MCLA Concice Review

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Contents

Part I

Multivariable Calculus

Chapter 11

Parametric Equations and Polar Coordinates

11.1 Curves Defined by Parametric Equations

x and y are given as functions of a third variable t, a **parameter**, by the equations

$$x = f(t) \qquad y = g(t)$$

Each value of t determines a point (x, y). As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, a **parametric curve**.

Example

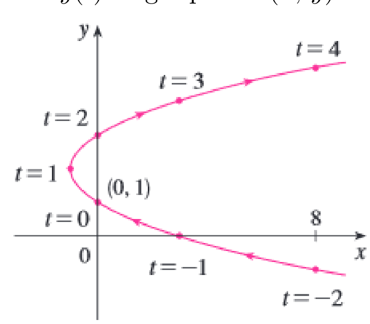
Sketch the curve defined by

$$x = t^2 - 2t \qquad y = t + 1$$

Solution

Use values for t and plug those into $x(t)$ and $y(t)$ to get points (x, y)

t	x	y
−2	8	−1
−1	3	0
0	0	1
1	−1	2
2	0	3
3	3	4
4	8	5



11.2 Calculus with Parametric Curves

Tangents

f and g are differentiable functions and we want to find the tangent line at a point $(f(t), g(t))$. The Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad if\ \frac{dx}{dt} \neq 0$$

d^2y/dx^2 can be found by replacing y with dy/dx in the equation above

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

Example

Find the tangent to the cycloid $x = r(\theta - sin\,\theta)$, $y = r(1 - cos\,\theta)$ at the point where $\theta = \pi/3$.

Solution

The slope of the tangent is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\,sin\,\theta}{r(1 - cos\,\theta)} = \frac{sin\,\theta}{1 - cos\,\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{sin(\pi/3)}{1 - cos(\pi/3)} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right)$$

Areas

Area under a curve $y = F(x)$ form a to b is $A = \int_a^b F(x)dx$. If the curve is traced out with the parametric equations $f(t)$ and $g(t)$, we can calculate the area by using the Substitution Rule for Definite Integrals:

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$$

Example

Find the area under one arch of the cycloid

$$x = r(\theta - sin\,\theta) \qquad y = r(1 - cos\,\theta)$$

Solution

One arch of the cycloid is given by $0 \le \theta \le 2\pi$. Using the substitution rule with $y = r(1 - cos\,\theta)$ and $dx = r(1 - cos\,\theta)d\theta$, we have

$$A = \int_0^{2\pi r} y\, dx = \int_0^{2\pi r} r(1 - cos\,\theta)r(1 - cos\,\theta)d\theta$$

$$= r^2(\frac{3}{2} \cdot 2\pi) = 3\pi r^2$$

Arc Length

To find the length L of a curve C in the from $y = F(x)$

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

Suppose C can also be described with parametric equations, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_\alpha^\beta \sqrt{1 + (\frac{dy/dt}{dx/dt})^2} \frac{dx}{dt} dt$$

Theorem

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example

Find the length of one arch of the cycloid $x = r(\theta - sin\theta)$, $y = r(1 - cos\,\theta)$

Solution

Since

$$\frac{dx}{d\theta} = r(1 - cos\,\theta) \qquad \frac{dy}{d\theta} = rsin\,\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - cos\,\theta)^2 + r^2 sin^2\theta} d\theta$$

$$= r \int_0^{2\pi} \sqrt{2(1 - cos\,\theta)} d\theta$$

$$= 2r[2 + 2] = 8r$$

Surface Area

Suppose a curve C is rotated about the x-axis. If C si traversed exactly once as t increases from α to β, then the area of the surface is given by

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y ds$ and $S = \int 2\pi x ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution

The sphere is obtained by rotating the semicircle

$$x = rcos\,t \qquad y = rsin\,t \qquad 0 \leq t \leq \pi$$

about the x-axis. Then, we get

$$S = \int_0^\pi 2\pi r\, sin\,t \sqrt{(-rsin\,t)^2 + (rcos\,t)^2} dt$$

$$= 2\pi r^2 \int_0^\pi sin\,t\, dt = 2\pi r^2 (-cos\,t)]_0^\pi = 4\pi r^2$$

11.3 Polar Coordinates

The Polar Coordinate System starts witha point in the plan called the **pole** and is labeled O. Then we draw a ray starting at O called the **polar axis**. This corresponds to the positive x-axis in Cartesian coordinates.

r is the ditsance from O to P and θ is the angle between the polar axis and the line OP. The **polar coordinates** of P are in the form $(r,\, \theta)$,

If a point P has Cartesian coordinates $(x,\, y)$ and polar coordinates $(r,\, \theta)$

$$cos\,\theta = \frac{x}{r} \qquad y = r sin\,\theta$$

and so

$$x = r cos\,\theta \qquad y = r sin\,\theta$$

To find r and θ when x and y are known, we use

$$r^2 = x^2 + y^2 \qquad tan\,\theta = \frac{y}{x}$$

Example

What curve is represented by the polar equation r=2?

Solution

The equation $r = a$ represents a circle with center O and radius $|a|$. The polar equation represents a circle about the origin with a radius of 2.

Example

What curve is represented by the polar equation θ=1?

Solution

This curve consists of all points $(r,\, \theta)$ such that θ is 1 radian. It is a straight line that passes through O and makes an angle of 1 radian with the polar axis.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we write its parametric equations as

$$x = r cos\,\theta = f(\theta)\, cos\,\theta \qquad y = r sin\,\theta = f(\theta)\, sin\,\theta$$

Then, using the method of finding slopes of parametric curves and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} sin\,\theta + r\, cos\,\theta}{\frac{dr}{d\theta} cos\,\theta - r\, sin\,\theta}$$

11.4 Areas and Lengths in Polar Coordinates

Using the formula for the area of a sector of a circle

$$A = \frac{1}{2}r^2\theta$$

the formula for the area A of the polar region R is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2}r^2\, d\theta$$

Arc Length

To find the length of a polar curve $r{=}f(\theta)$, we write the parametric equations as

$$x = r cos\,\theta = f(\theta)\, cos\,\theta \qquad y = r sin\,\theta = f(\theta)\, sin\,\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\, cos\,\theta - r\, sin\,\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\, sin\,\theta + r\, cos\,\theta$$

so, using $cos^2\theta + sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{dr}{d\theta})^2 + r^2$$

Therefore the length of a curve is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

11.5 Conic Sections

Parabolas

An equation of the parabola with focus $(0,\, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

Ellipses

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \leq b > 0$$

has focus $(\pm c,\, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a,\, 0)$
The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \leq b > 0$$

has focus $(0,\, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0,\, \pm a)$

Hyerpbolas

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci$(\pm c,\ 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a,\ 0)$, and asymptomes $y = \pm(b/a)x$.
The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci$(0,\ \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0,\ \pm a)$, and asymptomes $y = \pm(a/b)x$.

11.6 Conic Sections in Polar Coordinates

Theorem

Let P be a fixed point (the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

The conic is
(a) an ellipse if $e < 1$
(b) a parabola if $e = 1$
(c) a hyperbola if $e > 1$

Theorem

A polar equation of the form

$$r = \frac{ed}{1 \pm e\,cos\,\theta} \quad or \quad r = \frac{ed}{1 \pm e\,sin\,\theta}$$

represents a conic section the eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Polar

The polar equation of an ellipse with focus at the origin, emimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e\,cos\,\theta}$$

The **perihelion distance** from a planet to the sun is $a(1 - e)$ and the **aphelion distance** is $a(1 + e)$.

Chapter 12

Infinite Sequences and Series

12.1 Sequences

A **sequence** can be thought of as a list of numbers in definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

which can also be denoted by

$$\{a_n\} \qquad or \qquad \{a_n\}_{n=1}^{\infty}$$

Definition

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad or \qquad a_n \to L \quad as \quad n \to \inf$$

if $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges**). Otherwise, we say the sequence **diverges**.

Definition

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad or \qquad a_n \to L \quad as \quad n \to \inf$$

if for every $\epsilon > 0$ there is corresponding integer N such that

$$if \quad n > N \quad then \quad |a_n - L| < \epsilon$$

Theorem

if $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

Definition

$\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$if \quad n > N \quad then \quad a_n > M$$

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{a\to\infty} a_n = 0$

Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

A sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r \leq 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example

Find $\lim_{n\to\infty} sin(\pi/n)$.

Solution

Because the sine function is continuous at 0,

$$\lim_{n\to\infty} sin(\pi/n) = sin\left(\lim_{n\to\infty} (\pi/n)\right) = sin\, 0 = 0$$

Definition

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is $a_1 < a_2 < a_3 < \ldots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it's either increasing or decreasing.

Example

Show that the sequence $a_n - \dfrac{n}{n^2+1}$ is decreasing.

Solution

Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$$

Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore a_n is decreasing.

Definition

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad foralln \geq 1$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \qquad foralln \geq 1$$

It is bounded above and below, then $\{a_n\}$ is a **bounded sequence**

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

12.2 Series

If we try to add the terms if an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

which is an **infinite series** (or just a **series**) and is denoted by

$$\sum_{n=1}^{\infty} a_n \qquad or \qquad \sum a_n$$

Definition

Given a series $\sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + \ldots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad or \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

Example

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

Solution

Let's rewrite the nth term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty}(2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

This is a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges by (4).

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Solution

$$\lim_{n\to\infty} a_n = lim_{n\to\infty}\frac{n^2}{5n^2+4} = lim_{n\to\infty}\frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

12.3 The Integral Test and Estimates of Sums

The Integral Test

Suppose f is a continuous, positive, decreasing, function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integeral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution

The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1,\infty)$ so we use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty} tan^{-1}x]_1^t$$

$$= \lim_{t\to\infty} \left(tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The integral converges and so the series is convergent/

12.4 The Comparison Tests

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Solution

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

is convergent by part (i) of the Comparison Test

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or diverge.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

Solution

We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n-1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = 1 > 0$$

Since this limit exists and $\sum 1/2^n$ is a convergent geometric series, the given converges by the Limit Comparison Test.

12.5 Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n \qquad$ for all n

(ii) $lim_{n \to \infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0$

so the series is convergent by the Alternating Series Test.

Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies

$$(i)\, b_{n+1} \leq b_n \qquad and \qquad (ii)\, \lim_{n \to \infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

12.6 Absolute Convergence and the Ratio and Root Tests

Definition

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Definition

A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

The Ratio Test

(i) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ or $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$

Example

Test the convergence fo the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution

Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e$$

Since $e > 1$, the given series divergent by the Ratio Test.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Example

Test the convergence of the series $\sum_{n=1}^{\infty} (\frac{2n+3}{3n+2})^n$.

Solution

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series is absolutely convergent by the Root Test.

12.7 Strategy for Testing Series

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p < 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if an is a rational function or an algebraic function of n, then the series should be compared with a p-series. The comparison tests apply only to series with positive terms, but if $\sum a_n$ some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products are often conveniently tested using the Ratio Test. This test should not be used because $|\frac{a_{n+1}}{a_n}| \to 1$ as $n \to \infty$ for all p-series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the Integral Test is effective.

12.8 Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is the variable and the c_n's are constants called the **coefficients**.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is called a **power series in (x-a)** or a **power series centered at** a.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

Solution

We use the Ratio Test. If we let a_n denote the $n$the term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$.

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** in case (i) is a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) there are four possibilities

$$(a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R) \quad [a-R, a+R]$$

12.9 Representations of Functions as Power Series

We start with a familiar equation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Example

Find a power series representation for 1/(x+2)

Solution

In order to put this equation in the proper form, we factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2[1-(-\frac{x}{2})]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

The series converges when $|\frac{-x}{2}| < 1$, or $|x| < 2$. So the interval of convergence is $(-2, 2)$.

Term-by-Term Differentiation and Integration

If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and continuous on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a^2) + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in both Equations is R.

12.10 Taylor and Maclaurin Series

Theorem

If f has a power series expansion at a, which is

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting the formula for c_n into the series f must be of the following form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

This series is called the **Taylor series of the function** f **about** a For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

Theorem

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Example

Find the Maclaurin series for sin x and prove that it represents sin x for all x.

Solution

We arrange our computation in two columns:

$$\begin{array}{ll}
f(x) = sinx & f(0) = 0 \\
f(x) = cosx & f'(0) = 1 \\
f(x) = -sinx & f''(0) = 0 \\
f(x) = -cosx & f'''(0) = -1 \\
f(x) = sinx & f^4(0) = 0
\end{array}$$

Since the derivatives repeat in a cycle of four, we write the Maclaurin as follows:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{7^3}{7!} + \dots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Important Maclaurin Series and Their Radii of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

12.11 Applications of Taylor Polynomials

Approximating Functions by Polynomails

Example

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution

(a)

$$\begin{array}{ll} f(x) = x^{1/3} & f(8) = 2 \\ f'(x) = \frac{1}{3}x^{-2/3} & f'(8) = \frac{1}{12} \\ f''(x) = -\frac{2}{9}x^{-5/3} & f''(8) = \frac{1}{144} \\ f'''(x) = \frac{10}{27}x^{-8/3} & \end{array}$$

Using the second-degree Taylor polynomial, $T_2(x)$, the desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(b) The tailor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem. But we can use the Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \le \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \le M$. Because $x \ge 7$, we have $x^{8/3} \ge 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \le \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$

Thereform we can take $M = 0.0021$. Also $7 \le x \le 9$, so $-1 \le x - 8 \le 1$ and $|x - 8| \le 1$. Then Taylor's Inequality gives

$$|R_2(x)| \le \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \le x \le 9$, the approximation in part (a) is accurate within 0.0004.

Chapter 13

Vectors and Geometry of Space

13.1 Three-Dimensional Coordinate Systems

3D Space

The direction of the z-axis is determined by the **right-hand rule**. if you curl the fingers of your right hand around the z-axis, in the clockwise direction from the x-axis to the y-axis, your thumb points in the direction of the z-axis.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{x, y, z\}|x, y, z \in \mathbb{R}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$.

Surfaces

In three-dimensions, an equation in x, y, z represents a *surface* in $\mathbb{R}^3$.

Example

What surfaces in $\mathbb{R}^3$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$

Solution

(a) $z = 3$ represents the set $\{x, y, z|z = 3\}$, which all the points in $\mathbb{R}^3$ whose z-coordinate is 3. This is a horizontal plane parallel to the xy-plane and three units above it.
(b) $y = 5$ represents a vertical plane parallel to the xz-plane and five units to the right of it.

Distance Formula in Three Dimensions
The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere
An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

If the center is the origin O, then the equation is

$$x^2 + y^2 + z^2 = r^2$$

Example

What region in $\mathbb{R}^3$ is represents by the following inequalities?

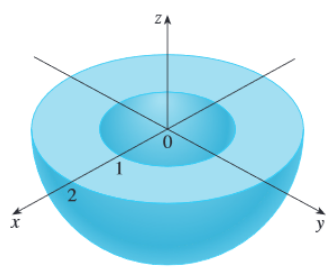
$$1 \leq x^2 + y^2 + z^2 \leq 4 \qquad z \leq 0$$

Solution

The inequalities can be rewritten as

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2$$

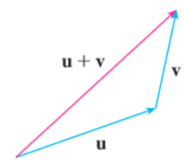
so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2.



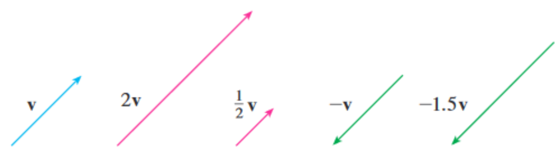
13.2 Vectors

Combining Vectors

Definition of Vector Addition
If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the intial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.



Definition of Scalar Multiplication
If c is scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$.



Components

The **components** of vector $\mathbf{a}$ are written as

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Example

Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Solution

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$
$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4 \rangle = \langle -4, 4, -3 \rangle$$

Example

If $\mathbf{a}$=$\langle 4, 0, 3 \rangle$, find $|\mathbf{a}|$.

Solution

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars

1. $\mathbf{a+b=b+a}$
2. $\mathbf{a+(b+c)=(a+b)+c}$
3. $\mathbf{a+0=a}$
4. $\mathbf{a+(-a)=0}$
5. $c\mathbf{(a+b)} = c\mathbf{a} + c\mathbf{b}$
6. $(c+d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Standard Basis Vectors

$$\mathbf{i} = \langle 1,0,0\rangle \qquad \mathbf{j} = \langle 0,1,0\rangle \qquad \mathbf{k} = \langle 0,0,1\rangle$$

Example

Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{k} - 2\mathbf{k}$.

Solution

The given vector has length

$$|2\mathbf{i} - \mathbf{k} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

and by the equation $\mathbf{u} = \dfrac{1}{|\mathbf{a}|}\mathbf{a} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$, the unit vector is

$$\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

13.3 The Dot Product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties of the Dot Product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is scalar

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^{\mathbf{2}}$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

Theorem

If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \, cos\, \theta$$

Two nonzero vectors are **perpendicular** or **orthogonal** if the angle between then is $\theta = \pi/2$. The dot product gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \, cos(\pi/2) = 0$$

Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Projections

Scalar projection of **b** onto **a**: $\quad comp_a\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad proj_a\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^{\mathbf{2}}}\mathbf{a}$

Example

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Solution

Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is:

$$comp_a\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$proj_a\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$

13.4 The Cross Product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

This Definition can be rewritten using second-order determinants and the standard basis vectors **i, j,** and **k**

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\
&= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}
\end{aligned}$$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Proof

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\, sin\, \theta$$

Corollary

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Example

Find the vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution

$$\overrightarrow{PQ} = (-2 - 1)\mathbf{i} + (5 - 4)\mathbf{j} + (-1 - 6)\mathbf{k} = \mathbf{-3i + j - 7k}$$

$$\overrightarrow{PR} = (1 - 1)\mathbf{i} + (-1 - 4)\mathbf{j} + (1 - 6)\mathbf{k} = \mathbf{-5j - 5k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

Properties of the Cross Product

If **a, b,** and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Triple Products

The volume of the parallelepiped determined by the vectors **a, b,** and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

When the volume is 0, the vectors lie in the same plane and are **coplanar**.

13.5 Equations of Lines and Planes

Lines

$$\mathbf{r} = \mathbf{r}_0 | t\mathbf{v}$$

is the **vector equation** of L. Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Example

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Solution

Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

Planes

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0 = 0)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either of the two equations are the **vector equation of the plane**.

A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which can be rewritten as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$.

Example

Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Solution

The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

Distances

The formula for the distance D can be written as

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the distance between the parallel planes $10x+2y-2z-5$ and $5x+y-z=1$

Solution

The planes are parallel because their normal vectors $\langle 10, 2, -2\rangle$ and $\langle 5, 1, -1\rangle$ are parallel. To find the distance between the planes, we choose any point on one plane and calculate its distance to the other plane. If we plug in $y = z = 0$ into the first equation, we get $10x = 5$ and so $(\frac{1}{2}, 0, 0)$ is a point in the plane.

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

13.6 Cylinders and Quadric Surfaces

Cylinders

A **cylinder** is a surface that consists of all lines that are parallel to a given line and pass through a given plane curve.

Quadric Surfaces

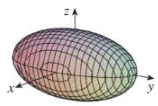
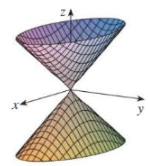
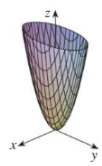
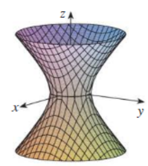
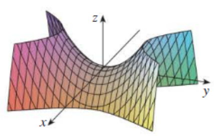
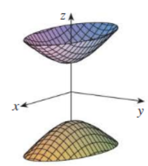
A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad Ax^2 + By^2 + Iz = 0$$

Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Chapter 14

Vector Functions

14.1 Vector Functions and Space Curves

A vector expression of the form $\langle f(t), g(t), h(t)\rangle$ is called a vector function. It can also be described as three separate functions, $x = f(t)$, $y = g(t)$, and $z = h(t)$, that describe points in space. In this case, we refer to the set of equations as parametric equations for the curve.

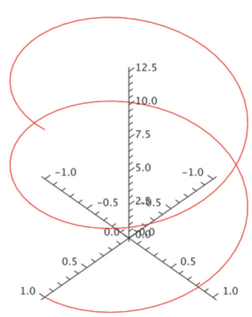
$$r(t) = \langle f(t), g(t), h(t)\rangle = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

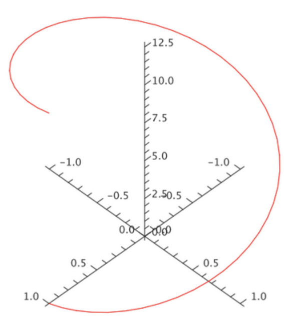
Example

Describe the curves $\langle cos\, t, sin\, t, 0\rangle$, $\langle cos\, t, sin\, t, t\rangle$, and $\langle cos\, t, sin\, t, 2t\rangle$

Solution

As t varies, the first two coordinates in all three functions trace out a unit circle. In the first curve, the z-coordinate is always 0, so this is a 2D unit circle in the xy plane. In the second curve, the z-coordinate varies with t which will produce a helix. In the third curve, the z coordinate varies twice as fast as t which produces a stretched out helix. Below are the two helixes:





If $r(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} r(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

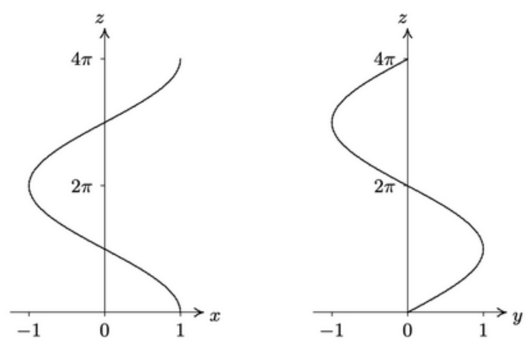
provided the limits of the component functions exist.

Example

Graph the projections of $\langle cos\, t, sin\, t, 2t \rangle$ onto the xz plane and the yz plane

Solution

The 2D vector function for the projection onto the xz plane is $\langle cos\, t, 2t \rangle$, or in parametric force: $x = cos\, t$, $z = 2t$. By substituting for t, we get $x = cos(z/2)$, which is the curve below on the left. For the projection onto the yz plane, we start with the vector function $\langle sin\, t, 2t \rangle$, which is $y = sin t$, $z = 2t$. Substituting for t gives $y = sin(z/2)$ as shown below on the right.



14.2 Derivatives and Integrals of Vector Functions

Derivatives

The derivative of a vector function is defined the same way as for real-valued functions:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

If $r(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ where f, g, and h are differentiable functions, then

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h'(t)\,\hat{\mathbf{k}}$$

Example

(a) Find the derivative of $r(t) = (1 + t^3)\,\hat{\mathbf{i}} + te^{-t}\,\hat{\mathbf{j}} + sin\ 2t\,\hat{\mathbf{k}}$

(b) Find the unit tangent vector at the point where $t = 0$

Solution

(a) $r'(t) = 3t^2\,\hat{\mathbf{i}} + (1 - t)e^{-t}\,\hat{\mathbf{j}} + 2cos\ 2t\,\hat{\mathbf{k}}$

(b) Since $r(0) = \hat{\mathbf{i}}$ and $r'(0) = \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$, the unit tangent vector at (1,0,0) is

$$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}} + \frac{2}{\sqrt{5}}\,\hat{\mathbf{k}}$$

Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\,\hat{\mathbf{i}} + \left(\int_a^b g(t)\,dt\right)\,\hat{\mathbf{j}} + \left(\int_a^b h(t)\,dt\right)\,\hat{\mathbf{k}}$$

Example

If $\mathbf{r}(t) = 2cos\ t\,\hat{\mathbf{i}} + sin\ t\,\hat{\mathbf{j}} + 2t\,\hat{\mathbf{k}}$, then

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \left(\int 2cos\ t\,dt\right)\,\hat{\mathbf{i}} + \left(\int cos\ t\,dt\right)\,\hat{\mathbf{j}} + \left(\int 2t\,dt\right)\,\hat{\mathbf{k}} \\ &= 2sin\ t\,\hat{\mathbf{i}} - cos\ t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}} + C\end{aligned}$$

Where C is a vector constant of integration, and

$$\int_0^{\pi/2} \mathbf{r}(t)\,dt = \Big[2sin\,t\,\hat{\mathbf{i}} - cos\,t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}\Big]_0^{\pi/2} = 2\,\hat{\mathbf{i}} + \,\hat{\mathbf{j}} + \frac{\pi^2}{4}\,\hat{\mathbf{k}}$$

14.3 Arc Length and Curvature

Length of a Curve

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt \\ &= \int_a^b \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2 + (\frac{\mathrm{d}z}{\mathrm{d}t})^2}\,dt \end{aligned}$$

The formula can be put into the compact form $L = \int_a^b |\mathbf{r}'(t)|\,dt$

Example

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ from the point $(1,0,0)$ to the point $(1,0,2\pi)$

Solution

Since $\mathbf{r}'(t) = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \,\hat{\mathbf{k}}$, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + cos^2 t + 1} = \sqrt{2}$$

The arc from (1,0,0) to (1,0,2π) is described by the parameter interval $0 \le t \le 2\pi$ and so we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$$

Proof

$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| \to \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\kappa = \frac{|d\mathbf{T}/dt|}{|ds/dt|}$$

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = |\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

14.4 Motion in Space: Velocity and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice that for small values of h, the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The vector gives the average velocity over a time interval of length and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Example

The position vector of an object moving in a place is given by $\mathbf{r}(t) = t^3\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$. Find its velocity, speed, and acceleration when $t = 1$.

Solution

The velocity and acceleration equations at time t are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

When $t = 1$, we have

$$\mathbf{v}(1) = 3\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} \qquad \mathbf{a}(1) = 6\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} \qquad |\mathbf{v}(1)| = \sqrt{13}$$

Parametric Equations of Trajectory

$$x = (v_0\cos\alpha)t \qquad y = (v_0\sin\alpha)t = \frac{1}{2}gt^2$$

Tangential and Normal Components of Acceleration

$$\mathbf{T}(t) = \frac{\mathbf{v}}{v} \qquad \mathbf{v} = v\mathbf{T}$$

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \qquad |\mathbf{T}'| = \kappa v$$

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} \qquad \mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v \mathbf{N}$$

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

Keplar's Laws

1. A planet revolves around the Sun in an elliptical orbit with the Sun at one focus.
2. The line joining the Sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Chapter 15

Partial Derivatives

15.1 Functions of Several Variables

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that takes on, that is, $f(x, y) \mid (x, y) \in D$.

Example

Find the domains of the following functions and evaluate $f(3, 2)$

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

(b) $f(x, y) = x \ln\big(y^2 - x\big)$

Solution

(a)

$$f(3, 2) == \frac{\sqrt{3 + 2 + 1}}{3 - 1} \frac{\sqrt{6}}{2}$$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$ describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the point on the line $x = 1$ must be excluded from the domain

(b)

$$f(3, 2) = 3 \ln\big(2^2 - 3\big) = 3 \ln(1) = 0$$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$ that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points on the left of the parabola $x = y^2$.

Definition

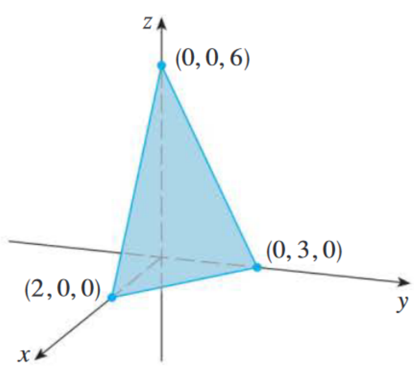
If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D

Example

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$

Solution

The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts. Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the y-intercept is 3 and the z-intercept is 6. This helps us sketch the portion of the graph that lies in the first octant.



Definition

The level curves of a function f of two variables are the curves with the equations $f(x, y) = k$, where k is a constant in the range of f.

Example

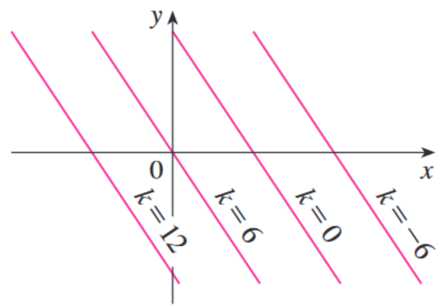
Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$

Solution

The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6, 12$ are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They're equally spaced because the graph of f is a plane.



Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the Earth depends on the longitude x nd the latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

15.2 Limits and Continuity

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

If for every number $\epsilon > 0$ there is a corresponding number $\sigma > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ and $|f(x, y) - L| < \epsilon$

Example

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach (0,0) along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \qquad \text{as} \qquad (x, y) \to (0, 0) \quad \text{along the x-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-\,y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \qquad \text{as} \qquad (x, y) \to (0, 0) \quad \text{along the y-axis}$$

Since f has two different limits along two different lines, the limit does not exist.

Definition

A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is continuous on D if f is continuous at every point (a, b) in D.

Example

Evaluate $\displaystyle\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

Solution

Since $f(x, y) = (x^2y^3 - x^3y^2 + 3x + 2y)$ is a polynomial, it is continuous everywhere, so we can find the limit through direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = (1)^2(2)^3 - (1)^3(2)^2 + (3)(1) + (2)(2) = 11$$

15.3 Partial Derivatives

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at** (a, b) and denote it by $f_x(a, b)$. Thus

$$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

$$\text{which becomes} \qquad f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the partial derivative of f with respect to y at (a,b), denoted by $f_y(a,b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a,y)$:

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Notations for Partial Derivatives

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rules for Finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y

Example

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$

Solution

Holding y cosntant and differentiating with respect to x, we get

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example

Find $\partial z/\partial x$ and find $\partial z/\partial y$ if z is defined implicitly as a function x and y by the equation

$$x^3 + y^3 + z^3 + 6xzy = 1$$

Solution

To find $\partial z/\partial x$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this equation for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$

Example

Calculate f_{xxyz} if $f(x, y, z) = sin(3x + yz)$

$f_x = 3cos(3x + yz)$

$f_{xx} = -9sin(3x + yz)$

$f_{xxy} = -9zcos(3x + yz)$

$f_{xxyz} = -9cos(3x + yz) + 9yzsin(3x + yz)$

15.4 Tangent Planes and Linear Approx

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_o, y_o, z_o)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find the tangent plane to the elliptic paraboloid $z = 2x2 + y2$ at the point $(1, 1, 3)$.

Solution

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

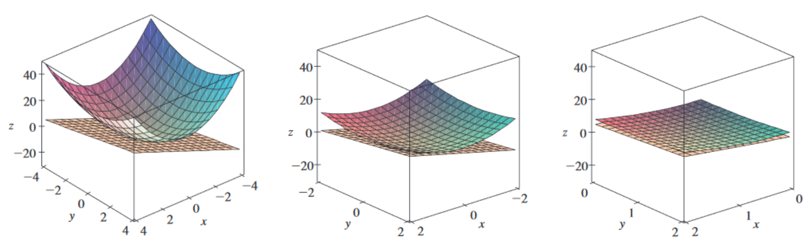
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

The equation of the tangent plane at $(1, 1, 3)$ is

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$\text{or} \qquad z = 4x + 2y - 3$$

Figure (a) shows the elliptic paraboloid and its tangent plane at $(1, 1, 3)$. Figures (b) and (c) are zoomed in towards the point $(1, 1, 3)$. The more we zoom, the flatter the graph appears.



(a) (b) (c)

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

Where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Differentials

For a differentiable function of two variables $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Example

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution

(a)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)](0.05) + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\Delta z = f(2.05, 2.96) - f(2, 3)$$

$$= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2] = 6.449$$

15.5 The Chain Rule

The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Since we often write $\frac{\partial z}{\partial x}$ in place of $\frac{\partial f}{\partial x}$, we can rewrite the Chain Rule in the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Example

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which pressure is chaning when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

Solution

If t represents the time elapsed in seconds, then at the given instant we have $T = 300$, $\frac{\mathrm{d}T}{\mathrm{d}t} = 0.1$, $V = 100$, and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.2$. Since

$$P = 8.31\frac{T}{V}$$

the Chain Rule gives

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial P}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\partial P}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{8.31}{V}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{8.31T}{V^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s.

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t)$, $y = h(s, t)$. Then z is indirectly a function of s and t and we wish to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$. Recall that in computing $\frac{\partial z}{\partial t}$ we hold s fixed and compute the ordinary derivative of z with respect to t. Therefore,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule (General Version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Example

If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

Solution

We have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

Example

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Solution

Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s, t) = f(x, y)$ and the Chain Rule gives

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right) = 0$$

Implicit Differentiation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid. It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by that equation.

15.6 Directional Derivatives and the Gradient Vector

Directional Derivatives

Definition

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

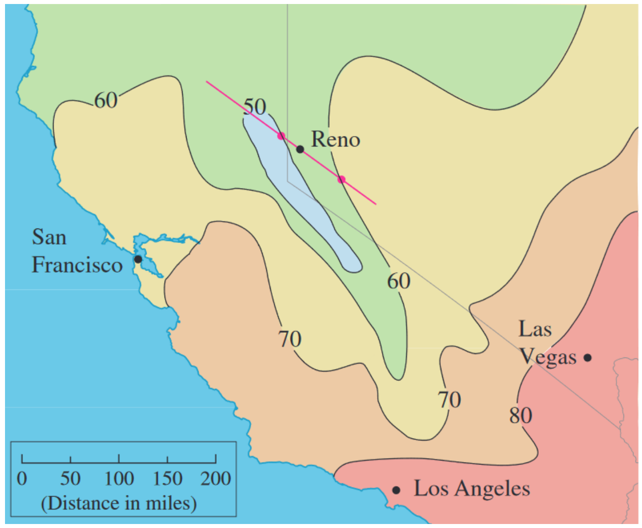
if this limit exists.

Example

Use the weather map in the figure to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

Solution

The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression. We start by drawing a line through Reno toward the southeast.



We approximate the directional derivative D_uT by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$. The temperature at the point southeast of Reno is $T = 60°F$ and the tempera-ture at the point northwest of Reno is $T = 50°F$. The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_uT \approx \frac{60-50}{75} = \frac{10}{75} \approx 0.13°F/mi$$

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_uT = f_x(x,y)a + f_y(x,y)b$$

Proof

If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then by the definition of a derivative we have

$$g'(0) = \lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

$$= D_uf(x_0,y_0)$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$, $y = y_0 + hb$, so the Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}h} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}h} = f_x(x, y)a + f_y(x, y)b$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$, and

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Comparing the two equations for $g'(0)$, we see that

$$D_u f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in the previous theorem becomes

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Definition

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}}\frac{\partial f}{\partial y}\,\hat{\mathbf{j}}$$

Definition

The directional derivative of f at $\langle x_0, y_0, z_0 \rangle$ in the direction of a unit vector $u = \langle a, b, c \rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

The compact form is

$$D_u f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hu) - f(x_0)}{h}$$

The Gradient Vector

Definition

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}}$$

For a function f of three variables, the **gradient vector** is

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}$$

Example

If $f(x,y,z) = x\sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1,3,0)$ in the direction of $\mathbf{v} = \hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Solution

(a) The gradient of f is

$$\begin{aligned}\nabla f(x,y,z) &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \\ &= \langle \sin yz, cz\cos yz, xy\cos yz\rangle\end{aligned}$$

(b) At $(1,3,0)$ we have $\nabla f(1,3,0) = \langle 0,0,3\rangle$. The unit vector in the direction of $\mathbf{v} = \hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{6}}\,\hat{\mathbf{j}} - \frac{1}{\sqrt{6}}\,\hat{\mathbf{k}}$$

Therefore, $D_u f(x,y,z) = \nabla f(x,y,z)\cdot\mathbf{u}$ gives us

$$\begin{aligned}D_u f(1,3,0) &= \nabla f(1,3,0)\cdot\mathbf{u} \\ &= 3\,\hat{\mathbf{k}}\cdot\left(\frac{1}{\sqrt{6}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{6}}\,\hat{\mathbf{j}} - \frac{1}{\sqrt{6}}\,\hat{\mathbf{k}}\right) \\ &= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}\end{aligned}$$

Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$

Proof

We have

$$D_u f = \nabla f\cdot\mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

Where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore, the maximum value of $D_u f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f

15.7 Maximum and Minimum Values

Definition

A function of two variables has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b). The number $f(a,b)$ is called a **local maximum value**, if $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b), then $f(a,b)$ is a **local minimum value**.

Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Example

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f. This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum

Example

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution

We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

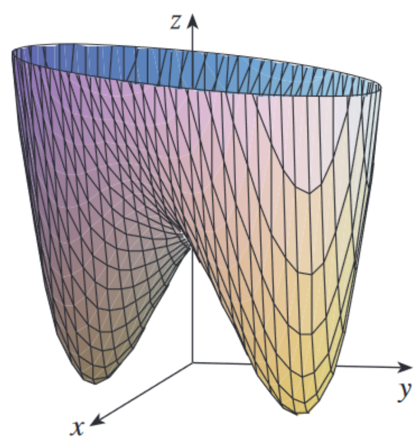
To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, \text{-}1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$
$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows that form case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarlym we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum. The graph of f is shown below.



Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D
2. Find the extreme values of f on the boundary of D
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

15.8 Lagrange Multipliers

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$subject to the constraint $g(x, y, z) = k$:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad \text{and} \qquad g(x, y, z) = k$$

(b) Evaluate f at all points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f

Example

A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box.

Solution

We let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x \qquad V_y = \lambda g_y \qquad V_z = \lambda g_z \qquad 2xz + 2yz + xy = 12$$

which become

(2) $yz = \lambda(2z + y)$

(3) $xz = \lambda(2z + x)$

(4) $xy = \lambda(2x + 2y)$

(5) $2xz + 2yz + xy = 12$

There are no general rules for solving systems of equations. Sometimes ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. We get

(6) $xyz = \lambda(2xz + xy)$

(7) $xyz = \lambda(2yz + xy)$

(8) $xyz = \lambda(2xz + 2yz)$

We observe that λ is not equal to 0 because $\lambda = 0$ would imply $yz - xz - xy = 0$. From (2), (3), and (4) and this would contradict (5). Therefore, from (6) and (7) we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.

Example

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of the intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

We maximum the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

(17) $1 = \lambda + 2x\mu$

(18) $2 = -\lambda + 2y\mu$

(19) $3 = \lambda$

(20) $x - y + z = 1$

(21) $x^2 + y^2 = 1$

Putting $\lambda = 3$ /[from (19)/] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/sqrt29$, $y = \pm 5/\sqrt{29}$, and, from (2), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

$$\mp\frac{2}{\sqrt{29}} + 2\left(\pm\frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore, the maximum value of f on the given curve is $3 + \sqrt{29}$.

Chapter 16

Multiple Integrals

16.1 Double Integrals over Rectangles

Definition

Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if this limit exists.

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

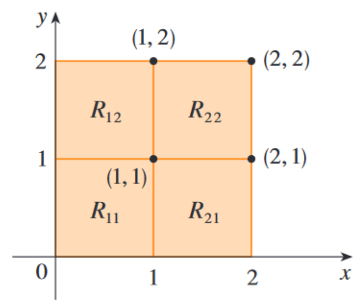
Example

Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

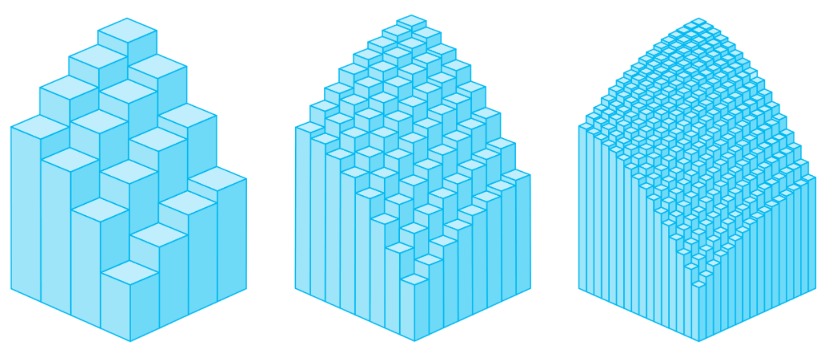
Solution

The squares are shown below. The paraboloid is the graph of $f(x,y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$
\begin{aligned}
V &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\delta A \\
&= f(1,1)\delta A + f(1,2)\delta A + f(2,1)\delta A + f(2,2)\delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) = 34
\end{aligned}
$$



We get better apprximations to the volume if we increase the number of squares. Shown below is how the columns look more like the actual solid when we use 16, 64, and 256 squares.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Midpoint Rules for Double Integrals

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint\limits_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Solution

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles shown below. So $\bar{x}_1 = \frac{1}{2}$, $\bar{x}_2 = \frac{3}{2}$, $\bar{y}_1 = \frac{5}{4}$, and $\bar{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus

$$\begin{aligned}
\iint\limits_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\Delta A \\
&= f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A \\
&= f\left(\frac{1}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{1}{2}, \frac{7}{4}\right)\Delta A + f\left(\frac{3}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{3}{2}, \frac{7}{4}\right)\Delta A \\
&= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2} \\
&= -\frac{95}{8} = -11.875
\end{aligned}$$

Thus we have

$$\iint\limits_R (x - 3y^2)\, dA \approx -11.875$$

Average Value

The average value of a function f of one variable is

$$f_{ave} = \frac{1}{b - a}\int_a^b f(x)\, dx$$

Similarly, the **average value** of a function f of two variables defined on a rectangle R is

$$f_{ave} = \frac{1}{A(R)}\iint\limits_R f(x, y)\, dA$$

where $A(R)$ is the area of R.

16.2 Iterated Integrals

Example

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2y \, dx \, dy$

Solution

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2y \, dy = \left[x^2 \frac{y^2}{2}\right]_{y=1}^{y=2}$$

$$= x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \frac{3}{2}x^2$$

Thus, the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example. We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2y \, dy\right] dx$$

$$= \int_0^3 \frac{3}{2}x^2 \, dx = \left[\frac{x^3}{2}\right]_0^3 = \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2y \, dx\right] dy = \int_1^2 \left[\frac{x^3}{3}y\right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = \left[9\frac{y^2}{2}\right]_1^2 = \frac{27}{2}$$

Notice that we obtained the same answer whether we integrated with respect to y or x first. The order of integration doesn't matter.

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x, y) \mid a \le x \le b, \ c \le y \le d\}$ then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

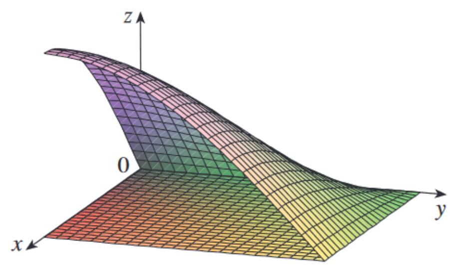
More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example

If $R = [0, \pi/2] \times [0, \pi/2]$, then

$$\iint\limits_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2} = 1 \cdot 1 = 1$$



16.3 Double Integrals over General Regions

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint\limits_D f(x,y) \, dA = \iint\limits_R F(x,y) \, dA$$

If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \le x \le b, \mid g_1(x) \le y \le g_2(x)\}$$

then

$$\iint\limits_D f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$$

Using the same methods as above, we can show that

$$\iint\limits_D f(x,y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy$$

where D is a type II region.

Example

Evaluate $\int\limits_D (x+2y) \, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$

Solution

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region D, is a type I region but not a type II region and we can write

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ 2x^2 \leq y \leq 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, we have

$$\begin{aligned}
\iint\limits_D (x + 2y)\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx \\
&= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx \\
&= \int_{-1}^{1} \left[x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2\right] dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \\
&= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}
\end{aligned}$$

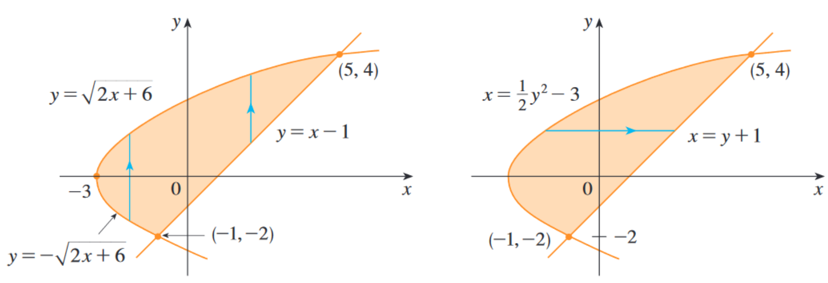
Example

Evaluate $\iint\limits_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution

The region D is shown below. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore, we prefer to express D as a type II region:

$$D = \{(x, y) \mid -2 \leq y \leq 4,\ \frac{1}{2}y^2 - 3 \leq x \leq y + 1\}$$



(a) D as a type I region (b) D as a type II region

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$
$$= \frac{1}{2}\int_{-2}^{4} y[(y+1)^2 - (\frac{1}{2}y^2-3)^2]\,dy$$
$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right)dy$$
$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

If we had expressed D as a type I region using (a), then we would have obtained

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

but this would have involved more work than the other method.

Properties of Double Integrals

(6) $\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

(7) $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$

(9) $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

(10) $\iint_D 1\,dA = A(D)$

(11) If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$$

Example

Use Property 11 to estimate the integral $\iint_D e^{\sin x\cos y}\,dA$, where D is the disk with center the origin and radius 2.

Solution

Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, we have $-1 \le \sin x\cos y \le 1$ and therefore

$$e^{-1} \le e^{\sin x\cos y} \le e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y}\, dA \leq 4\pi e$$

16.4 Double Integrals in Polar Coordinates

Change to Polar Coordinates in a Double Integral

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution

If we put $z = 0$ in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \leq 1$. In polar coordinates D is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ Since $1 - x^2 - y^2 = 1 - r^2$, the volume is

$$\begin{aligned} V &= \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2} \end{aligned}$$

If we had used rectangular coordinates instead of polar coordinates, then we would have obtained

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

which is not easy to evaluate because it involves finding the following integrals:

$$\int \sqrt{1 - x^2}\, dx \qquad \int x^2 \sqrt{1 - x^2}\, dx \qquad \int (1 - x^2)^{3/2}\, dx$$

If f is continuous on the polar region of the form

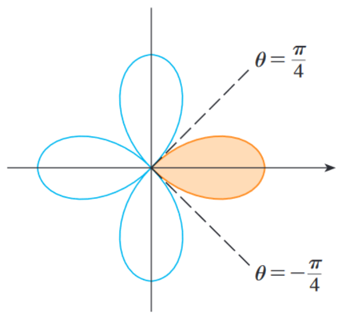
$$D = \{(r, \theta) \mid \alpha \leq \theta\beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example

Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



Solution

We see that the loop is given by the region

$$D = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4.\ 0 \le r \le \cos 2\theta\}$$

So the area is

$$\begin{aligned}
A(D) = \iint_D dA &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2\right]_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta \\
&= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}
\end{aligned}$$

16.5 Applications of Double Integrals

Density and Mass

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$$

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x,y)\,dA$$

Example

Charge is distributed over the triangular region D in the figure so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

Solution

$$\begin{aligned}
Q &= \iint_D \sigma(x, y)\, dA = \int_0^1 \int_{1-x}^1 xy\, dy\, dx \\
&= \int_0^1 \left[x \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} [1^2 - (1-x)^2]\, dx \\
&= \frac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}
\end{aligned}$$

Thus the total charge is $\frac{5}{24}$ C.

Moments and Centers of Mass

The **moment** of the entire lamina **about the x-axis**:

$$M_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y \rho(x, y)\, dA$$

Similarly, the **moment about the y-axis** is:

$$M_y = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(y_{ij}^*, x_{ij}^*) \Delta A = \iint_D x \rho(x, y)\, dA$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

Moment of Inertia

The **moment of inertia** of the lamina **about the x-axis**:

$$I_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D y^2 \rho(x, y)\, dA$$

Similarly, the **moment of inertia about the y-axis** is:

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (x_{ij}^*)^2 \rho(y_{ij}^*, x_{ij}^*)\Delta A = \iint_D x^2 \rho(x,y)\, dA$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2+y^2)\rho(x,y)\, dA$$

Note that $I_0 = I_x + I_y$.

Example

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x,y) = \rho$ center the origin, and radius a.

Solution

The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \le \theta \le 2\pi$, $0 \le r \le a$. Let's compute I_0 first:

$$I_0 = \iint_D (x^2+y^2)\rho\, dA = \rho \int_0^{2\pi}\int_0^a r^2 r\, dr\, d\theta$$

$$\rho \int_0^{2\pi} d\theta \int_0^a r^3\, dr = 2\pi\rho \left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{2}$$

Instead of computing I_x and I_y directly, we use the facts that $I_x + I_y = I_0$ and $I_x = I_y$ (from the symmetry of the problem). Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}$$

Example

The manager of a movie theater determines that the average time movie-goers wait in line to buy a ticket for this week's film is 10 minutes and the average time they wait to buy popcorn is 5 minutes. Assuming that the waiting times are independent, find the probability that a moviegoer waits a total of less than 20 minutes before taking his or her seat.

Solution

Assuming that both the waiting time X for the ticket purchase and the waiting time Y in the refreshment line are modeled by exponential probability density functions, we can write the individual density functions as

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10}e^{-x/10} & \text{if } x \geq 0 \end{cases} \qquad f_2(x) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5}e^{-y/5} & \text{if } y \geq 0 \end{cases}$$

Since X and Y are independent, the joint density function is the product:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{5}e^{-x/10}e^{-y/5} & \text{if } x \geq 0,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We are asked for the probability that $X + Y < 20$:

$$P(X + Y < 20) = P((X,Y) \in D)$$

where D is the triangular region given by the area under $y = -x + 20$. Thus

$$\begin{aligned} P(X+Y<20) &= \iint_D f(x,y)\,dA = \int_0^{20}\int_0^{20-x} \frac{1}{50}e^{-x/10}e^{-y/5}\,dy\,dx \\ &= \frac{1}{50}\int_0^{20}\left[e^{-x/10}(-5)e^{-y/5}\right]_{y=0}^{y=20-x}\,dx \\ &= \frac{1}{10}\int_0^{20} e^{-x/10}(1-e^{(x-20)/5})\,dx \\ &= \frac{1}{10}\int_0^{20}(e^{-x/10}-e^{-4}e^{x/10})\,dx \\ &= 1 + e^{-4} - 2e^{-2} \approx 0.7476 \end{aligned}$$

This means that about 75% of the moviegoers wait less than 20 minutes before taking their seats.

16.6 Surface Area

We define the **surface area** of S to be

$$\begin{aligned} A(S) &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta T_{ij} \\ &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{[f_x(x_i,y_j)]^2+[f_y(x_i,y_j)]^2+1}\,\Delta A \end{aligned}$$

The area of the surface with equation $z = f(x,y), (x,y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x_i,y_j)]^2+[f_y(x_i,y_j)]^2+1}\,\Delta A)$$

Rewrite it as follows

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with the vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Solution

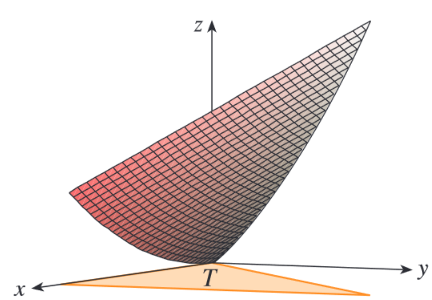
The region T is described by

$$T = (x, y) \mid 0 \le x \le 1,\ 0 \le y \le x$$

Using $f(x, y) = x^2 + 2y$, we get

$$\begin{aligned} A &= \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx \\ &= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \left[\frac{1}{8} \cdot \frac{2}{3}(4x^2 + 5)^{3/2}\right]_0^1 = \frac{1}{12}(27 - 5\sqrt{5}) \end{aligned}$$

The figure below shows the portion of the surface whose area we have just computed.



16.7 Triple Integrals

Definition

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Example

Evaluate $\iiint\limits_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

When we set up a triple integral it's wise to draw **two** diagrams: one of the solid region E and one of its projection D on the xy-plane. The lower boundary of the tetrahedron is the plane $z = 0$ and the upper boundary is the plane $x + y + z = 1$, so we use $u_1(x, y) =$ and $u_2(x, y) = 1 - x - y$. Notice that the planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ in the xy-plane. So the projection of E is the triangular region and we have

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}$$

This description of E as a type I region enables us to evaluate the integral as follows:

$$\begin{aligned}
\iiint\limits_E z\, dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y^2)\, dy\, dx = \frac{1}{2} \int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx \\
&= \frac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}
\end{aligned}$$

Applications of Triple Integrals

The case where $f(x, y, z) = 1$ for all points in E, the triple integral represents the volume of E:

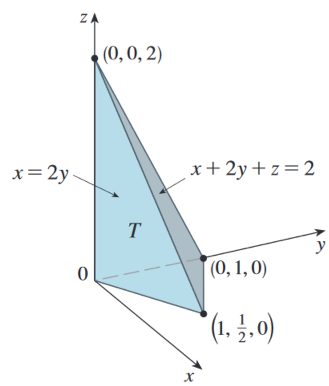
$$V(E) = \iiint\limits_E dV$$

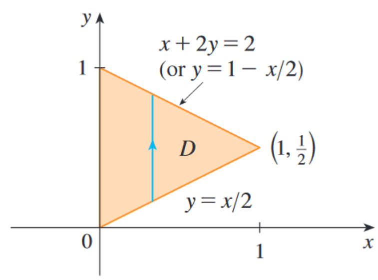
Example

Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution

The tetrahedron T and its projection D on the xy-plane are shown below. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.





Therefore, we have

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \frac{1}{3}$$

16.8 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

To convert form cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Triple Integration in Cylindrical Coordinates

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Example

A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution

In cylindrical coordinates the cylinder is $r=1$ and the paraboloid is $z=1-r^2$, so we can write

$$E=\{(r,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}$$

Since the density at (x,y,z) is proportional to the distance from the z-axis, the density function is

$$f(x,y,z)=K\sqrt{x^2+y^2}=Kr$$

where K is the proportionality constant. Therefore, the mass of E is

$$m=\iiint_E K\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 Kr^2[4-(1-r^2)]\,dr\,d\theta=K\int_0^{2\pi}d\theta\int_0^1(3r^2+r^4)\,dr$$

$$=2\pi K\left[r^3+\frac{r^5}{5}\right]_0^1=\frac{12\pi K}{5}$$

Spherical Coordinates

To convert from spherical to rectangular coordinates, we use the eqations

$$x=\rho\sin\phi\cos\theta \qquad y=\rho\sin\phi\sin\theta \qquad z=\rho\cos\phi$$

Also, the distance formula shows that

$$\rho^2=x^2+y^2+z^2$$

Triple Integration in Spherical Coordinates

$\iiint_E f(x,y,z)\,dV$

$$=\int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a spherical wedge given by

$$E=\{(\rho,\theta,\phi)\mid a\le\rho\le b,\ \alpha\le\theta\le\beta,\ c\le\phi\le d\}$$

Example

Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}} dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$

Solution

Since the boundary of B is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

In addition, spherical coordinates are appropriate because

$$x^2 + y^2 + z^2 = \rho^2$$

Thus

$$\begin{aligned}
\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}} dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} \, d\rho \\
&= [-\cos\phi]_0^{\pi} \, (2\pi) \left[\frac{1}{3} e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e-1)
\end{aligned}$$

16.9 Change of Variables in Multiple Integrals

Example

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$.

Solution

The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \le u \le 1)$. From the given equations we have $x = u^2$, $y = 0$, and so $0 \le x \le 1$. Thus S_1 is mapped onto the line segment from (0,0) to (1,0) in the xy-plane. The second side, S_2, is $u = 1$ $(0 \le v \le 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

$$x = 1 - \frac{y^2}{4} \qquad 0 \le x \le 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \le u \le 1)$, whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \le x \le 0$$

Finally, S_4 is given by $u = 0$ $(0 \le v \le 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \le x \le 0$. The image of S is the region R bounded by the x-axis and the parabolas given by the two equations we found.

Definition

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variables in a Double Integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(y, v)} \right| du\, dv$$

Example

Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint\limits_R y\, dA$ where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y = 4 + 4x$.

Solution

$T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by the Change in Variables in a Double Integral,

$$\iint\limits_R y\,dA = \iint\limits_S 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA = \int_0^1\int_0^1 (2uv)4(u^2+v^2)\,du\,dv$$

$$= 8\int_0^1\int_0^1 (u^3v+uv^3)\,du\,dv = 8\int_0^1 \left[\frac{1}{4}u^4v+\frac{1}{2}u^2v^3\right]_{u=0}^{u=1} dv$$

$$\int_0^1 (2v+4v^3)\,dv = \left[v^2+v^4\right]_0^1 = 2$$

Chapter 17

Vector Calculus

17.1 Vector Fields

A vector field in $\mathbb{R}^2$ is a function F that assigns a 2D vector $f(x, y)$ to each point (x, y) in 0.

Component functions P and Q: $F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$

A vector field on $\mathbb{R}^3$ is a function of F that assigns a 3D vector $f(x, y, z)$ to each point (x, y, z) in E.

Example

Suppose electric charge Q is located at the origin. According to Coulomb's Law, the electric force excerted at a point (x, y, z) by vector $\vec{x} = \langle x, y, z \rangle$ is

$$F(x) = \frac{\epsilon q Q}{|\vec{x}|^3}\vec{x}$$

The force is repulsive for like charges and attractive for unlike charges. These vector fields are known as **Force field**. Instead of considering the electric force F, physicists often consider the force per unit charge:

$$E(x) = \frac{1}{q}F(x) = \frac{\epsilon Q}{|\vec{x}|^3}x$$

Gradient Fields

A gradient of ∇f is defined by:

$$\nabla f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}}$$

Example

Find gradient vector of $f(x, y) = x^2 y - y^3$.

Solution

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = 2xy\,\hat{\mathbf{i}} + (x^2 - 3y^2)\,\hat{\mathbf{j}}$$

A vector field is called a **conservative vector** field if it is the gradient of some scalar function, that is, if there exists a function f such that $F = \nabla f$. In this situation, f, is a **potential function** for F.

17.2 Line Integrals

Definition

If f is defined on a smooth curve C given by $[x = x(t) \quad y = y(t) \quad a \le t \le b]$, then the line integral of f along curve C is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

The following formula can also be used to evaluate a line integral:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2}dt$$

Example

A wire takes the shape of the semicircle $x^2 + y^2 = 1, y \ge 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution Use parametrization $x = cos\,t$, $y = sin\,t$, $0 \le t \le \pi$, and find that $ds = dt$. The linear density is

$$\rho(x,y) = k(1-y)$$

where k is constant, and so the mass of the wire is

$$m = \int_C k(1-y)ds = \int_0^\pi k(1 - sin\,t)dt = k[t + cos\,t]_0^\pi = k(\pi - 2)$$

we know that the center of mass of the wire is

$$\overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds = \frac{1}{k(\pi-2)}\int_C yk(1-y)ds$$

$$= \frac{1}{\pi-2}\int_0^\pi (sin\,t - sin^2\,t)dt = \frac{1}{\pi-2}\left[-cos\,t - \frac{1}{2}t + \frac{1}{4}sin\,2t\right]_0^\pi = \frac{4-\pi}{2(\pi-2)}$$

By symmetry we see that $\overline{x} = 0$, so the center of mass is

$$\left(0, \frac{4-\pi}{2(\pi-2)}\right) \approx (0, 0.38)$$

The line integral with respect to arc length:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),t(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),t(t))\,y'(t)\,dt$$

When setting up a line integral, it is often difficult to think of a parametric representation for a curve. It's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leq t \leq 1$$

A given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t.

If $-C$ denotes the curve of the same points as C but with the opposite orientation, then we have:

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

Line Integrals in Space

Suppose that C is now a smooth curve. We define the line integral along C (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,dt$$

The integral can also be written as:

$$\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

Example

Evaluate $\int_C y sin\, z\, ds$, where C is the circular helix given by $x = cos\, t$, $y = sin\, t$, $z = t$, $0 \leq t \leq 2\pi$.

Solution

$$\begin{aligned}\int_C y sin\, z\, ds &= \int_0^{2\pi} (sin\, t) sin\, t \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, dt \\ &= \int_0^{2\pi} sin^2 t \sqrt{sin^2 t + cos^2 t + 1}\, dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - cos\, 2t)\, dt \\ &= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} sin\, 2t \right]_0^{2\pi} = \sqrt{2}\pi\end{aligned}$$

Line Integrals of Vector Fields

$$W = \int_C \mathbf{F}(x,y,z) \cdot \mathbf{T}(x,y,z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Work is the line integral with respect to arc length of the tangetial component for the force.

Definition

Let F be a continous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of** F **along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Example

Find the work done by the force field $\mathbf{F}(x,y) = x^2\, \hat{\mathbf{i}} - xy\, \hat{\mathbf{j}}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = cos\, t\, \hat{\mathbf{i}} + sin\, t\, \hat{\mathbf{j}}$, $0 \leq t \leq \pi/2$.

Solution

Since $x = cos\, t$ and $y = sin\, t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = cos^2 t\, \hat{\mathbf{i}} - cos\, t sin\, t\, \hat{\mathbf{j}}$$

$$\mathbf{r}'(t) = -sin\, t\, \hat{\mathbf{i}} + cos\, t\, \hat{\mathbf{j}}$$

Therefore the work done is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{\pi/2} (-2 cos^2 t\, sin\, t)\, dt = \left[2 \frac{cos^3 t}{3} \right]_0^{\pi/2} = -\frac{2}{3}$$

17.3 The Fundamental Theorem for Line Integrals

The Fundamental Theorem of Calculus is:

$$\int_a^b F'(x)\, dx = F(B) - F(a)$$

Theorem

Let C be a smooth curve given by the vector function $r(t)$, $a \leq t \leq b$. Let f be a differentiable functino of two or three variables whose gradient vector ∇f is continous.

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

Proof

Using the line integral, we have

$$\begin{aligned}
\int_C \nabla f \cdot dr &= \int_a^b \nabla f(r(t)) \cdot r'(t)\, dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \right) dt \\
&= \int_a^b \frac{d}{dt} f(r(t))\, dt \\
&= f(r(b)) - f(r(a))
\end{aligned}$$

Independence of Path

If F is a continous vector field with domain D, we say that the line integral $\int_c F \cdot dr$ is independent of path if $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$ for any two paths C_1 and C_2 in D that have the same initial and terminal paths.

A curve is called closed if $r(b) = r(a)$.

Theorem

$\int_C F \cdot dr$ is independent of path in D if and only if $\int_C F \cdot dr = 0$ for every closed path C in D.

Proof

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = \int_{C_1} F \cdot dr - \int_{-C_2} F \cdot dr = 0$$

Conversely, if that is true, do the same thing backwards to prove it is closed.

D is open $\rightarrow$ for every point P in D there is no disk with center P that lies entirely on D.

D is connected $\rightarrow$ any two points in D can be joined by a path in D.

Theorem

F is a vector field that is continous in an open connected region in D. If $\int_C F \cdot dr$ is independent of path in D, then F is a conservative vector field in D where a function f that $\nabla f = F$ exist.

Theorem

If $F(x, y) + P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}$ is a conservative vector field, where P and Q have continous first-order partial derivatives on a domain D, then through D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem

Let $F + P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$ be a vector field on an open simply-connected region D. Supposed that P and Q have continous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Then F is conservative.

Example

Determine whether or not $F(x, y) = (x, y)\,\hat{\mathbf{i}} + (x, z)\,\hat{\mathbf{j}}$ is conservative.

Solution

Let $P(x, y) = x - y$ and $Q(x, y) = x - z$

$$\text{then} \qquad \frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, F is not conservative.

17.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simpled curve C and a double integral over the plane region D bounded by C.

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continous partial derivatives on an open region that contains D, then

$$\int\_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

Proof

For the case in with D is a simple region

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$\iint\limits_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\, dy\, dx = \int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right]\, dx$

$\int_{C_1} P(x,y)\, dx = \int_a^b P(x, g(1(x)))\, dx$

$\int_{C_3} P(x,y)\, dx = -\int_{-C_3} P(x,y)\, dx = -\int_a^b P(x, g_2(x))\, dx$

$\int_{C_2} P(x,y)\, dx = 0 = \int_{C_4} P(x,y)\, dx$

$\int_C P(x,y)\, dx = \int_{C_1} P(x,y)\, dx + \int_{C_2} P(x,y)\, dx + \int_{C_3} P(x,y)\, dx$

$+ \int_{C_4} P(x,y)\, dx$

$\int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx$

$\int_C P(x,y)\, dx = -\iint\limits_D \frac{\partial P}{\partial y}\, dA$

Example

Evaluate $\oint_C (3y - c^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy$, where C is the circle $x^2 + y^2 = 9$.

Solution

The region D bounded by C is the disk $x^2 + y^2 \leq 9$ so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - c^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy$$

$$= \iint\limits_D \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right]\, dA$$

$$= \int_0^{2\pi} \int_0^3 (7-3)\, r\, dr\, d\theta = 4 \int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi$$

The Green's Theorem gives the following formulas for the area of D:

$$A = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx$$

Example

Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution

The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \le t \le 2\pi$. Using the formula for area from above, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$\frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$

$$\frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

17.5 Curl and Divergence

Curl

If $F + P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of F is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

$$\text{or} \qquad \text{curl F} = \nabla \times F$$

Example

$F(x, y, z) = xz\,\hat{\mathbf{i}} + xyz\,\hat{\mathbf{j}} - y^2\,\hat{\mathbf{k}}$, find curl F.

Solution

$$\begin{aligned}
\text{curl F} = \nabla \times F &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\hat{\mathbf{i}} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right] \\
&+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right] \\
&= (-2y - xy)\,\hat{\mathbf{i}} - (0 - x)\,\hat{\mathbf{j}} + (yz - 0)\,\hat{\mathbf{k}} \\
&= -y(2 + x)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + yz\,\hat{\mathbf{k}}
\end{aligned}$$

Theorem

If $F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continous second-order partial derivatives, then

$$\text{div curl F} = 0$$

Proof

Using the definitions of divergence and curl, we have

$$\begin{aligned}
\text{div curl F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0
\end{aligned}$$

Vector Forms of Green's Theorem

$F = P\,\hat{\mathbf{i}} + Q\,\hat{\mathbf{j}}$, its line integral is:

$$\oint_C F \cdot dr = \oint_C P\,dx + Q\,dy$$

and its curl is:

$$\text{curl F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,\hat{\mathbf{k}}$$

Therefore

$$\text{curl F} \cdot\, \hat{\mathbf{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,\hat{\mathbf{k}} \cdot\, \hat{\mathbf{k}} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

$$\oint_C F \cdot dr = \iint_D (\text{curl F}) \cdot\, \hat{\mathbf{k}}\,dA$$

A similar function involving the **normal** component of F:

$$\oint_C F \cdot n\,ds = \iint_D \text{div F}(x,y)\,dA$$

17.6 Parametric Surfaces and Their Areas

Parametric Surfaces

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies through D, is called a **parametric surface** S.

Example

Identify and sketch the surface with vector equation

$$r(u, v) = 2\cos u\,\hat{\mathbf{i}} + v\,\hat{\mathbf{j}} + 2\sin u\,\hat{\mathbf{k}}$$

Solution

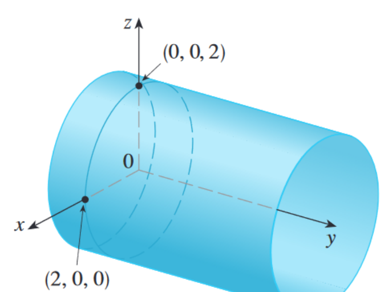
The parametric equations for this surface are

$$x = 2\cos u \qquad y = v \qquad z = 2\sin u$$

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis.



Tangent Planes

The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of r with respect to v:

$$r_v = \frac{\partial x}{\partial v}\,\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\,\hat{\mathbf{j}} + \frac{\partial z}{\partial v}\,\hat{\mathbf{k}}$$

Same thing fo r_u but ∂v changes to ∂u.

If $r_u \times r_v$ is not 0, the surface S is called smooth (has no corners). For a smooth surface, the **tangent plane** is the plane that contains r_u and r_v, and the vector $r_u \times r_v$ is a normal vector to the tangent plane.

Example

Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point (1, 1, 3).

Solution

We first compute the tangent vectors:

$$r_u = \frac{\partial x}{\partial u}\,\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\,\hat{\mathbf{j}} + \frac{\partial z}{\partial u}\,\hat{\mathbf{k}} = 2u\,\hat{\mathbf{i}} + \,\hat{\mathbf{k}}$$

$$r_v = \frac{\partial x}{\partial v}\,\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\,\hat{\mathbf{j}} + \frac{\partial z}{\partial v}\,\hat{\mathbf{k}} = 2v\,\hat{\mathbf{j}} + \,\hat{\mathbf{k}}$$

Thus a normal vector to the tangent plane is

$$r_u \times r_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\hat{\mathbf{i}} - 4u\,\hat{\mathbf{j}} + 4uv\,\hat{\mathbf{k}}$$

The point (1, 1, 3) corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$$

Therefore an equation of the tangent plane at (1, 1, 3) is

$$-2(x-1) - 4(y-1) + 4(z-3) = 0$$

$$\text{or} \qquad x + 2y - 2z + 3 = 0$$

Surface Area

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3.

$$\begin{aligned} A &= \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\ &= \iint\limits_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\ &= \iint\limits_D \sqrt{1 + 4(x^2 + y^2)}\, dA \end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr \\ &= \left[2\pi \left(\frac{1}{8}\right) \frac{2}{3}(1 + 4r^2)^{3/2}\right]_0^3 = \frac{\pi}{6}(37\sqrt{37} - 1) \end{aligned}$$

17.7 Surface Integrals

Surface integral of f over the surface S:

$$\iint\limits_S f(x, y, z)\, dS = \iint\limits_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Example

Evaluate $\iint\limits_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Solution

Since

$$\frac{\partial z}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2y$$

$$\iint_S y\,dS = \iint_D y\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \int_0^1 \int_0^2 y\sqrt{1+1+4y^2}\,dy\,dx$$
$$= \int_0^1 dx\sqrt{2}\int_0^2 y\sqrt{1+2y^2}\,dy$$
$$= \left[\sqrt{2}\left(\frac{1}{4}\right)\frac{2}{3}(1+2y^2)^{3/2}\right]_0^2 = \frac{13\sqrt{2}}{3}$$

Surface Integrals of Vector Fields

The mass of a fluid crossing S_{ij} in the direction of the normal n per unit time by the quantity

$$(pv \cdot n)A(S_{ij})$$

where p, v, and n are evaluated at some point on S_{ij}.

$$\iint_S pv \cdot n\,dS = \iint_S p(x,y,z)v(x,y,z)\cdot n(x,y,z)\,dS$$

is the rate of flow through S.

If we write $F = pv$, then F is also a vector field on $\mathbb{R}^3$ and the integral above becomes

$$\iint_S F \cdot n\,dS$$

Definition

If F is a continuous vector field defined on an oriented surface S with unit normal vector n, then the **surface integral of F over S** is

$$\iint_S F \cdot dS = \iint_S F \cdot n\,dS$$

This integral is also called the **flux** of F across F.

It can also be written as:

$$\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\,dA$$

Example

Evaluate $\iint_S F \cdot dS$, where $F(x,y,z) = y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution

$$P(x,y,z) = y \qquad Q(x,y,z) = x \qquad R(x,y,z) = z = 1 - x^2 - y^2$$

$$\frac{\partial g}{\partial x} = -2x \qquad \text{and} \qquad \frac{\partial g}{\partial y} = -2y$$

$$\begin{aligned}
\iint\limits_{S_1} F \cdot dS &= \iint\limits_{D} \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \\
&= \iint\limits_{D} [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA \\
&= \iint\limits_{D} (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} (\frac{1}{4} + \cos\theta \sin\theta)\, d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $n = -\,\hat{\mathbf{k}}$ and we have

$$\iint\limits_{S_2} F \cdot dS = \iint\limits_{S_2} F \cdot (-\,\hat{\mathbf{k}})\, dS = \iint\limits_{D} (-z)\, dA = \iint\limits_{D} 0\, dA = 0$$

$$\iint\limits_{S} F \cdot dS = \iint\limits_{S_1} F \cdot dS + \iint\limits_{S_2} F \cdot dS = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

17.8 Stokes Theorem

Stokes Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C F \cdot dr = \iint\limits_{S} \text{curl F} \cdot dS$$

Proof

$$\begin{aligned}
\oint F \cdot dr &= \iint (\nabla \times \bar{F}) \cdot dA = \int (F_x dx + F_y dy + F_z dz) \\
&= \int (\int dF_x dx) + \int (\int dF_y dy) + \int (\int dF_z dz) \\
&= \int (\int \frac{\partial F_x}{\partial y} dy + \frac{\partial F_x}{\partial z} dz) dx + \int (\frac{\partial F_y}{\partial x} dx + \frac{\partial F_y}{\partial z} dz) dy \\
&+ \int (\frac{\partial F_z}{\partial x} dx + \frac{\partial F_z}{\partial y} dy) dz) \\
&\text{order of integration matters: } dx \times dy = -dy \times dx \\
&= \int (\int -\frac{\partial F_x}{\partial y} dx dy + \frac{\partial F_x}{\partial z} dz dx) + (\int \frac{\partial F_y}{\partial x} dx dy - \frac{\partial F_y}{\partial z} dy dz) \\
&+ \int (-\frac{\partial F_z}{\partial x} dz dx + \frac{\partial F_z}{\partial y} dy dz) \\
&= \int (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) dy dz + \int (\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial x}) dz dx + \int (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) dx dy \\
&= \int (\int \nabla \times F)_x dA_x + (\int \nabla \times F)_y dA_y + (\int \nabla \times F)_z dA_z \\
&= \iint (\nabla \times \bar{F} \cdot d\bar{A})
\end{aligned}$$

Example

Use Stokes' Theorem to compute the integral $\iint_S \text{curl F} \cdot dS$, where $F(x, y, z) = xz\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Solution

To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$. Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$r(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \sqrt{3}\,\hat{\mathbf{k}} \qquad 0 \leq t \leq 2\pi$$

$$r'(t) = -\sin t\,\hat{\mathbf{i}} + cost\,\hat{\mathbf{j}}$$

$$F(r(t)) = \sqrt{3}\cos t\,\hat{\mathbf{i}} + \sqrt{3}\sin t\,\hat{\mathbf{j}} + \cos t \sin t\,\hat{\mathbf{k}}$$

Therefore, by Stokes' Theorem

$$\iint\limits_S \text{curl F} \cdot dS = \int_C F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t)\, dt$$

$$\sqrt{3}\int_0^{2\pi} 0\, dt = 0$$

17.9 Divergence Theorem

The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S F \cdot dS = \iiint_E \text{div F}\, dV$$

Proof

$$\begin{aligned}
\oiint \bar{F} \cdot d\bar{S} &= \iint (F_x dA_x + F_y dA_y + F_z dA_z) \\
&= \iint F_x dydz + F_y dzdx + F_z dxdy \\
&= \iiint (\frac{\partial F_x}{\partial x} dxdydz + \frac{\partial F_y}{\partial y} dydzdx + \frac{\partial F_z}{\partial z} dzdxdy) \\
&= \iiint \nabla \bar{F}\, dV
\end{aligned}$$

Example

Find the flux of the vector field $F(x, y, z) = z\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

First we compute the divergence of F:

$$\text{div F} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$. Thus the Divergence Theorem gives the flux as

$$\iint_S F \cdot dS = \iiint_B 1\, dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$

Chapter 18

Second-Order Differential Equations

18.1 Second-Order Differential Equations

A second-order differential equation has the form

$$P(x)\frac{d^y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

If given in the form

$$ay'' + by' + cy = 0$$

Replacing y'' with r^2, y' with r, and y with 1 lets us solve the equation using the auxiliary equation

$$ar^2 + br + c = 0$$

Case 1: $b^2 - 4ac > 0$ $\qquad y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: $b^2 - 4ac = 0$ $\qquad y = c_1 e^{r_1 x} + c_2 x e^{r_2 x}$

Case 3: $b^2 - 4ac < 0$ $\qquad y = e^{\alpha x}(c_1 cos\, \beta x + c_2 sin\, \beta x)$

$\alpha = \frac{-b}{(2a)}$ and $\beta = \frac{\sqrt{4ac-b^2}}{(2a)}$

Example

Solve the equation $y'' + y' - 6y = 0$ with the initial values $y(0) - 1$, $y'(0) = 0$

Solution

The auxiliary equation is

$$r^2 + r + 6 = (r-2)(r+3) = 0$$

So the solution is

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Differentiating this, we get

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

Satisfying initial conditions

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - 3c_2 = 0$$

$$c_1 = \frac{3}{5} \qquad c_2 = \frac{2}{5}$$

Giving us the solution

$$y = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}$$

18.2 Nonhomogenous Linear Equations

Method of Undetermined Coefficients

If the differential equation is given in the form $ay'' + by' + cy = G(x)$, then the solution is given by

$$y(x) = y_p(x) + y_c(x)$$

Where $y_c(x)$ is the general solution and $y_p(x)$ is the particular solution.

Proof

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = ay'' - ay_p'' + by' - by_p' + cy - cy_p$$

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p)$$

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p) = g(x) - g(x) = 0$$

Example

Solve the equation $y'' + y' - 2y = x^2$

Solution

The auxiliary equation becomes

$$r^2 + r - 2 = (r - 1)(r + 2) = 0$$

which gives a complementary solution of

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Since $G(x)$ is a polynomial of the 2nd degree, this gives a particular solution

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

Substituting into the original differential equation gives

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

$$A = -\frac{1}{2} \qquad B = -\frac{1}{2} \qquad C = -\frac{3}{4}$$

which gives the particular solution

$$y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

so the final solution is

$$y = c_1 e^x + c_2 e^{-2x} + -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$$

Method of Variation of Parameters

If we have already solved the homogeneous equation $ay'' + by' + cy = 0$ in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

We can replace the constants and write this with an arbitrary function $u(x)$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p'(x) = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

Example

Solve the equation $y'' + y = tan\, x$, $0 < x < \pi/2$

Solution

Solving for the complementary equation gives us

$$r^1 + 1 = 0$$

$$y_c(x) = c_1 sin\, x + c_2 cos\, x$$

Using variation of parameters means we need a solution of the form

$$y_p(x) = u_1(x) sin\, x + u_2(x) cos\, x$$

$$y_p'(x) = (u_1' sin\, x + u_2' cos\, x) + (u_1 sin\, x + u_2 cos\, x)$$

$$y_p''(x) = u_1' cos\, x - u_2' sin\, x - u_1 sin\, x - u_2 cos\, x$$

To get a solution we need

$$y_p'' + y_p = u_1' cos\, x - u_2' sin\, x = tan\, x$$
$$u_1'(sin^2 x + cos^2 x) = cos\, x\, tan\, x$$
$$u_1' = sin\, x \qquad u_1(x) = -cos\, x$$

Solving for u_2 gives

$$u_2'(x) = -\frac{sin\, x}{cos\, x} \qquad u_1'(x) = cos\, x - sec\, x$$
$$u_2(x) = sin\, x - lin(sec\, x + tan\, x)$$

Therefore

$$y_p(x) = -cos\, x\, ln(sec\, x + tan\, x)$$
$$y(x) = c_1 sin\, x + c_2 cos\, x - cos\, x\, ln(sec\, x + tan\, x)$$

18.3 Applications of Second-Order Differential Equations

Vibrating strings follow a second-order differential equation by satisfying Newton's second law

$$m\frac{d^2x}{dy^2} = -kx$$

The general solution can be written as

$$x(t) = c_1 cos\, wt + c_2 sin\, wt$$

or

$$x(t) = Acos(wt + \delta)$$

Where

$$w = \sqrt{k/m}$$
$$A = \sqrt{c_1^2 + c_2^2}$$
$$cos\, \delta = \frac{c_1}{A} \qquad sin\, \delta = -\frac{c_2}{A}$$

Damped vibrations in a string are subjected to a frictional force but still satisfies a second-order differential equation

$$m\frac{d^2x}{dy^2} + c\frac{dy}{dx} + kx = 0$$

In electrical circuits, voltage drop across resistors, inductors and conductors must be equal to the supplied voltage, but since $I = \frac{dQ}{dt}$, this turns into a second-order differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + Q/C = E(t)$$

Example

A spring with a mass of 2kg has a natural length of 0.5m. A force of 25.6 N is required to maintain it stretched to a length of 0.7m. If the spring is stretched to a length of 0.7m and then released with initial velocity 0, find the position of the mass at any time t.

Solution

From Hooke's law

$$k(0.2) = 25.6$$

Since $k = 128$ and $m = 2$

$$2\frac{d^x}{dy^2} + 128x = 0$$

which gives

$$x(t) = c_1 cos\, 8t + c_2 sin\, 8t$$

18.4 Series Solution

If differential equations aren't in the two forms given in 18.1 and 18.2, they won't be able to be solved using finite combinations of simple functions. Using a power series though lets us express the differential equations in another form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

Example

Use power series to solve the equation $y'' + y = 0$

Solution

Differentiating the power series in the form above gives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + \ldots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 x + 6c_3 x + \ldots$$

Substituting expressions gives

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+1} + c_n\right) x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal

$$(n+2)(n_1)c_{n+1} + c_n = 0$$

$$c_{n+2} = \frac{-c_0}{((n+1)(n+2))}$$

This gives a recursive relation. By plugging in values for n, we discover a pattern

For even coefficients: $c_{2n} = \frac{(-1)^n c_0}{(2n)!}$

For odd coefficients: $c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$

Plugging these back into the original form gives

$$y = c_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n c_0}{(2n)!}\right)$$

$$+c_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n c_1}{(2n+1)!}\right)$$

$$y = c_0 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n_1)!}$$

Part II

Linear Algebra

Chapter 1

Vectors

1.1 The Geometry and Algebra of Vectors

Vector: a directed line segment that corresponds to a displacement from one point A to another point B.

Vector Addition

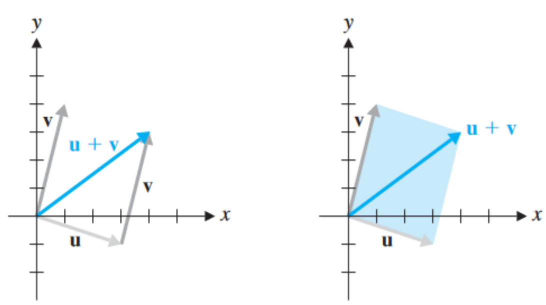
$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

Example

If $\mathbf{u} = [3, -1]$ and $\mathbf{v} = [1, 4]$ compute and draw $\mathbf{u} + \mathbf{v}$

Solution

We compute $\mathbf{u} + \mathbf{v} = [3 + 1, -1 + 4] = [4, 3]$



Scalar Multiplication

$$c\mathbf{v} = c[\mathbf{v}_1, \mathbf{v}_2] = [c\mathbf{v}_1, c\mathbf{v}_2]$$

Example

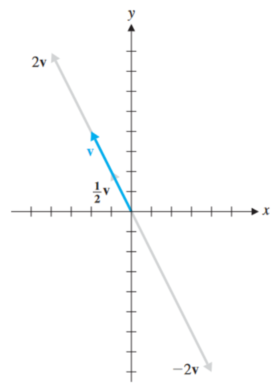
If $\mathbf{v} = [2, -4]$, compute and draw $2\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, and $-2\mathbf{v}$

Solution

$$2\mathbf{v} = [2(-2), 2(4)] = [-4, 8]$$

$$\tfrac{1}{2}\mathbf{v} = [\tfrac{1}{2}(-2), \tfrac{1}{2}(4)] = [1, -2]$$

$$-2\mathbf{v} = [-2(-2), -2(4)] = [4, -8]$$



Algebraic Properties of Vectors in $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(g) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

A vector $\mathbf{v}$ is a **linear combination** of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ if there are scalars $c_1, c_2, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1, c_2\mathbf{v}_2, \ldots, c_k\mathbf{v}_k$. The scalars $c_1, c_2, \ldots, c_k$ are called the coefficients of the linear combination.

1.2 Length and Angle: The Dot Product

The Dot Product

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}$$

then the dot product of $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Example

Compute $\mathbf{u} \cdot \mathbf{v}$ when

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

Solution

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-3) + 2 \cdot 5 + (-3) \cdot 2 = 1$$

Theorem

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

The **length** (or **norm**) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Theorem

Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$

(b) $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

Cauchy-Schwarz Inequality

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

The Triangle Inequality

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Proof

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$$

$$= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$$

$$\leq \|\mathbf{u}\|^2 + 2|\mathbf{u} \cdot \mathbf{v}| + \|\mathbf{v}\|^2$$

$$\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ and $u \neq 0$, then the **projection of v onto u** is the vector $\text{proj}_u(\mathbf{v})$ defined by

$$\text{proj}_u(\vec{v}) = \frac{(\vec{u} \cdot \vec{v})}{\vec{u} \cdot \vec{u}}\vec{u}$$

Example

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$ in each case

(a) $\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{u} = e_3$

(c) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$

Solution

(a) $\mathbf{u} \cdot \mathbf{v} = 1$ and $\mathbf{u} \cdot \mathbf{u} = 5$ so

$$\text{proj}_u(v) = \left(\frac{u \cdot v}{u \cdot u}\right) u = \frac{1}{5}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2/5\\1/5\end{bmatrix}$$

(b) Since $\mathbf{e}_3$ is a unit vector,

$$\text{proj}_{e_3}(v) = (e_3 \cdot v)e_3 = \begin{bmatrix}0\\0\\3\end{bmatrix}$$

(c) We see that $\|\mathbf{u}\| = 1$. Thus,

$$\text{proj}_u(v) = (u \cdot v)u = \frac{3(1+\sqrt{2})}{4}\begin{bmatrix}1\\1\\\sqrt{2}\end{bmatrix}$$

1.3 Lines and Planes

The vector $\mathbf{n}$ is perpendicular to the line - that is, it is **orthogonal** to any vector $\mathbf{x}$ that is parallel to the line – and it is called a **normal vector** to the line. The equation $\mathbf{n} \cdot \mathbf{x} = 0$ is the **normal form** of the equation of l.

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases}\mathbf{n}_1 \cdot \mathbf{x} = \mathbf{n}_1 \cdot \mathbf{p}_1 \\ \mathbf{n}_2 \cdot \mathbf{x} = \mathbf{n}_2 \cdot \mathbf{p}_2\end{cases}$	$\begin{cases}a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2\end{cases}$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases}x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3\end{cases}$
Planes	$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$	$ax + by + cz = d$	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$\begin{cases}x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3\end{cases}$

$$d(B, l) = \frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}}$$

$$d(B, P) = \frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2 + b^2 + c^2}}$$

1.4 Code Vectors and Module Arithmetic

Example

Let $\mathbf{u} = [1, 1, 0, 1, 0]$ and $\mathbf{v} = [0, 1, 1, 1, 0]$ be two binary vectors of length 5. Find $\mathbf{u} \cdot \mathbf{v}$.

Solution

The calculation of $\mathbf{u} \cdot \mathbf{v}$ takes place in , so we have

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + \mathbb{Z}_2 0 \cdot 0 = 0 + 1 + 0 + 1 + 0 = 0$$

Chapter 2

Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

A linear equation in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where the coefficients $a_1, a_2, \ldots, a_n$ and the constant term b are constants.

Example

Solve the system

$$x - y - z = 2$$

$$y + 3z = 5$$

$$5z = 10$$

Solution

Starting from the last equation and working backward, we find successively that $z = 2$, $y = 5 - 3(2) = -1$, and $x = 2 + (-1) + 2 = 3$. So the unique solution is $[3, -1, 2]$.

2.2 Direct Methods for Solving Linear Systems

The **coefficient matrix** contains the coefficients of the variables, and the **augmented matrix** is the coefficient matrix augmented by an extra column containing the constant terms.

A matrix is in **row echelon form** if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom
2. In each nonzero row, the first nonzero entry (called the leading entry) is in a column to the left of any leading entries below it

Example

Reduce the following matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \begin{array}{c} \xrightarrow{R_2-2R_1} \\ \xrightarrow{R_3-2R_1} \\ \xrightarrow{R_4+R_1} \end{array} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 0 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow{R_4+3R_2} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & 5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

$$\xrightarrow{R_4-29R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

Gaussian Elimination

When row reduction is applied to the augmented matrix of a system of linear equations, we create an equivalent system that can be solved by back substitution.

1. Write the augmented matrix of the system of linear equations.

2. Use elementary row operations to reduce the augmented matrix to row echelon form.

3. Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

The **rank** of a matrix is the number of nonzero rows in its row echelon form.

Example

Solve the system:

$$x_1 - x_2 + 2x_3 = 3$$

$$x_1 + 2x_2 - x_3 = -3$$

$$2x_2 - 2x_3 = 1$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Leading to the impossible equation $0 = 5$. Thus, the system has no solutions.

Gauss-Jordan Elimination

Modification of Gaussian elimination greatly simplifies the back substitution phase and is particularly helpful when calculations are being done by hand or on a system with infinitely many solutions. This method relies on reducing the augmented matrix even further.

1. Write the augmented matrix of the system of linear equations.

2. Use elementary row operations to reduce the augmented matrix to row echelon form.

3. If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

A matrix is in **reduced row echelon form** if it satisfies the following properties:

1. It is in row echelon form.

2. The leading entry in each nonzero row is a 1 (called a leading 1).

3. Each column containing a leading 1 has zeroes everywhere else.

Example

Determine whether the lines $\mathbf{x} = \mathbf{p} + s\mathbf{u}$ and $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ intersect and, if so find their point of intersection when

$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

Solution

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v} \quad \text{or} \quad s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$$

$s - 3t = -1$

$s + t = 2$

$s + t = 2$

From this, we find that $s = \frac{5}{4}$, $t = \frac{3}{4}$.

The point of intersection is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 5/4 \\ 1/4 \end{bmatrix}$$

A system of linear equations is called **homogeneous** if the constant term in each equation is zero.

2.3 Spanning Sets and Linear Independence

Example

Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

Solution

We want to find scalars x and y such that

$$x \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Expanding, we obtain the system

$x - y = 1$

$y = 2$

$3x - 3y = 3$

Whose augmented matrix is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{bmatrix} \xrightarrow{ref} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So the solution is $x = 3$, $y = 2$, and the corresponding linear combination is

$$3 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. If $\text{span}(S) = \mathbb{R}^n$, then S is called a **spanning set** for $\mathbb{R}^n$.

Example

Show that

$$\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

Solution

We need to show that an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Using the augmented matrix and row reduction we produce:

$$\begin{bmatrix} 2 & 1 & a \\ -1 & 3 & b \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 3 & b \\ 2 & 1 & a \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} -1 & 0 & b \\ 0 & 7 & a + 2b \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2} \begin{bmatrix} -1 & 3 & b \\ 0 & 1 & (a + 2b)/7 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} -1 & 0 & (b - 3a)/7 \\ 0 & 1 & (a + 2b)/7 \end{bmatrix}$$

$$x = \frac{3a - b}{7} \qquad \text{and} \qquad y = \frac{(a + 2b)}{7}$$

Thus,

$$\frac{3a - b}{7} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{a + 2b}{7} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$$

A set of vectors that is not linearly dependent is called **linearly independent**.

Example

Determine whether the following set of vectors are linearly independent:

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\4\\2\end{bmatrix}$$

Solution

$$\begin{bmatrix}1 & 1 & 1 & 0\\2 & 1 & 4 & 0\\0 & -1 & 2 & 0\end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix}1 & 1 & 1 & 0\\0 & -1 & 2 & 0\\0 & -1 & 2 & 0\end{bmatrix} \begin{matrix}\xrightarrow{R_1+R_2}\\ \xrightarrow{R_3-R_2}\\ \xrightarrow{-R_2}\end{matrix} \begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & -2 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$c_1 + 3c_3 = 0 \qquad c_2 - 2c_3 = 0$

$c_1 = -3c_3 \qquad c_2 = 2c_3$

$$-3c_3\begin{bmatrix}1\\2\\0\end{bmatrix} + 2c_3\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\4\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

2.4 Applications

Example

The combustion of ammonia (NH_3) in oxygen produces nitrogen (N_2) and water. Find a balanced chemical equation for this reaction.

Solution

$$wNH_3 + xO_2 \to yN_2 + zH_2O$$

Nitrogen: $w = 2y$

Hydrogen: $3w = 2z$

Oxygen: $2x = z$

$$\begin{matrix} w - 2y = 0\\ 3w - 2z = 0\\ 2x - z = 0\end{matrix} \to \begin{bmatrix}1 & 0 & -2 & 0 & 0\\3 & 0 & 0 & -2 & 0\\0 & 2 & 0 & -1 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0 & -2/3 & 0\\0 & 1 & 0 & -1/2 & 0\\0 & 0 & 1 & -1/3 & 0\end{bmatrix}$$

$$w = \frac{2}{3}z \qquad x = \frac{1}{2}z \qquad y = \frac{1}{3}z$$

$$w = 4 \qquad x = 3 \qquad y = 2 \qquad z = 6$$

$$4NH_3 + 3O_2 \to 2N_2 + 6H_2O$$

Chapter 3

Matrices

3.1 Matrix Operations

Matrix Addition and Scalar Multiplication

The **sum** of matrix A and B is obtained by adding the corresponding entries.

$$A + B = [a_{ij} + b_{ij}]$$

Example

Let

$$A = \begin{bmatrix} 1 & 4 & 0 \\ -2 & 6 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 5 & -1 \\ 1 & 6 & 7 \end{bmatrix}$$

but neither $A + C$ not $B + C$ is defined.

The **scalar multiple** cA is the matrix obtained by multiplying each entry of A by c.

$$cA = c[a_{ij}] = [ca_{ij}]$$

Example

$$2A = \begin{bmatrix} 2 & 8 & 0 \\ -4 & 12 & 10 \end{bmatrix} \qquad \frac{1}{2}A = \begin{bmatrix} 1/2 & 2 & 0 \\ -1 & 3 & 5/2 \end{bmatrix} \qquad (-1)A = \begin{bmatrix} -1 & -4 & 0 \\ 2 & -6 & -5 \end{bmatrix}$$

Matrix Multiplication

If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the **product** $C = AB$ is an $m \times r$ matrix. The (i, j) entry of the product is computed as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

or

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example

Compute AB if

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & 0 & 3 & -1 \\ 5 & -2 & -1 & 1 \\ -1 & 2 & 0 & 6 \end{bmatrix}$$

Solution

Since A is $2\times$ and B is $3\times$, AB will be a 2×4 matrix.

$$\begin{aligned}
c_{11} &= 1(-4) + 3(5) + (-1)(-1) = 12 \\
c_{12} &= 1(0) + 3(-2) + (-1)(2) = -8 \\
c_{13} &= 1(3) + 3(-1) + (-1)(0) = 0 \\
c_{14} &= 1(-1) + 3(1) + (-1)(6) = -4 \\
c_{21} &= (-2)(-4) + (-1)(5) + (1)(-1) = 2 \\
c_{22} &= (-2)(0) + (-1)(-2) + (1)(2) = 4 \\
c_{23} &= (-2)(3) + (-1)(-1) + (1)(0) = -5 \\
c_{24} &= (-2)(-1) + (-1)(1) + (1)(6) = 7
\end{aligned}$$

Thus, product matrix is given by

$$AB = \begin{bmatrix} 12 & -8 & 0 & -4 \\ 2 & 4 & -5 & 7 \end{bmatrix}$$

Matrix Powers

$$A^k = AA \ldots A$$

If A is a square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

Transpose of a Matrix

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A. That is, the $i$the column of A^T is the ith row of A for all i.

Example

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad C = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix}$$

Then their transposes are

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad C = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

The dot product of two vectors $\mathbf{u}$ and $\mathbf{b}$ can be denoted by $\mathbf{u}^T\mathbf{v}$. A square matrix is **symmetric** if $A^T = A$ (if A is equal to its own transpose).

3.2 Matrix Algebra

Properties of Addition and Scalar Multiplication

Let A, B, and C be matrices of the same size and let c and d be scalars. Then

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $A + 0 = A$

(d) $A + (-A) = 0$

(e) $c(A + B) = cA + cB$

(f) $(c + d)A = cA + dA$

(g) $c(dA) = (cd)A$

(h) $1A = A$

A linear combination of matrices looks like

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

Where $c_1, c_2, \ldots, c_k$ are the **coefficicents** of the linear combination.

Example

Let $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
Is $B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$ a linear combination of A_1, A_2, and A_3?

Solution

We want to find scalars c_1, c_2, and c_3 such that $c_1A_1 + c_2A_2 + c_3A_3 = B$. Thus,

$$c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

The left-hand side of this equation can be rewritten as

$$\begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix}$$

Comparing entries and using the definition of the matrix equality, we have four linear equations

$$\begin{aligned} c_2 + c_3 &= 1 \\ c_1 + c_3 &= 4 \\ -c_1 + c_3 &= 2 \\ c_2 + c_3 &= 1 \end{aligned}$$

Gauss-Jordan elimination easily gives

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$. Thus, $A_1 - 2A_2 + 3A_3 = B$.

The **span** of a set of matrices is the set of all linear combinations of the matrices.

Example

Describe the span of the matrices A_1, A_2, and A_3 from the previous example.

Solution

Write out a general linear combination of A_1, A_2, and A_3.

$$\begin{aligned} c_1A_1 + c_2A_2 + c_3A_3 &= c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix} \end{aligned}$$

Suppose we want to know when the matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in span (A_1, A_2, A_3). We know that it is when

$$\begin{bmatrix} c_2 + c_3 & c_1 + c_3 \\ -c_1 + c_3 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

for some choice of scalars c_1, c_2, c_3. This gives a system of lienar equations whose left-hand side is exactly the same as in the previous example but whose right-hand side is general. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & w \\ 1 & 0 & 1 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right]$$

and row reduction produces

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & w \\ 1 & 0 & 1 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}x - \frac{1}{2}y \\ 0 & 1 & 0 & -\frac{1}{2}x - \frac{1}{2}y + w \\ 0 & 0 & 1 & \frac{1}{2}x + \frac{1}{2}y \\ 0 & 0 & 0 & w - z \end{array}\right]$$

The only restriction comes from the last row, where we must have $w - z = 0$ to have a solution. Thus, the span of A_1, A_2, and A_3 consists of all matrices $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ for which $w = z$. That is, $\operatorname{span}(A_1, A_2, A_3) = \left\{ \begin{bmatrix} w & x \\ y & w \end{bmatrix} \right\}$

Matrices $A_1, A_2, \ldots, A_k$ are **linearly independent** if the only solution of

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k = 0$$

Is the trivial one: $c_1 = c_2 = \cdots = c_k = 0$. If not, the matrices are **linearly dependent**.

Properties of Matrix Multiplication

Let A, B, and C be matrices and let k be a scalar. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(A + B)C = AC + BC$

(d) $k(AB) = (kA)B = A(kB)$

(e) $I_m A = A = AI_n$ if A is $m \times n$

Properties of Transpose

Let A and B be matrices and let k be a scalar. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(kA)^T = k(A^T)$

(d) $(AB)^T = B^T A^T$

(e) $(A^r)^T = (A^T)^r$ for all nonnegative integers r

Theorem

(a) If A is a square matrix, then $A + A^T$ is a symmetric matrix.

(b) For any matrix A, AA^T, and A^TA are symmetric matrices.

3.3 Inverse of a Matrix

Definition

If A is an $n \times n$ matrix, an **inverse** of A is an $n \times n$ matrix A' with the property that

$$AA' = I \qquad \text{and} \qquad A'A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an A' exists, then A is called **invertible.**

Example

If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is an inverse of A, since
$AA' = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A'A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem

If A is an invertible matrix, then its inverse is unique.

Proof

A standard way to show that there is just one of something is to show that there cannot be more than one. So, suppose A has two inverses, A' and A''. Then

$$AA' = I = A'A \qquad \text{and} \qquad AA'' = I = A''A$$

$$A' = A'I = A'(A'') = (A'A)A'' = IA'' = A''$$

Hence, $A' = A''$, and the inverse is unique.

Theorem

If A is an invertible $n \times n$ matrix, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^n$.

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The expression $ad - bc$ is called the **determinant** of A.

Example

Find the inverses of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 12 & -15 \\ 4 & -5 \end{bmatrix}$, if they exist.

Solution

We have $\det A = 1(4) - 2(3) = -2 \neq 0$, so A is invertible, with

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

On the other hand, $\det B = 12(-5) - (-15)(4) = 0$, so B is not invertible.

Properties of Invertible Matrices

(a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

(b) If A is an invertible matrix and c is a nonzero scalar, then cA is an invertible matrix and

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

(c) If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1} A^{-1}$$

(d) If A is an invertible matrix, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

(e) If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and

$$(A^n)^{-1} = (A^{-1})^n$$

The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

(c) $A\mathbf{x} = 0$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is a product of elementary matrices.

Proof

We will establish the theorem by proving the circular chain of implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$$

$(a) \Rightarrow (b)$ We have already shown that if A is invertible, then $A\mathbf{x} - \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^n$.

$(b) \Rightarrow (c)$ Assume that $A\mathbf{x} - \mathbf{b}$ has a unique solution for any $\mathbf{b}$ in $\mathbb{R}^n$. This implies that $A\mathbf{x} = 0$ has a unique solution. But a homogeneous system $A\mathbf{x} = 0$ always has $\mathbf{x} = 0$ as one solution. So $\mathbf{x} = 0$ must be the solution.

$(c) \Rightarrow (d)$ Suppose that $A\mathbf{x} = 0$ has only the trivial solution. The corresponding system of equations is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned}$$

and we are assuming that its solution is

$$x_1 = 0 \quad x_2 = 0 \quad \ldots \quad x_n = 0$$

In other words, Gauss-Jordan elimination applied to the augmented matrix of the system gives

$$[A|0] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{array}\right] = [I_n|0]$$

Thus, the reduced row echelon form of A is I_n.

3.4 LU Factorization

Definition

Let A be a square matrix. A factorization of A as $A = LU$, where L is unit lower triangular and U is upper triangular, is called **LU factorization** of A.

Example

Row reduction of A proceeds as follows:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \underset{\xrightarrow{R_3+R_1}}{\xrightarrow{R_2-2R_1}} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = U$$

The three elementary matrices E_1, E_2, E_3 that accomplish this reduction of A to echelon form U are

$$E_1 = \begin{bmatrix} 1 & 0 & \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Hence,

$$E_3E_2E_1A = U$$

Solving for A, we get

$$\begin{aligned} A = E_1^{-1}E_2^{-1}E_3^{-1}U &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} U \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} U = LU \end{aligned}$$

Thus, A can be factored as

$$A = LU$$

The elementary row operations that were used are, in order:

$$\begin{array}{ll} R_2 - 2R_1 & \text{(multiplier = 2)} \\ R_3 + R_1 = R_3 - (-1)R_1 & \text{(multiplier = -1)} \\ R_3 + 2R_2 = R_3 - (-2)R_2 & \text{(multiplier = -2)} \end{array}$$

The multipliers are precisely the entries of L that are below its diagonal.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

and $L_{21} - 2$, $L_{31} = -1$, and $L_{32} = -2$. Notice the elementary row operations $R_i = kR_j$ has its multiplier k placed in the (i, j) entry of L.

$\mathbf{P}^T$LU Factorization

This method is an adaptation of the LU factorization which handles row interchanges during Gaussian elimination. P is called the **permutation matrix**.

Theorem

If P is a permutation matrix, then $P^{-1} = P^T$.

Definition

Let A be a square matrix. A factorization of A as $A = P^T LU$, where P is a permutation matrix, L is unit lower triangular, and U is upper triangular, is called a $P^T LU$ factorization of A.

Example

Find a $P^T LU$ factorization of $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

Solution

First we reduce A to row echelon form.

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

We have used two row interchanges ($R_1 \leftrightarrow R_2$ and then $R_2 \leftrightarrow R_3$), so the required permutation matrix is

$$P = P_2 P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We now find an LU factorization of PA.

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix} = U$$

Hence $L_{21} = 2$, and so

$$A = P^T LU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix}$$

3.5 Subspaces, Basis, Dimension, and Rank

Definition

A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector 0 is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u}+\mathbf{v}$ is in S.
3. If $\mathbf{u}$ is in S and c is a scalar, then $c\mathbf{u}$ is in S.

Example

Every line and place through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. It should be clear geometrically that properties (1) through (3) are satisfied. Here is an algebriac proof in the case of a plane through the origin. You are asked to give the corresponding proof for a line.

Let $\mathscr{P}$ be a plane through the origin with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence, $\mathscr{P} = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. Ther zero vector 0 is in $\mathscr{P}$, since $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2$. Now let

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$$

be two vectors in $\mathscr{P}$. Then

$$\mathbf{u}+\mathbf{v} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2) = (c_1+d_1)\mathbf{v}_1 + (c_2+d_2)\mathbf{v}_2$$

Thus, $\mathbf{u}+\mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so is in $\mathscr{P}$.

Now let c be a scalar. Then

$$c\mathbf{u} = c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2$$

which shows that $c\mathbf{u}$ is also a linear combination of $*v_1$ and $*v_2$ and is therefore in $\mathscr{P}$. We have shown that $\mathscr{P}$ satisfies properties (1) through (3) and hence is a subspace of $\mathbb{R}^3$.

Theorem

Let $v_1, v_2, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Then $\text{span}(v_1, v_2, \ldots, v_k)$ is a subspace of $\mathbb{R}^n$.

Example

Show that the set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfy the conditions $x = 3y$ and $z = -2y$ forms a subspace of $\mathbb{R}^3$.

Solution

Substituting the two conditions into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields

$$\begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Since y is arbitrary, the given set of vectors is span$\left(\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\right)$ and is thus a subspace of $\mathbb{R}^3$.

Subspaces Associated with Matrices

Definition

Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace row(A) of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** of A is the subspace col(A) of $\mathbb{R}^m$ spanned by the columns of A.

Example

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$$

(a) Determine whether $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of A.

(b) Determine whether $\mathbf{w} = \begin{bmatrix} 4 & 5 \end{bmatrix}$ is in the row space of A.

(c) Describe row(A) and col(A).

Solution

(a) $\mathbf{b}$ is a linear combination of the columns of A if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. We row reduce the augmented matrix as follows:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the system is consistent. Therefore, $\mathbf{b}$ is in col(A).

(b) Elementary row operations simply create linear combinations of the rows of a matrix. That is, they produce vectors only in the row space of the matrix. If the vector $\mathbf{w}$ is in row(A), then $\mathbf{w}$ is a linear combination of the rows of A, so if we augmente A by $\mathbf{w}$ as $\left[\frac{A}{\mathbf{w}}\right]$, it will be possible to apply elementary row operations to this augmented matrix to reduce it to $\left[\frac{A'}{0}\right]$ using only elementary row operations of the form $R_i + kR_j$, where $i > j$

In this example, we have

$$\left[\frac{A}{\mathbf{w}}\right] = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right] \xrightarrow[R-4-4R_1]{R_3-3R_1} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{array}\right] \xrightarrow{R_4-9R_2} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Therefore, $\mathbf{w}$ is a linear combination of the rows of A, and thus $\mathbf{w}$ is in row(A).

(c) It is easy to check that, for any vector $\mathbf{w} = \begin{bmatrix} x & y \end{bmatrix}$, the augmented matrix $\left[\frac{A}{\mathbf{w}}\right]$ reduces two

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

in a similar fashion. Therefore, every vector in $\mathbb{R}^2$ is in row(A), and so row$(A) = \mathbb{R}^2$.

Definition

Let A be an $m \times n$ matrix. The **null space** of A is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = 0$. It is denoted by null(A).

Basis

Definition

A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors in A that

1. spans A and
2. is linearly independent

Example

The standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$. Therefore, they form a basis for $\mathbb{R}^n$, called the **standard basis**.

Following is a summary of the most effective procedure to use to find bases for the row space, the column space, and the null space of a matrix A.

1. Find the reduced row echelon form R of A.
2. Use the nonzero row vectors of R to form a basis for row(A).
3. Use the column vectors of A that correspond to the columns of R containing leading 1s to form a basis for col(A).
4. Solve for the leading variables of $R\mathbf{x} = 0$ in terms of the free variables, set the free variables equal to the parameters, substitute back into $\mathbf{x}$, and write the result as a linera combination of f vectors. These f vectors form a basis for null(A).

Dimension and Rank

The Basis Theorem

Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

Definition

If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted by dim S.

Example

Since the standard basis for $\mathbb{R}^n$ has n vectors, dim $\mathbb{R}^n = n$.

Definition

The **rank** of a matrix A is the dimension of its row and column spaces and is denoted by rank(A).

Definition

The **nullity** of a matrix A is the dimension of its null space and is denoted by nullity(A).

The Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Proof

Let R be the reduced row echelon form of A, and suppose that rank(A)=r. Then R has r leading 1s, so there are r leading variables and $n-r$ free variables in the solution to $A\mathbf{x}=0$. Since dim(null(A))=$n-r$, we have

$$\text{rank}(A)+\text{nullity}(A)=r+(n-r)=n$$

The Fundamental Theorem of Invertible Matrices

Let A be an $n\times n$ matrix. THe following statements are equivalent:

(a) A is invertible.

(b) $A\mathbf{x}=\mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

(c) $A\mathbf{x}=0$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is a product of elementary matrices.

(f) rank(A)=n

(g) nullity(A)=0

(h) The column vectors of A are linearly independent.

(i) The column vectors of A span $\mathbb{R}^n$.

(j) The column vectors of A form a basis for $\mathbb{R}^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

Coordinates

Definition

Let S be a subspace of $\mathbb{R}^n$ and let $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ be a basis for S. Let $\mathbf{v}$ be a vector in S, and write $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k$. Then $c_1+c_2+\cdots+c_k$ are called the **coordinates of v with respect to** $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}}\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix}$$

is called the **coordinate vectors of v with respect to** $\mathcal{B}$.

Example

Let $\mathcal{E} = \{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Find the coordinate vector of

$$\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

with respect to $\mathcal{E}$.

Solution

Since $\mathbf{v} = 2\mathbf{e}_1 + 7\mathbf{e}_2 + 4\mathbf{e}_3$,

$$[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

3.6 Intro to Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^n$ and all scalars c.

Example

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that sends each point to its reflection in the x-axis. Show that F is a linear transformation.

Solution

F sends the point (x, y) to the point $(x, -y)$. Thus, we write

$$F \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

We could proceed to check that F is linear, but it's faster to observe that

$$\begin{bmatrix} x \\ -y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, $F \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so F is a matrix transformation. It now follows, that F is a linear transformation.

Theorem

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$ be linear transformations. Then $S \circ T : \mathbb{R}^m \to \mathbb{R}^p$ is a linear transformation. Moreover, their standard matrices are related by

$$[S \circ T] = [S][T]$$

Inverse Transformations

Definition

Let S and S be linearl transformations form $\mathbb{R}^n$ to $\mathbb{R}^n$. Then S and T are **inverse transformations** if $S \circ T = I_n$ and $T \circ S = I_n$.

Example

Find the standard matrix of a $60°$ clockwise rotation about the origin in $\mathbb{R}^n$.

Solution

The matrix of a $60°$ counterclockwise rotation about the origin is

$$[R_{60}] \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Since a $60°$ clockwise rotation is the inverse of a $60°$ counterclockwise rotation

$$[R_{-60}] = [(R_{60})^{-1}] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

Chapter 4

Eigenvalues

4.1 Intro to Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Example

Show that $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and find the corresponding eigenvalue.

Solution

We compute

$$A\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\mathbf{x}$$

from which it follows that $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 4.

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ and is denoted by E_λ.

Example

Show that $\lambda = 6$ is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ and find a basis for its eigenspace.

Solution

$$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we see that the null space of $A - 6I$ is nonzero. Hence, 6 is an eigenvalue of A, and the eigenvectors corresponding to this eigenvalue satisfy $x_1 + x_2 - 2x_3 = 0$, or $x_1 = -x_2 + 2x_3$. It follows that

$$E_6 = \left\{ \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

4.2 Determinants

Definition

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det\ A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Compute the determinant of

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution

We compute

$$\begin{aligned} \det\ A &= 5 \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 5(0 - (-2)) + 3(3 - 4) + 2(-1 - 0) \\ &= 5(2) + 3(-1) + 2(-1) = 5 \end{aligned}$$

The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\begin{aligned} \det &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \\ &= \sum_{j=1}^{n} a_{ij}C_{ij} \end{aligned}$$

(which is the **cofactor expansion along the ith row**) and also as

$$\begin{aligned} \det &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \\ &= \sum_{j=1}^{n} a_{ij}C_{ij} \end{aligned}$$

(the **cofactor expansion along the $j$the column**).

Example

Compute the determinant of the matrix

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

by (a) cofactor expansion along the third row and (b) cofactor expansion along the second column.

Solution

(a) We compute

$$\begin{aligned} \det &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= 2\begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & -3 \\ 1 & 0 \end{vmatrix} \\ &= 2(-6) + 8 + 3(3) \\ &= 5 \end{aligned}$$

(b) In this case, we have

$$\begin{aligned} \det &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= -(-3)\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} \\ &= 3(-1) + 0 + 8 \\ &= 5 \end{aligned}$$

Cramer's Rule

Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det\ A} \quad \text{for } i = 1, \ldots, n$$

Example

Use Cramer's Rule to solve the system

$$x_1 + 2x_2 = 2 \qquad -x_1 + 4x_2 = 1$$

Solution

We compute

$$\det\ A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6 \qquad \det(A_1(\mathbf{b})) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6 \qquad \det(A_2(\mathbf{b})) = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

By Cramer's Rule,

$$x_1 = \frac{\det(A_1(\mathbf{b}))}{\det\ A} = \frac{6}{6} = 1 \qquad \text{and} \qquad x_2 = \frac{\det(A_2(\mathbf{b}))}{\det\ A} = \frac{3}{6} = \frac{1}{2}$$

is called the **adjoint** of A and is denoted by adj A.

Theorem

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det\ A} \text{adj}\ A$$

Example

Use the adjoint method to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

Solution

We compute $\det\ A = -2$ and the nine cofactors

$$C_{11} = +\begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18 \qquad C_{12} = -\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10 \qquad C_{13} = +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3 \qquad C_{22} = +\begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2 \qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{31} = +\begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10 \qquad C_{32} = -\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6 \qquad C_{33} = +\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

THe adjoint is the **transpose** of the matrix of cofactors

$$\text{adj } A = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

Then

$$A^{-1} = \frac{1}{\det A}\text{adj } A = -\frac{1}{2}\begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -3/2 & 5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

4.3 Eigenvalues and Eigenvectors of $n \times n$ Matrices

The eigenvalues of a square matrix A are precisely the solutions λ of the equation

$$\det(A - \lambda I) = 0$$

Example

Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution

The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & - & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$
$$= -\lambda(\lambda^2 + 2\lambda) = -\lambda^2(\lambda + 2)$$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$. The eigenvalue 0 has algebriac multiplicity 2 and the eigenvalue -2 has algebriac multiplicity 1.

For $\lambda_1 = \lambda_2 = 0$, we compute

$$[A - 0I|0] = [A|0] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

from which it follows that an eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in E_0 satisfies $x_1 = x_3$. Therefore, both x_2 and x_3 are free. Setting $x_2 = s$ and $x_3 = t$, we have

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

For $\lambda_3 = -2$,

$$[A - (-2)I|0] = [A + 2I|0] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

so $x_3 = t$ is free and $x_1 = -x_3 = -t$ and $x_2 = 3x_3 = 3t$. Consequently,

$$E_{-2} = \left\{ \begin{bmatrix} -t \\ 3t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \operatorname{span} \left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right)$$

It follows that $\lambda_1 = \lambda_2 = 0$ has geometric multiplicity 2 and $\lambda_3 = -2$ has geometric multiplicity 1.

The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is the prodcut of elementary matrices.

(f) $\operatorname{rank}(A) = n$

(g) $\operatorname{nullity}(A) = 0$

(h) The column vectors of A are linearly independent.

(i) The column vectors of A span $\mathbb{R}^n$.

(j) The column vectors of A form a basis for $\mathbb{R}^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

(n) $\det A \neq 0$

(o) 0 is not an eigenvalue of A.

4.4 Similarity and Diagonalization

Definition

Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A\ B$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A\ B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus, $AP = PB$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Definition

An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D. That is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Example

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is diagonalizable since, if $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, then $P^{-1}AP = D$, as can be easily checked.

The Diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following statements are equivalent:

(a) A is diagonalizable.

(b) The unique $\mathcal{B}$ of the bases of the eigenspaces of A contains n vectors.

(c) The algebriac multiplicity of each eigenvalue equals its geometric multiplicity.

4.5 Iterative Methods for Computing Eigenvalues

Power Method

Let A be a diagonalizable $n \times n$ matrix with a corresponding dominant eigenvalue λ_1.

1. Let $\mathbf{x}_0 = \mathbf{y}_0$ be any initial vector in $\mathbb{R}^n$ whose largest component is 1.
2. Repeat the following sets for $k = 1, 2, \cdots$:
   (a) Compute $\mathbf{x}_k = A\mathbf{y}_{k-1}$.
   (b) Let m_k be the component of $\mathbf{x}_k$ with the largest absolute value.
   (c) Set $\mathbf{y}_k = (1/m_k)\mathbf{x}_k$.

For most choices of $\mathbf{x}_0$, m_k converges to the dominant eigenvalue λ_1 and $\mathbf{y}_k$ converges to a dominant eigenvector.

Example

Use the power method to approximate the dominant eigenvalue and a dominant eigenvector of

$$A = \begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$$

Solution

Taking as our initial vector

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You can see that the vectors $\mathbf{y}_k$ are approaching $\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$ and the scalars m_k are approaching 16. This suggests that they are, respectively, a dominant eigenvector and the dominant eigenvalue of A.

Gerschgorin's Disk Theorem

Let A be an $n \times n$ matrix. Then every eigenvalue of A is contained within a Gerschgorin disk.

Definition

Let $A = [a_{ij}]$ be a $n \times n$ matrix, and let r_i denote the sum of the absolute values of the off-diagonal entires in the ith row of A; that is, $r_i = \sum_{j \neq i} |a_{ij}|$. The ***i*th Gerschgorin disk** is the circular disk D_i in the complex plane with the center a_{ij} and radius r_i. That is,

$$D = \{z \text{ in } \mathbb{C} : |z - a_{ij}| \leq r_i\}$$

Chapter 5

Orthogonality

5.1 Orthogonality in $\mathbb{R}^n$

Orthogonal and Orthonormal Sets of Vectors

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal.

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \qquad \text{whenever} \quad i \neq j \quad \text{for } i, j = 1, 2, \ldots, k$$

Theorem

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then these vectors are linearly independent.

Proof

If $c_1, \ldots, c_k$ are scalars such that $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0$, then

$$(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i = 0 \cdot \mathbf{v}_i = 0$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthogonal set, all of the dot products form the equation above are zero except for $\mathbf{v}_i \cdot \mathbf{v}_i$ Therefore, $(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i = 0 \cdot \mathbf{v}_i = 0$ reduces to $c_i(\mathbf{v}_i \cdot \mathbf{v}_i) = 0$. Now, $\mathbf{v}_i \neq 0$ by the hypothesis and $\mathbf{v}_i \cdot \mathbf{v}_i \neq 0$ by the definition of orthogonality, so $c_i = 0$. Since this is true for all $i = 1, \ldots, k$, $*v_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ must be a linearly independent set.

Definition

A set of vectors in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors. An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

Orthogonal Matrices

Theorem

The columns of an $m \times n$ matrix Q form an orthonormal set if and only if

$$Q^T Q = I_n$$

Proof

$(Q^T Q)_{ij} = q_i \cdot q_j$ by the definition of matrix multiplication.

The columns Q form an orthogonal set if and only if

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

which holds if and only if

$$(Q^T Q)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Definition

An $n \times n$ matrix Q whose columns form an orthonormal set is called an **orthogonal matrix**.

Theorem

A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Example

Show that the following matrices are orthogonal:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

Solution

$$A^{-1} = A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B^{-1} = B^T = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

5.2 Orthogonal Complements and Orthogonal Projections

Definition

Let W be a subspace of $\mathbb{R}^n$ and let $u_1, \ldots, u_k$ be an orthogonal basis for W. For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the **orthogonal projection of v onto W** is given by

$$\text{proj}_W(\mathbf{v}) = \left(\frac{u_1 \cdot \mathbf{v}}{u_1 \cdot u_1}\right) u_1 + \cdots + \left(\frac{u_k \cdot \mathbf{v}}{u_k \cdot u_k}\right) u_k$$

The **component of v orthogonal to W** is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$

Example

Let $v = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and an orthogonal basis for W be $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ Find the orthogonal projection of v onto W and the component of v orthogonal to W.

Solution

$$\text{proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right) u_1 + \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right) u_2$$

$$u_1 \cdot v = 5 \quad u_2 \cdot v = -1 \quad u_1 \cdot u_1 = 2 \quad u_2 \cdot u_2 = 3$$

$$\text{proj}_W v = \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/8 \\ 17/6 \\ 18/6 \end{bmatrix}$$

$$\text{perp}_W(v) = v - \text{proj}_W(v) = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 17/6 \\ 13/6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 7/3 \\ -7/6 \end{bmatrix}$$

5.3 The Gram-Schmidt Process and QR Factorization

Theorem

Let $\{x_1, \ldots, x_k\}$ be a basis for a subspace of W of $\mathbb{R}^n$, then $v_1, \ldots, v_k$ is an orthogonal basis for W given by:

$$v_1 = x_2$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$$v_k = x_k - \left(\frac{v_1 \cdot x_k}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_k}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{v_{k-1} \cdot x_k}{v_{k-1} \cdot v_{k-1}}\right) v_{k-1}$$

$$W_1 = \operatorname{span}(x_1)$$
$$W_2 = \operatorname{span}(x_1, x_2)$$
$$W_3 = \operatorname{span}(x_1, x_2, x_3)$$
$$W_k = \operatorname{span}(x_1, \ldots, x_k)$$

Example

Apply the Gram-Schmidt Process to construct an orthonormal basis for the subspace $W = \operatorname{span}(x_1, x_2, x_3)$ of $\mathbb{R}^3$, where

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Solution

$$v_1 = x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 = \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2 = \begin{bmatrix} 1/14 \\ -1/7 \\ 3/14 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/14 \\ -1/7 \\ 3/14 \end{bmatrix} \right\} \text{is an orthogonal basis for W}$$

The QR Factorization

Theorem

Let A be an $m \times n$ matrix with linearly independent columns. Then A can be factored as $A = QR$, where Q is a $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix.

Example

Find a QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Solution

An orthogonal basis for col(A) is

$$v_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad v_2 \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix} \quad v_3 \begin{bmatrix} 1/14 \\ -1/7 \\ 3/14 \end{bmatrix}$$

$$q_1 = \frac{v_1}{\|v_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$q_2 = \frac{v_2}{\|v_2} = \sqrt{\frac{3}{14}} \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3}\sqrt{\frac{3}{14}} \\ \frac{4}{3}\sqrt{\frac{3}{14}} \\ \frac{1}{3}\sqrt{\frac{3}{14}} \end{bmatrix}$$

$$q_3 = \frac{v_3}{\|v_3} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1/14 \\ -1/7 \\ 3/14 \end{bmatrix} = \begin{bmatrix} \frac{1}{14\sqrt{14}} & -\frac{1}{7\sqrt{14}} & \frac{3}{14\sqrt{14}} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{5}{3}\sqrt{\frac{3}{14}} & \frac{1}{14\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{4}{3}\sqrt{\frac{3}{14}} & -\frac{1}{7\sqrt{14}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{3}\sqrt{\frac{3}{14}} & \frac{3}{14\sqrt{14}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{5}{3}\sqrt{\frac{3}{14}} & \frac{4}{3}\sqrt{\frac{3}{14}} & \frac{1}{3}\sqrt{\frac{3}{14}} \\ \frac{1}{14\sqrt{14}} & -\frac{1}{7\sqrt{14}} & \frac{3}{14\sqrt{14}} \end{bmatrix} \begin{bmatrix} 1 & 5/3 & 1/14 \\ -1 & 4/3 & -1/7 \\ -1 & 1/5 & 8/14 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{14}{3}\sqrt{\frac{3}{14}} & 0 \\ 0 & 0 & \frac{1}{14\sqrt{14}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5/3 & 1/14 \\ -1 & 4/3 & -1/7 \\ -1 & 1/5 & 8/14 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{5}{3}\sqrt{\frac{3}{14}} & \frac{1}{14\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{4}{3}\sqrt{\frac{3}{14}} & -\frac{1}{7\sqrt{14}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{3}\sqrt{\frac{3}{14}} & \frac{3}{14\sqrt{14}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{14}{3}\sqrt{\frac{3}{14}} & 0 \\ 0 & 0 & \frac{1}{14\sqrt{14}} \end{bmatrix}$$

5.4 Orthogonal Diagonalization of Symmetric Matrices

A square matrix A is **orthogonally diagonalizable** if there is an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Example

Prove A is symmetric given that A is orthogonally diagonalizable.

Solution

Since A is orthogonally diagonalizable, then $Q^T AQ = D$.

$$Q^T Q = QQ^T = I \qquad \text{since} \qquad Q^{-1} = Q^T$$

$$QDQ^T = QQ^T AQQ^T = IAI = A$$

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$$

Therefore, A is symmetric.

5.5 Applications

Spectral Decomposition

Theorem

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

Derivation of the Spectral Decomposition

$$A = QDQ^T = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = \begin{bmatrix} \lambda_1 q_1 & \dots & \lambda_n q_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

This is the **spectral decomposition** of A.

Example

Given $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

(a) Orthogonally diagonalize the matrix A.

(b) Find the spectral decomposition of the matrix A.

Solution

(a)

$$\det(A - \lambda I) = \det \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda - 5)(\lambda - 3) = 0$$

$$\lambda = 3, 5 \to D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$(A - 5I)v_1 = 0 \qquad v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - 3I)v_1 = 0 \qquad v_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad D = Q^T A Q = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

(b)

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T = 5 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + 3 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Quadratic Forms

Definition

A quadratic form in n variables is a function of the form

$$f(x) = x^T A x$$

where A is a symmetric $n \times n$ matrix and x is in $\mathbb{R}^n$. We can represent quadratic forms using matrices as follows:

$$ax^2 + by^2 + cxy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$ax^2 + by^2 + cz^2 + dx + exz + fyz = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example

What is the quadratic form with associated matrix $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$?

Solution

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

$$a = 2 \quad b = 6 \quad c = -6$$

$$f(x) = x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 5x_2^2 - 6x_1x_2$$

Example

Find the the matrix associated with the quadratic form

$$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$$

Solution

$$A \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} = \begin{bmatrix} 2 & 3 & -3/2 \\ 3 & -1 & 0 \\ -3/2 & 0 & 5 \end{bmatrix}$$

Chapter 6

Vector Spaces

6.1 Vector Spaces and Subspaces

Definition

Let V be a set on which addition and scalar multiplication have been defined. If u and v are in V, their sum is denoted by u + v, and if c is a scalar multiple of u it is denoted by cu. If the following axioms hold for all u, v, and w in V and for all scalars c and d, then V is called a vector space and its elements are called vectors.

1. $u + v$ is in V
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There exists an element 0 in V, called a zero vector, such that $u + 0 = u$
5. For each u in V, there is an element $-u$ in V such that $u + (-u) = 0$
6. cu is in V
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

Example

Let V be a vector space. Prove the following:

(a) $0u = 0$

(b) $c0 = 0$

(c) $(-1)u = -u$

(d) If $cu = 0$, then $c = 0$ or $u = 0$

Solution

Let V be a vector space. Prove the following:

(a)

$$0u = (0+0)u = 0u + 0u \qquad 0u + (0(-u)) = 0u + (0u + (0(-u)))$$
$$0 = 0u + 0 \rightarrow 0u = 0$$

(b)

$$c0 = c(0+0)c0 + c0 \qquad c0 + (-c0) = c0 + (c0 + (-c0)) \quad 0 = c0 + 0 \quad c0 = 0$$

(c)

$$(-1)u + u = (-1)u + (1)u = (-1+1)u = 0u = 0$$
$$(-1)u + u = 0 \quad \text{and} \quad u + (-u) = 0 \rightarrow (-1)u = -u$$

(d)

$$\text{assume } c \neq 0 \qquad u = 1u = \left(\frac{1}{c}c\right)u = \frac{1}{c}(cu) = \frac{1}{c}(0) = 0$$

Theorem

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

(a) If u and v are in W, then $u + v$ is in W

(b) If u is in W and c is scalar, then cu is in W

Example

Let $\mathscr{D}$ be the set of all differentiable real-valued functions defined on $\mathbb{R}$. Show that $\mathscr{D}$ is a subspace of $\mathscr{F}$, the vector space of all real-valued functions defined on $\mathbb{R}$.

Solution

$$(f+g)' = f' + g'$$

so $\mathscr{D}$ is closed under addition.

$$(cf)' = c(f')$$

so $\mathscr{D}$ is closed under scalar multiplication. Therefore, $\mathscr{D}$ is a subspace of $\mathscr{F}$.

Example

Show that the set of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

Solution

Let u and v be in W where $u = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$ and $v = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$

$$u + v = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} + \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ -b-d \\ a+c \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ -(b+d) \\ a+c \end{bmatrix}$$

so $u + v$ is in W.

Let k be a scalar

$$ku = k \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so ku is in W.

W is closed under addition and scalar multiplication, so W is a subspace of $\mathbb{R}^4$.

6.2 Linear Independence, Basis, and Dimension

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that

$$\{c_1\mathbf{v}_1, c_2\mathbf{v}_2, \ldots, c_k\mathbf{v}_k = 0\}$$

A set of vectors that is not linearly dependent is said to be **linearly independent**.

Theorem

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Example

Show that set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathscr{P}_n$.

Solution

$$c + 0 \cdot 1 + c_1 x + c_2 s^2 + \cdots + c_n x^n = 0$$

Let $x = 0 \to c_0 = 0$ and repeat with each derivative

$$c_1 + 2c_2 x + \cdots + (n-1)c_n x^{n-1} = 0$$

$c_1 = 0$ when $x = 0$
We find that $c_0 = c_1 = c_2 = \cdots = c_n = 0$ so $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

Bases

Definition

A subset $\mathcal{B}$ of a vector space V is a basis for V if

1. $\mathcal{B}$ spans V
2. $\mathcal{B}$ is linearly independent

Example

Show that $\mathcal{B} = \{1 + x, x + x^2, 1 + x^2\}$ is a basis for $\mathscr{P}_2$.

Solution

$$a(1+x) + b(x+x^2) + c(1+x^2) = 0 \qquad a + ax + bx + bx^2 + c + cx^2 = 1$$

$$(a+c) + (a+b)x + (b+c)x^2 = 0 \quad a + c = 0 \quad a + b = 0 \quad b + c = 0$$

$$a = -c \quad -c+b = 0 \quad c = b \quad 2b = 0 \quad b = 0 \quad a+0 = 0 \quad a = 0 \quad 0+c = 0 \quad c = 0$$

$$a = b = c = 0, \text{ so } \mathcal{B} \text{ is linearly independent}$$

$$(a+c) + (a+b)x + (b+c)x^2 = \alpha + \beta x + \gamma x^2$$

$$a + c = \alpha \qquad a + b = \beta \qquad b + c = \gamma$$

Since a solution exists, $\mathcal{B}$ spans $V \to \mathcal{B}$ is a basis for $\mathscr{P}_2$.

Coordinates

Theorem

Let V be a vector space and let $\mathcal{B}$ be a basis for V. For every vector $\mathbf{v}$ in V, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in $\mathcal{B}$.

Definition

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{v}$ be a vectors in V, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Then $c_1, c_2, \ldots, c_n$ are called the **coordinates of v with respect to** $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of v with respect to** $\mathcal{B}$.

Example

Find the coordinate vector $[p(x)]_{\mathcal{B}}$ of $p(x) = 2 - 3x + 5x^2$ with respect to the standard basis $\mathcal{B} = 1, x, x^2$ of $\mathscr{P}_2$.

Solution

$$a(1) + b(x) + c(x^2) = 2 - 3x + 5x^2$$

$$a = 2 \qquad b = -3 \qquad c = 5$$

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

Dimension

Theorem

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for vector space V.

(a) Any set of more than n vectors in V must be linearly dependent.

(b) Any set of fewer than n vectors in V cannot span V.

Definition

A vector space V is called **finite-dimensional** if it has a basis consisting of finitely many vectors. The dimension of V, denoted by dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. A vector space that has no finite basis is called **infinite-dimensional**.

Theorem

Let V be a vector space with dim $V = n$. Then

(a) Any linearly independent set in V contains at most n vectors.

(b) Any spanning set for V contains at least n vectors.

(c) Any linearly independent set of exactly n vectors in V is a basis for V.

(d) Any spanning set for V consisting of exactly n vectors is a basis for V.

(e) Any linearly independent set in V can be extended to a basis for V.

(f) Any spanning set for V can be reduced to a basis for V.

Example

In each case determine whether S is a basis for V.

(a)

$$V = \mathscr{P}_2 \qquad S = \{1+x, 2-x+x^2, 3x-2x^2, -1+3x+x^2\}$$

(b)

$$V = M_{22} \qquad S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

(c)

$$V = \mathscr{P}_2 \qquad S = \{1+x, x+x^2, 1+x^2\}$$

Solution

(a) Since dim $(\mathscr{P}_2) = 3$ and S contains four vectors, S is linearly dependent. Therefore, S is not a basis for $\mathscr{P}_2$.

(b) Since dim $(M_{22}) = 4$ and S contains three vectors, S cannot span M_{22}. Therefore, S is not a basis for M_{22}.

(c) Since dim $(\mathscr{P}_2) = 3$ and S contains three vectors, S will be a basis for $\mathscr{P}_2$ if it is linearly independent or if it spans. It is easier to show that S is linearly independent as we did in the first example under the subtopic of bases. Therefore, S is a basis for $\mathscr{P}_2$.

6.3 Change of Basis

Definition

Let $B = u_1, \ldots, u_n$ and $C = v_1, \ldots, v_n$ be bases for a vector space V. The $n \times n$ matrix whose columns are the coordinate vectors $[u_1]_C, \ldots, [u_n]_C$ of the vectors in B with respect to C is denoted by $P_{C \leftarrow B}$ and is called the **change-of-basis matrix** from B to C. That is,

$$P_{C\leftarrow B} = \begin{bmatrix} [u_1]_C & [u_2]_C & \ldots & [u_n]_C \end{bmatrix}$$

Theorem

Let $B = u_1, \ldots, u_n$ and $C = v_1, \ldots, v_n$ be bases for a vector space V and let $P_{C \leftarrow B}$ be the change-of-basis matrix from B to C. Then

(a) $P_{C \leftarrow B}[x]_b = [x]_c$ for all x in V

(b) $P_{C \leftarrow B}$ is the unique matrix P with the property that $P[x]_B = [x]_C$ for all x in V

(c) $P_{C \leftarrow B}$ is invertible and $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$

Example

Find the change-of-basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B = 1, x, x^2$ and $C = 1 + x, x + x^2, 1 + x^2$ of $\mathscr{P}_2$. Then find the coordinate vector of $p(x) = 1 + 2x - x^2$ with respect to C.

Solution

Changing toa standard basis is easy, so we find $P_{B \leftarrow C}$ first. Observe that thecoordinate vectors for C in terms of B are

$$[1 + x]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad [x + x^2]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad [1 + x^2]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

To find $P_{C \leftarrow B}$ we could express each vector in B as a linear combination of the vectors in C, but it is much easier to use the fact that $P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1}$. We find that

$$P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

It now follows that

$$\begin{aligned} [p(x)]_C &= P_{C \leftarrow B}[p(x)]_B \\ &= \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

Theorem

$B = \mathbf{u}_1, \ldots, \mathbf{u}_n$ and $C = \mathbf{v}_1, \ldots, \mathbf{v}_n$ be bases for a vector space V. Let $B = [[\mathbf{u}_1]E \ldots [\mathbf{u}_n]E]$ and $C = [[\mathbf{v}_1]E \ldots [\mathbf{v}_n]E]$, where ϵ is any basis for V. Then row reduction applied to the $n \times 2n$ augmented matrix $[C \mid B]$ produces

$$[C \mid B] \to [I \mid P_{C \leftarrow B}]$$

Example

In M_{22}, let B be the basis $\{E_{11}, E_{21}, E_{12}, E_{22}\}$ and let C be the basis $\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Find the change-of-basis matrix $P_{C \leftarrow B}$ using the Gauss-Jordan method.

Solution

$$B = P_{\epsilon \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[C|B] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.4 Linear Transformations

Definition

A **linear transformation** from a vector space V to a vector space W is a mapping $T : V \to W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars c

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

$T : V \to W$ is a linear transformation if and only if

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k)$$

for all $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V and scalars $c_1, \ldots, c_k$.

Example

Define $T : M_{nn} \to M_{nn}$ by $T(A) = AT$. Show that T is a linear transformation.

Solution

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$
$$T(cA) = (cA)^T = CA^T = cT(A)$$

Therefore, T is a linear transformation.

Theorem

Let $T : V \to W$ be a linear transformation. Then

(a) $T(0) = 0$

(b) $T(-v) = -T(v)$

(c) $T(u - v) = T(u) - T(v)$ for all u and v in V

Definition

If $T : U \to V$ and $S : V \to W$ are linear transformations, then the **composition of S with T** is the mapping $S \circ T$, defined by

$$(S \circ T)(i) = S(T(u))$$

where u is in U.

Definition

A linear transformation $T : V \to W$ is **invertible** if there is a linear transformation $T' : W \to V$ such that

$$T' \circ T = I_V \quad \text{and} \quad T \circ T' = I_W$$

In this case T' is called an **inverse** for T.

Example

Verify that the mappings $T : \mathbb{R}^2 \to \mathscr{P}_1$ and $T' : \mathscr{P}_1 \to \mathbb{R}^2$ defined by

$$T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)xx \quad \text{and} \quad T'(c + dx = \begin{bmatrix} c \\ d - c \end{bmatrix})$$

are inverses.

Solution

We compute

$$(T \circ T') \begin{bmatrix} a \\ b \end{bmatrix} = T' \left(T \begin{bmatrix} a \\ b \end{bmatrix} \right) = T'(a + (a+b)x) = \begin{bmatrix} a \\ (a+b) - c \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$(T \circ T')(c + dx) = T(T'(c + dx)) = T \begin{bmatrix} c \\ d - c \end{bmatrix} = c + (c + (d - c))x = c + dx$$

Hence, $T' \circ T = I_{\mathbb{R}^2}$ and $T \circ T' = I_{\mathscr{P}_1}$. Therefore, T and T' are inverses of each other.

6.5 Kernel and Range

Definition

Let $T : V \to W$ be a linear transformation. The **kernel** of T denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to 0 in W. That is,

$$\ker(T) = \{v \text{ in } V \ : \ T(v) = 0\}$$

The **range** of T, denoted by $\text{range}(T)$, is the set of all vectors in W that are images of vectors in V under T. That is,

$$\text{range}(T) = \{T(v) \ : \ v \text{ in } V\} = \{w \text{ in } W \ : \ w = T(v) \text{ for some } v \text{ in } V\}$$

Example

Find the kernel and range of the differential operator $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution

$$\begin{aligned} \ker(D) &= \{a + bx + cx^2 + dx^3 : D(a + bx + cx^2 + dx^3) = 0\} \\ &= \{a + bx + cx^2 + dx^3 : b + 2cx + 3dx^2 = 0\} \\ &= \{a + bx + cx^2 + dx^3 : b = c = d = 0\} \\ &= \{a : a \text{ in } \mathbb{R}\} \end{aligned}$$

$\text{range}(D)$ is all polynomials in $\mathscr{P}_2$

Definition

Let $T : V \to W$ be a linear transformation. The **rank** of T is the dimension of range of T and is denoted by $\text{rank}(T)$. The **nullity** of T is dimension of the kernel of T and is denoted by $\text{nullity}(T)$.

Example

Find the rank and the nullity of the linear transformation $D : \mathscr{P}_3 \to \mathscr{P}_2$ defined by $D(p(x)) = p'(x)$.

Solution

$$\operatorname{rank}(D) = \dim \mathscr{P}_2 = 3$$

$$\operatorname{nullity}(D) = \dim(\ker(D)) = 1$$

The Rank Theorem

Let $T : V \to W$ be a linear transformation from a finite-dimensional vector space V into a vector space W. Then

$$\operatorname{rank}(T) + \text{nullity} = \dim\, V$$

Definition

A linear transformation $T : V \to W$ is called **one-to-one** if T maps distinct vectors in V to distinct vectors in W. If $\operatorname{range}(T) = W$, then T is called **onto**.

1. $T : V \to W$ is one-to-one if, for all u and v in V $u \neq v$ implies that $T(u) \neq T(v)$
2. $T : V \to W$ is one-to-one if, for all u and v in V $T(v) = T(v)$ implies that $u = v$
3. $T : V \to W$ is onto if, for all u and v in V, there is at least one v in V such that $w = T(v)$

Theorem

A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{0\}$.

Example

Show that the linear transformation $T : \mathbb{R}^2 \to \mathscr{P}_1$ defined by

$$T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a + b)x$$

is one-to-one and onto.

Solution

If $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the kernel of T, then

$$0 = T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$$

It follows that $a =$ and $a + b = 0$. Hence, $b = 0$, and therefore $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Consequently, $\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, and T is one-to-one.

By the Rank Theorem,

$$\operatorname{rank}(T) = \dim \mathbb{R}^2 - \operatorname{nullity}(T) = 2 - 0 = 2$$

Therefore, the range of T is a two-dimensional subspace of $\mathbb{R}^2$, and hence $\operatorname{range}(T) = \mathbb{R}^2$. It follows that T is onto.

Theorem

Let $\dim V = \dim W = n$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.

Theorem

A linear transformation $T : V \to W$ is one-to-one if and only if it is onto.

Definition

A linear transformation $T : V \to W$ is called an isomorphism if it is one-to-one and onto. If V and W are two vector spaces such that there is an isomorphism from V to W, then we say V is isomorphic to W and write $V \cong W$.

Theorem

Let V and W be two finite-dimensional vector spaces. Then V is isomorphic to W if and only if $\dim V = \dim W$.

Example

Let W be the vector space of all symmetric 2×2 matrices. Show that W is isomorphic to $\mathbb{R}^3$.

Solution

W is represented by the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, so $\dim W = 3$. Hence, $\dim W = \dim \mathbb{R}^3$, so $W \cong \mathbb{R}^3$.

6.6 Matrix of a Linear Transformation

Theorem

Let V and W be two finite-dimensional vector spaces with bases B and C respectively, where $B = v_1, \ldots, v_n$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = [[T(\mathbf{v}_1)_C][T(\mathbf{v}_2)_C] \ddots [T(\mathbf{v}_n)_C]]$$

satisifes

$$A[\mathbf{v}]_B = [T(\mathbf{v})]_C$$

for every vector $\mathbf{v}$ in V.

$$[T]_{C \leftarrow B}[\mathbf{v}]_B = [T(\mathbf{v})]_C$$

$$[T]_B[\mathbf{v}]_B = [T(\mathbf{v})]_B$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$$

and let $B = \{e_1, e_2, e_3\}$ and $C = \{e_2, e_1\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Find the matrix of T with respect to B and C and verify the previous Theorem for $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

Solution

First, we compute

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad T(e_2) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad T(e_3) \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Next, we need their coordinate vectors with respect to C. Since

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_2 + e_1 \qquad \begin{bmatrix} -2 \\ 1 \end{bmatrix} = e_2 - 2e_1 \qquad \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3e_2 + 0e_1$$

we have

$$[T(e_1)]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [T(e_2)]_C = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad [T(e_3)]_C = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Therefore, the matrix of T with respect to B and C is

$$\begin{aligned} A = [T]_{C\leftarrow B} &= \begin{bmatrix} [T(e_1)]_C & [T(e_2)]_C & [T(e_3)]_C \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \end{aligned}$$

To verify the Theorem for $\mathbf{v}$, we first compute

$$T(\mathbf{v}) = T \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

Then

$$[\mathbf{v}]_B = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

and

$$[T(\mathbf{v})]_C \begin{bmatrix} -5 \\ 10 \end{bmatrix}_C = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

Using all of these facts, we confirm that

$$A[\mathbf{v}]_B = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = [T(\mathbf{v})]_C$$

Matrices of Composite and Inverse Linear Transformations

Theorem

Let U, V, and W be finite-dimensional vector spaces with bases B, C, and D, respectively. A linear transformation $T : U \to V$ and $S : V \to W$ be linear transformation. Then

$$[S \circ T]_{D\leftarrow B} = [S]_{D\leftarrow C}[T]_{C\leftarrow B}$$

Theorem

Let $T : V \to W$ be a linear transformation between n-dimensional vector spaces V and W and let B and C be bases for V and W, respectively. Then T is invertible if and only if the matrix $[T]_{C\leftarrow B}$ if invertible. In this case,

$$([T]_{C\leftarrow B})^{-1} = [T^{-1}]_{B\leftarrow C}$$

Example

The linear transformation $T : \mathbb{R}^2 \to \mathscr{P}_1$ defined by

$$T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$$

was shown to be one-to-one and onto and hence invertible. Find T^{-1}.

Solution

Using ϵ and ϵ' for $\mathbb{R}^2$ and $\mathscr{P}_2$

$$[T]_{\epsilon'\leftarrow\epsilon} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

It follows that the matrix of T^{-1} with respect to ϵ' and ϵ is

$$[T^{-1}]_{\epsilon\leftarrow\epsilon'} = ([T]_{\epsilon'\leftarrow\epsilon})^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} [T^{-1}(a+bx)]_{\epsilon} &= [T^{-1}]_{\epsilon\leftarrow\epsilon'}[a+bx]_{\epsilon'} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a \\ b-a \end{bmatrix} \end{aligned}$$

This means that

$$T^{-1}(a+bx) = ae_1 + (b-a)e_2 = \begin{bmatrix} a \\ b-a \end{bmatrix}$$

6.7 Applications

Homogeneous Linear Differential Equations

Theorem

Let S be the solution space of

$$y'' + ay' + by = 0$$

and let λ_1 and λ_2 be the roots of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

(a) If $\lambda_1 \neq \lambda_2$, then $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ is a basis for S.

(b) If $\lambda_1 = \lambda_2$, then $\{e^{\lambda_1 t}, e^{\lambda_1 t}\}$ is a basis for S.

Therefore, the solutions are

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \qquad \text{and} \qquad y = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$$

Example

Find all solutions of $y'' - 5y' + 6y = 0$.

Solution

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

Thus, the roots are 2 and 3, so $\{e^{2t}, e^{3t}\}$ is a basis for the solution space. The solutions to the given equation are of the form

$$y = c_1 e^{2t} + c_2 e^{3t}$$

The constants c_1 and c_2 can be determined if additional equations, called **boundary conditions**, are specified.

Chapter 7

Distance and Approximation

7.1 Inner Product Spaces

We can define the dot product $u \cdot v$ of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. An inner product on a vector space V is an operation that assigns to every pair of vectors $\vec{u}$ and $\vec{v}$ in V a real number $\langle u, v \rangle$ such that the following properties hold for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and all scalars c.

Properties of Inner Products

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in an inner product space V and let c be a scalar.

(a) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(b) $\langle u, cv \rangle = c \langle uv \rangle$

(c) $\langle u, 0 \rangle = \langle 0, v \rangle = 0$

Definition

Let u and v be vectors in an inner product space V

1. The **length** of v is $\|V\| = \sqrt{\langle v, v \rangle}$
2. The **distance** between u and v is $d(u, v) = \|u - v\|$
3. u and v are **orthogonal** if $\langle u, v \rangle = 0$

Example

Consider the inner product on $\mathscr{L}[0, 1]$ If $f(x) = x$ and $g(x) = 3x - 2$, find

(a) $\|f\|$

(b) $d(f,g)$

(c) $\langle f,g\rangle$

Solution

(a)

$$\langle f,f\rangle = \int_0^1 f^2(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$$

$$\|f\| = \sqrt{\langle f,f\rangle} = \frac{1}{\sqrt{3}}$$

(b)

$$d(f,g) = \|f-g\| = \sqrt{\langle f-g, f-g\rangle}$$

$$f(x)-g(x) = x-(3x-2) = 2-2x$$

$$\langle f-g, f-g\rangle = \int_0^1 (f(x)-g(x))^2\,dx = 4\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{4}{3}$$

$$d(f,g) = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

(c)

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 (3x^2-2x)\,dx = \left[x^3-x^2\right]_0^1 = 0$$

Thus, f and g are orthogonal.

Pythagoras' Theorem

Let u and v be vectors in an inner product space V. Then u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

Example

Construct an orthogonal basis for $\mathscr{P}_2$ with respect to the inner product

$$\langle f,g\rangle = \int_{-1}^1 f(x)g(x)\,dx$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Solution

Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, and $\mathbf{x}_3 = x^2$. We begin by setting $\mathbf{v}_1 = x_1 = 1$. Next we compute

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \int_{-1}^{1} dx = [x]_{-1}^{1} = 2 \qquad \text{and} \qquad \langle \mathbf{v}_1, \mathbf{x}_2 \rangle = \int_{-1}^{1} x\, dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Therefore,

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{v}_1, \mathbf{x}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = x - \frac{0}{2}(1) = x$$

To find $\mathbf{v}_3$, we first compute

$$\langle \mathbf{v}_1, \mathbf{x}_3 \rangle = \int_{-1}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \qquad \langle \mathbf{v}_2, \mathbf{x}_3 \rangle = \int_{-1}^{1} x^3\, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$$

Then

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{v}_1, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{x}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 = x^2 - \frac{\frac{2}{3}}{2}(1) - \frac{0}{\frac{2}{3}}x = x^2 - \frac{1}{3}$$

It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathscr{P}_2$ on the interval [-1, 1]. The polynomials

$$1, \quad x, \quad x^2 - \frac{1}{3}$$

are the first three **Legendre polynomials**. If we divide each of these polynomials byits length relative to the same inner product, we obtain **normalized Legendre polynomials**.

The Cauchy-Schwarz Inequality

Let u and v vectors in an inner product space V. Then

$$|\langle u, v \rangle| \leq \|u\|\|v\|$$

with equality holding if and only if u and v are scalar multiples of each other

The Triangle Inequality

Let u and v vectors in an inner product space V. Then

$$\|u + v\| \leq \|u\| + \|v\|$$

7.2 Norms and Distance Functions

A norm on a vector space V is a mapping that associates with each vector $\vec{v}$ a real number $\|v\|$, called the **norm** of $\vec{v}$, such that the following properties are satisfied for all vectors $\vec{u}$ and $\vec{v}$ and all scalars c:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$
2. $\|cv\| = |c|\|v\|$
3. $\|u + v\| \leq \|u\| + \|v\|$

A vector space with a norm is called a normed linear space.

Vector Norms

1. **Sum Norm** is the sum of the absolute values of its components
$$\|v_s\| = |v_1| + \cdots + |v_n|$$
2. **Max Norm** is the largest number along the absolute values of its components
$$\|v_m\| = \max(|v_1|, \ldots, |v_n|)$$
3. **Euclidean Norm** is the value of the distance between the two vectors

Example

Let $u = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Copmute $d(u, v)$ relative to (a) the Euclidean norm, (b) the sum norm, and (c) the max norm.

Solution

Each calculation requires knowing that $u - v = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

(a) $d_E(u, v) = \|u - v\|_E = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

(b) $d_S(u, v) = \|u - v\|_S = |4| + |-3| = 7$

(c) $d_m(u, v) = \|u - v\|_m = \max(|4|, |-3|) = 4$

Matrix Norms

A Matrix Norm on M_{nn} is a mapping that associates with each matrix A, called the norm of A, and satisfies the following properties.

1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$

2. $\|cA\| = |c|\|A\|$

3. $\|A + B\| \leq \|A\| + \|B\|$

4. $\|AB\| \leq \|A\|\|B\|$

7.3 Least Squares Approximation

If W is a subscape of a normed linear space V and if v is a vector in V, then the best approximation to v in W such that

$$\|v - v\| < \|v - w\|$$

for every vector w in W different from v.

Least Squares Theorem

Let A be an $m \times n$ and let b be in $\mathbb{R}^n$. Then $A\mathbf{x} = b$ always has at least one least squares solution. Moreover,

1. $\bar{x}$ is a least square solution of $A\mathbf{x} = b$ if and only if $\bar{x}$ is a solution of the normal equations $A^T A\bar{x} = A^t\mathbf{b}$

2. A has linearly independent coloumns if any only if $A^T A$ is invertible. In this case, the least squares solution is unique and is given by $\bar{x} = (A^T A)^{-1}A^T\mathbf{b}$

Penrose Conditions

The **pseudoinverse** of A is defined by $A^+ = (A^T A)^{-1}A^T$. Note that if A is $m \times n$, then A^+ is $n \times m$. Then **Penrose Conditions** for A are

1. $AA^+A = A$

2. $A^+AA^+ = A^+$

3. AA^+ and A^+A are symmetric

7.4 The Singular Value Decomposition

If A is an $m \times n$, the **singular values** of A are the square roots of the eigenvalues of $A^T A$ and are denoted by $\sigma_1, \dots, \sigma_n$.

Singular Value Decomposition

Let A be an $m \times n$ with singular values $\sigma_1, \dots, \sigma_n$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_{r+n} = 0$. There exists an $m \times n$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ such that $A = U\Sigma V^T$.

Example

Find a singular value deconomposition for the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

We compute

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find that its eigenvalues are $\lambda_1 = 2$, $\lambda_1 = 2$, and $\lambda_3 = 0$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

These vectors are orthogonal, so we normalize them to obtain

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

The singular values of A are $\sigma_1 = \sqrt{2}$, $\sigma_2 = \sqrt{1}$, and $\sigma_3 = \sqrt{0} = 0$. Thus,

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

To find U, we compute

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These vectors already form an orthonormal basis for $\mathbb{R}^2$, so we have

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This yields the SVD

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = U\Sigma V^T$$

The Outer Product Form of the SVD

$$a = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

Byproduct of Penrose Conditions

$A^+ = V\Sigma^+ U^T$ where Σ^+ is the $n \times m$ matrix $\begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}$

The Fundamental Theorem of Invertible Matricies

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is the prodcut of elementary matrices.

(f) $\text{rank}(A) = n$

(g) $\text{nullity}(A) = 0$

(h) The column vectors of A are linearly independent.

(i) The column vectors of A span $\mathbb{R}^n$.

(j) The column vectors of A form a basis for $\mathbb{R}^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

(n) $\det A \neq 0$

(o) 0 is not an eigenvalue of A.

(p) T is invertible.

(q) T is one-to-one.

(r) T is onto.

(s) $\ker(T) = 0$

(t) $\text{range}(T) = W$

(u) 0 is not a singular value of A

7.5 Applications

Example

Find the best linear approximation to $f(x) = e^x$ on the interval $[-1, 1]$.

Solution

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

A basis for $\mathscr{P}_1[-1, 1]$ is given by $\{1, x\}$. Since

$$\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0$$

this is an orthogonal basis, so the best approximation to f in W is

$$\begin{aligned} g(x) = \text{proj}(e^x) &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x, e^x\rangle}{\langle x, x\rangle}x \\ &= \frac{\int_{-1}^{1}(1 \cdot e^x)\,dx}{\int_{-1}^{1}(1 \cdot 1)\,dx} + \frac{\int_{-1}^{1} xe^x\,dx}{\int_{-1}^{1} x^2\,dx} \\ &= \frac{e - e^{-1}}{2} + \frac{2e^{-1}}{\frac{2}{3}}x \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}x \approx 1.18 + 1.10x \end{aligned}$$

Example

Find the fourth-order Fourier approximation to $f(x) = x$ on $[-\pi, \pi]$

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

and for $k \geq 1$, integration by parts yields

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx = \frac{1}{\pi}\left[\frac{x}{k}\sin kx + \frac{1}{k^2}\cos kx\right]_{-\pi}^{\pi} = 0$$

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx \, dx = \frac{1}{\pi}\left[-\frac{x}{k}\cos kx + \frac{1}{k^2}\sin kx\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\frac{-\pi\cos k\pi - \pi\cos(-k\pi)}{k}\right] \\
&= \begin{cases} -\frac{2}{k} & \text{if } k \text{ is even} \\ \frac{2}{k} & \text{if } k \text{ is odd} \end{cases} \\
&= \frac{2(-1)^{k+1}}{k}
\end{aligned}
$$

It follows that the fourth-order Fourier approximation to $f(x) = x$ on $[-\pi, \pi]$ is

$$
2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x\right)
$$