

Chapter 5

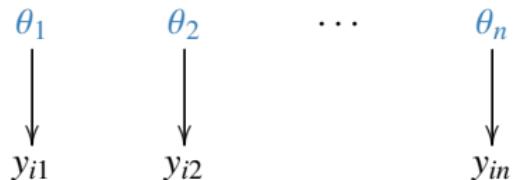
- 5.1 Lead-in to hierarchical models
- 5.2 Exchangeability (useful concept)
- 5.3 Bayesian analysis of hierarchical models (we use Stan/brms for computation)
- 5.4 Hierarchical normal model (we use Stan/brms for computation)
- 5.5 Example: parallel experiments in eight schools (useful discussion on benefits of hierarchical model)
- 5.6 Meta-analysis
- 5.7 Weakly informative priors for hierarchical variance parameters

Hierarchical model

- In simple model: posterior for the parameters
- In hierarchical model: posterior for the prior parameters

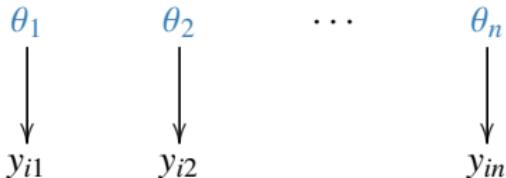
Hierarchical model

- Example: CVD treatment effectiveness
 - in hospital j the survival probability is θ_j
 - observations y_{ij} tell whether patient i survived in hospital j

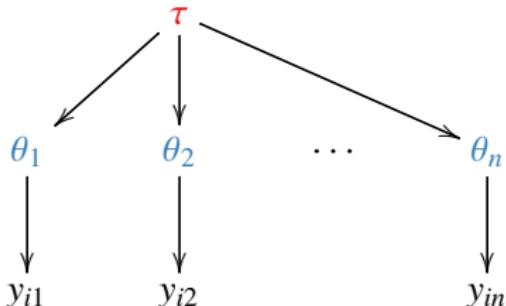


Hierarchical model

- Example: CVD treatment effectiveness
 - in hospital j the survival probability is θ_j
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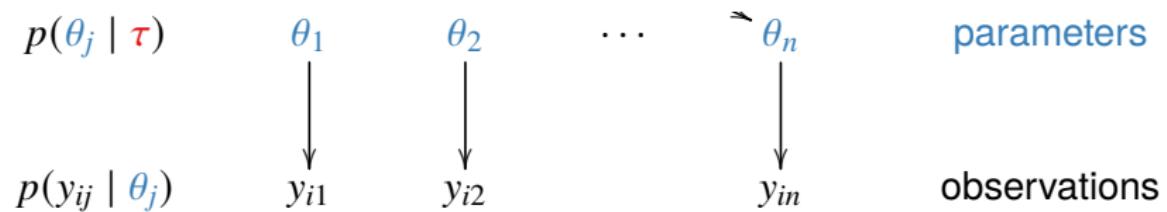
- sensible to assume that θ_j are similar



- natural to think that θ_j have common population distribution
- θ_j is not directly observed and the population distribution is unknown

Hierarchical model: terms

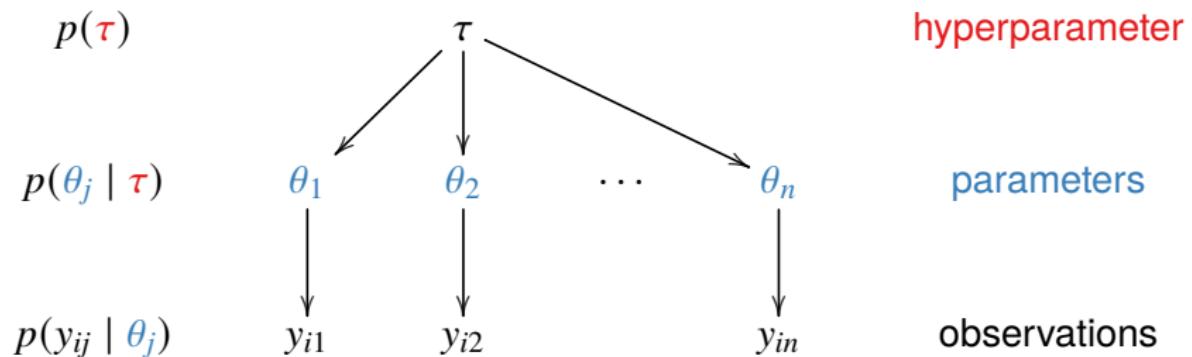
Level 1: observations given parameters $p(y_{ij} \mid \theta_j)$



Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} | \theta_j)$

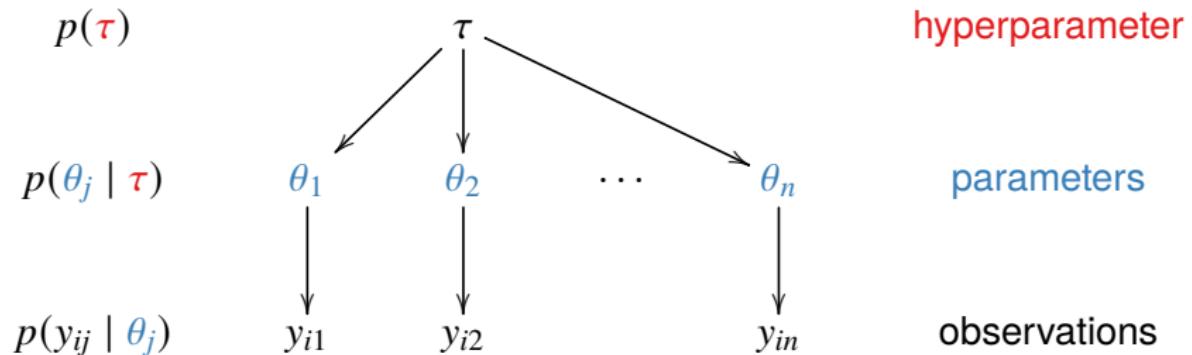
Level 2: parameters given hyperparameters $p(\theta_j | \tau)$



Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} | \theta_j)$

Level 2: parameters given hyperparameters $p(\theta_j | \tau)$

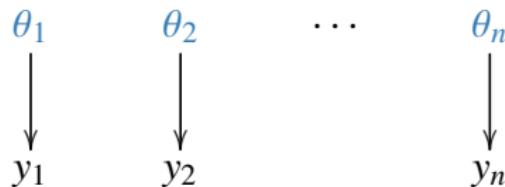


Joint posterior

$$\begin{aligned} p(\theta, \tau | y) &\propto p(y | \theta, \tau)p(\theta, \tau) \\ &\propto p(y | \theta)p(\theta | \tau)p(\tau) \end{aligned}$$

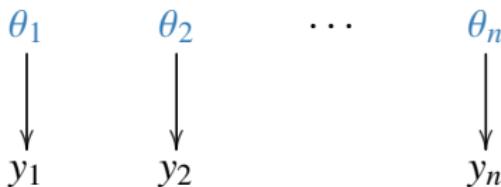
Compare

- "Separate model" (model with separate/independent effects)

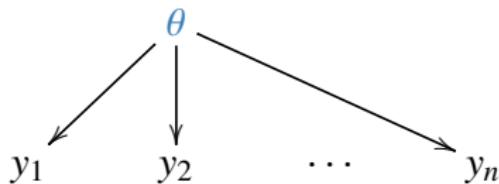


Compare

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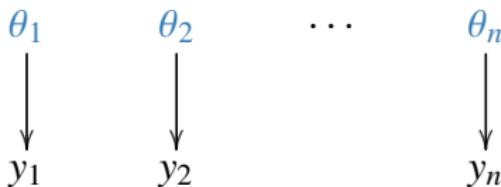


- "Joint model" (model with a common effect / pooled model)

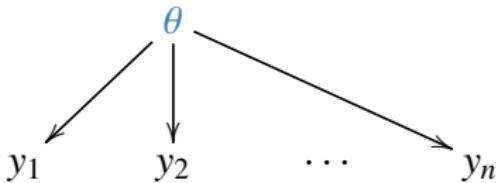


Compare

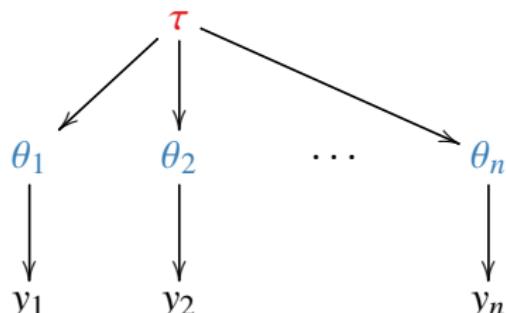
- "Separate model" (model with separate/independent effects)



- "Joint model" (model with a common effect / pooled model)



- Hierarchical model



Hierarchical binomial model: rats

- Medicine testing
- Type F344 female rats in control group given placebo
 - count how many get endometrial stromal polyps
 - familiar binomial model example

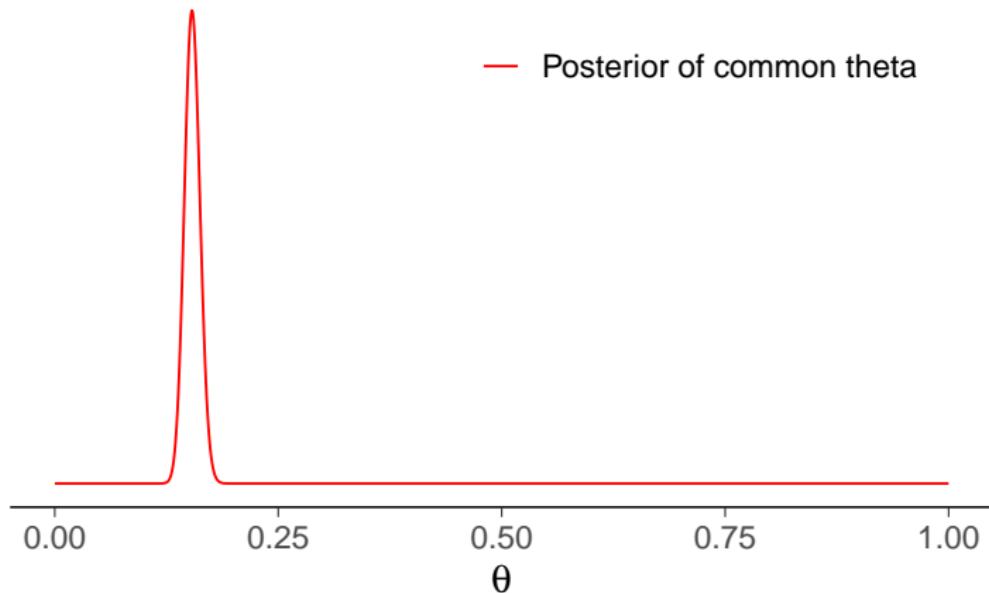
Hierarchical binomial model: rats

- Medicine testing
- Type F344 female rats in control group given placebo
 - count how many get endometrial stromal polyps
 - familiar binomial model example
- Experiment has been repeated 71 times

0/20	0/20	0/20	0/20	0/20	0/20	0/20	0/19	0/19	0/19
0/19	0/18	0/18	0/17	1/20	1/20	1/20	1/20	1/19	1/19
1/18	1/18	2/25	2/24	2/23	2/20	2/20	2/20	2/20	2/20
2/20	1/10	5/49	2/19	5/46	3/27	2/17	7/49	7/47	3/20
3/20	2/13	9/48	10/50	4/20	4/20	4/20	4/20	4/20	4/20
4/20	10/48	4/19	4/19	4/19	5/22	11/46	12/49	5/20	5/20
6/23	5/19	6/22	6/20	6/20	6/20	16/52	15/46	15/47	9/24
4/14									

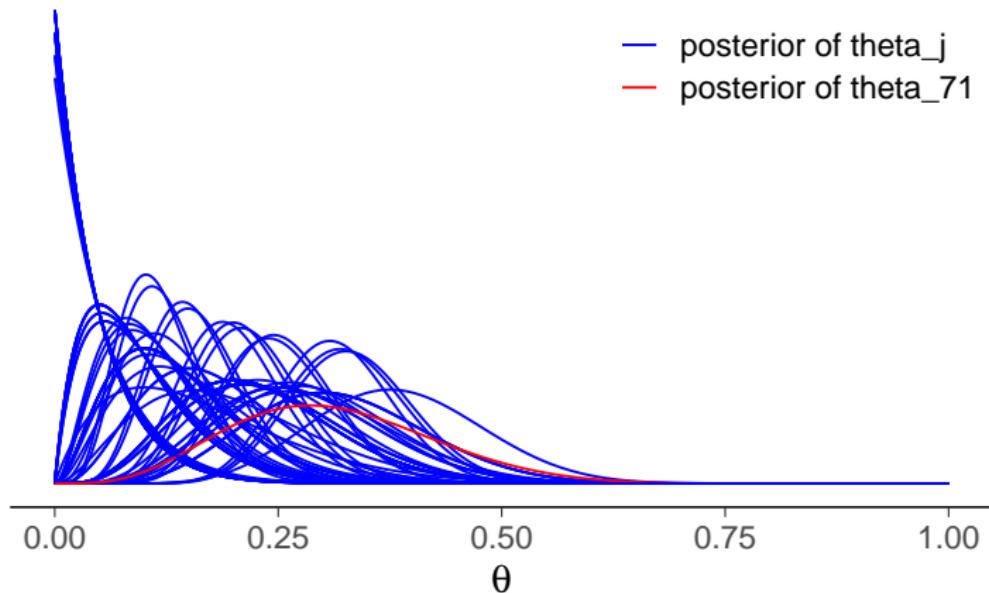
Hierarchical binomial model: rats

Pooled model



Hierarchical binomial model: rats

Separate model

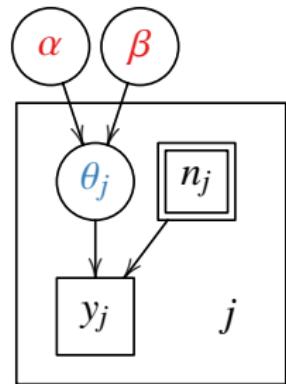


Hierarchical binomial model: rats

- Hierarchical binomial model for rats
prior parameters α and β are unknown

$$\theta_j \mid \alpha, \beta \sim \text{Beta}(\theta_j \mid \alpha, \beta)$$

$$y_j \mid n_j, \theta_j \sim \text{Bin}(y_j \mid n_j, \theta_j)$$



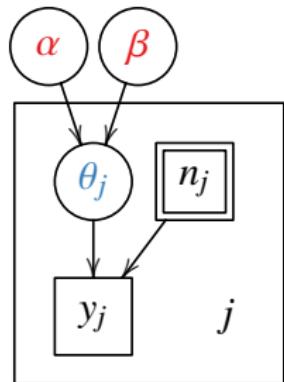
- Joint posterior $p(\theta_1, \dots, \theta_J, \alpha, \beta \mid y)$
 - multiple parameters

Hierarchical binomial model: rats

- Hierarchical binomial model for rats
prior parameters α and β are unknown

$$\theta_j \mid \alpha, \beta \sim \text{Beta}(\theta_j \mid \alpha, \beta)$$

$$y_j \mid n_j, \theta_j \sim \text{Bin}(y_j \mid n_j, \theta_j)$$



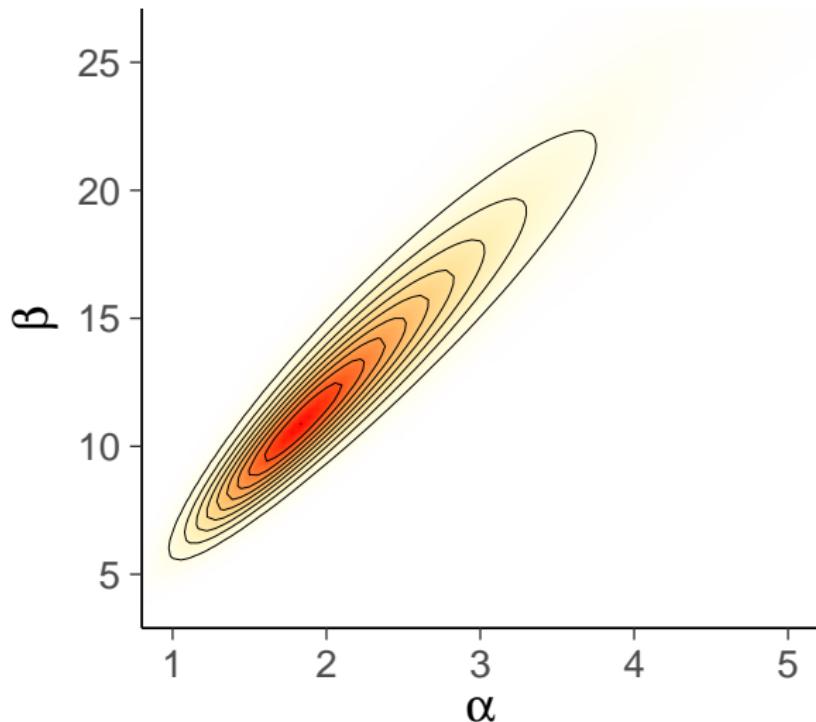
- Joint posterior $p(\theta_1, \dots, \theta_J, \alpha, \beta \mid y)$
 - multiple parameters
 - factorize $\prod_{j=1}^J p(\theta_j \mid \alpha, \beta, y) p(\alpha, \beta \mid y)$

Hierarchical binomial model: rats

- Population prior $\text{Beta}(\theta_j \mid \alpha, \beta)$
- Hyperprior $p(\alpha, \beta)$?
 - α, β both affect the location and scale
 - BDA3 has $p(\alpha, \beta) \propto (\alpha + \beta)^{-5/2}$
 - diffuse prior for location and scale (BDA3 p. 110)
- demo5_1

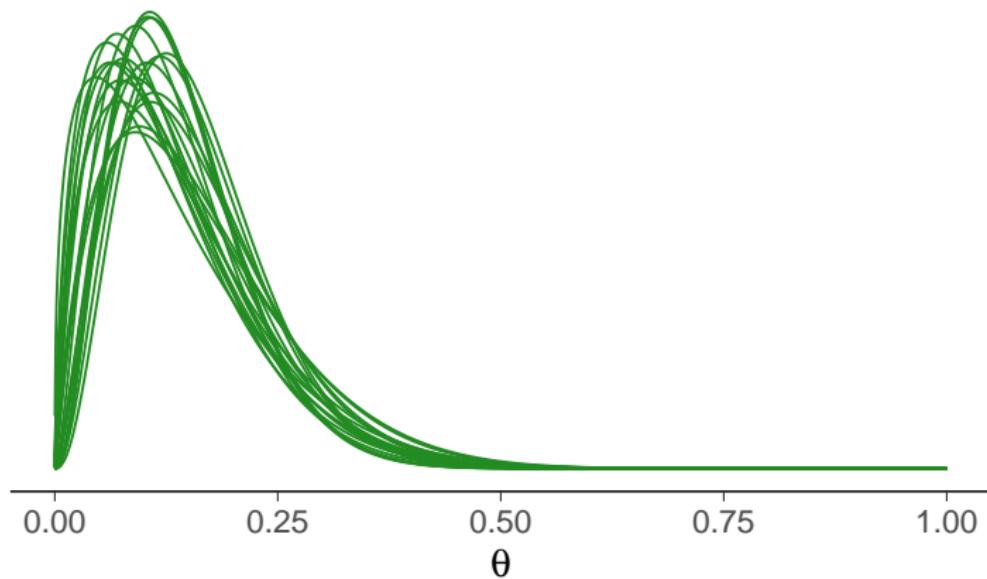
Hierarchical binomial model: rats

The marginal of α and β



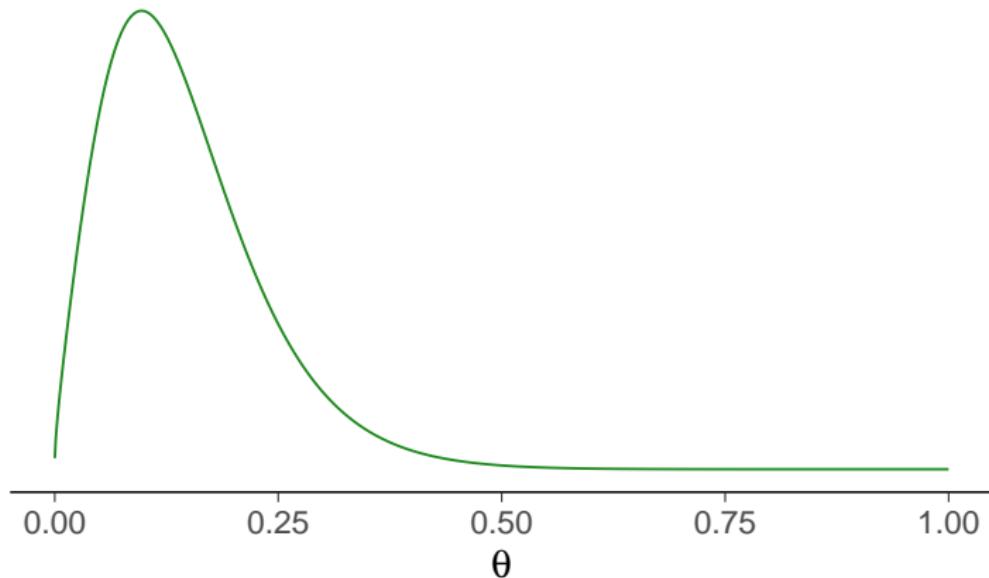
Hierarchical binomial model: rats

Beta(α, β) given posterior draws of α and β



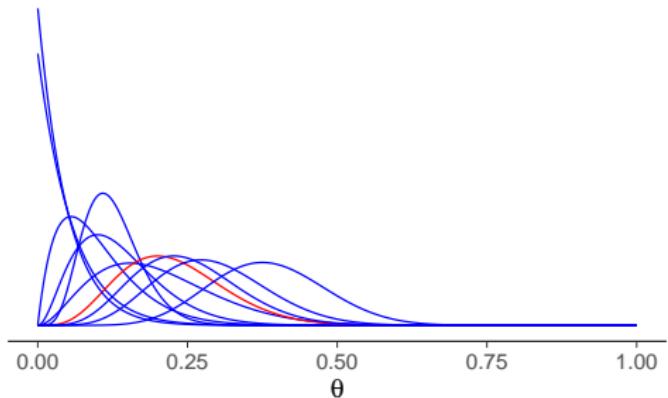
Hierarchical binomial model: rats

Population distribution (prior) for θ_j



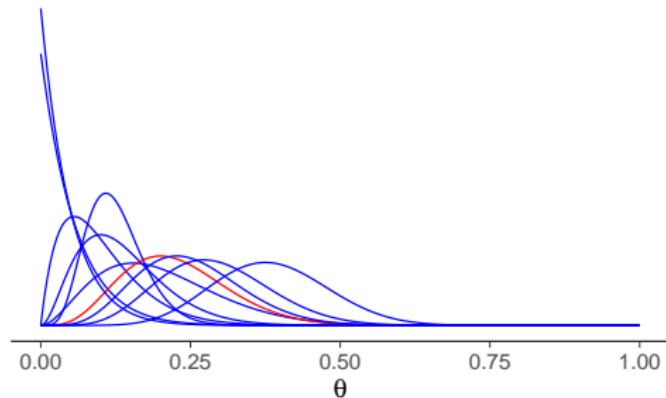
Hierarchical binomial model: rats

Separate model

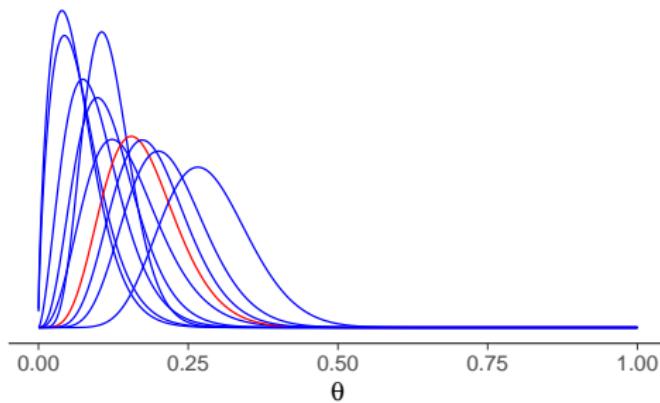


Hierarchical binomial model: rats

Separate model

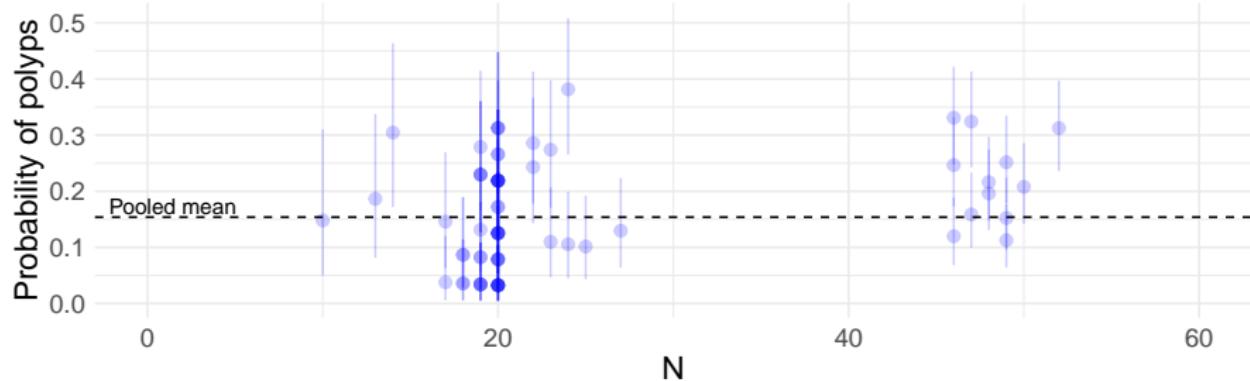


Hierarchical model

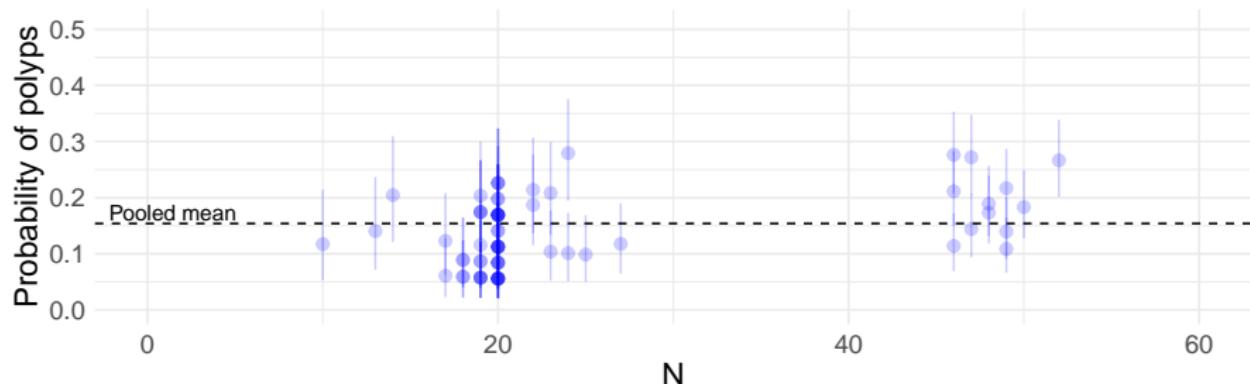


Hierarchical model and group size: Rats

Separate



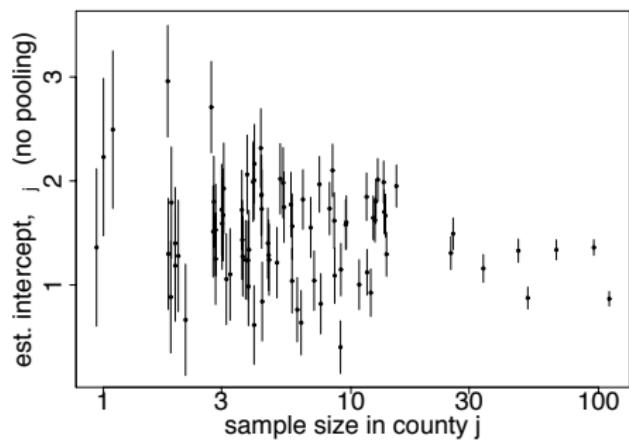
Hierarchical



Hierarchical model and group size: Radon

919 home radon levels in 85 counties in Minnesota:

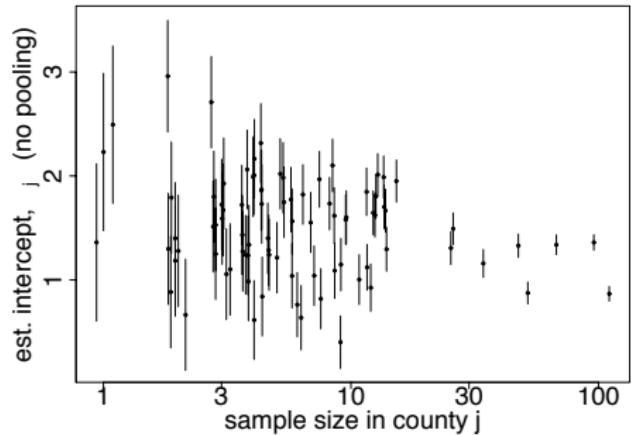
Separate



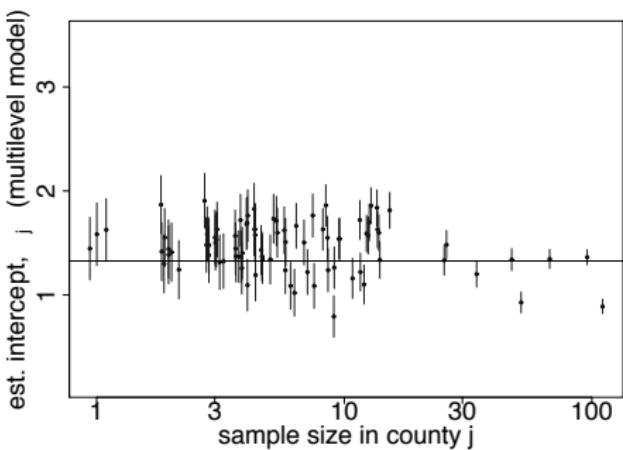
Hierarchical model and group size: Radon

919 home radon levels in 85 counties in Minnesota:

Separate



Hierarchical



Stan code for hierarchical model

Diet effect on chicken weights (at age 12 days)

```
data {
  int<lower=1> N_observations;
  int<lower=1> N_diets;

  vector[N_observations] weight;
  vector[N_observations] diet_idx;
}
```

Diet effect on chicken weights (at age 12 days)

Hierarchical model with

- unknown mean weight per diet
- unknown prior mean and sd
- unknown residual sigma

```
parameters {
  vector[N_diets] mu_diet;
  real mu_0;
  real<lower=0> sd;
  real<lower=0> sigma;
}
```

Stan hierarchical model

diet_idx is a vector with each element indicating the group

```
model {
    // Priors
    for (diet in 1:N_diets) {
        mu_diet[diet] ~ normal(mu_0, sd);
    }
    mu_0 ~ normal(0, 1);
    sd ~ normal(0, 1)
    sigma ~ normal(0, 1);

    // Observation model
    for (obs in 1:N_observations) {
        weight[obs] ~ normal(mu_diet[diet_idx[obs]], sigma);
    }
}
```

Stan hierarchical model

mu_0 and sd are the population mean and sd

```
model {
    // Priors
    for (diet in 1:N_diets) {
        mu_diet[diet] ~ normal(mu_0, sd);
    }
    mu_0 ~ normal(0, 1);
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    for (obs in 1:N_observations) {
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Stan hierarchical model

sd is constrained to be positive and thus the prior is half-normal

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    sd ~ normal(0, 1)
    sigma ~ normal(0, 1);
```



```
// Observation model
for (obs in 1:N_observations) {
    weight[obs] ~ normal(mu_diet[diet_idx[obs]], sigma);
}
```

Stan without loops

Vectorized statements

```
model {
    // Priors
    mu_diet ~ normal(mu_0, sd);
    mu_0 ~ normal(0, 1);
    sd ~ normal(0, 1)
    sigma ~ normal(0, 1);

    // Observation model
    weight ~ normal(mu_diet[diet_idx], sigma);
}
```

Stan vs brms

Stan

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    // Priors
    mu_diet ~ normal(mu_0, sd);
    mu_0 ~ normal(0, 1);
    sd ~ normal(0, 1)
    sigma ~ normal(0, 1);

    // Observation model
    weight ~ normal(mu_diet[diet_idx], sigma);
}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet),
```

Stan vs brms

Stan

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model {
    // Priors
    mu_diet ~ normal(mu_0, sd);
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Stan vs brms

Stan

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  sd ~ normal(0, 1)
  sigma ~ normal(0, 1);

  // Observation model
  weight ~ normal(mu_diet[diet_idx], sigma);
}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet), data=Chick12,
```

Stan vs brms

Stan

```
model {
  // Priors
  mu_diet ~ normal(mu_0, sd);
  mu_0 ~ normal(0, 1);
  sd ~ normal(0, 1)
  sigma ~ normal(0, 1);

  // Observation model
  weight ~ normal(mu_diet[diet_idx], sigma);
}
```

brms formula

```
brm(weight ~ 1 + (1 | Diet), data=Chick12,
     prior=c(prior(normal(0,1), class="Intercept"), # p(mu_0)
             prior(normal(0,1), class="sd"),        # p(tau)
             prior(normal(0,1), class="sigma")))) # p(sigma)
```

Stan vs brms

Stan

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model {
  // Priors
  mu_diet ~ normal(mu_0, sd);
  mu_0 ~ normal(0, 1);
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```

brms generated Stan code

```
// generated with brms 2.22.1
data {
  int<lower=1> N; // total number of observations
  vector[N] Y; // response variable
  // data for group-level effects of ID 1
  int<lower=1> N_1; // number of grouping levels
  int<lower=1> M_1; // number of coefficients per level
  array[N] int<lower=1> J_1; // grouping indicator per observation
  // group-level predictor values
  vector[N] Z_1_1;
  int prior_only; // should the likelihood be ignored?
}
```

brms generated Stan code

```
parameters {
    real Intercept; // temporary intercept for centered predictors
    real<lower=0> sigma; // dispersion parameter
    vector<lower=0>[M_1] sd_1; // group-level standard deviations
    array[M_1] vector[N_1] z_1; // standardized group-level effects
}
transformed parameters {
    vector[N_1] r_1_1; // actual group-level effects
    real lprior = 0; // prior contributions to the log posterior
    r_1_1 = (sd_1[1] * (z_1[1]));
    lprior += normal_lpdf(Intercept | 0, 1);
    lprior += normal_lpdf(sigma | 0, 1)
        - 1 * normal_lccdf(0 | 0, 1);
    lprior += normal_lpdf(sd_1 | 0, 1)
        - 1 * normal_lccdf(0 | 0, 1);
}
```

brms generated Stan code

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parameters {
    real Intercept; // temporary intercept for centered predictors
    real<lower=0> sigma; // dispersion parameter
    vector<lower=0>[M_1] sd_1; // group-level standard deviations
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    real lprior = 0; // prior contributions to the log posterior
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    lprior += normal_lpdf(sigma | 0, 1)
        - 1 * normal_lccdf(0 | 0, 1);
    lprior += normal_lpdf(sd_1 | 0, 1)
        - 1 * normal_lccdf(0 | 0, 1);
}
```

brms generated Stan code

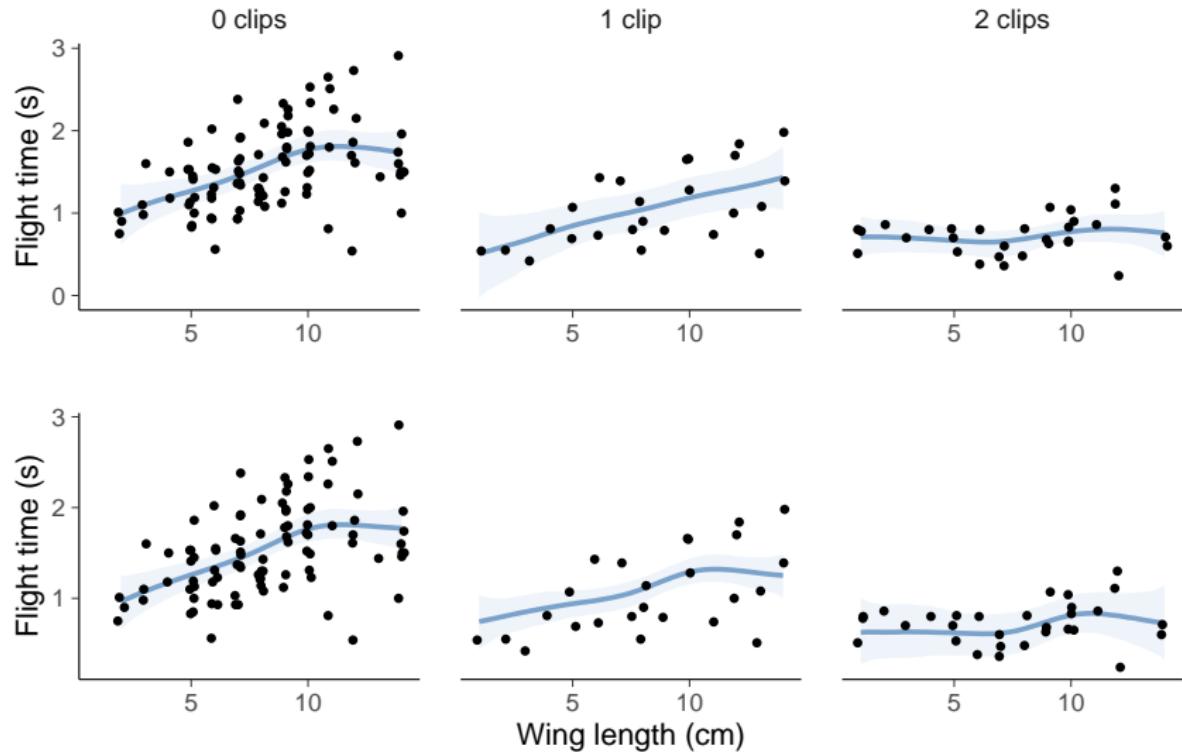
```
model {
  // likelihood including constants
  if (!prior_only) {
    // initialize linear predictor term
    vector[N] mu = rep_vector(0.0, N);
    mu += Intercept;
    for (n in 1:N) {
      // add more terms to the linear predictor
      mu[n] += r_1_1[J_1[n]] * Z_1_1[n];
    }
    target += normal_lpdf(Y | mu, sigma);
  }
  // priors including constants
  target += lprior;
  target += std_normal_lpdf(z_1[1]);
}
generated quantities {
  // actual population-level intercept
  real b_Intercept = Intercept;
}
```

brms generated Stan code

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model {
  // likelihood including constants
  if (!prior_only) {
    // initialize linear predictor term
    vector[N] mu = rep_vector(0.0, N);
    mu += Intercept;
    for (n in 1:N) {
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    }
    target += normal_lpdf(Y | mu, sigma);
  }
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  target += std_normal_lpdf(z_1[1]);
}
generated quantities {
  // actual population-level intercept
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}
```

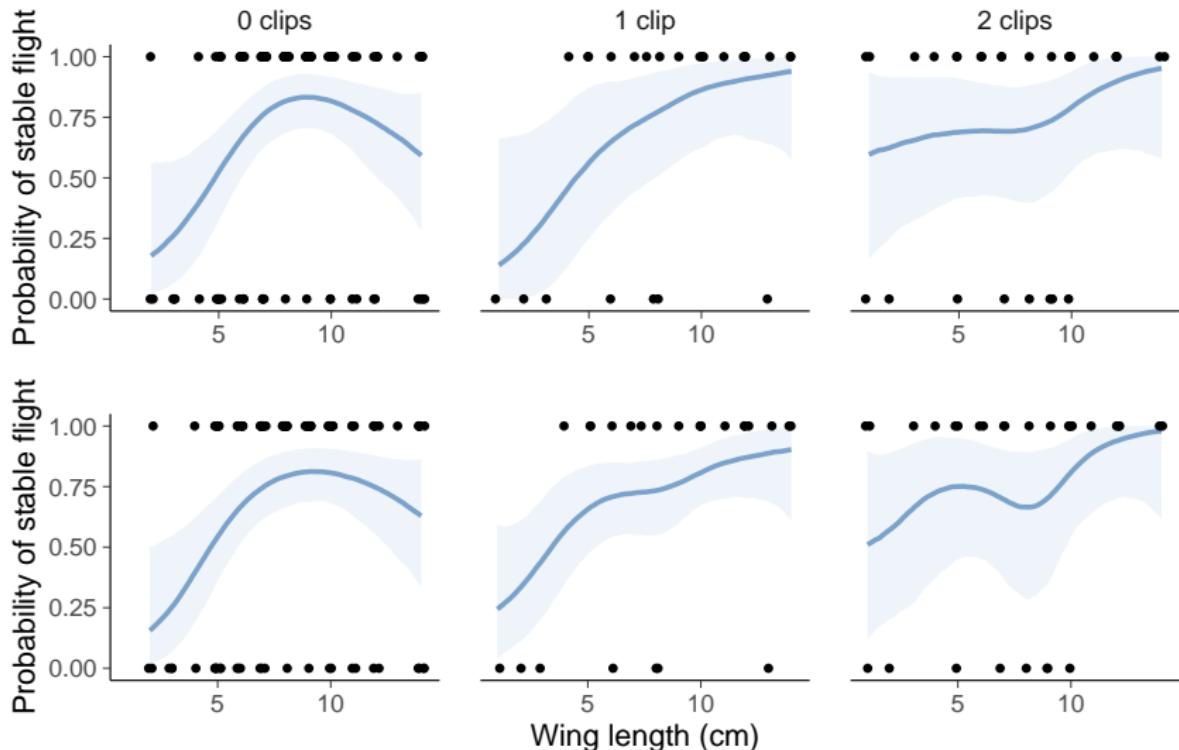
Paper helicopters: flight time

Separate model vs. hierarchical model



Paper helicopters: stability

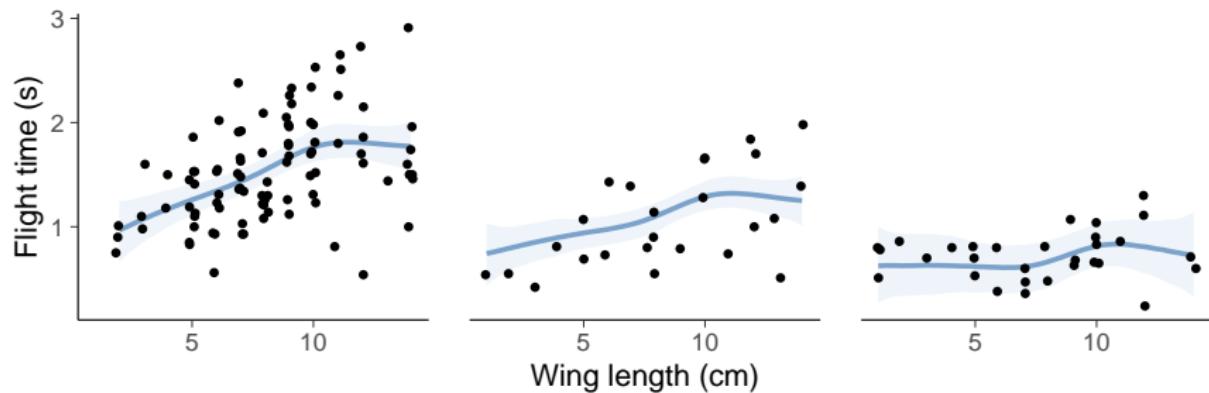
Separate model vs. hierarchical model



Paper helicopters: brms

Flight time

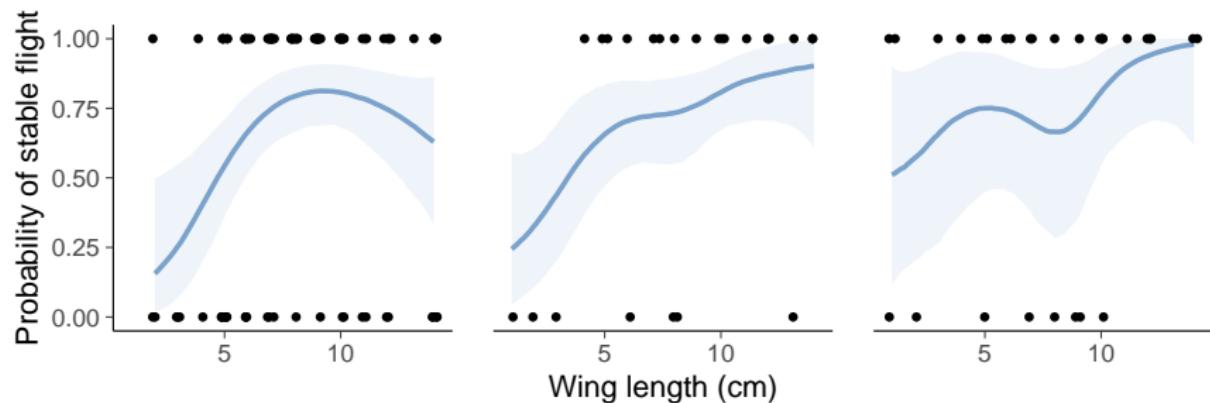
```
flight_time ~ s(wing_length) + s(wing_length, by = nclips)
```



Paper helicopters: brms

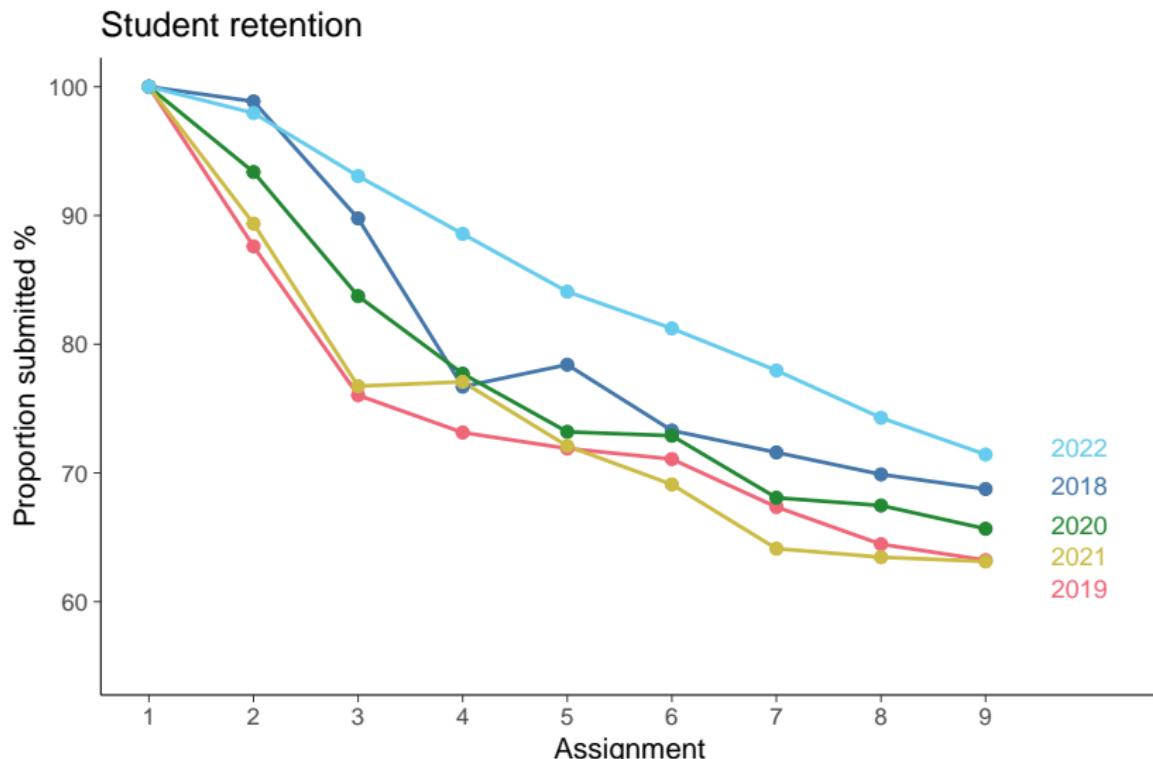
Stability

```
stable_flight ~ s(wing_length) + s(wing_length, by = nclips),  
family = bernoulli()
```

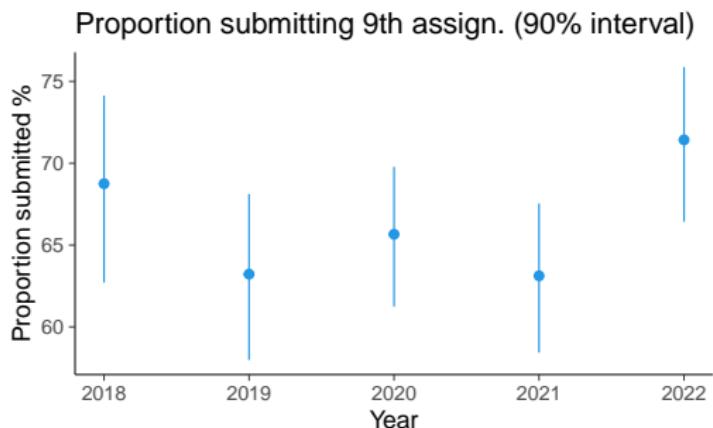


Student retention

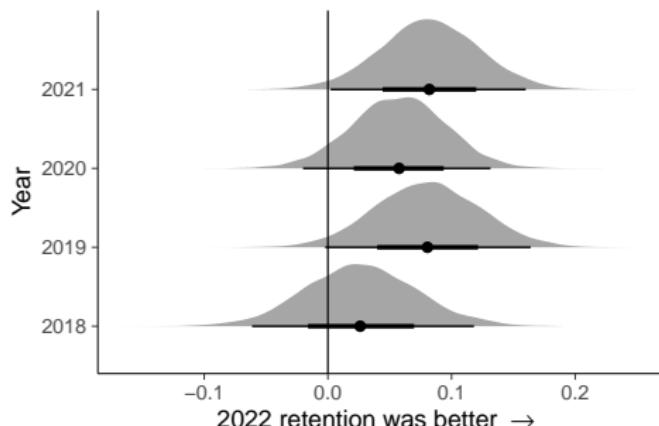
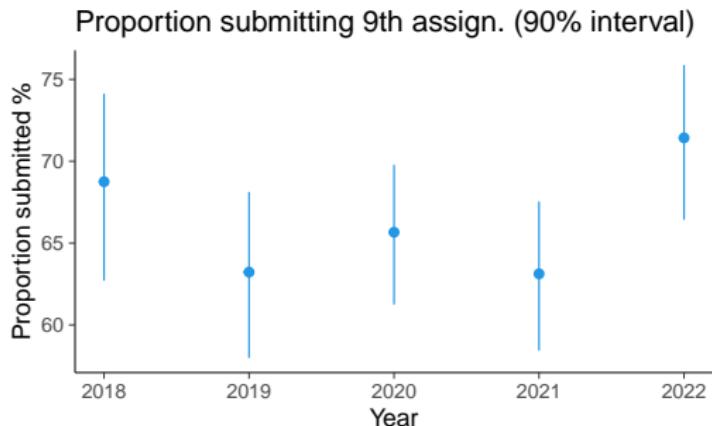
Was year 2022 better than earlier year?



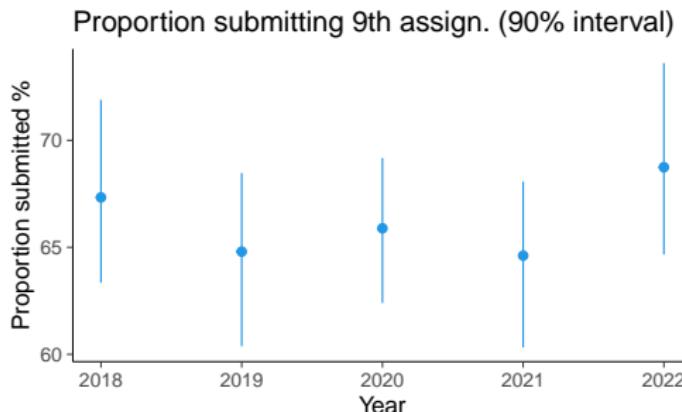
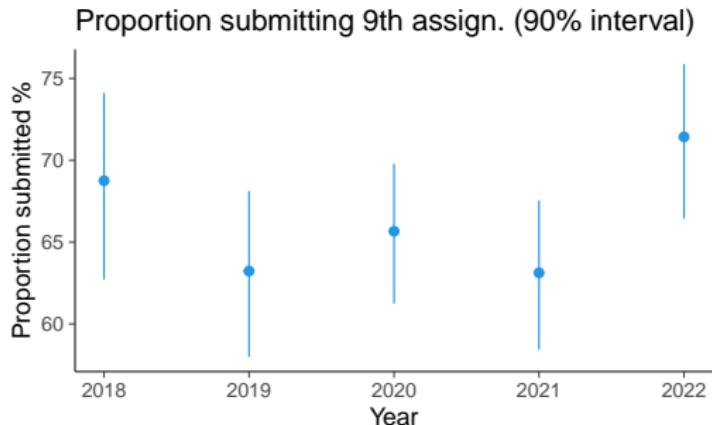
Student retention separate model



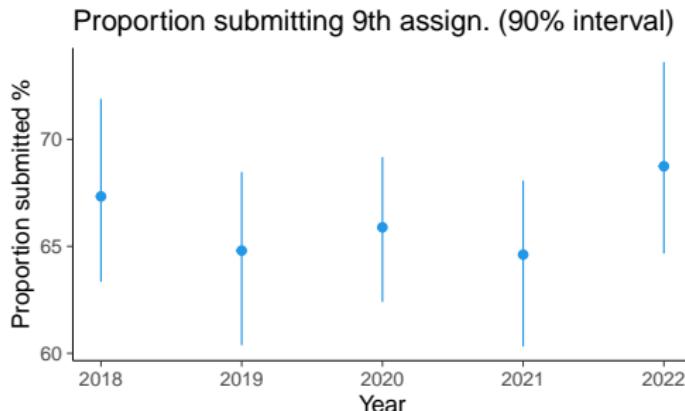
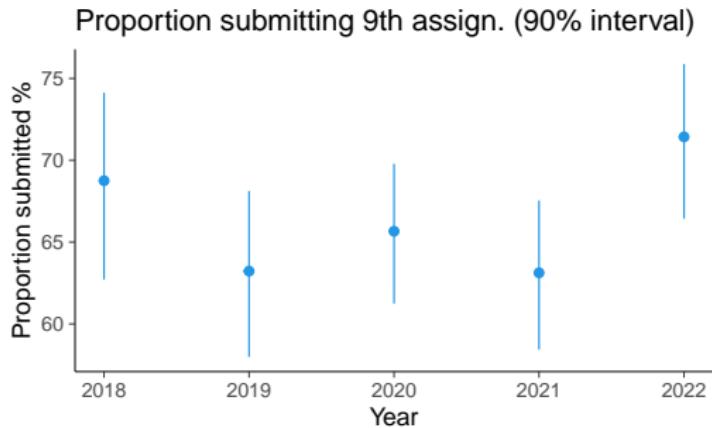
Student retention separate model



Student retention separate vs hierarchical model

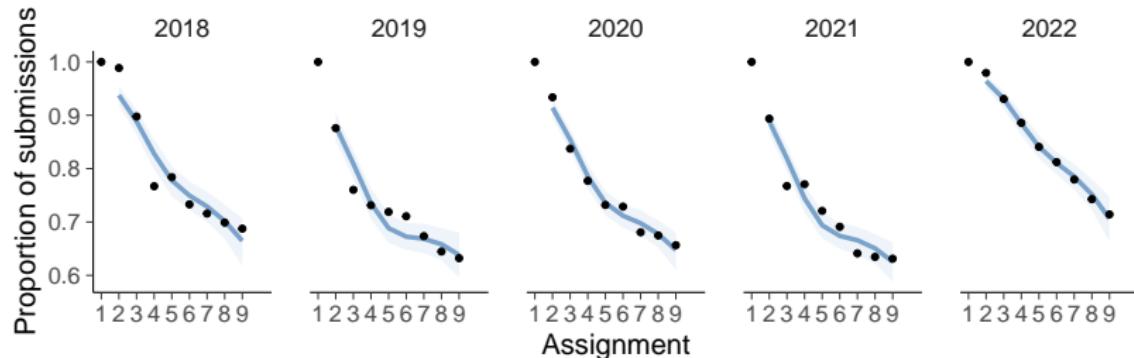


Student retention separate vs hierarchical model



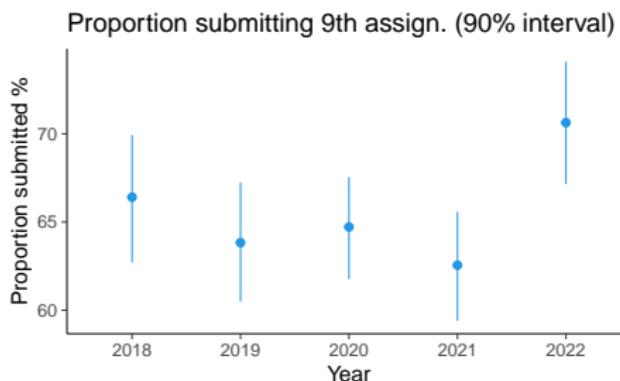
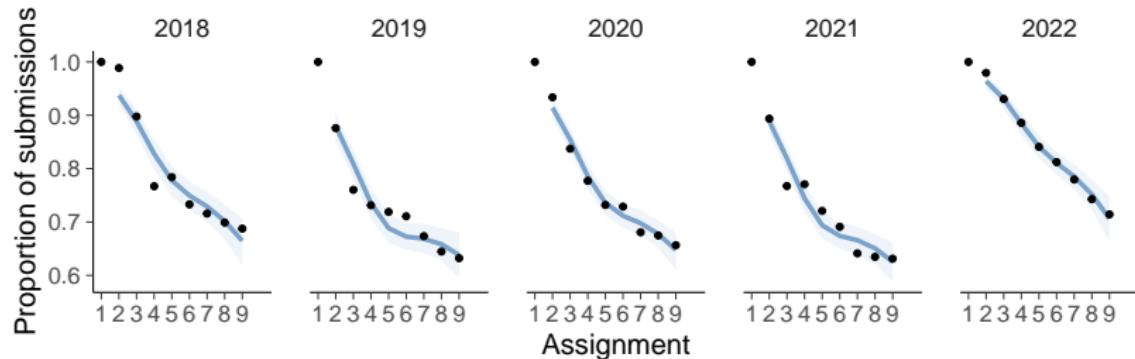
nstudents | trials(nstudents1) ~ 1 + (1 | year), family=binomial()

Student retention latent spline model

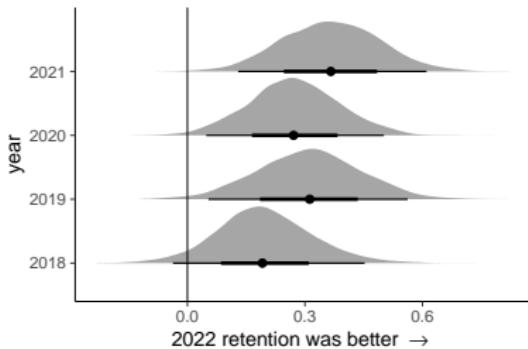
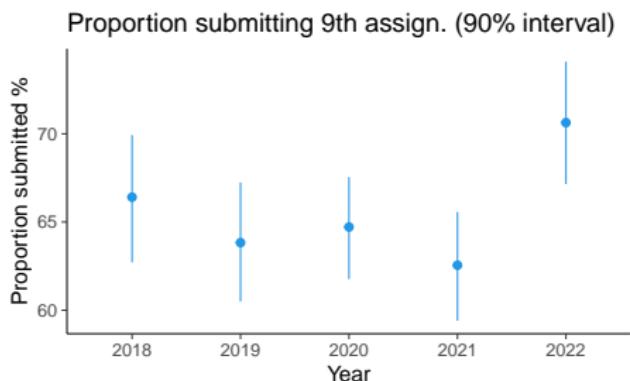
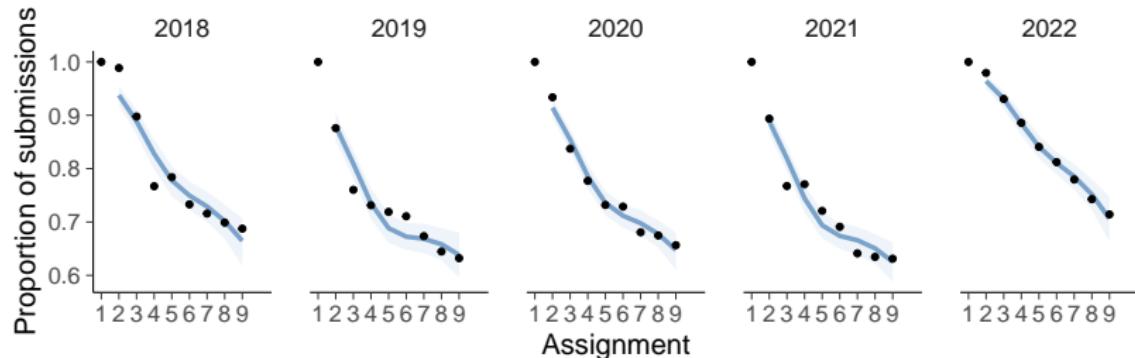


```
nstudents | trials(nstudents1) ~ s(assignment, k=4) + (assignment | year),  
family=binomial()
```

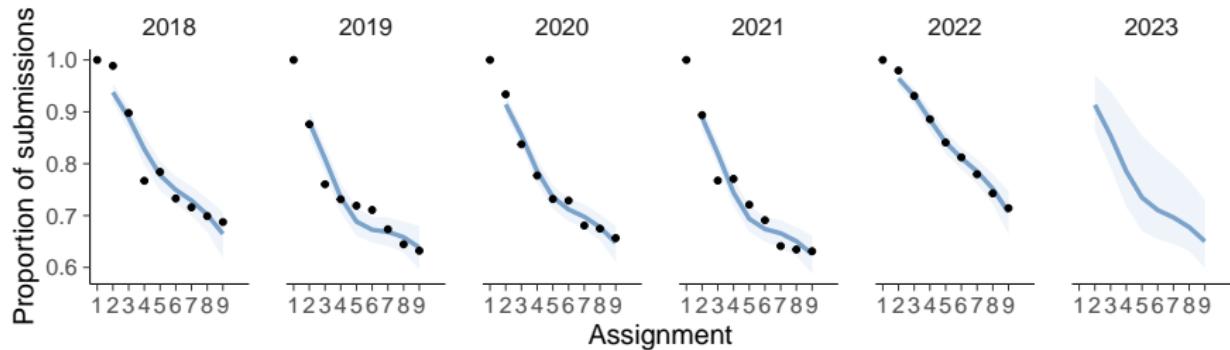
Student retention latent spline model



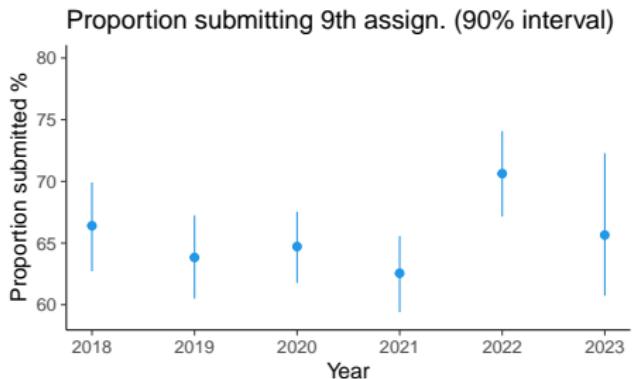
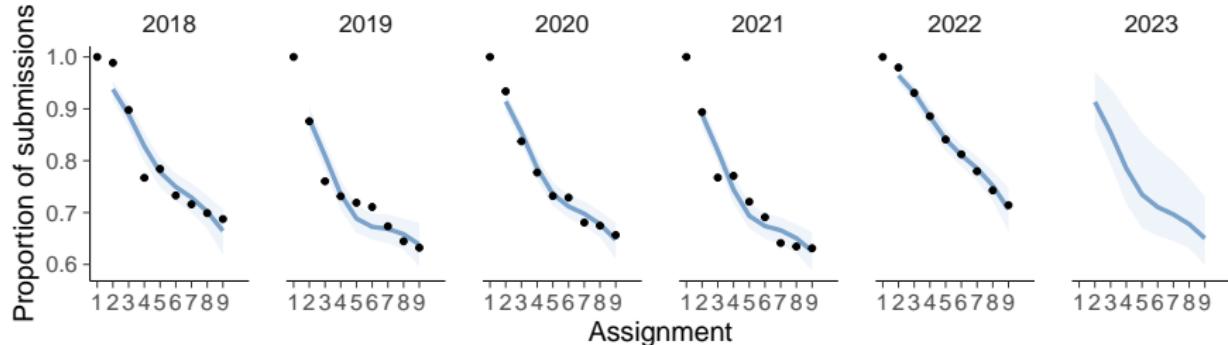
Student retention latent spline model



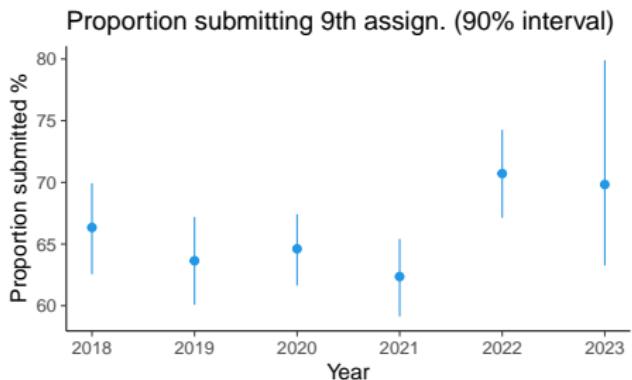
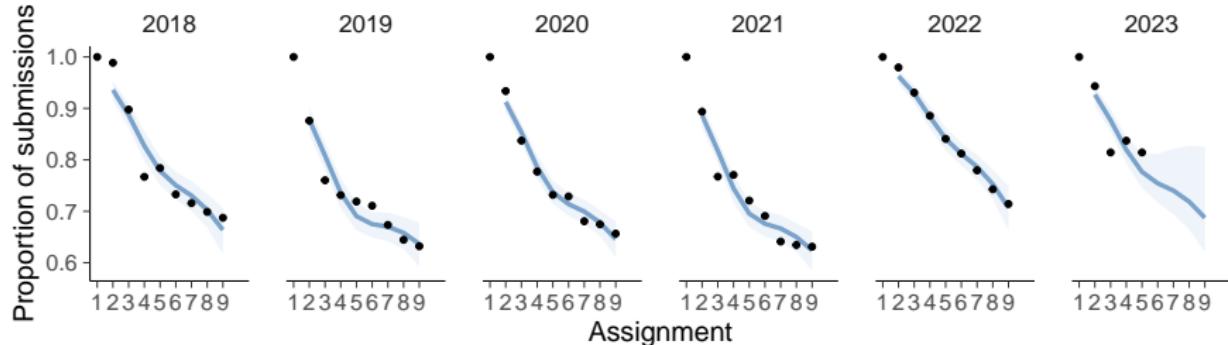
Student retention latent spline model, year 2023?



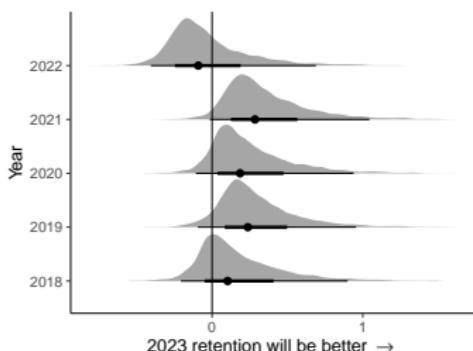
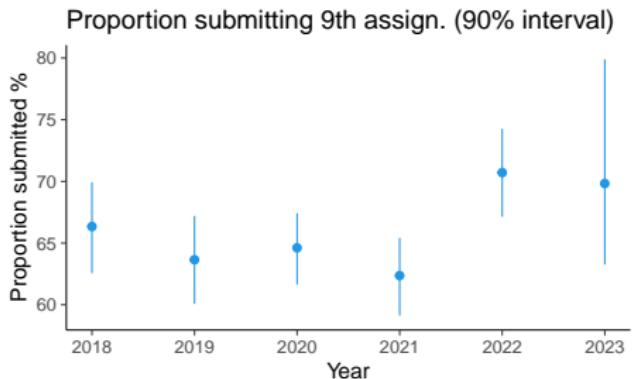
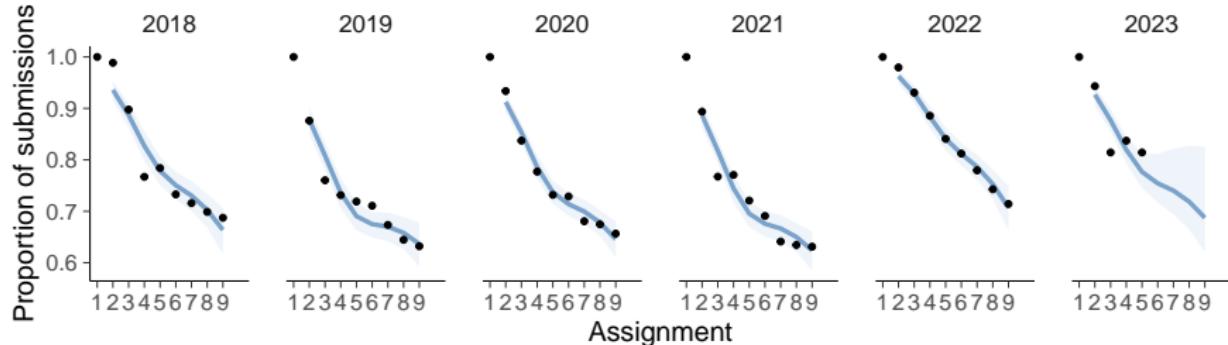
Student retention latent spline model, year 2023?



Student retention latent spline model, year 2023?



Student retention latent spline model, year 2023?



brms summary: one varying coefficient

```
nstudents | trials(nstudents1) ~ 1 + (1 | year), family=binomial()
```

Year specific intercepts have prior normal($0, \sigma_{\text{intercept}}$)

Family: binomial

Links: mu = logit

Formula: nstudents | trials(nstudents1) ~ 1 + (1 | year)

Data: filter(tb, assignment == 9) (Number of observations: 5)

Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
total post-warmup draws = 4000

Multilevel Hyperparameters:

~year (Number of levels: 5)

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sd(Intercept)	0.16	0.13	0.01	0.50	1.00	667	878

Regression Coefficients:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	0.68	0.11	0.45	0.92	1.01	390	242

brms summary: two varying coefficients, nocor

```
nstudents | trials(nstudents1) ~ assignment + (assignment || year), ...
```

Year specific intercepts and slopes have independent priors
 $\text{normal}(0, \sigma_{\text{intercept}})$ and $\text{normal}(0, \sigma_{\text{assignment}})$

```
Family: binomial
Links: mu = logit
Formula: nstudents | trials(nstudents1) ~ assignment + (assignment || year)
Data: filter(tb, assignment > 1) (Number of observations: 40)
Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
      total post-warmup draws = 4000
```

Multilevel Hyperparameters:

`~year` (Number of levels: 5)

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
<code>sd(Intercept)</code>	0.79	0.45	0.30	1.95	1.00	972	1470
<code>sd(assignment)</code>	0.07	0.06	0.01	0.23	1.01	592	720

Regression Coefficients:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	2.37	0.38	1.58	3.14	1.01	1114	1471
assignment	-0.21	0.04	-0.30	-0.14	1.00	1056	803

brms summary: two varying coefficients

```
nstudents | trials(nstudents1) ~ assignment + (assignment | year), ...
```

Year specific intercepts and slopes have a joint prior

$$N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\text{intercept}} & \rho \\ \rho & \sigma_{\text{slope}} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma_{\text{intercept}} \\ \sigma_{\text{slope}} \end{pmatrix}^T \right]$$

Family: binomial

Links: mu = logit

Formula: nstudents | trials(nstudents1) ~ assignment + (assignment | year)

Data: filter(tb, assignment > 1) (Number of observations: 40)

Draws: 4 chains, each with iter = 2000; warmup = 1000; thin = 1;
total post-warmup draws = 4000

Multilevel Hyperparameters:

~year (Number of levels: 5)

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sd(Intercept)	0.76	0.35	0.34	1.69	1.00	1079	1487
sd(assignment)	0.06	0.03	0.02	0.15	1.00	1211	1979
cor(Intercept,assignment)	-0.85	0.23	-1.00	-0.20	1.00	1730	2265

Regression Coefficients:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	2.36	0.37	1.60	3.10	1.00	860	1179
assignment	-0.21	0.03	-0.27	-0.14	1.00	1073	1464

brms multivariate population prior

```
nstudents | trials(nstudents1) ~ assignment + (assignment | year), ...
```

Year specific intercepts and slopes have a joint prior

$$N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\text{intercept}} & \rho \\ \rho & \sigma_{\text{slope}} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma_{\text{intercept}} \\ \sigma_{\text{slope}} \end{pmatrix}^T \right]$$

- brms uses LKJ prior for the correlation matrix
- LKJ is a good default weakly informative prior for correlation matrices

Centered vs non-centered parameterization

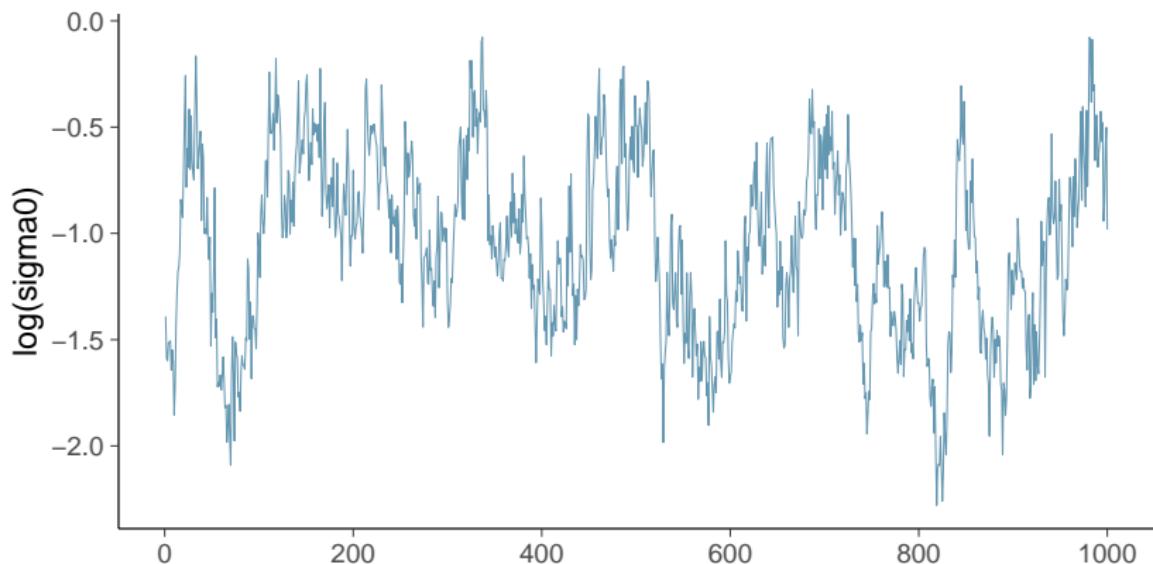
HMC divergences are more likely when using hierarchical models

Divergences

- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
 - indicates possibility of biased estimates

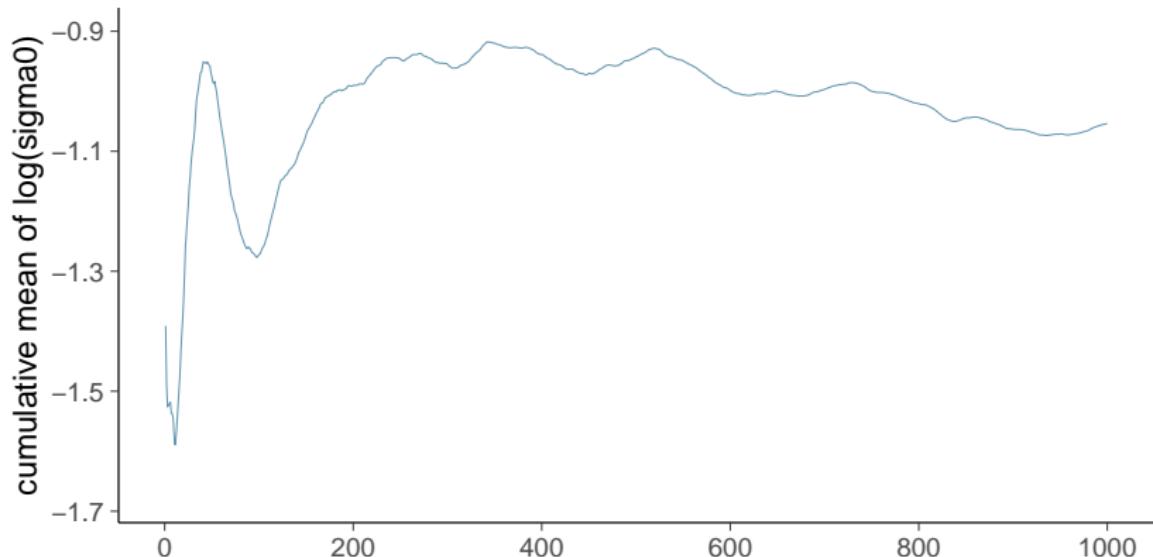
Divergences

- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
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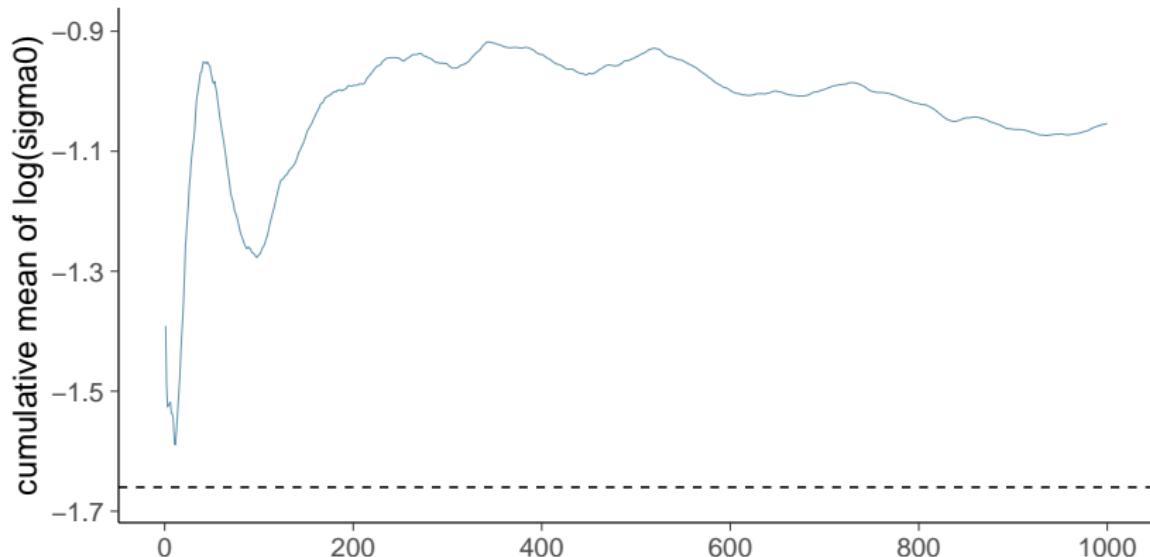
Divergences

- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
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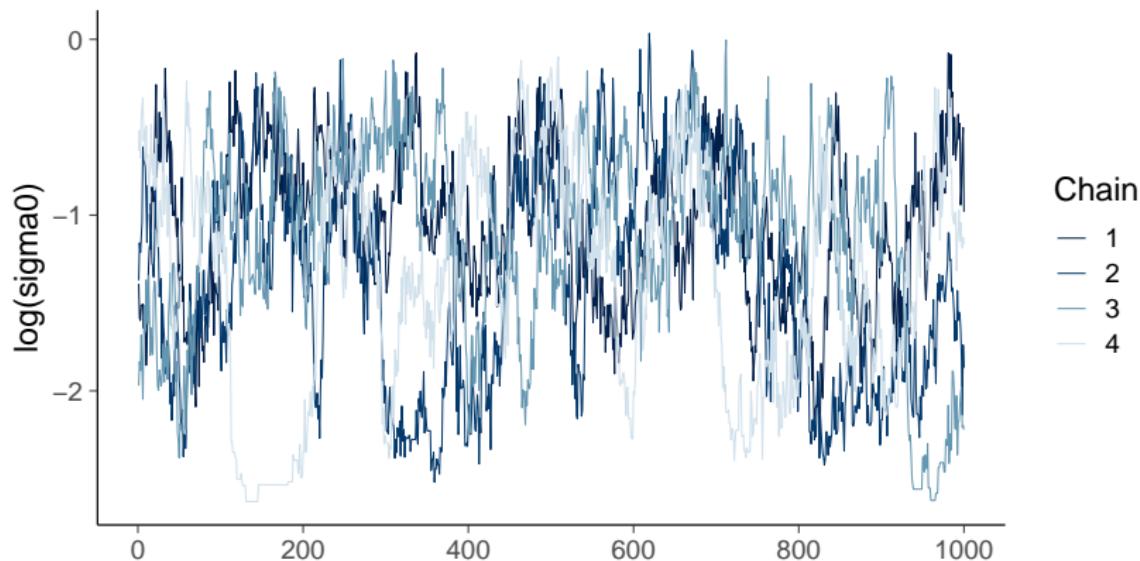
Divergences

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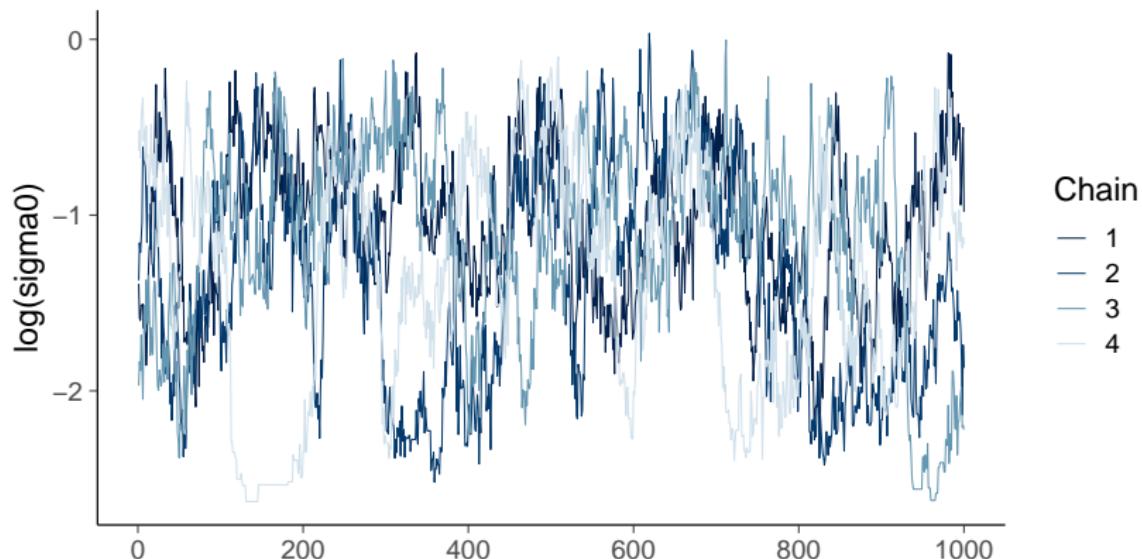
Divergences

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Divergences

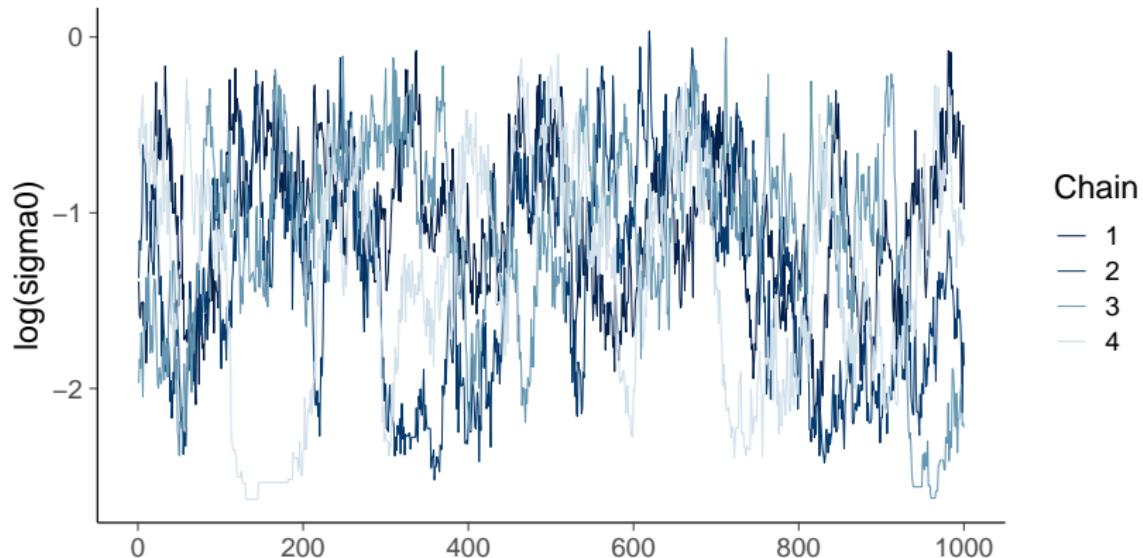
- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
 - indicates possibility of biased estimates



variable	mean	median	sd	mad	q5	q95	rhat	ess_bulk	ess_tail
lsigma0	-1.0	-1.0	0.10	0.070	-1.2	-0.93	1.2	7.2	19.

Divergences

- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
 - indicates possibility of biased estimates

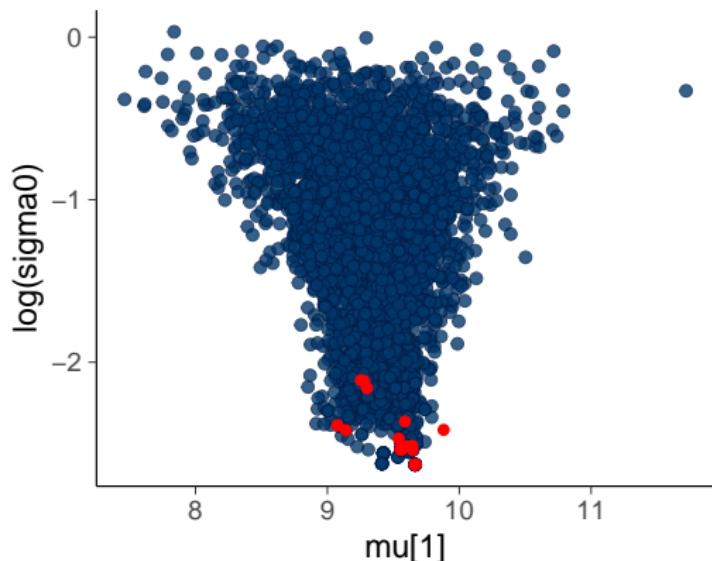


Warning: 29 of 4000 (1.0%) transitions ended with a divergence.

See <https://mc-stan.org/misc/warnings> for details.

Divergences

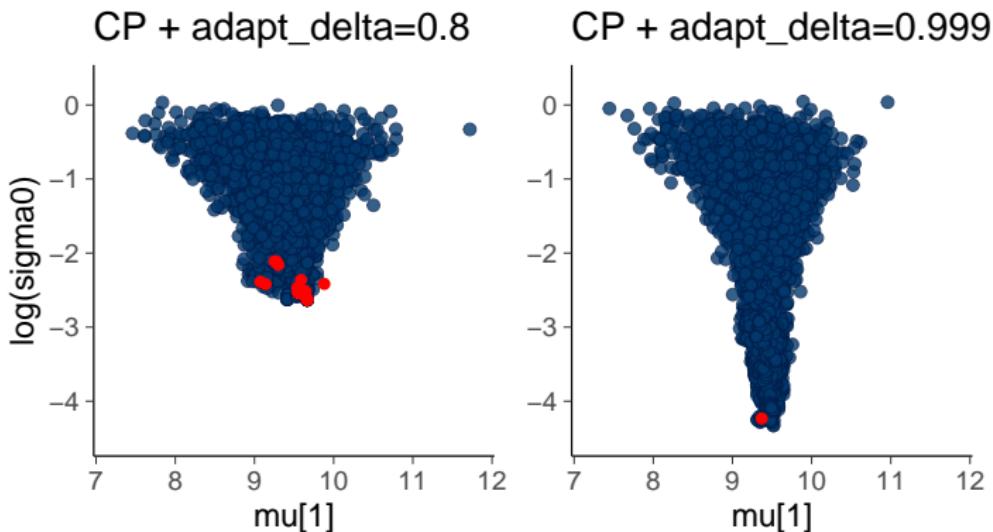
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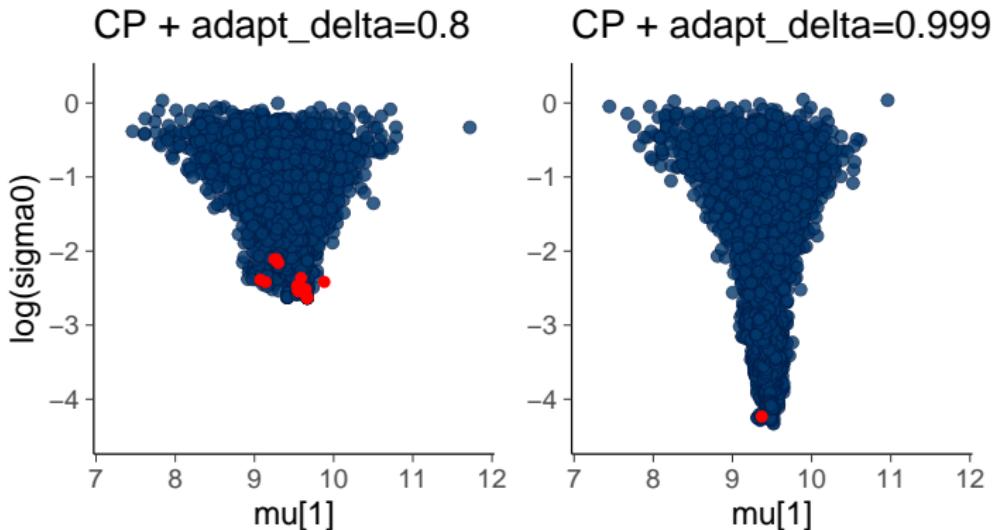
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Divergences

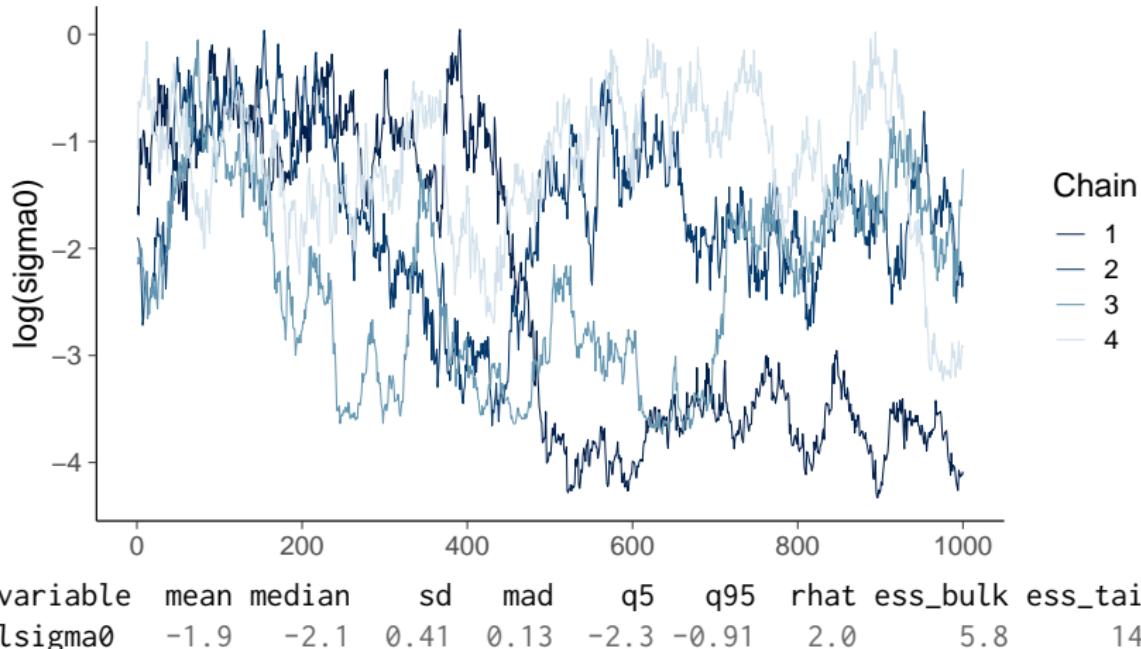
- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
 - indicates possibility of biased estimates



variable	mean	median	sd	mad	q5	q95	rhat	ess_bulk	ess_tail
lsigma0	-1.9	-2.1	0.41	0.13	-2.3	-0.91	2.0	5.8	14.

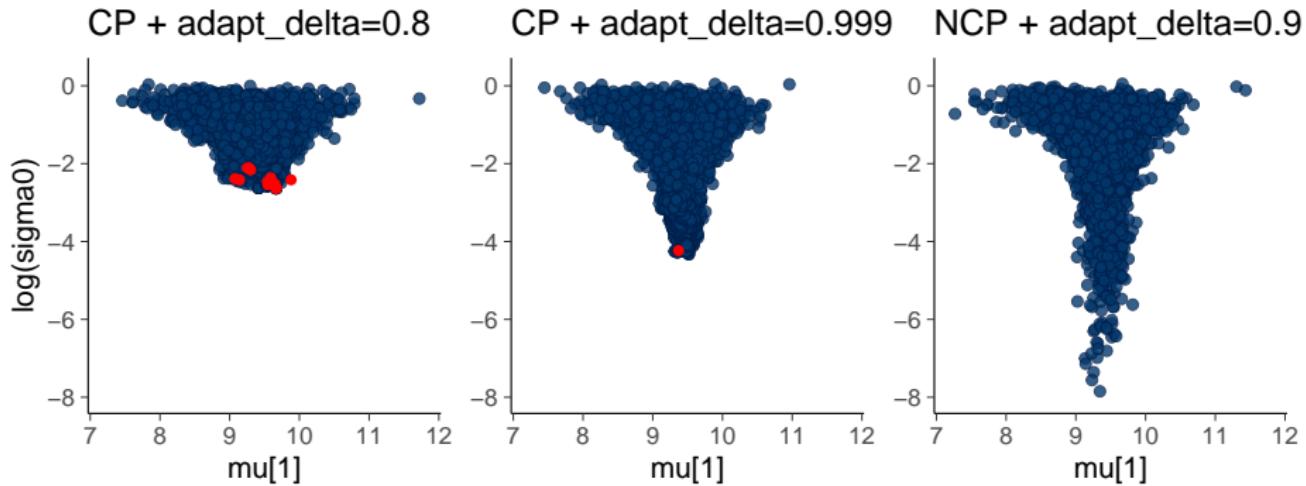
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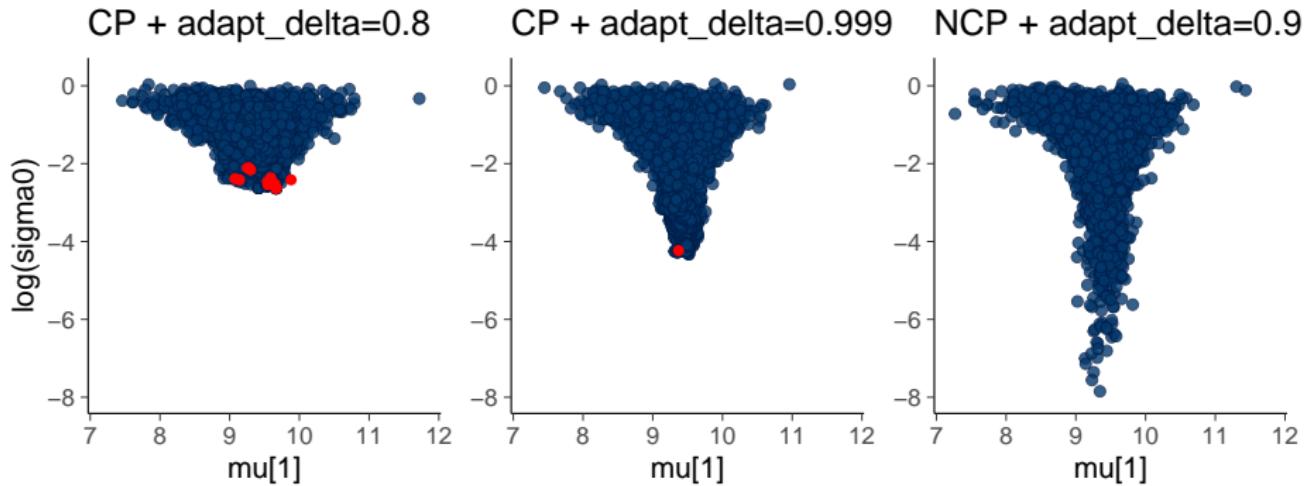
Divergences

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 - indicates possibility of biased estimates



Divergences

- HMC specific: indicates that Hamiltonian dynamic simulation has problems with unexpected fast changes in log-density
 - indicates possibility of biased estimates



With NCP parameterization

variable	mean	median	sd	mad	q5	q95	rhat	ess_bulk	ess_tail
σ_0	0.27	0.24	0.19	0.20	0.023	0.64	1.0	1571.	1737.
$\ln \sigma_0$	-1.7	-1.4	1.1	0.87	-3.8	-0.45	1.0	1571.	1737.

Centered parameterization

Hierarchical model code from the course demos

```
data {  
    int<lower=0> N;                      // number of observations  
    int<lower=0> K;                      // number of groups  
    array[N] int<lower=1, upper=K> x; // discrete group indicators  
    vector[N] y;                         // real valued observations  
}  
}
```

Centered parameterization

Hierarchical model code from the course demos

```
data {
    int<lower=0> N;                                // number of observations
    int<lower=0> K;                                // number of groups
    array[N] int<lower=1, upper=K> x; // discrete group indicators
    vector[N] y;                                    // real valued observations
}
parameters {
    real mu0;                                     // prior mean
    real<lower=0> sigma0;                         // prior std constrained to be pos.
    vector[K] mu;                                  // group means
    real<lower=0> sigma;                           // common std constrained to be pos.
}
```

Centered parameterization

Hierarchical model code from the course demos

```
data {
    int<lower=0> N;                                // number of observations
    int<lower=0> K;                                // number of groups
    array[N] int<lower=1, upper=K> x;              // discrete group indicators
    vector[N] y;                                   // real valued observations
}
parameters {
    real mu0;                                     // prior mean
    real<lower=0> sigma0;                         // prior std constrained to be pos.
    vector[K] mu;                                 // group means
    real<lower=0> sigma;                          // common std constrained to be pos.
}
model {
    mu0 ~ normal(10, 10);                         // weakly informative prior
    sigma0 ~ normal(0, 10);                        // weakly informative prior
    mu ~ normal(mu0, sigma0);                     // population prior with unknown param.
    sigma ~ lognormal(0, .5);                      // weakly informative prior
    y ~ normal(mu[x], sigma);                     // observation model
}
```

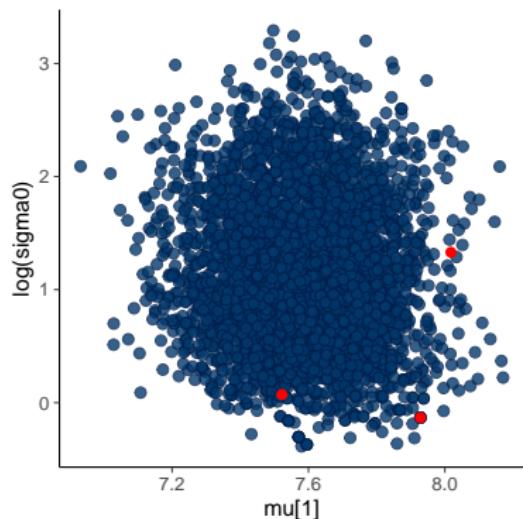
Centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

Centered parameterization

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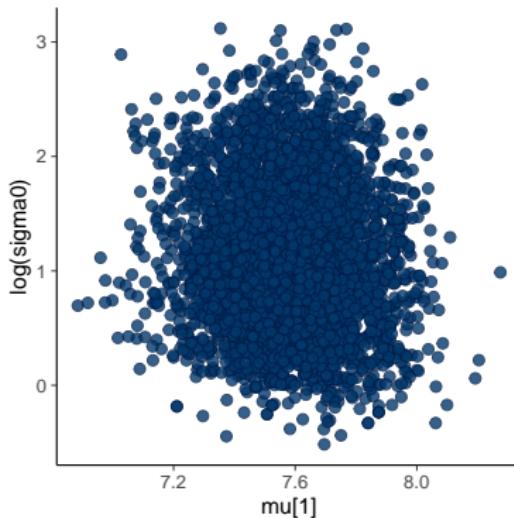
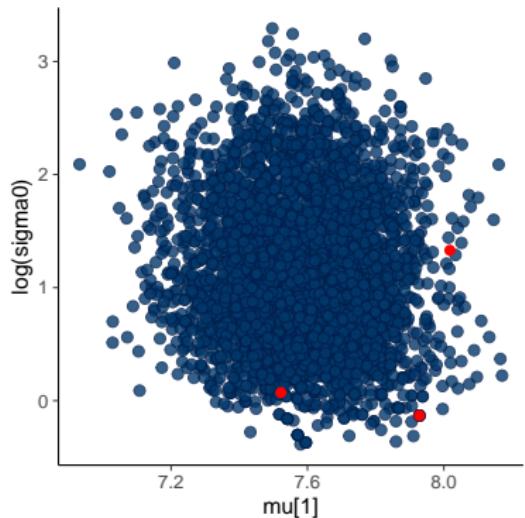
A few divergences that are not clustered.



Centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

And decreasing step size a little helps.



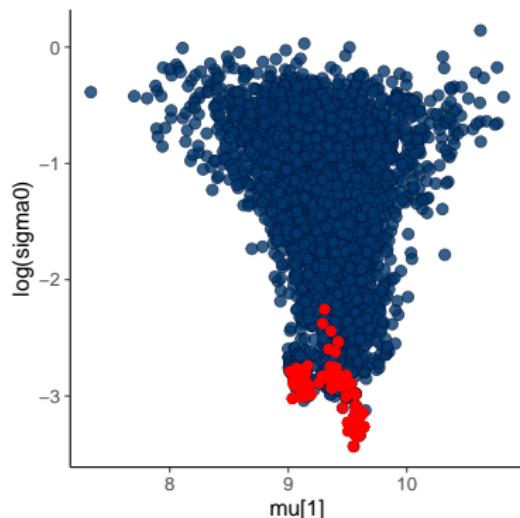
Centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

Centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

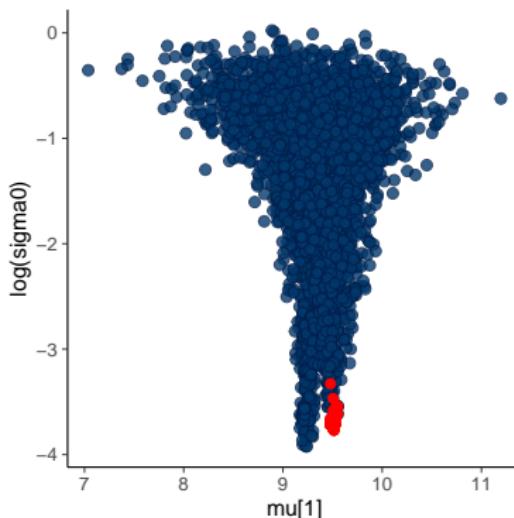
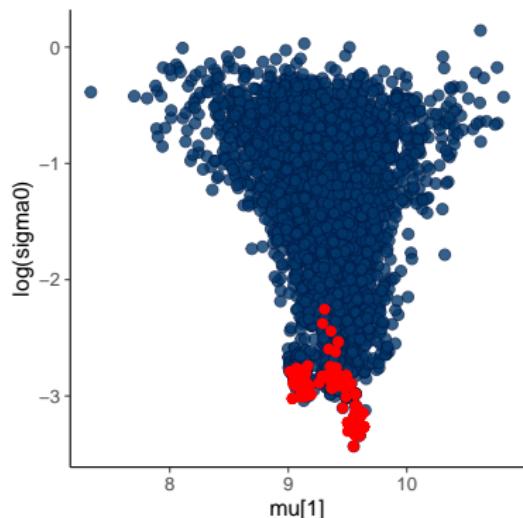
Many divergences that are clustered.



Centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

And decreasing step size doesn't remove the problem.



Centered parameterization

Hierarchical model code from the course demos

```
parameters {
    real mu0;                                // prior mean
    real<lower=0> sigma0;                    // prior std constrained to be pos.
    vector[K] mu;                            // group means
    real<lower=0> sigma;                     // common std constrained to be pos.
}

model {
    mu0 ~ normal(10, 10);                  // weakly informative prior
    sigma0 ~ normal(0, 10);                // weakly informative prior
    mu ~ normal(mu0, sigma0);            // population prior with unknown param.
    sigma ~ lognormal(0, .5);             // weakly informative prior
    y ~ normal(mu[x], sigma);           // observation model
}
```

Non-centered parameterization

Transformation

```
parameters {
    real mu0;                                // prior mean
    real<lower=0> sigma0;                    // prior std constrained to be pos.
    vector[K] z;                             // latent variable
    real<lower=0> sigma;                     // common std constrained to be pos.
}

transformed parameters {
    vector[K] mu = mu0 + sigma0 * z; // group means
}

model {
    mu0 ~ normal(10, 10);      // weakly informative prior
    sigma0 ~ normal(0, 10);    // weakly informative prior
    z ~ normal(0, 1);          // unit normal
    sigma ~ lognormal(0, .5); // weakly informative prior
    y ~ normal(mu[x], sigma); // observation model
}
```

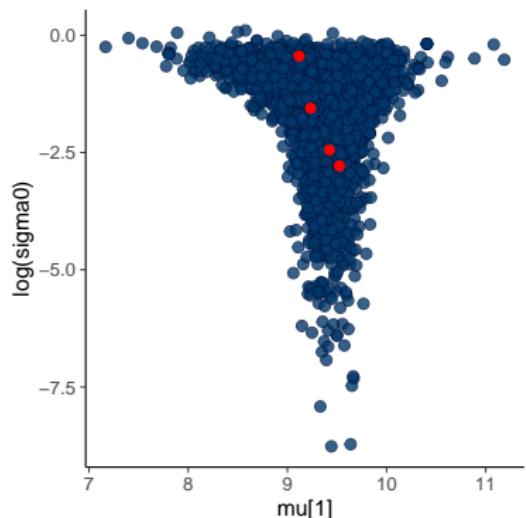
Non-centered parameterization

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Non-centered parameterization

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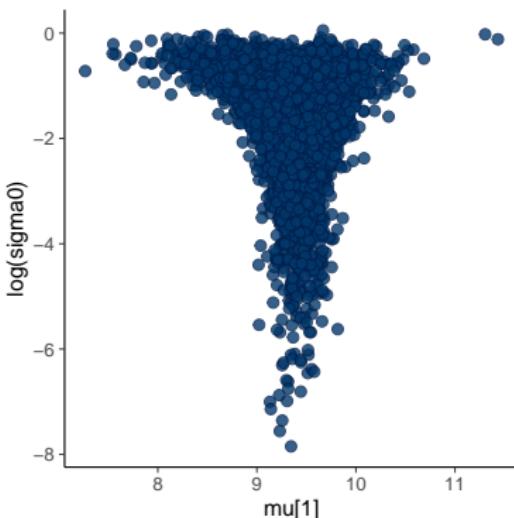
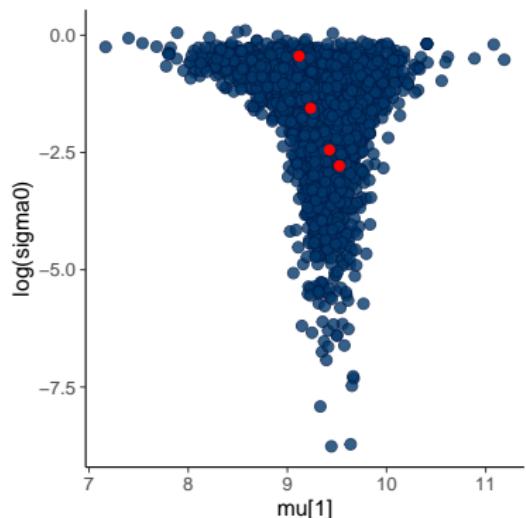
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Non-centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

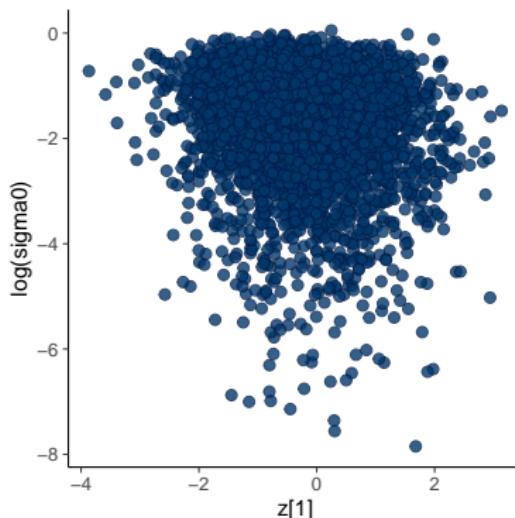
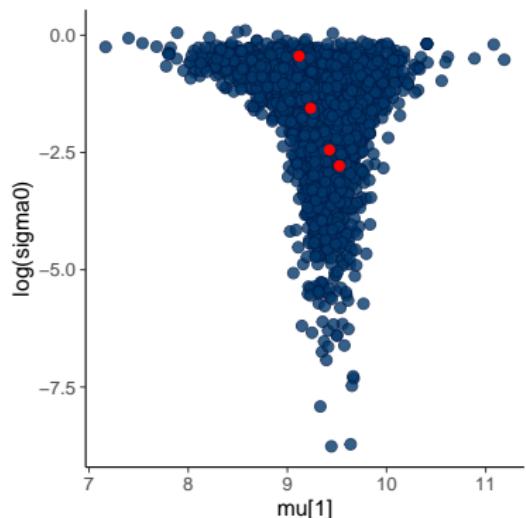
And decreasing step size a little helps.



Non-centered parameterization

Second data with a few observations per group: 71 years with each having 3 observations.

Because we're actually sampling z and not μ



Non-centered parameterization

No free lunch

- non-centered parameterization is good when likelihood is weak
- non-centered parameterization is bad when likelihood is strong

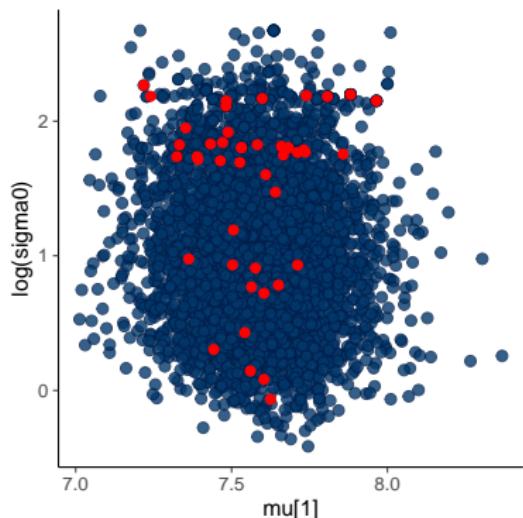
Non-centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

Non-centered parameterization

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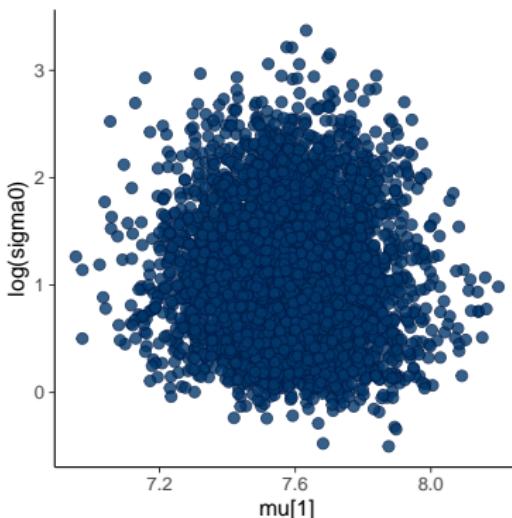
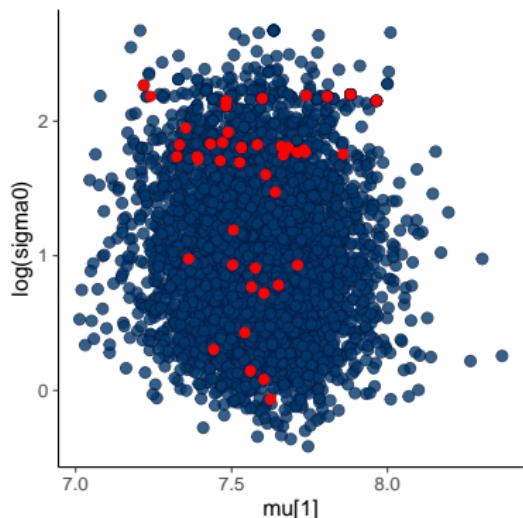
Many divergences that are not clustered.



Non-centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

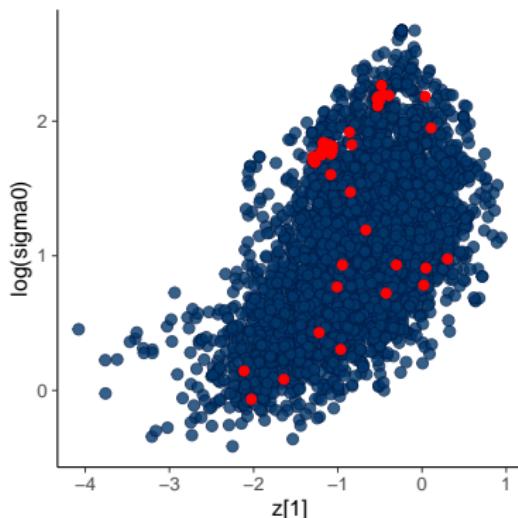
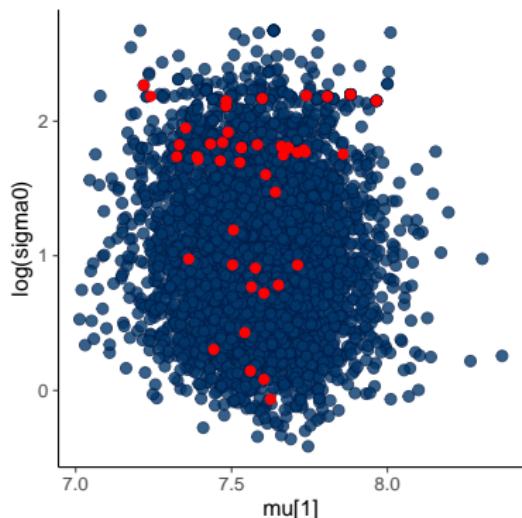
But decreasing step size a lot helps.



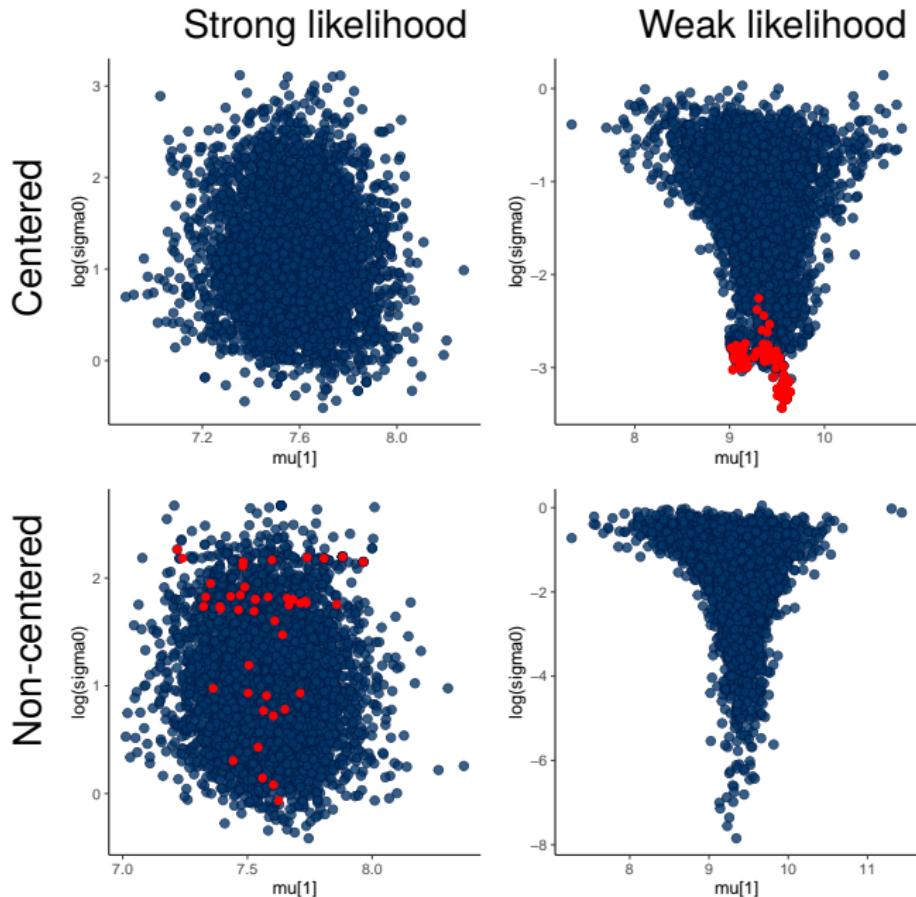
Non-centered parameterization

First data with many observations per group: 3 summer months with each having 71 observations.

Now the posterior for z is problematic.



Centered vs. non-centered parameterization



brms and rstanarm

- brms and rstanarm use non-centered parameterization
 - as hierarchical models and Bayesian inference is most useful when likelihood is weak

brms and rstanarm

- brms and rstanarm use non-centered parameterization
 - as hierarchical models and Bayesian inference is most useful when likelihood is weak
- If the likelihood is highly informative for all groups, and the non-centered parameterization is causing divergences,
 - use fixed population prior model as the hierarchical part is not that important with strong likelihood. Instead of

$y \sim 1 + (1 \mid \text{group})$

use, for example,

$y \sim 1 + \text{group}$,

`prior = prior(normal(0, 10), class="b", coeff="group")`

brms and rstanarm

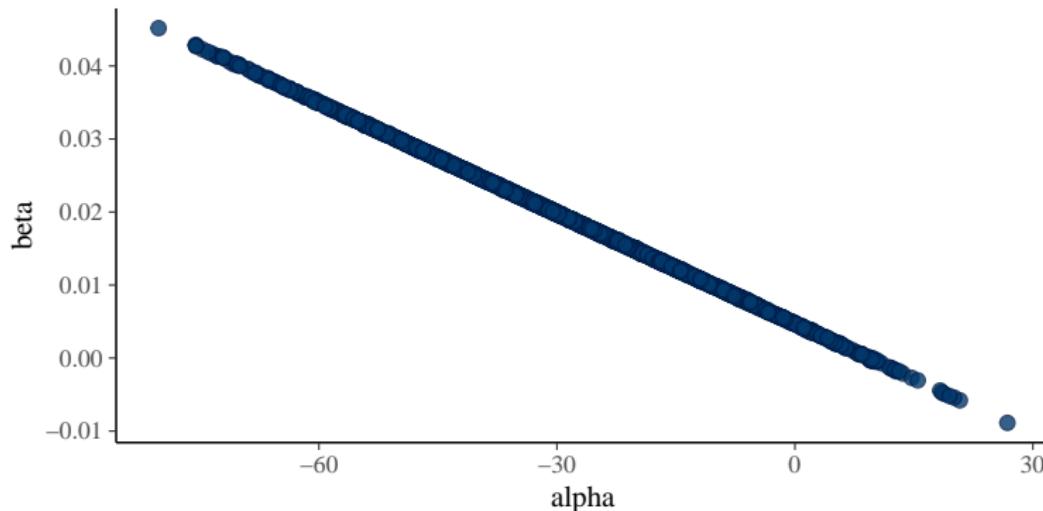
- brms and rstanarm use non-centered parameterization
 - as hierarchical models and Bayesian inference is most useful when likelihood is weak
- If the likelihood is highly informative for all groups, and the non-centered parameterization is causing divergences,
 - use fixed population prior model as the hierarchical part is not that important with strong likelihood. Instead of $y \sim 1 + (1 \mid \text{group})$ use, for example, $y \sim 1 + \text{group},$ `prior = prior(normal(0, 10), class="b", coeff="group")`
- There can be need for both centered and non-centered parameterization in the same model
 - automation not easy, but research goes on

brms non-centered parameterization

```
parameters {
    real Intercept; // temporary intercept for centered predictors
    real<lower=0> sigma; // dispersion parameter
    vector<lower=0>[M_1] sd_1; // group-level standard deviations
    array[M_1] vector[N_1] z_1; // standardized group-level effects
}
transformed parameters {
    vector[N_1] r_1_1; // actual group-level effects
    r_1_1 = (sd_1[1] * (z_1[1]));
//...
}
model {
//...
    for (n in 1:N) {
        // add more terms to the linear predictor
        mu[n] += r_1_1[J_1[n]] * Z_1_1[n];
    }
//...
    target += std_normal_lpdf(z_1[1]);
}
```

Kilpisjärvi summer temperature

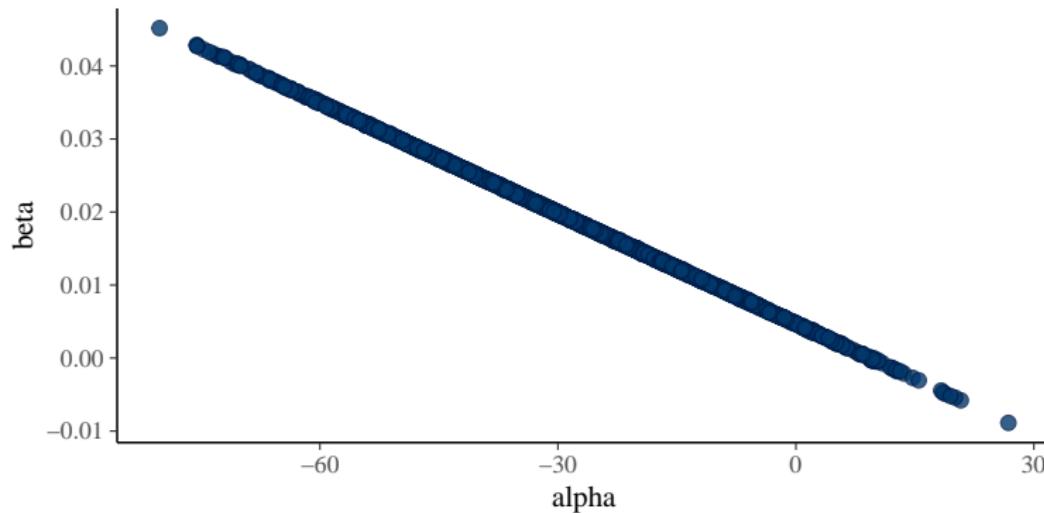
Posterior draws of alpha and beta



Warning: 1 of 4000 (0.0%) transitions hit the maximum treedepth limit of 10.
See <https://mc-stan.org/mcmc/warnings> for details.

Kilpisjärvi summer temperature

Posterior draws of alpha and beta



Warning: 1 of 4000 (0.0%) transitions hit the maximum treedepth limit of 10.
See <https://mc-stan.org/misc/warnings> for details.

Solution was to center the covariate time to have mean 0, so that the intercept is the expected temperature in the middle of the range

brms covariate centering by default

```
fit_lin <- brm(temp ~ year, data = data_lin)
```

brms covariate centering by default

```
fit_lin <- brm(temp ~ year, data = data_lin)

transformed data {
  matrix[N, Kc] Xc; // centered version of X without an intercept
  vector[Kc] means_X; // column means of X before centering
  for (i in 2:K) {
    means_X[i - 1] = mean(X[, i]);
    Xc[, i - 1] = X[, i] - means_X[i - 1];
  }
}
parameters {
  vector[Kc] b; // regression coefficients
  real Intercept; // temporary intercept for centered predictors
  real<lower=0> sigma; // dispersion parameter
}
model {
//...
  target += normal_id_glm_lpdf(Y | Xc, Intercept, b, sigma);
//...
}
generated quantities {
  // actual population-level intercept
  real b_Intercept = Intercept - dot_product(means_X, b);
}
```

brms covariate centering by default

Sometimes we want to turn off the covariate centering, which can be done with additional argument for brms formula function `bf()`

```
fit_lin <- brm(bf(temp ~ year, center=FALSE), data = data_lin)
```

brms formulas

model	formula	alternative formula
intercept only $y \sim N(\alpha, \sigma)$	$y \sim 1$	

brms formulas

model	formula	alternative formula
intercept only		
$y \sim N(\alpha, \sigma)$	$y \sim 1$	
linear models		
$y \sim N(\alpha + \beta x, \sigma)$	$y \sim x$	$y \sim 1 + x$

brms formulas

model	formula	alternative formula
intercept only		
$y \sim N(\alpha, \sigma)$	$y \sim 1$	
linear models		
$y \sim N(\alpha + \beta x, \sigma)$	$y \sim x$	$y \sim 1 + x$
$y \sim N(\beta x, \sigma)$	$y \sim 0 + x$	$y \sim -1 + x$

brms formulas

model	formula	alternative formula
intercept only		
$y \sim N(\alpha, \sigma)$	$y \sim 1$	
linear models		
$y \sim N(\alpha + \beta x, \sigma)$	$y \sim x$	$y \sim 1 + x$
$y \sim N(\beta x, \sigma)$	$y \sim 0 + x$	$y \sim -1 + x$
hierarchical models		
$y \sim N(\alpha_0 + \alpha_g + \beta x, \sigma)$	$y \sim x + (1 \mid g)$	

brms formulas

model	formula	alternative formula
intercept only		
$y \sim N(\alpha, \sigma)$	$y \sim 1$	
linear models		
$y \sim N(\alpha + \beta x, \sigma)$	$y \sim x$	$y \sim 1 + x$
$y \sim N(\beta x, \sigma)$	$y \sim 0 + x$	$y \sim -1 + x$
hierarchical models		
$y \sim N(\alpha_0 + \alpha_g + \beta x, \sigma)$	$y \sim x + (1 \mid g)$	
$y \sim N(\alpha_0 + \alpha_g + \beta_0 x + \beta_g x, \sigma)$	$y \sim x + (x \mid g)$	(see above)

brms formulas

model	formula
heteroskedastic $y \sim N(\alpha_\mu + \beta_\mu x, \exp(\alpha_\sigma + \beta_\sigma x))$	<code>bf(y ~ x, sigma ~ x)</code>

brms families

family argument determines the observation model family

model	brms
$y \sim t_v(\alpha + \beta x, \sigma)$	$y \sim x, \text{family} = \text{student}()$
$y \sim \text{Bin}(\text{logit}^{-1}(\alpha + \beta x), N)$	$y \mid \text{trials}(N) \sim x, \text{family} = \text{binomial}()$
$y \sim \text{Neg-bin}(\exp(\alpha + \beta x), \phi)$	$y \sim x, \text{family} = \text{negbinomial}()$

Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

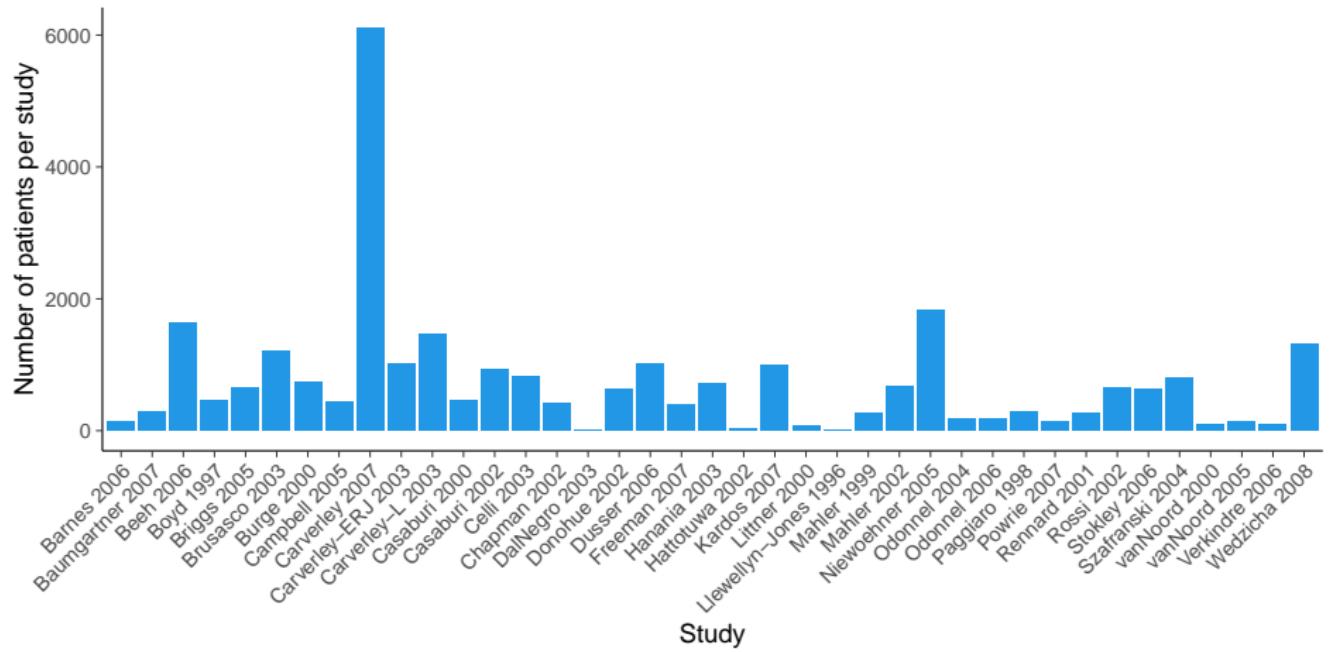
```
> head(dat.baker2009)
```

	study	year	id	treatment	exac	total
1	Llewellyn-Jones	1996	1996	1 Fluticasone	0	8
2	Llewellyn-Jones	1996	1996	1 Placebo	3	8
3	Boyd	1997	1997	2 Salmeterol	47	229
4	Boyd	1997	1997	2 Placebo	59	227
5	Paggiaro	1998	1998	3 Fluticasone	45	142
6	Paggiaro	1998	1998	3 Placebo	51	139

Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

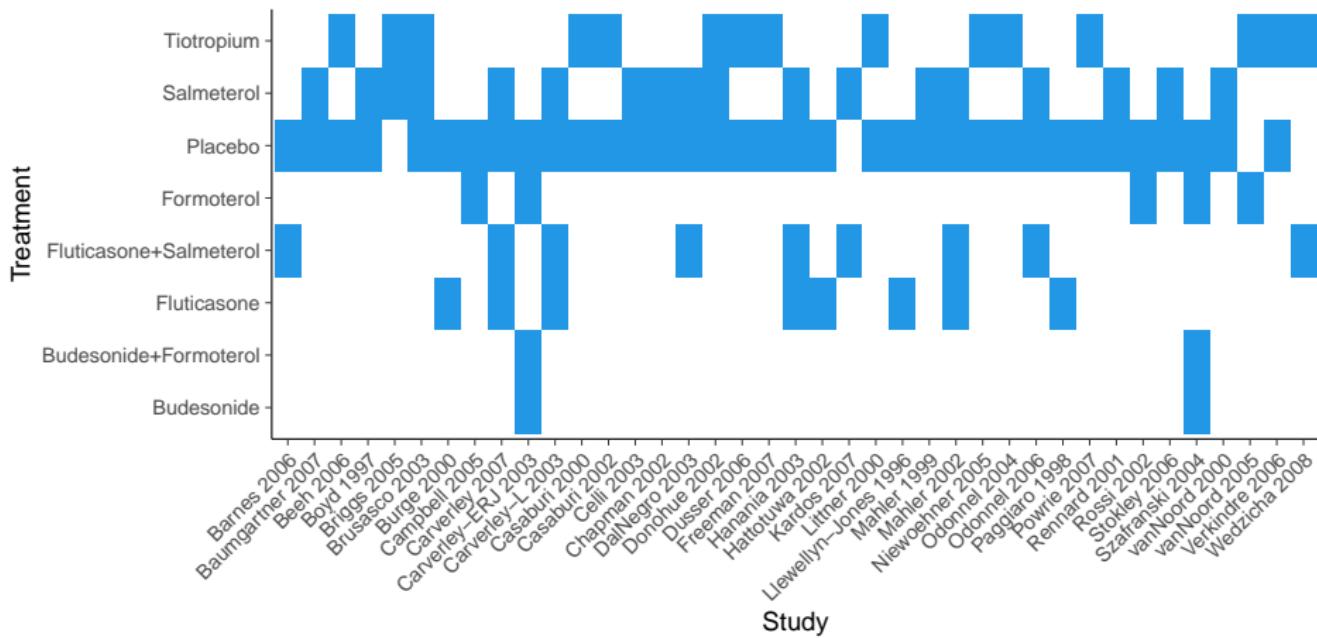
A big variation in the number of patients per study



Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

2–4 drugs tested per study, and some not including placebo



Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

Pooled over studies, separate for treatments

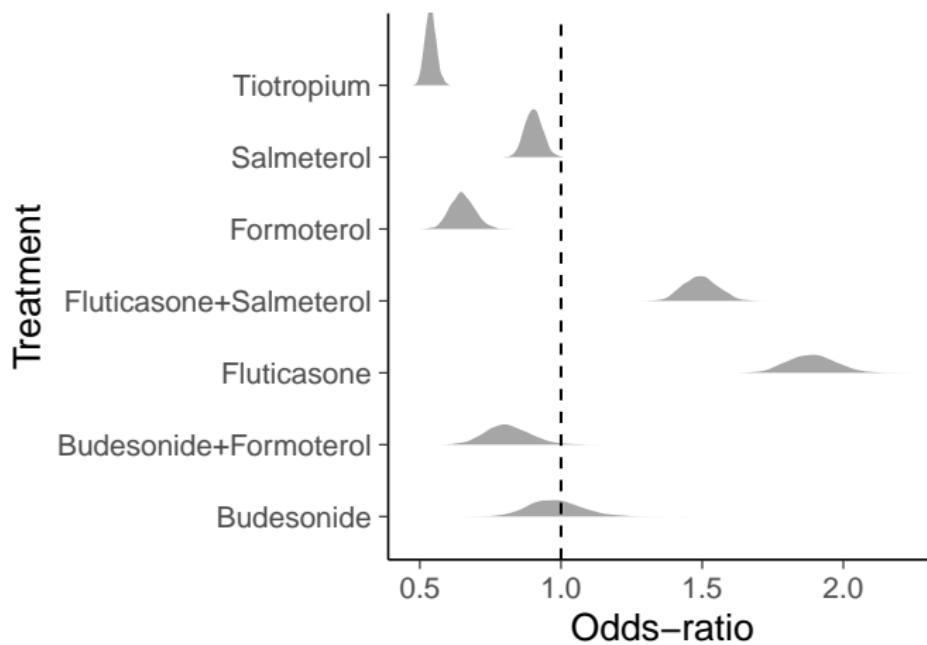
```
fit_pooled <- brm(exac | trials(total) ~ 0 + treatment,  
                    prior = prior(student_t(7, 0, 1.5), class='b'),  
                    family = binomial(), data = dat.baker2009)
```

Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

Pooled over studies, separate for treatments

```
fit_pooled <- brm(exac | trials(total) ~ 0 + treatment,  
                    prior = prior(student_t(7, 0, 1.5), class='b'),  
                    family = binomial(), data = dat.baker2009)
```



Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

Hierarchical for studies, hierarchical for treatments

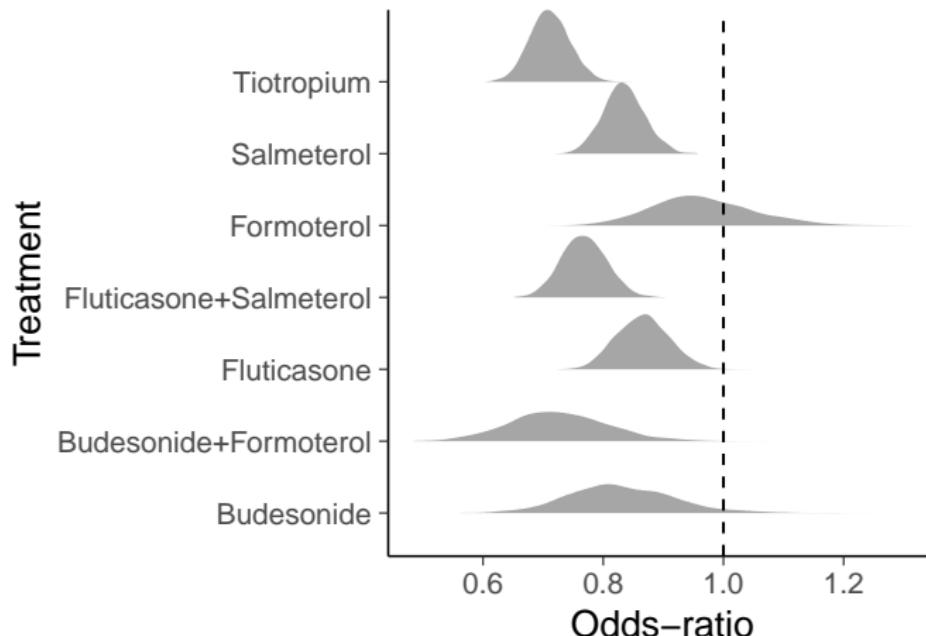
```
fit_hier <- brm(exac | trials(total) ~ (1 | treatment) + (1 | study),  
                  family = binomial(), data = dat.baker2009)
```

Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

Hierarchical for studies, hierarchical for treatments

```
fit_hier <- brm(exac | trials(total) ~ (1 | treatment) + (1 | study),  
family = binomial(), data = dat.baker2009)
```

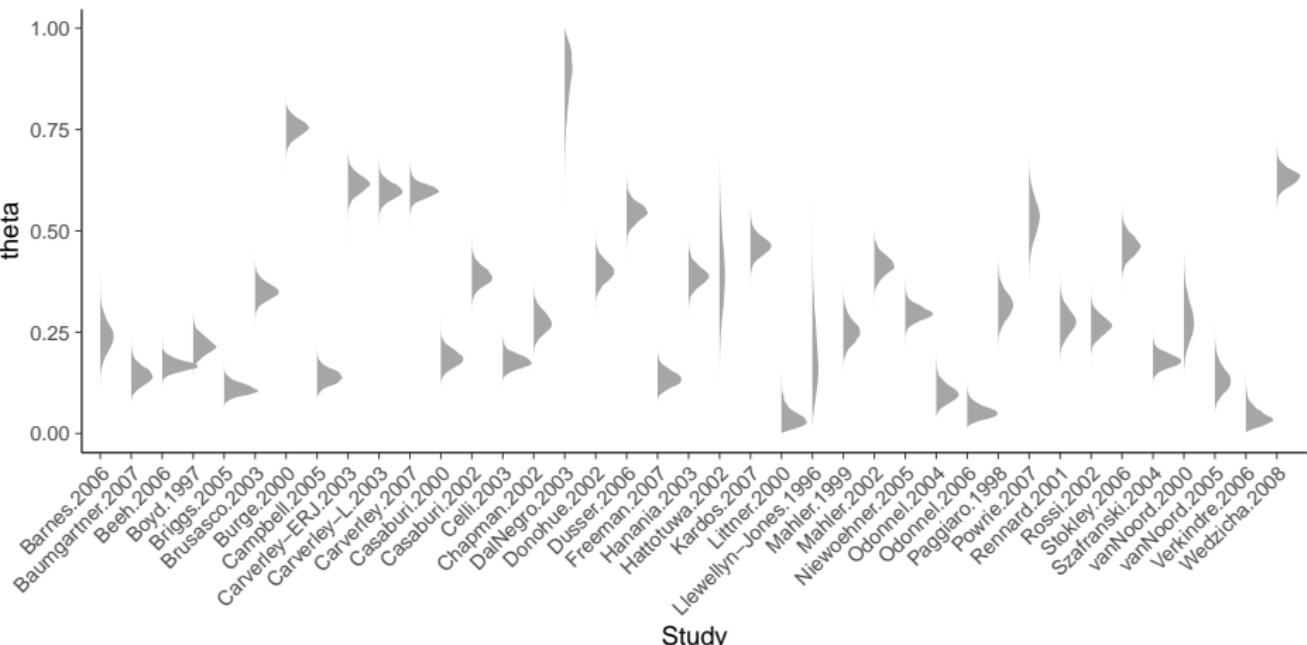


Meta-analysis

Pharmacologic treatments for chronic obstructive pulmonary disease

Hierarchical for studies, hierarchical for treatments

```
fit_hier <- brm(exac | trials(total) ~ (1 | treatment) + (1 | study),  
family = binomial(), data = dat.baker2009)
```



BDA course demo for brms

Link in the course web site or directly

https://avehtari.github.io/BDA_R_demos/demos_rstan/brms_demo.html

Exchangeability

- Justifies why we can use
 - a joint model for data
 - a joint prior for a set of parameters
- Less strict than independence

Exchangeability

- *Exchangeability*: Parameters $\theta_1, \dots, \theta_J$ (or observations y_1, \dots, y_J) are exchangeable if the joint distribution p is invariant to the permutation of indices $(1, \dots, J)$
- e.g.

$$p(\theta_1, \theta_2, \theta_3) = p(\theta_2, \theta_3, \theta_1)$$

- Exchangeability implies symmetry: If there is no information which can be used *a priori* to separate θ_j from each other, we can assume exchangeability.
 - "Ignorance implies exchangeability"

Exchangeability

- Exchangeability does not mean that the results of the experiments could not be different
 - e.g. if we know that the experiments have been in two different laboratories, and we know that the other laboratory has better conditions for the rats, but we do not know which experiments have been made in which laboratory
 - a priori experiments are exchangeable
 - model could have unknown parameter for the laboratory with a conditional prior for rats assumed to come from the same place (clustering model)

Exchangeability and additional information

- Example: bioassay
 - y_i number of dead animals are not exchangeable alone

Exchangeability and additional information

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 - y_i number of dead animals are not exchangeable alone
 - x_i dose is additional information

Exchangeability and additional information

- Example: bioassay
 - y_i number of dead animals are not exchangeable alone
 - x_i dose is additional information
 - (x_i, y_i) exchangeable and logistic regression was used

$$p(\alpha, \beta \mid y, n, x) \propto \prod_{i=1}^n p(y_i \mid \alpha, \beta, n_i, x_i) p(\alpha, \beta)$$

Hierarchical exchangeability

- Example: hierarchical rats example
 - all rats not exchangeable

Hierarchical exchangeability

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 - all rats not exchangeable
 - in a single laboratory rats exchangeable

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 - → hierarchical model

Hierarchical exchangeability

- Example: hierarchical rats example
 - all rats not exchangeable
 - in a single laboratory rats exchangeable
 - laboratories exchangeable
 - → hierarchical model
- Example: paper helicopters
 - we may assume that the groups had variation in dropping height, dropping technique, timing, and how to define unstable

Partial or conditional exchangeability

- Conditional exchangeability
 - if y_i is connected to an additional information x_i , so that y_i are not exchangeable, but (y_i, x_i) exchangeable use joint model or conditional model $(y_i | x_i)$.

Partial or conditional exchangeability

- Conditional exchangeability
 - if y_i is connected to an additional information x_i , so that y_i are not exchangeable, but (y_i, x_i) exchangeable use joint model or conditional model $(y_i | x_i)$.
- Partial exchangeability
 - if the observations can be grouped (a priori), then use hierarchical model

Exchangeability

- The simplest form of the exchangeability (but not the only one) for the parameters θ conditional independence

$$p(x_1, \dots, x_J \mid \theta) = \prod_{j=1}^J p(x_j \mid \theta)$$

Exchangeability - Counter example

- A six sided die with probabilities $\theta_1, \dots, \theta_6$
 - without additional knowledge $\theta_1, \dots, \theta_6$ exchangeable
 - due to the constraint $\sum_{j=1}^6 \theta_j$, parameters are not independent and thus joint distribution can not be presented as iid

Exchangeability

- See more examples in the BDA3 notes - Exchangeability vs. independence