UNIVERSITA’ DEL SALENTO

PROJECT CHARTER

Estimation and Data Analysis with applications

Kalman Filtering with State Equality Constraints

(land-based vehicle that is equipped to measure its range relative to two reference points)

Dipartimento di Ingegneria dell’Innovazione

CDLM Computer Engineering

Prof. Daniela De Palma | Student
Prof. Gianfranco Parlangeli | *Andrea Vedruccio, 20051313*

A.Y. 2020 - 2021

Contents

1. Introduction

Kalman filters are commonly used to estimate the states of a dynamic system. However, in the application of Kalman filters there is often known model or signal information that is either ignored or dealt with heuristically. For instance, constraints on state values (which may be based on physical considerations) are often neglected because they do not fit easily into the structure of the Kalman filter. A rigorous analytic method of incorporating state equality constraints in the Kalman filter is developed here.

The constraints may be time varying. At each time step the unconstrained Kalman filter solution is projected onto the state constraint surface. This significantly improves the prediction accuracy of the filter. The use of this algorithm is demonstrated on a simple nonlinear vehicle tracking problem.

2.Kalman filter with State Equality Constraints

There are some applications where other information can be included to improve the estimation. For example, based on physical considerations related to the nature of the problem, the state variables of the problem satisfy equality constraints

Dxk = dk (1)

where D is a known s x n constant matrix, dk is a known s x 1 vector, s is the number of constraints, n is the number of states, and s <= n. It is assumed in this paper that D is full rank, i.e., that D has rank s. This is an easily satisfied assumption. If D is not full rank that means we have redundant state constraints.

3.Navigation Problem

In this section we present a simple example to illustrate the efficacy of the constrained Kalman filter. Consider a land-based vehicle that is equipped to measure its range relative to two reference points, (rn1, re1) and (rn2, re2), where each reference point is specified by its northerly and easterly positions. The vehicle dynamics and measurements can be approximated by the equations

$$x_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ T\sin\theta \\ T\cos\theta \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} (x_1 - r_{n1})^2 + (x_2 - r_{e1})^2 \\ (x_1 - r_{n2})^2 + (x_2 - r_{e2})^2 \end{bmatrix} + e_k$$

where the first two elements of x are the northerly and easterly positions, the last two elements of x are the northerly and easterly velocities, w represents process disturbances due to potholes and the like, e represents measurement errors, and u is the commanded acceleration. T is the sample period of the position estimator, ϴ is the heading angle (measured counter clockwise from due east), and e is the measurement error.

The navigation reference points are at (0,0) and (173210,100000) meters, while the covariances of the process and measurement noise are

$$Q = \mathrm{Diag}(4\ \mathrm{m/s}, 4\ \mathrm{m/s}, 1\ \mathrm{m/s}^2, 1\ \mathrm{m/s}^2)$$

$$R = \mathrm{Diag}(900\ \mathrm{m}^2, 900\ \mathrm{m}^2).$$

We can use a Kalman filter to estimate the position of the vehicle. During certain times the vehicle may be travelling off-road, or on an unknown road, in which case the problem is unconstrained. At other times it may be known that the vehicle is travelling on a given road, in which case the state estimation problem is constrained. For instance, if it is known that the vehicle is travelling on a road with a heading of ϴ then the matrix D and the vector d of (1) can be given by

$$D = \begin{bmatrix} 1 & -\tan\theta & 0 & 0 \\ 0 & 0 & 1 & -\tan\theta \end{bmatrix}$$

$$d = [0 \quad 0]^T$$

The sample period T is 3 s and the heading ϴ is set to a constant 60 deg.

If we know that the vehicle is on a road with a heading of ϴ then we have

tanϴ =x(1)/x(2) =x(3)/x(4)

The commanded acceleration is alternately set to +-1 m/s2, as if the vehicle was alternately accelerating and decelerating in traffic. Note that with the 60 deg heading, the vehicle and the two reference points form a straight line, which makes the state estimation problem more difficult.

The initial conditions are set to

$$x = \hat{x}_0 = [0 \quad 0 \quad 17 \quad 10]^T$$

$$P_0 = \mathrm{Diag}[900 \quad 900 \quad 4 \quad 4]^T.$$

This scenario is depicted in Fig. 1.

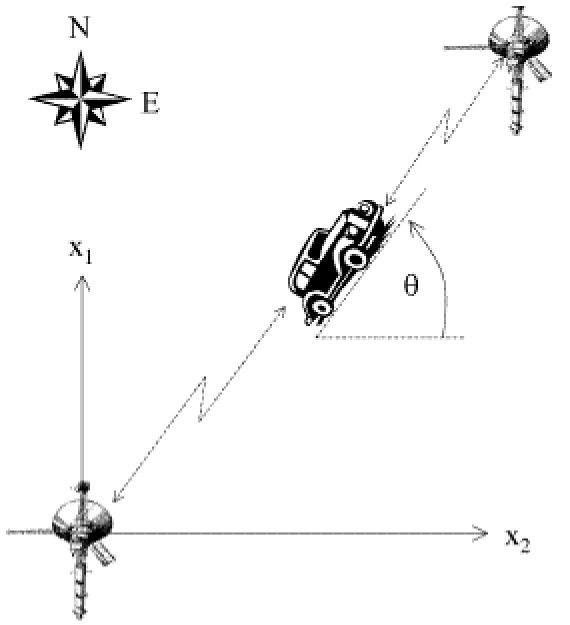

Fig. 1. Land vehicle and 2 transponders.

The unconstrained and constrained Kalman filters were simulated using MATLAB for 300 s.

4. *Constrained Kalman Filter Approach*

In this work we will evaluate four Kalman filter estimates: Unconstrained, *Perfect Measurements approach* , *Maximum probability and Mean Square (General Projection Approach)*

4.1 Perfect Measurements

The simplest solution could be to resort to Perfect Measurements (for equality constraints).

State equality constraints can be treated as perfect measurements with zero measurement noise . If the constraints are given by (1) we can augment

$$y_k = Hx_k + v_k,$$

with s perfect measurements of the state.

$$\begin{bmatrix} y_k \\ d \end{bmatrix} = \begin{bmatrix} H \\ D \end{bmatrix} x_k + \begin{bmatrix} v_k \\ 0 \end{bmatrix}.$$

The state equation is not changed, but the measurement equation is augmented. The fact that the last s components of the measurement equation are noise free means that the a posteriori Kalman filter estimate of the state is consistent with these s measurements.

4.2 General Projection Approach

In this section we derive the constrained Kalman filter by directly projecting the unconstrained state estimate ˆx onto the constraint surface. That is, we solve the problem

$$\min_{\tilde{x}} (\tilde{x} - \hat{x})^T W (\tilde{x} - \hat{x}) \quad \text{such that} \quad D\tilde{x} = d$$

where W is any symmetric positive definite weighting matrix. The solution of this problem is given by

$$\tilde{x} = \hat{x} - W^{-1} D^T (D W^{-1} D^T)^{-1} (D\hat{x} - d).$$

The constrained estimates derived by the *Maximum Probability Method* and *The Mean Square Method* can be obtained from this equation by setting

W = P^(-1) and W = I, respectively.

5. *Results*

The figures in this section show simulation results.

Figure 1 shows the true vehicle position and the unconstrained estiomation

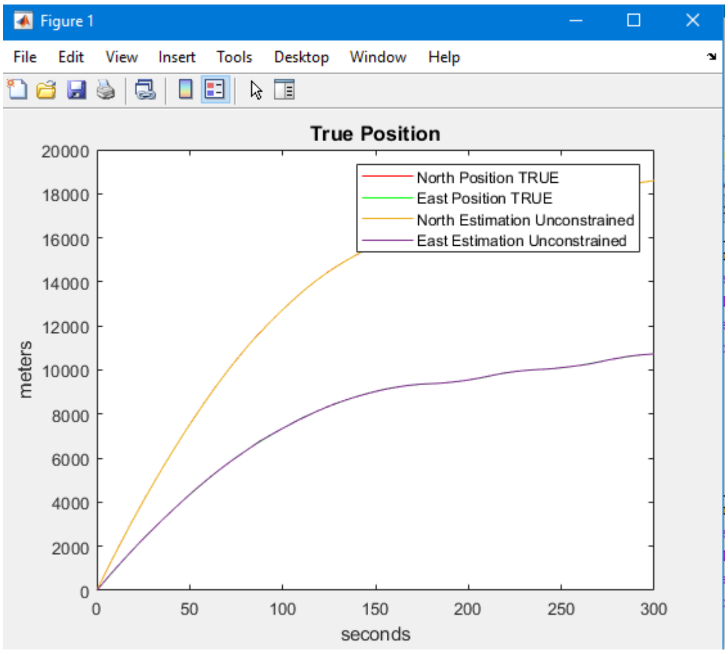

FIGURE 1 TRUE POSITION AND ESTIMATION UNCOSTRAINED

Figure 2 shows the position estimation error of the unconstrained Kalman filter, and Figure 3 shows the position estimation error of the constrained Kalman filter (Perfect Measurements approach).

Figure 4 and Figure 5 shows the position estimation error of the constrained Kalman filter (General Projection Approach) , Mean Square and Maximum probability, respectively.

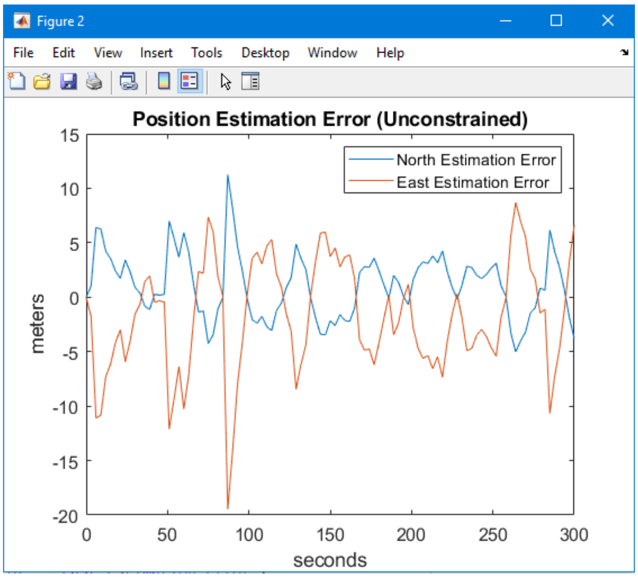

FIGURE 2-UNCOSTRAINED POSITION ESTIMATION ERROR. NORTH BLUE , EAST RED

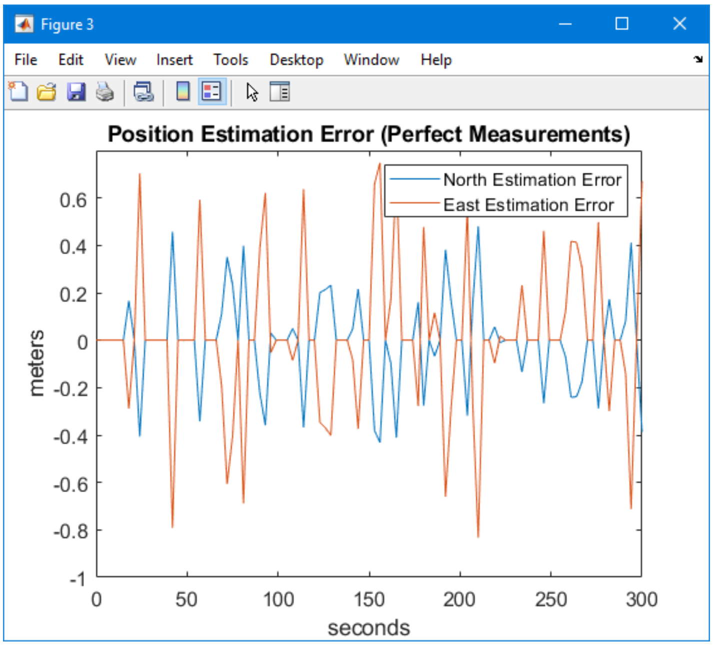

FIGURE 3-PERFECT MEASUREMENTS POSITION ESTIMATION ERROR. NORTH BLUE , EAST RED

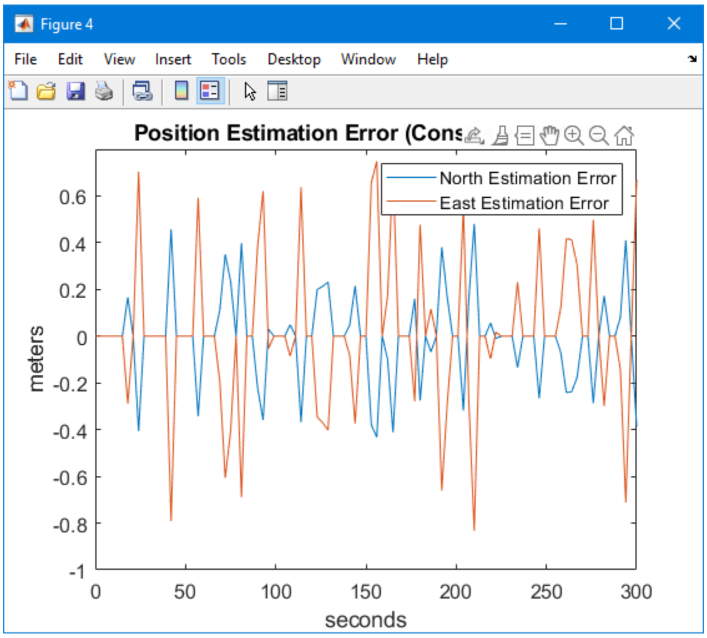

FIGURE 4-MEAN SQUARE POSITION ESTIMATION ERROR. NORTH BLUE , EAST RED

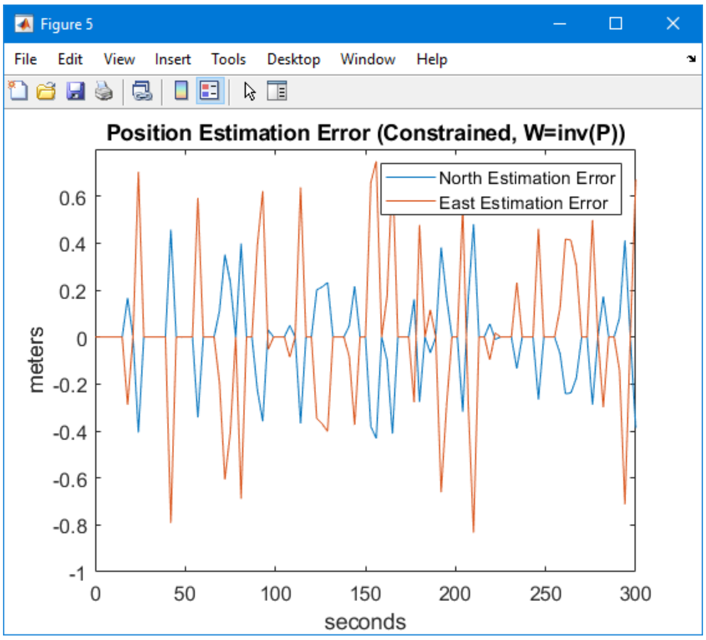

FIGURE 5-MAXIMUM PROBABILITY POSITION ESTIMATION ERROR. NORTH BLUE , EAST RED

```
>> ConstrainedKalmanFiltering
Average Constraint Error (Unconstrained) = 13.5764
Average Constraint Error (Perfect Meas) = 9.6512e-09
Average Constraint Error (W=I) = 0.144
Average Constraint Error (W=inv(P)) = 0.144
>>
```

FIGURE 6- AVERAGE CONSTRAINT ERROR RESULT

6. Conclusion

It can be that the constrained filter results in much more accurate estimates than the unconstrained filter. The unconstrained filter results in average position errors of about 12.1886 m, while the constrained filter (Perfect Measur) results in position errors of about 8.4872e.09 m. General Projection Approach average position errors Mean Square approach is equals to Maximum Probability approach (0.12609), this means they are equivalent.

In this navigation problem, the Kalman filter performs identically whether we incorporate state constraints using perfect measurements, or whether we incorporate state constraints using the approach presented here with either $W = I$ or $W = P^{-1}$.

A MATLAB m-file that implements the algorithms in this work and that was used to produce these simulation results can be downloaded from the email sent on date 13/09/2021