Chapter 6

- 6.1 The place of model checking in applied Bayesian statistics
- 6.2 Do the inferences from the model make sense?
- 6.3 Posterior predictive checking
- 6.4 Graphical posterior predictive checks (can be skimmed)
- 6.5 Model checking for the educational testing example

Model checking

- demo6_1: Posterior predictive checking - light speed
- demo6_2: Posterior predictive checking - sequential dependence
- demo6_3: Posterior predictive checking - poor test statistic
- demo6_4: Posterior predictive checking - marginal predictive p-value

Model checking – overview

- Sensibility with respect to additional information not used in modeling
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- Internal validation
    - posterior predictive checking
    - cross-validation predictive checking

Posterior predictive checking – example

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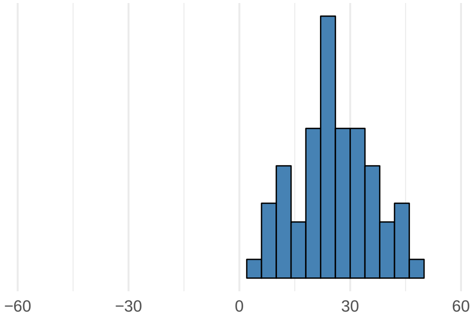

Replicates vs. future observation

- Predictive $\tilde{y}$ is the next not yet observed possible observation. y^{rep} refers to replicating the whole experiment (potentially with same values of x) and obtaining as many replicated observations as in the original data.

Posterior predictive checking – example

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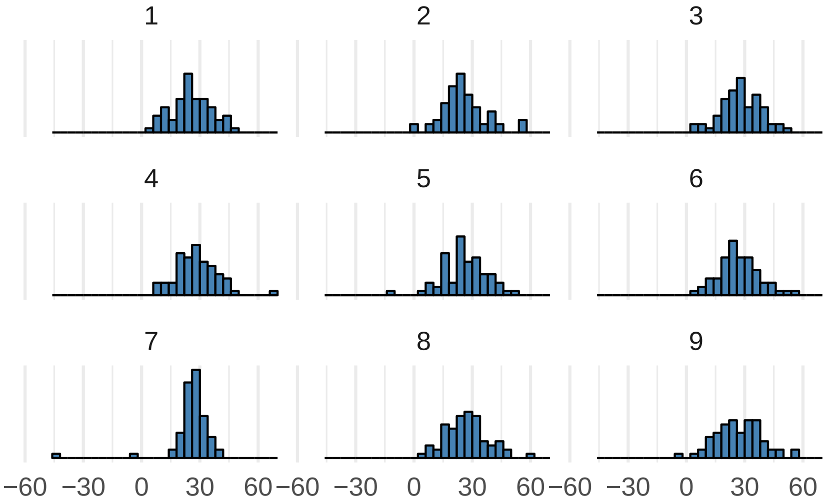
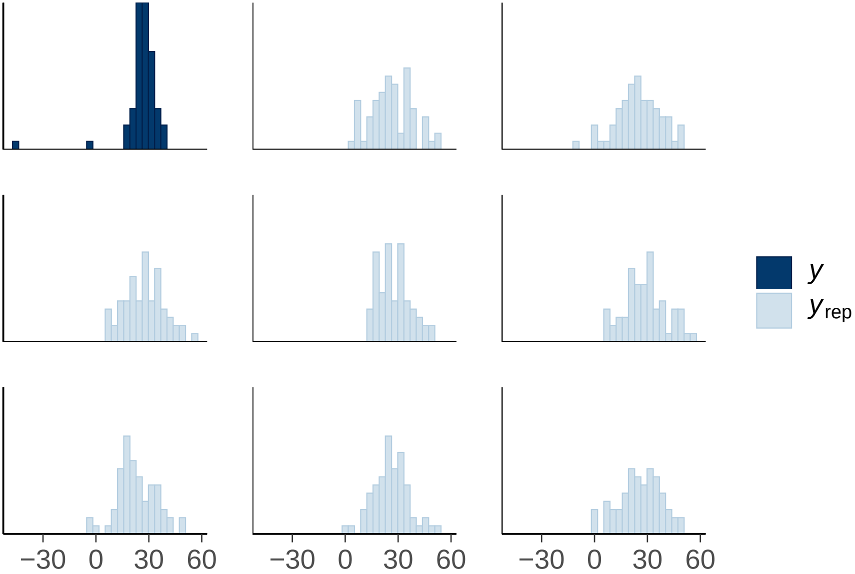


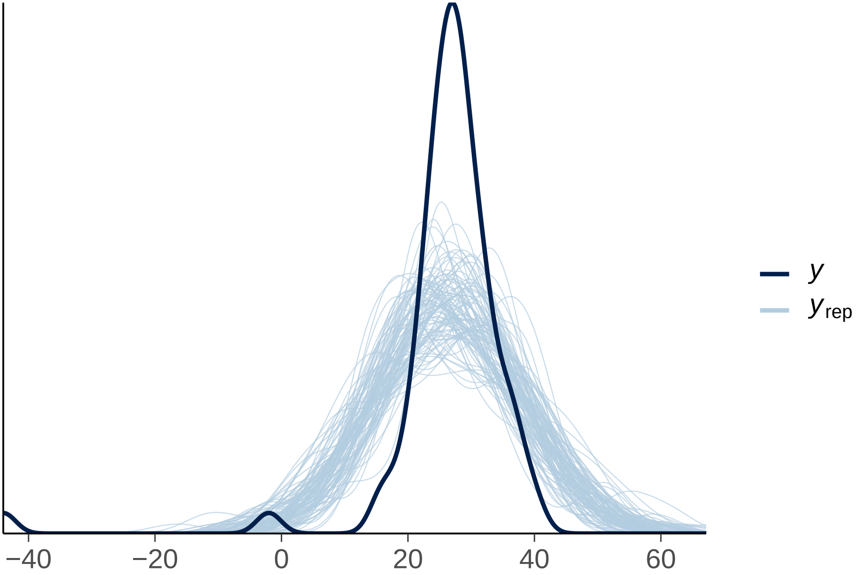
Posterior predictive checking – bayesplot

```
ppc_hist(y, yrep[1:8,])
```



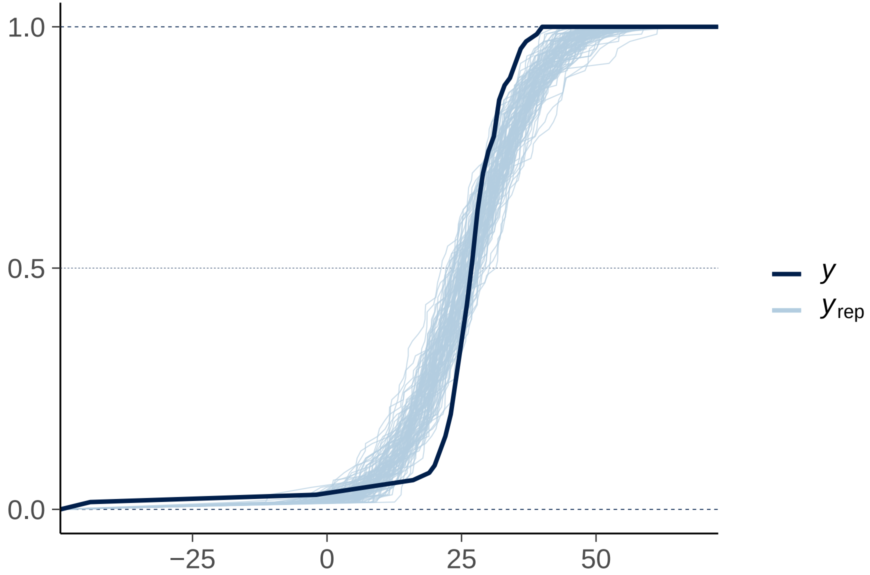
Posterior predictive checking – bayesplot

```
ppc_dens_overlay(y, yrep[1:100,])
```



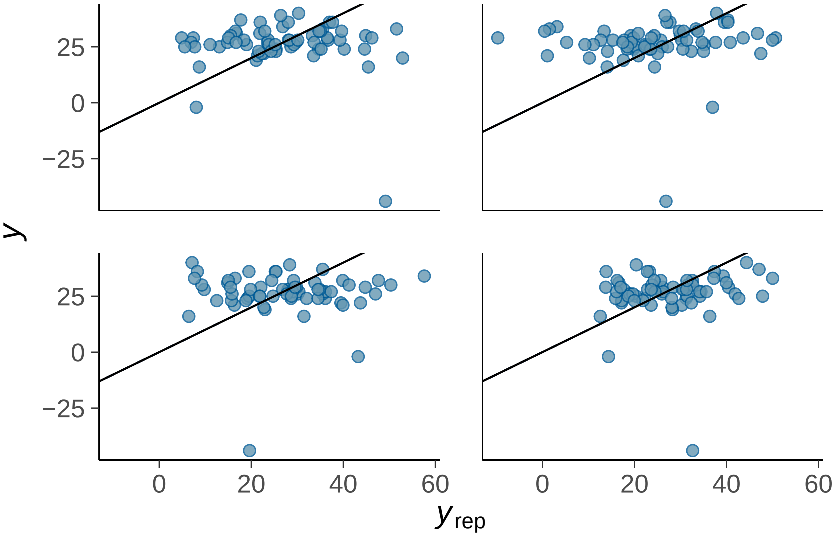
Posterior predictive checking – bayesplot

```
ppc_ecdf_overlay(y, yrep[1:100,])
```



Posterior predictive checking – bayesplot

```
ppc_scatter(y, yrep[1:4,]) + geom_abline()
```



Posterior predictive checking with test statistic

- Replicated data sets y^{rep}
- Test quantity (or discrepancy measure) $T(y, \theta)$
  - summary quantity for the observed data $T(y, \theta)$
  - summary quantity for a replicated data $T(y^{\mathrm{rep}}, \theta)$
  - can be easier to compare summary quantities than data sets

Posterior predictive checking – example

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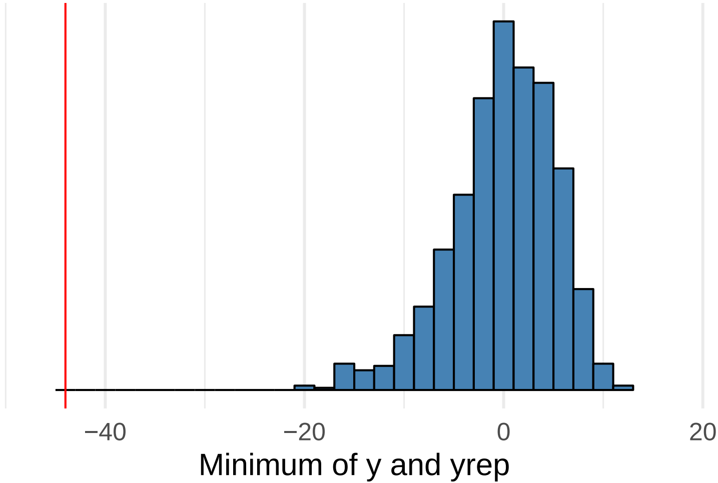

Posterior predictive checking – example

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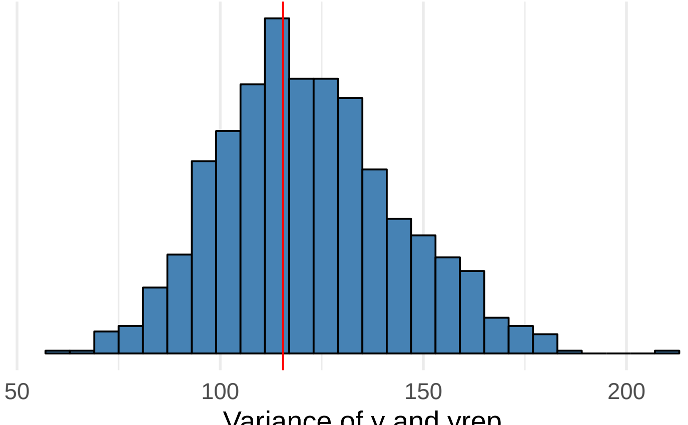
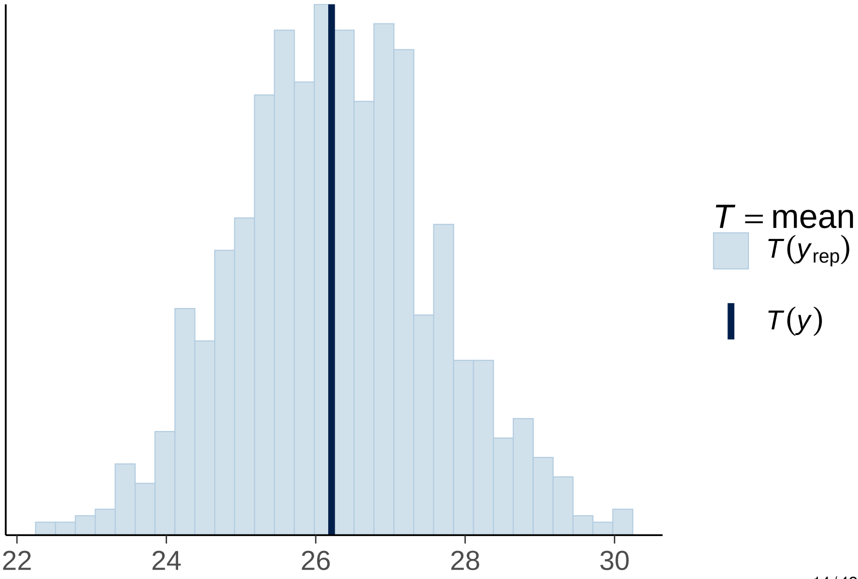


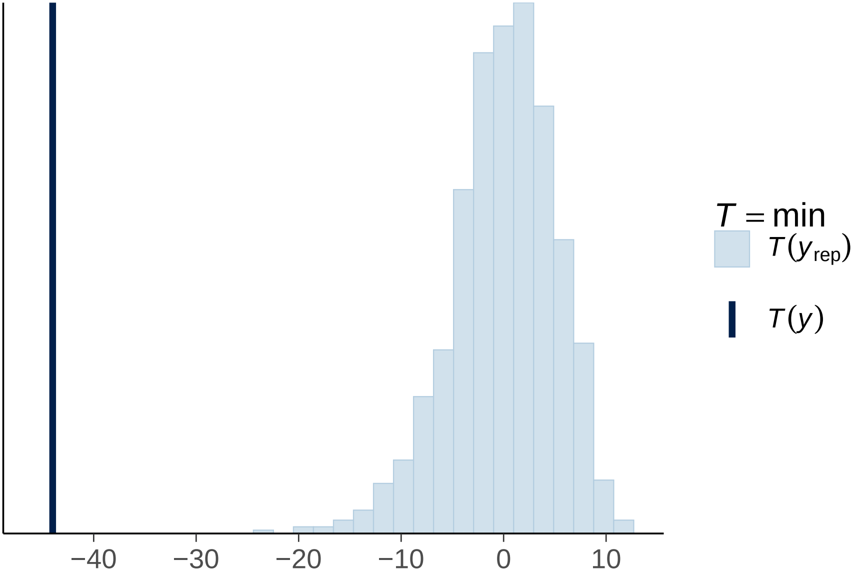
Posterior predictive checking – bayesplot

`ppc_stat(y, yrep)`, the default statistic "mean" is usually bad



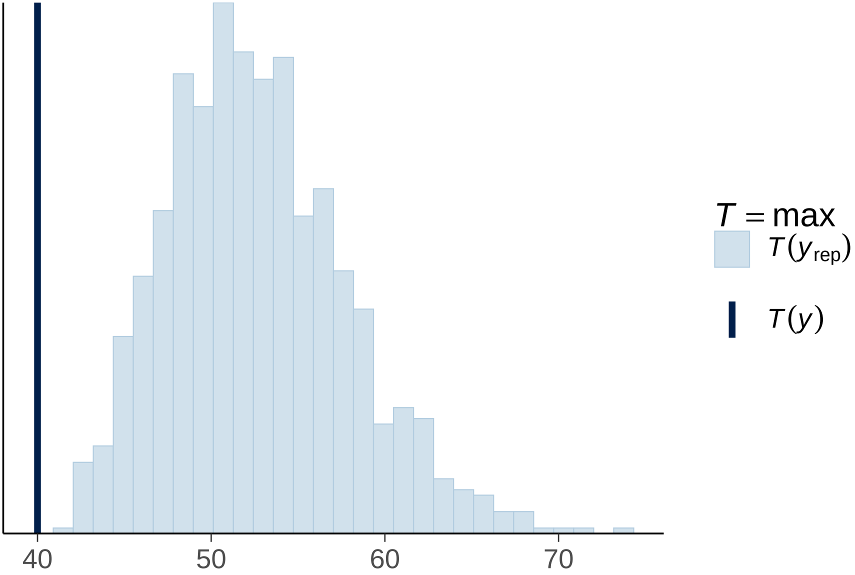
Posterior predictive checking – bayesplot

```
ppc_stat(y, yrep, stat="min")
```



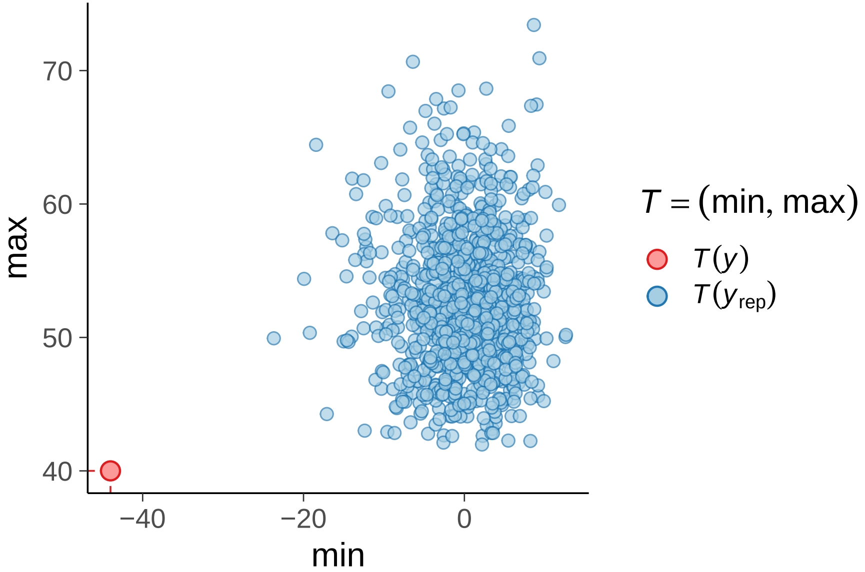
Posterior predictive checking – bayesplot

```
ppc_stat(y, yrep, stat="max")
```



Posterior predictive checking – bayesplot

```
ppc_stat2d(y, yrep, stat=c("min","max"))
```



Posterior predictive checking

- *Posterior predictive p-value*

$$
\begin{aligned}
p &= \Pr(T(y^{\text{rep}}, \theta) \geq T(y, \theta) | y) \\
&= \int \int I_{T(y^{\text{rep}}, \theta) \geq T(y, \theta)} p(y^{\text{rep}} | \theta) p(\theta | y) dy^{\text{rep}} d\theta
\end{aligned}
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where I is an indicator function

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- having $(y^{\mathrm{rep}\,(s)}, \theta^{(s)})$ from the posterior predictive distribution, easy to compute

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T(y^{\mathrm{rep}(s)}, \theta^{(s)}) \geq T(y, \theta^{(s)}), \quad s = 1, \ldots, S
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- Posterior predictive p-value (ppp-value) estimates whether difference between the model and data could arise by chance
- Not commonly used, as
  - the distribution of test statistic has more information
  - not calibrated in case of non-ancillary statistic

Posterior predictive checking – example

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- There can be cases where after the first model fit, it is clear that some data are clearly incorrectly measured, but there has to be really strong justifications for dropping out observations

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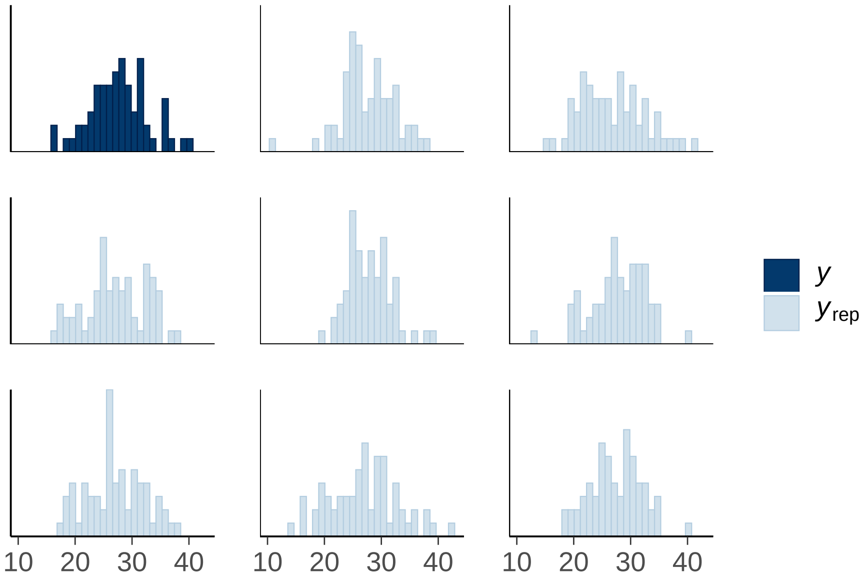
- There can be cases where after the first model fit, it is clear that some data are clearly incorrectly measured, but there has to be really strong justifications for dropping out observations
- Let's assume that in Newcomb experiment, the two observations with negative values are clearly incorrect and let's refit by removing them

Posterior predictive checking (good fit)

```
ppc_hist(y2, y2rep[1:8,])
```



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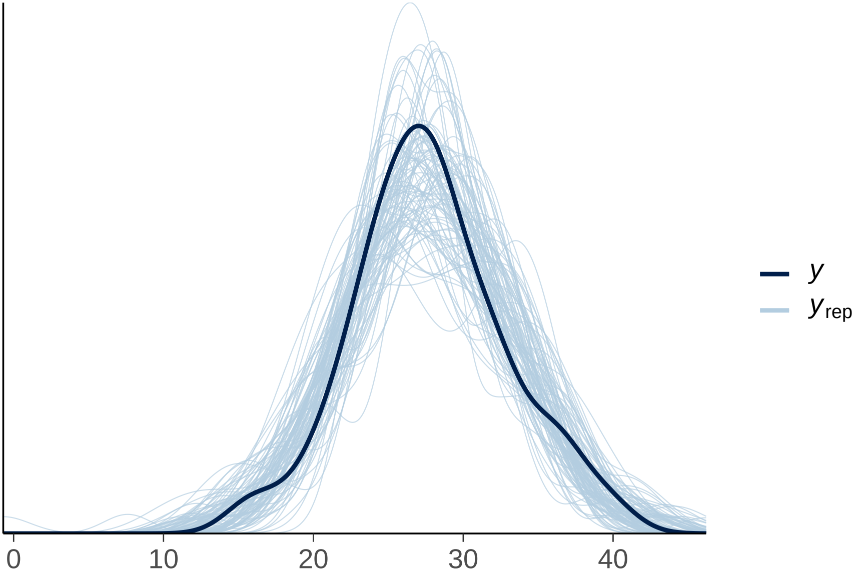

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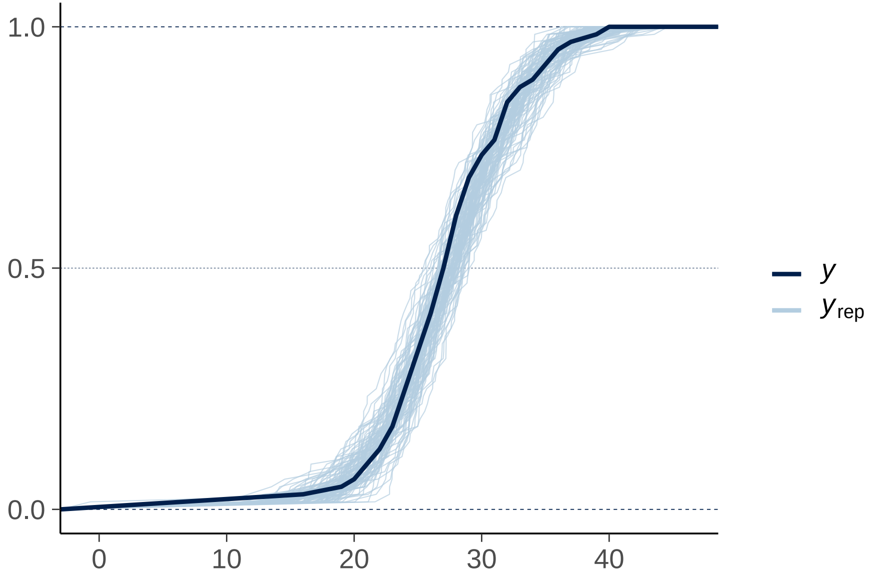

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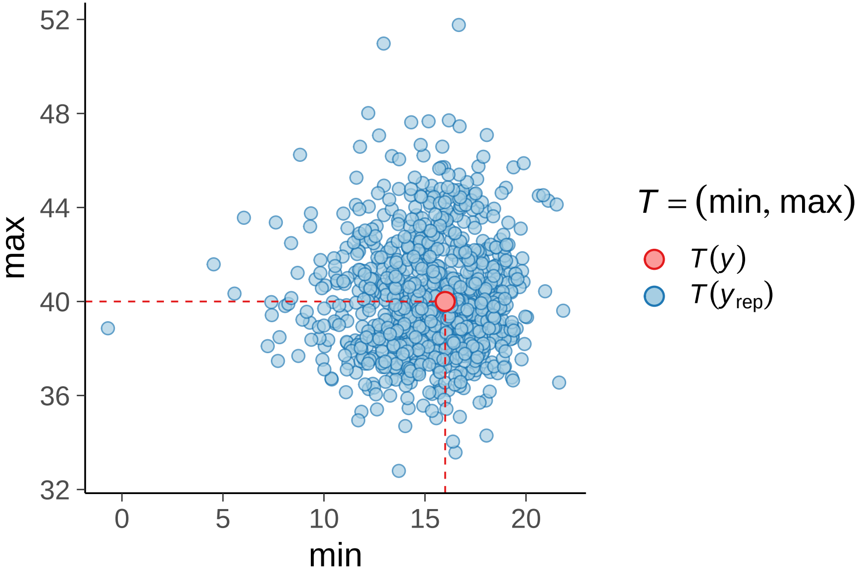

Marginal and CV predictive checking

- Consider marginal predictive distributions $p(\tilde{y}_i|y)$ and each observation separately
  - marginal posterior p-values

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if $T(y_i) = y_i$

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- if $Pr(\tilde{y}_i|y)$ well calibrated, distribution of p_i would be uniform between 0 and 1
  - holds better for cross-validation predictive tests (cross-validation BDA3 Ch 7)

Marginal predictive checking - Example

- Marginal tail area or Probability integral transform (PIT)
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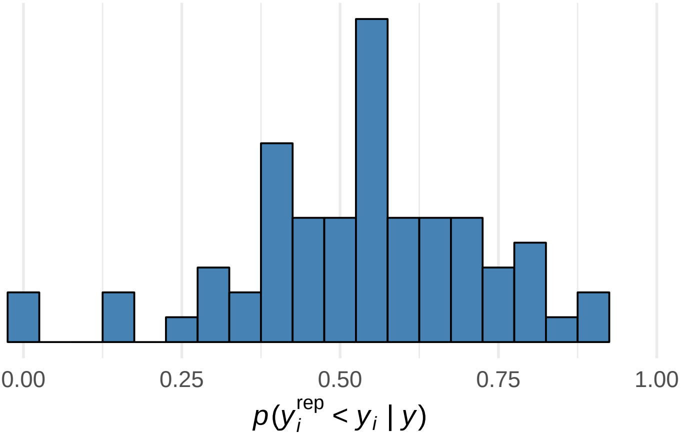

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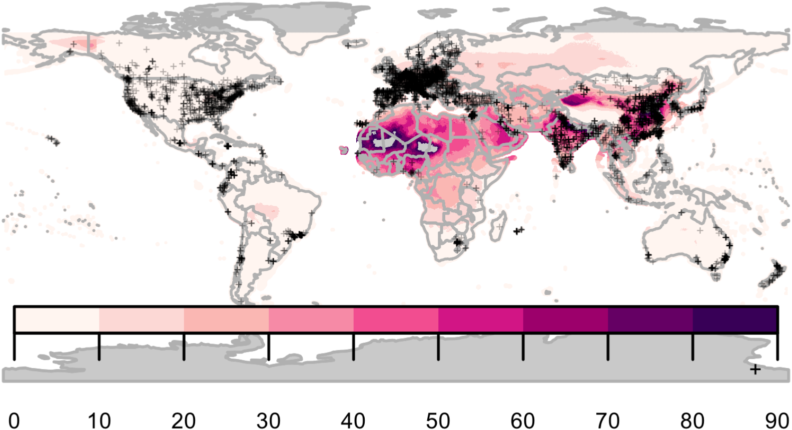
  - robust models are good for testing sensitivity to "outliers"
    - e.g. t instead of Gaussian
- Compare sensitivity of essential inference quantities
  - extreme quantiles are more sensitive than means and medians
  - extrapolation is more sensitive than interpolation

Example: Exposure to air pollution

- Example from Jonah Gabry, Daniel Simpson, Aki Vehtari, Michael Betancourt, and Andrew Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378
- Estimation of human exposure to air pollution from particulate matter measuring less than 2.5 microns in diameter ($\mathrm{PM}_{2.5}$)
  - Exposure to $\mathrm{PM}_{2.5}$ is linked to a number of poor health outcomes and a recent report estimated that $\mathrm{PM}_{2.5}$ is responsible for three million deaths worldwide each year (Shaddick et al., 2017)
  - In order to estimate the public health effect of ambient $\mathrm{PM}_{2.5}$, we need a good estimate of the $\mathrm{PM}_{2.5}$ concentration at the same spatial resolution as our population estimates.

Example: Exposure to air pollution

- Direct measurements of PM 2.5 from ground monitors at 2980 locations
- High-resolution satellite data of aerosol optical depth



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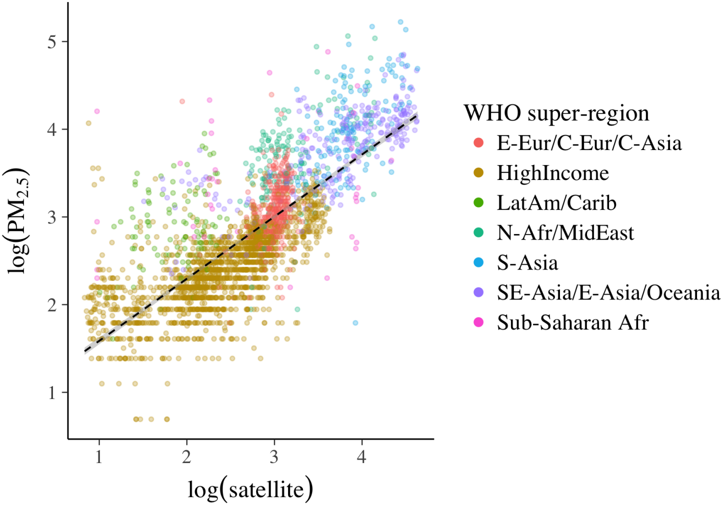

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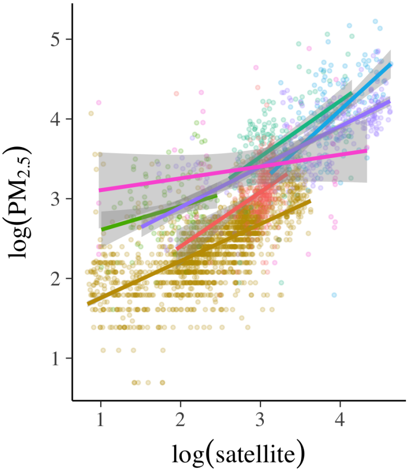
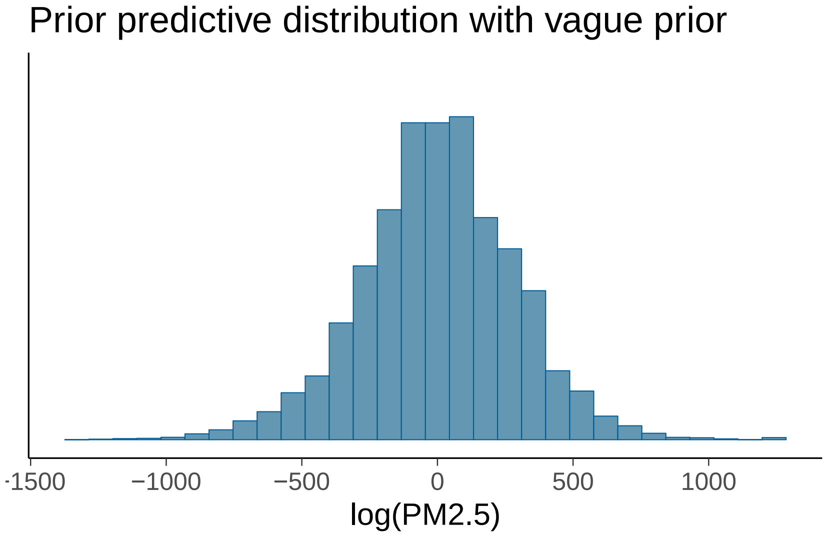


Example: Exposure to air pollution

Prior predictive checking



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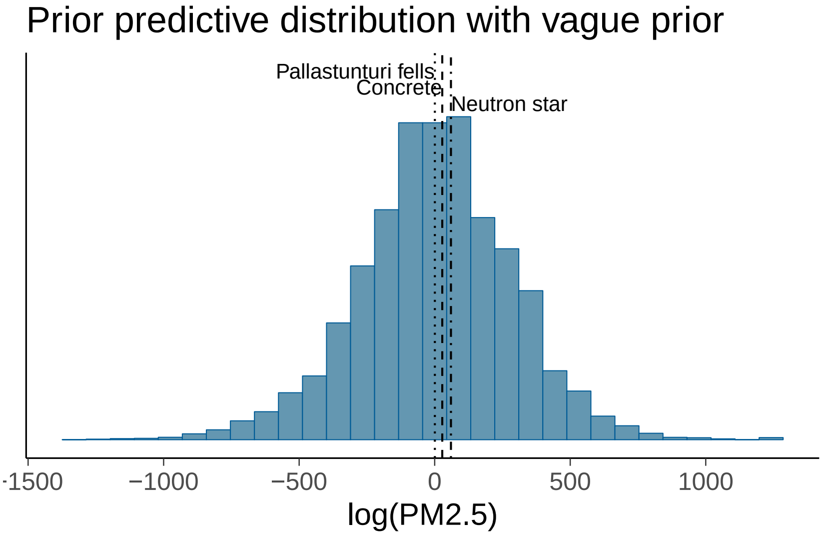
Prior predictive checking



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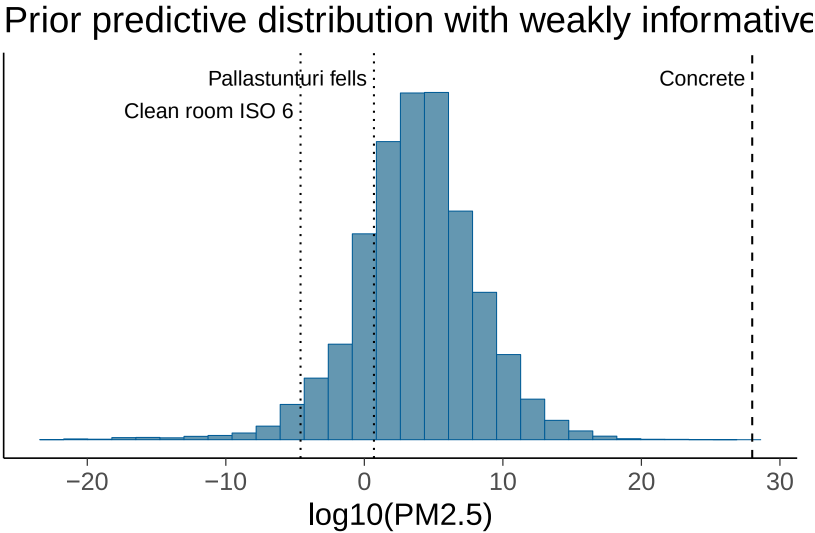
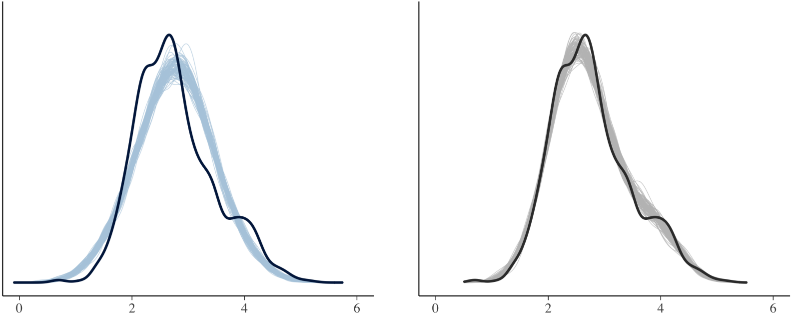


Example: Exposure to air pollution

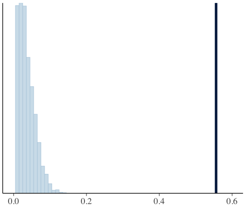
Posterior predictive checking – marginal predictive distributions



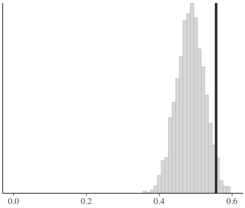
(a) Model 1 (b) Model 2

Example: Exposure to air pollution

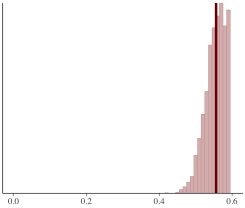
Posterior predictive checking – test statistic (skewness)



(a) Model 1



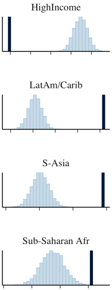
(b) Model 2



(c) Model 3

Example: Exposure to air pollution

Posterior predictive checking – test statistic (median for groups)



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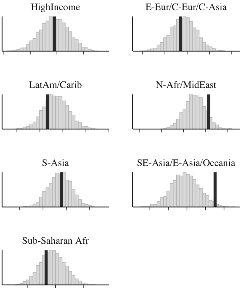
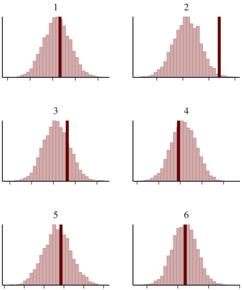


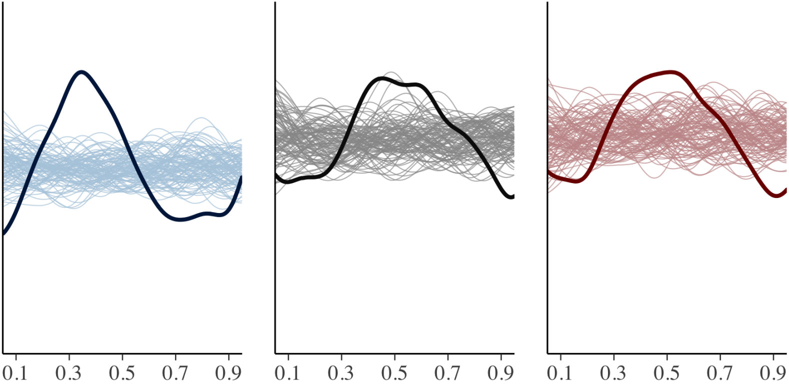
(b) Model 2



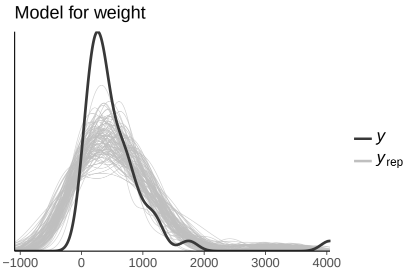
(c) Model 3

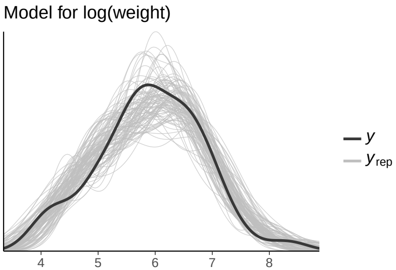
Example: Exposure to air pollution

LOO predictive checking – LOO-PIT



PPC for binary target



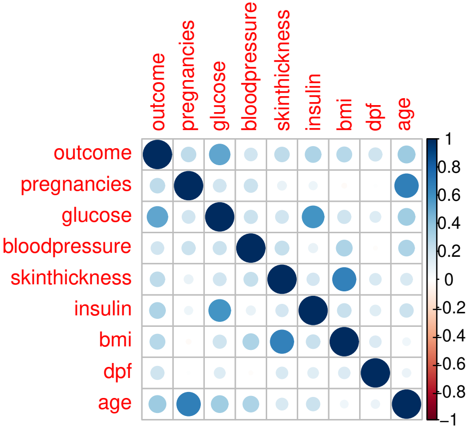


Predicting the yields of mesquite bushes.

Gelman, Hill & Vehtari (2020): Regression and Other Stories, Chapter 11.

PPC for binary target

Diabetes prediction with logistic regression - diabetes demo



PPC for binary target

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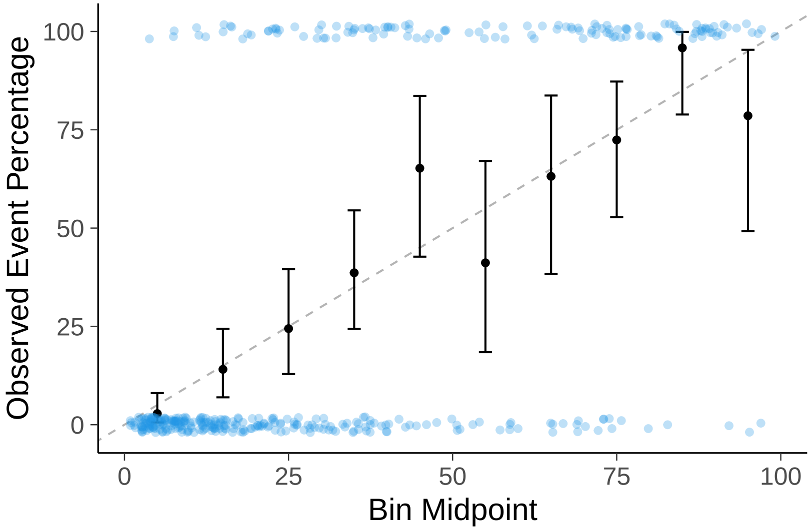
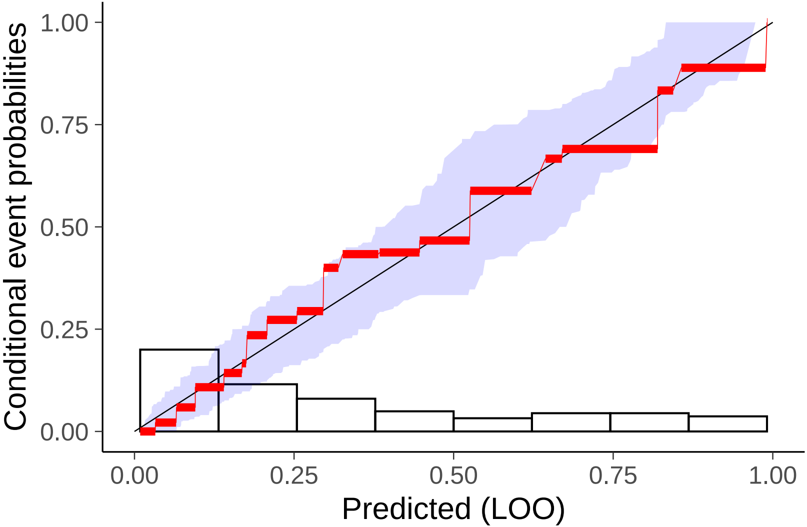
PPC with binning for binary data



PPC for binary target

Diabetes prediction with logistic regression - diabetes demo

PPC with monotonic regression for binary data



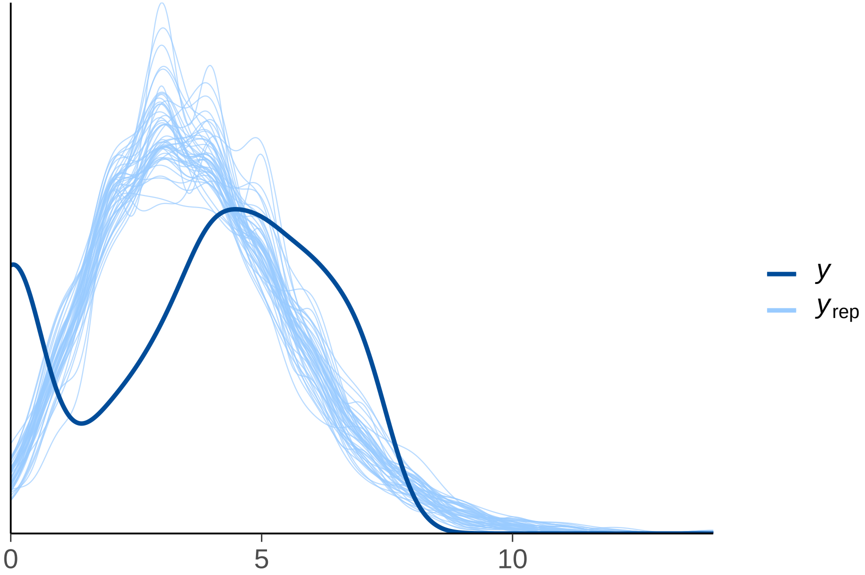
Posterior predictive checking

- demo demos_rstan/ppc/poisson-ppc.Rmd

```
data {
  int<lower=1> N;
  int<lower=0> y[N];
}
parameters {
  real<lower=0> lambda;
}
model {
  lambda ~ exponential(0.2);
  y ~ poisson(lambda);
}
generated quantities {
  real log_lik[N];
  int y_rep[N];
  for (n in 1:N) {
    y_rep[n] = poisson_rng(lambda);
    log_lik[n] = poisson_lpmf(y[n] | lambda);
    }
}
```

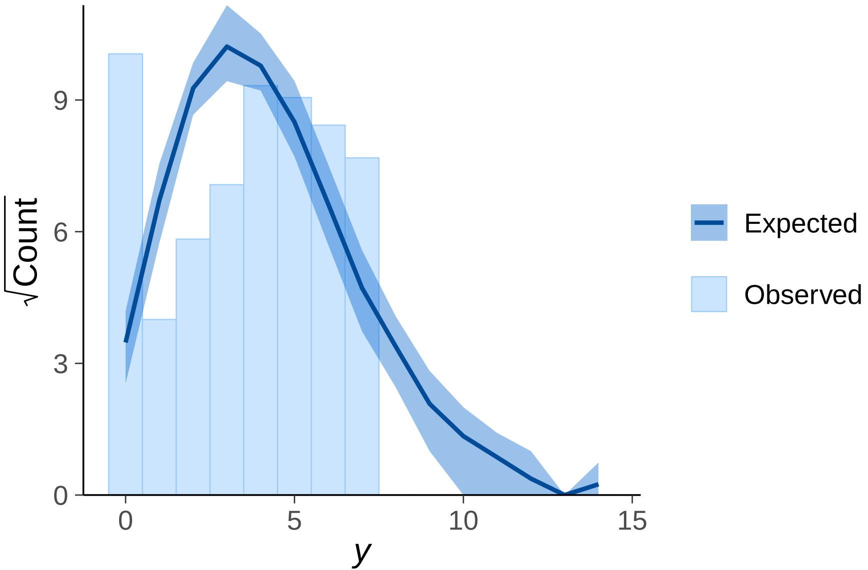
PPC for count data – Poisson model

```
ppc_dens_overlay(y, yrep[1:50,])
```



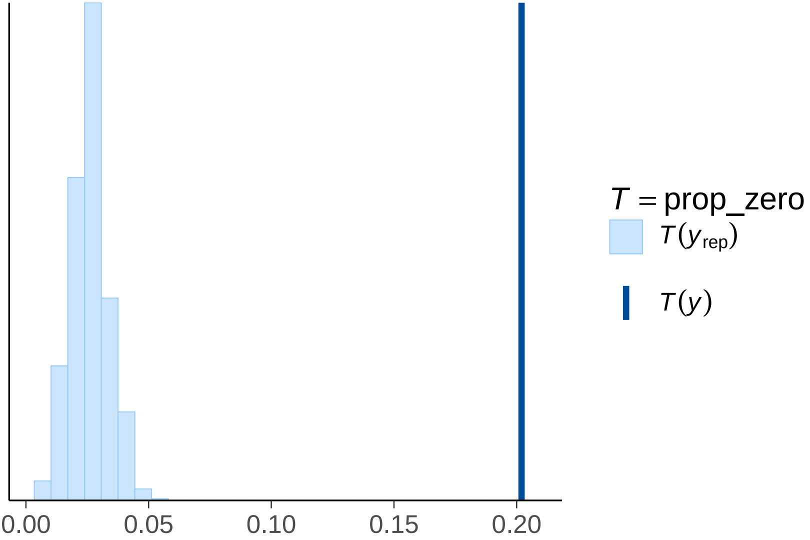
PPC for count data – Poisson model

```
ppc_rootogram(y, yrep)
```



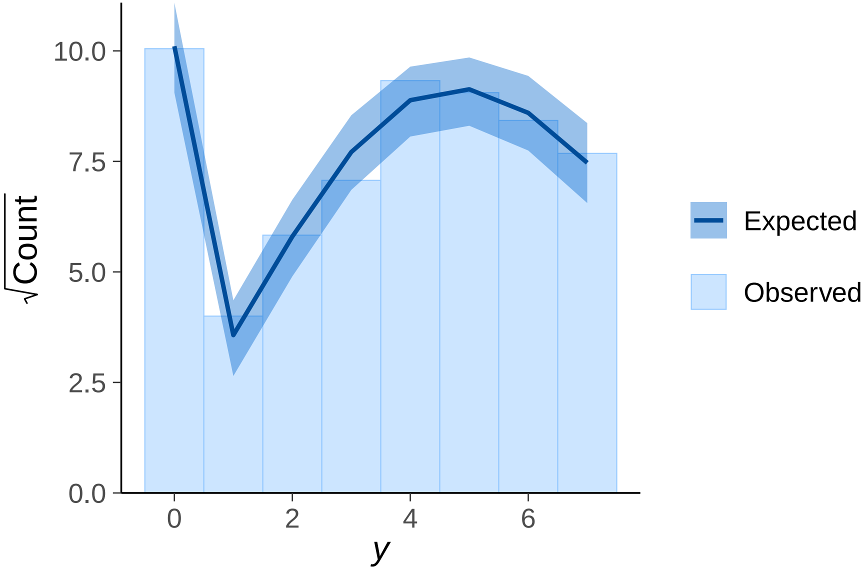
PPC for count data – Poisson model

```
prop_zero <- function(x) mean(x == 0)
ppc_stat(y, yrep, stat = "prop_zero")
```



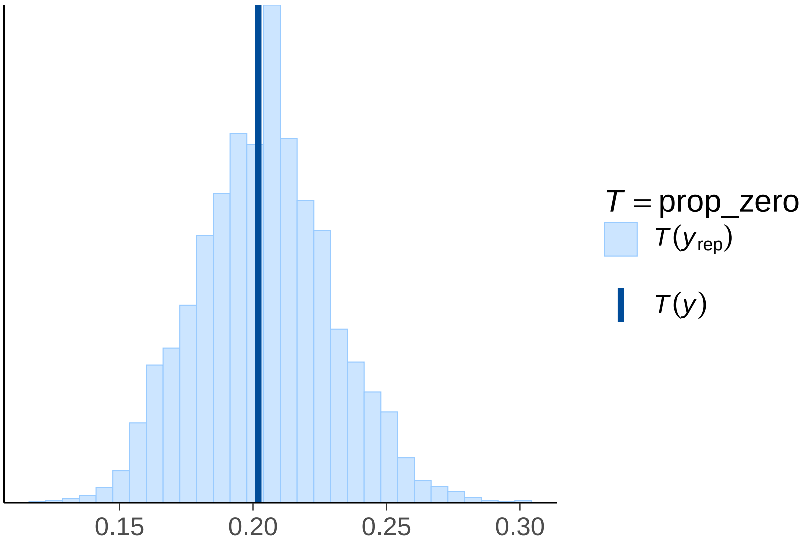
PPC for count data – hurdle truncated Poisson model

```
ppc_rootogram(y, yrep2)
```



PPC for count data – hurdle truncated Poisson model

```
prop_zero <- function(x) mean(x == 0)
ppc_stat(y, yrep2, stat = "prop_zero")
```



Further reading and examples

- Gabry, Simpson, Vehtari, Betancourt, and Gelman (2019). Visualization in Bayesian workflow. https://doi.org/10.1111/rssa.12378.
- Graphical posterior predictive checks using the bayesplot package
http://mc-stan.org/bayesplot/articles/graphical-ppcs.html
- Another demo demos_rstan/ppc/poisson-ppc.Rmd