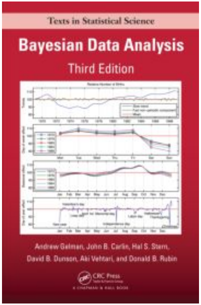
Bayesian data analysis (Aalto fall 2022)

- Book: Gelman, Carlin, Stern, Dunson, Vehtari & Rubin: Bayesian Data Analysis, Third Edition. (online pdf available)
- The course website has more detailed information than these slides
  https://avehtari.github.io/BDA_course_Aalto/Aalto2022.html
- Timetable: see the course website
- TAs: Anna Riha, Elena Shaw, Kunal Ghosh, Andrew Johnson, Noa Kallioinen, David Kohns, Leevi Lindgren, Yann McLatchie, Teemu Sailynoja, Niko Siccha



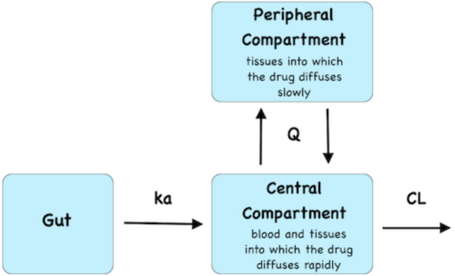
Uncertainty and decision making

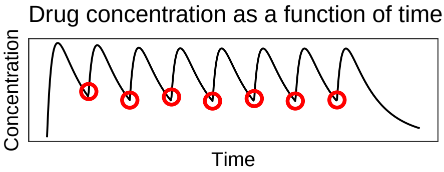
- Predicting concrete quality

$\rightarrow$

Uncertainty and decision making[1]

- Everolimus is immunosuppressant to prevent rejection of organ transplants
- Pharmacokinetic model of drug and body, optimal dosage depends on weight





[1] with E. Siivola, Aalto and S. Weber, Novartis Pharma

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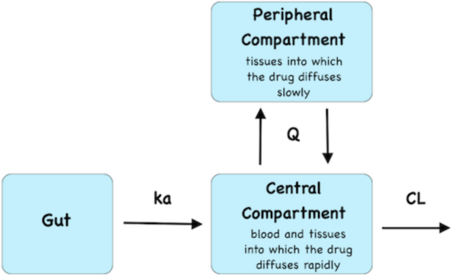
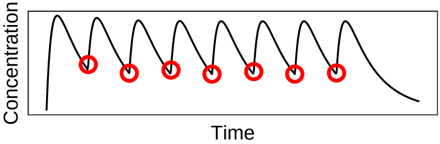


Drug concentration as a function of time



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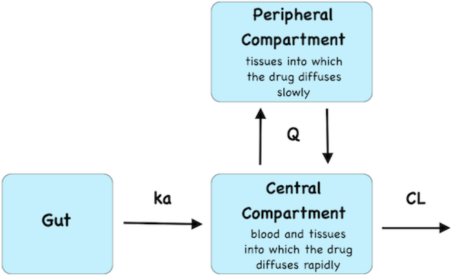

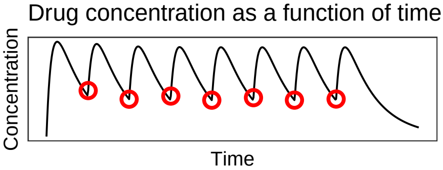

- Model fitted with 500 adults, extrapolation to children?
- Maturation effect, 17 observations from children

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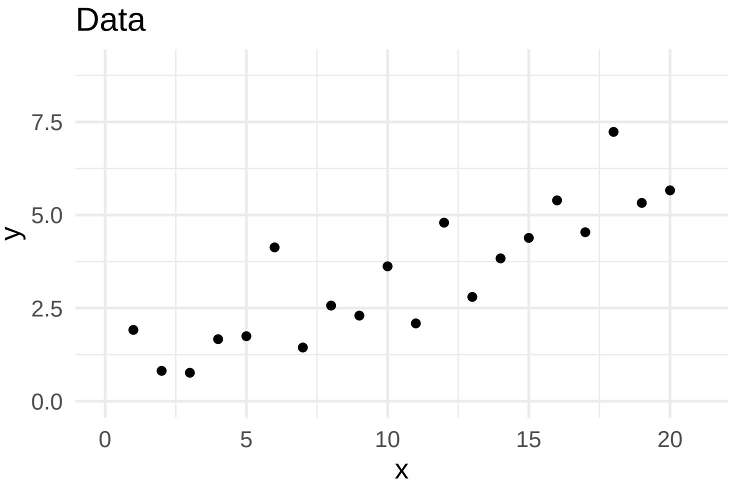
Uncertainty in modeling



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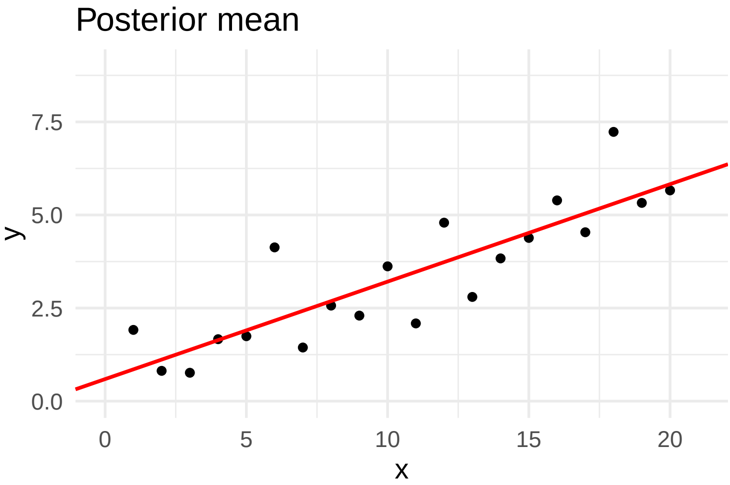

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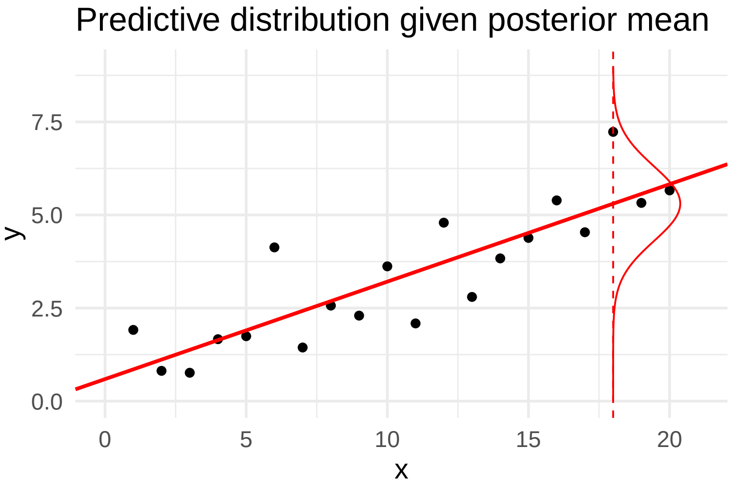

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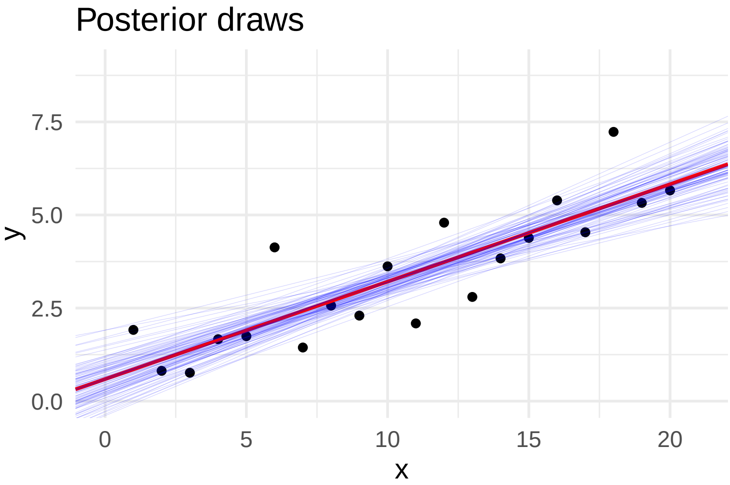

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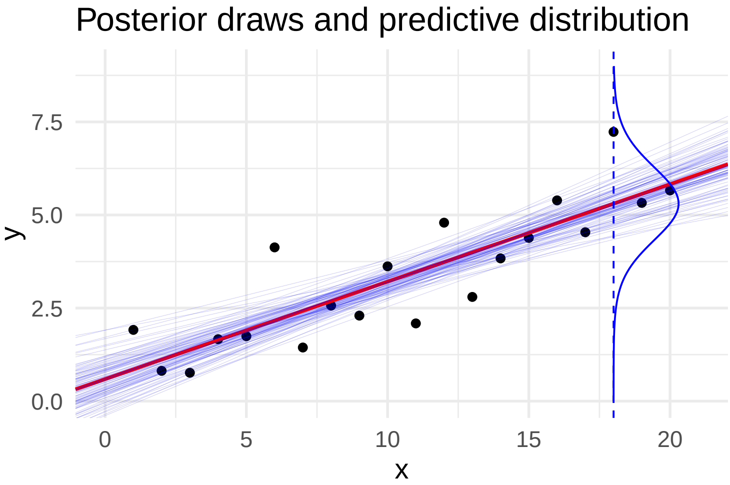

Bayesian probability theory

expert information

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expert information

$\searrow$

mathematical model

+

uncertainty with probabilities

Bayesian probability theory

expert information $\searrow$ mathematical model + uncertainty with probabilities

data

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data $\swarrow$ Bayesian probability theory

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understandable trusted models

Bayesian inference with computers

mathematical model to computer

probabilistic programming

computation, automatic inference algorithms

limitations of computers

Yes, but did it work?

computation + inference diagnostics

model diagnostics

limitations of mathematical models

improve and iterate

Probabilistic programming and Stan

Stan is a probabilistic programming framework and ecosystem
40+ developers, 100+ contributors, 100K+ users

mc-stan.org

Bayesian Data Analysis course

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  - continuous vs discrete (prereq.)
  - observation model, likelihood, prior
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  - We need to be able to compute expectations

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  - grid, importance sampling, Monte Carlo, Markov chain Monte Carlo

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- Workflow
  - steps of model building, inference, and diagnostics

Impact on society

Better modelling and quantification of uncertainty

$\rightarrow$ better science

$\rightarrow$ better informed decision making
in companies, government, and NGOs

Bayesian probability theory

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- A nice book about history: Sharon Bertsch McGrayne, *The Theory That Would Not Die*, 2012.

Term Bayesian used first time in mid 20th century

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- R. A. Fisher used in 1950 first time term "Bayesian" to emphasize the difference to general term "probability theory"
  - term became quickly popular, because alternative descriptions were longer

Uncertainty and probabilistic modeling

- Two types of uncertainty: aleatoric and epistemic
- Representing uncertainty with probabilities
- Updating uncertainty

Two types of uncertainty

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  - two observers may have different epistemic uncertainty

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- $p(\theta|\text{y} = \#\text{red}, \#\text{yellow}, \#\text{red}, \#\text{red}, \ldots) = ?$

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- Bayes rule $p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$

Model vs. likelihood

- Bayes rule $p(\theta|y) \propto p(y|\theta)p(\theta)$

- Model: $p(\mathbf{y}|\theta)$ as a function of $\mathbf{y}$ given fixed θ describes the aleatoric uncertainty

- Likelihood: $p(y|\boldsymbol{\theta})$ as a function of $\boldsymbol{\theta}$ given fixed y provides information about epistemic uncertainty, but is not a probability distribution

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- Bayes rule combines the likelihood with prior uncertainty $p(\theta)$ and transforms them to updated posterior uncertainty

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- "Easy" part is to use Bayes rule to update the uncertainties
  - computational challenges
- Other parts of the art of probabilistic modeling are, for example,
  - model checking: is data in conflict with our prior knowledge?
  - presentation: presenting the model and the results to the application experts

Modeling nature

- Drop a ball from different heights and measure time

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- Taking into account the accuracy of the measurements, how accurate model is needed?
  - often simple models are adequate and useful
  - *All models are wrong, but some of them are useful*, George P. Box

Reminder: Uncertainty and probabilistic modeling

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Questions

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- Is the quantum uncertainty aleatoric or epistemic?
- What is your own example with both aleatoric and epistemic uncertainty?

Pre-requisites

- probability
- probability density
- probability mass
- probability density function (pdf)
- probability mass function (pmf)
- probability distribution
- discrete probability distribution
- continuous probability distribution
- cumulative distribution function (cdf)
- likelihood

Ambiguous notation in statistics

$\ln p(y|\theta)$

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- Due to the sloppines sometimes likelihood is used to refer $P_{Y,\theta}(Y|\Theta)$, $p_{Y,\theta}(Y|\Theta)$

Chapter 1

Reading instructions

- 1.1-1.3 important terms
- 1.4 a useful example
- 1.5 foundations
- 1.6 & 1.7 examples (can be skipped, but may be useful to read)
- 1.8 & 1.9 background material, good to read before doing the exercises
- 1.10 a point of view for using Bayesian inference