MIR - Modeling of Mechanical System
Double pendulum Assignment

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Double Pendulum

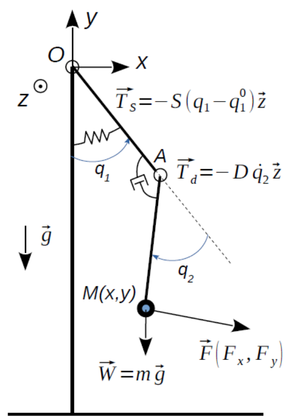

Figure 1: Double pendulum with stiffness and damping

Question1:

In the International System of Units (SI), the units for the stiffness coefficient S and the damping coefficient D are as follows:

1. The stiffness coefficient S is associated with the spring-like restoring torque, which is torque per unit angle (since $T_s = -S(q_1 - q_1')$. Since torque is measured in Newton-meters $N \cdot m$ in SI units, and angle is dimensionless (as it's often measured in radians), the unit for S is Newton-meters per radian $N \cdot m/rad$. However, radians are dimensionless, so it is often simplified to just Newton-meters $N \cdot m$.

2. The damping coefficient D is associated with the damping torque, which is torque per unit angular velocity (since $T_d = -D\dot{q}_2$). Angular velocity is measured in radians per second (rad/s), so the unit for D is Newton-meters per radian per second $N \cdot m \cdot s/rad$. As radians are dimensionless, this could be simplified to Newton-meters-seconds $N \cdot m \cdot s$.

Therefore, the units are:

- S: $N \cdot m$ or $N \cdot m/rad$
- D: $N \cdot m \cdot s$ or $N \cdot m \cdot s/rad$

Question 2:

The forward kinematics for a bi-articular pendulum system prescribe the transformation from the mass's Cartesian coordinates (x, y) to its rotational degrees of freedom, denoted by the angles q_1 and q_2. These angles correspond to the displacements of the pendulum segments relative to the plumb line.
Let L_1 and L_2 be the lengths of the proximal and distal segments of the pendulum, respectively. The spatial positioning of the mass M in Cartesian coordinates is deduced as follows:

The lateral displacement, or the x-coordinate, is described by:

$$x = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)$$

Conversely, the vertical displacement, or the y-coordinate, is articulated as:

$$y = -L_1 \cos(q_1) - L_2 \cos(q_1 + q_2)$$

Herein, q_1 is the angle of deviation for the proximal pendulum segment from the vertical axis, while the sum $q_1 + q_2$ represents the total angular deviation of the distal segment. The negative sign preceding the vertical displacement term accounts for the upward positive direction of the y-axis, opposite to the gravitational direction that the pendulum mass naturally assumes.

Here, q_1 is the angle of the first pendulum arm from the vertical, and $q_1 + q_2$ is the total angle of the second pendulum arm from the vertical. The negative sign in the y-coordinate equation accounts for the fact that the y-coordinate is measured positively upwards from the origin O, and the pendulum hangs downwards.
These equations assume that the pivot point O is at the origin of the coordinate system, and the angles are measured from the negative y-axis, counterclockwise.

Question 3:

The Jacobian matrix for this system provides a relationship between the velocities in the generalized coordinates and the velocities in the Cartesian coordinates. It is a matrix of partial derivatives of the Cartesian coordinates concerning the generalized coordinates.

Given the direct geometric model for the double pendulum:

$$x = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)$$

$$y = -L_1 \cos(q_1) - L_2 \cos(q_1 + q_2)$$

The Jacobian matrix J is defined as:

$$J = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{bmatrix}$$

$$R1 = \begin{bmatrix} \cos(q1) & -\sin(q1) & 0 \\ \sin(q1) & \cos(q1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T1 = \begin{bmatrix} 1 & 0 & L1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R2 = \begin{bmatrix} \cos(q2) & -\sin(q2) & 0 \\ \sin(q2) & \cos(q2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T2 = \begin{bmatrix} 1 & 0 & L2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_t = R1 \cdot T1 \cdot R2 \cdot T2$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & L_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) & L_2 \cos(q_1 + q_2) \\ L_1 \sin(q_1) + L_2 \sin(q_1 + q_2) & L_2 \sin(q_1 + q_2) \end{bmatrix}$$

Question 4:

The generalized forces, Q_1 for $\vec{W}$ and Q_2 for $\vec{\vec{W}}$, concerning the weight of the sphere, are determined by projecting the gravitational force onto the directions of the generalized coordinates q_1 and q_2. Since the gravitational force acts downward, its effect on the motion of the pendulum arms will depend on the configuration of the system, specifically how the arms are positioned relative to the vertical.

The weight of the sphere $\vec{W}$ can be expressed as $\vec{W} = m\vec{g}$, where m is the mass of the sphere and $\vec{g}$ is the acceleration due to gravity. To find the generalized forces due to the weight, we calculate the virtual work done by the weight for virtual displacements in the directions of q_1 and q_2. The virtual work δW done by the force $\vec{F}$ for a virtual displacement $\delta\vec{r}$ is given by:

$$\delta W = \vec{F} \cdot \delta\vec{r}$$

For the generalized force Q_i, in the direction of a generalized coordinate q_i, we use:

$$Q_i = \frac{\partial \delta W}{\partial q_i}$$

Considering the pendulum system, the weight will only do work during the movement of the mass M in the vertical direction (y-direction). Therefore, we consider the vertical component of the virtual displacement associated with each q_i.

The virtual displacement $\delta \vec{r}$ in the y-direction for a small change in q_1 and q_2 is:

$$\delta y_{q_1} = \frac{\partial y}{\partial q_1} \delta q_1$$
$$\delta y_{q_2} = \frac{\partial y}{\partial q_2} \delta q_2$$

From the earlier equations, we have:

$$\frac{\partial y}{\partial q_1} = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)$$
$$\frac{\partial y}{\partial q_2} = L_2 \sin(q_1 + q_2)$$

The virtual work done by the weight for these virtual displacements is:

$$\delta W_{q_1} = -mg \frac{\partial y}{\partial q_1} \delta q_1$$

The generalized forces $Q_{q1}\left(\vec{T_s}\right)$ and $Q_{q2}\left(\vec{T_s}\right)$ relative to the Weight torque are:

$$Q_{q_1}(\vec{W}) = -mg(L_1 \sin(q_1) + L_2 \sin(q_1 + q_2))$$
$$Q_{q_2}(\vec{W}) = -mgL_2 \sin(q_1 + q_2)$$

These expressions represent the torques due to the gravitational force acting on the mass M at the end of the pendulum, as a function of the angular positions q_1 and Q. The negative signs indicate that the gravitational force acts in the opposite direction to the positive angles defined by q_1 and q_2.

Question 5:

To express the generalized forces $Q_{q1}\left(\vec{T_s}\right)$ and $Q_{q2}\left(\vec{T_s}\right)$ relative to the stiffness torque, we need to consider the torque's contribution to the virtual work done due to small displacements in $q1$ and

$q2$. The stiffness torque, given by $\vec{T}_s = -S_{q1}\vec{z}$, will only do work concerning changes in the angle $q1$ because it is a function of $q1$ alone.
The work done by a torque $\vec{T}$ during a small angular displacement $d\vec{\theta}$ is $dW = \vec{T} \cdot d\vec{\theta}$, where $d\vec{\theta}$ is the vector of small angular displacements. Since $\vec{T}_s$ is in the direction of $\vec{z}$, and we are considering rotations about the z-axis (the axis perpendicular to the plane of motion), the work done by $\vec{T}_s$ for small changes in $q1$ and $q2$ is:

$$dW_{q1} = \vec{T}_s \cdot d\vec{\theta}_{q1} = -S_{q1} d\theta_{q1}$$

The generalized forces $Q_{q1}\left(\vec{T}_s\right)$ and $Q_{q2}\left(\vec{T}_s\right)$ relative to the stiffness torque are:

$$Q_{q1}\left(\vec{T}s\right) = -Sq1$$
$$Q_{q2}\left(\vec{T}_s\right) = 0$$

Question 6:

The damping torque $\vec{T}_d$ given by $\vec{T}_d = -D\dot{q}_2\vec{z}$ affects the motion associated with the angular velocity q_2 of the second pendulum arm. Because this torque is proportional to the angular velocity $\dot{q}_2$ and acts in the direction that opposes the motion, it does not contribute to work done concerning q_1 (since it does not depend on q_1 or $\dot{q}_1$).

To find the generalized force due to the damping torque for q_2, we consider the virtual work done by $\vec{T}_d$ which is the product of the torque and the angular displacement in the direction of q_2. The generalized force for q_2 due to the damping torque can be expressed as:

$$Q_{q_2}\left(\vec{T}_d\right) = -D\dot{q}_2$$

Since $\vec{T}_d$ is only a function of $\dot{q}_2$, it will not contribute to the generalized force for q_1, and thus:

$$Q_{q_1}\left(\vec{T}_d\right) = 0$$

So, we have:

$$Q_{q1}\left(\vec{T}_d\right) = 0$$

$$Q_{q2}\left(\vec{T}_d\right) = -D\dot{q}_2$$

These expressions represent the generalized forces due to the damping torque, which will appear in the equations of motion when we apply the Euler-Lagrange equation to account for non-conservative forces.

Question 7:

To express the generalized forces $Q_{q1}\left(\vec{F}\right)$ and $Q_{q2}\left(\vec{F}\right)$ relative to the external force $\vec{F}$, we consider the work done by this force for small displacements in the directions of the generalized coordinates $q1$ and $q2$. The external force $\vec{F}$ is given by $\vec{F} = F_x\hat{x} + F_y\hat{y}$, where F_x and F_y are the components of the force in the Cartesian coordinate system.

The virtual work done by the force $\vec{F}$ for a virtual displacement $\delta\vec{r}$ in Cartesian space is given by $\delta W = \vec{F}\cdot\delta\vec{r}$. We need to express this virtual work in terms of the virtual displacements associated with $q1$ and $q2$.

Given the x and y positions of the mass M as functions of $q1$ and $q2$:

$$x = L_1\sin(q1) + L_2\sin(q1+q2)$$
$$y = -L_1\cos(q1) - L_2\cos(q1+q2)$$

The virtual displacements in x and y for small changes in $q1$ and $q2$ are:

$$\delta x_{q1} = \frac{\partial x}{\partial q1}\delta q1$$
$$\delta x_{q2} = \frac{\partial x}{\partial q2}\delta q2$$
$$\delta y_{q1} = \frac{\partial y}{\partial q1}\delta q1$$
$$\delta y_{q2} = \frac{\partial y}{\partial q2}\delta q2$$

The virtual work done by $\vec{F}$ for these virtual displacements is:

$$\delta W_{q1} = F_x \delta x_{q1} + F_y \delta y_{q1}$$
$$\delta W_{q2} = F_x \delta x_{q2} + F_y \delta y_{q2}$$

The generalized forces $Q_{q1}\left(\vec{F}\right)$ and $Q_{q2}\left(\vec{F}\right)$ are then obtained by differentiating the virtual work concerning the virtual displacements:

$$Q_{q1}\left(\vec{F}\right) = \frac{\partial(\delta W_{q1})}{\partial(\delta q1)} = F_x \frac{\partial x}{\partial q1} + F_y \frac{\partial y}{\partial q1}$$
$$Q_{q2}\left(\vec{F}\right) = \frac{\partial(\delta W_{q2})}{\partial(\delta q2)} = F_x \frac{\partial x}{\partial q2} + F_y \frac{\partial y}{\partial q2}$$

These expressions represent the generalized forces due to the external force $\vec{F}$, in terms of the generalized coordinates $q1$ and $q2$. Let's calculate these explicitly.

The generalized forces $Q_{q1}\left(\vec{F}\right)$ and $Q_{q2}\left(\vec{F}\right)$, relative to the external force $\vec{F}$, are expressed as follows:

$$Q_{q1}\left(\vec{F}\right) = F_x(L_1\cos(q1) + L_2\cos(q1+q2)) + F_y(L_1\sin(q1) + L_2\sin(q1+q2))$$

$$Q_{q2}\left(\vec{F}\right) = F_x L_2 \cos(q1+q2) + F_y L_2 \sin(q1+q2)$$

These formulas denote the torques or generalized forces exerted on the double pendulum by the external force vector $\vec{F}$ in the horizontal (x) and vertical (y) directions. These forces are functions of the generalized coordinates q_1 and q_2.

Question 8:

In the examination of forces exerted upon the pendulum, two primary forces are delineated: gravitational force and Stiffness torque.

Gravitational Force (Conservative): This force is delineated by $F_g = -mg$, where m epitomizes the pendulum's mass and g the acceleration due to gravity. The force's negative magnitude conveys its direction as antithetical to the displacement vector.

Stiffness Torque (Conservative): Emanating from the material's reluctance to deform, this torque is designated by $T_s = -S \cdot q_1$, with S representing the stiffness coefficient and q_1 the angular displacement.

To compute the potential energy (U) associated with these conservative forces, the subsequent methodology is employed:

For the *Gravitational Potential Energy* (E_{pg}), it is calculated as the negative of the work conducted by the gravitational force, adhering to the rudimentary work equation:

$$\delta W = -dE \quad \Rightarrow \quad \vec{F} \cdot d\vec{M} = -dE \tag{0.1}$$

Given that $\vec{F} = -mg$ symbolizes the gravitational force and $d\vec{M}$ the infinitesimal displacement in the vertical direction dy, we deduce:

$$-mg \cdot dy = -dE \tag{0.2}$$

Integrating both expressions, we derive:

$$E_{pg} = \int mg \cdot dy = mgy + C \tag{0.3}$$

Herein, y denotes the vertical displacement, and C is the constant of integration.

The total potential energy U of the system is the sum of these potential energies:

$$U = V_g + V_s$$

$$U = mgy + \frac{1}{2}Sq_1^2$$

$$U = -mg(L_1\cos(q_1) + L_2\cos(q_1 + q_2)) + \frac{1}{2}Sq_1^2$$

The adjusted total potential energy U of the system, with the condition that $U = 0$ when $q_1 = \frac{\pi}{2}$ and $q_2 = 0$, is given by:

$$U = \frac{1}{2}Sq_1^2 - \frac{1}{8}\pi^2 S + gm(L_1\cos(q_1) + L_2\cos(q_1 + q_2))$$

This expression represents the total potential energy of the double pendulum system, accounting for both the gravitational potential energy and the potential energy due to the stiffness. The term $\frac{1}{8}\pi^2 S$ is a constant that ensures the potential energy is zero at the specified angles.

Question 9:

The translational kinetic energy T_t of the mass M at the end of the double pendulum is given by the standard formula for kinetic energy:

$$T_t = \frac{1}{2}mv^2$$

where m is the mass of the pendulum bob and v is its velocity. The velocity v can be expressed in terms of the Cartesian coordinates x and y as a function of generalized coordinates q_1 and q_2, their time derivatives $\dot{q}_1$ and $\dot{q}_2$.

The velocity components $\dot{x}$ and $\dot{y}$ can be found by differentiating expressions for x and y with respect to time:

$$x = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)$$

$$y = -L_1 \cos(q_1) - L_2 \cos(q_1 + q_2)$$

Thus, velocity components are:

$$\dot{x} = \frac{\partial x}{\partial t} = \frac{\partial x}{\partial q_1} \cdot \frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2} \cdot \frac{\partial q_2}{\partial t}$$

$$\dot{y} = \frac{\partial y}{\partial t} = \frac{\partial y}{\partial q_1} \cdot \frac{\partial q_1}{\partial t} + \frac{\partial y}{\partial q_2} \cdot \frac{\partial q_2}{\partial t}$$

The total velocity v is then the square root of the sum of the squares of these components:

$$v = \sqrt{x^2 + y^2}$$

Substituting v into the kinetic energy formula gives the translational kinetic energy. Let's calculate it.

the translational kinetic energy for a mass m is:

$$T_t = \frac{1}{2}mv^2$$

$$T_t = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T_t = \frac{1}{2}m((L_1 \cos(q_1)\dot{q}_1 + L_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2))^2 + (L_1 \sin(q_1)\dot{q}_1 + L_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2))^2)$$

Expanding the squared terms separately for $\dot{x}^2$ and $\dot{y}^2$, we would get:

$$\dot{x}^2 = (L_1\cos(q_1)\dot{q}_1)^2 + 2L_1L_2\cos(q_1)\cos(q_1+q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2) + (L_2\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2))^2$$

$$\dot{y}^2 = (L_1\sin(q_1)\dot{q}_1)^2 + 2L_1L_2\sin(q_1)\sin(q_1+q_2)\dot{q}_1(\dot{q}_1+\dot{q}_2) + (L_2\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2))^2$$

Adding $\dot{x}^2$ and $\dot{y}^2$:

$$\dot{x}^2+\dot{y}^2 = L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)(\cos(q_1)\cos(q_1+q_2)+\sin(q_1)\sin(q_1+q_2)) + L_2^2(\dot{q}_1+\dot{q}_2)^2$$

Using the trigonometric identity $\cos(a)\cos(b)+\sin(a)\sin(b) = \cos(a-b)$, the expression simplifies to:

$$\dot{x}^2+\dot{y}^2 = L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2) + L_2^2(\dot{q}_1+\dot{q}_2)^2$$

Substituting this into the expression for T_t, we get:

$$T_t = \frac{1}{2}m[L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2) + L_2^2(\dot{q}_1+\dot{q}_2)^2]$$

This expression represents the kinetic energy associated with the linear motion of pendulum mass, taking into account contributions from both angular velocities i_1 and i_q and lengths of pendulum arms L_1 and L_2. The terms involving $\cos(q)$ arise from the coupled nature motion of two pendulum segments.

Question 10:

The rotational kinetic energy T_r of the sphere at the end of the double pendulum can be calculated using the formula for rotational kinetic energy:

$$T_r = \frac{1}{2}I\omega^2$$

where I is the moment of inertia of the sphere with respect to its diameter, and ω is the angular velocity of the sphere.

For a sphere, the moment of inertia about any diameter is given by:

$$I = \frac{2}{5}mr^2$$

where m is the mass of the sphere, and r is its radius.

The angular velocity ω of the sphere is the same as the angular velocity of the second segment pendulum, which is q_2. This is because the sphere is fixed to the end of the pendulum, and thus rotates with the same angular velocity as the pendulum arm it is attached to.

Therefore, the rotational kinetic energy T_r is:

$$T_r = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\dot{q}_2^2$$

The rotational kinetic energy T_r of the sphere at the end of the double pendulum, calculated using the moment of inertia to its diameter, is given by:

$$T_r = \frac{1}{5}mr^2\dot{q_2}^2$$

Question 11:

The Lagrangian L of a mechanical system is defined as the difference between the total kinetic energy T and the total potential energy U. In the case of our double pendulum system with a rotating sphere at the end, the total kinetic energy T is the sum of the translational kinetic energy T_t and the rotational kinetic energy T_r.

The total kinetic energy T is thus:

$$T = T_t + T_r$$

From previous calculations, we have:

$$T_t = \frac{1}{2}m\left(L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1\dot{q}_2\cos(q_2) + 2L_1L_2\dot{q}_1^2\cos(q_2) + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2\right)$$

$$T_r = \frac{1}{5}mr^2\dot{q}_2^2$$

The total potential energy U was calculated as:

$$U = \frac{1}{2}Sq_1^2 - \frac{1}{8}Tq_1^2 + gm\left(L_1\cos(q_1) + L_2\cos(q_1+q_2)\right)$$

The simplified expression for the Lagrangian L of the double pendulum system is: The Lagrangian L is:

$$L = T_r + T_t - U$$

$$L = \frac{1}{2}m\left[L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2) + L_2^2(\dot{q}_1+\dot{q}_2)^2\right]$$
$$+\frac{1}{2}\left(\frac{5}{2}mR^2\right)(\dot{q}_1+\dot{q}_2)^2$$
$$-\left[-mg(L_1\cos(q_1)+L_2\cos(q_1+q_2)) + \frac{1}{2}S\left(q_1-\frac{\pi}{2}\right)^2\right]$$
$$L = \frac{1}{2}m\left[L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2) + \left(L_2^2+\frac{5}{2}R^2\right)(\dot{q}_1+\dot{q}_2)^2\right]$$
$$+mg(L_1\cos(q_1)+L_2\cos(q_1+q_2)) - \frac{1}{2}S\left(q_1-\frac{\pi}{2}\right)^2$$

This Lagrangian captures the dynamics of the system, accounting for the kinetic energy from both translational and rotational motion, as well as the potential energy due to gravity and stiffness. It serves as the basis for deriving the equations of motion using the Euler-Lagrange equations.

Question 12:

To find the differential equations of motion using the Lagrangian formalism, we need to calculate the Euler-Lagrange equations for each generalized coordinate. The Euler-Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

where Q_i is a generalized coordinate, $\dot{q}_i$ is the time derivative of q_i (the generalized velocity), and L is the Lagrangian.

To calculate the partial derivatives of the Lagrangian (L) with respect to the generalized coordinates (q_1 and q_2) and their time derivatives ($\dot{q}_1$ and $\dot{q}_2$). Partial Derivative with respect to q_1:

$$\frac{\partial L}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{1}{2}m\left[L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2) + \left(L_2^2+\frac{5}{2}R^2\right)(\dot{q}_1+\dot{q}_2)^2\right]\right.$$
$$+mg(L_1\cos(q_1)+L_2\cos(q_1+q_2)) - \frac{1}{2}S\left(q_1-\frac{\pi}{2}\right)^2$$
$$\frac{\partial L}{\partial q_1} = -S_{q1} - gm(L_2\sin(q_1+q_2) + L_1\sin(q_1))$$

Partial Derivative with respect to q_2:

$$\frac{\partial L}{\partial q_2} = \frac{\partial}{\partial q_2}\left(-\frac{1}{2}m[L_1^2\dot{q}_1^2 + 2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2)\right.$$

$$+\left(L_2^2+\frac{2}{5}R^2\right)(\dot{q}_1+\dot{q}_2)^2]+[mg(L_1\cos(q_1)+L_2\cos(q_1+q_2))]-\frac{1}{2}S\left(\dot{q}_1^2-\left(\frac{\pi}{2}\right)^2\right)$$

$$\frac{\partial L}{\partial q_2}=-L_2gm\sin(q_1+q_2)$$
$$-L_1L_2m\dot{q}_1\sin(q_2)(\dot{q}_1+\dot{q}_2)$$

These derivatives contain contributions from both the potential energy and the kinetic energy components of the Lagrangian. The terms involving $\dot{q}_1$ and $\dot{q}_2$ in the derivatives of L with respect to q_1 and q_2 arise from the coupled nature of the motion of the two pendulum segments and the rotational motion of the sphere.

Partial Derivative with respect to $\dot{q}_1$:

$$\frac{\partial L}{\partial \dot{q}_1}=\frac{\partial}{\partial \dot{q}_1}\left(\frac{1}{2}m\left[L_1^2\dot{q}_1^2+2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2)+\left(L_2^2+\frac{2}{5}R^2\right)(\dot{q}_1+\dot{q}_2)^2\right]\right)$$

$$\frac{\partial L}{\partial \dot{q}_1}=m\left((2\dot{q}_1+2\dot{q}_2)\left(L_2^2+\frac{2}{5}R^2\right)+2L_1^2\dot{q}_1+2L_1L_2\cos(q_2)(\dot{q}_1+\dot{q}_2)+2L_1L_2\dot{q}_1\cos(q_2)\right)\frac{1}{2}$$

Partial Derivative with respect to $\dot{q}_2$:

$$\frac{\partial L}{\partial \dot{q}_2}=\frac{\partial}{\partial \dot{q}_2}\left(\frac{1}{2}m\left[L_1^2\dot{q}_1^2+2L_1L_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos(q_2)+\left(L_2^2+\frac{2}{5}R^2\right)(\dot{q}_1+\dot{q}_2)^2\right]\right)$$

$$\frac{\partial L}{\partial \dot{q}_2}=m\left((2\dot{q}_1+2\dot{q}_2)\left(L_2^2+\frac{2}{5}R^2\right)+2L_1L_2\dot{q}_1\cos(q_2)\right)\frac{1}{2}$$

$$\frac{\partial L}{\partial \dot{q}_2}=0.4mr^2\dot{q}_2+0.5m\left(2L_2\left[L_2\dot{q}_2\sin(q_1+q_2)\right]+\dot{q}_1\left[L_1\sin(q_1)+L_2\sin(q_1+q_2)\right]\sin(q_1+q_2)\right)$$

These derivatives contain contributions from both the potential energy and the kinetic energy components of the Lagrangian. The terms involving $\dot{q}_1$ and $\dot{q}_2$ in the derivatives of L with respect to q_1 and q_2 arise from the coupled nature of the motion of the two pendulum segments and the rotational motion of the sphere.

Question 13:

To express the differential equations of Lagrange, we will use the partial derivatives we calculated earlier and apply the Euler-Lagrange equation for each generalized coordinate q_i. The Euler-Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_{q_i}(F)$$

Differential Equation for q_1:

Using the Euler-Lagrange equation for q_1:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_{q_1}(F)$$

$\mathrm{S}_{q_1} - \frac{m\ddot{q}_2(2L_1L_2\sin(q_2)(\dot{q}_1+\dot{q}_2)+2L_1L_2\dot{q}_1\sin(q_2))}{2} + \frac{gm(L_2\sin(q_1+q_2)+L_1\sin(q_1))}{2} + \frac{m\ddot{q}_2(2L_2^2+2L_1\cos(q_2)L_2+\sigma_1)}{2} + \frac{m\ddot{q}_1(2L_1^2+4\cos(q_2)L_1L_2+2L_2^2+\sigma_1)}{2} = F_x(L_1\cos(q_1)+L_2\cos(q_1+q_2)) + F_y(L_1\sin(q_1)+L_2\sin(q_1+q_2))$

After simplifying:

$$2A\ddot{q}_1 + C(q_2)\dot{q}_2 - 2mL_1L_2\dot{q}_1^2\sin(q_2) - mL_1L_2\ddot{q}_2\sin(q_2) + S_{q1}$$

$$+mg(L_1\sin(q_1)+L_2\sin(q_1+q_2)) = F_x(L_1\cos(q_1)+L_2\cos(q_1+q_2)) + F_y(L_1\sin(q_1)+L_2\sin(q_1+q_2))$$

Where,

$$A = \frac{m(L_1L_2 + L_1L_2\cos^2\theta + R^2)}{2}$$

$$B = \frac{m(L_2 + L_2\cos^2\theta + R^2)}{2}$$

$$C = \frac{m(L_2 + 2L_1L_2\cos\theta + R^2)}{2}$$

Differential Equation for q_2:

Using the Euler-Lagrange equation for q_2:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = Q_{q_2}(F)$$

$$m\ddot{q}_1\left(2L_2^2+2L_1\cos(q_2)L_2+\frac{4R^2}{5}\right)+m\ddot{q}_2\left(2L_2^2\right.$$
$$\left.+\frac{4R^2}{5}+L_2gm\sin(q_1+q_2)-L_1L_2m\dot{q}_1\dot{q}_2\sin(q_2)\right.$$
$$+L_1L_2m\dot{q}_1\sin(q_2)(\dot{q}_1+\dot{q}_2)=F_x(L_2\cos(q_1+q_2))+F_y(L_2\sin(q_1+q_2))$$

After simplification

$$2B\ddot{q}_2+C(q_2)\dot{q}_1+mL_1L_1\dot{q}_1^2\sin(q_2)+mgL_2\sin(q_1+q_2)$$
$$=F_x(L_1\cos(q_1)+L_2\cos(q_1+q_2))+F_y(L_1\sin(q_1)+L_2\sin(q_1+q_2))-D\dot{q}_2$$

These two differential equations, along with any initial conditions, will fully describe the motion of the system for the generalized coordinates q_1 and q_2.

Question 14:

Presenting the equations of Lagrange in matrix format is an effective way to streamline the representation, especially for systems with multiple degrees of freedom like the one you're working with. For your system with generalized coordinates q_1 and q_2, Lagrange's equations can be represented in a matrix form that highlights the mass matrix, damping matrix, stiffness matrix, and external force vector.

The general form of the matrix equation is:

$$\mathbf{M}(q)\ddot{q}+\mathbf{C}(q,\dot{q})\dot{q}+\mathbf{K}(q)=\mathbf{F}$$

To represent the equations of motion derived from the Euler-Lagrange equations in matrix format, we express these equations in a more structured form. The matrix equations can be seen as a system of second-order differential equations in terms of q_1 and q_2. Let's express the Euler-Lagrange equations from the previous calculation in this matrix format.

The Euler-Lagrange equations for the double pendulum system, expressed in matrix format, can be represented as $\mathbf{A}\ddot{\mathbf{q}}=\mathbf{B}$, where $\mathbf{A}$ is a matrix of coefficients for the second-order derivatives, and $\mathbf{B}$ is a vector of the remaining terms.

The matrix **A** and vector **B** are given by:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0.4mr^2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1.0L_1L_2m(2\dot{q}_1 + \dot{q}_2)\sin(q_2)\dot{q}_2 - S_{q_1} + gm(L_1\sin(q_1) + L_2\sin(q_1+q_2)) \\ -L_2m(L_1\sin(q_2)\dot{q}_1^2 - g\sin(q_1+q_2)) \end{bmatrix}$$

Question 15:

The geometric constraint is given as:

$$y = d$$

This leads to the constraint function:

$$g(q_1, q_2) = -L_1 \cdot \cos(q_1) - L_2 \cdot \cos(q_1+q_2) - d = 0$$

We can express the constraint equation as:

$$L_1 \cdot \cos q_1 + L_2 \cdot \cos(q_1+q_2) + d = 0$$

Now, let's calculate the Lagrange multipliers associated with this constraint. The Lagrange multipliers are obtained by taking the partial derivatives of the constraint function concerning the generalized coordinates q_1 and q_2 Introducing the Lagrange multiplier λ, the augmented Lagrangian L' becomes:

$$L' = L - \lambda\left[-L_1\cos(q_1) - L_2\cos(q_1+q_2) - d\right]$$

Where λ is the Lagrange multiplier. multiplier λ accounts for the influence of the constraint on the system's dynamics. The sin terms indicate the trigonometric part of the derivatives, considering the angles q_1 and q_2.

For $q_1(t)$:

$$S_{q_1}(t) - gm(L_1\sin(q_1(t)) + L_2\sin(q_1(t)+q_2(t))) + \lambda(L_1\sin(q_1(t)) + L_2\sin(q_1(t)+q_2(t)))2L_1L_2\cos(q_2(t))\dot{q}_1(t)$$

$$+L_1L_2\cos(q_2(t))\dot{q}_2(t) + \frac{1}{2}L_1^2\dot{q}_1^2(t) + \frac{1}{2}L_2^2\dot{q}_2^2(t)$$

For $q_2(t)$:

$$L_1 L_2 m \sin(q_2(t))\dot{q}_1^2(t) + L_1 L_2 m \cos(q_2(t))\dot{q}_1(t)\dot{q}_2(t)$$

$$+\frac{1}{2}L_2^2 m\ddot{q}_2(t) - L_2 g m \sin(q_1(t)+q_2(t)) + L_2\lambda\sin(q_1(t)+q_2(t)) + 0.4mr^2\ddot{q}_2(t) = 0$$

After Simplification it will be :

For $q_1(t)$:

$$2A\dot{q}_1 + C(q_2)\dot{q}_2 + S\cdot q_1 + m\cdot g\cdot(L_1\cdot\sin q_1 + L_2\cdot\sin(q_1+q_2))$$

$$= -\lambda\cdot(L_1\cdot\sin q_1 + L_2\cdot\sin(q_1+q_2)) + F_x\cdot(L_1\cdot\cos q_1 + L_2\cdot\cos(q_1+q_2)) + F_y\cdot(L_1\cdot\sin q_1 + L_2\cdot\sin(q_1+q_2))$$

For $q_2(t)$:

$$2B\ddot{q}_2 + C(q_2)\dot{q}_1 + mL_1L_1\dot{q}_1^2\sin q_2 + mgL_2\sin(q_1+q_2)\dot{q}_2$$

$$= \lambda\cdot(L_2\cdot\sin(q_1+q_2)) + F_x\cdot(L_1\cos q_1 + L_2\cos(q_1+q_2)) + F_y\cdot(L_1\sin q_1 + L_2\sin(q_1+q_2)) - D\dot{q}_2$$

Question 16:

We analyze how the standard Lagrange equations are modified when actuated torques τ_1 and τ_2 are applied to pivot joints.

For a system with two degrees of freedom, described by generalized coordinates q_1 and q_2, the Lagrange's equations are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_{q_1}(F)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = Q_{q_2}(F)$$

When external torquesτ_1 and τ_2 are applied, they act as non-conservative generalized forces and modify the right-hand side of Lagrange's equations. The modified equations, accounting for the torques, become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \tau_2$$

Taking into account the system outlined by the given equations, the Lagrange equations adjusted for the applied torques can be articulated as follows:

for τ_1:

$$\tau_1 = Sq_1(t) - gm\left(L_1\sin(q_1(t)) + L_2\sin(q_1(t)+q_2(t))\right) + m\left(L_1^2\ddot{q}_1(t) - 2L_1L_2\sin(q_2(t))\dot{q}_1(t)\dot{q}_2(t)\right)$$

$$-L_1L_2\sin(q_2(t))\dot{q}_2(t)^2 + 2L_1L_2\cos(q_2(t))\ddot{q}_1(t) + L_1L_2\cos(q_2(t))\ddot{q}_2(t) + L_2^2\ddot{q}_1(t) + L_2^2\ddot{q}_2(t)$$

we can say that after simplification

$$\tau_1 = 2A\ddot{q}_1 + C(q_2)\dot{q}_1 + S_{q1}(t) + mg(L_1\sin q_1 + L_2\sin(q_1+q_2))$$

$$-(L_{1c1} - L_{2c12}) - F_y(L_1\sin q_1 - L_2\sin(q_1+q_2))$$

for τ_2:

$$\tau_2 = m\left(L_1L_2\sin(q_2(t))\dot{q}_1(t)^2 + L_1L_2\cos(q_2(t))\ddot{q}_1(t)\right)$$

$$+L_2^2\ddot{q}_1(t) + L_2^2\ddot{q}_2(t) - L_2g\sin(q_1(t)+q_2(t)) + 0.4r^2\ddot{q}_2(t))$$

As we did previously, we can also put

$$\tau_2 = 2B\ddot{q}_2 + C(q_2)\dot{q}_1 + mL_1L_2\sin q_1\dot{q}_1\dot{q}_2 + mg(L_2\sin(q_1+q_2))\dot{q}_2$$

$$-F_x(L_1\cos q_1 - L_2\cos(q_1+q_2)) - F_y(L_2\sin(q_1+q_2)) + D\dot{q}_2$$

The equations incorporate inertia via terms $2A\ddot{q}_1$ and $2B\ddot{q}_2$. The term $C(q_2)\dot{q}_1$ signifies coordinate coupling due to dynamics like Coriolis effects. Sq_1 is a linear restorative force, mg terms denote gravitational potential, while F_x, F_y, and damping $D\dot{q}_2$ represent external forces and damping. Torques τ_1 and τ_2 are external inputs that alter the system's energy, thus modifying its dynamics.