Notes for Algebraic Structures

Spring 2016

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Notes for Algebraic Structures, taught Spring 2016 at Carnegie Mellon University, by Professor Clinton Conley.

Administrativia

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Grading. 20% HW, 20% × 2 midterms, 40% Final

Homework. Wednesday-Wednesday. Graded for completeness, one starred problem for which no collaboration of any type is allowed.
Most homework out of textbook ("D&F").

Contents

The Integers

Lecture 1 (2016–01–11)

NOTATION. $\mathbb{N} := \{1, 2, 3, \dots\}$ in this class.

Properties: Order, other things. Least element in a set S: $x \in S$ s.t. $\forall y \in S, x \leq y$
Addition $(\mathbb{Z}, +)$:

- Associativity $(x + y) + z = x + (y + z)$
- Identity $x + 0 = 0 + x = 0$
- Inversion $x + (-x) = (-x) + x = 0$
- Commutativity $x + y = y + x$

Multiplication $(\mathbb{Z}, +, \cdot)$:

- Associative
- Distributive
- Identity ("1")

Integer division: Assume x an integer and $y \in \mathbb{Z}^+$ then $\exists! d \in \mathbb{Z}, \exists! r \in \mathbb{Z} : 0 \leq r < y, x = d \cdot y + r$

DEFINITION 1. $y | x$ "y divides x" iff $\exists d \in \mathbb{Z} : x = d \cdot y$.

E.g. $3 | 9, 4 \nmid 7$.

DEFINITION 2. d is a gcd of x and y if

- $d | x, d | y$
- If $c | x$ and $c | y$ then $c | d$

Lecture 2 (2016–01–13)

DEFINITION 3. Given $a, b \in \mathbb{Z}$, denote by $\mathbb{Z}(a, b)$ the set $\{ax + by | x, y \in \mathbb{Z}\}$.

THEOREM 4 (Euclid, Bezout). *Suppose $a, b \in \mathbb{Z}$ are nonzero and let d be the smallest positive element of $\mathbb{Z}(a, b)$, then d is the unique positive GCD of a and b.*

PROOF. d is a gcd of a, b

(1) (Existance of positive GCD)
    (a) By integer division, $\exists q \in \mathbb{Z}, \exists r \in \mathbb{Z}$ with $0 \leq r < d$ such that $a = qd + r$. If $r = 0$ then $d | a$, so done. Otherwise, suppose $0 < r < d$, so $r = a - qd$ since $d \in \mathbb{Z}(a, b)$, we may fix x, y st $d = ax + by$, meaning $r = a - q(ax + by) = a(1 - qx) + b(-qy)$, so $r \in \mathbb{Z}(a, b)$, meaning d was not the minimal positive element in $\mathbb{Z}(a, b)$, RAA. Thus, $d | a$
    (b) HW: If $c | a$ and $c | b$ then $c | (ax + by)$ for all $x, y \in \mathbb{Z}$ Hence $c | d$

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(2) (Uniqueness of positive GCD) Suppose d_1, d_2 are both positive gcds of a and b. $d_1 | d_2$ and $d_2 | d_1$ as they are both gcds. i.e., $\exists m, n \in \mathbb{Z}$ such that $d_2 = md_1$ and $d_1 = nd_2$. As $\text{sgn}(d_1) = \text{sgn}(d_2)$, $m \geq 0$ and $n \geq 0$. As $d_1 = mnd_1$, $m = n = 1$. Thus $d_1 = d_2$.

$\square$

DEFINITION 5. Relatively prime $\iff$ $\gcd(a, b) = 1$

THEOREM 6. *Suppose p is prime and $a, b \in \mathbb{Z}$ are nonzero, and $p | (ab)$ then $p | a$ or $p | b$.*

PROOF. Consider $d = \gcd(p, a)$. Since $d | p$, we know $d = p$ or $d = 1$.

If $d = p$: By def of GCD, $d | p$ and $d | a$ ie. $p | p$ and $p | a$ so we're done.

If $d = 1$: Fix integers x and y such that $px + ay = 1$. $b = p(xb) + (ab)y$ as $p | p(xb)$ and $p | \underbrace{(ab)}_{\uparrow} y$, $p | b$.

$\square$

THEOREM 7 (Unique Prime Factorization). *Suppose that $a > 1$ an integer, $m, n \geq 1$ and $p_1 \leq p_2 \leq \ldots \leq p_m, q_1 \leq q_2 \leq \ldots \leq q_m$ are positive primes.*

Then $m = n$ and $p_i = q_i$ for all i.

PROOF. By induction, it suffices to show $p_1 = q_1$. Suppose not. WLOG, assume $p_1 < q_1$. We know that $p_1 | a$ (as $p_1 | q_1 q_2 \ldots q_n$) Hence, $\exists i \leq n$ such that $p_i | q_i$. since p_i and q_i prime, $p_1 = q_i$. However, $p_1 < q_1 \leq q_i = p_1$ so $p_1 < p_2$ contradiction. Hence $p_1 = q_1$ so by induction, we're done. $\square$

Lecture 3 (2016–01–15)

Teaser: Construct numbers of the form $a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$.
Notion of addition still exists: (similar to complex numbers, coefficients remain integers)
Same with multiplication
Among these "numbers", 2 is irreducible. But, 2 is not prime, as $2 \nmid (1 + \sqrt{-5})$ and $2 \nmid (1 + \sqrt{-5})$, but $2 | \underbrace{(1 + \sqrt{-5})(1 - \sqrt{-5})}_{=6=2\cdot 3}$.

The Integers (mod n)

For today, $n > 0$.

DEFINITION 8. For $a, b \in \mathbb{Z}$ we say $a \equiv b \pmod{n}$ iff $n \mid (b - a)$.

$\equiv$ is an *equivalence relation*

- Reflexivity: $a \equiv a$
- Symmetry: $a \equiv b \iff b \equiv a$
- Transitivity: $a \equiv b \wedge b \equiv c \implies a \equiv c$

PROOF. We know that $a \equiv b$ and $b \equiv c$, i.e. $n \mid (b - a)$ and $n \mid (c - b)$ We want $a \equiv c$, i.e., $n \mid (c - a)$

$$c - a = c + (-b + b) - a = \underbrace{(c - b) + (b - a)}_{n \text{ divides these}}$$

$\square$

DEFINITION 9. Denote by $\bar{a}$ or $[a]_n$ the equivalence class of a with respect to $\equiv \pmod{n}$ (I.e., The set $\{b \in \mathbb{Z} : a \equiv b \pmod{n}\} = \overline{\{a + kn : k \in \mathbb{Z}\}}$).

EXAMPLE. If $n = 2$, there are 2 equivalence classes:
$\bar{0} = \{\dots, -4, -2, 0, 2, 4, \dots\} = \bar{2} = \overline{-36}$
$\bar{1} = \{\dots, -3, -1, 1, 3, \dots\}$

DEFINITION 10. Denote by $\mathbb{Z}/n\mathbb{Z}$ the collection of all $\equiv \pmod{n}$ equivalence classes.

E.g. $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$
"Define" addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$ as follows:

$$\bar{a} + \bar{b} = \overline{a + b}$$
$$\bar{a} \cdot \bar{b} = \overline{ab}$$

Makes sense, but we need to check that this definition makes any sense at all (make sure it's *well-defined*). Specifically, we need to make sure that the results of these operations doesn't depend on the representatives of the equivalence classes we chose (e.g. check that $\bar{x} + \bar{z} \equiv \bar{y} + \bar{z}$ if $x \equiv y$).
For brevity, we just show addition.

THEOREM 11. $+$ *and* $\cdot$ *are well-defined on* $\mathbb{Z}/n\mathbb{Z}$

PROOF. (of $\cdot$) Assume that $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$. Then, we want to show that $a_1 b_1 \equiv a_2 b_2 \pmod{n}$.
We know: $n \mid (a_2 - a_1)$ and $n \mid (b_2 - b_1)$.

We want: $n \mid (a_2 b_2 - a_1 b_1)$.

$$
\begin{aligned}
a_2 b_2 - a_1 b_1 &= a_2 b_2 + (-a_1 b_2 + a_1 b_2) - a_1 b_1 \\
&= (a_2 b_2 - a_1 b_2) + (a_1 b_2 - a_1 b_1) \\
&= \underbrace{(a_2 - a_1) b_2 + a_1 (b_2 - b_1)}_{n \text{ divides these}}
\end{aligned}
$$

So, $n \mid (a_2 b_2 - a_1 b_1)$ as desired □

Remark: This is a special case of a "quotient construction," in which you start with a set and an equivalence relation on it and operations on the set that "respect" the equivalence relations (i.e. equivalent inputs yield equivalent outputs)

Moar notes: Mutiplicative inverses are uncommon in the integers (only for 1 and -1). However, it's "more prevalent" in $\mathbb{Z}/n\mathbb{Z}$ in the following sense:

THEOREM 12. *Suppose $n > 0$ is an integer, $a \in \mathbb{Z}$ such that $\gcd(n, a) = 1$ (they're coprime). Then there is $b \in \mathbb{Z}$ such that $ab = 1 \pmod{n}$ (alternatively, $\bar{a} \cdot \bar{b} = \bar{1}$)*

PROOF. Use Bezout's theorem (from last lecture) Take integers x, y such that $nx + ay = \gcd(a, n) = 1$. Then, $nx = 1 - ay$, so $n \mid (1 - ay)$, so $1 \equiv ay \pmod{n}$, Choose $b = y$ and we're done $(\bar{a}\bar{b} \equiv \bar{1})$. □

Groups

DEFINITION 13. We say that $*$ is a <u>binary operation</u> on some set X if it is a function $* : X \times X \to X$. (That is, $*$ accepts two (ordered) inputs from X and it outputs one element of X.)

Remark: usually write $a * b$ for the output of $*$ on the input (a, b).

DEFINITION 14. A <u>group</u> is a set G with a binary operation $*$ (often abbreviated $(G, *)$) satisfying the following 3 axioms.

  i. Associativity: $\forall a, b, c \in G : (a * b) * c = a * (b * c)$
 ii. Identity: There is some $e \in G$ such that $\forall a \in G : a * e = e * a = a$
iii. Inversion: $\forall a \in G (\exists b \in G (a * b = b * a = e))$ (where e is as described in ii)

Lecture 4 (2016–01–20)

Recall the definition of a group.

DEFINITION 15. $(G, *)$ is an <u>abelian</u> (commutative) group if it is a group and

   iv. $(G, *)$ is commutative $(\forall x, y \in G : x * y = y * x)$

Let $(G, *)$ be an arbitrary but fixed group.

PROPOSITION 16. *There is a unique identity element.*

PROOF. Suppose e and f both satisfy the second group property. we compute $e * f$ in two ways. $e * f = f$ and $e * f = e$, so by transitivity, $e = f$. $\qquad \square$

PROPOSITION 17. *If $a \in G$, a has a unique inverse.*

PROOF. Suppose that b and c are both inverses for a, $b * a = e$, $a * c = e$. Then,
$$b = b * e = b * (a * c) = (b * a) * c = e * c = c$$

$\qquad \square$

Notational Conventions.
- We will often just call a group G instead of $(G, *)$
- We abbreviate multiplication $(x * y)$ as $x \cdot y$ or just xy
- We will often write xyz for $(x * y) * z$ (due to associativity)
- When working with $(\mathbb{Z}, +)$, we'll just use $+$
- We'll denote the (unique) identity of G by 1 or by e.
- We'll denote the inverse of x by x^{-1}
- Given an integer exponent $n \in \mathbb{Z}$ and $x \in G$, define
$$x^n = \begin{cases} \prod_{i=1}^{n} x, & \text{if } n > 0 \\ e, & \text{if } n = 0 \\ \prod_{i=1}^{-n} (x^{-1}), & \text{if } n < 0 \end{cases}$$

Group Examples. "Definition:" $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}^+ \right\}$

(1) $(\mathbb{Z}, +)$ is an abelian group
(2) $(\mathbb{Z}, \times)$ is not a group
    Why? 2 has no inverse in $\mathbb{Z}$. ($\nexists x \in \mathbb{Z} : (2x = 1)$)
(3) $(\mathbb{Q}, +)$ is an abelian group
(4) $(\mathbb{Q}, \times)$ is not a group (0 has no inverse)
(5) $(\mathbb{Q} \setminus \{0\}, \times)$ is an abelian group.
(6) $GL(n)$ is the set of matricies $A_{n \times n}$ for which $\det A_{n \times n} \neq 0$
(7) The set G of 2×2 matrices with determinant 1, along with matrix multiplication, is a group. Called the "special linear group."
Closure:
$$\det \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix} \right) \right) = (bc - ad)(fg - eh) = 1$$

    i. Associativity: proof left for the reader.
    ii. Identity: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$
    iii. Given $a = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, take $a^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix} \right)$, which you can verify is still in G.
    The group is *not* abelian. Take $a = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), b = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$. Verify that $ab \neq ba$
(8) Suppose that $X \neq \emptyset$ is some set, and denote by S_x the set of bijections $f : X \to X$. Then $(S_x, \circ)$ is a group, where $\circ$ is function composition. $((f \circ g)$ is the function $x \mapsto f(g(x))$.)
    Identity is $x \mapsto x$. Inversion $f^{-1} = f^{-1}$.

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