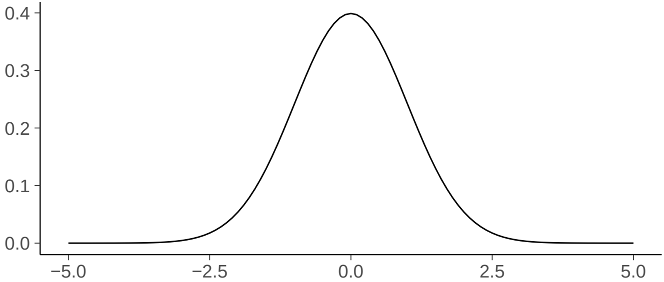
# Chapter 3

- 3.1 Marginalization
- 3.2 Normal distribution with a noninformative prior (important)
- 3.3 Normal distribution with a conjugate prior (important)
- 3.4 Multinomial model (can be skipped)
- 3.5 Multivariate normal with known variance (useful for chapter 4)
- 3.6 Multivariate normal with unknown variance (glance through)
- 3.7 Bioassay example (very important, related to one of the exercises)
- 3.8 Summary (summary)

# Normal / Gaussian

observation $y$, and parameters $\mu$ and $\sigma$

$$p(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$



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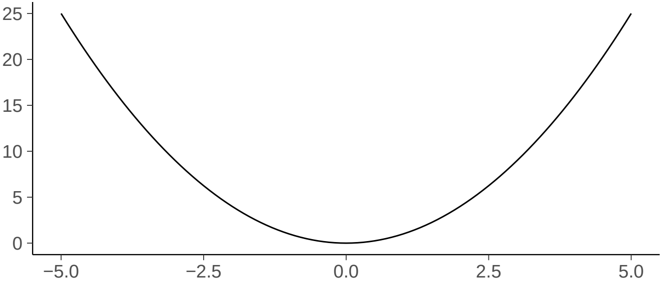

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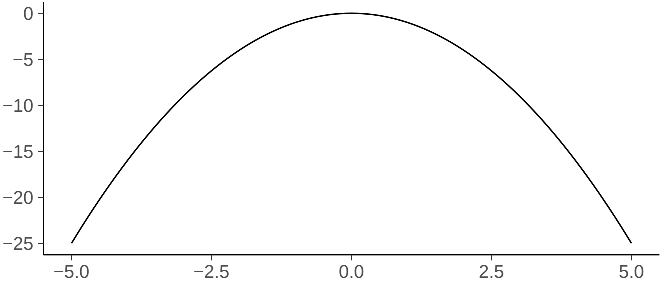

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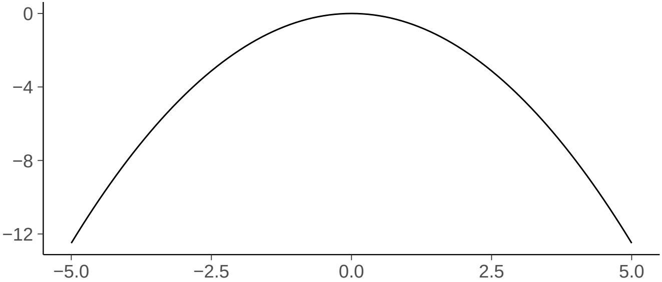

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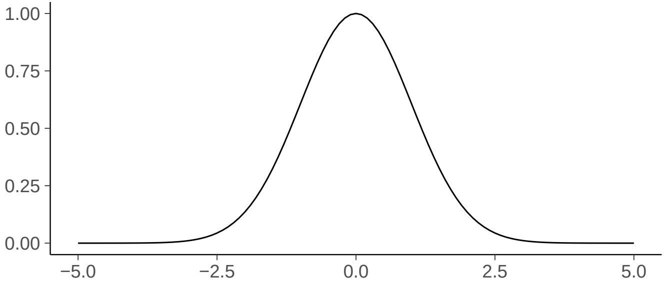

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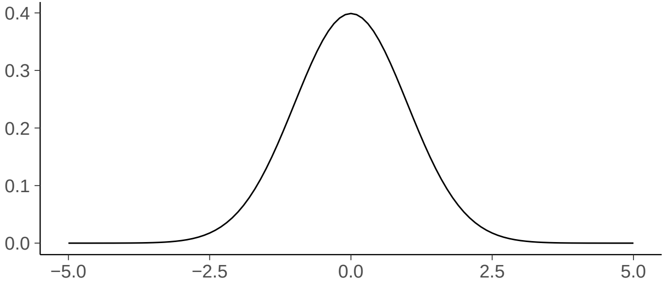

# Normal / Gaussian

observation $y$, and parameters $\mu = 1$ and $\sigma = 1$

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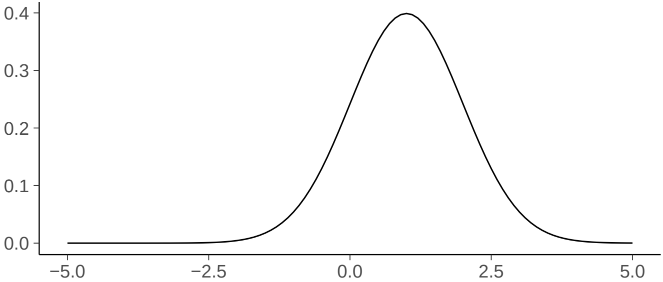

# Normal / Gaussian

observation $y$, and parameters $\mu = 1$ and $\sigma = 1/2$

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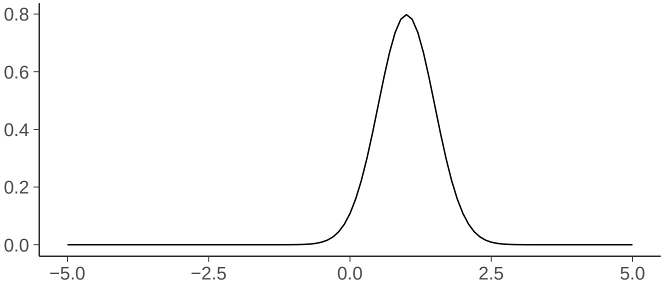
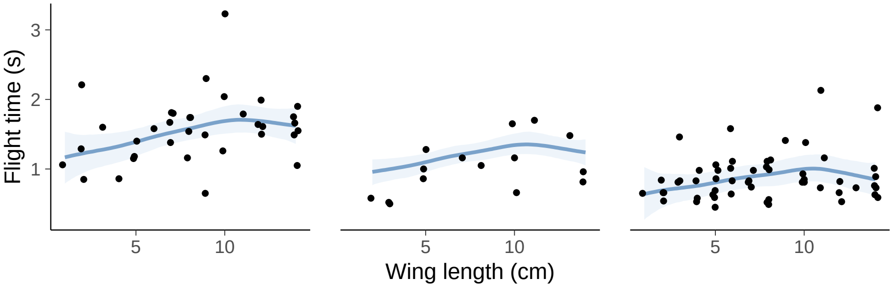


# Variation in helicopter flight times



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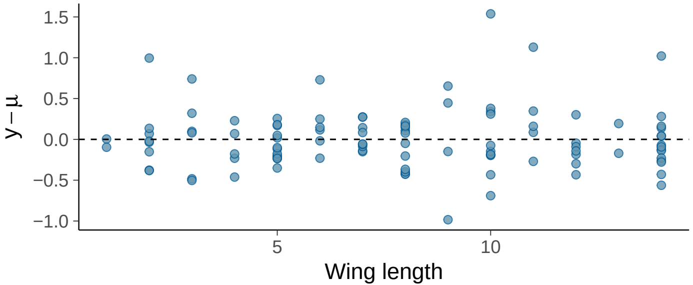

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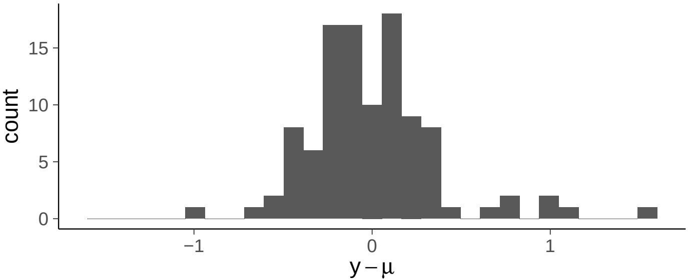

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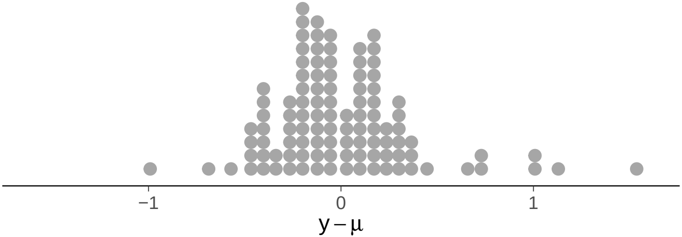

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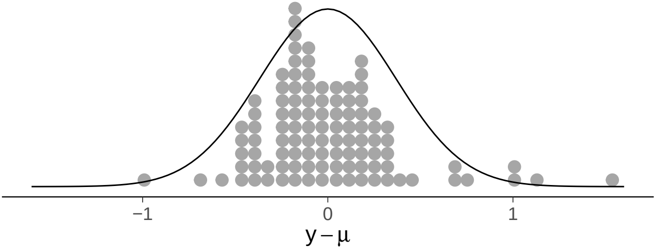

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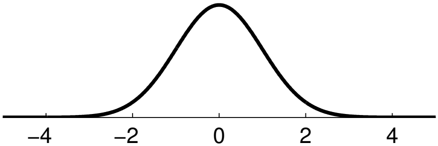


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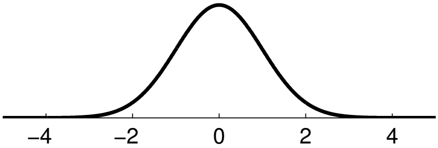


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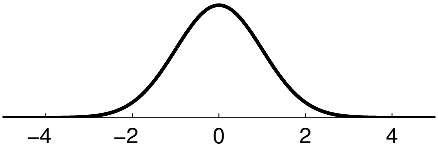


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- Shorthand notation
  - $y \sim \mathrm{N}(\mu, \sigma^2)$ with variance $\sigma^2$ (useful in derivations)
  - $y \sim \mathrm{normal}(\mu, \sigma)$ with deviation $\sigma$ (useful for interpreting prior and posterior scales, used in Stan)

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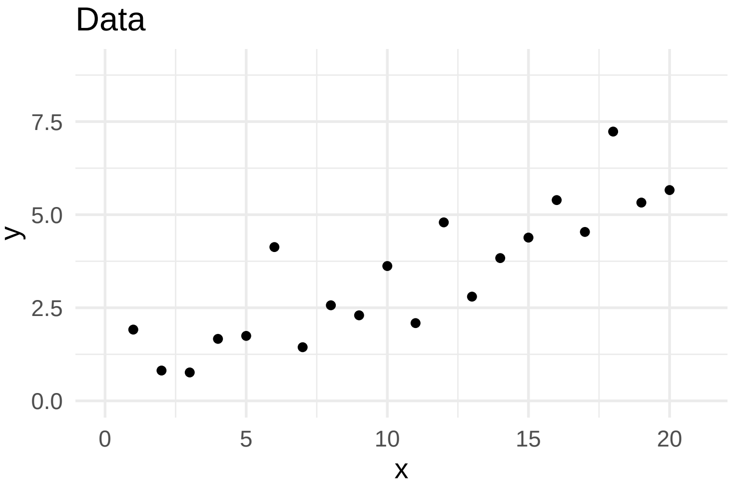
- Posterior distribution approximation with Laplace, variational inference, expectation propagation

# Example of uncertainty in modeling



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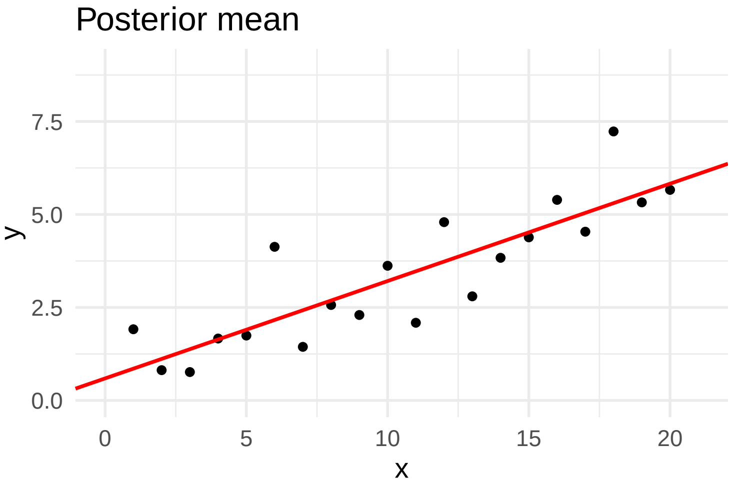

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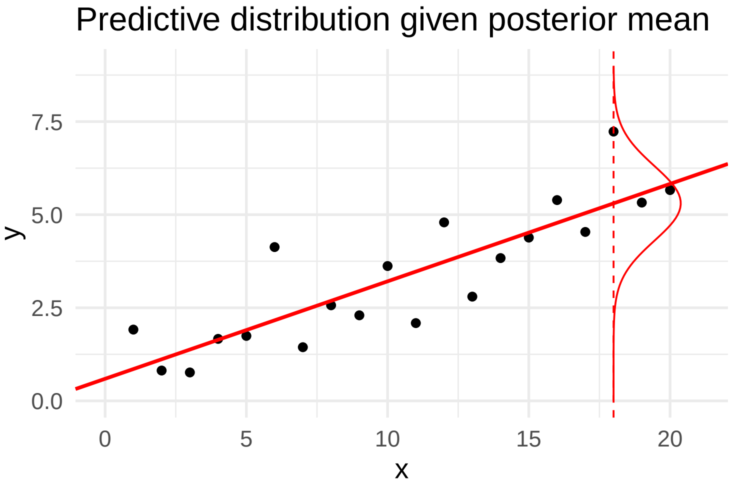

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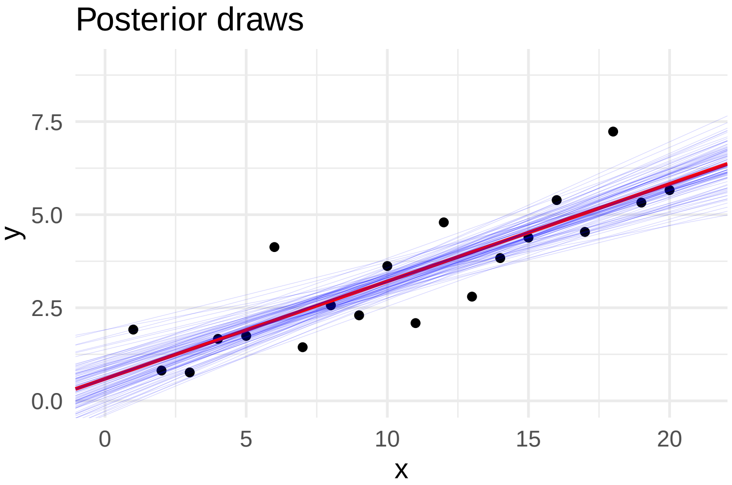

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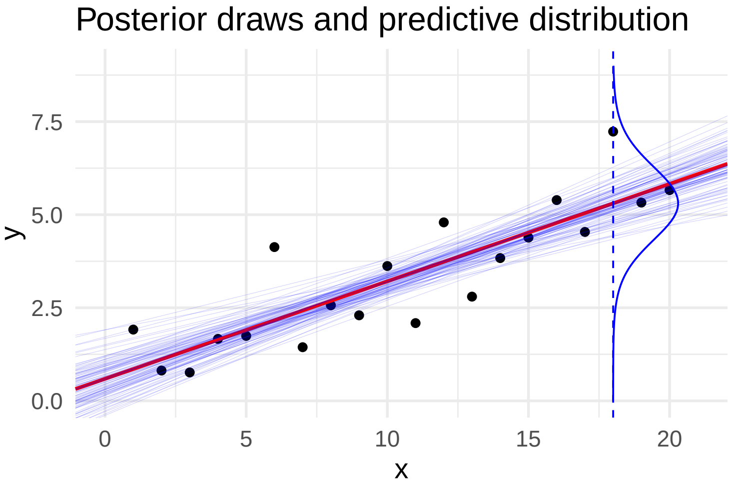

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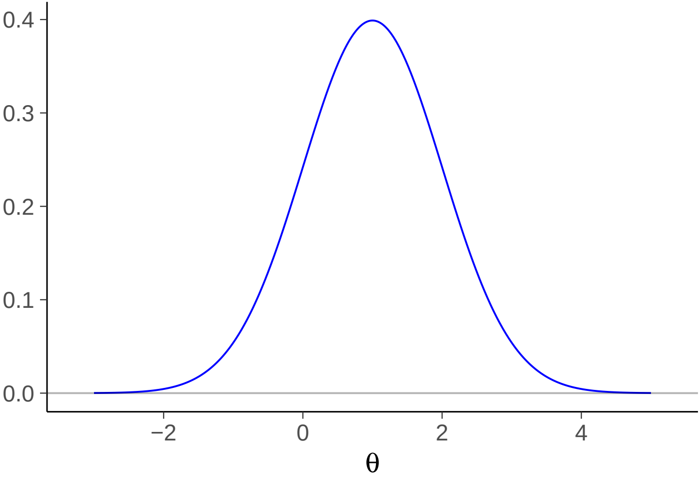
- For some priors there is analytic solution for posterior, and Monte Carlo is a general approach
- Before looking at the posterior $p(\mu, \sigma \mid y)$, connection between mean, cumulative density and Monte Carlo

# Monte Carlo and posterior draws

Density $p(\theta|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(\theta-\mu)^2\right)$ (`dnorm()`)



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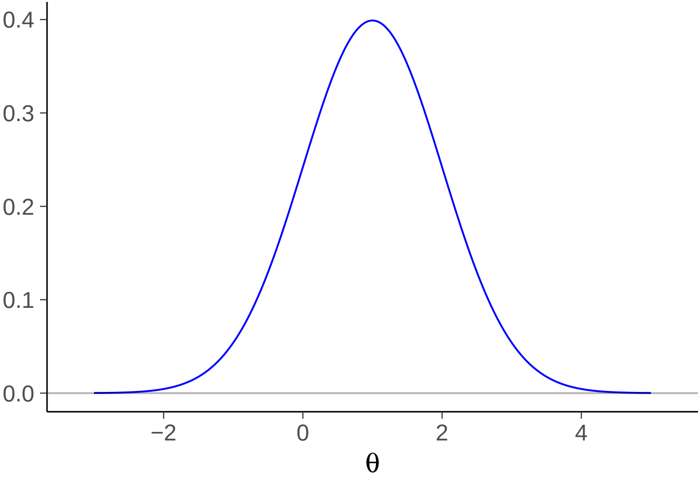


$E(\theta) = \int \theta p(\theta|\mu,\sigma) d\theta = \mu$

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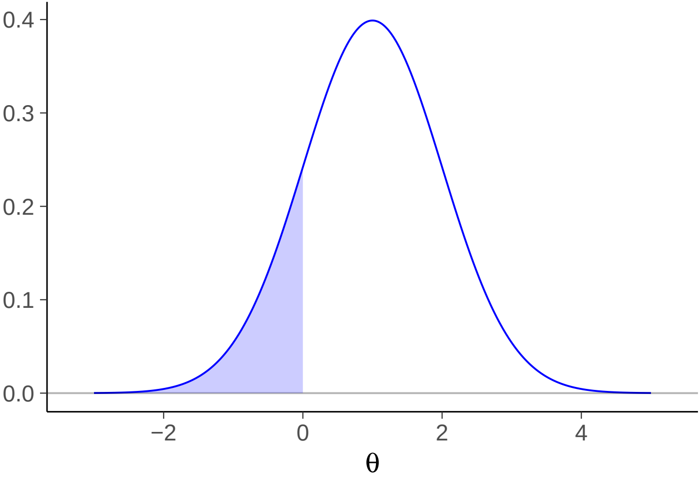
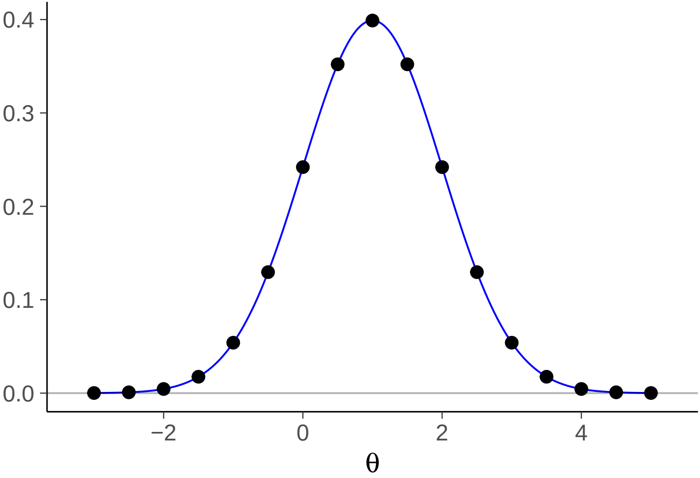



$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta|\mu,\sigma) d\theta$,
many numerical approximations (pnorm())

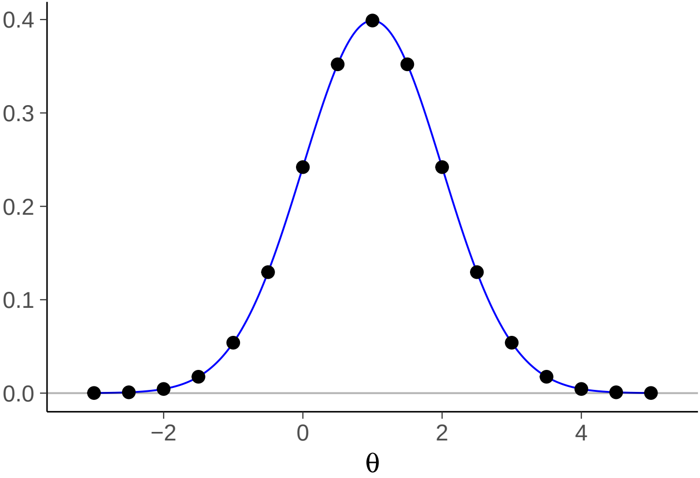
# Monte Carlo and posterior draws

In practice evaluate in finite number of locations (dnorm())



# Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.5



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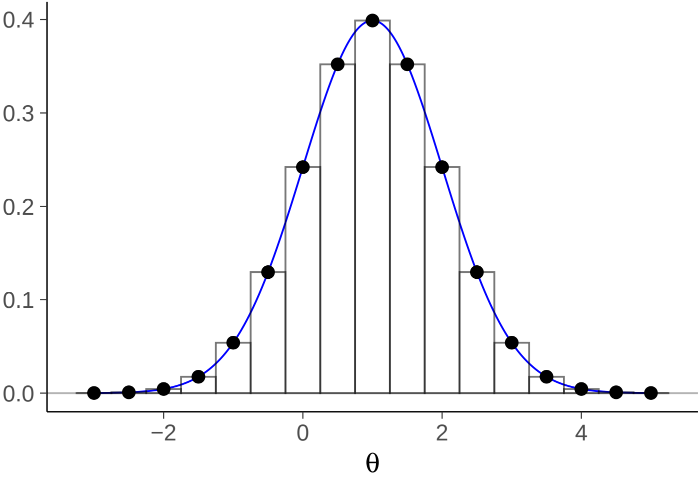

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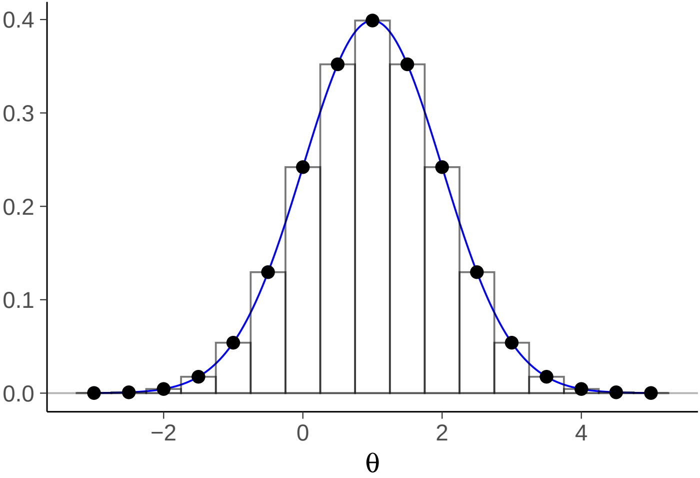


$\mathrm{E}(\theta) = \int \theta p(\theta) d\theta \approx \sum_s^S \theta^{(s)} w_s \approx 1,$ where $w_s = 0.5 p(\theta)$

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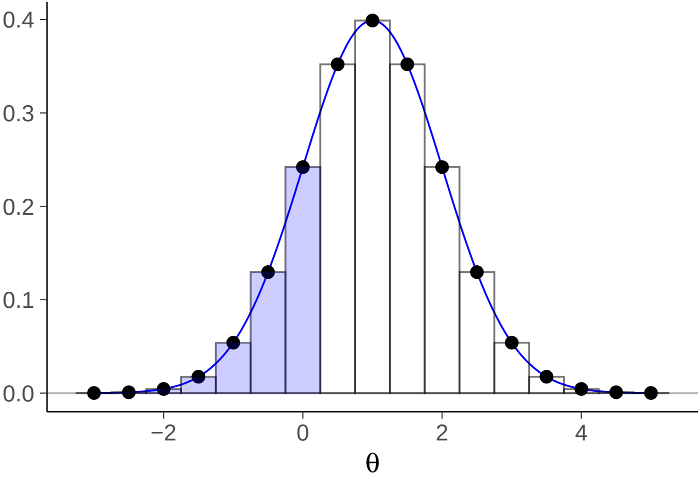
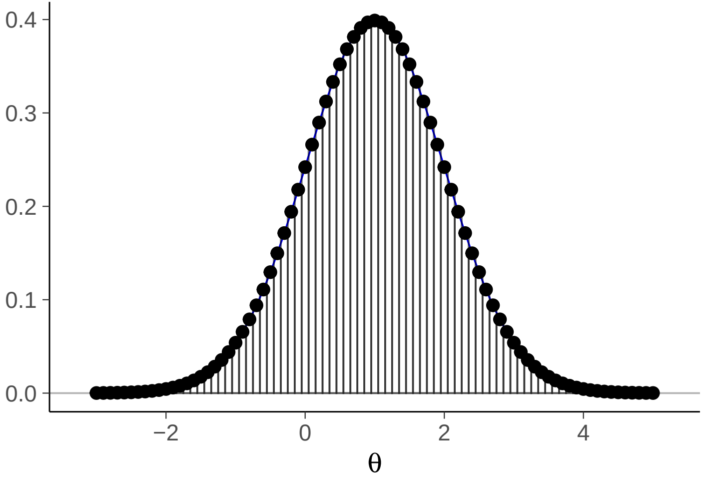



$$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta) d\theta \approx \sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) w_s \approx 0.22$$

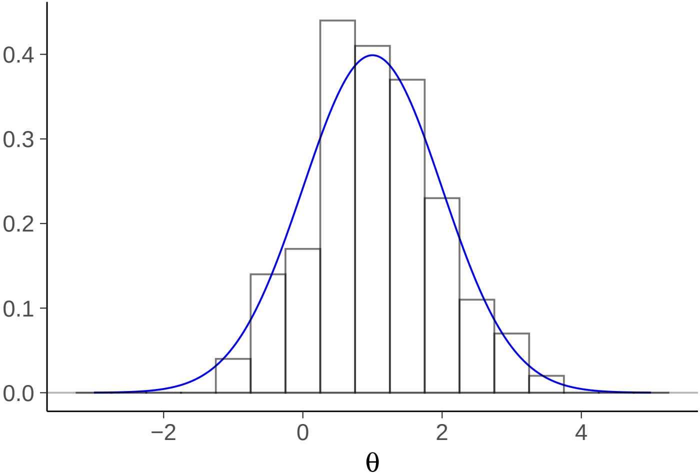
# Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.1



# Monte Carlo and posterior draws

Histogram of 200 random draws (`rnorm()`), bin width 0.5



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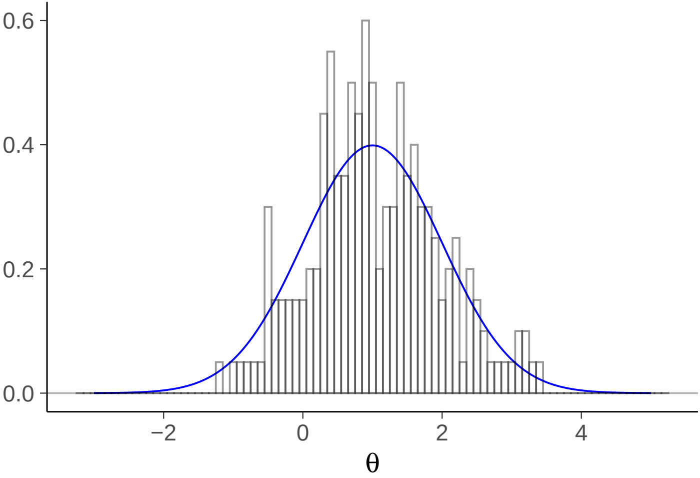

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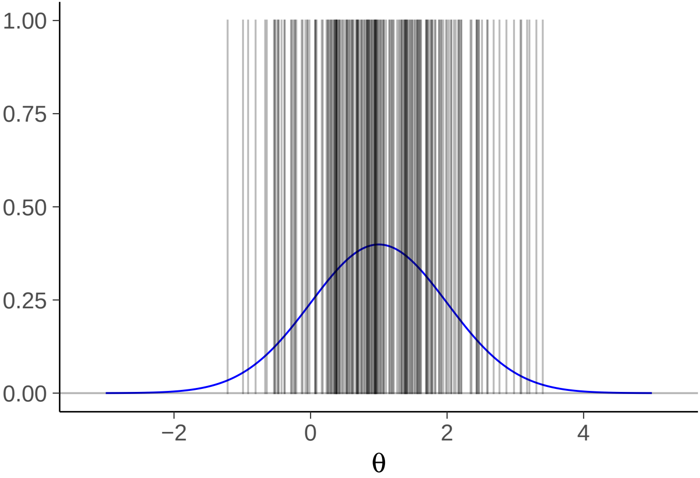


each bin has either 0 or 1 draw (and 0's can be ignored)

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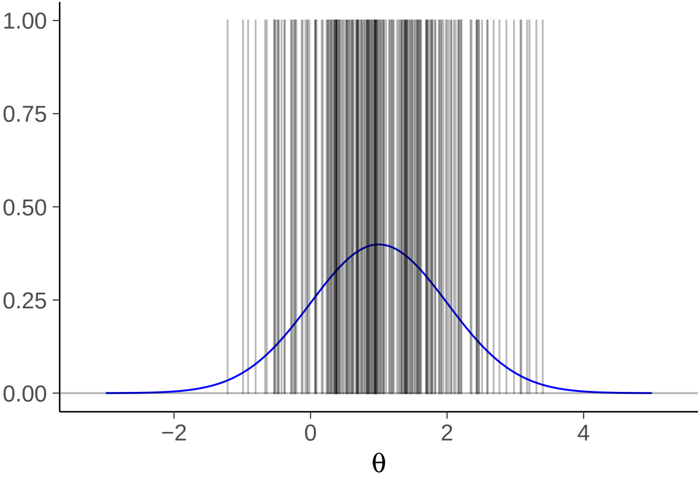


each bin with 1 draw has weight $1/S$

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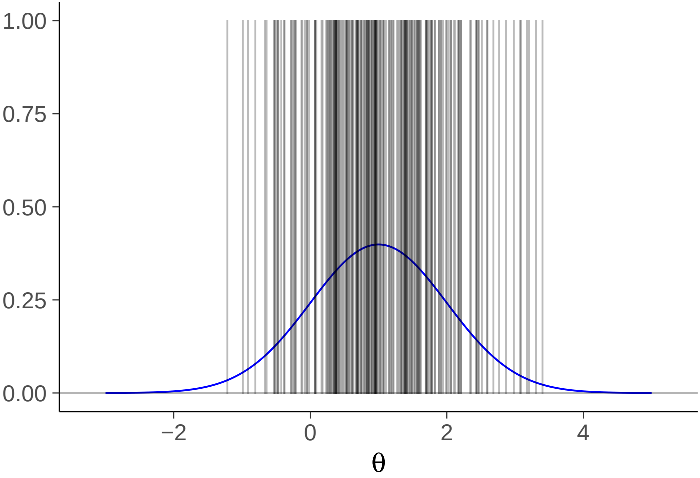


$E(\theta) \approx \frac{1}{S}\sum_{s}^{S} \theta^{(s)} \approx 1$

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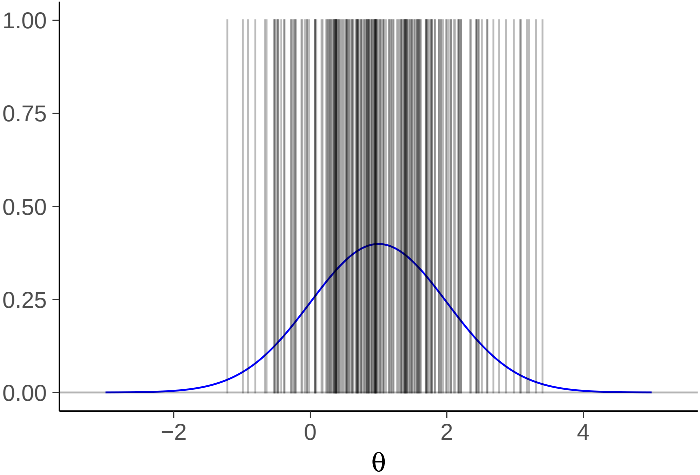


$E(\theta) \approx \frac{1}{S}\sum_{s}^{S}\theta^{(s)} \approx 1$, Monte Carlo estimate

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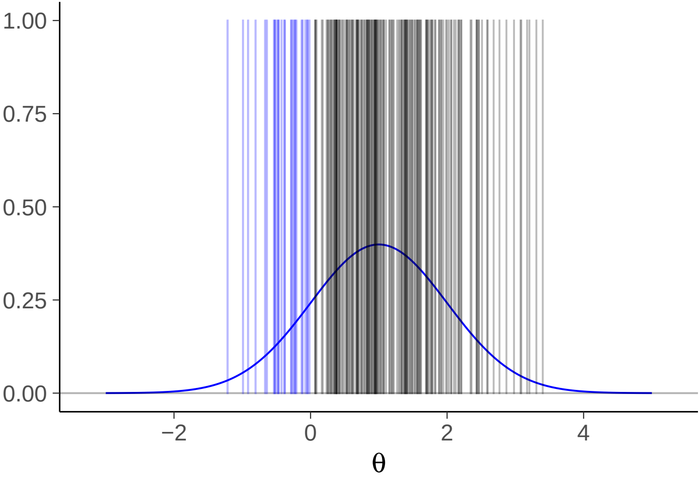


$p(\theta \leq 0) \approx \frac{1}{S}\sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) \approx 0.14$

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- If $p(g(\theta))$ has finite variance, then the Monte Carlo estimate is unbiased and the error approaches 0 with increasing $S$ based on the central limit theorem (CLT)
  - more about this later

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- Monte Carlo approximation

$$\begin{aligned} \text{if} \quad & (\theta_1^{(s)}, \theta_2^{(s)}) \sim p(\theta_1, \theta_2 \mid y) \\ \text{then} \quad & \theta_1^{(s)} \sim p(\theta_1 \mid y) \end{aligned}$$

# Marginalization - predictive distribution

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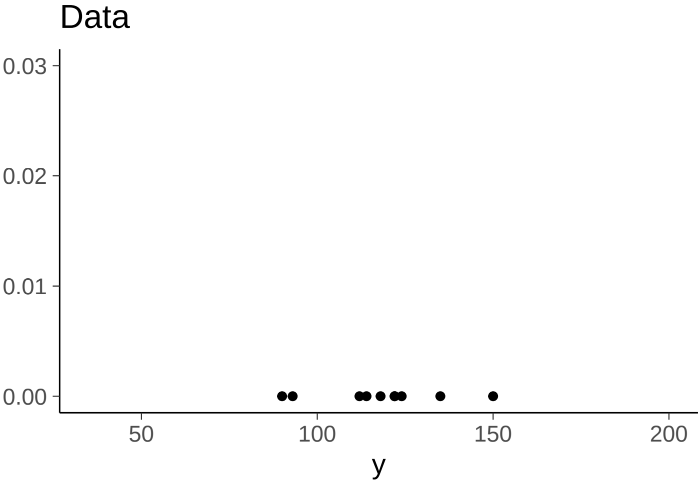
&= \int p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta
\end{aligned}
$$

# Normal distribution example



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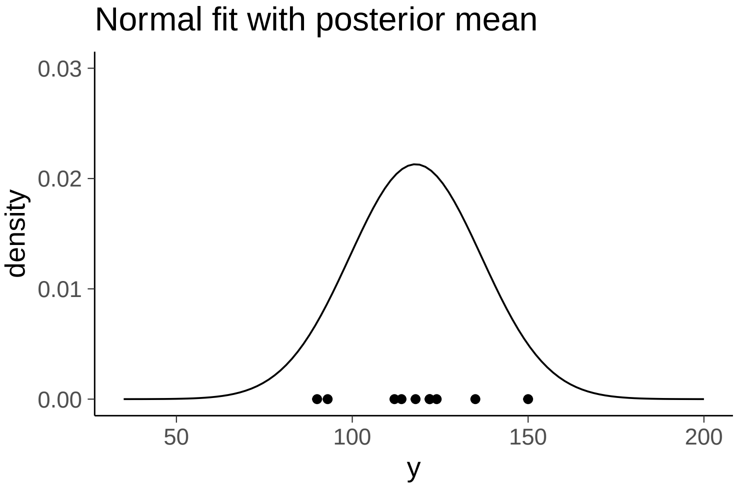
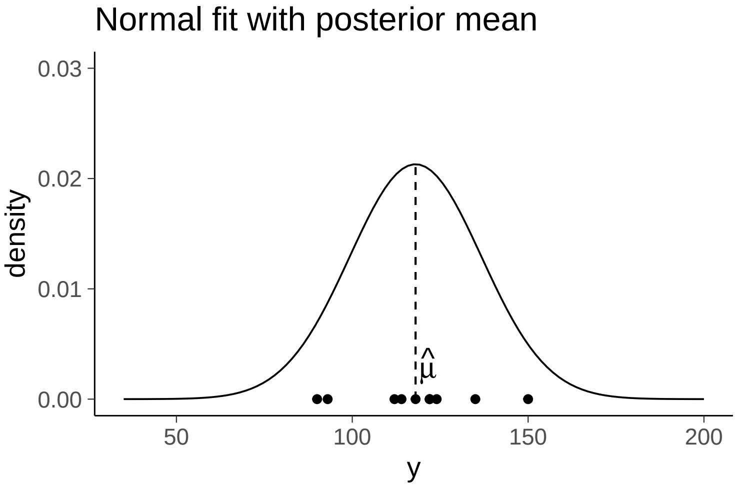


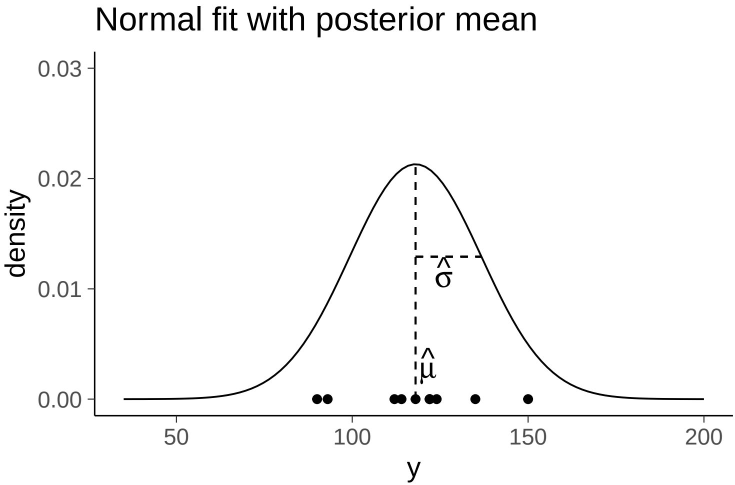

$$p({\color{red}y} \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}({\color{red}y} - \mu)^2\right)$$

# Normal distribution example



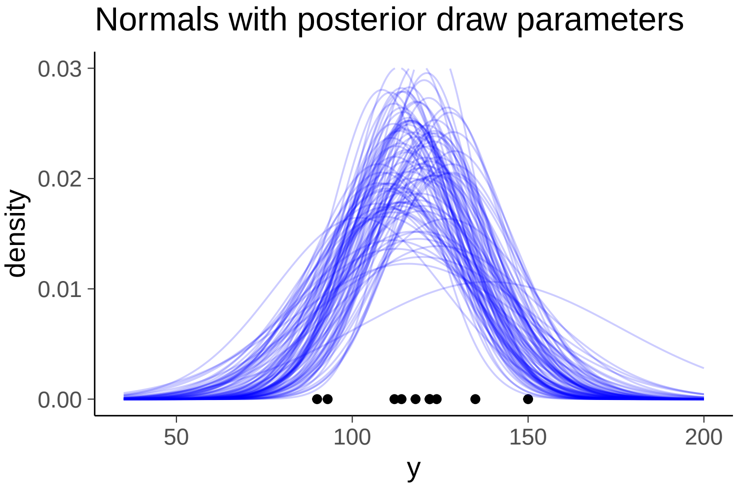

$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

# Normal distribution example



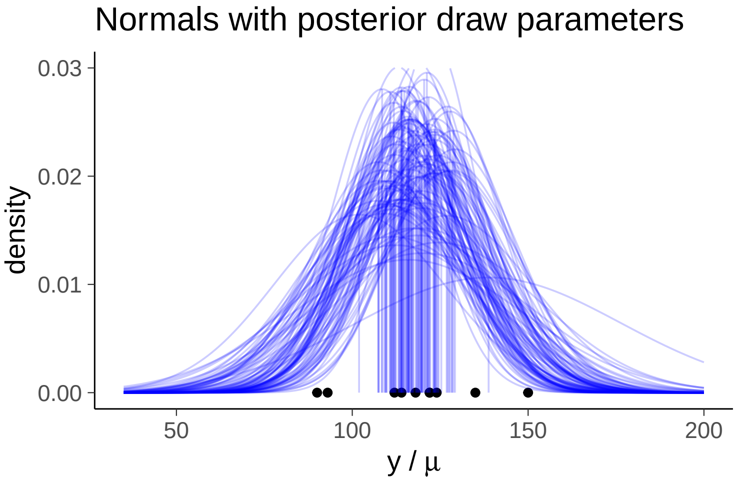

$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

# Normal distribution example



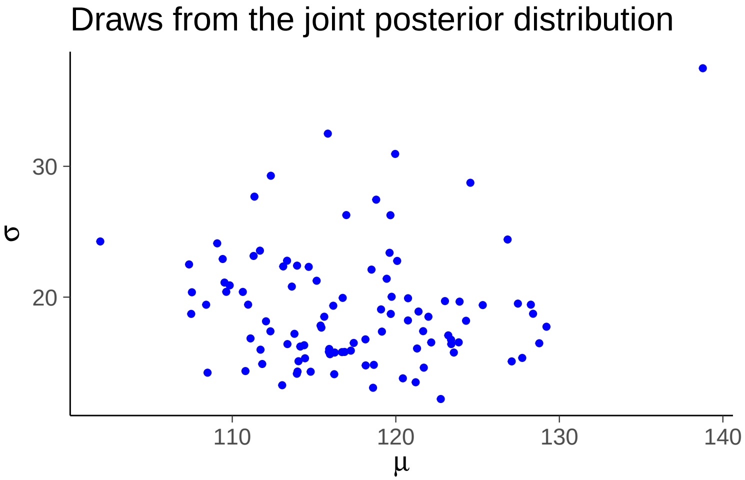

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

# Normal distribution example



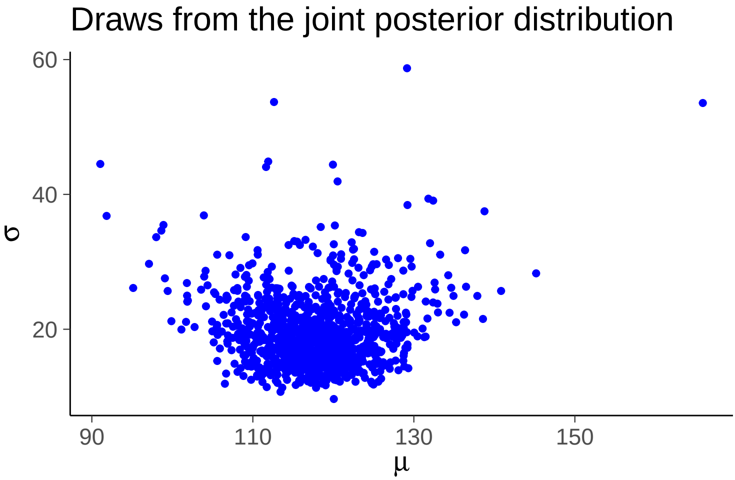

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

# Normal distribution example




$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

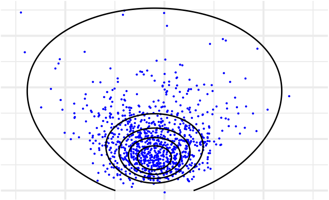
# Normal distribution example




$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior

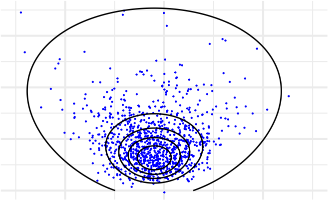
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$



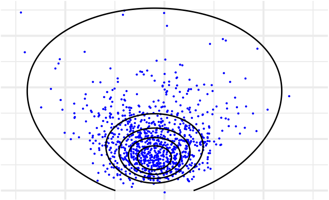
Joint posterior

$$\textcolor{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$



Joint posterior

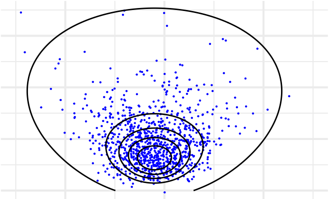


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

with $p(\mu, \sigma) \propto \sigma^{-1}$ (see BDA3 p. 21 transformation of variables)

Joint posterior

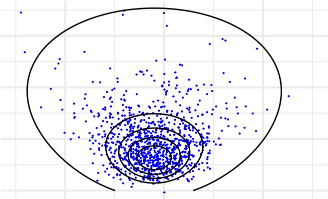


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

Joint posterior

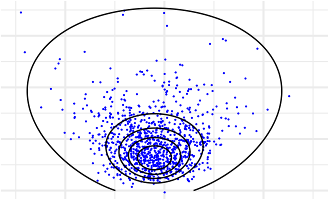


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

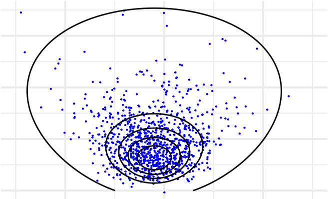
with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

# Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

# Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

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$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

# Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

# Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\sum_{i=1}^{n}(y_i^2-2y_i\mu+\mu^2)$$

$$\sum_{i=1}^{n}(y_i^2-2y_i\mu+\mu^2-\bar{y}^2+\bar{y}^2-2y_i\bar{y}+2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2-2y_i\bar{y}+\bar{y}^2)+\sum_{i=1}^{n}(\mu^2-2y_i\mu-\bar{y}^2+2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i-\bar{y})^2+n(\mu^2-2\bar{y}\mu-\bar{y}^2+2\bar{y}\bar{y})$$

# Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

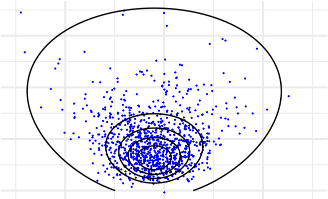
$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$



Joint posterior

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

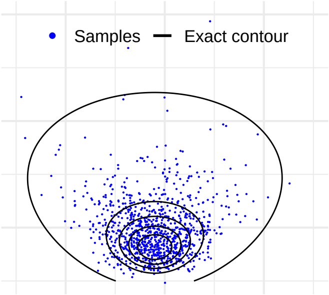
$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

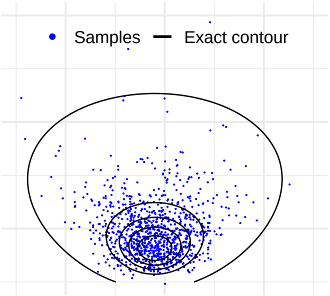
where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

## Joint posterior

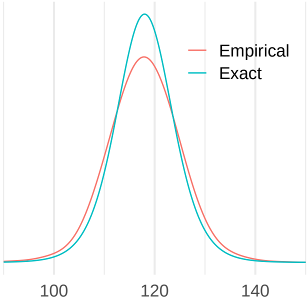



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior




Marginal of mu

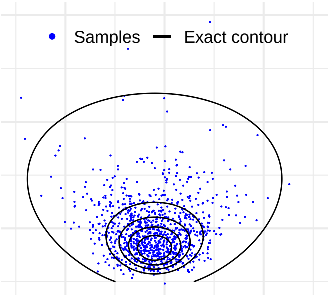



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

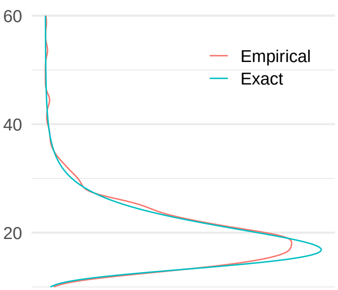
marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

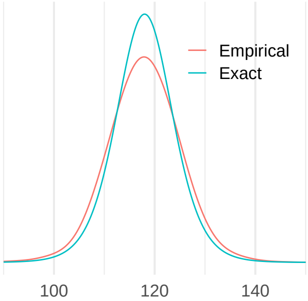
Joint posterior




Marginal of sigma




Marginal of mu




$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$p(\sigma^2 \mid y) \quad \propto \quad \int p(\mu, \sigma^2 \mid y) d\mu$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu
\end{aligned}
$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$
\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1
$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1$$

$$\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n}$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)
\end{aligned}
$$

## Marginal posterior $p(\sigma^2 \mid y)$ (easier for $\sigma^2$ than $\sigma$)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\
p(\sigma^2 \mid y) &= \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)
\end{aligned}
$$

# Normal - non-informative prior

Known mean

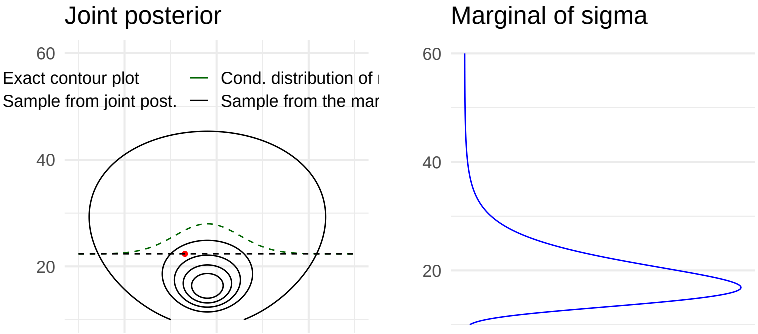
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, v)$$

$$\text{where} \quad v = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta)^2$$

Unknown mean

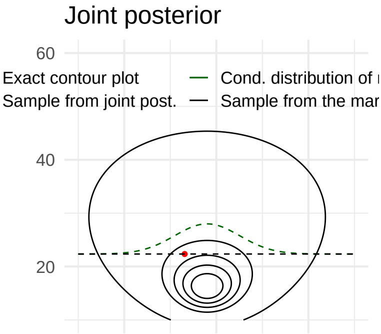
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$

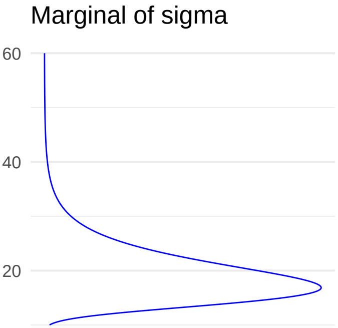
$$\text{where} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$




## Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$



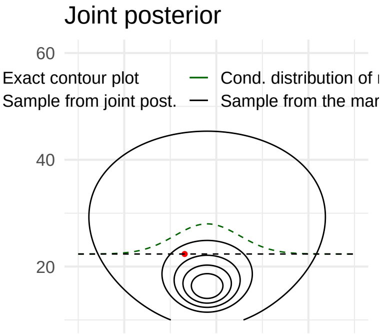



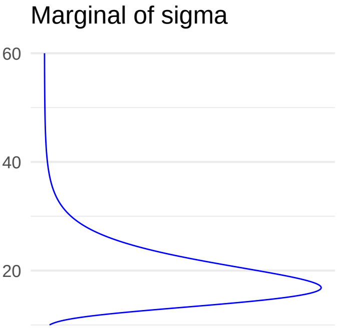

## Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$



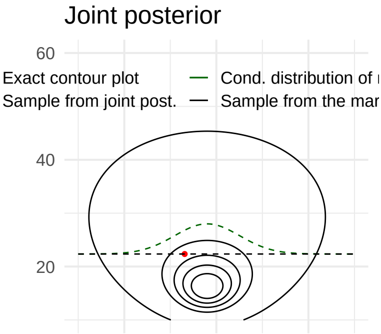

## Factorization

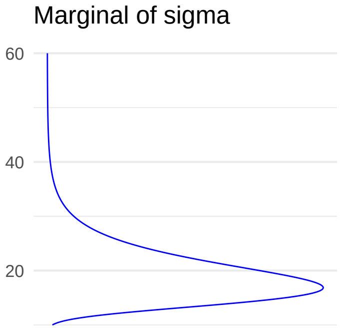
$$p(\mu, \sigma^2 \mid y) = {\color{green}p(\mu \mid \sigma^2, y)}{\color{blue}p(\sigma^2 \mid y)}$$

$${\color{blue}p(\sigma^2 \mid y)} = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim {\color{blue}p(\sigma^2 \mid y)}$$

$${\color{green}p(\mu \mid \sigma^2, y)} = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$



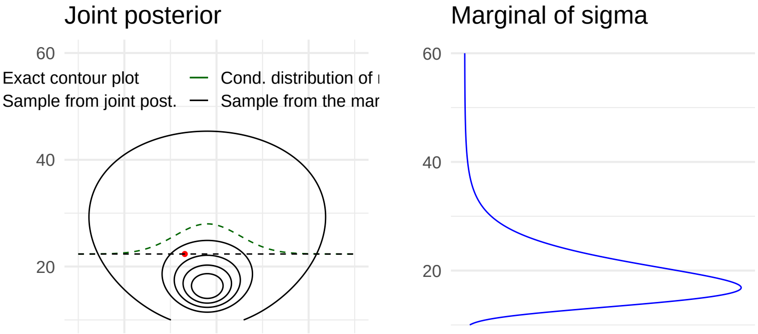

## Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$




## Factorization

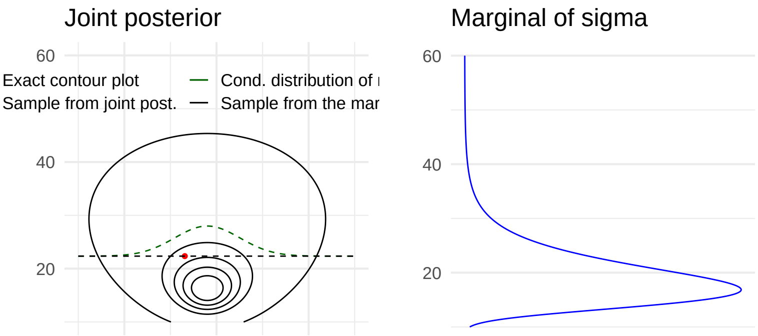
$$p(\mu, \sigma^2 \mid y) = {\color{green} p(\mu \mid \sigma^2, y)}{\color{blue} p(\sigma^2 \mid y)}$$

$${\color{blue} p(\sigma^2 \mid y)} = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim {\color{blue} p(\sigma^2 \mid y)}$$

$${\color{green} p(\mu \mid \sigma^2, y)} = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim {\color{green} p(\mu \mid (\sigma^2)^{(s)}, y)}$$




## Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

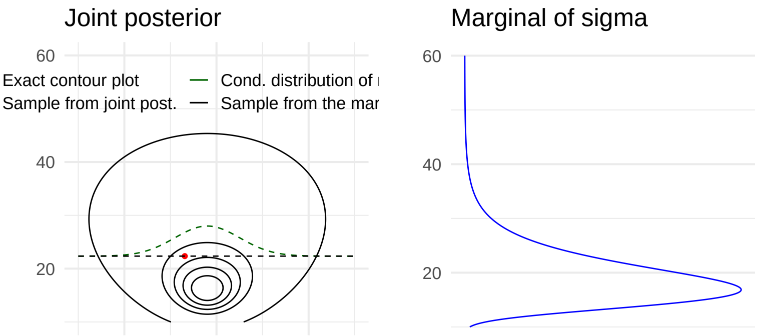
$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

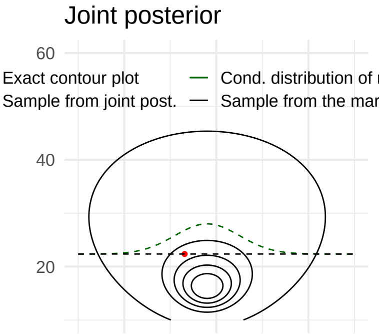
$$\mu^{(s)} \sim p(\mu \mid (\sigma^2)^{(s)}, y)$$

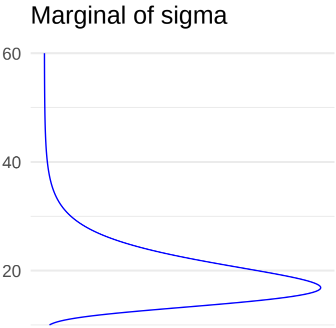
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$




## Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

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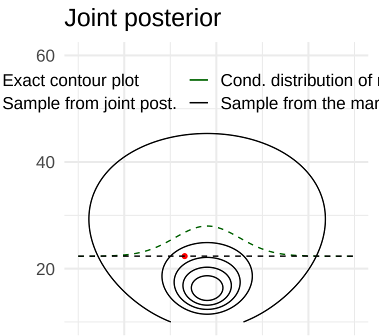


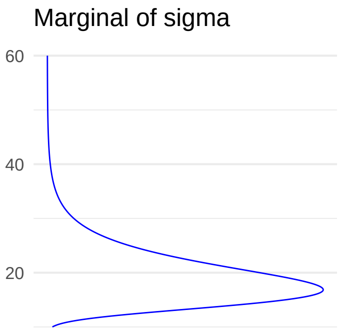


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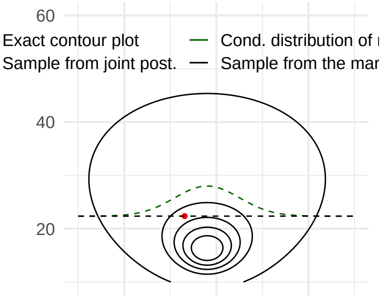
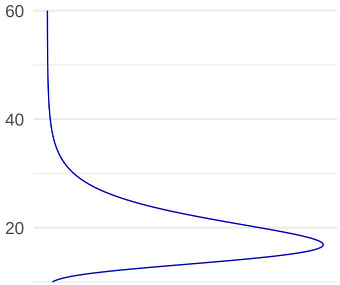
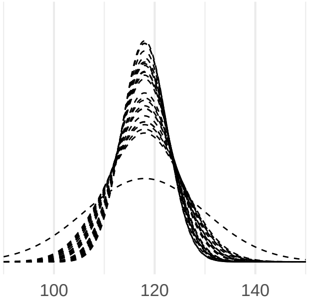
## Joint posterior




## Marginal of sigma




## Cond distr of mu for 25 draws

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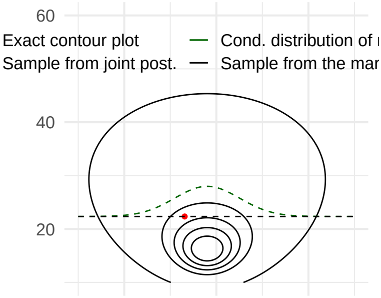


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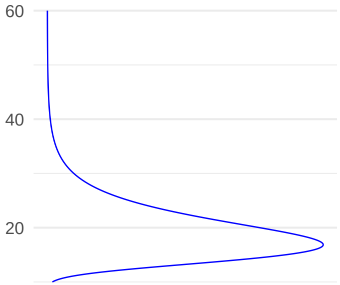


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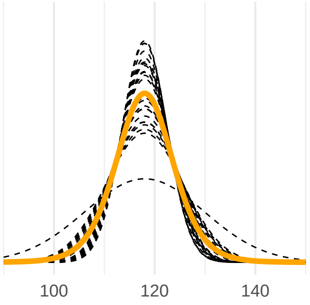


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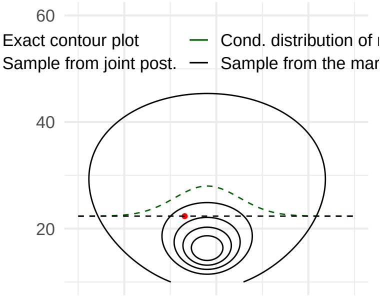


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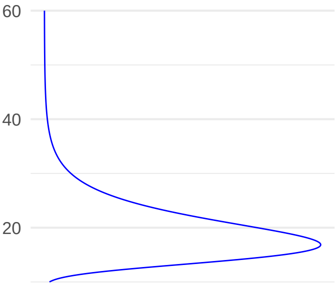


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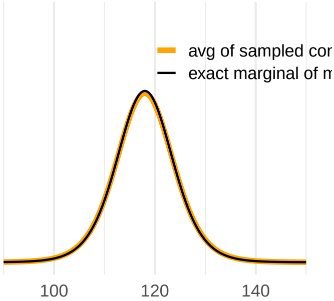


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$$A = (n-1)s^2 + n(\mu - \bar{y})^2$$

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\end{aligned}
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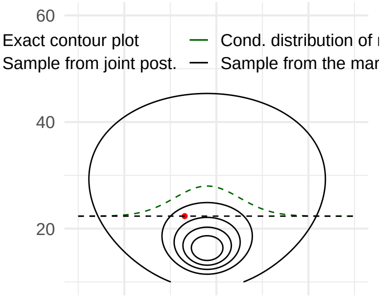
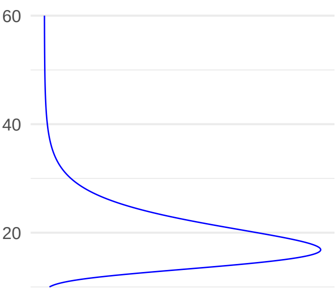
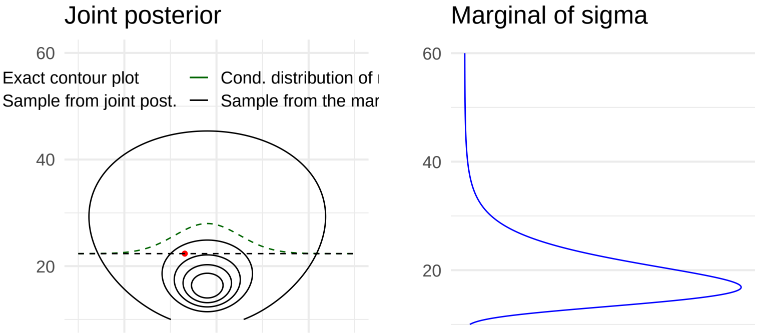



## Marginal of sigma




### Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

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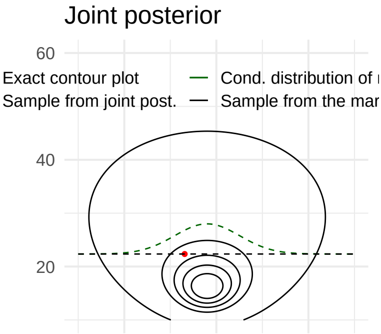


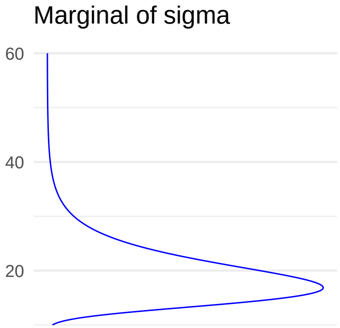


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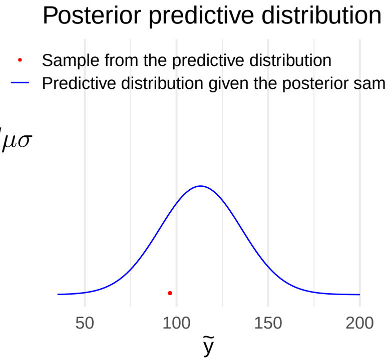

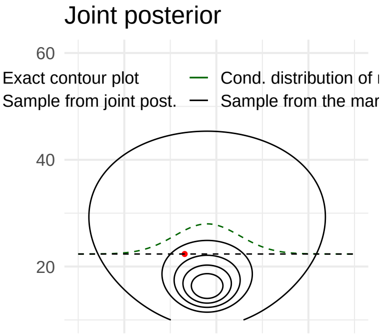


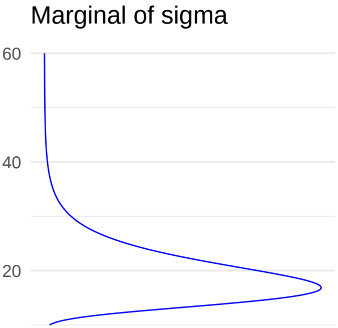


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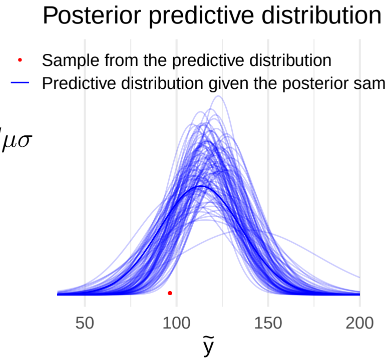

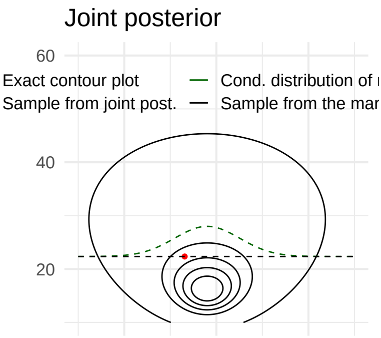


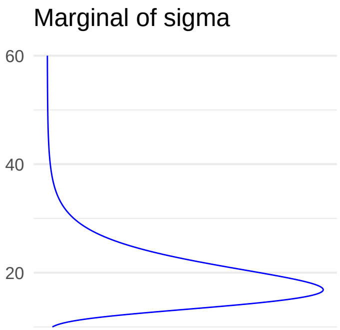


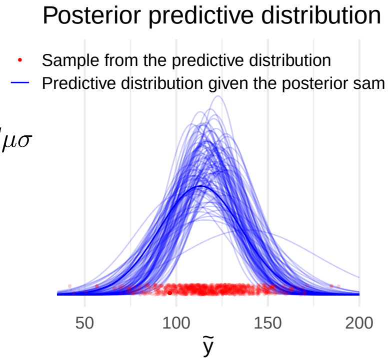


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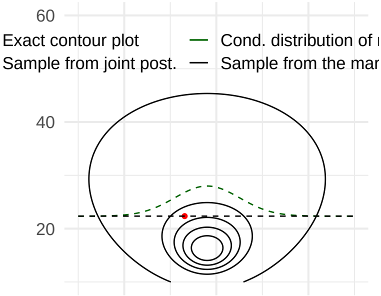


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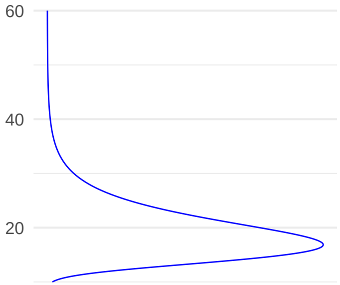
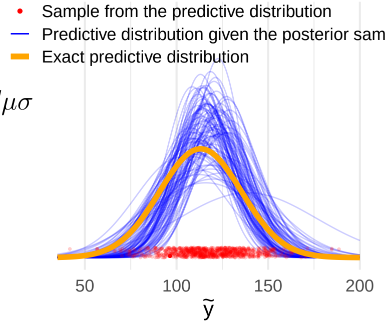



## Posterior predictive distribution

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# Normal - posterior predictive distribution

Posterior predictive distribution given known variance

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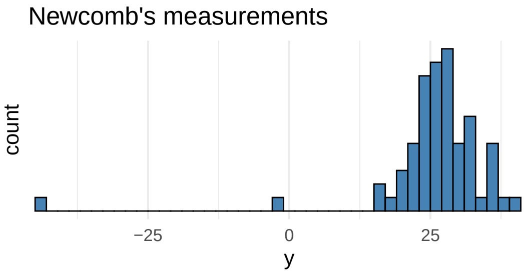
## Simon Newcomb's light of speed experiment in 1882

Newcomb measured ($n = 66$) the time required for light to travel from his laboratory on the Potomac River to a mirror at the base of the Washington Monument and back, a total distance of 7422 meters.



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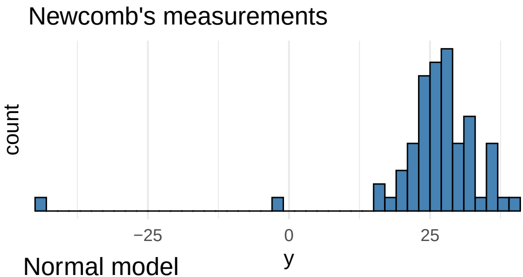


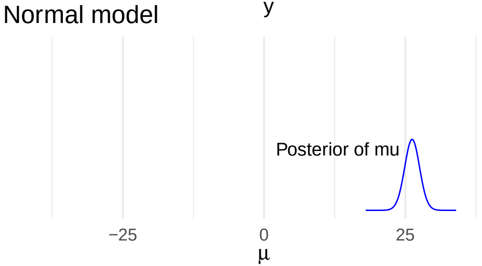

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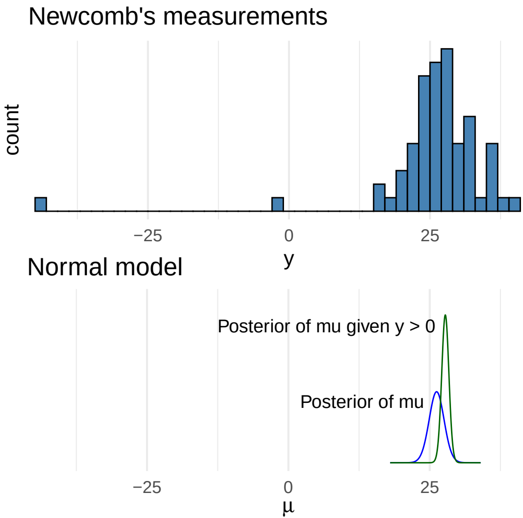

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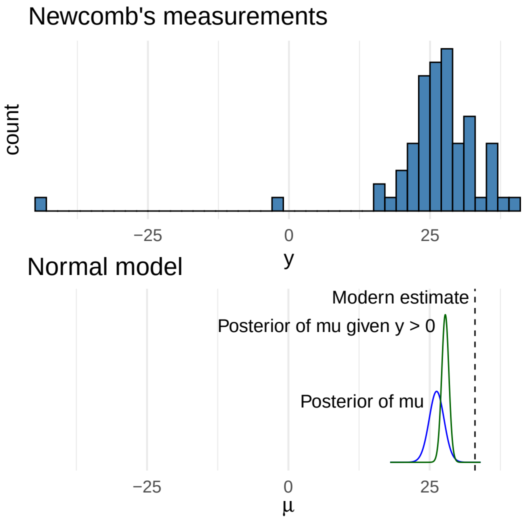

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which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

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$$\begin{aligned} \mu \mid \sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

  which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

- $\mu$ and $\sigma^2$ are a priori dependent
  - if $\sigma^2$ is large, then $\mu$ has wide prior

# Normal - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}$$

# Comparison of means of two normals

- The difference of two normally distributed variables is normally distributed
- The difference of two $t$ distributed variables with different variances and degrees of freedom doesn't have a closed form
  - but easy to sample from the two distributions, and obtain samples of the differences

$$
\begin{aligned}
\text{if} \quad & \mu_1^{(s)} \sim p(\mu_1 \mid y_1) \\
& \mu_2^{(s)} \sim p(\mu_2 \mid y_2) \\
& \delta^{(s)} = \mu_1^{(s)} - \mu_2^{(s)} \\
\text{then} \quad & \delta^{(s)} \sim p(\delta \mid y_1, y_2)
\end{aligned}
$$

# Multivariate normal

- Observation model

$$p(y \mid \mu, \Sigma) \propto \mid \Sigma \mid^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)^T\Sigma^{-1}(y-\mu)\right)$$

- BDA3 p. 72–
- Recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

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- BDA3 p. 72–
- Recommended LKJ-prior mentioned in Appendix A, see more in Stan manual
- Gaussian process and Gaussian Markov random field models are in practice computed with multivariate normals
    - GPs in BDA3 Chapter 21, and a course in spring
    - GPs and GMRFs often used also as priors for latent functions and combined with non-normal observation models

# Normal linear regression

- $y_i \sim \mathrm{N}(\alpha + \beta x_i, \sigma^2), \quad i = 1, \ldots, N$

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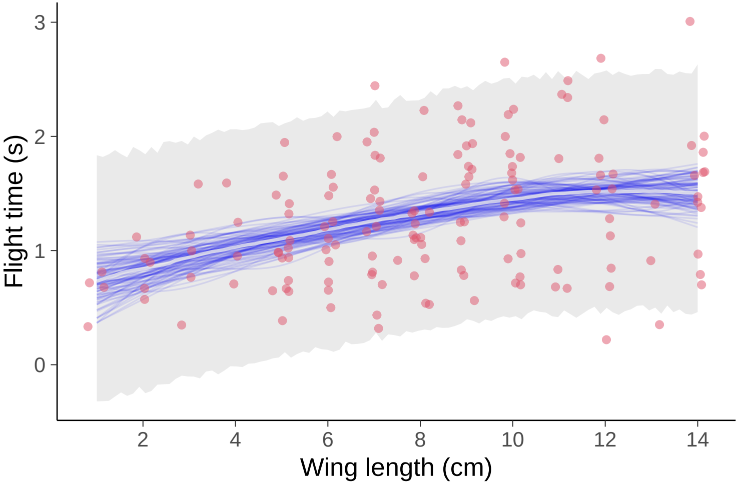
- with unknown $\sigma^2$, the posterior is multivariate $\mathrm{N\text{-}Inv\text{-}}\chi^2$
- with unknown prior scales and $\sigma^2$, numerical integration needed
- more in BDA3 Chapter 14 (not part of the course) and Regression and Other Stories book

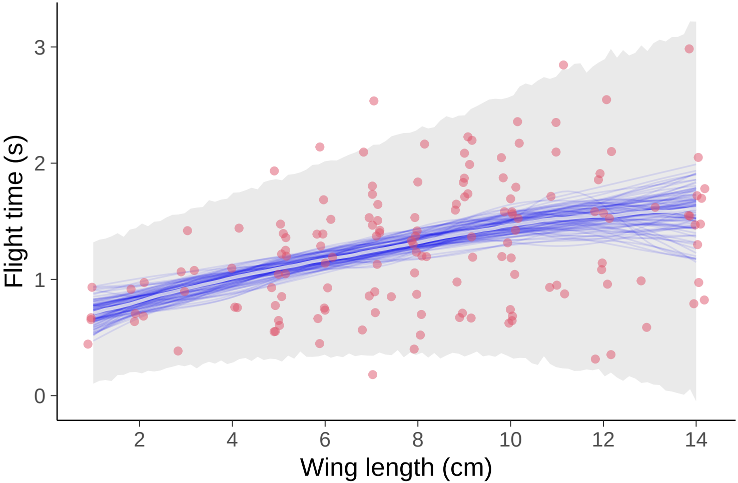
# Paper helicopter flight time

$$y \sim \text{normal}(f, \sigma)$$
$$f \sim GP(0, K(x, \theta))$$



# Paper helicopter flight time

$$y \sim \text{normal}(f, \sigma)$$
$$f \sim GP(0, K_f(x, \theta_f))$$
$$\log(\sigma) \sim GP(0, K_g(x, \theta_g))$$



# Scale mixture of normals

- Many useful distributions can be presented as scale mixture of normals, e.g.
  - Student's *t*
  - Cauchy
  - Double exponential aka Laplace
  - Horseshoe
  - R2-D2

# Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^{k} \theta_j^{y_j},$$

- BDA3 p. 69–

# Generalized linear model (GLM)

- $y_i \sim p(g^{-1}(\alpha + \beta x_i), \phi), \quad i = 1, \ldots, N$
    - where $p$ is non-normal (in original definition in exponential family)
    - and $g$ is a link function

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- Bioassay analysis is used as an example

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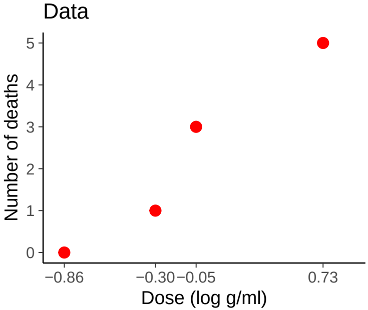
- Bioassay analysis is used as an example
- More in BDA3 Chapter 16 and Regression and other stories book

# Bioassay

| Dose, $x_i$ (log g/ml) | Number of animals, $n_i$ | Number of deaths, $y_i$ |
|---|---|---|
| -0.86 | 5 | 0 |
| -0.30 | 5 | 1 |
| -0.05 | 5 | 3 |
| 0.73 | 5 | 5 |



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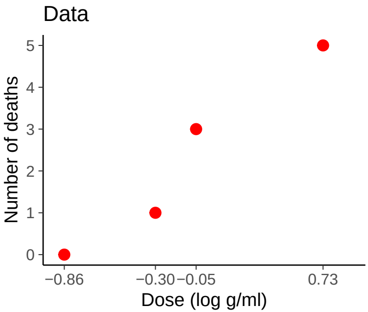


Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

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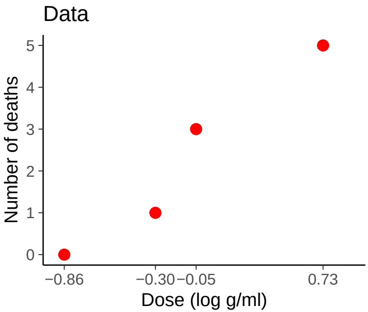


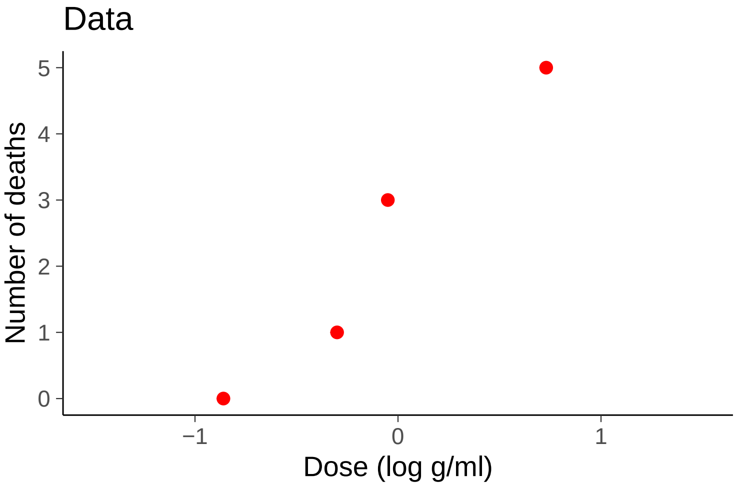

Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

# Bioassay



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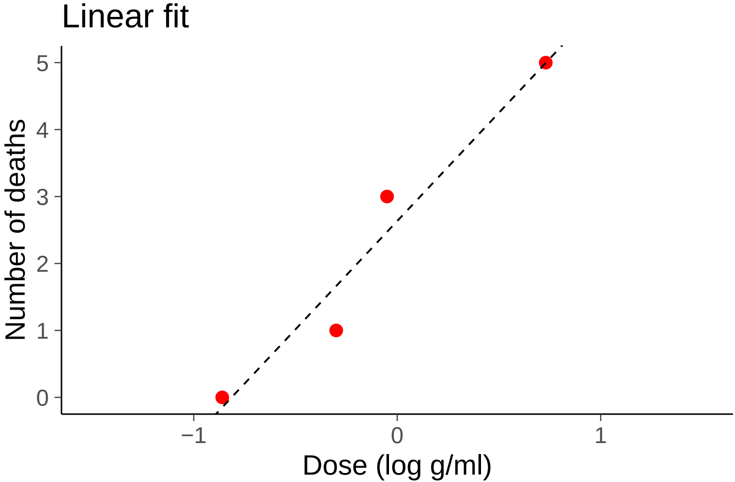

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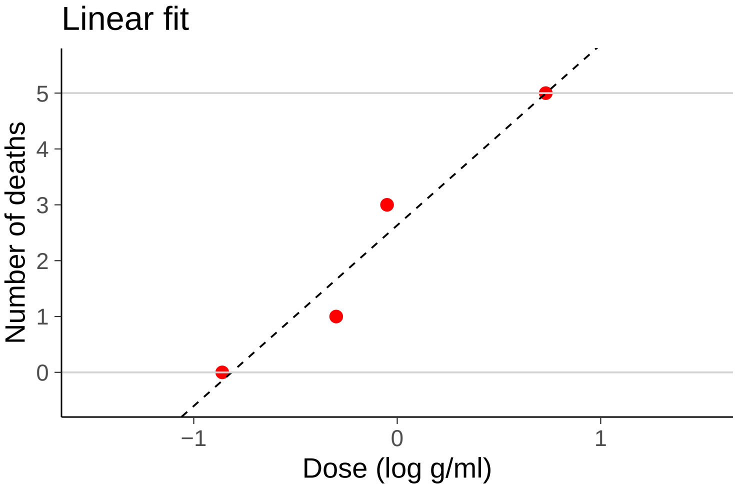

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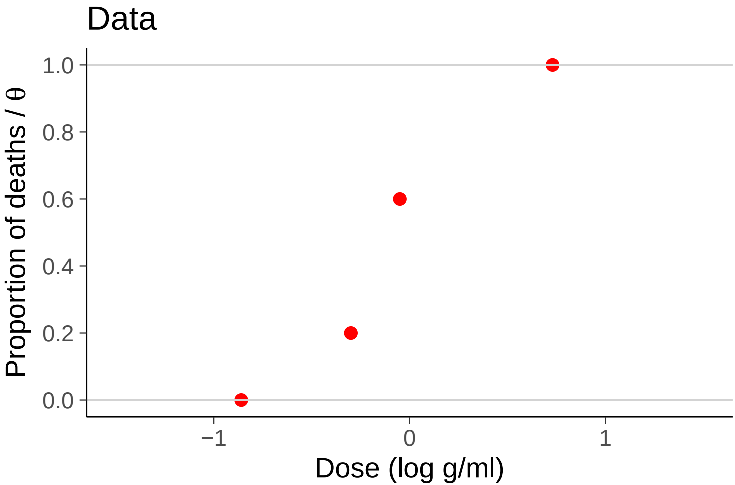

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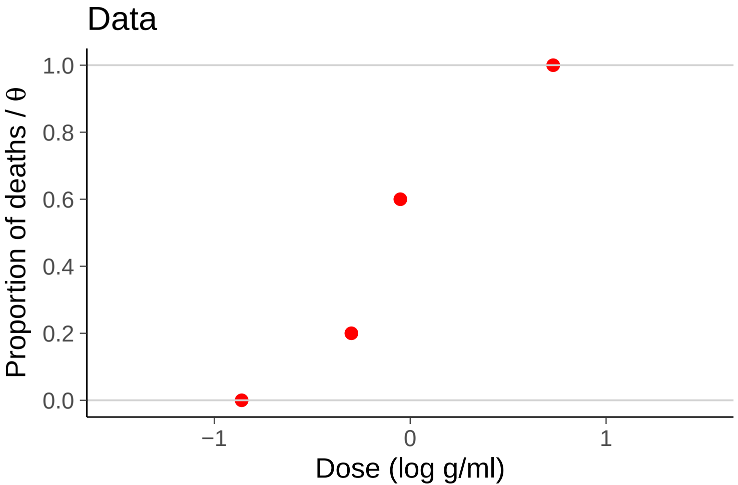
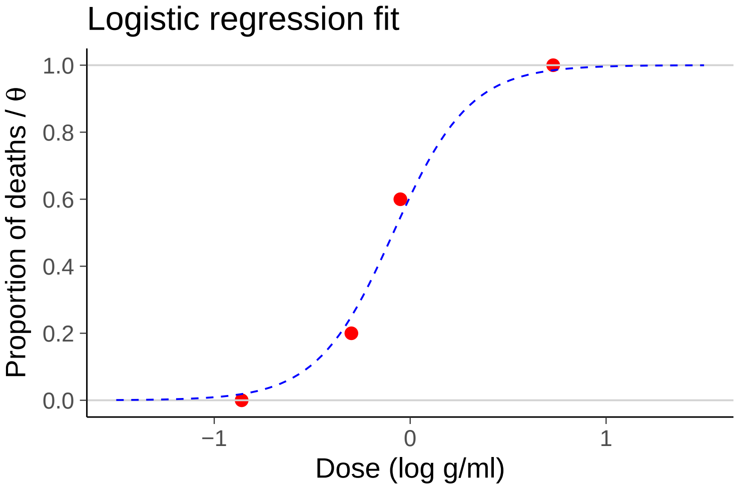



Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

# Bioassay



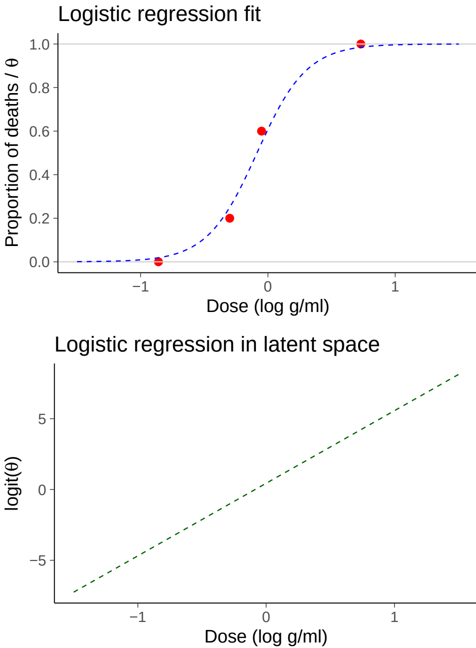

Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i), \quad \text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

# Bioassay

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

$$\begin{aligned}\mathrm{logit}(\theta_i) &= \log\left(\frac{\theta_i}{1-\theta_i}\right)\\ &= \alpha + \beta x_i\end{aligned}$$



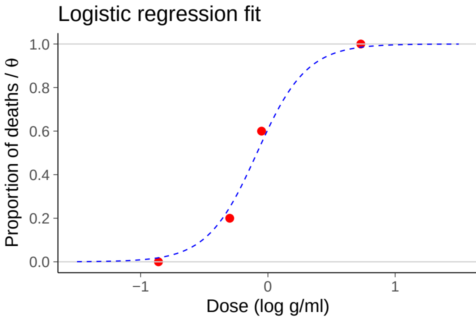
# Bioassay

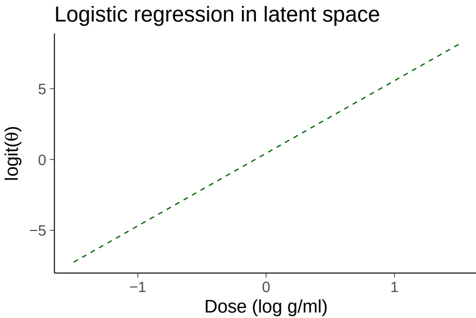
$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right)$$

$$= \alpha + \beta x_i$$

$$\theta_i = \frac{1}{1+\exp(-(\alpha+\beta x_i))}$$



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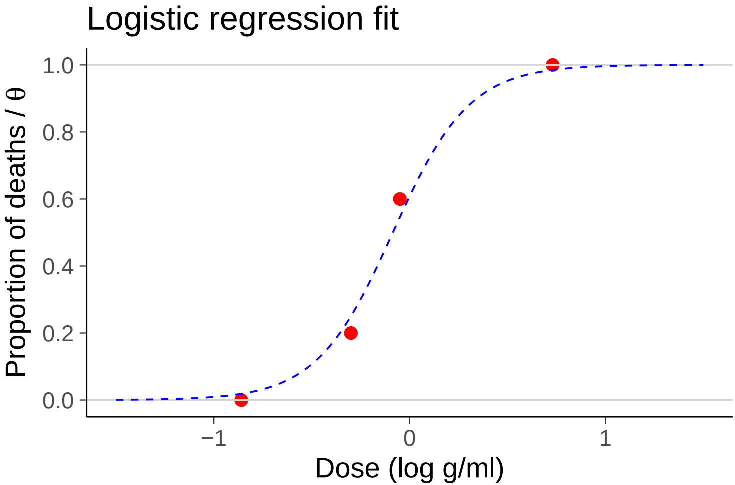
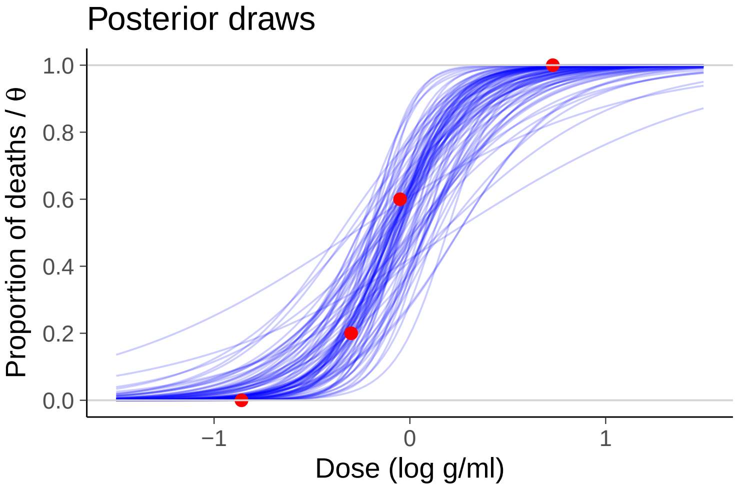


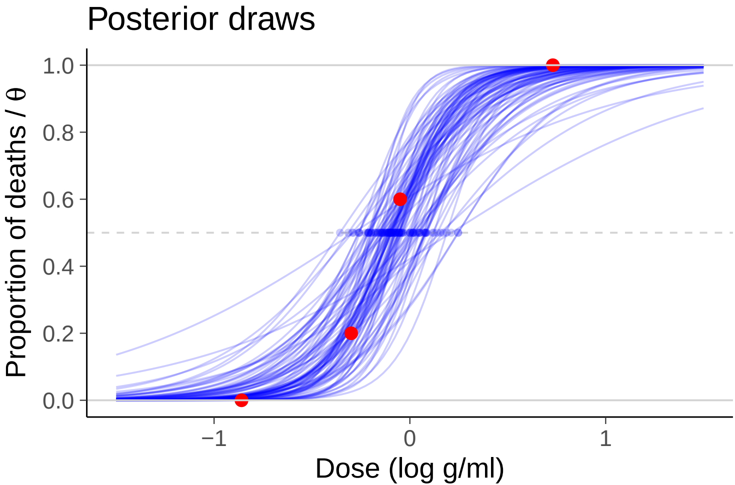
Logistic regression fit
1.0
0.8
0.6
0.4
0.2
0.0
Proportion of deaths / θ
−1
0
1
Dose (log g/ml)

# Bioassay



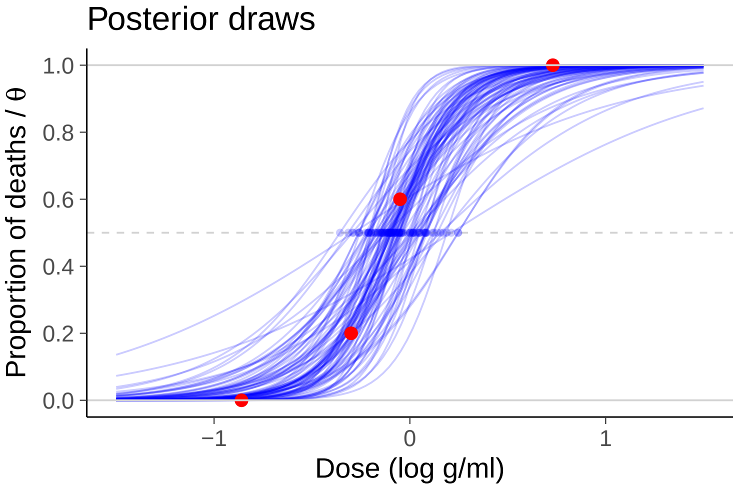
Posterior draws
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

# Bioassay



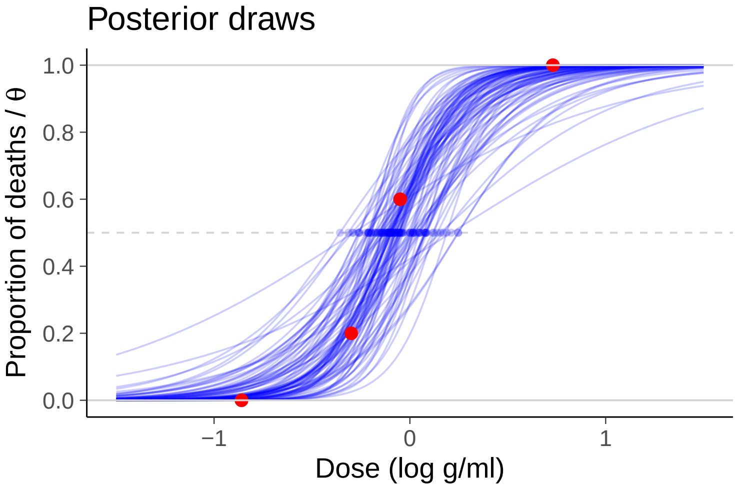

LD50: $\mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5$

# Bioassay




LD50: $\quad \mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

# Bioassay

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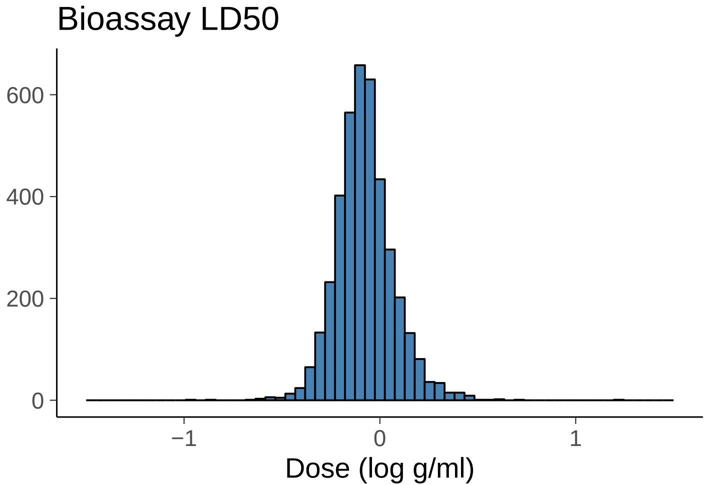
$$x^{(s)}_{\mathrm{LD50}} = -\alpha^{(s)}/\beta^{(s)}$$

# Bioassay

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$$x_{\text{LD50}}^{(s)} = -\alpha^{(s)}/\beta^{(s)}$$

# Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

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Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

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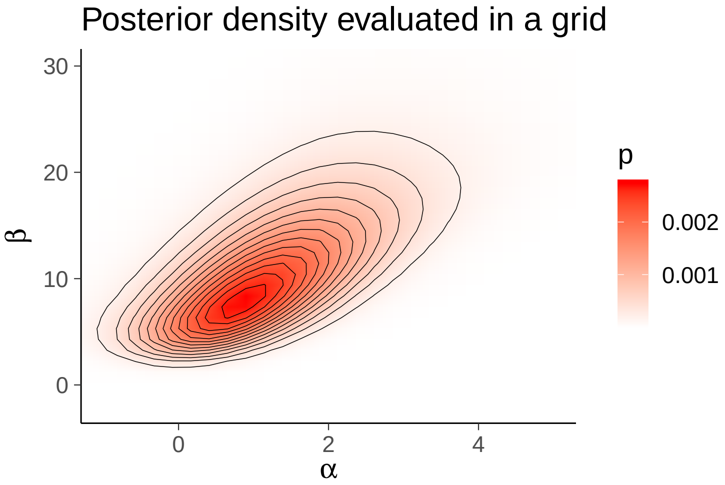
Posterior (with uniform prior on $\alpha, \beta$)

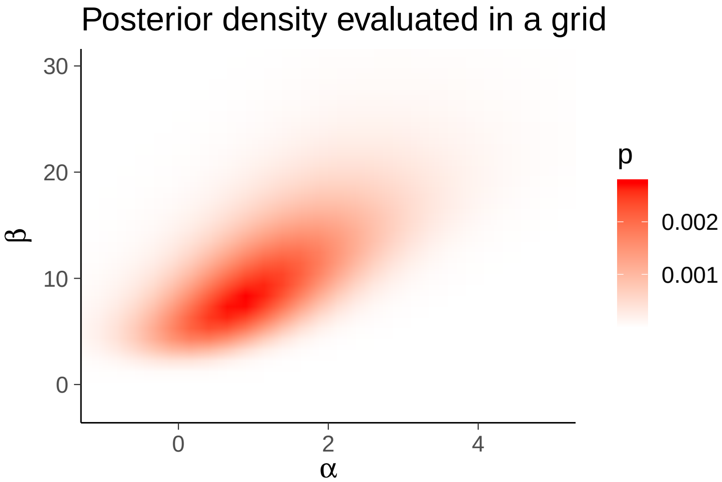
$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i)$$

# Bioassay



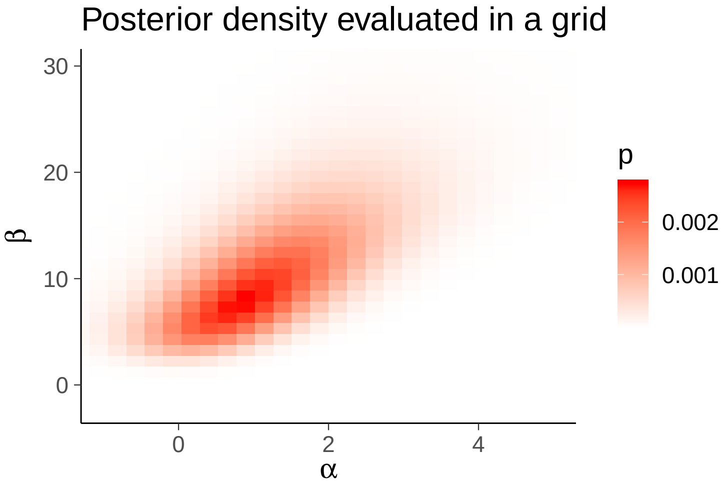
Posterior density evaluated in a grid
30
20
10
0
β
0
2
4
α
p
0.002
0.001

# Bioassay



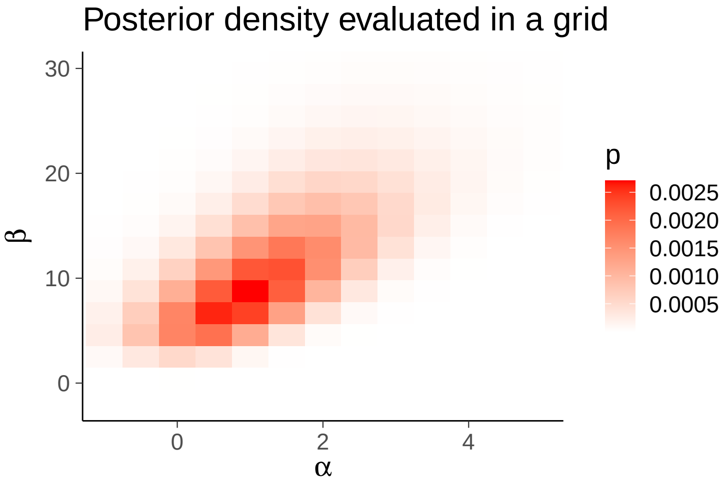

Density evaluated in grid, but plotted using interpolation

# Bioassay



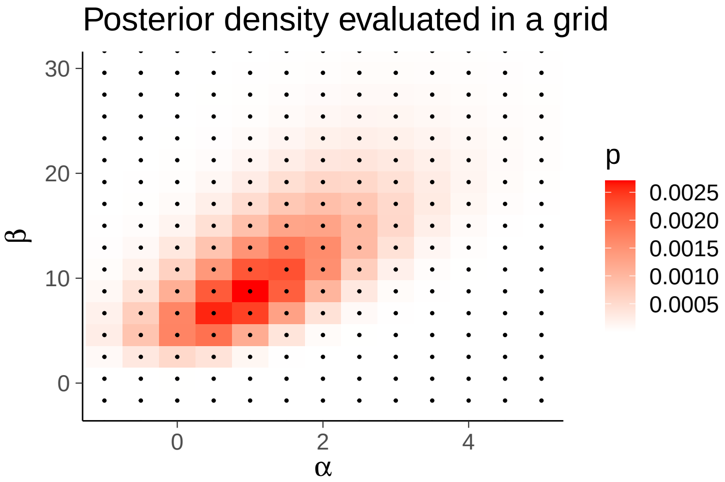

Density evaluated in grid, and plotted without interpolation

# Bioassay



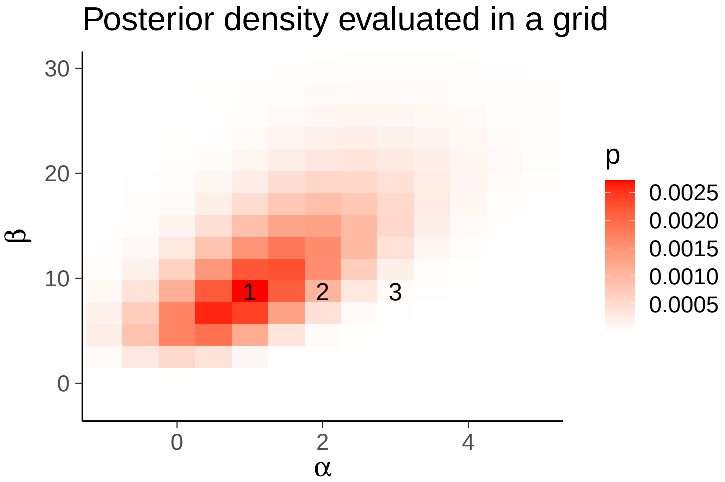

Density evaluated in a coarser grid

# Bioassay



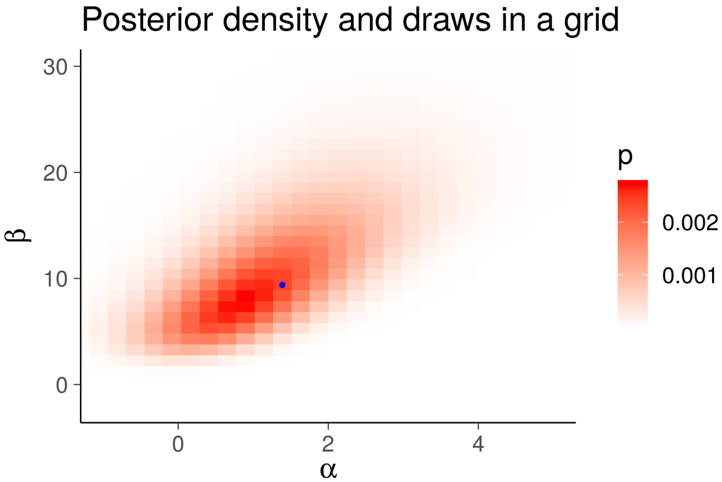

- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell

# Bioassay



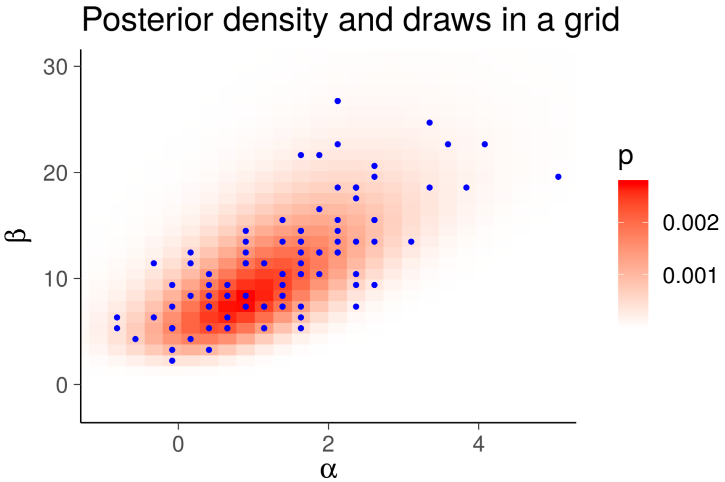

- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

# Bioassay




- Sample according to grid cell probabilities

# Bioassay

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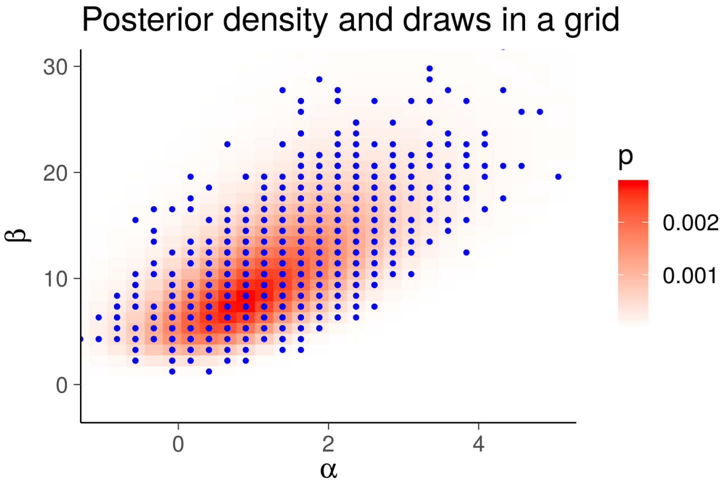
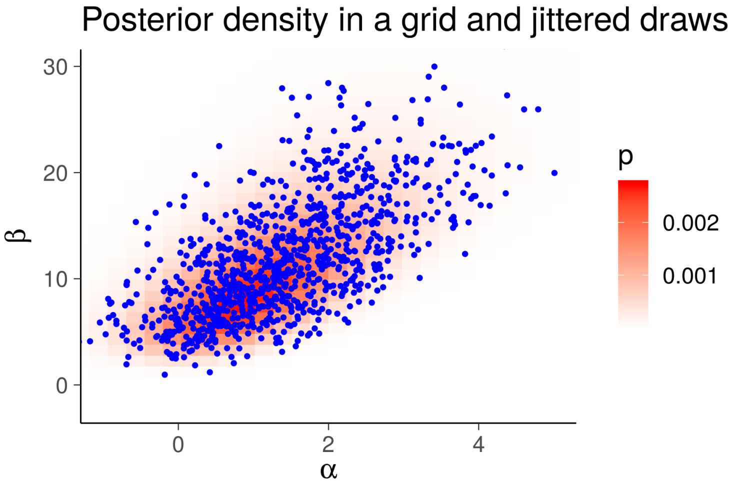



- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

# Bioassay




- Jitter can be added to improve visualization

# Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S} \sum_{s=1}^{S} -\frac{\alpha^{(s)}}{\beta^{(s)}}$$

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- Instead of sampling, grid could be used to evaluate functions directly, for example

$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} -\frac{\alpha^{(t)}}{\beta^{(t)}} w_{\mathrm{cell}}^{(t)},$$

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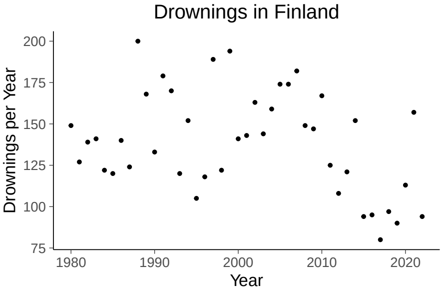
- Grid sampling gets computationally too expensive in high dimensions

# Example GLM

| Count of deaths, $y_i$ | Year |
|---|---|
| 149 | 1980 |
| 127 | 1981 |
| 139 | 1982 |
| $\vdots$ | $\vdots$ |
| 157 | 2021 |
| 94 | 2022 |



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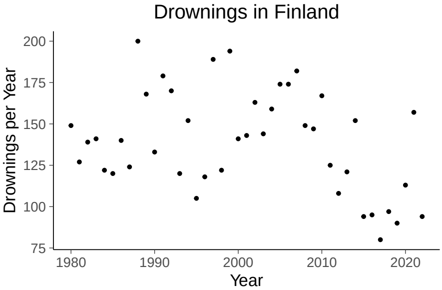


Swimming is popular in Finland, but also hazardous

- On average $\sim$140 drownings per year
- Finnish government has invested in measures for reducing deaths
- Recent narrative based on effectiveness of education

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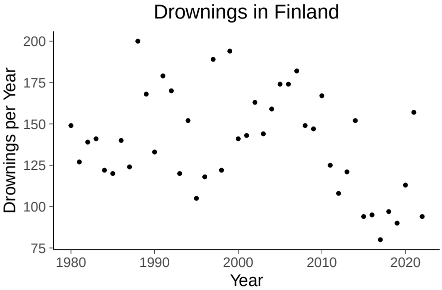


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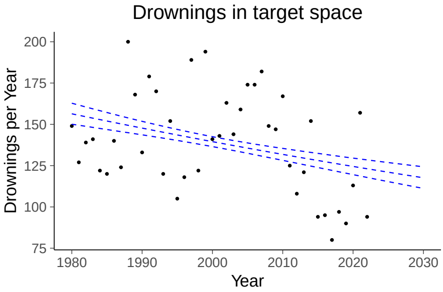
- Finnish government has invested in measures for reducing deaths
- Recent narrative based on effectiveness of education

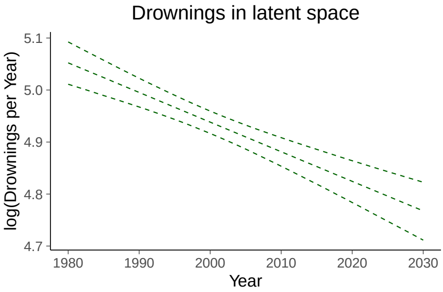
Bayesian methods help

- Describe trends over time
- Evaluate uncertainty

# Example GLM: Poisson Linear Model

$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i = e^{\alpha + \beta x_i}$$

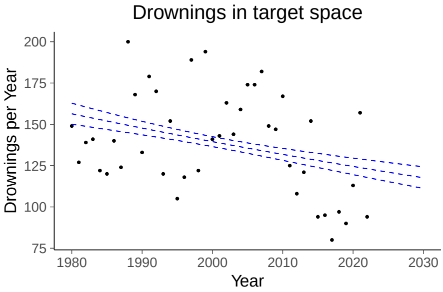


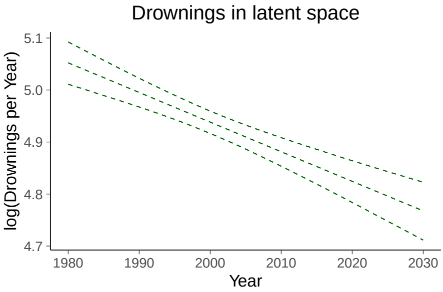



# Example GLM: Poisson Linear Model

$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i = e^{\alpha + \beta x_i}$$

$$\text{Poisson}(y_i|\mu_i) = \frac{1}{y_i!}\mu_i^{y_i} e^{-\mu_i}$$



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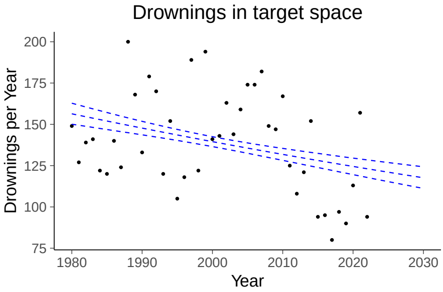


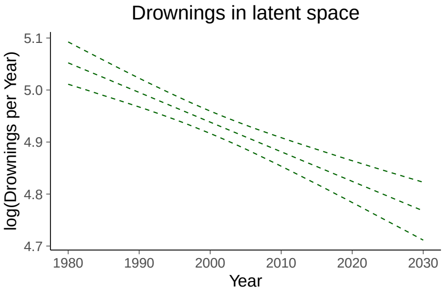

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Alternatively :

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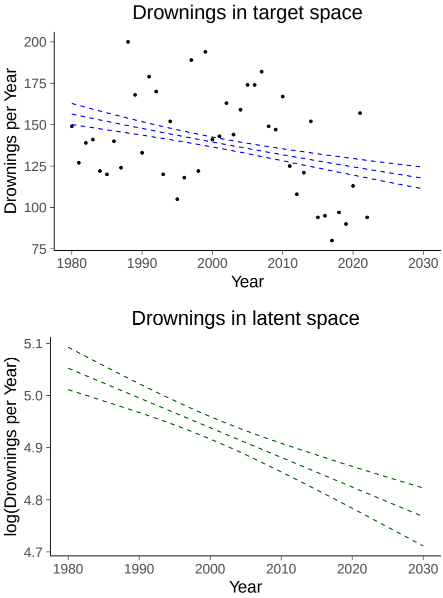
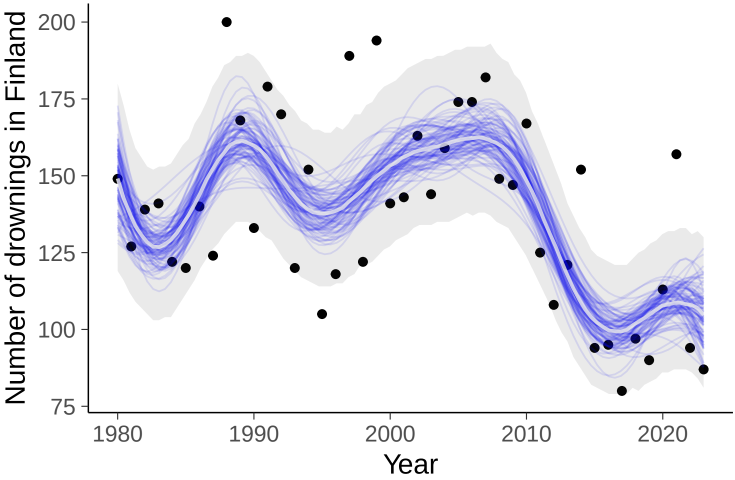
$$\text{Neg-bin}(y_i \mid \mu_i, \phi) = \frac{\Gamma(y_i + \phi)}{y_i!\Gamma(\phi)}\left(\frac{\mu_i}{\mu_i + \phi}\right)^{y_i}\left(\frac{\phi}{y_i + \phi}\right)^{\phi}$$

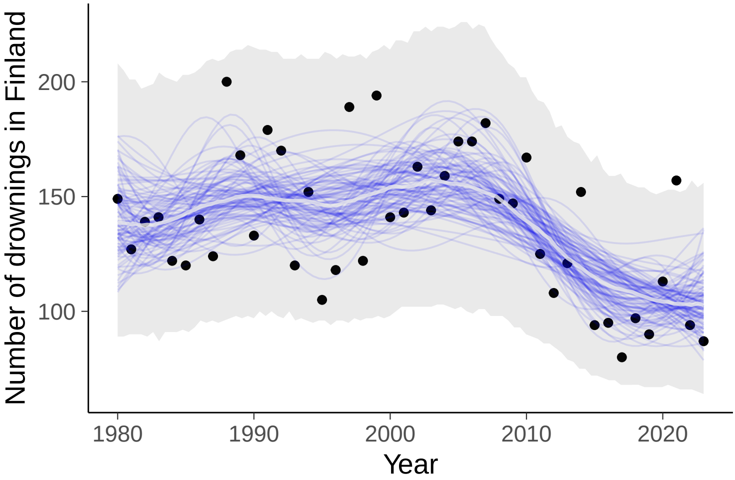


# Example GLM: Gaussian Process Models




$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i \sim e^{f_i},\ f \sim \text{GP}(0, \text{k}(\text{Year}, \theta))$$

# Example GLM: Gaussian Process Models

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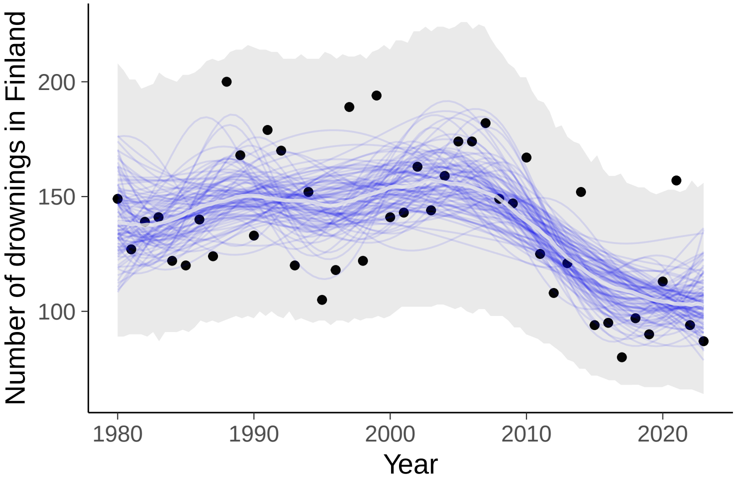
# Example GLM: Gaussian Process Models




- Clear overdispersion
  - later we use posterior predictive checking and cross-validation to confirm this
- Trend interpretations shouldn't be based on one observation

# Thinking counts

- For simplicity of exposition, we often start learning with normal observation models
- But we observe count data on a daily basis
- Very relevant in industry (number of sold products, ad views, customer count, etc.)
- Can you think of such examples from the class room?
  - Think of how many students attend BDA lectures over the course
  - Number of students who report getting sick over time until Christmas
  - Number of dropouts
  - Would you expect overdispersion?