A.

Proposition A.1. *If $F(\boldsymbol{\alpha})$ and $G(\boldsymbol{\alpha})$ are generating functions related by $F(\boldsymbol{\alpha}) = \phi(G(\boldsymbol{\alpha}) - G(\mathbf{0}))$ where ϕ is a scalar function with Taylor expansion $\phi(x) = \phi_0 + \sum_{k=1}^{\infty} \frac{1}{k!}\phi_k\, x^k$, their generalized moments are related by*

$$f(Q) = \sum_{k=1}^{n} \phi_k \sum_{(Q_1 \cdots Q_k)}^{\mathcal{P}_k(Q)} (-)^{t_{\mathbf{Q}}} g(Q_1) \cdots g(Q_k)$$

Proof: Denote the Taylor expansions of F and G as

$$F(\boldsymbol{\alpha}) = f_0 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{i_1 \cdots i_n} \alpha_{i_1} \cdots \alpha_{i_n} f(q_{i_1} \cdots q_{i_n}) \qquad G(\boldsymbol{\alpha}) = g_0 + \sum_{n=1}^{\infty} \tfrac{1}{n!} \sum_{i_1 \cdots i_n} \alpha_{i_1} \cdots \alpha_{i_n} g(q_{i_1} \cdots q_{i_n})$$

and define $G_n \equiv \frac{1}{n!} \sum_{i_1 \cdots i_n} \alpha_{i_1} \cdots \alpha_{i_n} g(q_{i_1} \cdots q_{i_n})$ so that $G(\boldsymbol{\alpha}) - G(\mathbf{0}) = \sum_{n=1}^{\infty} G_n$. Then

$$F = \phi(\textstyle\sum_n G_n) = \phi_0 + \displaystyle\sum_{k=1}^{\infty} \tfrac{1}{k!}\phi_k \left(\textstyle\sum_n G_n\right)^k = \phi_0 + \displaystyle\sum_{k=1}^{\infty} \tfrac{1}{k!}\phi_k \sum_{n_1 \cdots n_k} G_{n_1} \cdots G_{n_k} = \phi_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \tfrac{1}{k!}\phi_k \sum_{(n_1 \cdots n_k)}^{\mathcal{C}_k(n)} G_{n_1} \cdots G_{n_k}$$

where the last step sorts the sum into powers of $\boldsymbol{\alpha}$. $\mathcal{C}(n)$ denotes the set of integer compositions of n, i.e. the set of all ordered tuples $(n_1, n_2, \cdots)$ of positive integers that add up to n. $\mathcal{C}_k(n) \subset \mathcal{C}(n)$ is the set of k-tuple integer compositions of n, i.e. all $(n_1, \ldots, n_k)$ of fixed length k such that $n_1 + \cdots + n_k = n$. For each term in the sum