Notes for Algebraic Structures

Spring 2016

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Notes for Algebraic Structures, taught Spring 2016 at Carnegie Mellon University, by Professor Clinton Conley.

Administrativia

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Grading. 20% HW, 20% $\times$ 2 midterms, 40% Final

Homework. Wednesday-Wednesday. Graded for completeness, one starred problem for which no collaboration of any type is allowed.
Most homework out of textbook ("D&F").

Contents

The Integers

Lecture 1 (2016–01–11)

Notation. $\mathbb{N} := \{1, 2, 3, \ldots\}$ in this class.

Properties: Order, other things. Least element in a set S: $x \in S$ s.t. $\forall y \in S, x \leq y$
Addition $(\mathbb{Z}, +)$:

- Associativity $(x + y) + z = x + (y + z)$
- Identity $x + 0 = 0 + x = 0$
- Inversion $x + (-x) = (-x) + x = 0$
- Commutativity $x + y = y + x$

Multiplication $(\mathbb{Z}, +, \cdot)$:

- Associative
- Distributive
- Identity ("1")

Integer division: Assume x an integer and $y \in \mathbb{Z}^+$ then $\exists! d \in \mathbb{Z}, \exists! r \in \mathbb{Z} : 0 \leq r < y, x = d \cdot y + r$

Definition 1. $y|x$ "y divides x" iff $\exists d \in \mathbb{Z} : x = d \cdot y$.

E.g. $3|9, 4 \not| 7$.

Definition 2. d is a gcd of x and y if

- $d|x, d|y$
- If $c|x$ and $c|y$ then $c|d$

Lecture 2 (2016–01–13)

Definition 3. Given $a, b \in \mathbb{Z}$, denote by $\mathbb{Z}(a, b)$ the set $\{ax + by | x, y \in \mathbb{Z}\}$.

Theorem 4 (Euclid, Bezout). *Suppose $a, b \in \mathbb{Z}$ are nonzero and let d be the smallest positive element of $\mathbb{Z}(a, b)$, then d is the unique positive GCD of a and b.*

Proof. d is a gcd of a, b

(1) (Existance of positive GCD)
   (a) By integer division, $\exists q \in \mathbb{Z}, \exists r \in \mathbb{Z}$ with $0 \leq r < d$ such that $a = qd + r$. If $r = 0$ then $d|a$, so done. Otherwise, suppose $0 < r < d$, so $r = a - qd$ since $d \in \mathbb{Z}(a, b)$, we may fix x, y st $d = ax + by$, meaning $r = a - q(ax + by) = a(1 - qx) + b(-qy)$, so $r \in \mathbb{Z}(a, b)$, meaning d was not the minimal positive element in $\mathbb{Z}(a, b)$, RAA. Thus, $d|a$
   (b) HW: If $c|a$ and $c|b$ then $c|(ax + by)$ for all $x, y \in \mathbb{Z}$ Hence $c|d$

(2) (Uniqueness of positive GCD) Suppose d_1, d_2 are both positive gcds of a and b. $d_1 | d_2$ and $d_2 | d_1$ as they are both gcds. i.e., $\exists m, n \in \mathbb{Z}$ such that $d_2 = md_1$ and $d_1 = nd_2$. As $\operatorname{sgn}(d_1) = \operatorname{sgn}(d_2)$, $m \geq 0$ and $n \geq 0$. As $d_1 = mnd_1$, $m = n = 1$. Thus $d_1 = d_2$. □

Definition 5. <u>Relatively prime</u> $\iff$ $\gcd(a, b) = 1$

Theorem 6. *Suppose p is prime and $a, b \in \mathbb{Z}$ are nonzero, and $p | (ab)$ then $p | a$ or $p | b$.*

Proof. Consider $d = \gcd(p, a)$. Since $d | p$, we know $d = p$ or $d = 1$.

If $d = p$: By def of GCD, $d | p$ and $d | a$ ie. $p | p$ and $p | a$ so we're done.

If $d = 1$: Fix integers x and y such that $px + ay = 1$. $b = p(xb) + (ab)y$ as $p | p(xb)$ and $p | \underbrace{(ab)}_{\uparrow} y$, $p | b$.

□

Theorem 7 (Unique Prime Factorization). *Suppose that $a > 1$ an integer, $m, n \geq 1$ and $p_1 \leq p_2 \leq \ldots \leq p_m, q_1 \leq q_2 \leq \ldots \leq q_m$ are positive primes.*

Then $m = n$ and $p_i = q_i$ for all i.

Proof. By induction, it suffices to show $p_1 = q_1$. Suppose not. WLOG, assume $p_1 < q_1$. We know that $p_1 | a$ (as $p_1 | q_1 q_2 \ldots q_n$) Hence, $\exists i \leq n$ such that $p_i | q_i$. since p_i and q_i prime, $p_1 = q_i$. However, $p_1 < q_1 \leq q_i = p_1$ so $p_1 < p_2$ contradiction.
Hence $p_1 = q_1$ so by induction, we're done. □

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