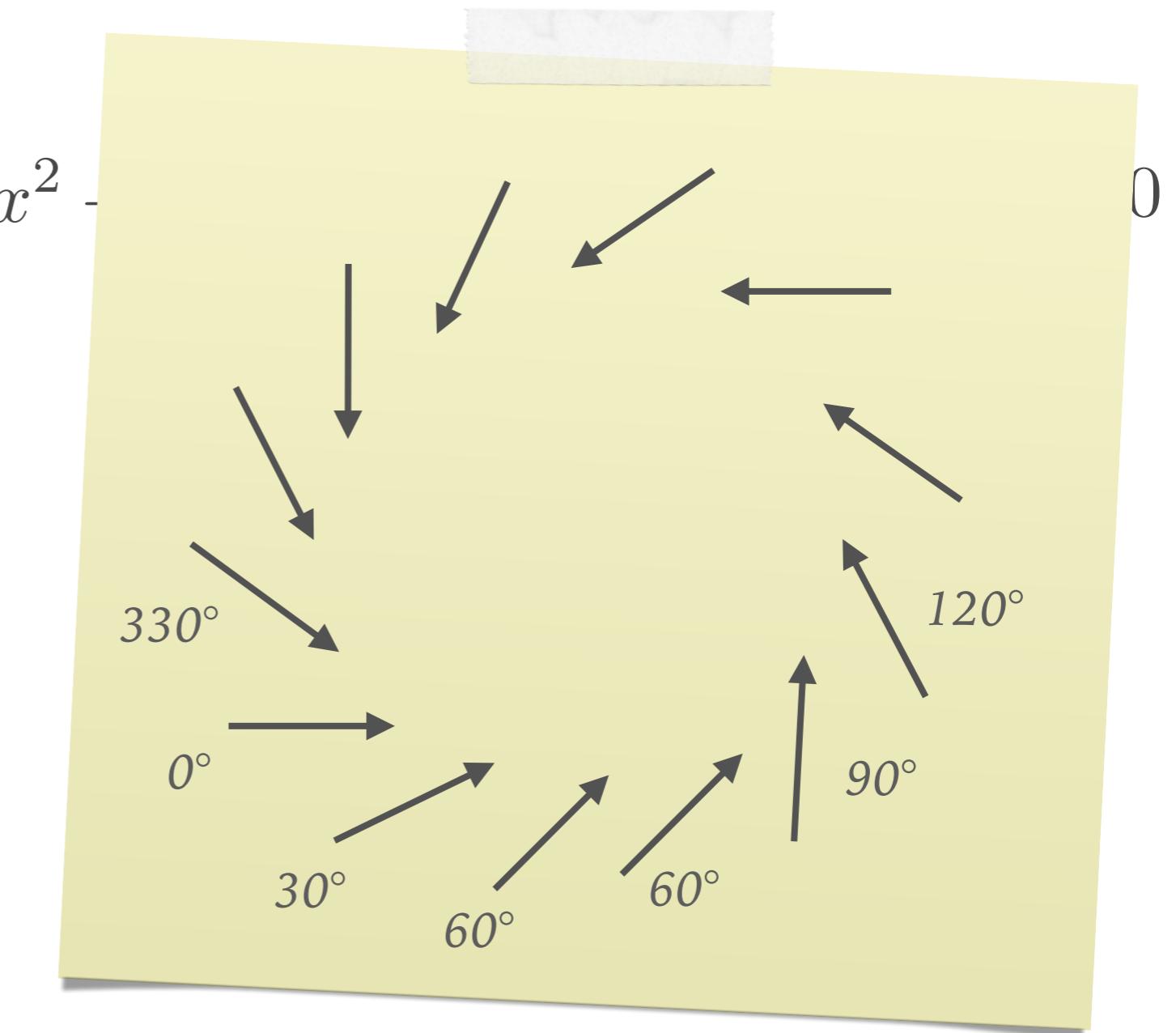
direction field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$



REPRESENTATION

- 1 directional



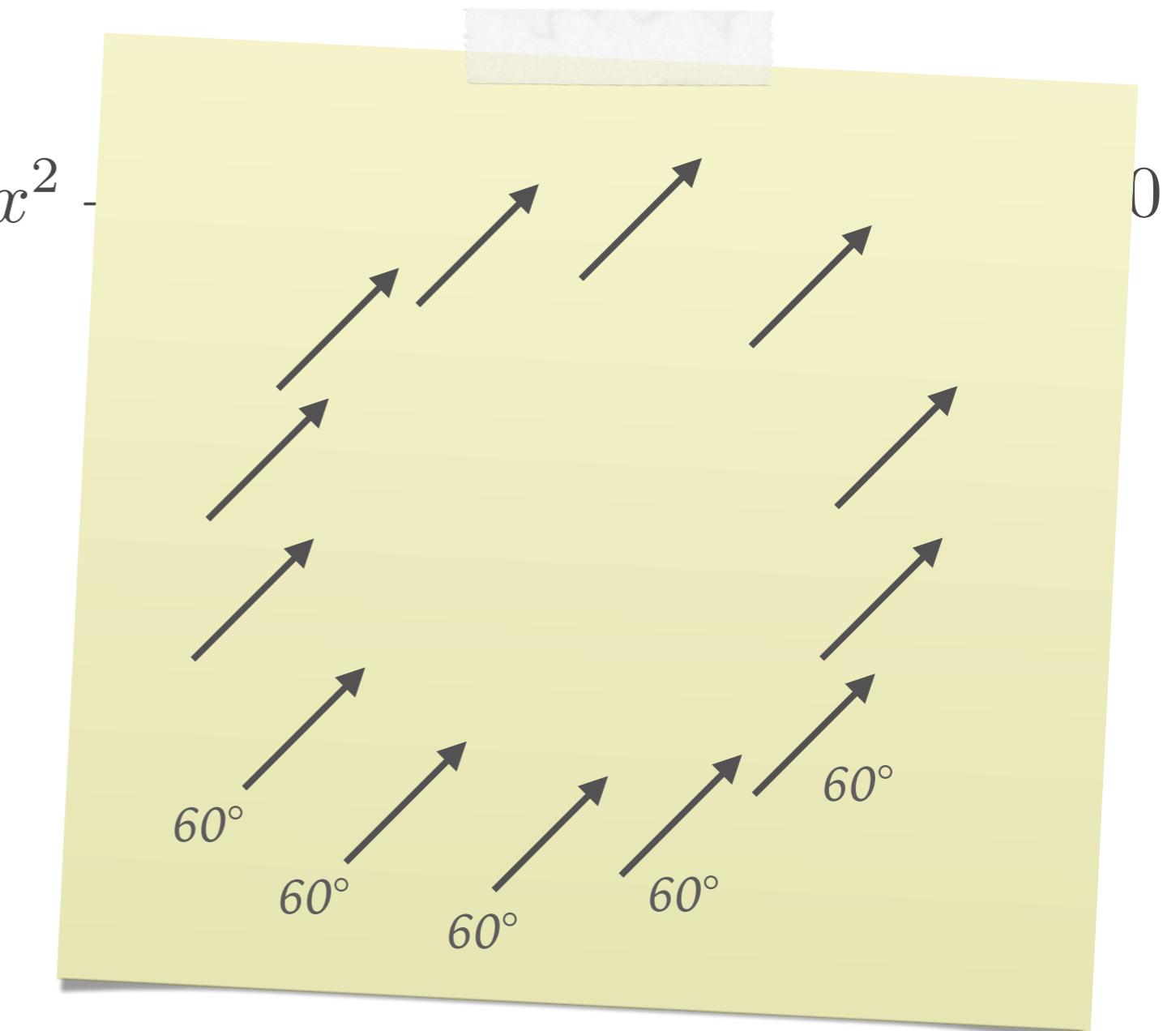
direction field

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$



REPRESENTATION

- 1 directional



direction field

2π -invariance

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x^2 + y^2 = 1$$

built-in

- Polar

$$\begin{pmatrix} \theta \\ r \end{pmatrix}$$

$$r e^{i\theta}$$

$$\begin{pmatrix} \theta \end{pmatrix}$$

$$\theta_i - \theta_j + k2\pi$$

k integer

explicit choice of period
 \Rightarrow control over topology

REPRESENTATION

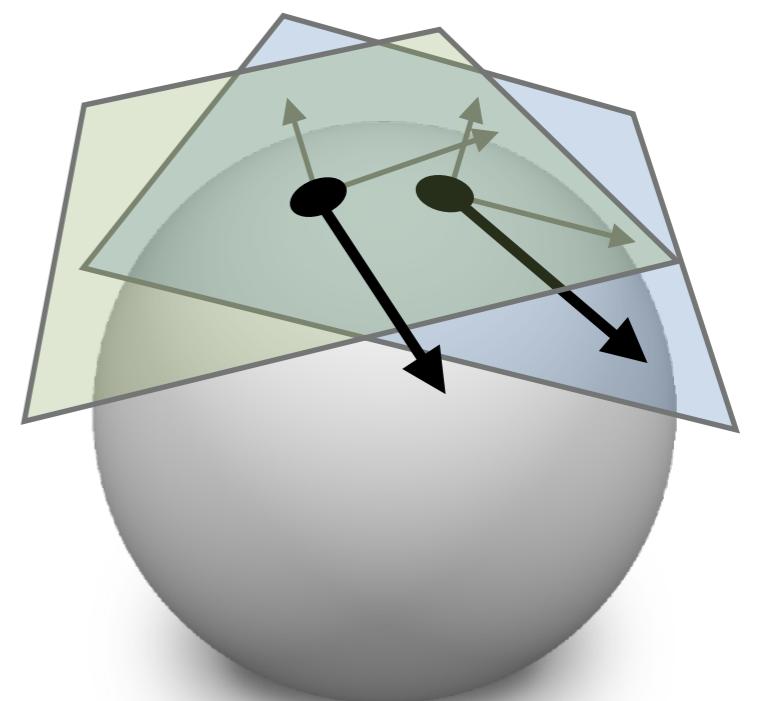
- Differences between tangent vectors?

- Cartesian

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

- Angle

$$\theta$$



REPRESENTATION

- Differences between tangent vectors?

- Cartesian

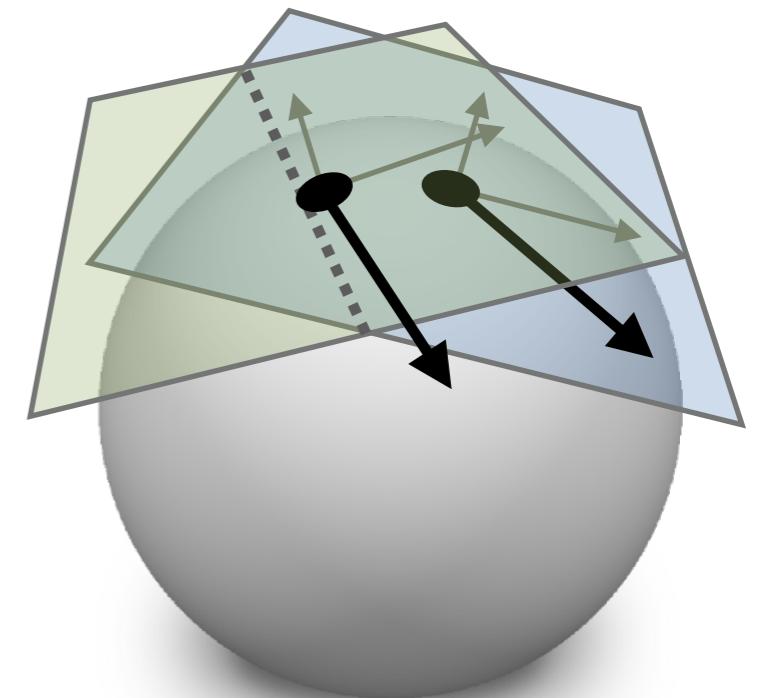
$$\begin{pmatrix} x \\ y \end{pmatrix}_j - \begin{pmatrix} \cos X_{ij} & -\sin X_{ij} \\ \sin X_{ij} & \cos X_{ij} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_i$$

- Angle

Period Jump

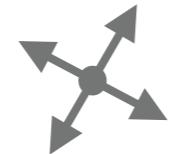
$$\theta_j - \theta_i + \cancel{X_{ij}} + \cancel{p_{ij}} 2\pi$$

Transition angle

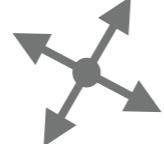


REPRESENTATION

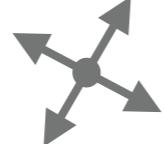
- N directionals



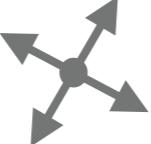
REPRESENTATION

- N directionals 
- simply use multiple $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} \theta \\ r \end{pmatrix}$ per location?

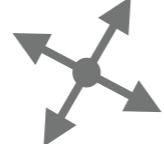
REPRESENTATION

- N directionals 
- simply use multiple $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} \theta \\ r \end{pmatrix}$ per location?
 - perhaps okay for mere representation,
but problematic for synthesis, optimization, ...

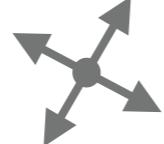
REPRESENTATION

- N directionals 
- simply use multiple $\begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} \theta \\ r \end{pmatrix}$ per location?
 - perhaps okay for mere representation,
but problematic for synthesis, optimization, ...
 - symmetries \Rightarrow additional constraints

REPRESENTATION

- N directionals 
- symmetric
 - just use one representative

REPRESENTATION

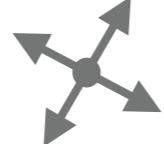
- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$

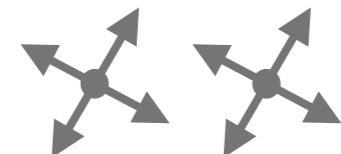


[Palacios & Zhang 2007]

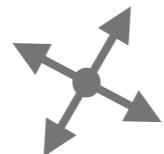
[Ray et al. 2008]

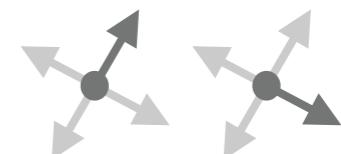
REPRESENTATION

- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$



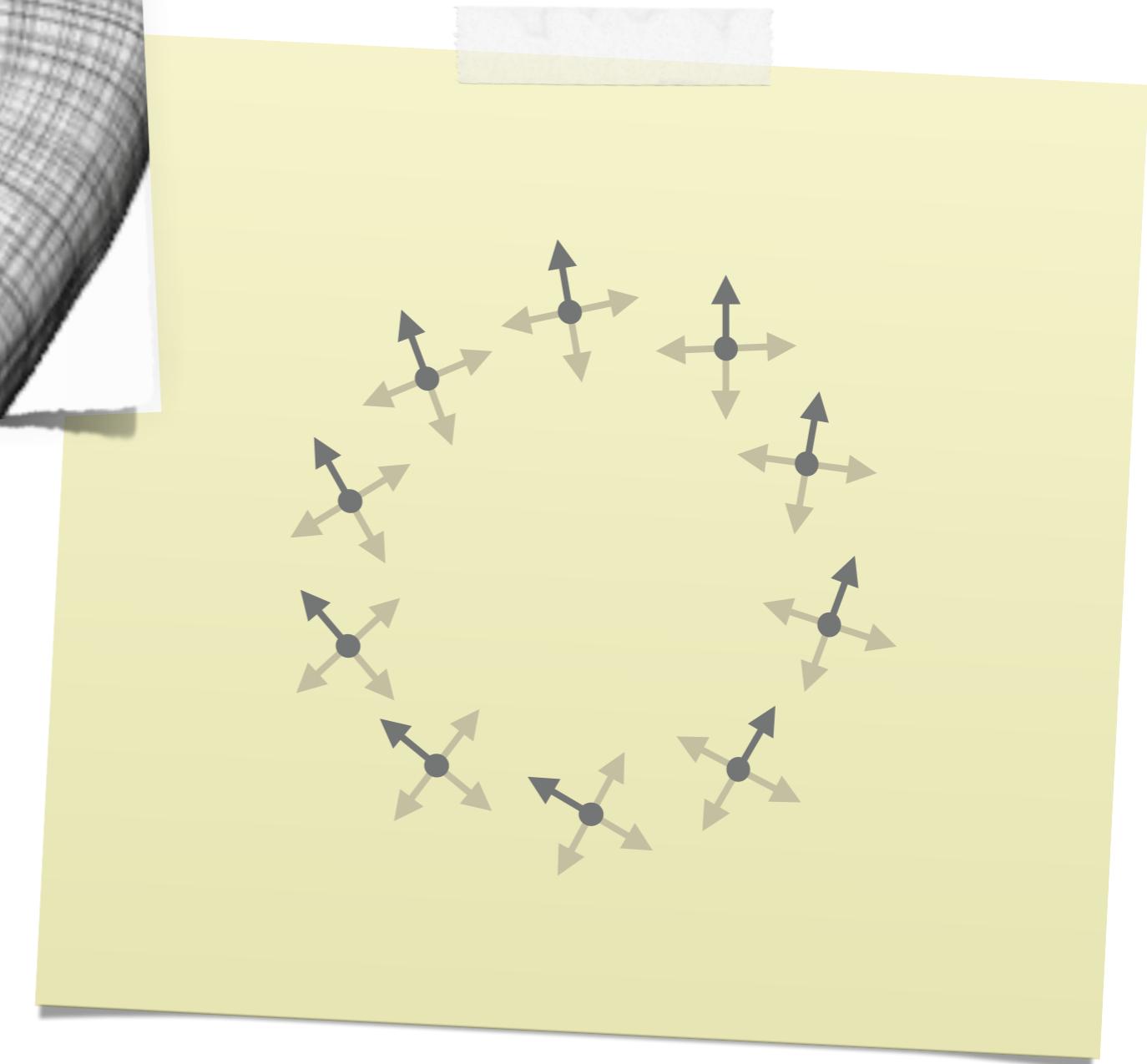
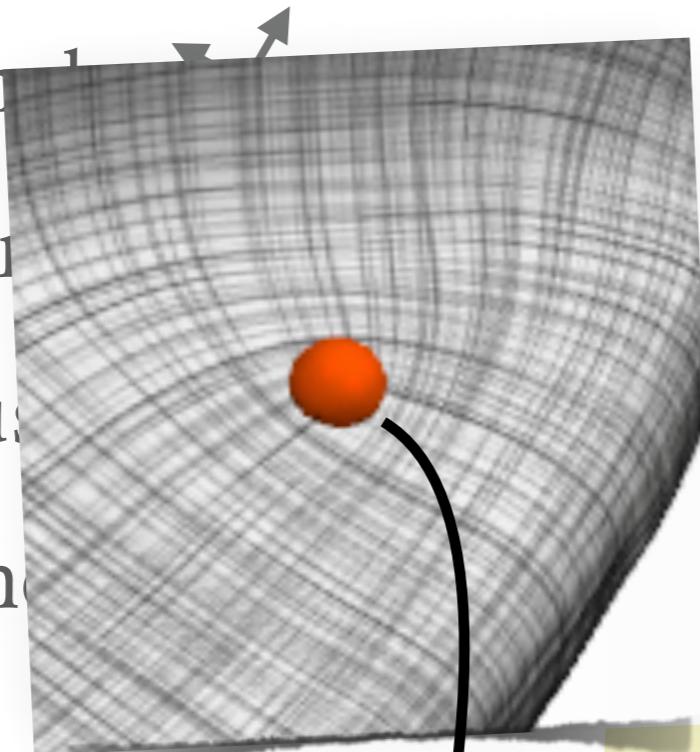
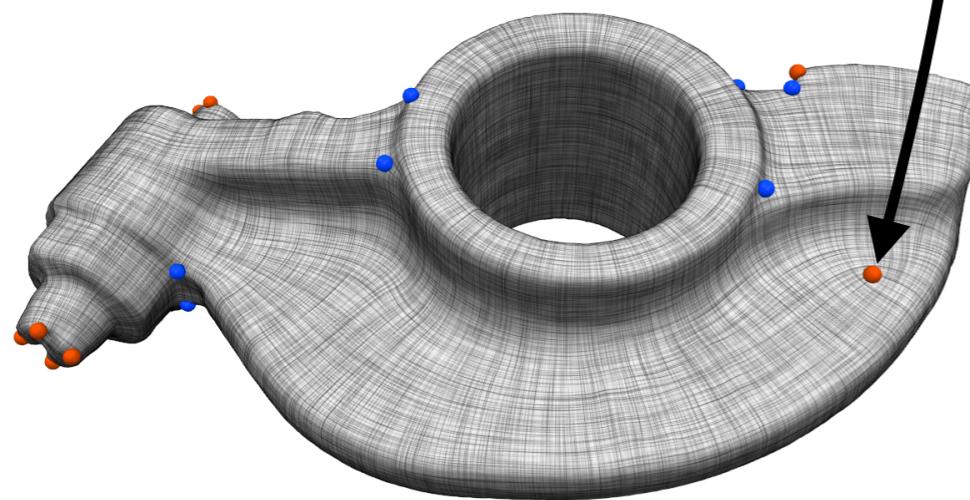
REPRESENTATION

- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$

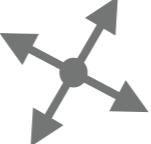


REPRESENTATION

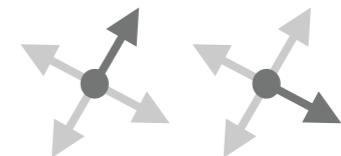
- N directions
 - symmetric
 - just use one
 - other



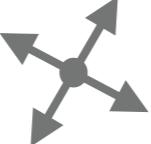
REPRESENTATION

- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$

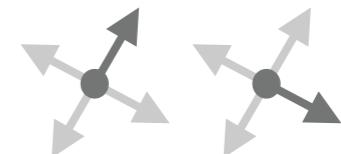
$$\theta_j - \theta_i + X_{ij} + p_{ij}2\pi/N$$



REPRESENTATION

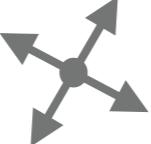
- N directionals 
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$$\theta_j - \theta_i + X_{ij} + p_{ij}2\pi/N$$

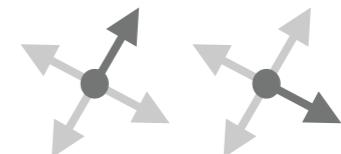


- Cartesian:
 - $\begin{pmatrix} x \\ y \end{pmatrix}$

REPRESENTATION

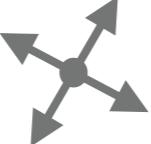
- N directionals 
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$$\theta_j - \theta_i + X_{ij} + p_{ij}2\pi/N$$

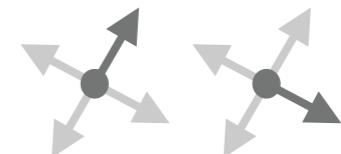


- Cartesian:
 - $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \theta = \text{atan2}(y, x)$

REPRESENTATION

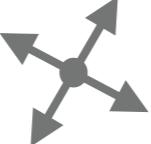
- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$

$$\theta_j - \theta_i + X_{ij} + p_{ij}2\pi/N$$

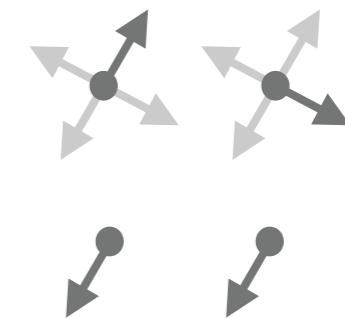


- Cartesian:
 - $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \theta = \text{atan2}(y, x) \rightarrow \begin{pmatrix} \sin N\theta \\ \cos N\theta \end{pmatrix}$

REPRESENTATION

- N directionals 
- symmetric
 - just use one representative
 - others implied by rotation by $k2\pi/N$

$$\theta_j - \theta_i + X_{ij} + p_{ij}2\pi/N$$



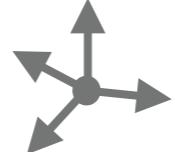
- Cartesian:

- $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \theta = \text{atan2}(y, x) \rightarrow \begin{pmatrix} \sin N\theta \\ \cos N\theta \end{pmatrix}$

“representation/power vector”

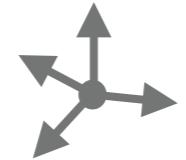
[Hertzmann & Zorin 2000]

REPRESENTATION

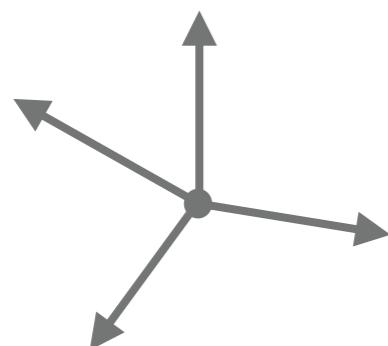
- N directionals 
- non-symmetric

REPRESENTATION

- N directionals

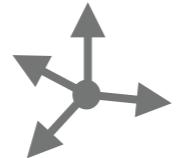


- non-symmetric

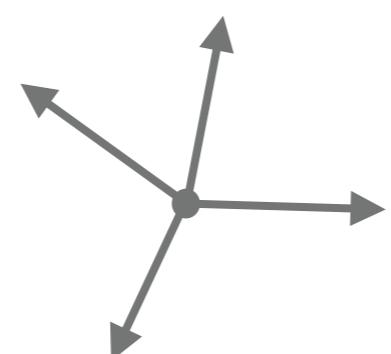
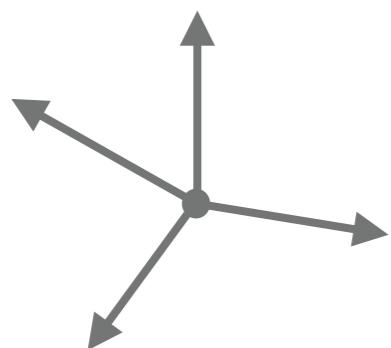


REPRESENTATION

- N directionals

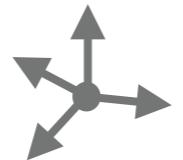


- non-symmetric

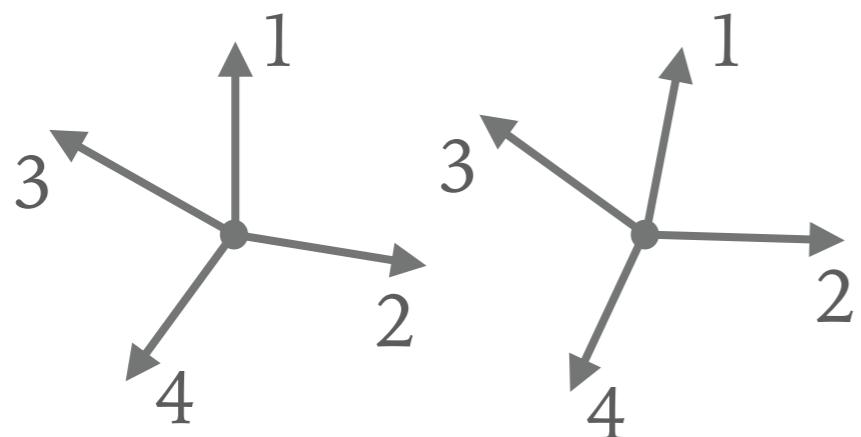


REPRESENTATION

- N directionals

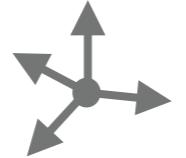


- non-symmetric

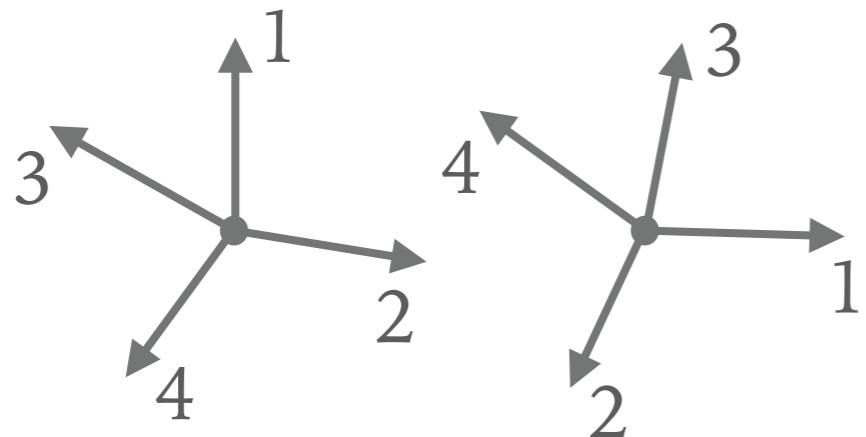


REPRESENTATION

- N directionals

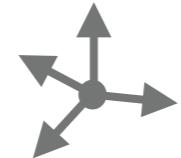


- non-symmetric

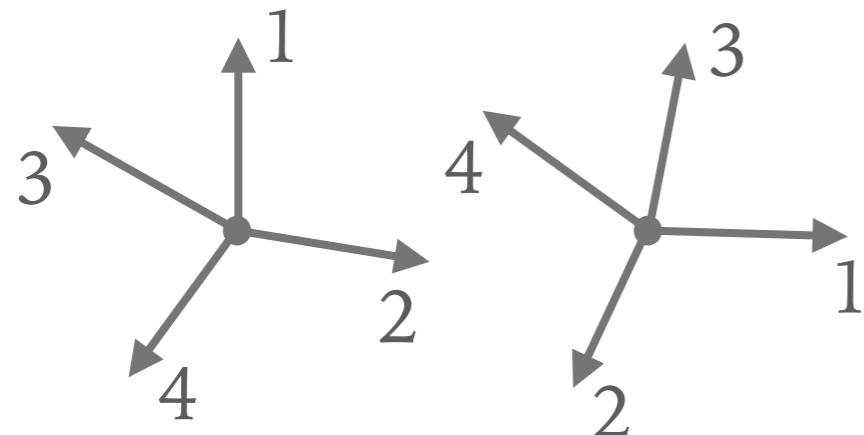


REPRESENTATION

- N directionals



- non-symmetric



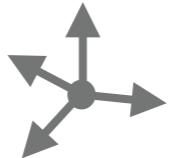
$$F(v_1, v_2, v_3, v_4) = u$$

F symmetric

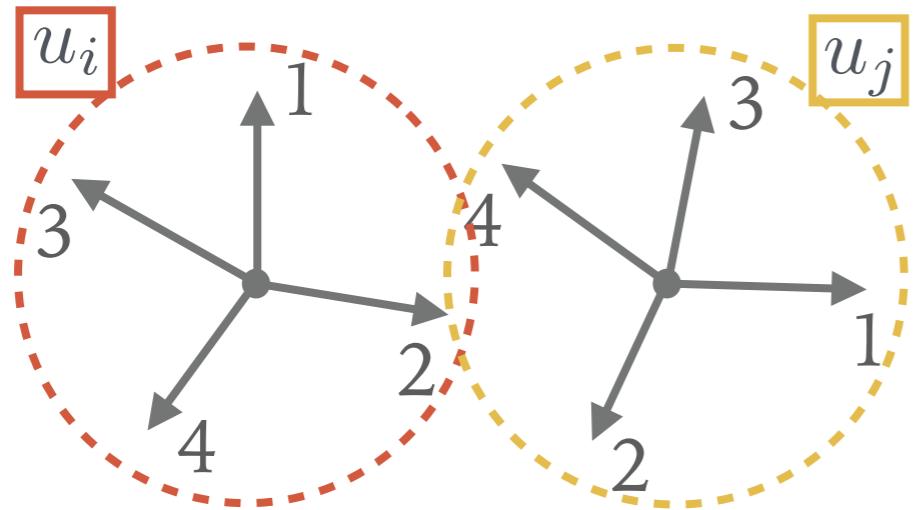
$$F(v_1, v_2, v_3, v_4) = F(v_3, v_2, v_4, v_1)$$

REPRESENTATION

- N directionals



- non-symmetric



$$F(v_1, v_2, v_3, v_4) = u$$

F symmetric

F invertible (up to symmetry)

$$(x - a)(x - b)(x - c)$$

$$= x^3 + (a + b + c)x^2 + (ab + ac + bc)x + (abc)$$

$$\rightarrow (a + b + c \mid ab + ac + bc \mid abc)$$

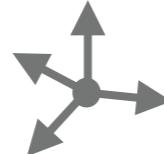
N -PolyVector

$$(\quad u \quad , \quad v \quad , \quad w \quad) =: F(a, b, c)$$

[Diamanti et al. 2014]

$$(a + b + c + d \mid ab + ac + ad + bc + bd + cd \mid abc + abd + acd + bcd \mid abcd)$$

REPRESENTATION

- N directionals 
- use complex numbers, complex polynomials

$$a = x + iy$$

$$a = re^{i\theta}$$

- non-symmetric

$$(a + b + c + d \mid ab + ac + ad + bc + bd + cd \mid abc + abd + acd + bcd \mid abcd)$$

- symmetric

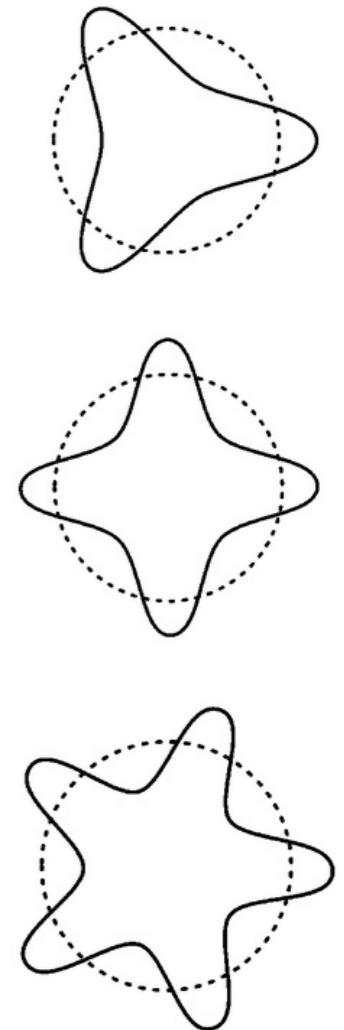
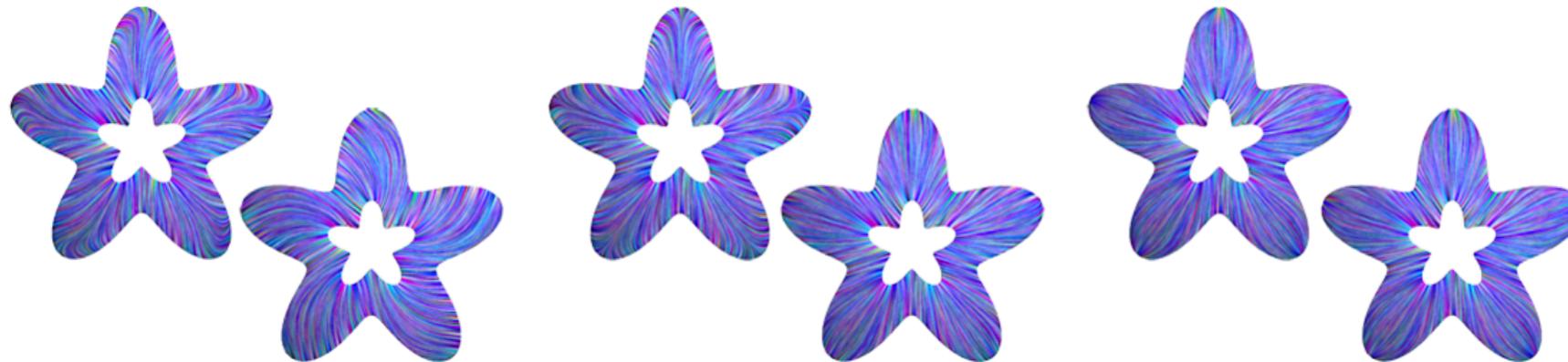
$$(a + b + c + d \mid ab + ac + ad + bc + bd + cd \mid abc + abd + acd + bcd \mid abcd)$$

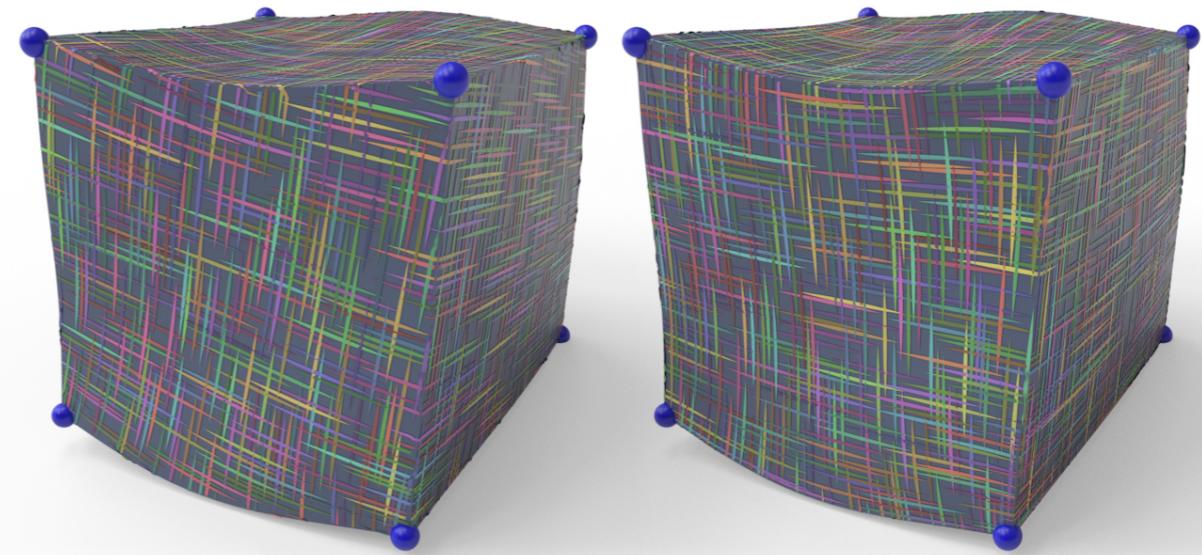
$$\underbrace{\hspace{1cm}}_0 \quad \underbrace{\hspace{1cm}}_0 \quad \underbrace{\hspace{1cm}}_0 \quad \underbrace{\hspace{1cm}}_{-a^4}$$

$$\begin{array}{ccc} \xrightarrow{\hspace{1cm}} & \rightarrow & \left(\begin{array}{c} \sin N\theta \\ \cos N\theta \end{array} \right) \\ & & = -re^{i4\theta} \end{array}$$

REPRESENTATION

- Alternatives
 - Extrema of periodic circular functions (SH)
 - Also spherical harmonics in 3D [Ray *et al.* 2016]
 - Eigenvectors of tensors (spd matrices)
 - Functional representation [Azencot *et al.* 2013]

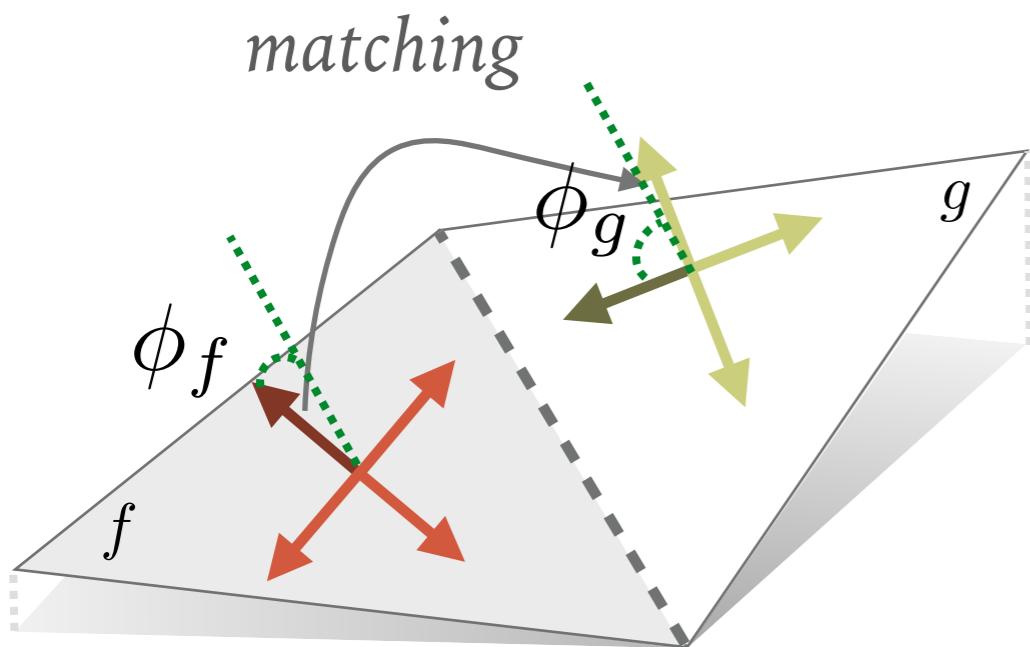




OBJECTIVES

OBJECTIVES - FIELD FAIRNESS

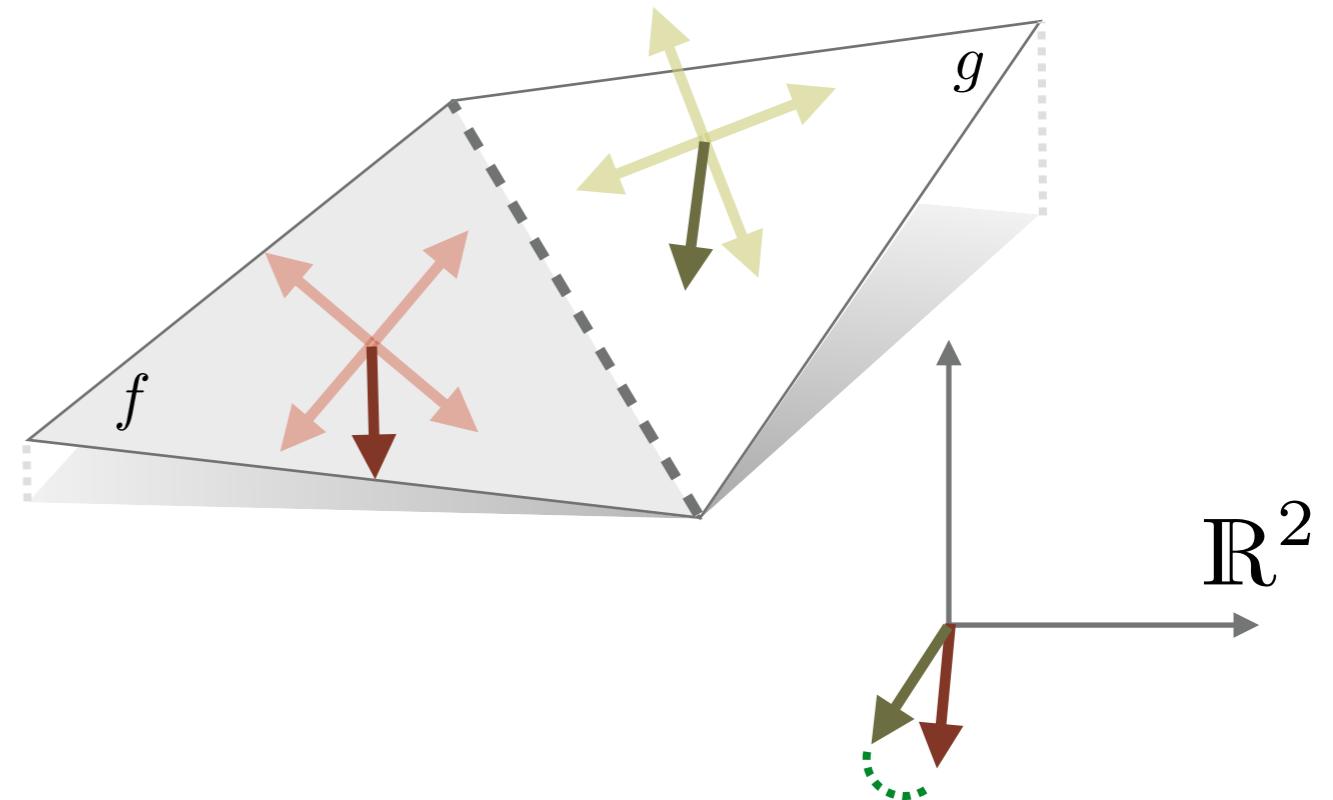
- “As-parallel-as-possible”



angle-based approaches

compare matching angles

[Hertzmann et al. 2000] [Crane et al. 2010]
[Ray et al. 2008] [Ray et al. 2009]
[Bommes et al. 2009] [Jakob et al. 2015]



cartesian/complex approaches

compare representative vectors

[Ray et al. 2006] [Knöppel et al. 2013]
[Palacios et al. 2007] [Diamanti et al. 2014]

DIRICHLET ENERGY

- Smoothest field - minimizing the Dirichlet energy:

$$E_D = \int_{\Omega} |\nabla v|^2 d\Omega = \langle \Delta v, v \rangle_{L^2}$$

- Discrete version (in Cartesian discretizations/representations):

$$E_D = \sum_{i,j} w_{ij} |v_i - R_{ij}v_j|^2 = v^T L v$$

Metric
(stiffness)

Connection
matrix

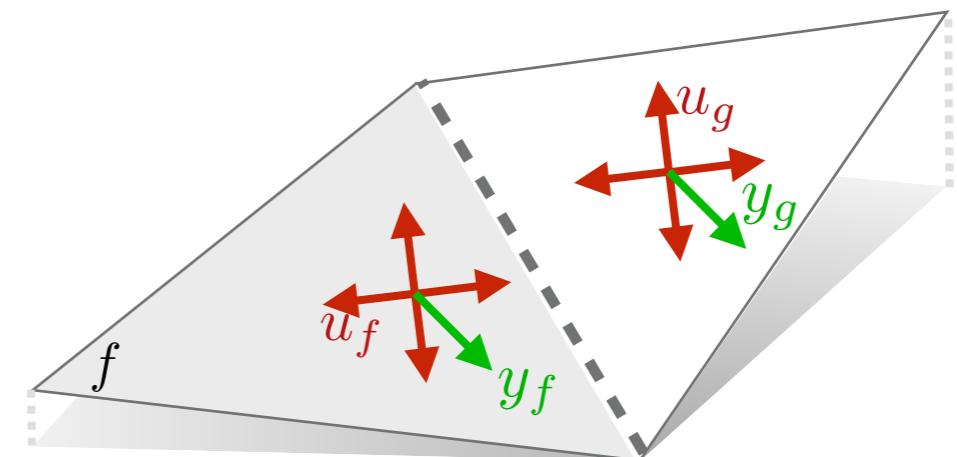
Discrete
(vector)
Laplacian

EXAMPLE AND DEMO: POWER FIELDS

- Input: N-RoSy as single complex number per constrained face
$$\forall f \in B, y_f = (u_f)^N$$
- Output: interpolation to all free faces $B = F \setminus U, y_U = (u_U)^N$
- Minimizing Dirichlet energy:

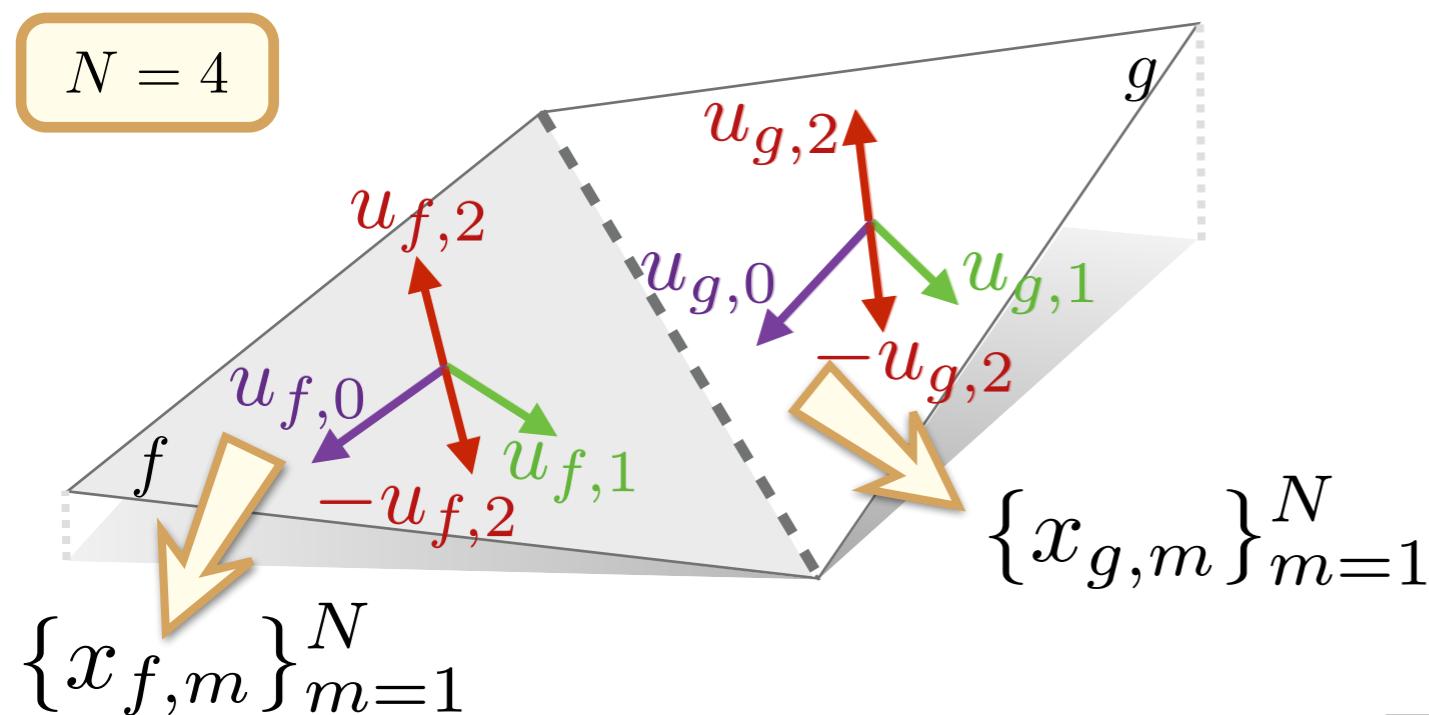
$$\sum_{e=(f,g)} w_e \left| y_f \bar{e}_f^N - y_g \bar{e}_g^N \right|^2 = y^T L_N y$$

- Take roots of all y_f
- Choice of $w_e : \frac{3l_e^2}{A_f + A_g}$



EXAMPLE DEMO: POLYVECTORS

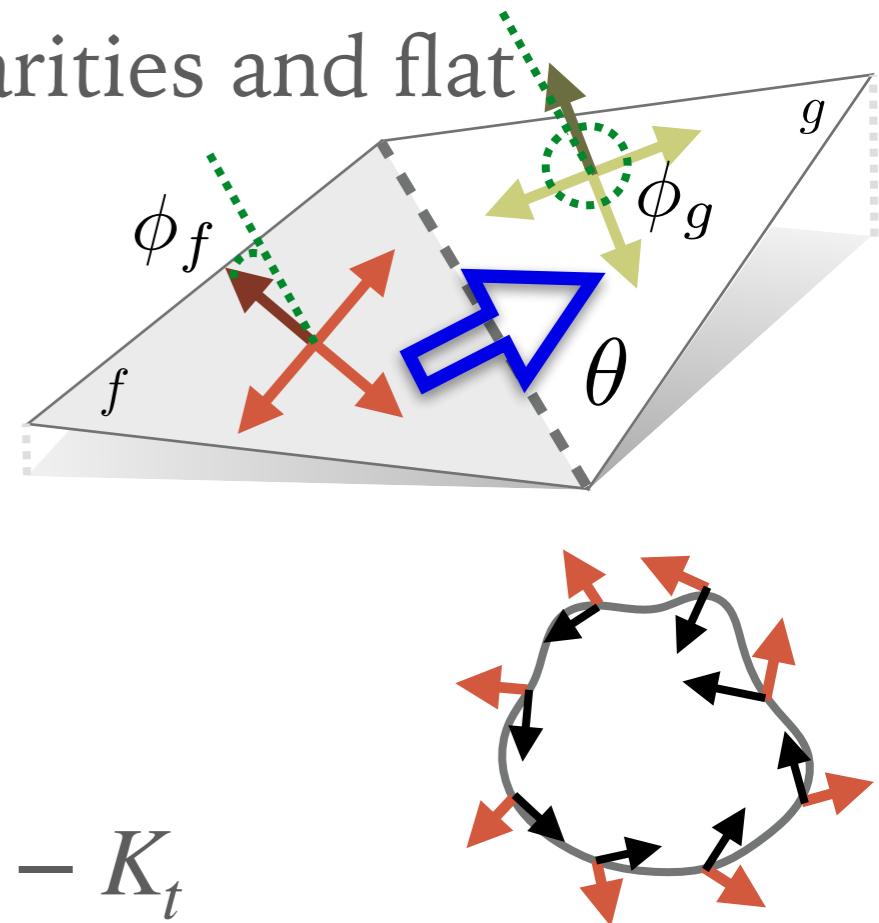
- Representing set of polynomial coefficients instead of single N-RoSy.
- Roots of polynomial = directional field in face.
- Interpolating each like in power fields.



$$\forall m \in [1, N], \sum_{e=(f, g)} w_e \left| y_f \bar{e}_f^m - y_g \bar{e}_g^m \right|^2 = y^T L_m y$$

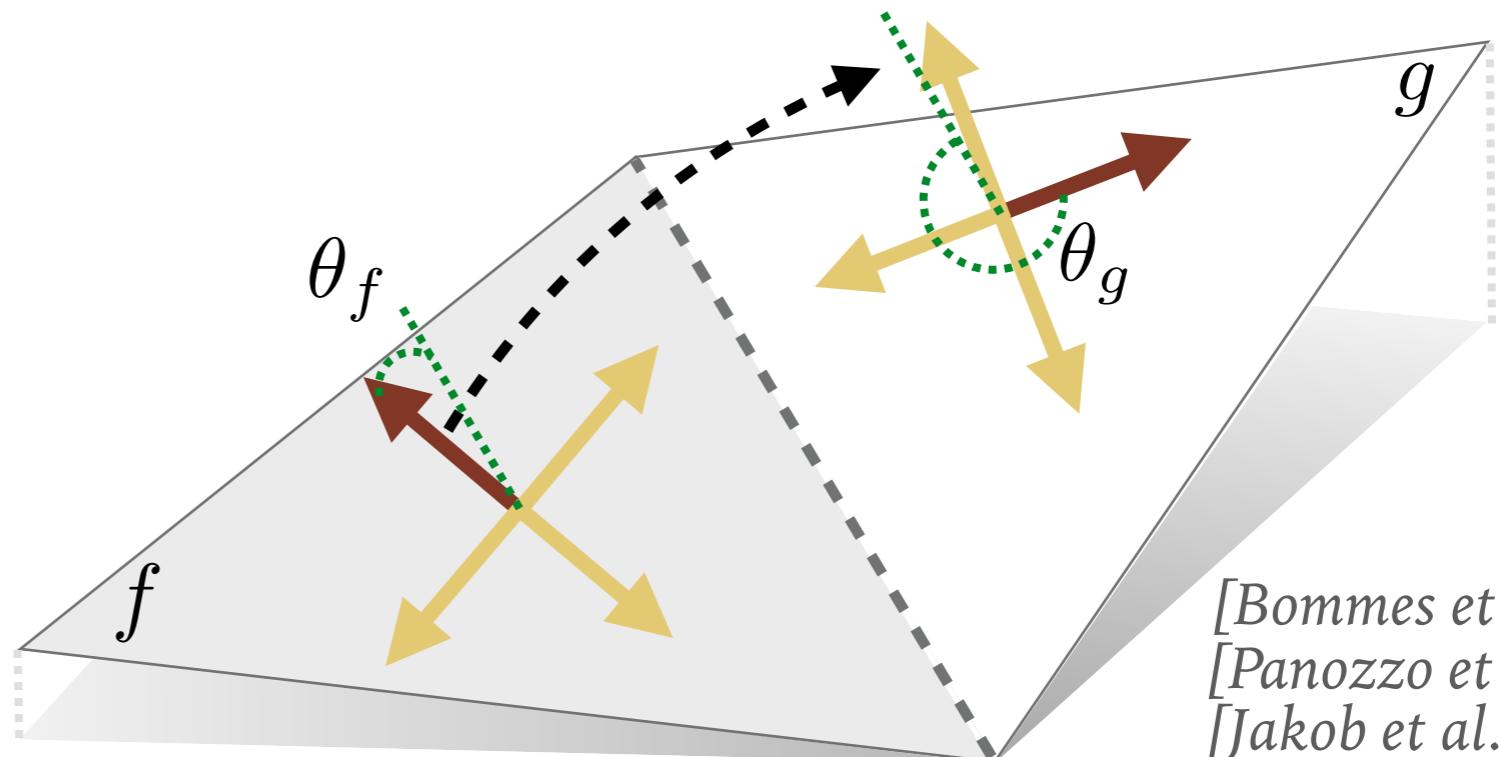
INDEX PRESCRIPTION

- Input:
 - Original vertex curvatures K_s .
 - Prescribed curvatures K_t (cone singularities and flat vertices).
- Polar Dirichlet energy:
$$E_D(\theta_E) = \sum_{e \in E} |\theta_e|^2$$
- Under the linear constraint: $(d_0)^T \theta_E = K_s - K_t$



POLAR DIRICHLET: FREE TOPOLOGY

- Explicitly model topology (typically angle-based)
 - Matchings are explicitly modeled
 - Mixed Integer Optimization
 - Local minima

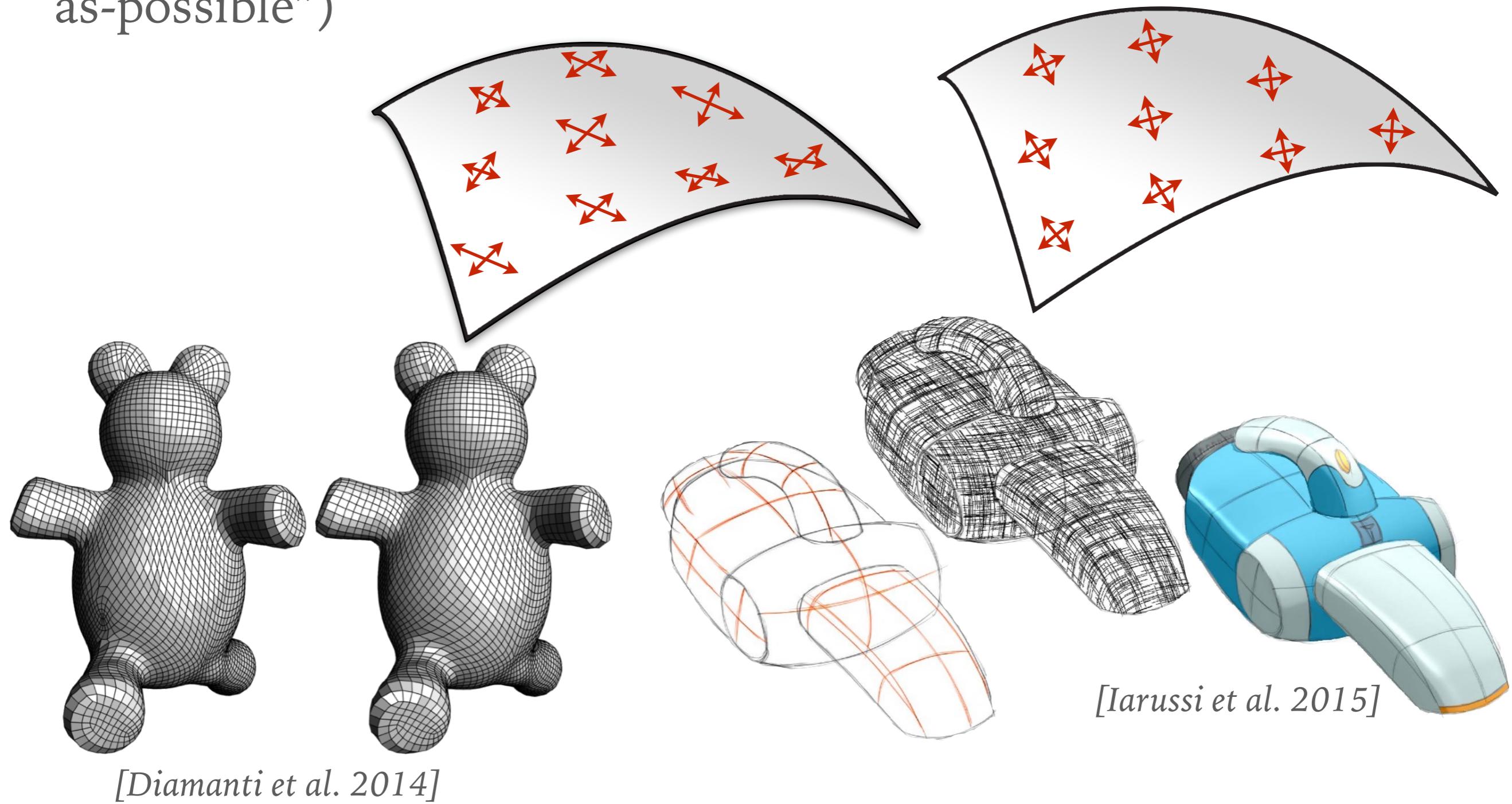


[Bommes et al. 2009]
[Panozzo et al. 2012]
[Jakob et al. 2015]

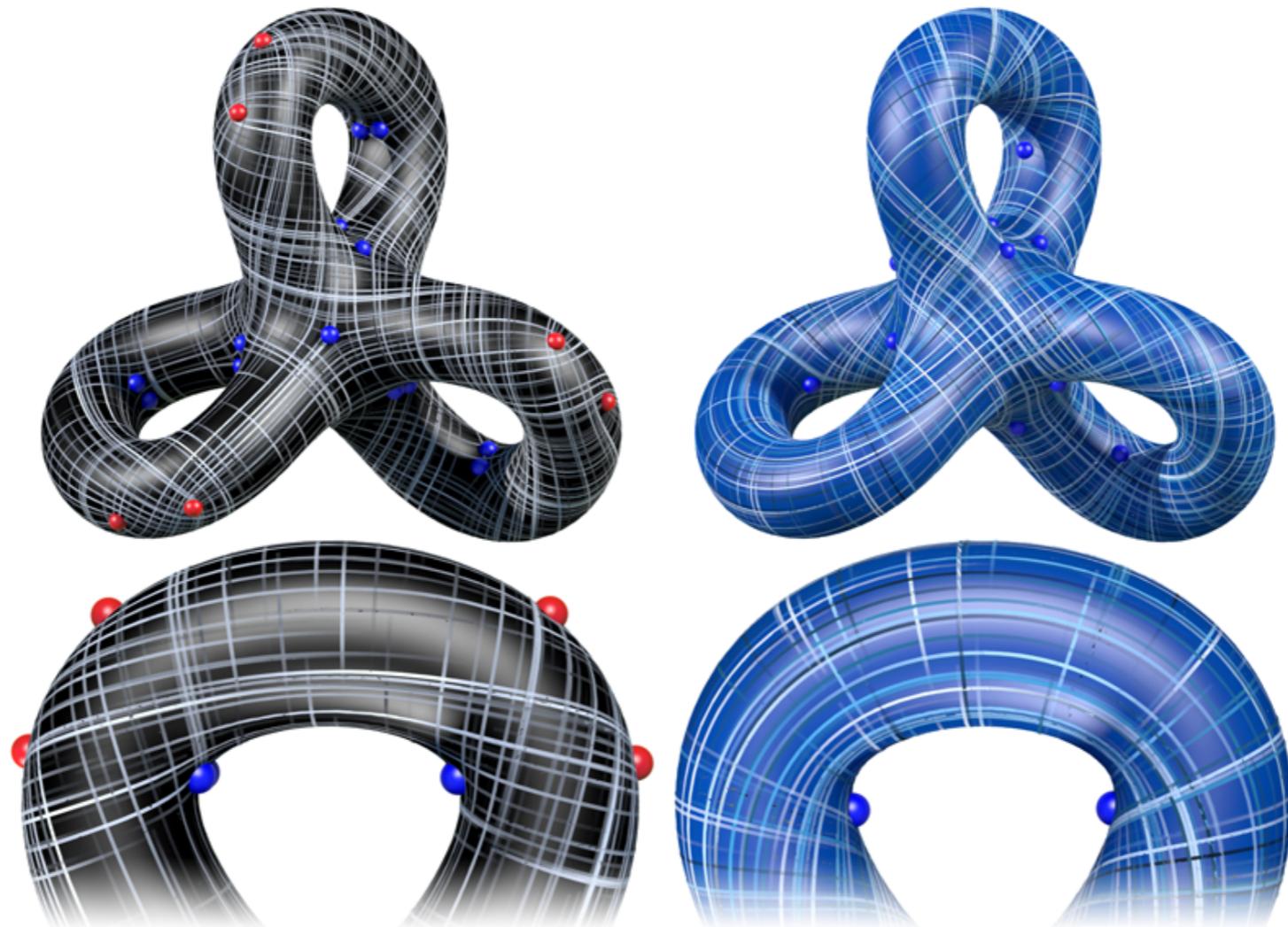
$$(\theta_f + \rho_{fg} \frac{\pi}{2} - \theta_g)^2 \quad \rho_{fg} \in \mathbb{I}$$

OBJECTIVES - ORTHOGONALITY

- In PolyVectors - minimize all non-free coefficients (“as-power-field-as-possible”)

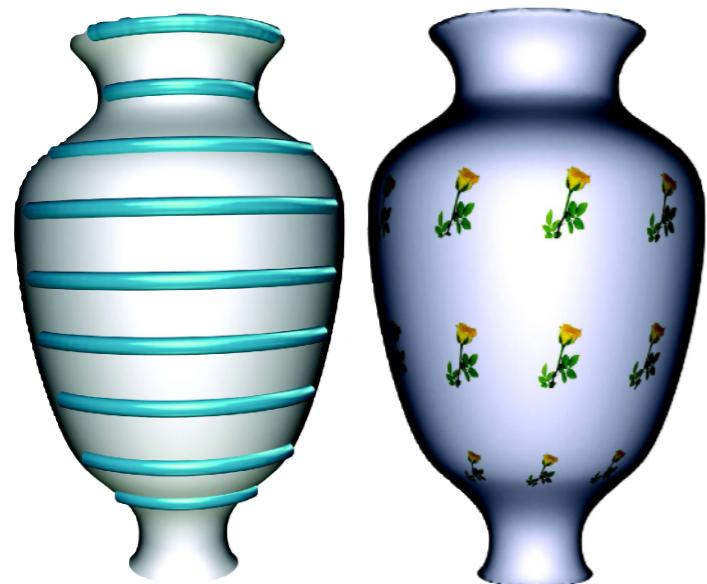


OBJECTIVES – SINGULARITY CONTROL



[Knöppel et al. 2013]

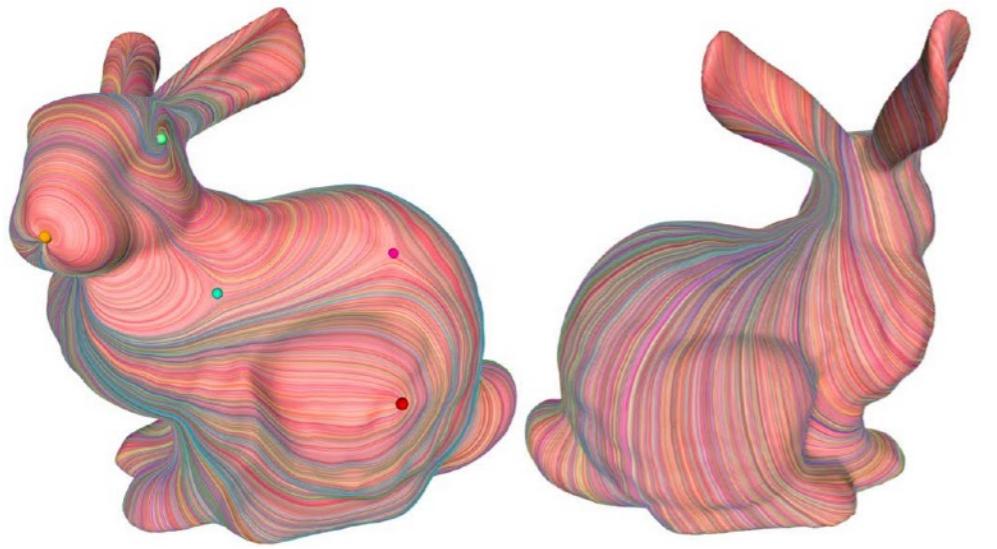
OBJECTIVES – ISOMETRY INDUCING



[Ben-Chen et al. 2010]

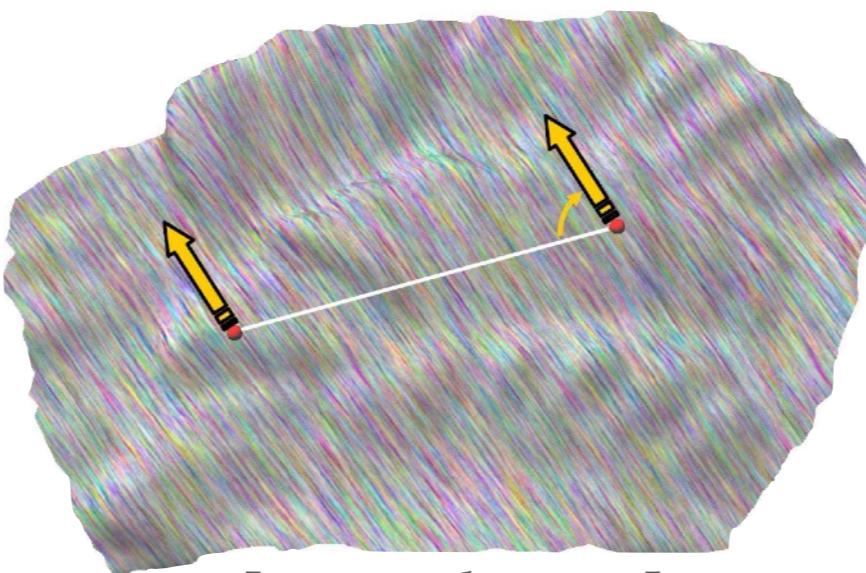
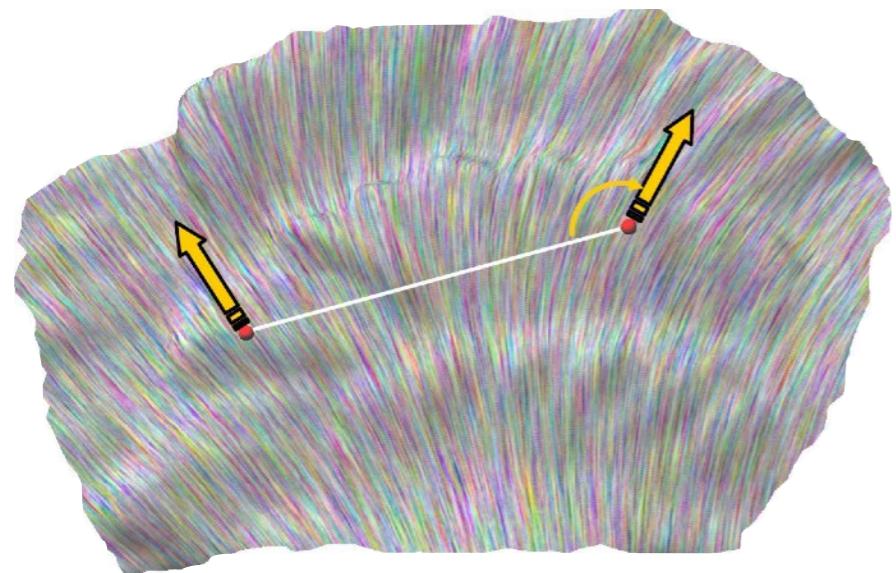


[Solomon et al. 2011]

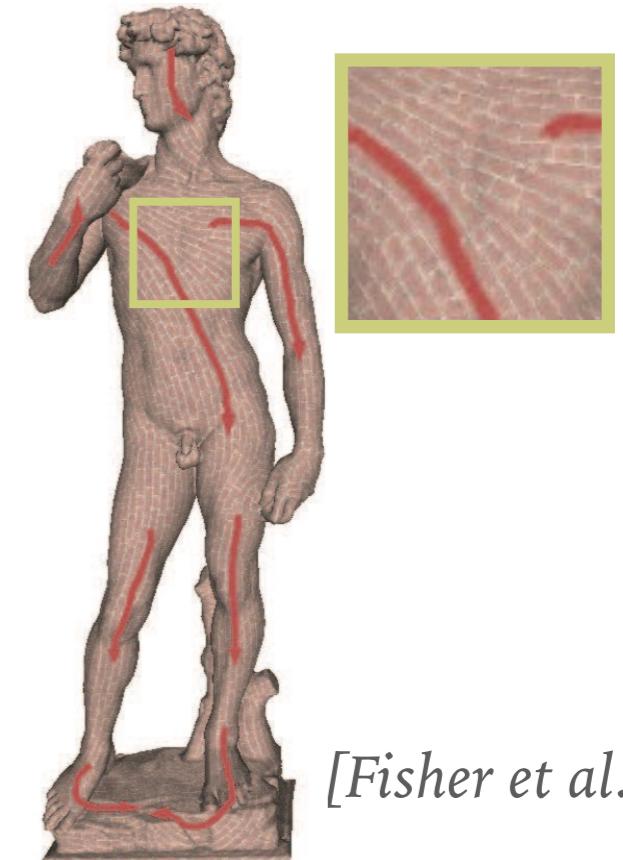


CONSTRAINTS

CONSTRAINTS – ALIGNMENT



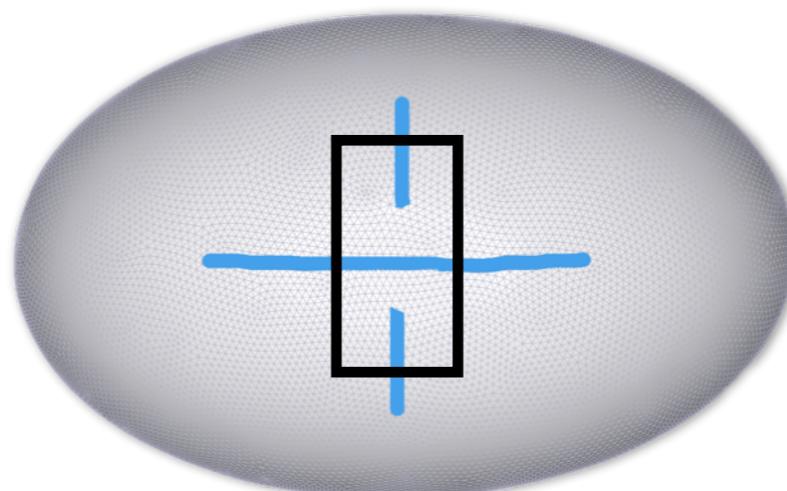
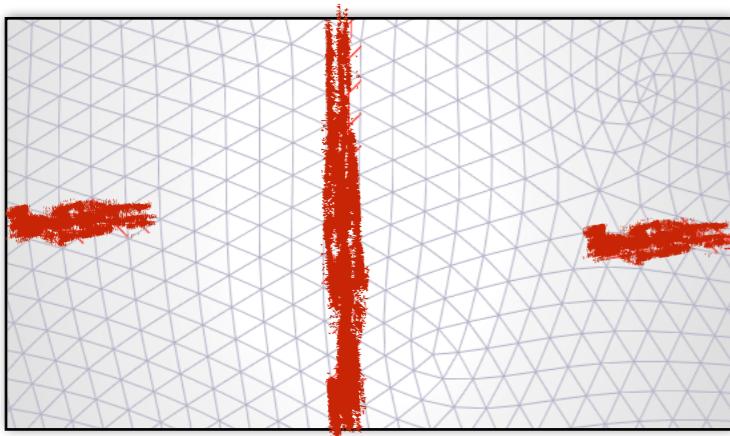
[Ray et al. 2008]



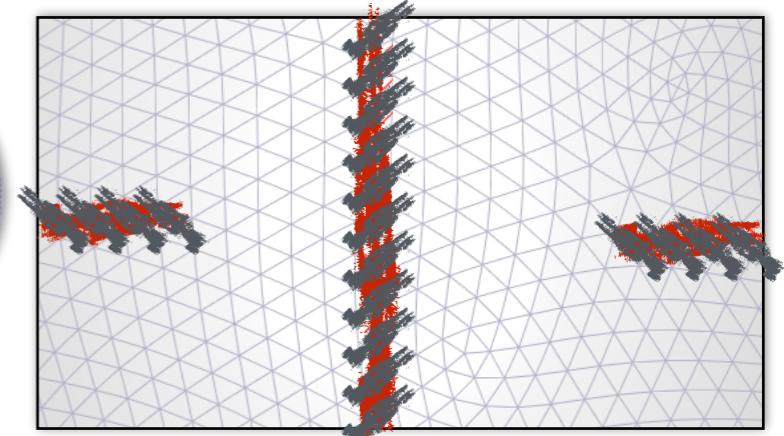
[Fisher et al. 2007]

- Complete or Partial

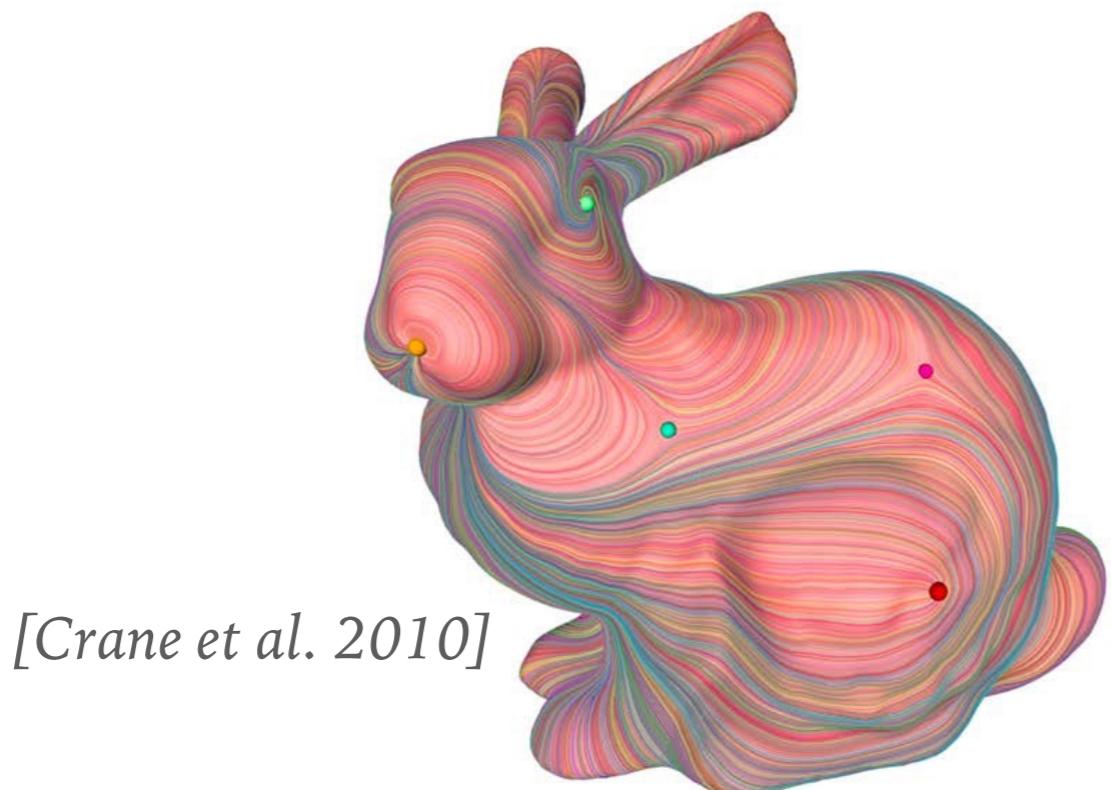
partial constraints



full constraints



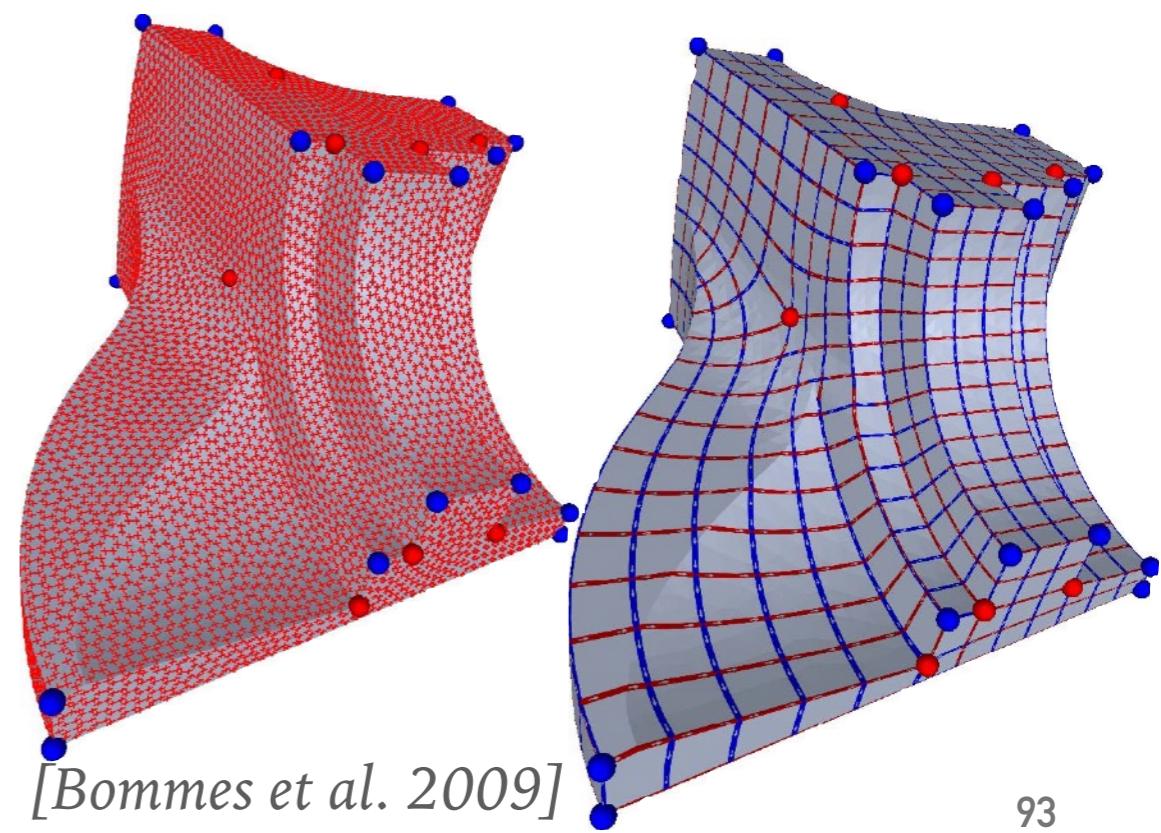
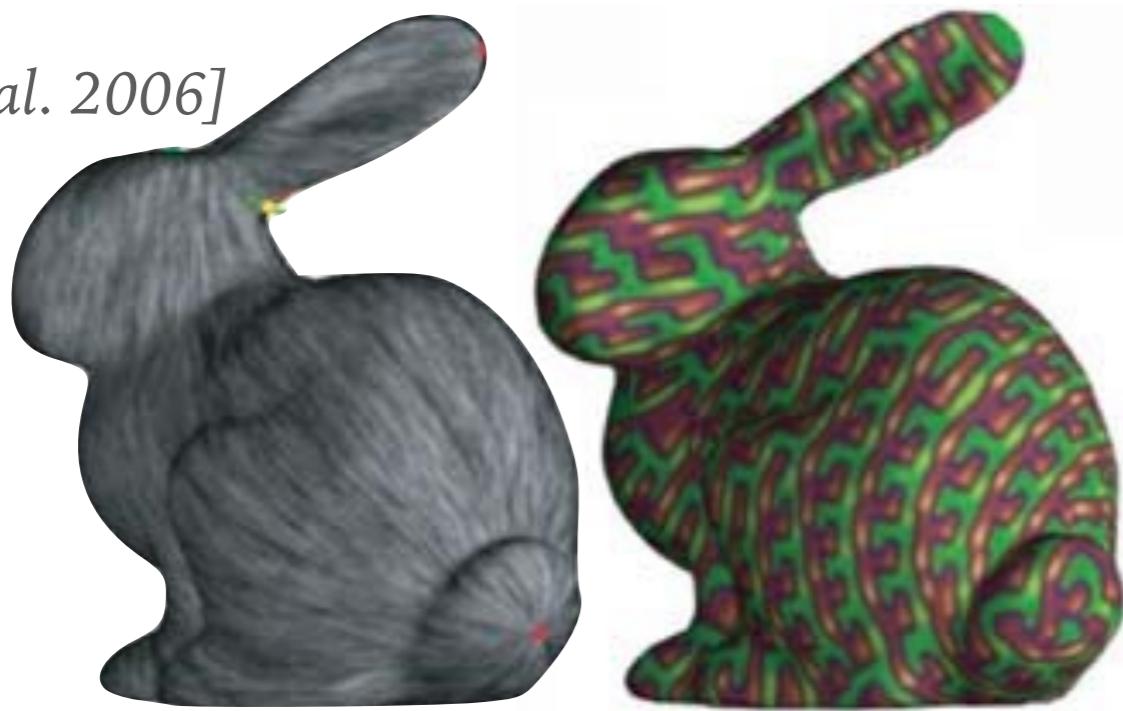
CONSTRAINTS - TOPOLOGY



[Crane et al. 2010]

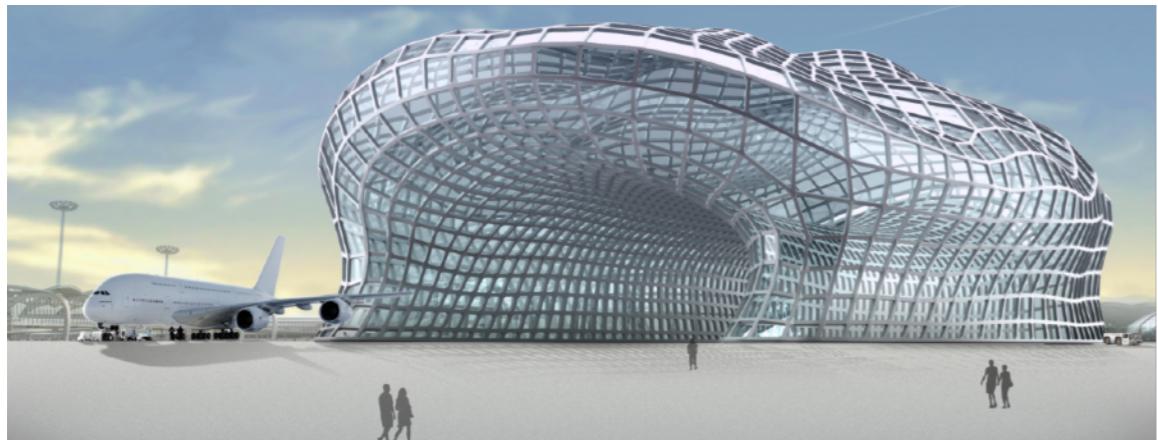


[Zhang et al. 2006]

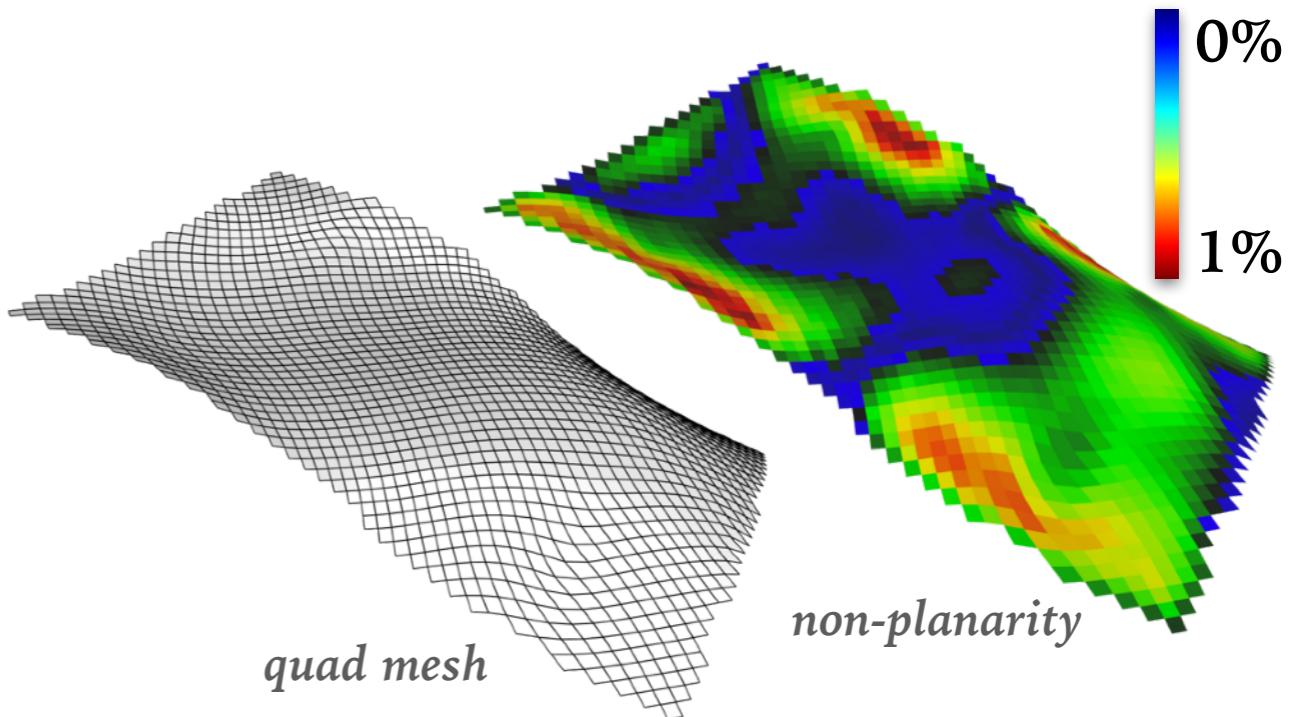


[Bommes et al. 2009]

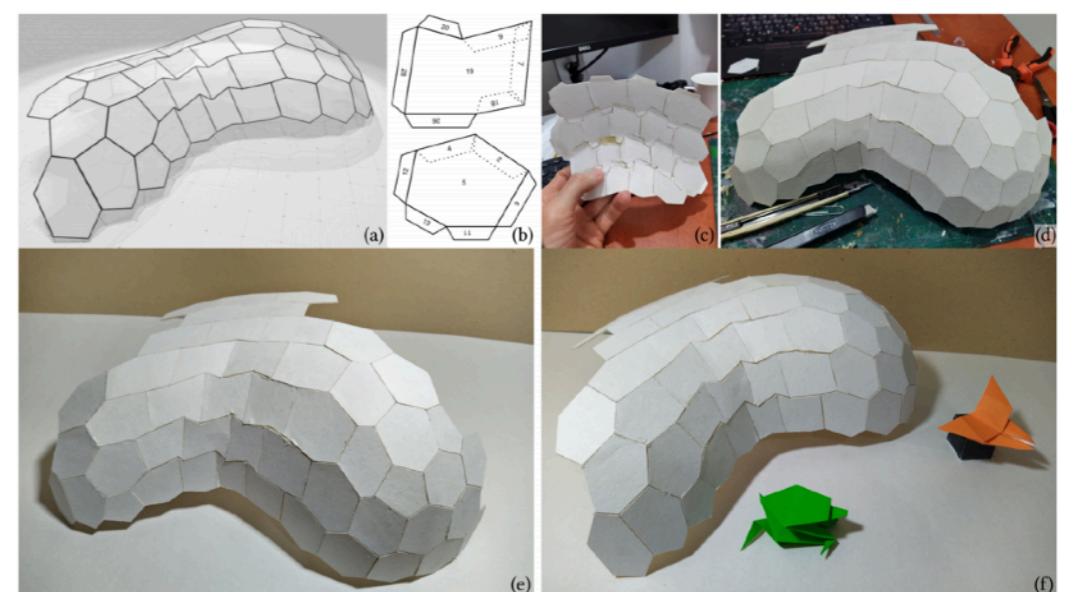
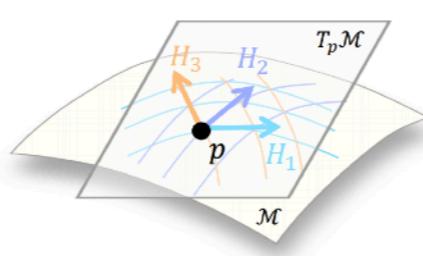
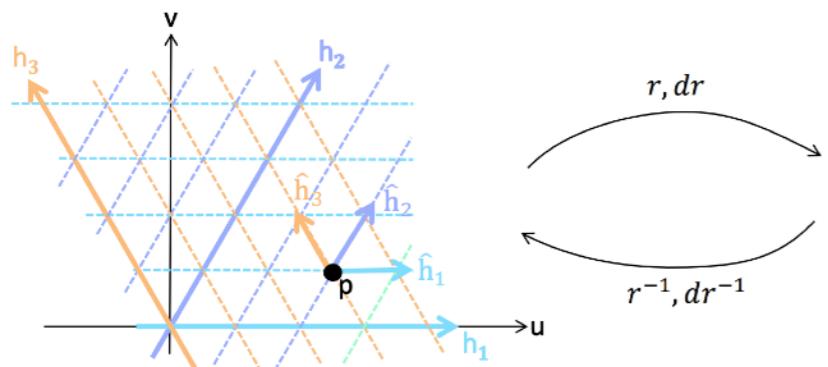
CONSTRAINTS - CONJUGACY (PLANARITY)



[Liu et al. 2011]

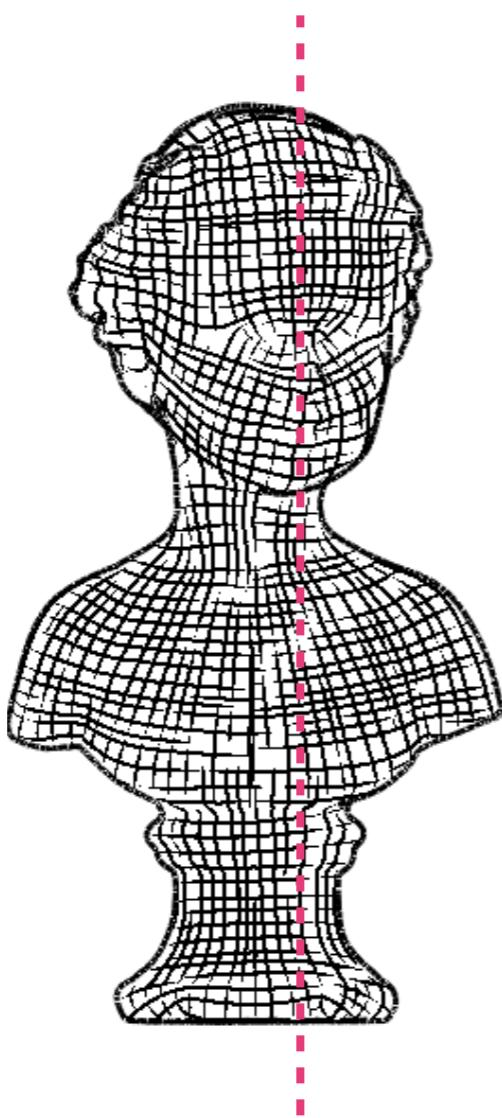


[Diamanti et al. 2014]

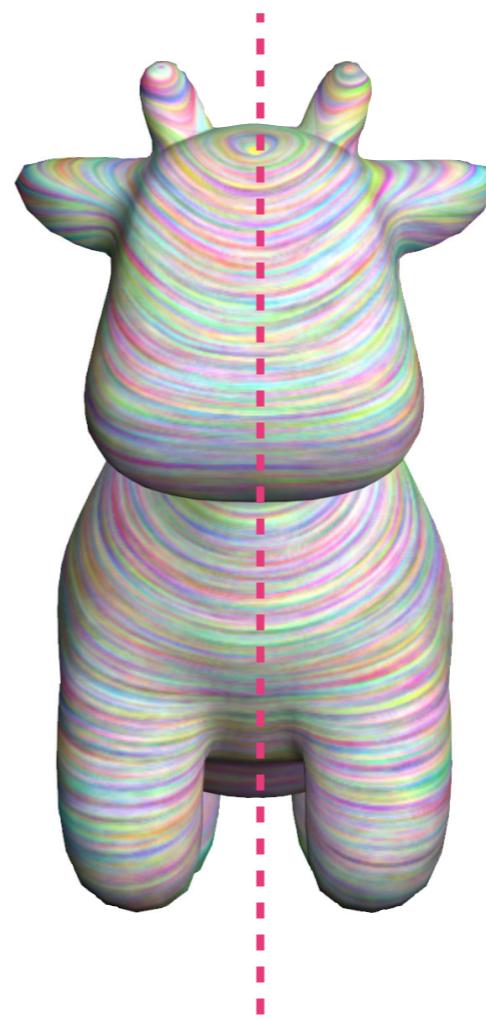


[Pluta et al. 2021]

CONSTRAINTS - SYMMETRY



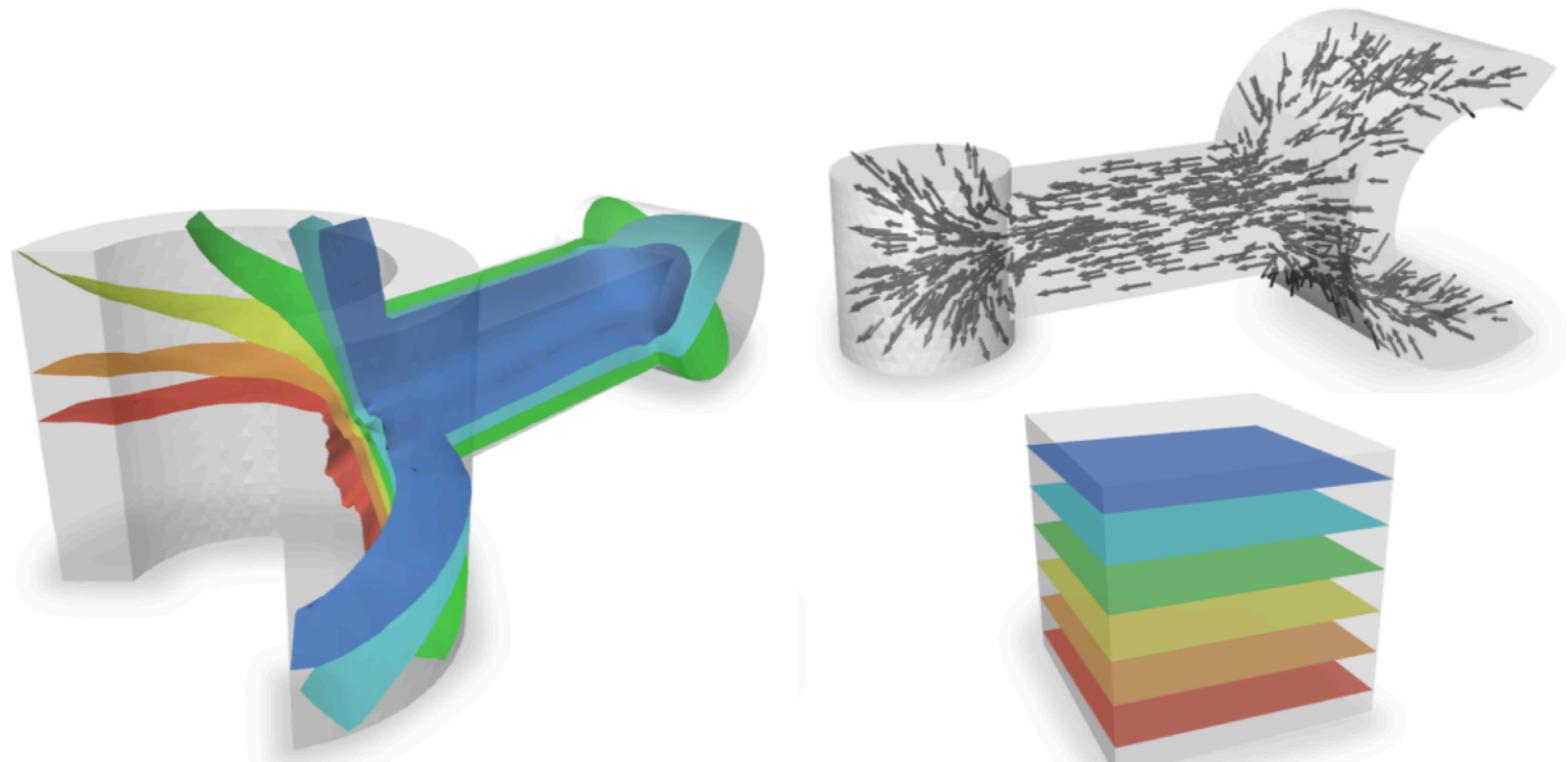
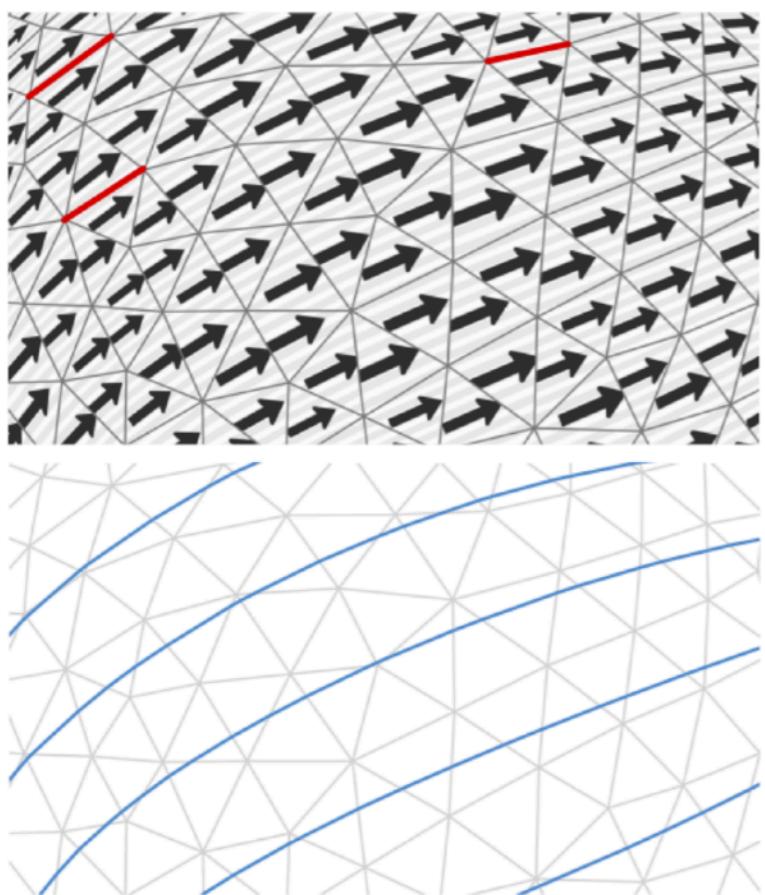
[Panozzo et al. 2012]



[Azencot et al. 2013]

CONSTRAINT: FOLIATION

- The field should trace lines (and dual surfaces) that foliate space



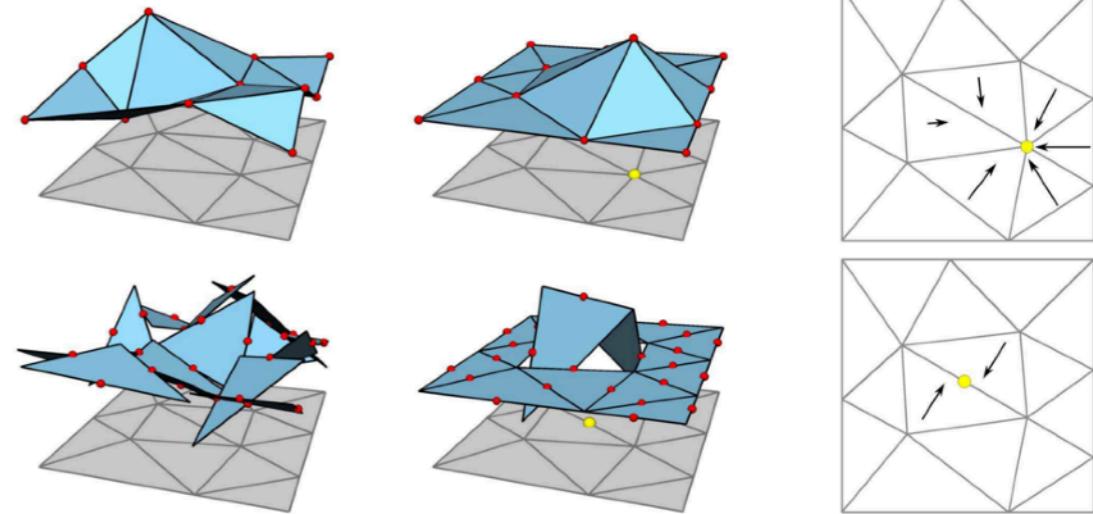
[Campen *et al.* 2017]

CONSTRAINT: STRUCTURE-PRESERVATION

- Discrete setting must respect:

$$v = \nabla f$$

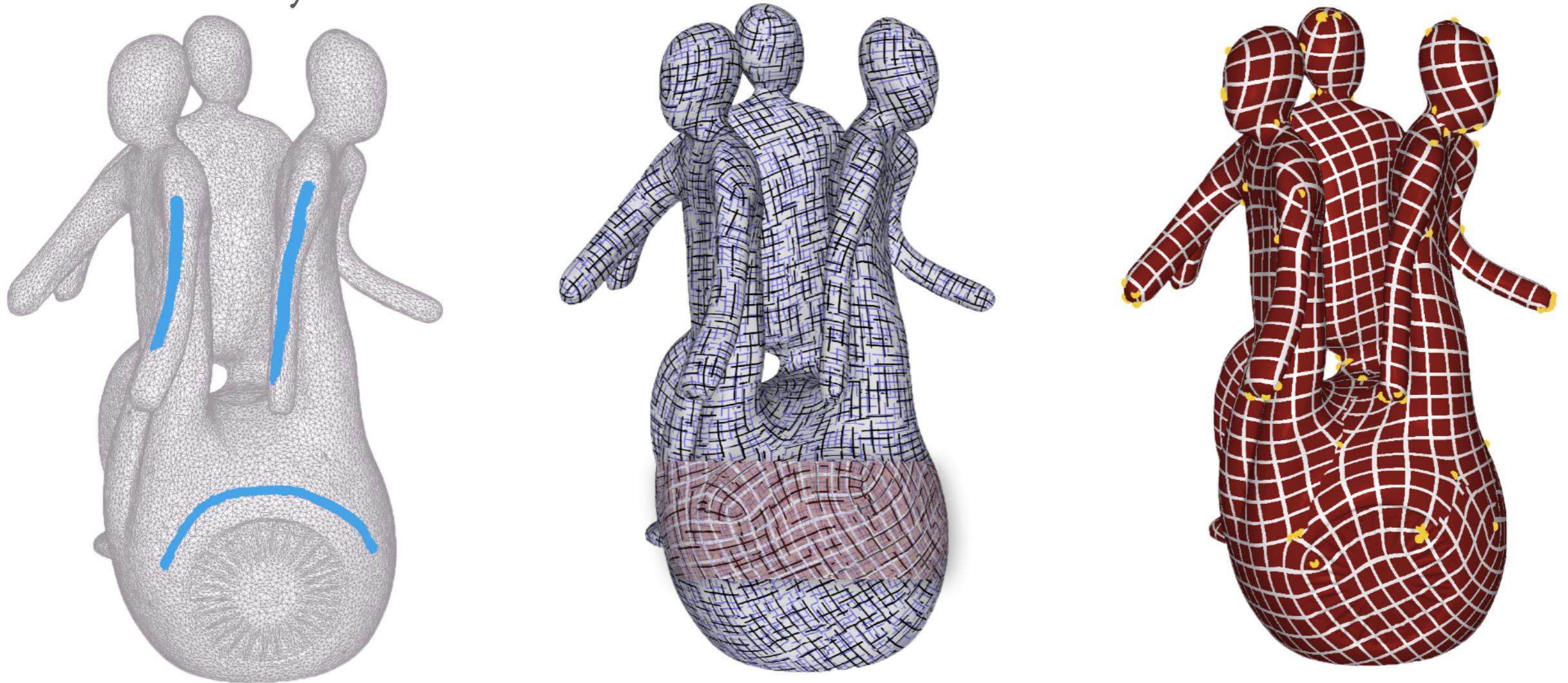
$$v = \nabla \times g$$



- By mixing conf/non-conf elements [Wardetzky 2006].
- Allowing for the existence and uniqueness of the Hodge decomposition.

CONSTRAINTS – INTEGRABILITY

- We want field to serve as *candidate gradients* for solving parameterizations.
- Necessary condition: to be curl free.

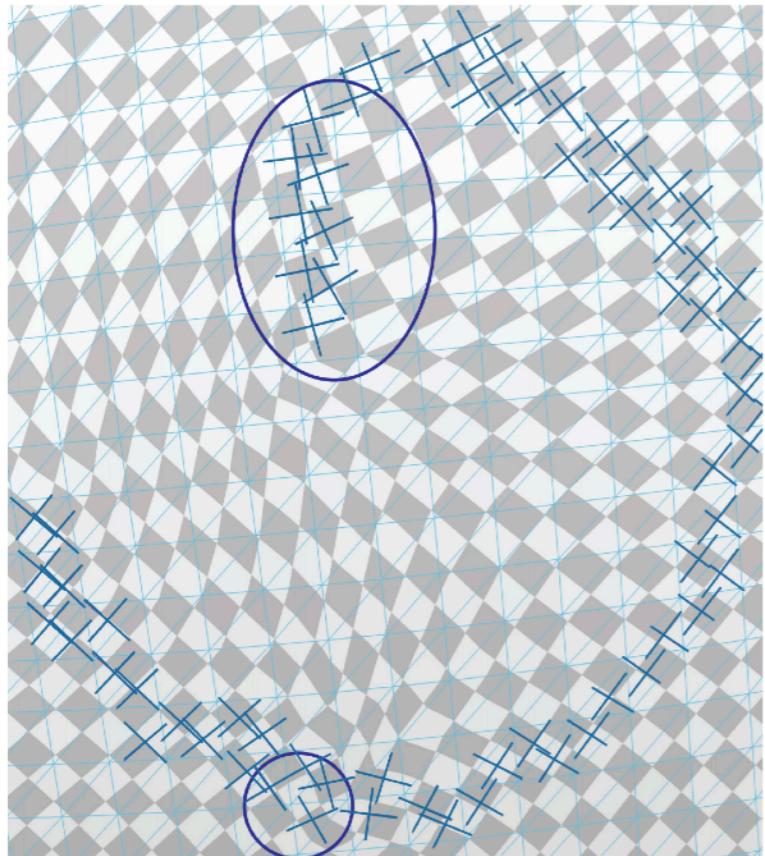


[Diamanti et al. 2015, Sageman-Furnas et al. 2019, Lu et al. 2020, Meekes et al. 2021...]

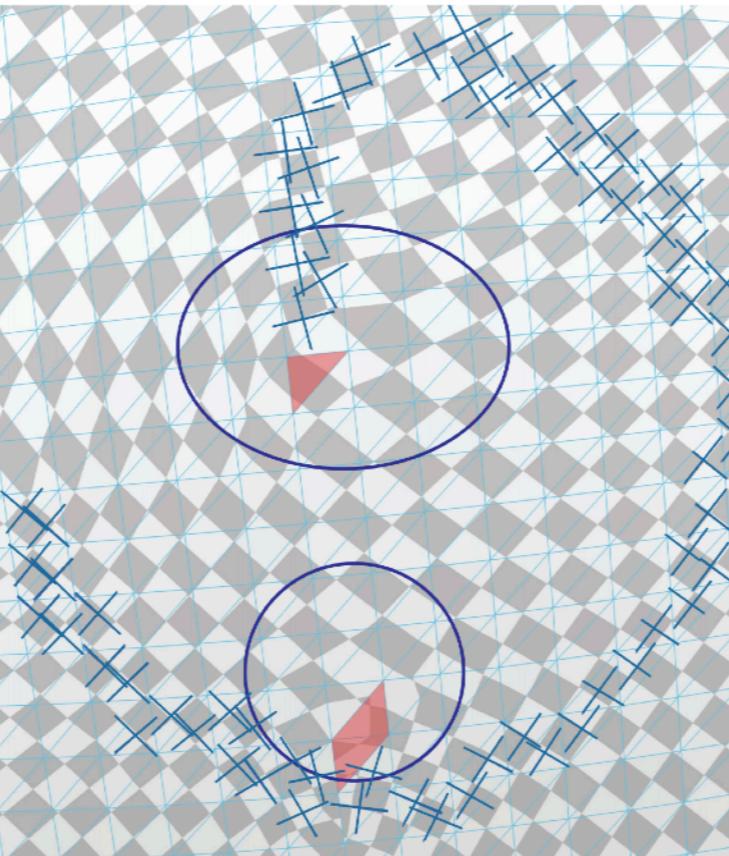
INTEGRABILITY

- Field to parameterization with minimum alignment error.

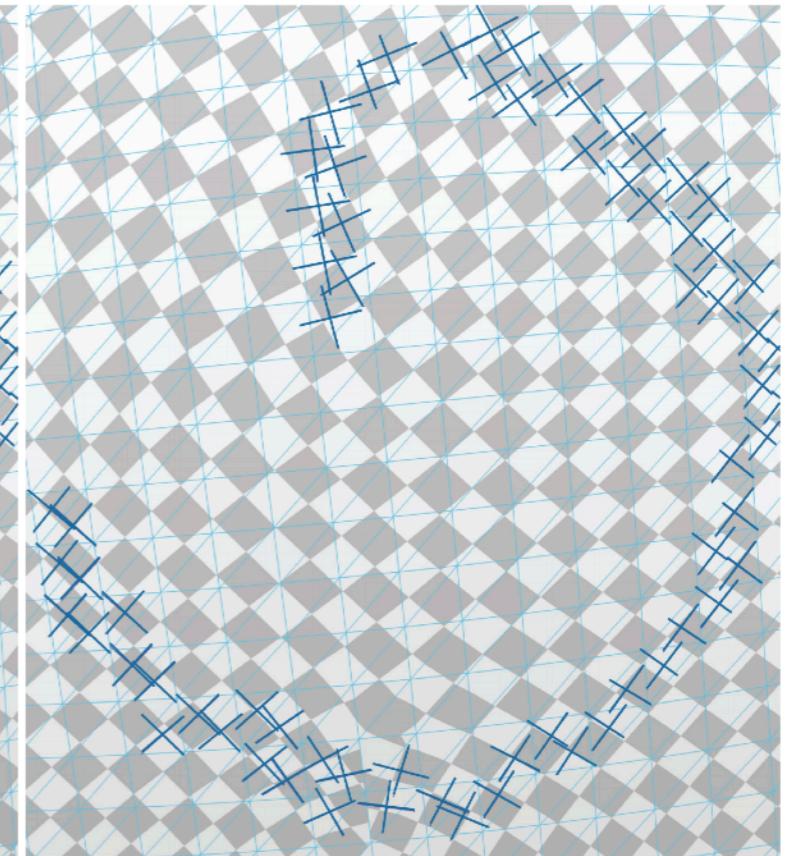
Non-integrable: misalignment



Non-integrable: flip-overs

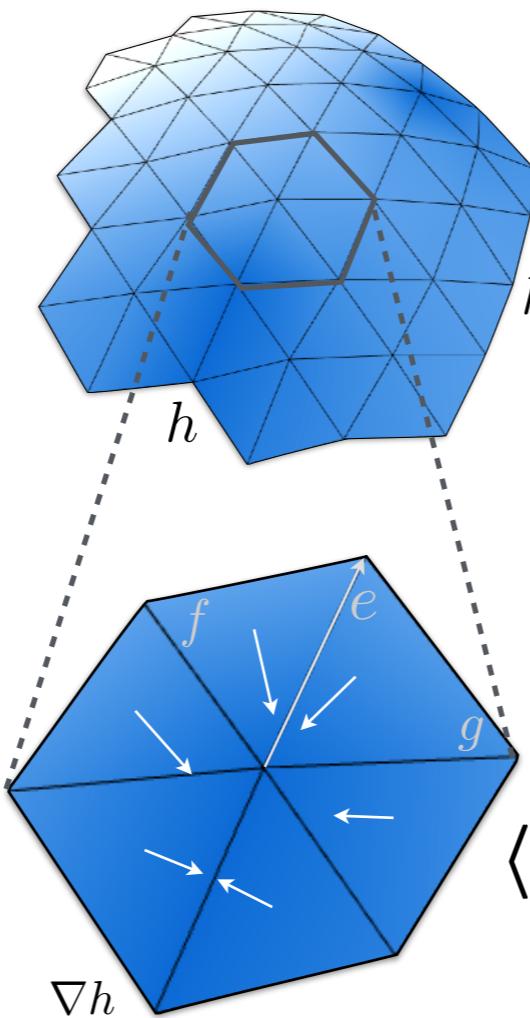


Integrable: perfect alignment



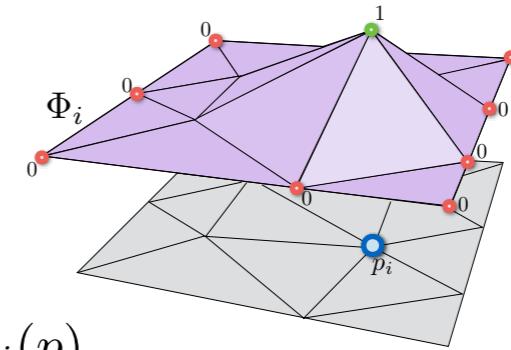
INTEGRABILITY AND DISCRETE CURL

Piecewise linear
function



Gradient

$$h(p) = \sum_{i \in V} h_i \Phi_i(p)$$



$$\langle \nabla h_f, e \rangle = \langle \nabla h_g, e \rangle$$

Discrete curl (per edge)
for vector field u

$$\langle u_f, e \rangle - \langle u_g, e \rangle$$

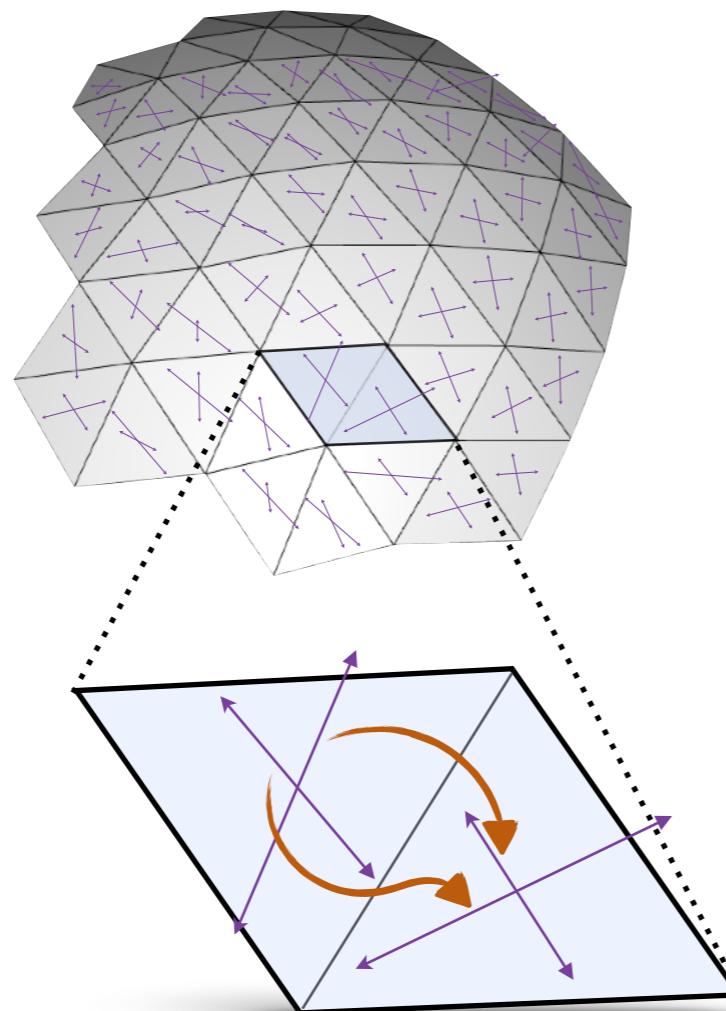
[Polthier and Preuß 2003]

[Kälberer *et al.* 2007]

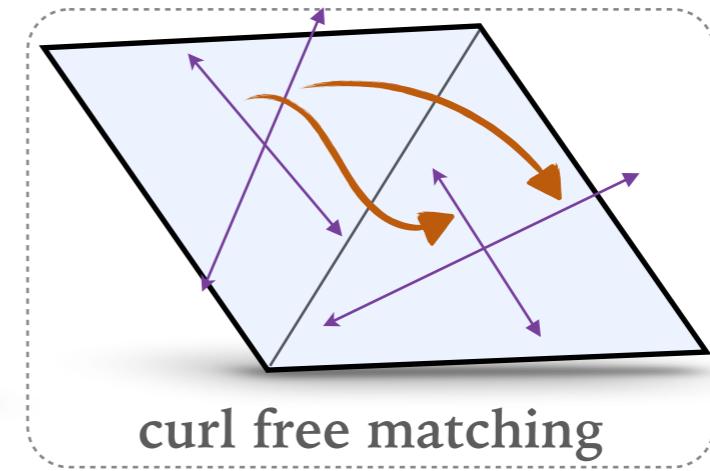
CURL-FREE FRAME FIELDS

frame field

matchings



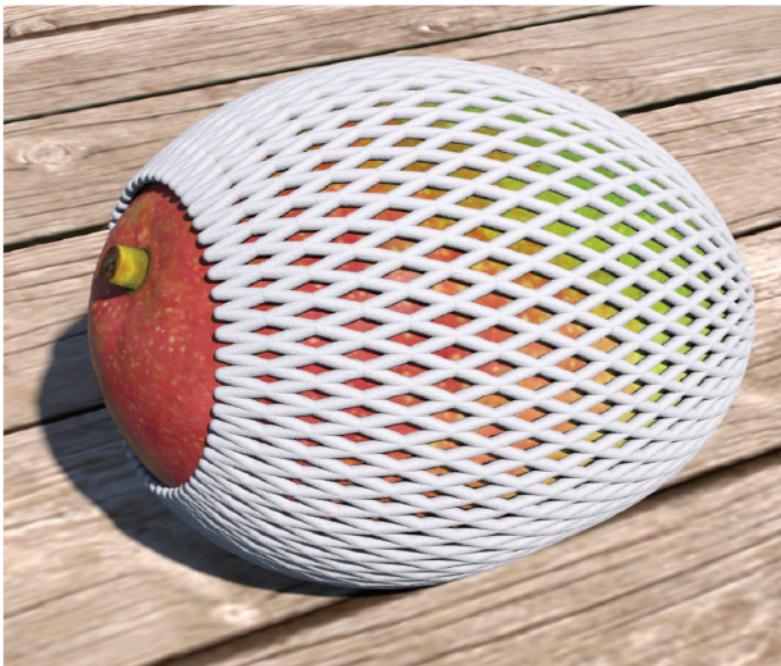
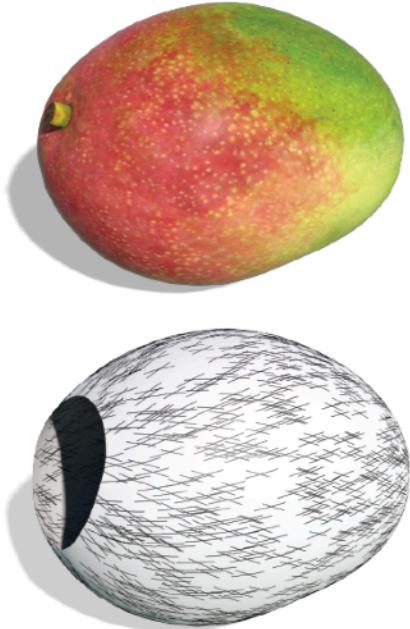
Discrete curl
 $\langle u_f, e \rangle - \langle u_g, e \rangle$



Curl-free directional field → Zero-error seamless parameterization!*

CONSTRAINTS - UNIFORM (UNIT) LENGTH

- $|u_i| = 1$
- Application: Chebyshev nets

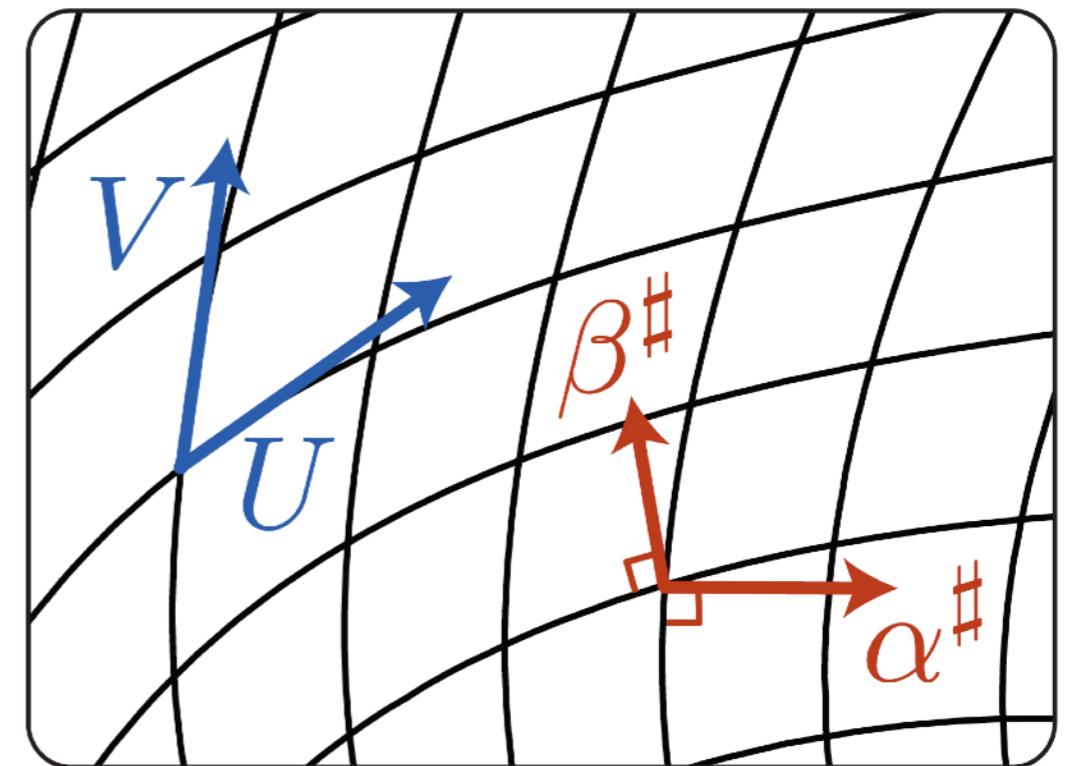


[Sageman-Furnas et al. 2019]

[Liu et al. 2020]

CHEBYSHEV PARAMETERIZATION

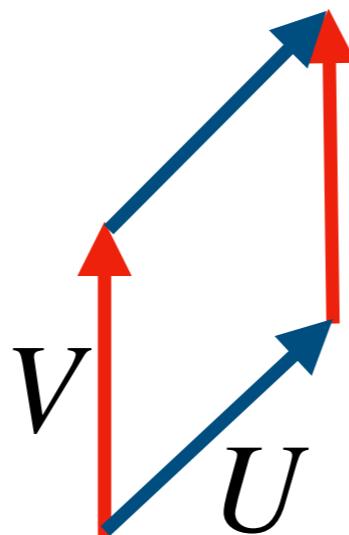
- Problem: traditional gradient design methods fail.
 - Gradients are normal to lines. We need tangents.
 - Gradients are not unit length!



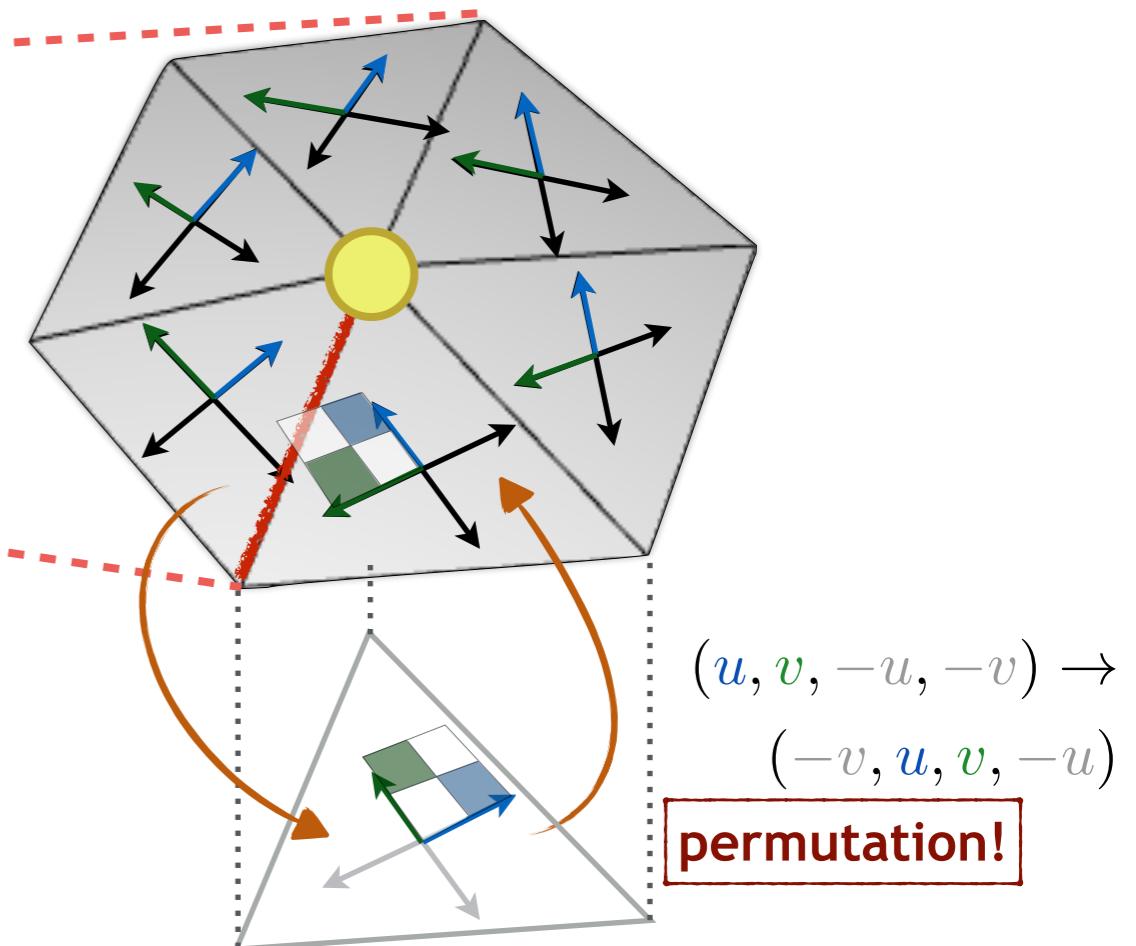
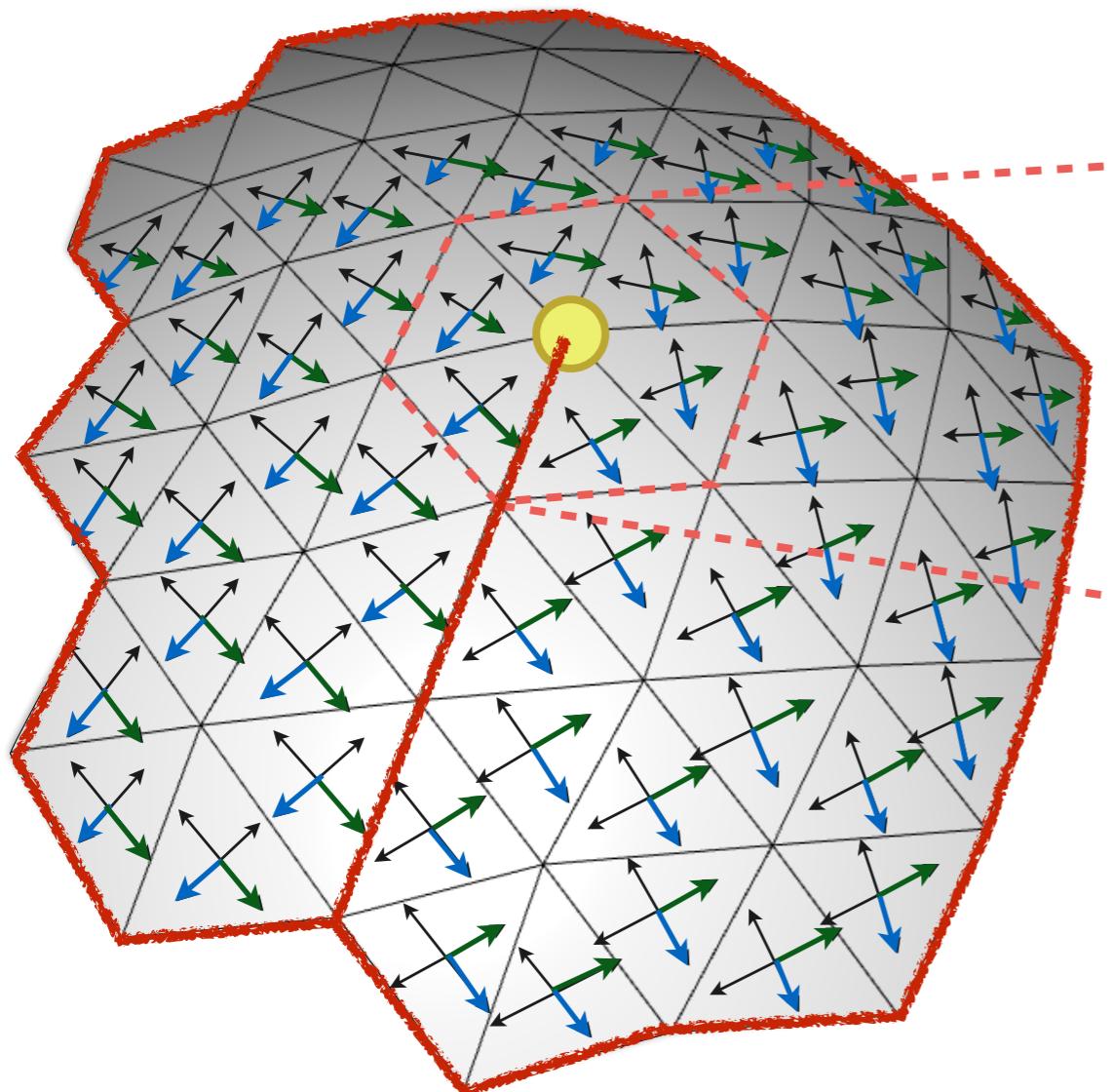
CONSTRAINTS: LIE-BRACKET INTEGRABILITY

- Alternative: Design unit-length tangents.
- Integrability condition: Optimize for zero Lie bracket:

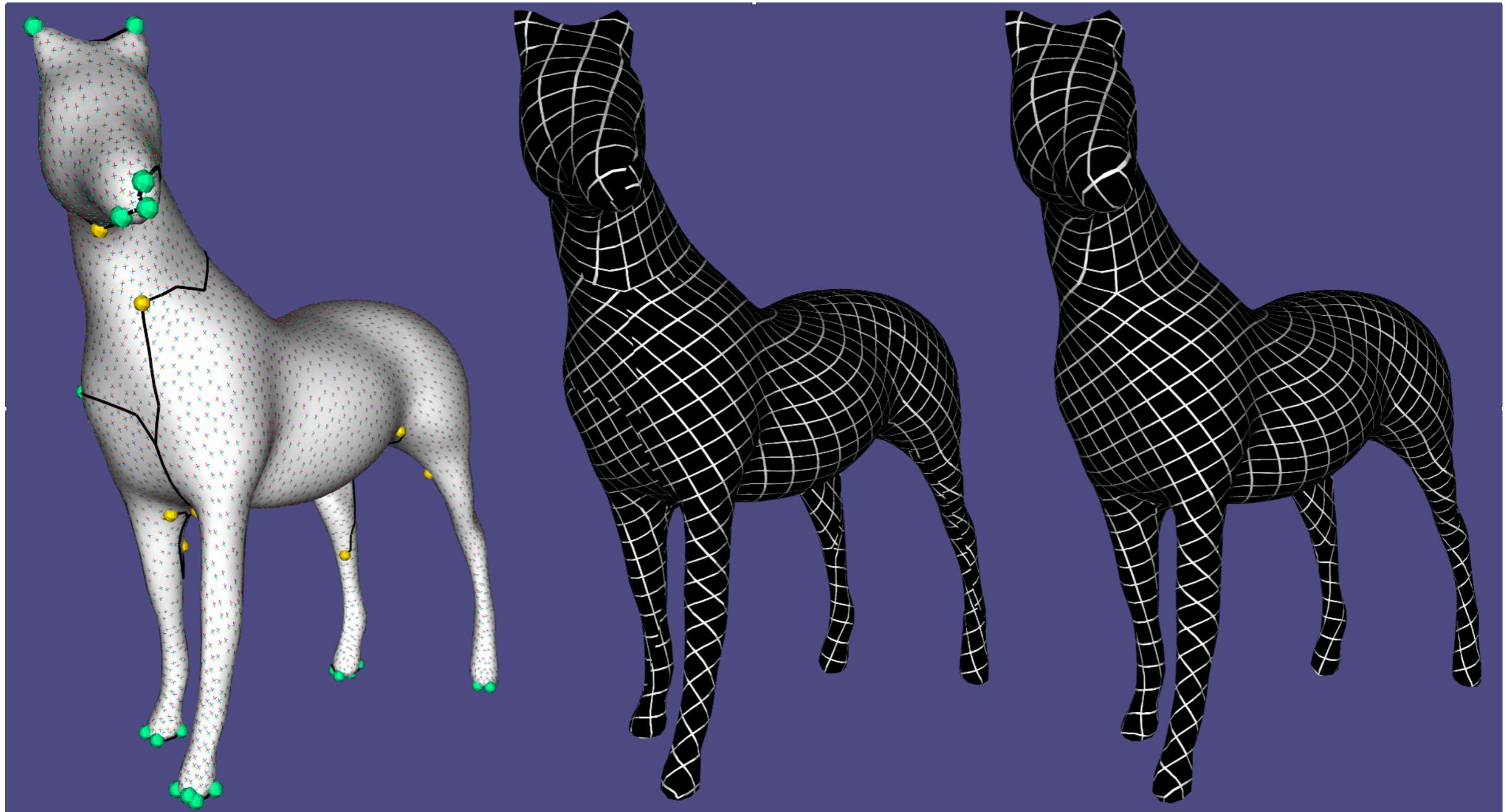
$$\nabla_U V - \nabla_V U = [U, V] = 0$$



RECOVERING THE SCALAR FUNCTIONS



SEAMLESS PARAMETERIZATION



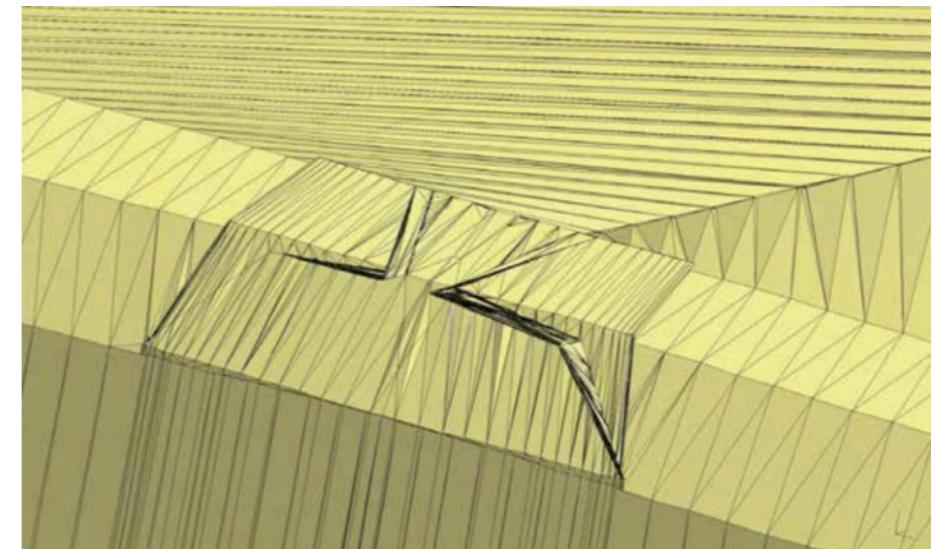
Field

Rot. Seamless

Fully seamless

CHALLENGES: DIRECTIONAL FIELDS “IN THE WILD”

- Many meshes are self-intersecting and not water-tight
- Most existing algorithms would fail!
- Interesting directions:
 - Envelopes [Hu et al. 2019]
 - Tufted covers [Sharp and Crane 2020]
 - Level sets



[Hu et al. 2019]

CHALLENGES: BIG GEOMETRIC DATA

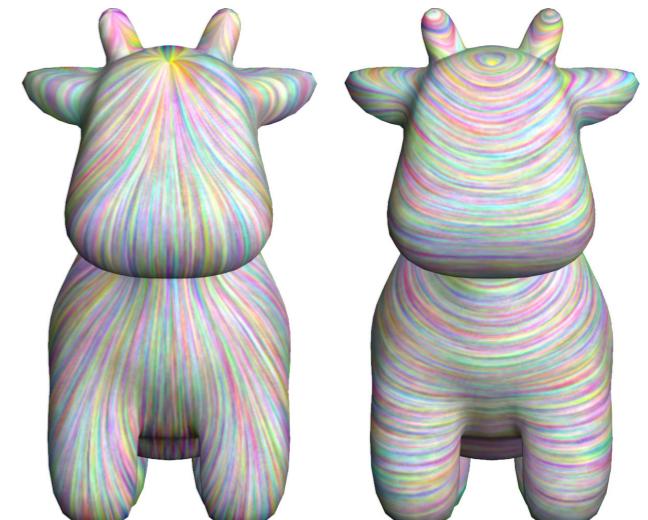
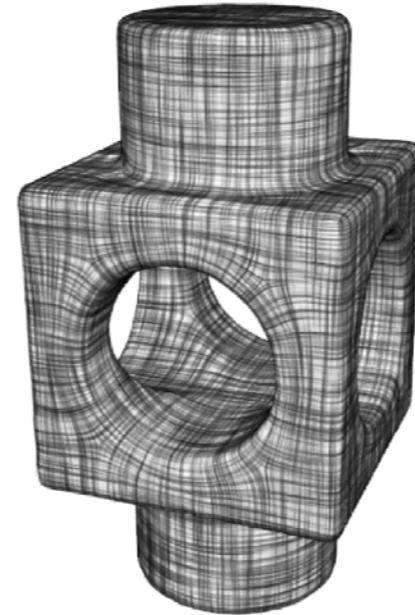
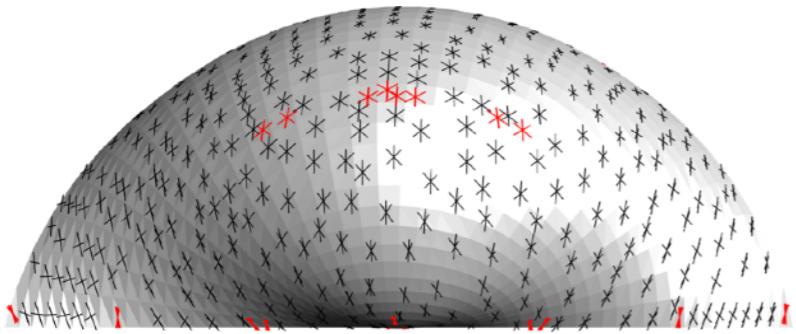
- Currently relying on sparse linear systems for design.
- Computing mostly smooth solutions.
- Interesting direction: multigrid.



[Liu *et al.* 2021]

DIRECTIONAL-FIELD PROCESSING SOURCES

- Vector field processing on triangle meshes: [https://
authors.library.caltech.edu/73395/1/a27-degoes.pdf](https://authors.library.caltech.edu/73395/1/a27-degoes.pdf)
- Directional: <https://github.com/avaxman/Directional>
- Geometry Central: [http://geometry-central.net/tutorials/
direction_fields/](http://geometry-central.net/tutorials/_direction_fields/)
- DGtal: <https://dgtal.org/doc/stable/packageDEC.html>
- CoMISo: [https://www.graphics.rwth-aachen.de/software/
comiso/](https://www.graphics.rwth-aachen.de/software/comiso/)
- ...Code for all individual papers mentioned!



Symmetry

Anti-symmetry

THANK YOU!

