Chapter 3

- 3.1 Marginalization
- 3.2 Normal distribution with a noninformative prior (important)
- 3.3 Normal distribution with a conjugate prior (important)
- 3.4 Multinomial model (can be skipped)
- 3.5 Multivariate normal with known variance (useful for chapter 4)
- 3.6 Multivariate normal with unknown variance (glance through)
- 3.7 Bioassay example (very important, related to one of the exercises)
- 3.8 Summary (summary)

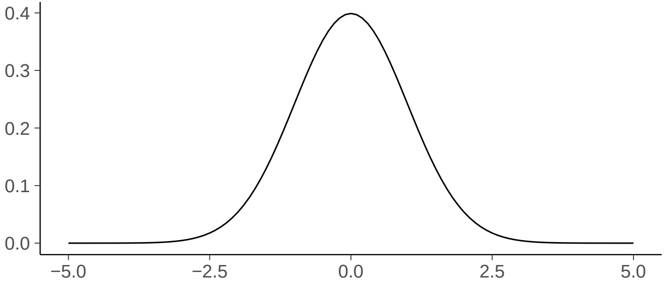
This lecture

- Normal / Gaussian distribution with unknown mean and scale
- Connection between analytic probability density and Monte Carlo approximation
- Marginalization by analytic integration and by Monte Carlo
- Posterior predictive distribution by analytic integration and by Monte Carlo
- Generalized Linear Models (GLM) (Binomial and Poisson)
- Grid Monte Carlo sampling

Normal / Gaussian

observation y, and parameters μ and σ

$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$



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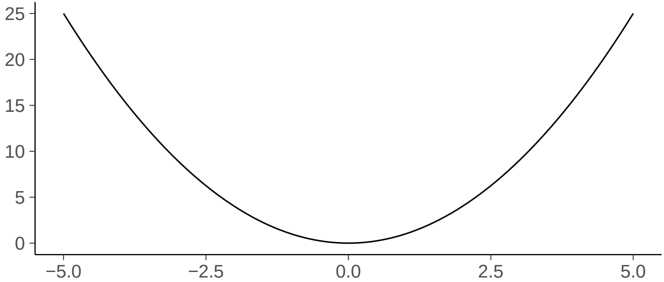

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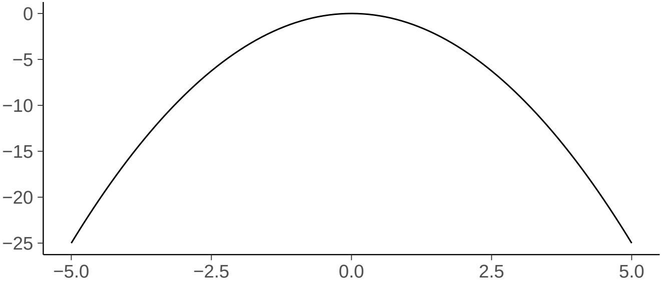

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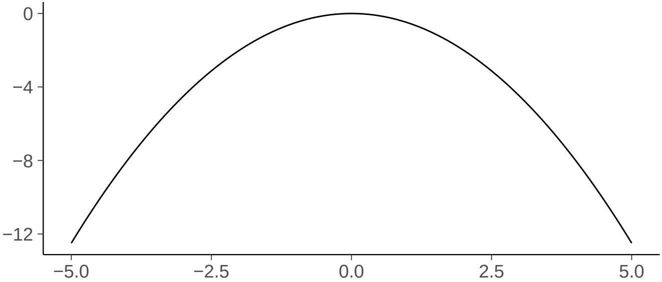

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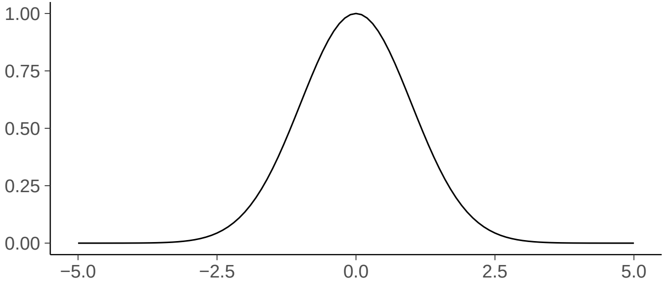

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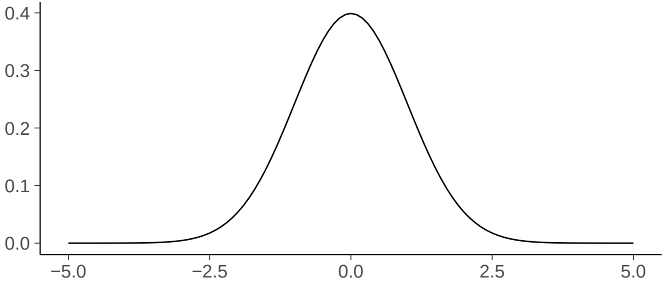

Normal / Gaussian

observation y, and parameters $\mu = 1$ and $\sigma = 1$

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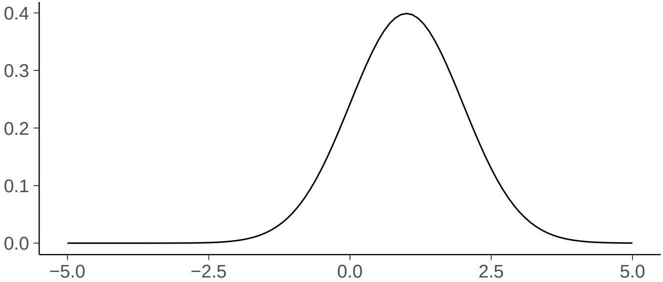

Normal / Gaussian

observation *y*, and parameters $\mu = 1$ and $\sigma = 1/2$

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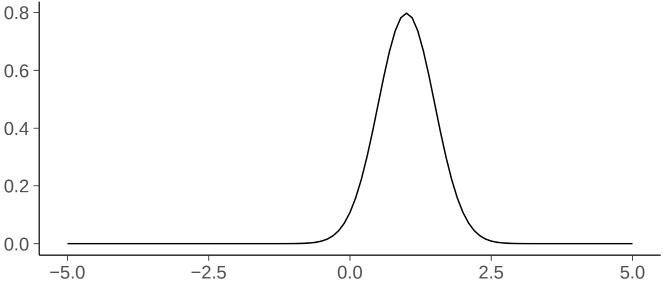
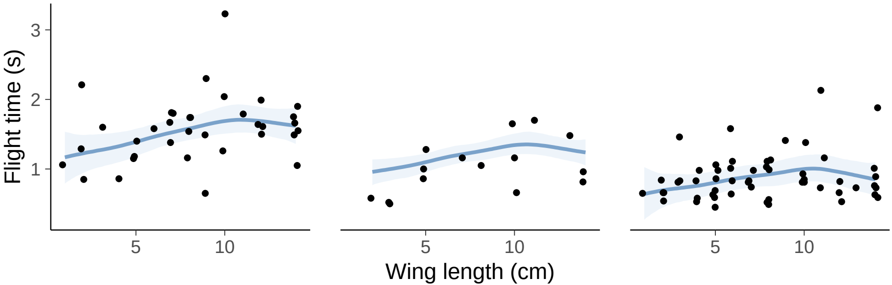


Variation in helicopter flight times



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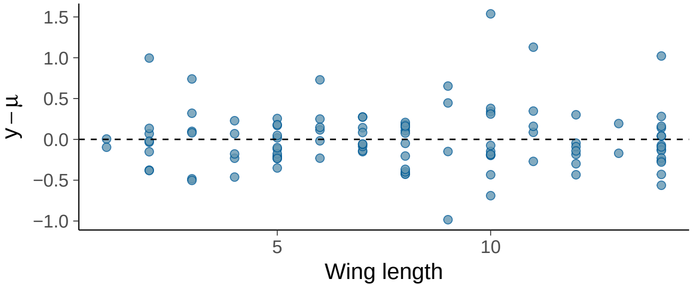

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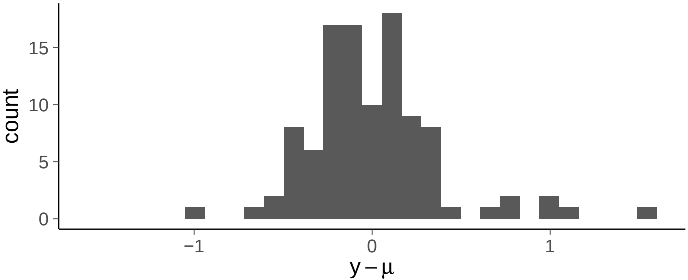

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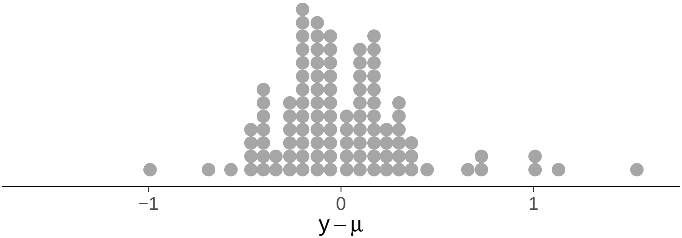

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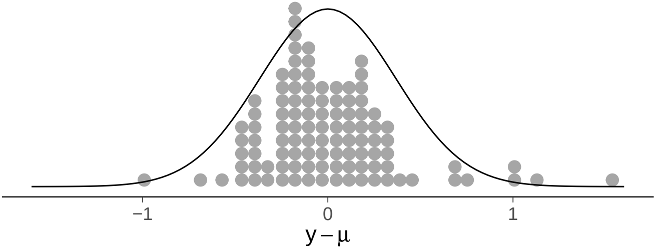

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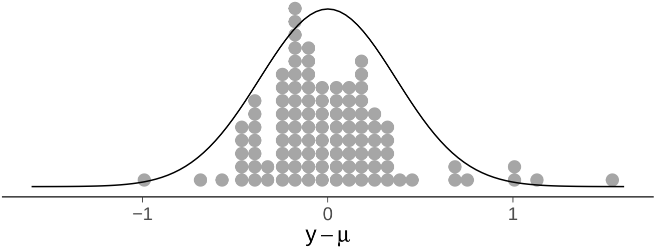
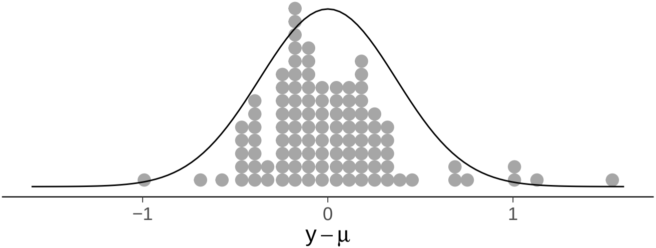


- Why close to normally distributed?

Variation in helicopter flight times



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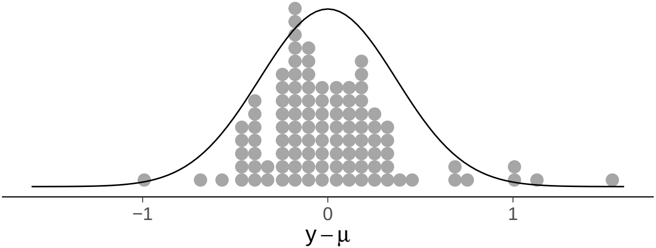

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  - Central Limit Theorem (CLT)

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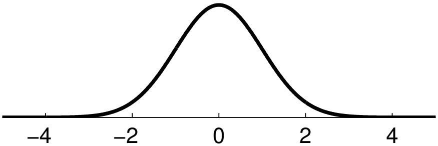

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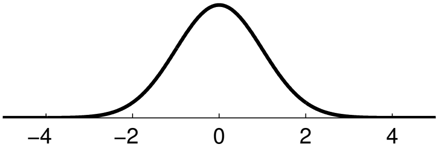

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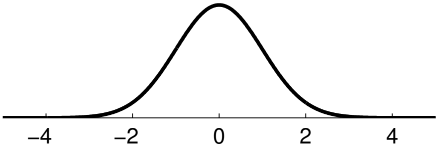

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- Gaussian: favored by Germans (but it was studied earlier by De Moivre and Laplace, and Gauss avoided using it)
- Shorthand notation
  - $y \sim \mathrm{N}(\mu, \sigma^2)$ with variance σ^2 (useful in derivations)
  - $y \sim \texttt{normal}(\mu, \sigma)$ with deviation σ (useful for interpreting prior and posterior scales, used in Stan)

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  - part of Assignment 3

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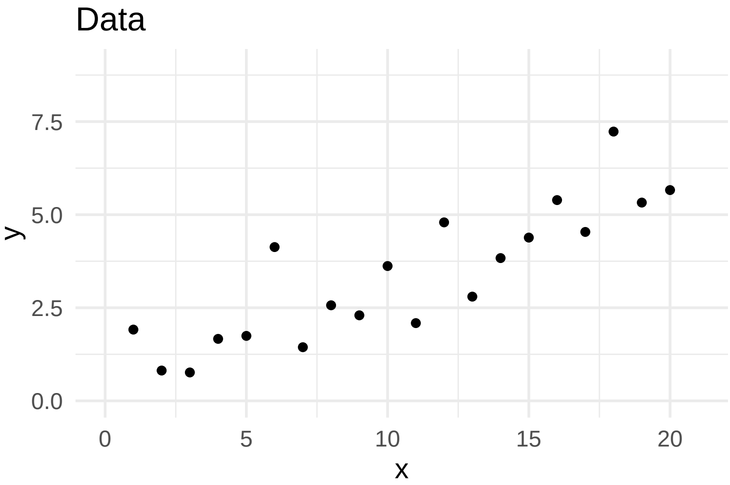
- Posterior distribution approximation with Laplace, variational inference, expectation propagation

Example of uncertainty in modeling



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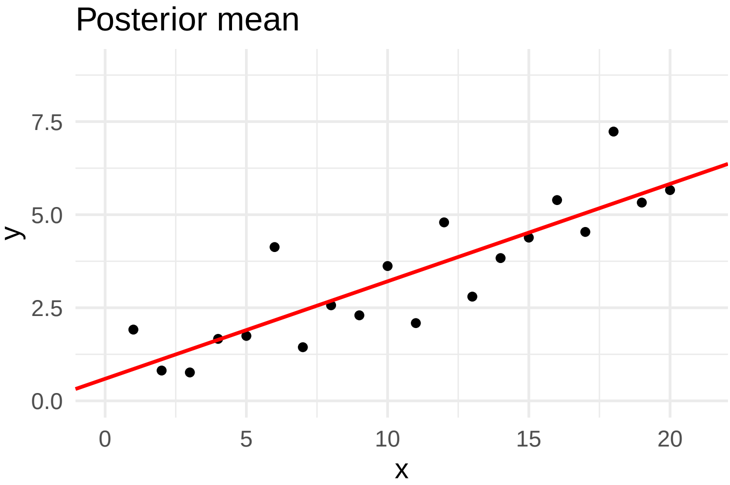

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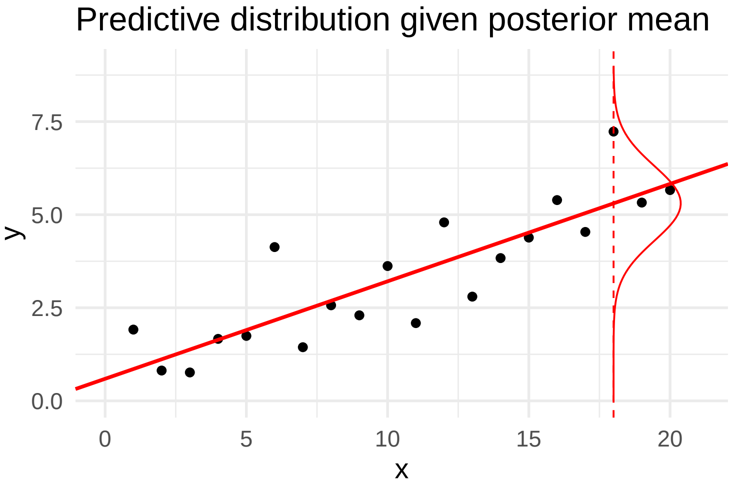

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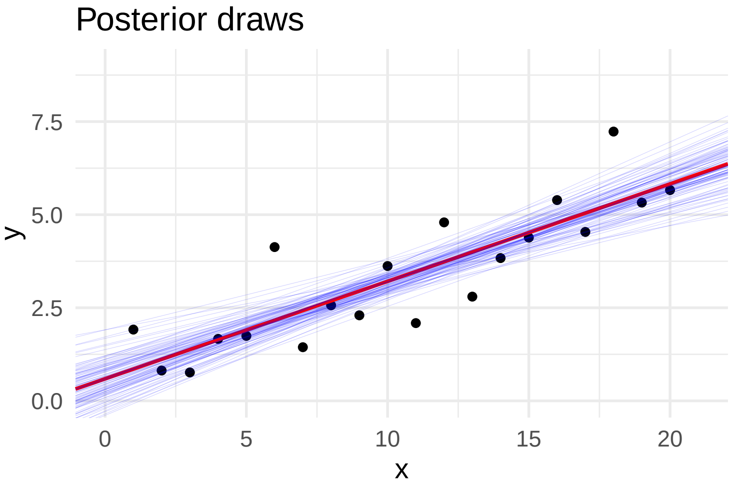

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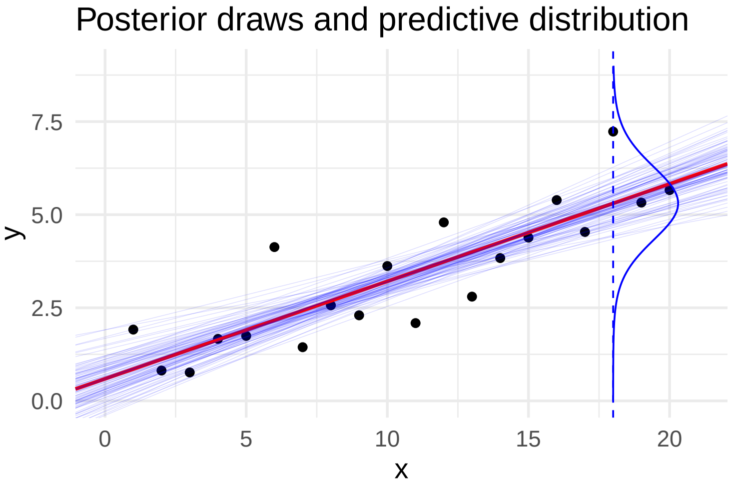

Posterior for μ and σ given *y*

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- Prior for μ and σ: $p(\mu, \sigma)$

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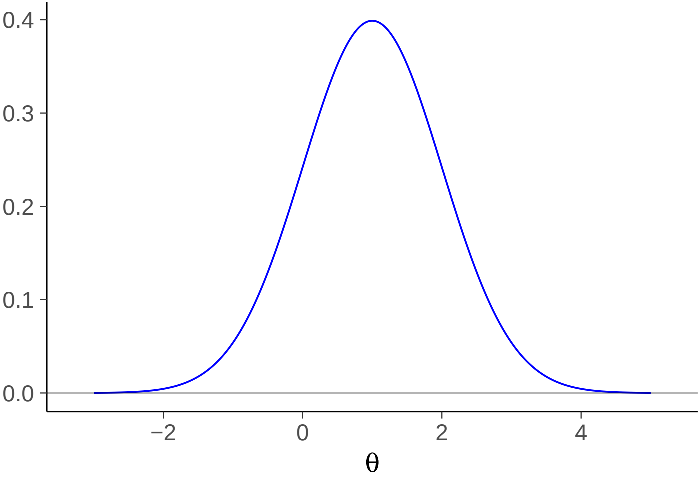
- For some priors there is analytic solution for posterior, and Monte Carlo is a general approach
- Before looking at the posterior $p(\mu, \sigma \mid y)$, connection between mean, cumulative density and Monte Carlo

Monte Carlo and posterior draws

Density $p(\theta \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(\theta - \mu)^2\right)$ (`dnorm()`)



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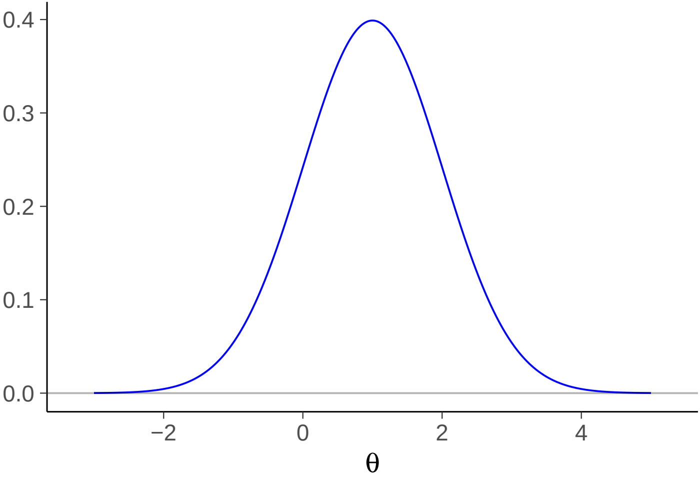

$\mathrm{E}(\theta) = \int \theta p(\theta \mid \mu, \sigma) d\theta = \mu$

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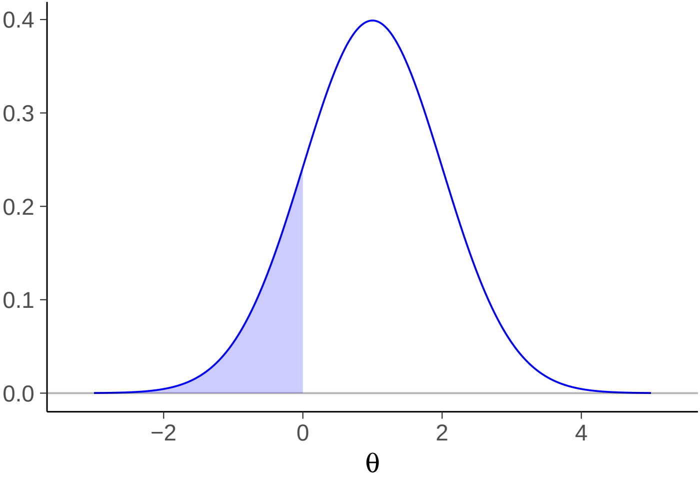
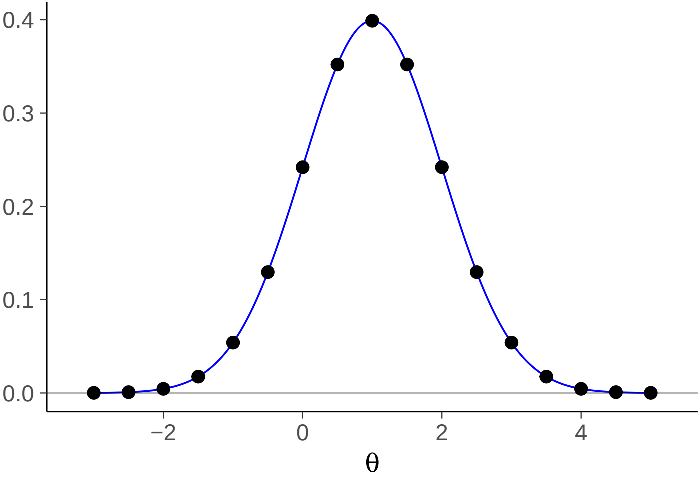


$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta \mid \mu, \sigma) d\theta$,
many numerical approximations (pnorm())

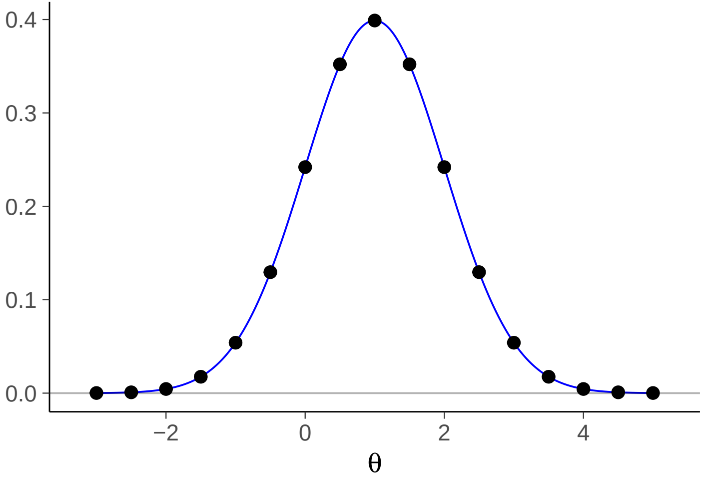
Monte Carlo and posterior draws

In practice evaluate in finite number of locations (dnorm())



Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.5



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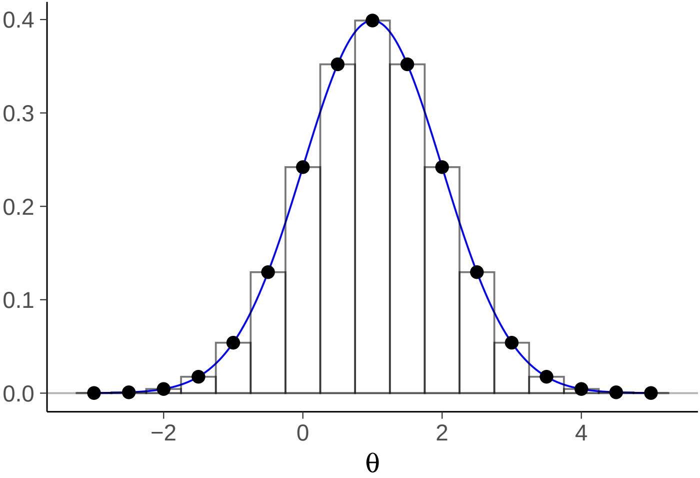

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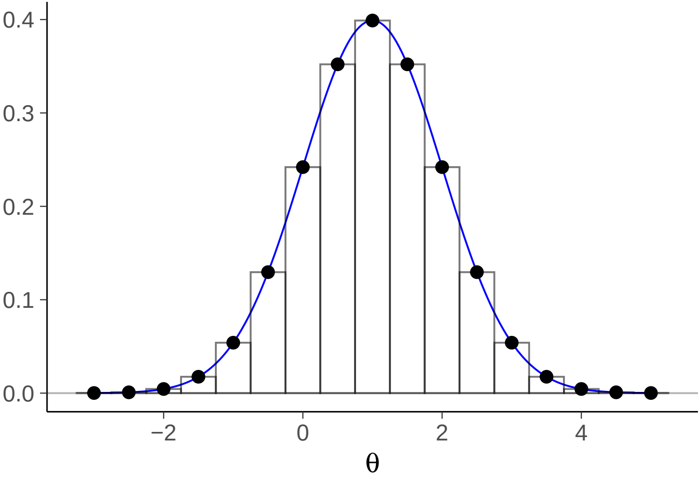
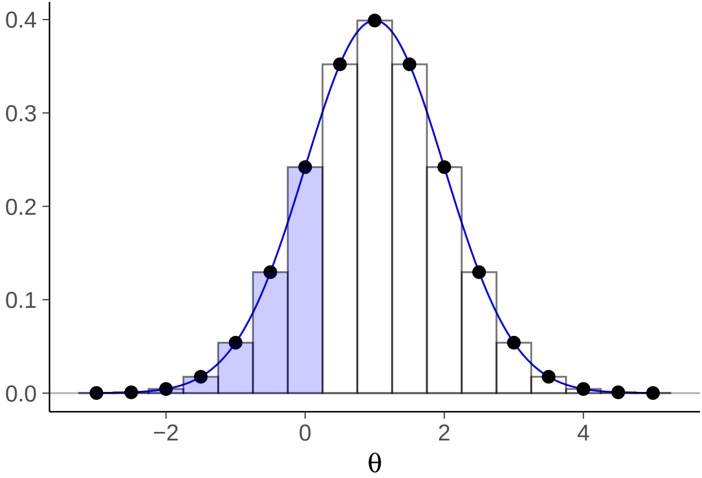


$\mathrm{E}(\theta) = \int \theta p(\theta) d\theta \approx \sum_s^S \theta^{(s)} w_s \approx 1,$ where $w_s = 0.5 p(\theta)$

Monte Carlo and posterior draws

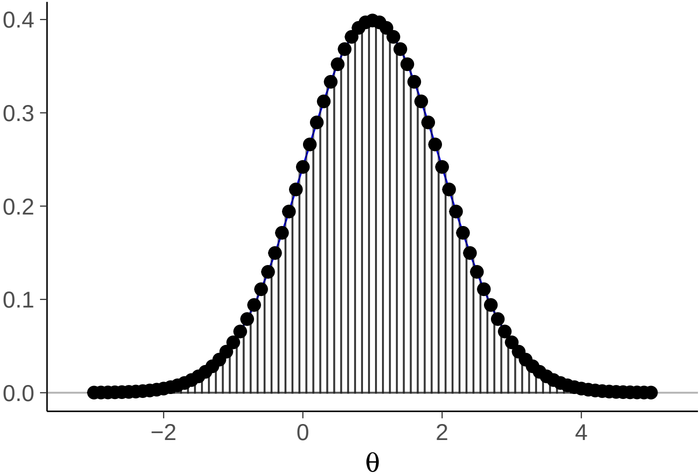
Here evaluated in grid with bin width 0.5



$$p(\theta \leq 0) = \int_{-\infty}^{0} p(\theta) d\theta \approx \sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) w_s \approx 0.22$$

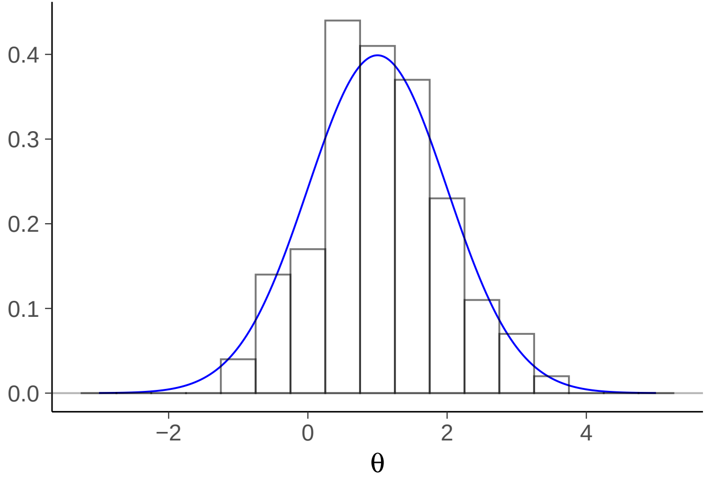
Monte Carlo and posterior draws

Here evaluated in grid with bin width 0.1



Monte Carlo and posterior draws

Histogram of 200 random draws (`rnorm()`), bin width 0.5



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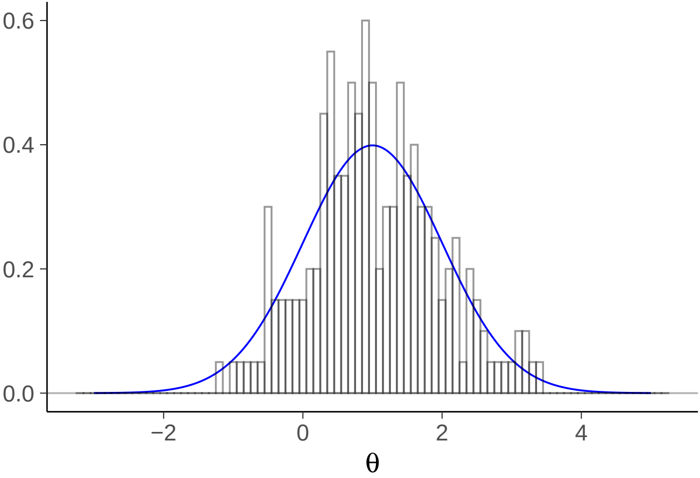
Histogram of 200 random draws (`rnorm()`), bin width 0.1



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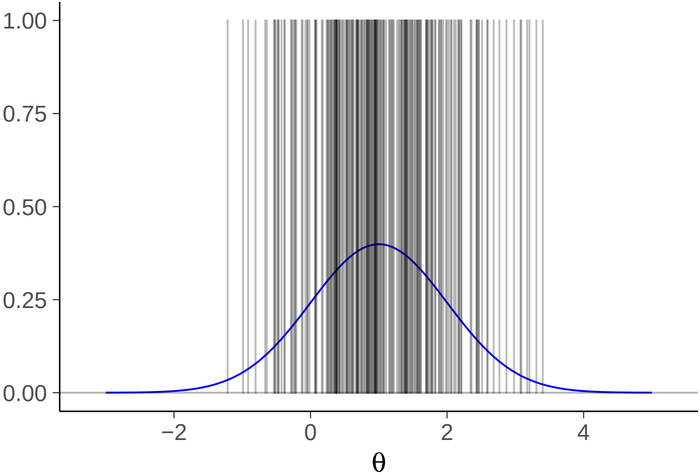

each bin has either 0 or 1 draw (and 0's can be ignored)

Monte Carlo and posterior draws

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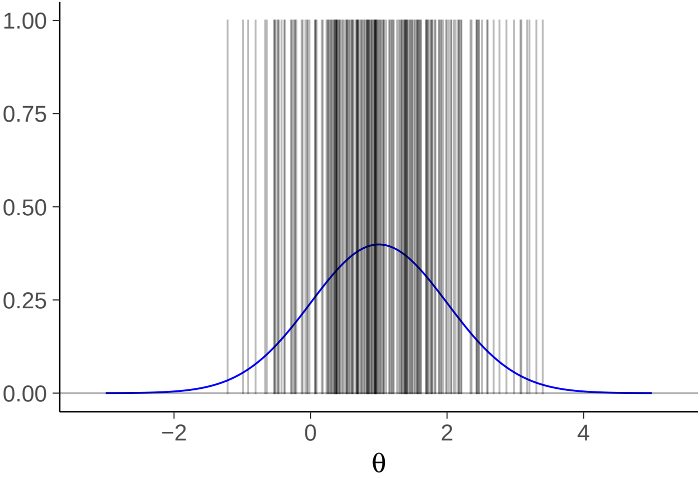

each bin with 1 draw has weight $1/S$

Monte Carlo and posterior draws

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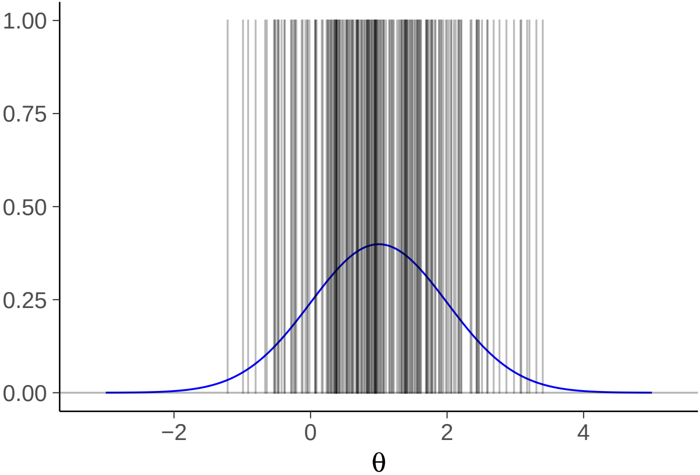

$E(\theta) \approx \frac{1}{S}\sum_s^S \theta^{(s)} \approx 1$

Monte Carlo and posterior draws

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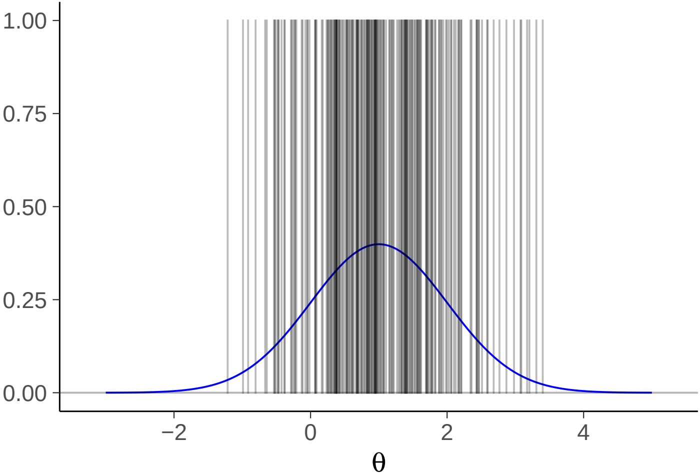

$E(\theta) \approx \frac{1}{S}\sum_{s}^{S}\theta^{(s)} \approx 1$, Monte Carlo estimate

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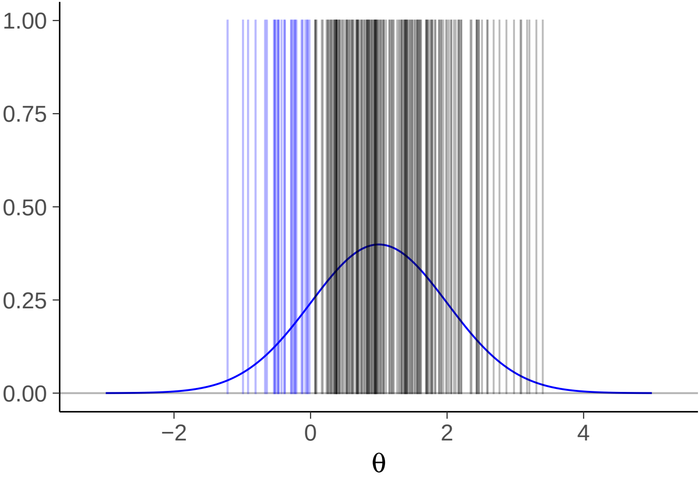

$p(\theta \leq 0) \approx \frac{1}{S}\sum_{s}^{S} \mathrm{I}(\theta^{(s)} \leq 0) \approx 0.14$

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  - to approximate expectations (integrals)

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$$E_{p(\theta|y)}[g(\theta)] = \int g(\theta) p(\theta \mid y) d\theta \approx \frac{1}{S} \sum_{s=1}^{S} g(\theta^{(s)})$$

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- If $p(g(\theta))$ has finite variance, then the Monte Carlo estimate is unbiased and the error approaches 0 with increasing S based on the central limit theorem (CLT)
  - more about this later

Marginalization

- Joint distribution of parameters

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$p(\mu \mid y)$ is a marginal distribution

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  - sometimes the integral has an analytic solution
- Monte Carlo approximation

$$\begin{aligned} \text{if} \quad & (\mu^{(s)}, \sigma^{(s)}) \sim p(\mu, \sigma \mid y) \\ \text{then} \quad & \mu^{(s)} \sim p(\mu \mid y) \end{aligned}$$

Marginalization - predictive distribution

- Posterior predictive distribution for a future $\tilde{y}$ is obtained by marginalizing the joint distribution of unknowns

$$p(\tilde{y}, \mu, \sigma, \mid y)$$

Marginalization - predictive distribution

- Posterior predictive distribution for a future $\tilde{y}$ is obtained by marginalizing the joint distribution of unknowns

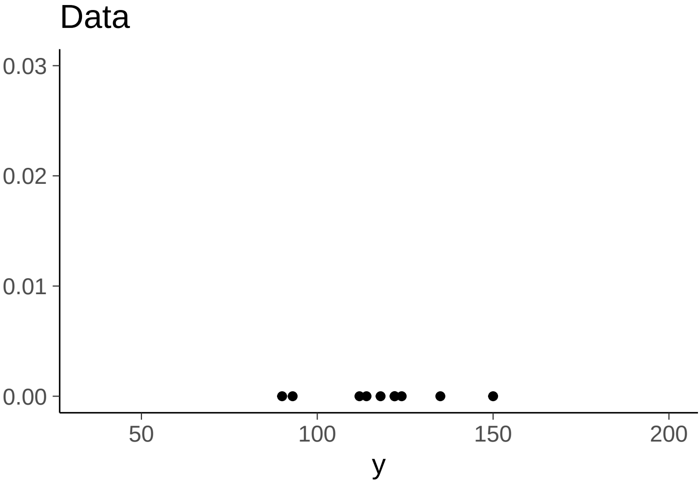
$$p(\tilde{y} \mid y) = \int p(\tilde{y}, \mu, \sigma, \mid y) d\mu\sigma$$

Marginalization - predictive distribution

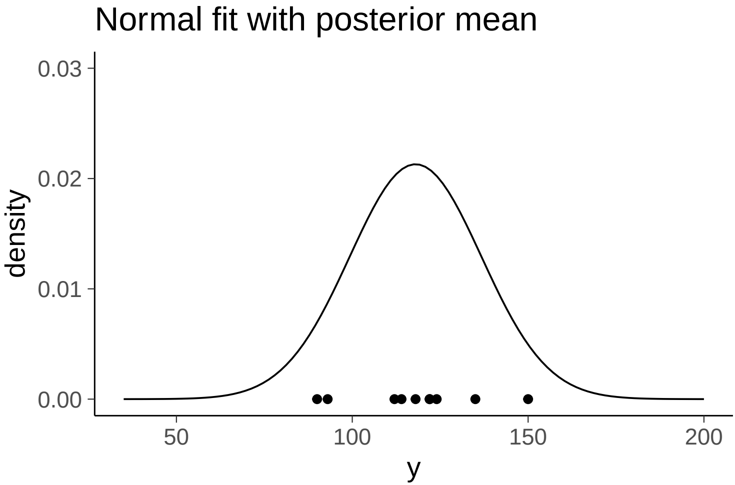
- Posterior predictive distribution for a future $\tilde{y}$ is obtained by marginalizing the joint distribution of unknowns

$$
\begin{aligned}
p(\tilde{y} \mid y) &= \int p(\tilde{y}, \mu, \sigma, \mid y) d\mu\sigma \\
&= \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma
\end{aligned}
$$

Normal distribution example

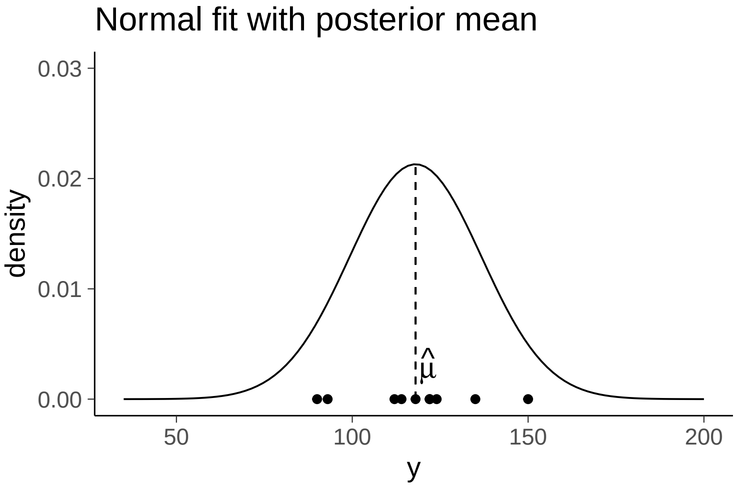


Normal distribution example



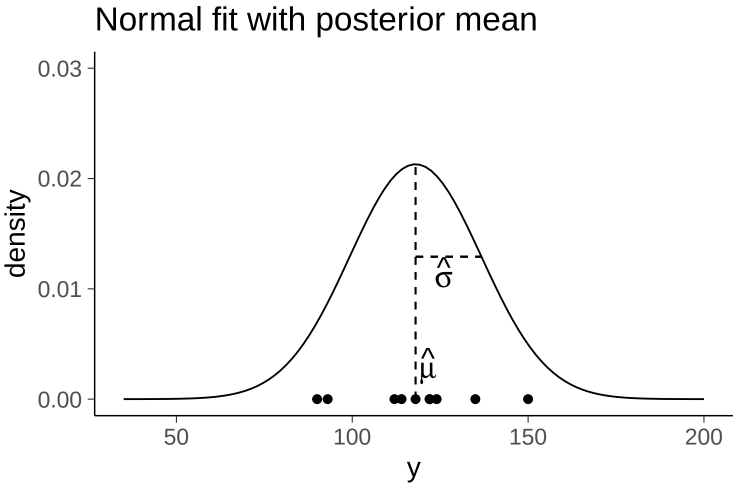
$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \mu)^2\right)$$

Normal distribution example



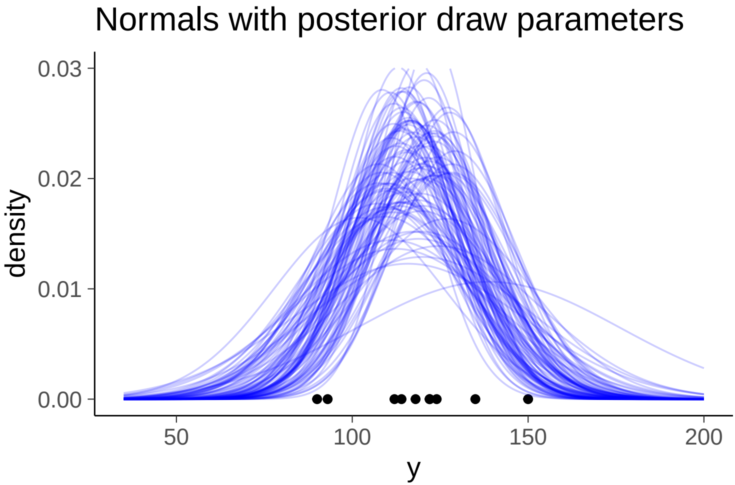
$$p(y \mid \color{red}{\mu}\color{black}{}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y - \color{red}{\mu}\color{black}{})^2\right)$$

Normal distribution example



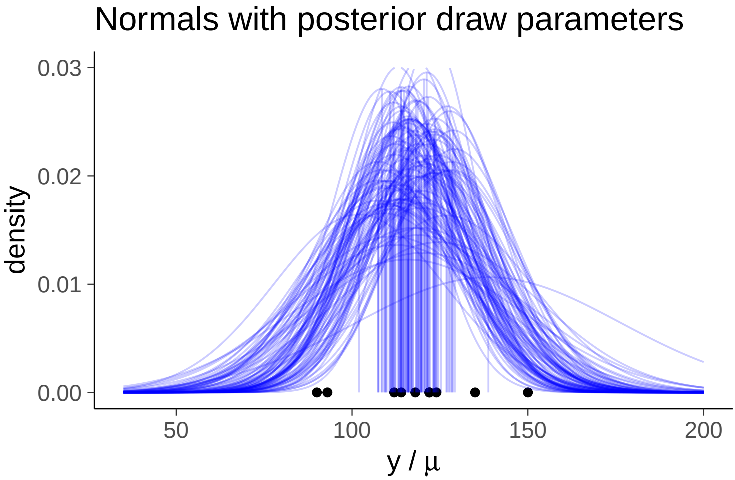
$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

Normal distribution example



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Normal distribution example



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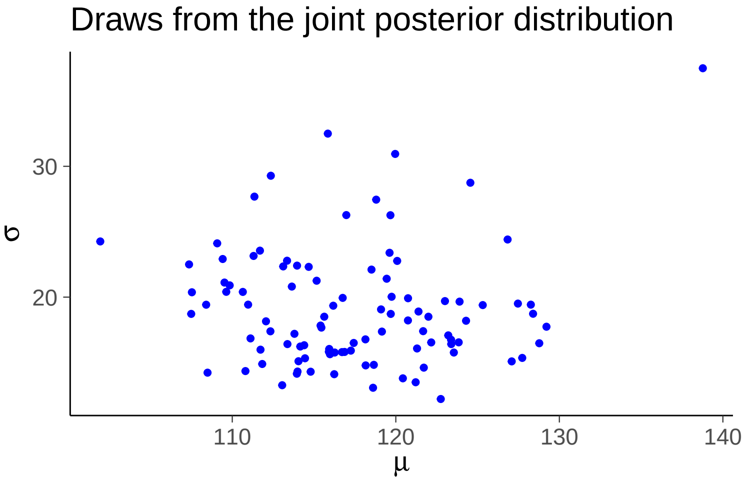
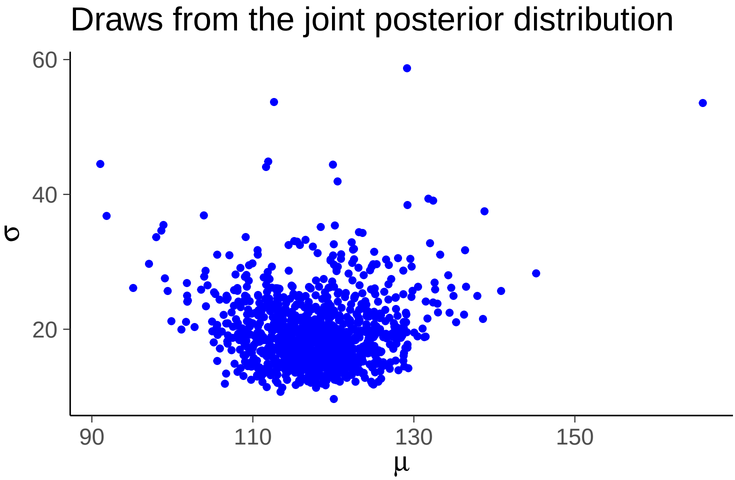
Normal distribution example



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

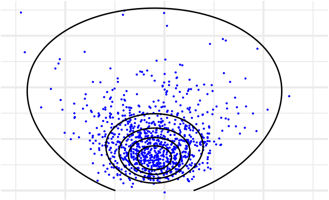
Normal distribution example



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior

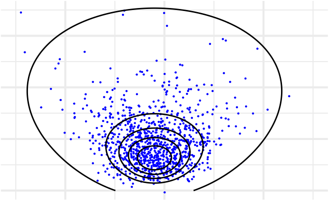
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$



Joint posterior

$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$



Joint posterior

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

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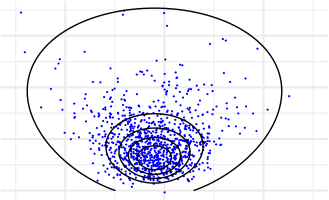
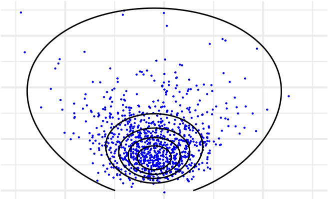


with $p(\mu, \sigma) \propto \sigma^{-1}$ (see BDA3 p. 21 transformation of variables)

Joint posterior

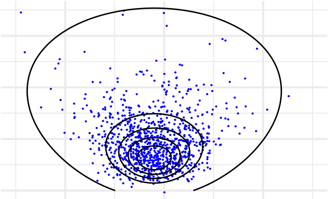


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

Joint posterior

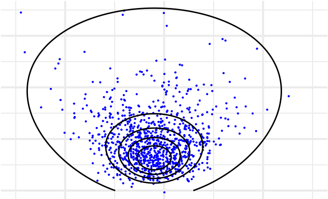


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right)$$

Joint posterior



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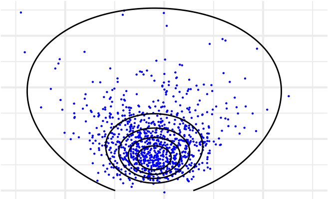
$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

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$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

Normal - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

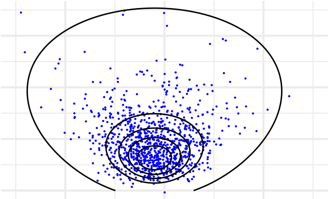
$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

Joint posterior



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

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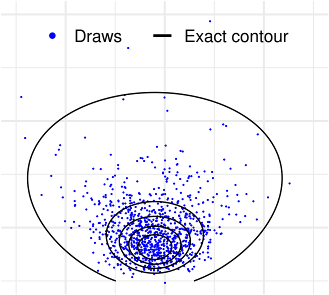
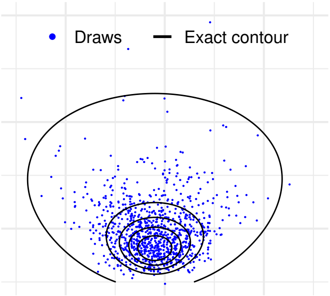
where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$

Joint posterior

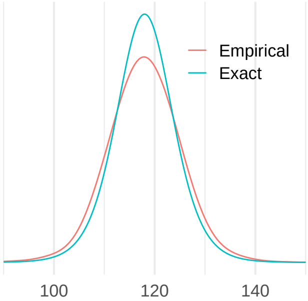


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



Marginal of mu

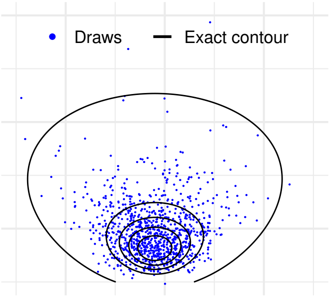


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

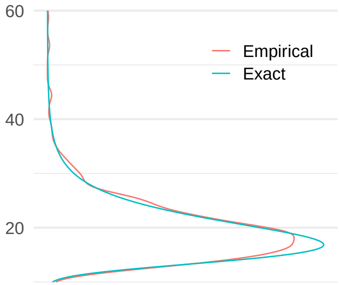
marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

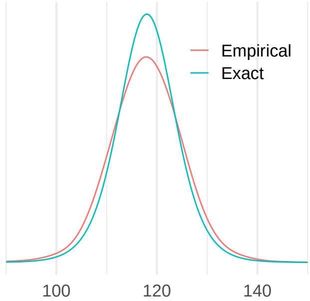
Joint posterior



Marginal of sigma



Marginal of mu



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$p(\sigma^2 \mid y) \quad \propto \quad \int p(\mu, \sigma^2 \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
\end{aligned}
$$

$$
\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1
$$

$$
\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\qquad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\
p(\sigma^2 \mid y) &= \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)
\end{aligned}
$$

Normal - non-informative prior

Known mean

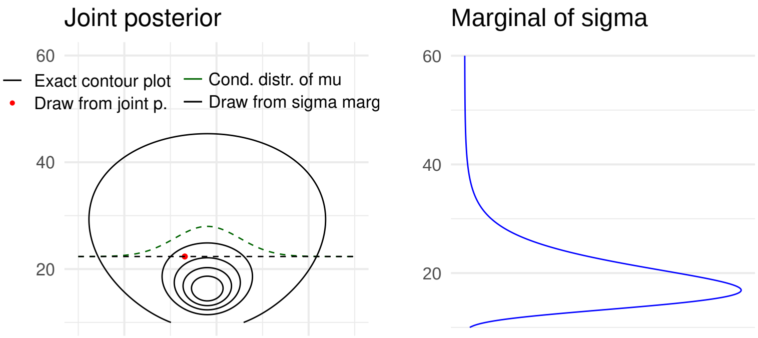
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, v)$$

$$\text{where} \quad v = \frac{1}{n}\sum_{i=1}^{n}(y_i - \theta)^2$$

Unknown mean

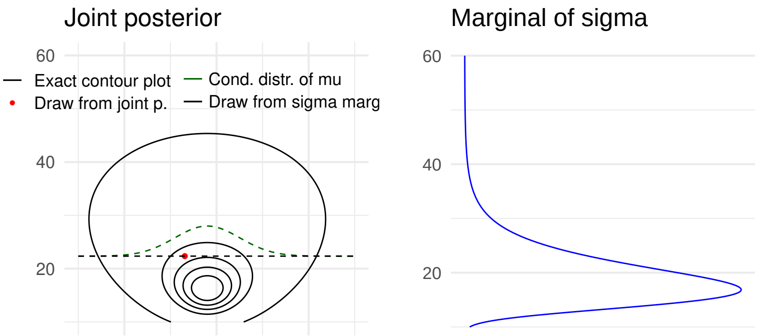
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$

$$\text{where} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$$



Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$



Factorization

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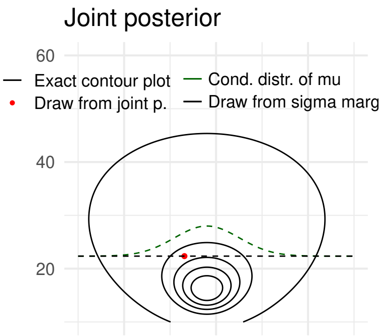
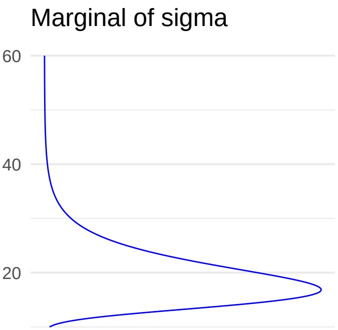
$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$





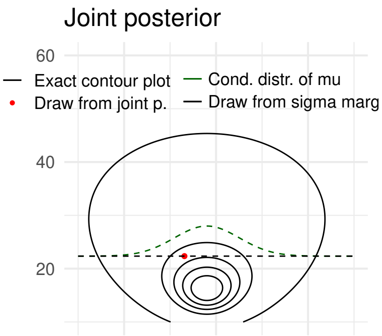
Factorization

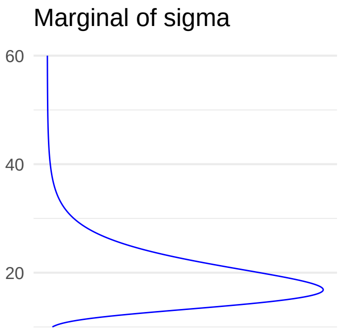
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n - 1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$





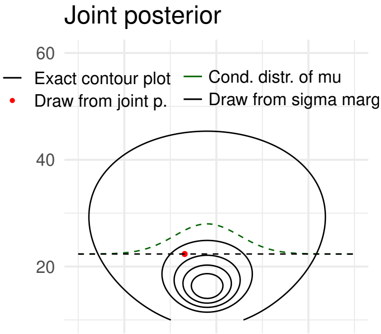
Factorization

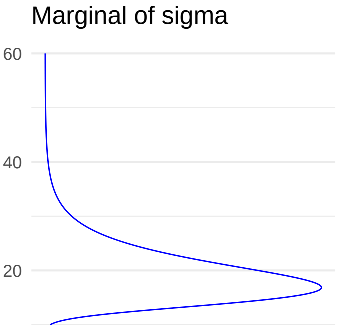
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) \propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right)$$





Factorization

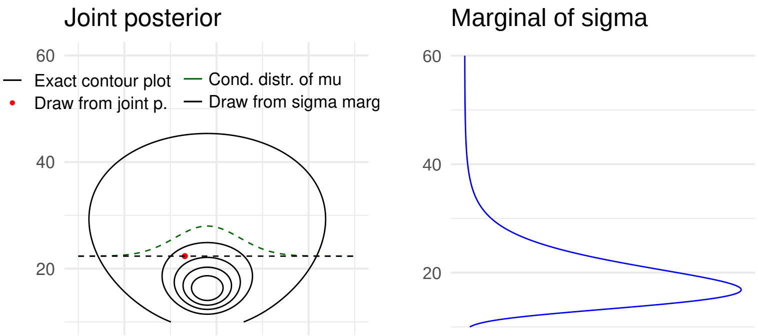
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

$$\mu^{(s)} \sim p(\mu \mid (\sigma^2)^{(s)}, y)$$



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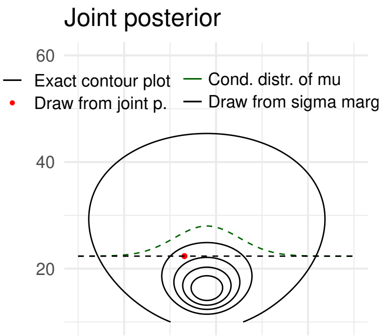

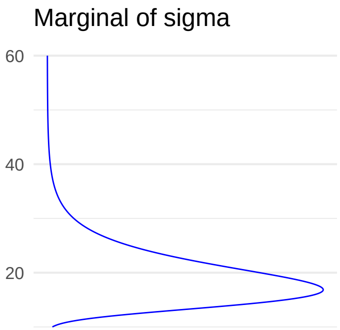

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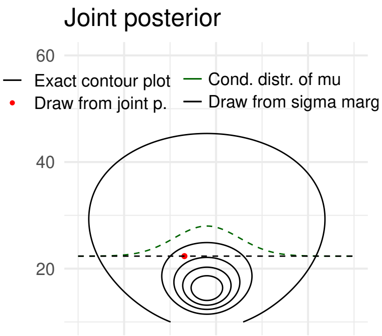
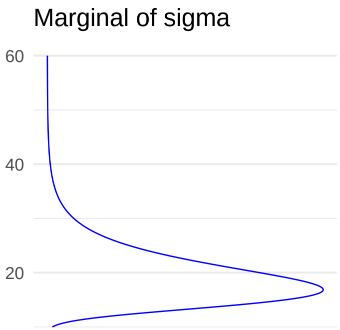
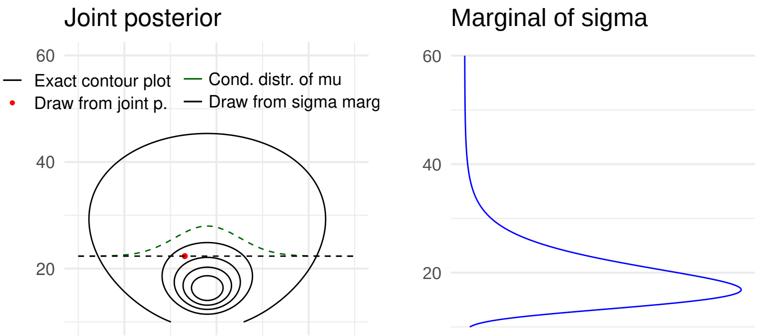




Factorization

$$p(\mu, \sigma^2 \mid y) = {\color{green}p(\mu \mid \sigma^2, y)}{\color{blue}p(\sigma^2 \mid y)}$$
$$(\sigma^2)^{(s)} \sim {\color{blue}p(\sigma^2 \mid y)}$$



Factorization

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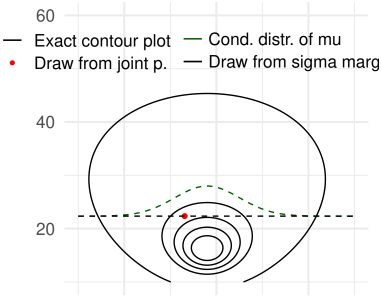
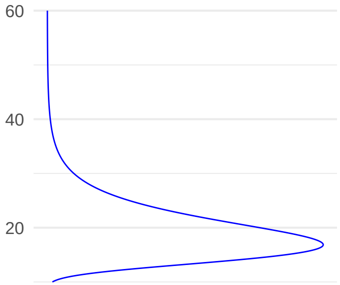
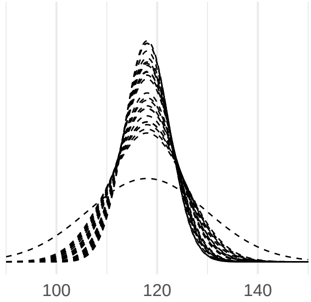
Joint posterior



Marginal of sigma



Cond distr of mu for 25 draws



Factorization

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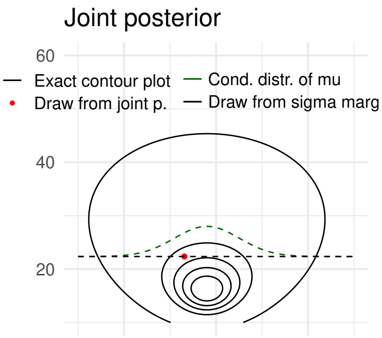

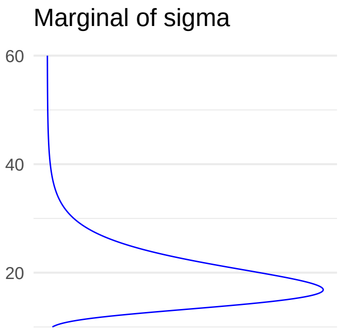

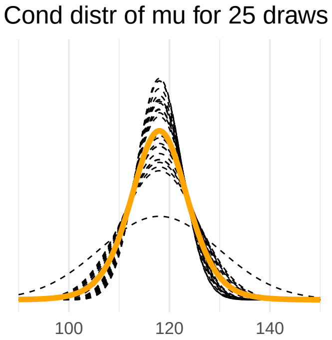

Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

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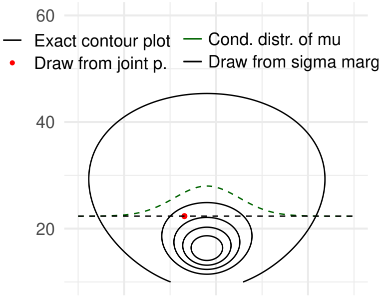
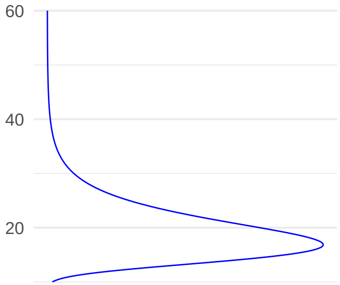
$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

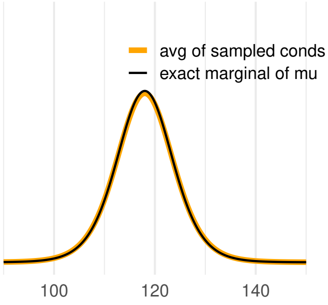
Joint posterior



Marginal of sigma



Cond. distr of mu



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Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

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\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
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Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2$$

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Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

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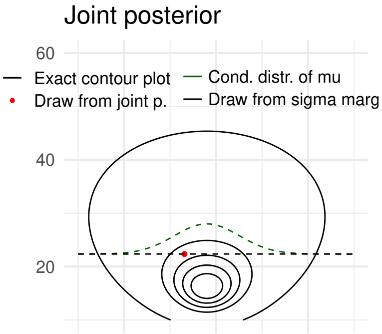
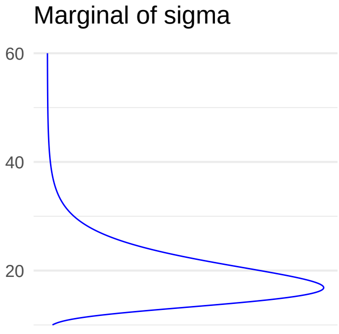
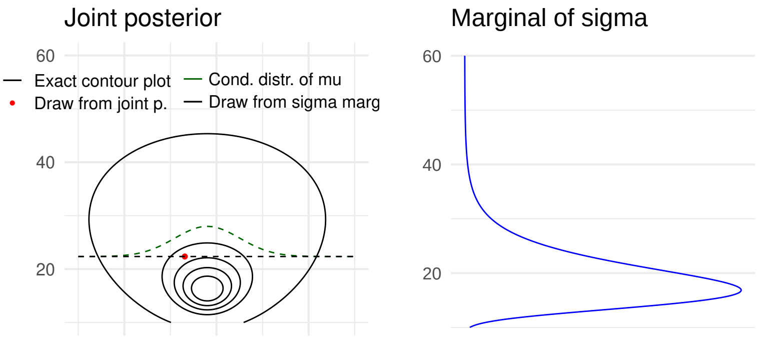
$$p(\mu \mid y) = t_{n-1}(\mu \mid \bar{y}, s^2/n) \quad \text{Student's } t$$





Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$



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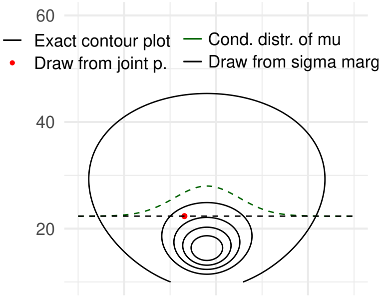
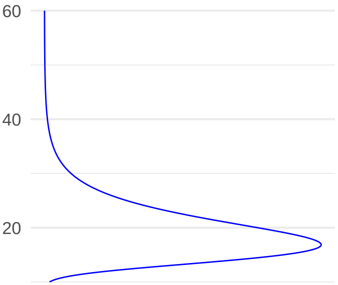
Joint posterior



Marginal of sigma



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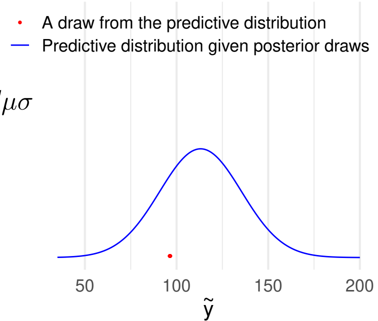
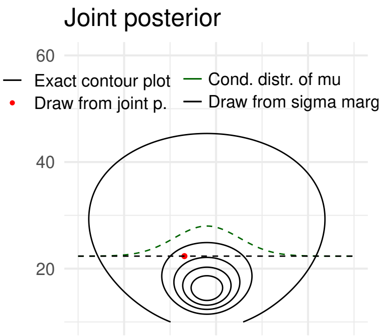
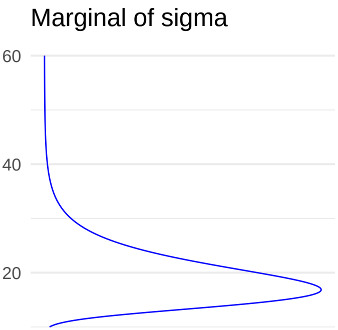
Posterior predictive distribution







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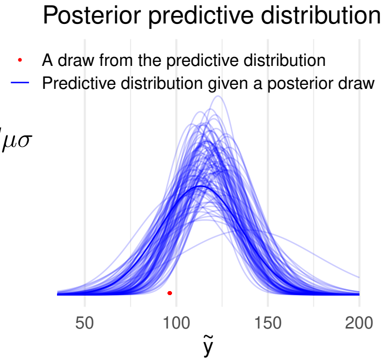

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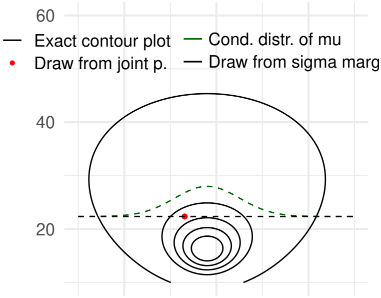
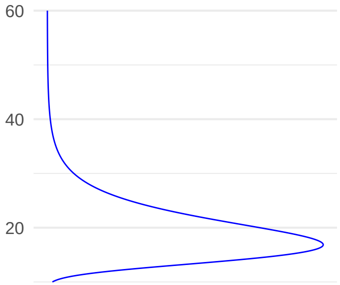


Marginal of sigma



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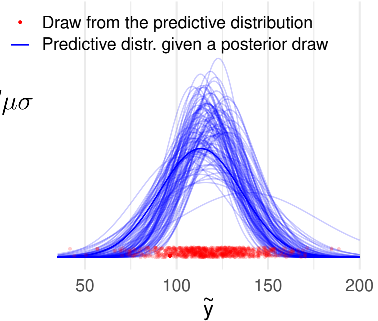
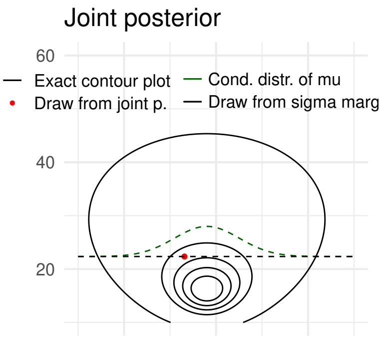
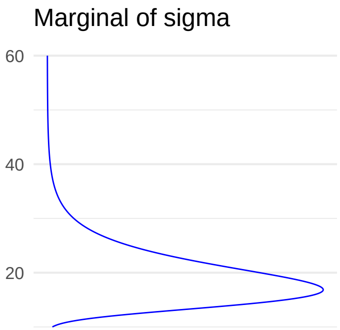
Posterior predictive distribution







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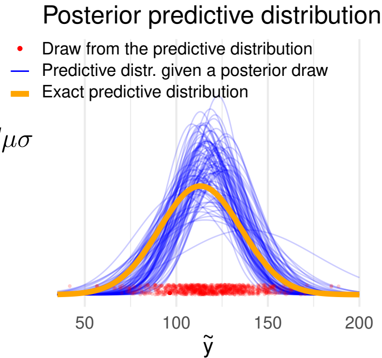

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$p(\tilde{y} \mid \sigma^2, y) = \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu$$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$
\begin{aligned}
p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\
&= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu
\end{aligned}
$$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$
\begin{aligned}
p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\
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&= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2)
\end{aligned}
$$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\ &= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2) \end{aligned}$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

Normal - posterior predictive distribution

Posterior predictive distribution given known variance

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\end{aligned}
$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

$$
p(\tilde{y} \mid y) = t_{n-1}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n}) s^2)
$$

Normal - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)

Normal - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)
- Handy parameterization

$$
\begin{aligned}
\mu \mid \sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0) \\
\sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)
\end{aligned}
$$

which can be written as

$$
p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)
$$

Normal - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)
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$$\begin{aligned} \mu \mid \sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

- μ and σ^2 are a priori dependent
  - if σ^2 is large, then μ has wide prior

Normal - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}$$

Comparison of means of two normals

- The difference of two normally distributed variables is normally distributed
- The difference of two t distributed variables with different variances and degrees of freedom doesn't have a closed form
    - but easy to sample from the two distributions, and obtain draws of the differences

$$
\begin{aligned}
\text{if} \quad \mu_1^{(s)} &\sim p(\mu_1 \mid y_1) \\
\mu_2^{(s)} &\sim p(\mu_2 \mid y_2) \\
\delta^{(s)} &= \mu_1^{(s)} - \mu_2^{(s)} \\
\text{then} \quad \delta^{(s)} &\sim p(\delta \mid y_1, y_2)
\end{aligned}
$$

Multivariate normal

- Observation model

$$p(y \mid \mu, \Sigma) \propto \mid \Sigma \mid^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)^T \Sigma^{-1} (y-\mu)\right)$$

- BDA3 p. 72–
- Recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

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- BDA3 p. 72–
- Recommended LKJ-prior mentioned in Appendix A, see more in Stan manual
- Gaussian process and Gaussian Markov random field models are in practice computed with multivariate normals
  - GPs in BDA3 Chapter 21, and a course in spring
  - GPs and GMRFs often used also as priors for latent functions and combined with non-normal observation models

Normal linear regression

- $y_i \sim \mathrm{N}(\alpha + \beta x_i, \sigma^2), \quad i = 1, \ldots, N$

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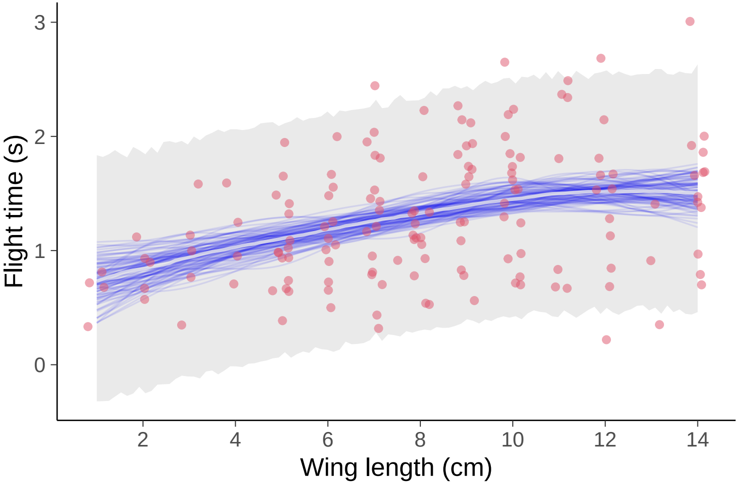
- more in BDA3 Chapter 14 (not part of the course) and Regression and Other Stories book

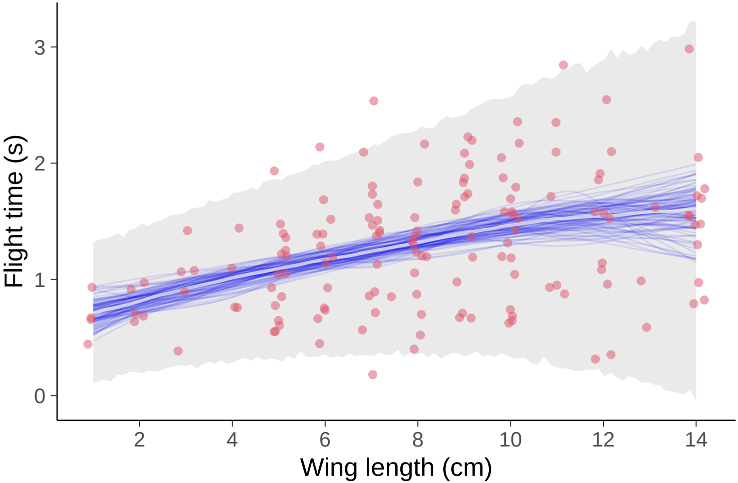
Paper helicopter flight time

$$y \sim \text{normal}(f, \sigma)$$
$$f \sim GP(0, K(x, \theta))$$



Paper helicopter flight time

$$y \sim \text{normal}(f, \sigma)$$
$$f \sim GP(0, K_f(x, \theta_f))$$
$$\log(\sigma) \sim GP(0, K_g(x, \theta_g))$$



Scale mixture of normals

- Many useful distributions can be presented as scale mixture of normals, e.g.
  - Student's *t*
  - Cauchy
  - Double exponential aka Laplace
  - Horseshoe
  - R2-D2

Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^{k} \theta_j^{y_j},$$

- BDA3 p. 69–

Generalized linear model (GLM)

- $y_i \sim p(g^{-1}(\alpha + \beta x_i), \phi), \quad i = 1, \ldots, N$
  - where p is non-normal (in original definition in exponential family)
  - and g is a link function

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- Bioassay analysis is used as an example

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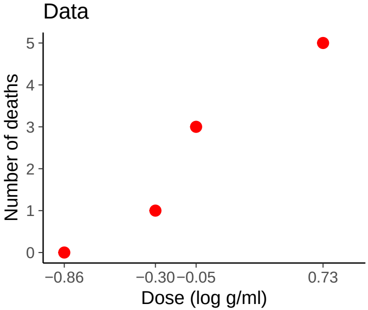
    - and g is a link function
- Bioassay analysis is used as an example
- More in BDA3 Chapter 16 and Regression and other stories book

Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



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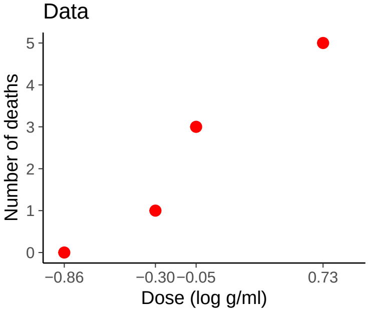

Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

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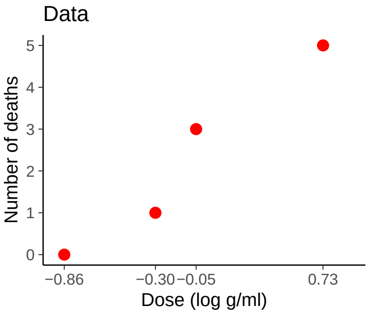


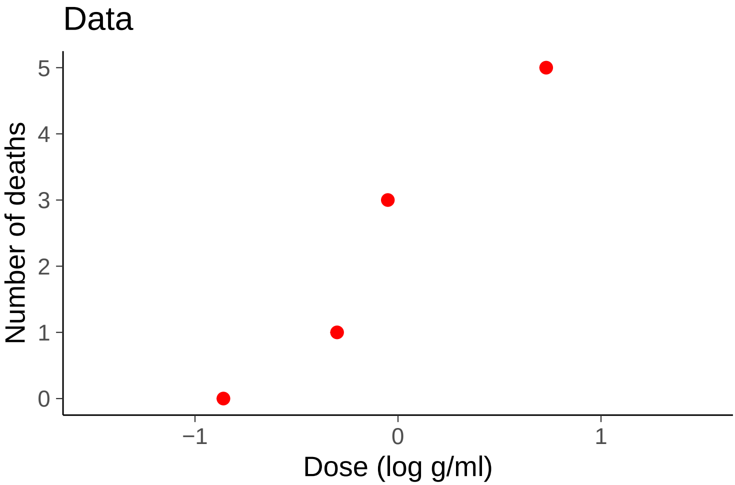
Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

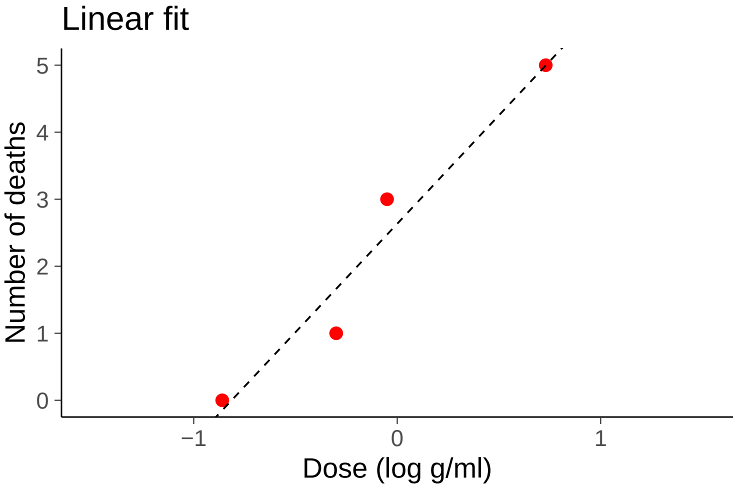
Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

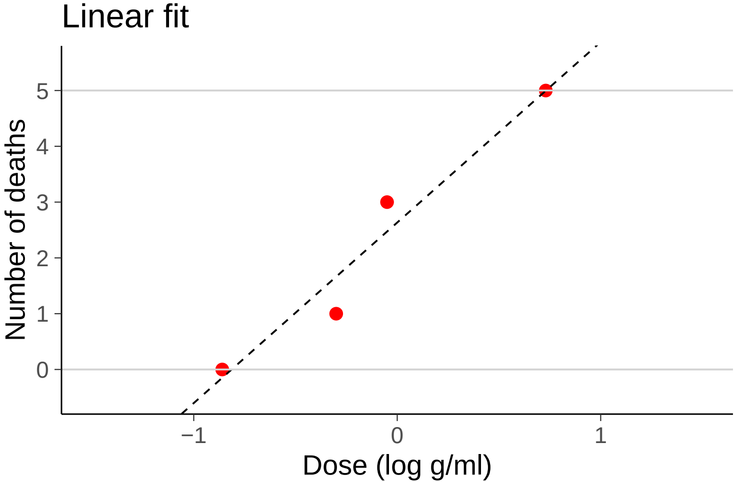
Bioassay



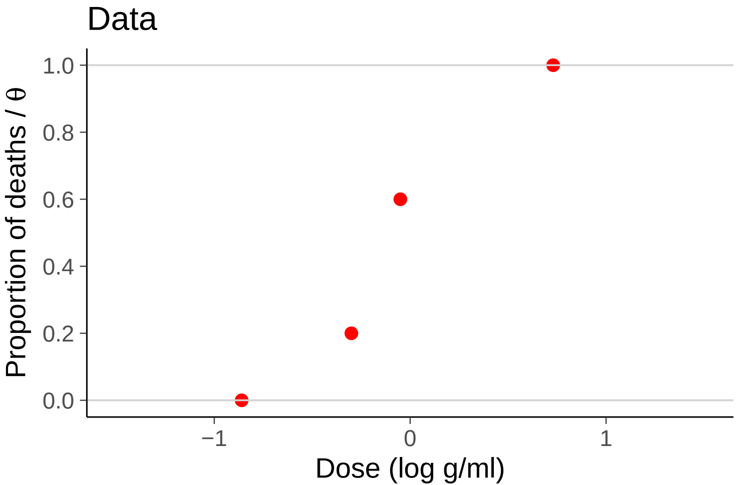
Bioassay



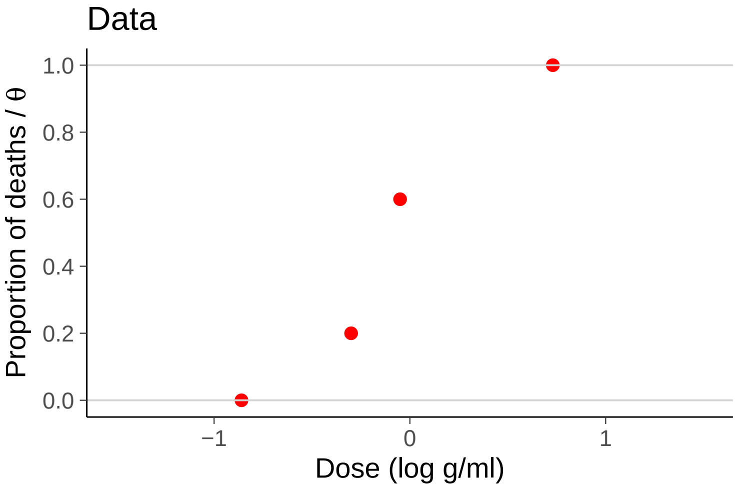
Bioassay



Bioassay



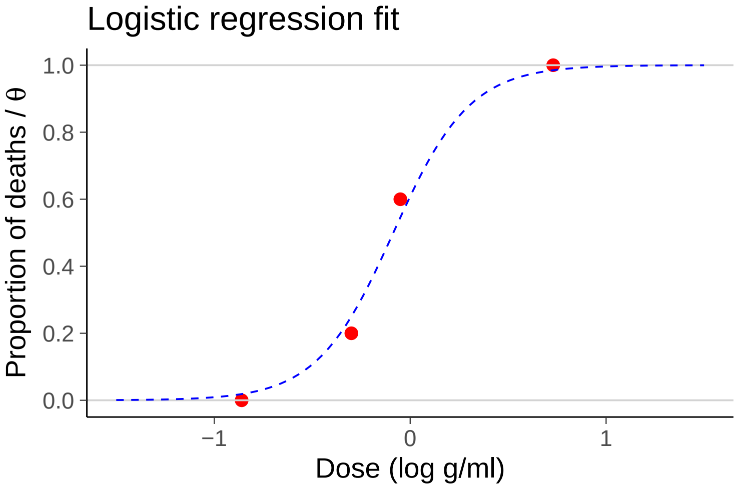
Bioassay



Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Bioassay



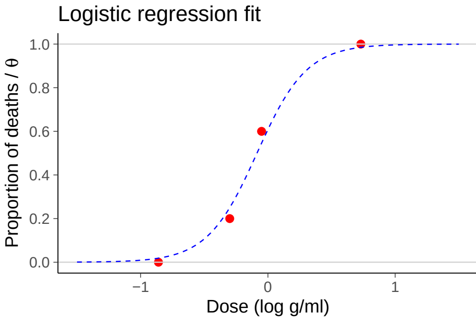
Binomial model

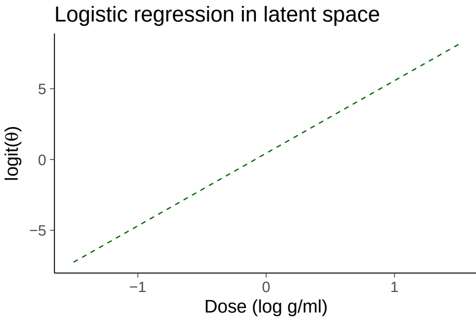
$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i), \quad \text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

Bioassay

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

$$\begin{aligned}\mathrm{logit}(\theta_i) &= \log\left(\frac{\theta_i}{1-\theta_i}\right)\\ &= \alpha + \beta x_i\end{aligned}$$



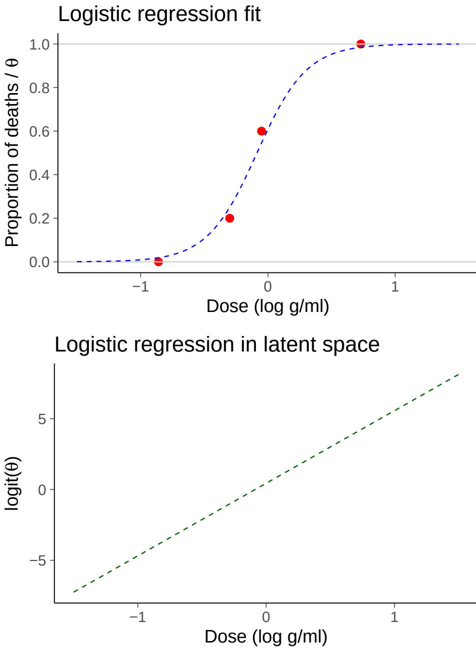


Bioassay

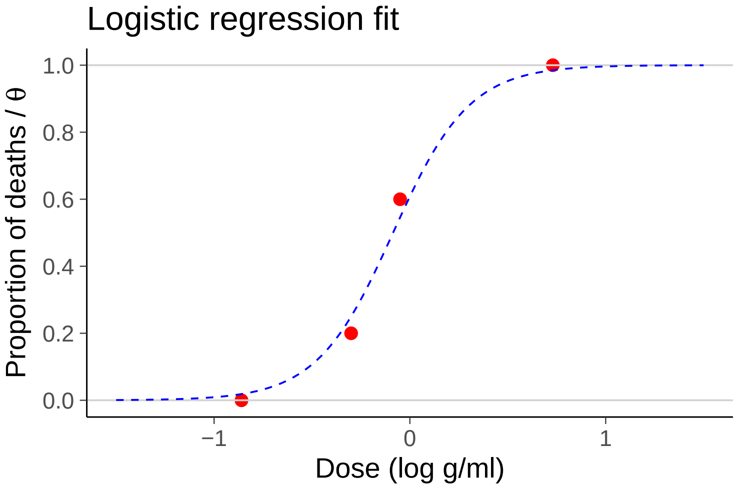
$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

$$\mathrm{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

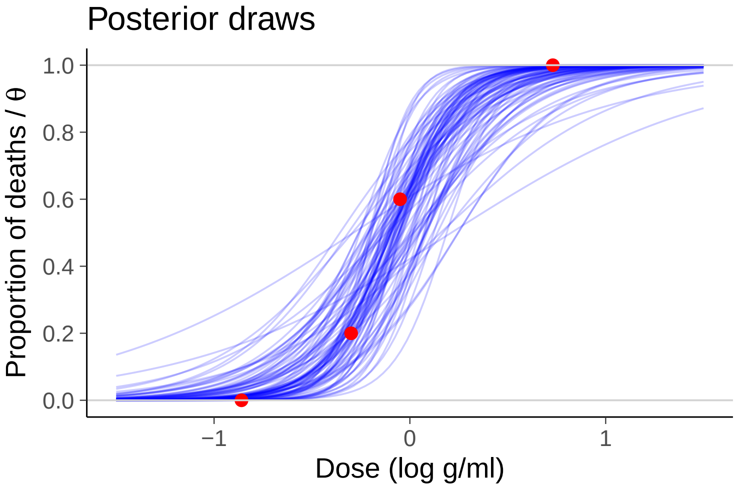
$$\theta_i = \frac{1}{1+\exp(-(\alpha+\beta x_i))}$$



Bioassay

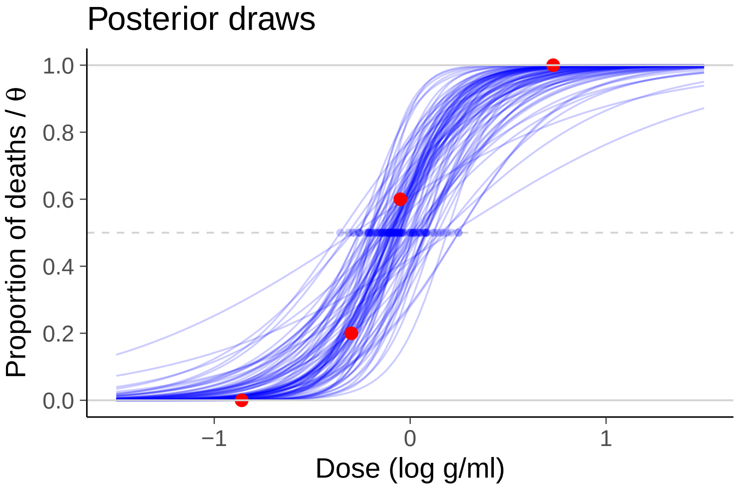

Logistic regression fit
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



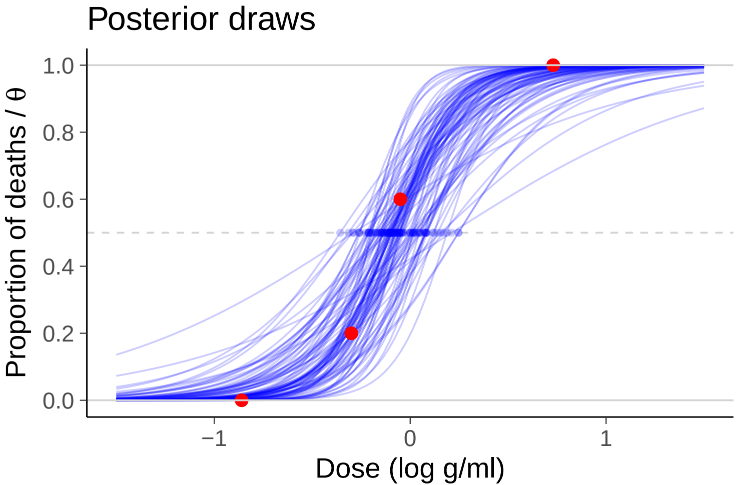
Posterior draws
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



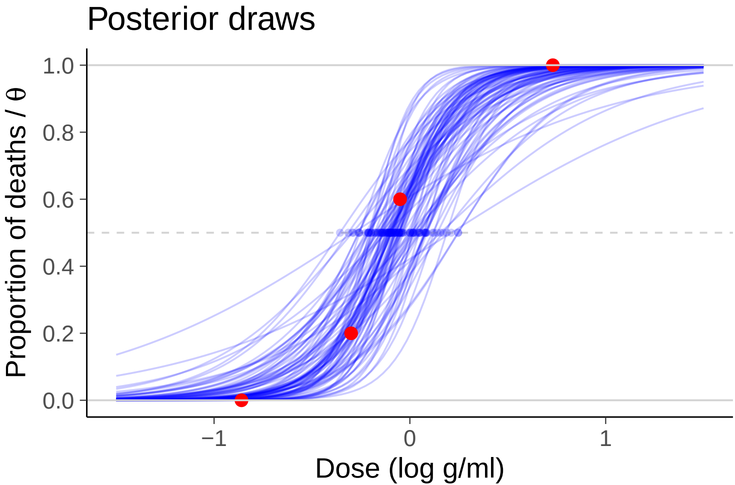
LD50: $\mathrm{E}[\theta] = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5$

Bioassay



LD50: $\mathrm{E}[\theta] = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

Bioassay



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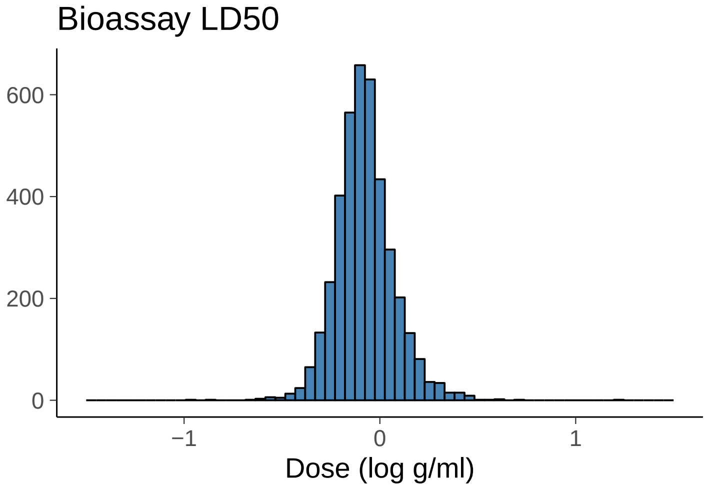
$$x^{(s)}_{\mathrm{LD50}} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay



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Bioassay posterior

Binomial model

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Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

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Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

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$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha+\beta x_i)]^{y_i}[1-\mathrm{logit}^{-1}(\alpha+\beta x_i)]^{n_i-y_i}$$

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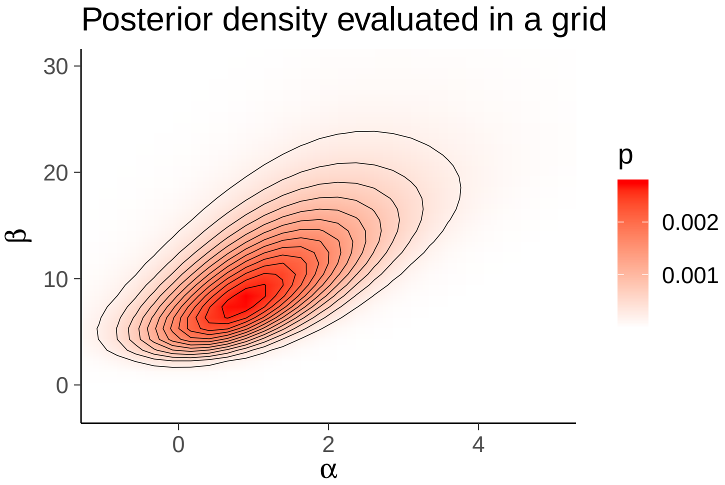
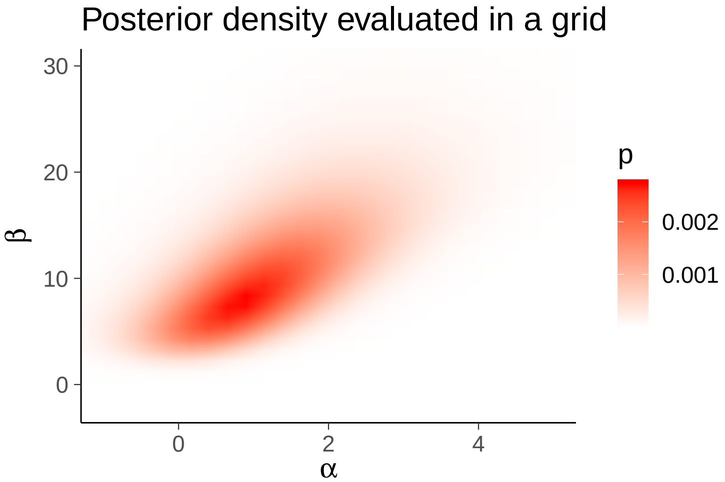
Posterior (with uniform prior on α, β)

$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i)$$

Bioassay

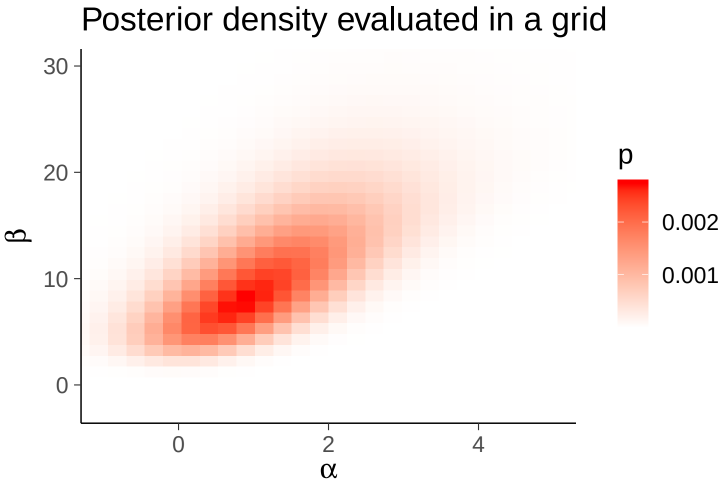

Posterior density evaluated in a grid
30
20
10
0
β
0
2
4
α
p
0.002
0.001

Bioassay



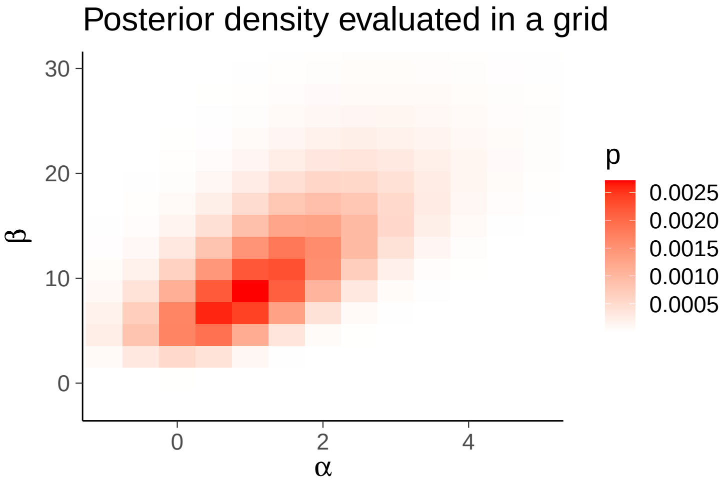
Density evaluated in grid, but plotted using interpolation

Bioassay



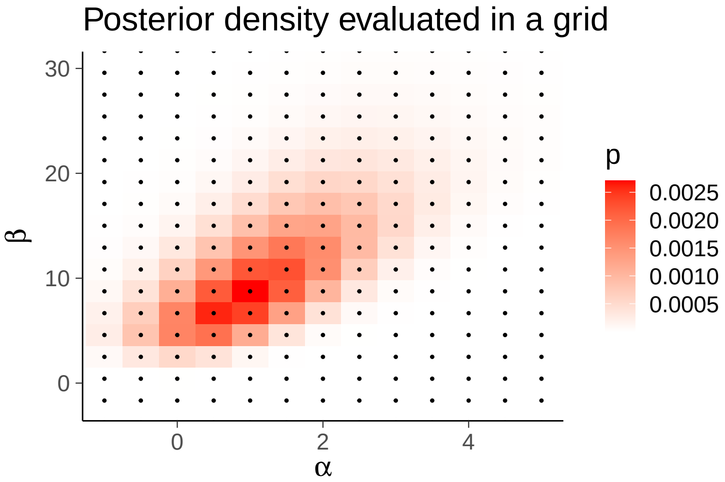
Density evaluated in grid, and plotted without interpolation

Bioassay



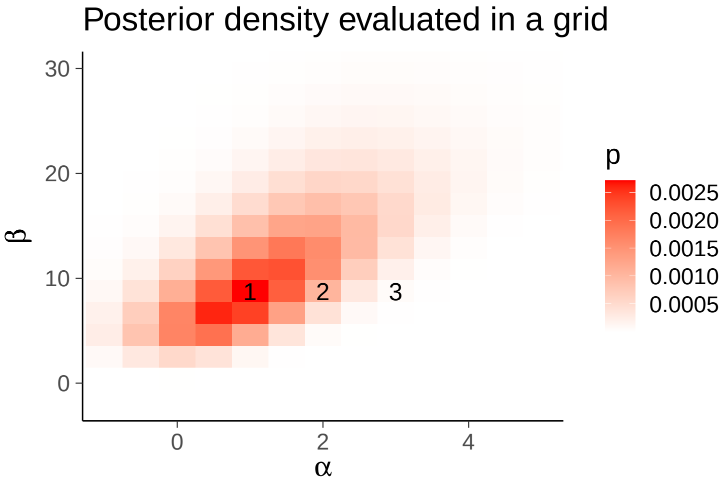
Density evaluated in a coarser grid

Bioassay



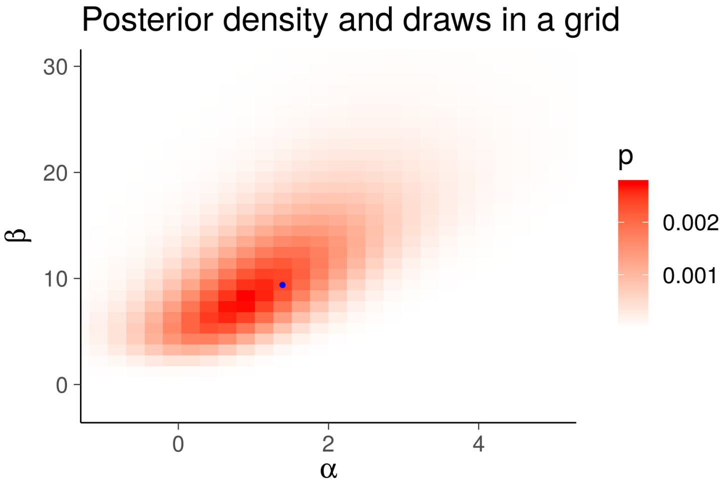
- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell

Bioassay



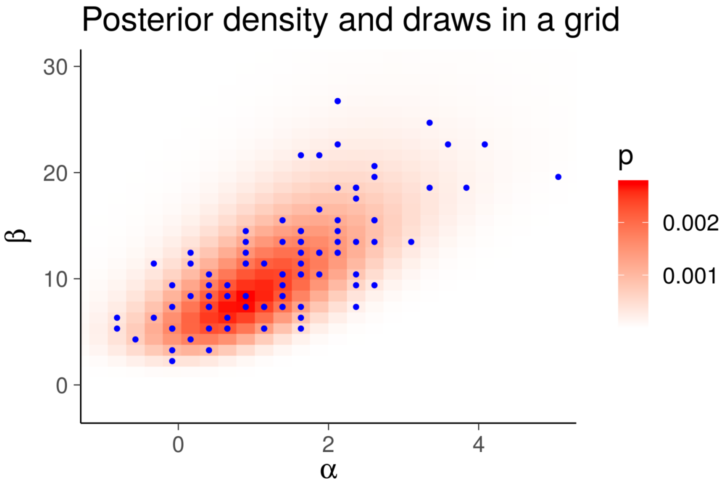
- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

Bioassay



- Sample according to grid cell probabilities

Bioassay



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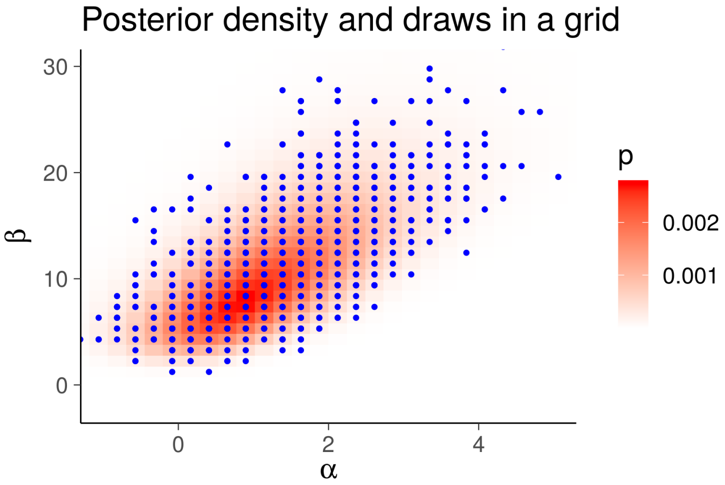
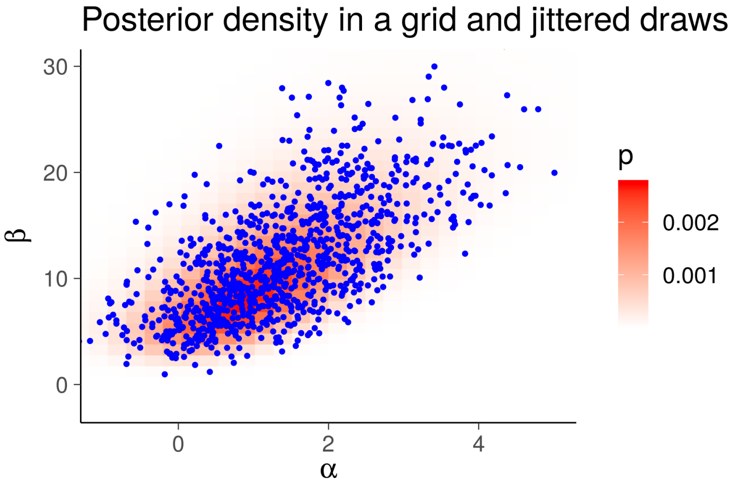


- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

Bioassay



- Jitter can be added to improve visualization

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S} -\frac{\alpha^{(s)}}{\beta^{(s)}}$$

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- Instead of sampling, grid could be used to evaluate functions directly, for example

$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} -\frac{\alpha^{(t)}}{\beta^{(t)}} w_{\mathrm{cell}}^{(t)},$$

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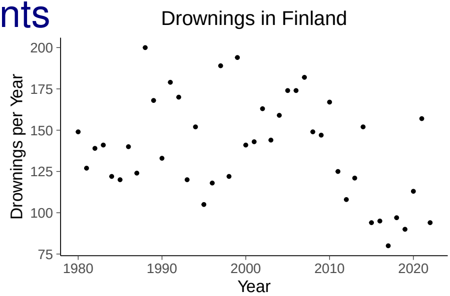
- Grid sampling gets computationally too expensive in high dimensions

Example: GLM for counts

Count of deaths, y_i	Year
149	1980
127	1981
139	1982
$\vdots$	$\vdots$
157	2021
94	2022



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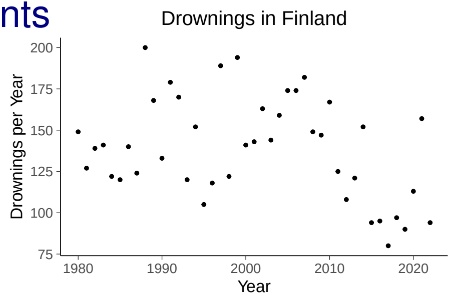

Swimming is popular in Finland, but also hazardous

- On average $\sim$140 drownings per year
- Finnish government has invested in measures for reducing deaths
- Recent narrative based on effectiveness of education

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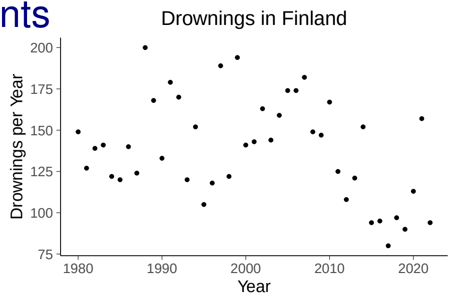

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- On average $\sim$140 drownings per year
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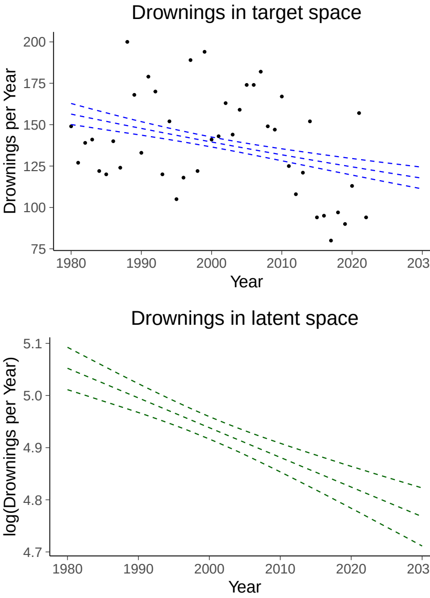
Bayesian methods help

- Describe trends over time
- Evaluate uncertainty

Example: GLM for counts

$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i = e^{\alpha+\beta x_i}$$



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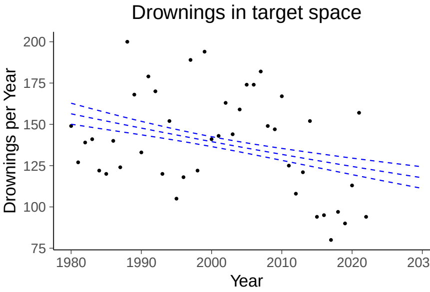
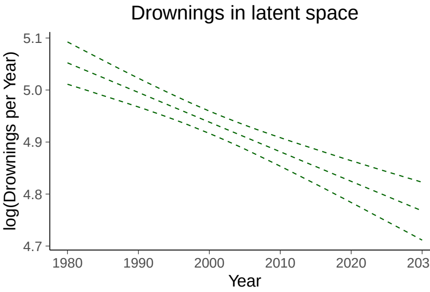
$$\text{Poisson}(y_i \mid \mu_i) = \frac{1}{y_i!} \mu_i^{y_i} e^{-\mu_i}$$





Example: GLM for counts

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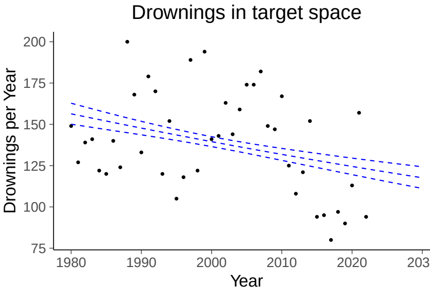
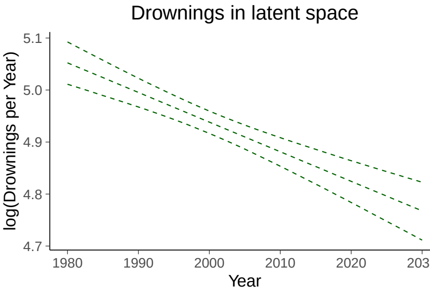
$$\mu_i = e^{\alpha + \beta x_i}$$

$$\text{Poisson}(y_i \mid \mu_i) = \frac{1}{y_i!}\mu_i^{y_i} e^{-\mu_i}$$

Alternatively :

$$y_i \mid \mu_i, \phi \sim \text{Neg-bin}(y_i \mid \mu_i, \phi)$$





Example: GLM for counts

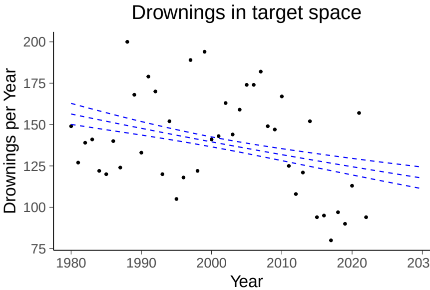
$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i = e^{\alpha + \beta x_i}$$

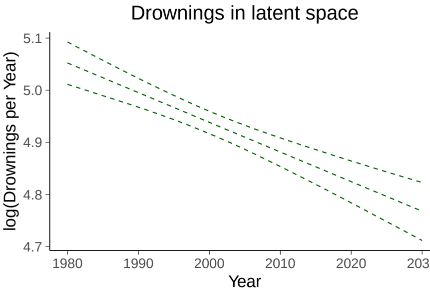
$$\text{Poisson}(y_i \mid \mu_i) = \frac{1}{y_i!}\mu_i^{y_i} e^{-\mu_i}$$

Alternatively :

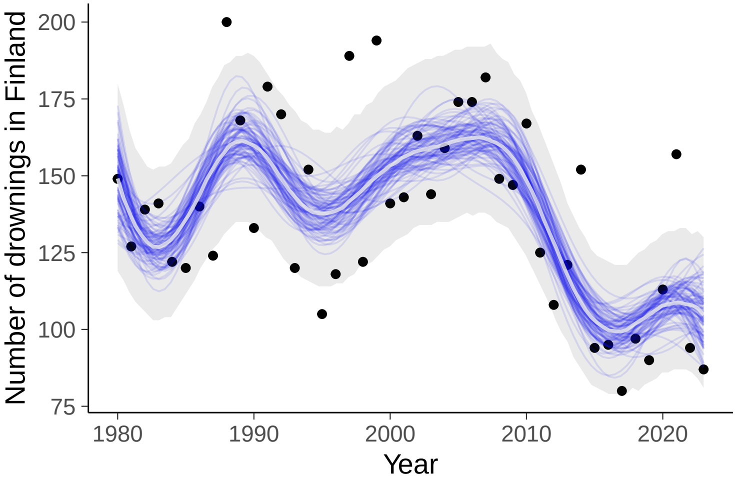
$$y_i \mid \mu_i, \phi \sim \text{Neg-bin}(y_i \mid \mu_i, \phi)$$

$$\text{Neg-bin}(y_i \mid \mu_i, \phi) = \frac{\Gamma(y_i + \phi)}{y_i!\Gamma(\phi)}\left(\frac{\mu_i}{\mu_i + \phi}\right)^{y_i}\left(\frac{\phi}{y_i + \phi}\right)^{\phi}$$



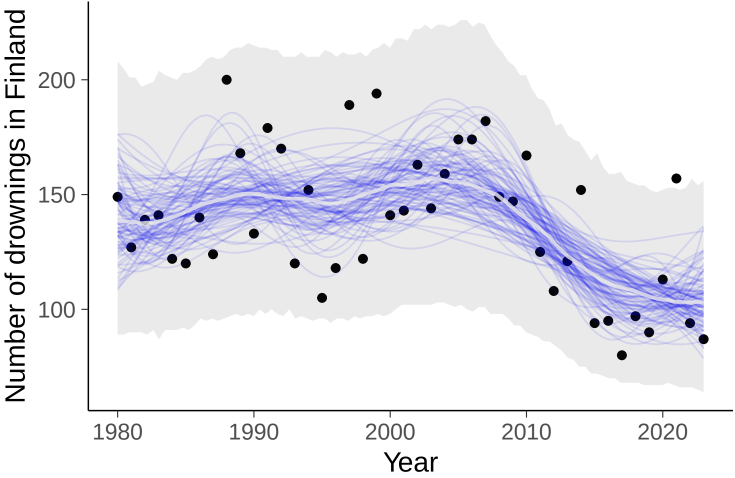


Example GLM: Gaussian Process Models



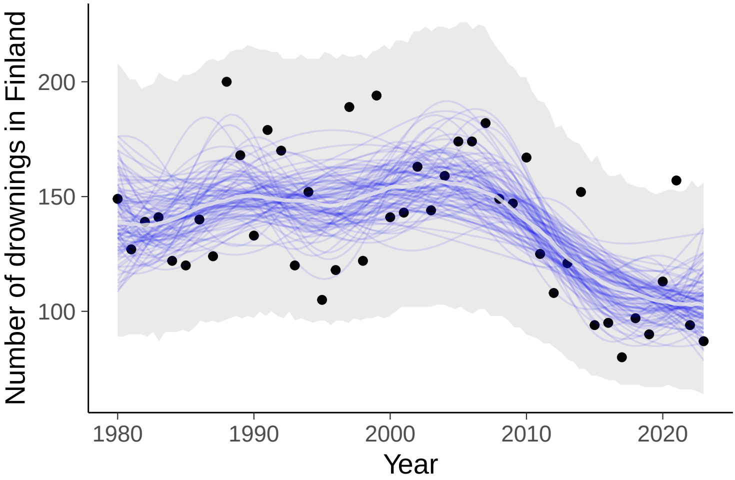
$$y_i \mid \mu_i \sim \text{Poisson}(\mu_i)$$
$$\mu_i \sim e^{f_i},\ f \sim \text{GP}(0, \text{k}(\text{Year}, \theta))$$

Example GLM: Gaussian Process Models



$$y_i \mid \mu_i \sim \text{Neg-bin}(\mu_i, \phi)$$
$$\mu_i \sim e^{f_i},\ f \sim \text{GP}(0, \text{k}(\text{Year}, \theta))$$

Example GLM: Gaussian Process Models



- Clear overdispersion
  - later we use posterior predictive checking and cross-validation to confirm this
- Trend interpretations shouldn't be based on one observation

Thinking counts

- For simplicity of exposition, we often start learning with normal observation models
- But we observe count data on a daily basis
- Very relevant in industry (number of sold products, ad views, customer count, etc.)
- Can you think of such examples from the class room?
  - Think of how many students attend BDA lectures over the course
  - Number of students who report getting sick over time until Christmas
  - Number of dropouts
  - Would you expect overdispersion?