Chapter 10

- 10.1 Numerical integration (overview)
- 10.2 Distributional approximations (overview, more in Chapter 4 and 13)
- 10.3 Direct simulation and rejection sampling (overview)
- 10.4 **Importance sampling**
  - used in PSIS-LOO (Lecture 9) and prior sensitivity analysis (Lecture ?)
- 10.5 **How many simulation draws are needed?**
  - **see chapter notes for how many significant digits to report**
  - **this week focus on independent draws and importance sampling, next week necessary adjustments needed for Markov chain Monte Carlo**
- 10.6 Software (can be skipped)
- 10.7 Debugging (can be skipped)

Notation

- In this chapter, generic $p(\theta)$ is used instead of $p(\theta|y)$
- Unnormalized distribution is denoted by $q(\cdot)$
  - $\int q(\theta)d\theta \neq 1$, but finite
  - $q(\cdot) \propto p(\cdot)$
- Proposal distribution is denoted by $g(\cdot)$

Numerical accuracy – floating point

- Floating point presentation of numbers. e.g. with 64bits
  - closest value to zero is $\approx 2.2 \cdot 10^{-308}$
    - generate sample of 600 from normal distribution: qr=rnorm(600)
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    - Laplace and ratio of girl and boy babies
    - pbeta(0.5, 241945, 251527) $\longrightarrow$ 1 (rounding)

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    - pbeta(0.5, 241945, 251527, lower.tail=FALSE) $\approx -1.2 \cdot 10^{-42}$ there is more accuracy near 0

Numerical accuracy – log scale

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  - convenience functions
    - `matrixStats::logSumExp(lx)` computes `log(sum(exp(lx)))` using the above rule
    - `log1p(x)` computes `log(1+x)` accurately also for $|x| \ll 1$
    - `expm1(x)` computes `exp(x) - 1` accurately also for $|x| \ll 1$

It's all about expectations

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$$\text{where} \quad p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

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- Monte Carlo methods which can sample from $p(\theta^{(s)}|y)$ using only $q(\theta^{(s)}|y)$ (each draw has weight 1/S)

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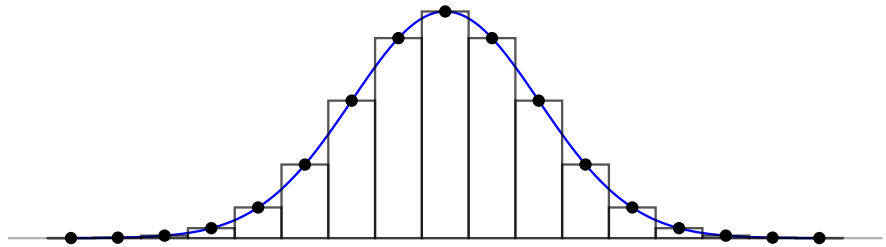
$$E_\theta[h(\theta)] = \int h(\theta)p(\theta|y)d\theta$$

- Conjugate priors and analytic solutions (Ch 1-5, Lec 2–3)
- Grid integration and other quadrature rules (Ch 3, 10, Lec 3–4)
- Independent Monte Carlo, rejection and importance sampling (Ch 10, Lec 4)
- Markov Chain Monte Carlo (Ch 11-12, Lec 5–6)
- Distributional approximations (Laplace, VB, EP) (Ch 4, 13)

Quadrature integration

- The simplest quadrature integration is grid integration

$$\mathrm{E}[\theta] \approx \sum_{t=1}^{T} \theta^{(t)} w^{(t)},$$



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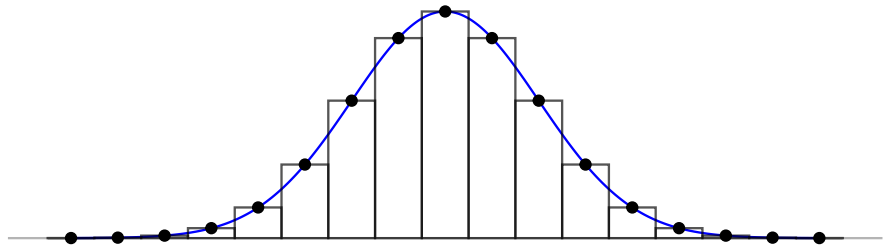

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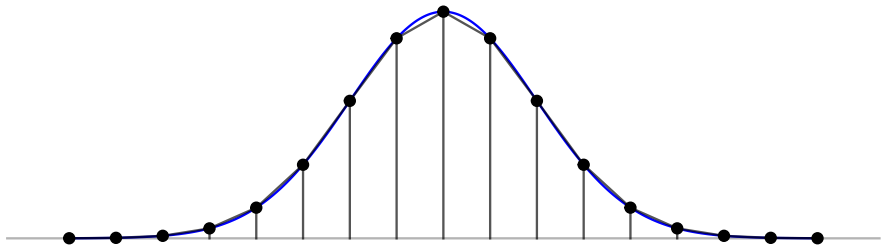

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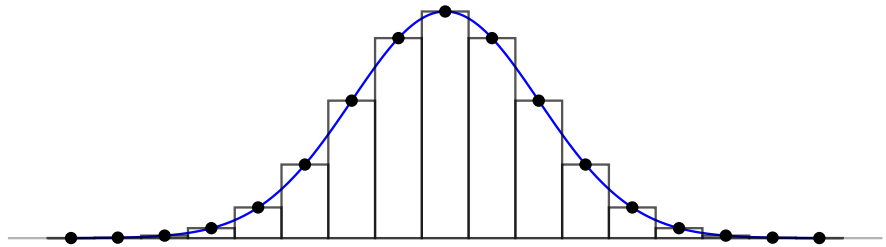

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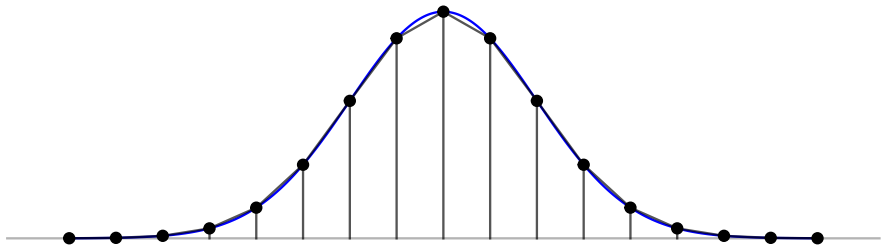

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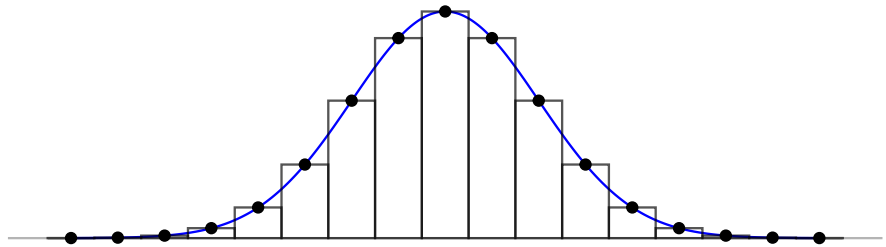

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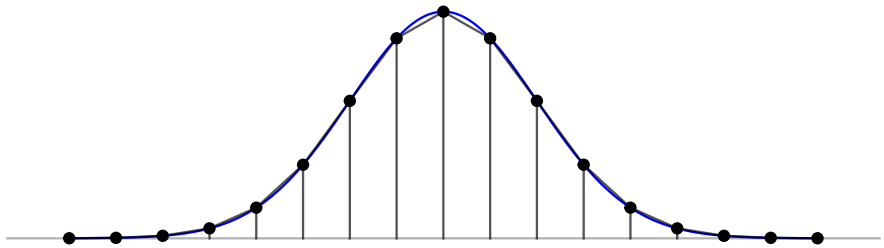
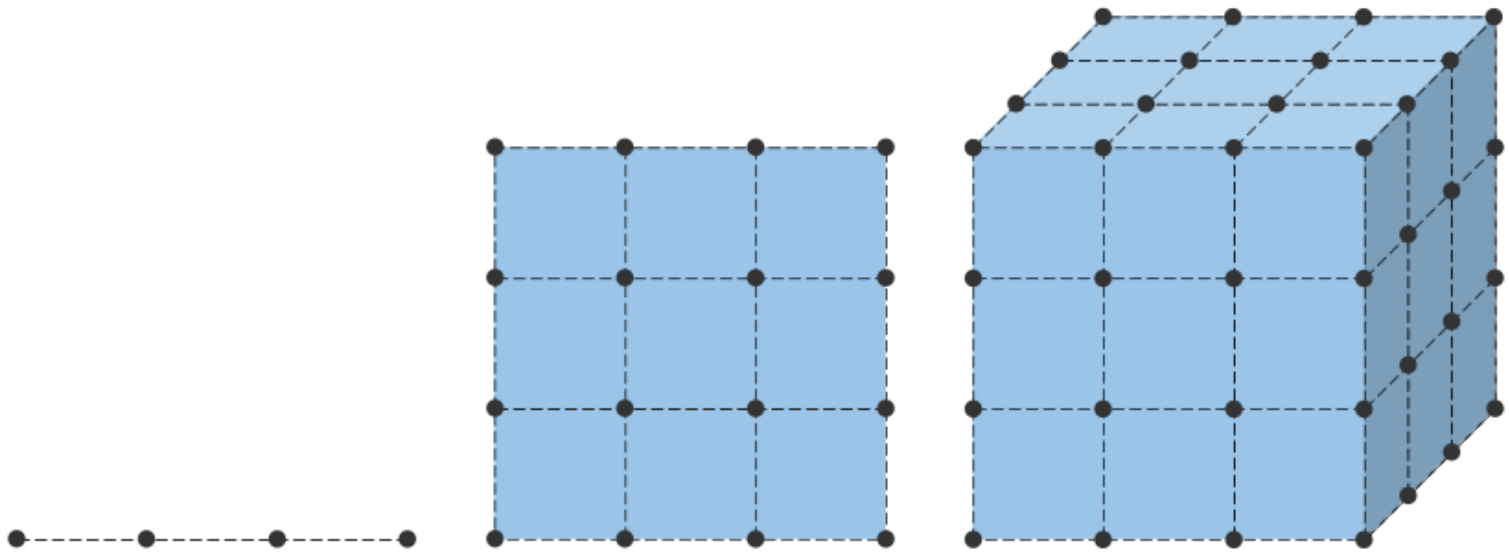


- Adaptive quadrature methods add evaluation points where needed, e.g., R function `integrate()`
- In 2D and higher
  - nested quadrature
  - product rules

Grid sampling and curse of dimensionality

- In general the number of evaluations increase exponentially c^D



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- e.g. 50 or 1000 grid points per dimension, and 10 dimensions
  - $\rightarrow$ $50^{10} \approx$ 1e17 grid points
  - $\rightarrow$ $1000^{10} \approx$ 1e30 grid points
- R and my current laptop can compute density of normal distribution about 50 million times per second
  - $\rightarrow$ evaluation in 1e17 grid points would take 60 years
  - $\rightarrow$ evaluation in 1e30 grid points would take 600 billion years

Monte Carlo - history

- Used already before computers
  - Buffon (18th century; needles)
  - De Forest, Darwin, Galton (19th century)
  - Pearson (19th century; roulette)
  - Gosset (Student, 1908; hat)

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- "Monte Carlo method" term was proposed by Metropolis, von Neumann or Ulam in the end of 1940s
  - they worked together in atomic bomb project
  - Metropolis and Ulam, "The Monte Carlo Method", 1949

Markov Chain Monte Carlo - history

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- Check this https://www.youtube.com/watch?v=KZeIEiBrT_w by Veritasium.

Monte Carlo

- Simulate draws from the target distribution
  - these draws can be treated as any observations
  - a collection of draws is sample
- Use these draws, for example,
  - to compute means, deviations, quantiles
  - to draw histograms
  - to marginalize
  - etc.

Monte Carlo vs. deterministic

- Monte Carlo = simulation methods
  - evaluation points are selected stochastically (randomly)
- Deterministic methods (e.g. grid)
  - evaluation points are selected by some deterministic rule
  - good deterministic methods converge faster (need less function evaluations for the same accuracy)

How many simulation draws are needed?

- How many draws or how big sample size?
- If draws are independent
  - usual methods to estimate the uncertainty due to a finite number of observations (finite sample size)
- Markov chain Monte Carlo produces dependent draws
  - requires additional work to estimate the effective sample size
  - next week

How many simulation draws are needed?

- Expectation of unknown quantity $\mathrm{E}(\theta) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$
  - If S is big,
  - $\theta^{(s)}$ are independent,
  - $p(\theta)$ has finite variance,

  then the central limit theorem (CLT) states that the distribution of the expectation approaches normal distribution (see BDA3 Ch 4) with variance σ_θ^2 / S

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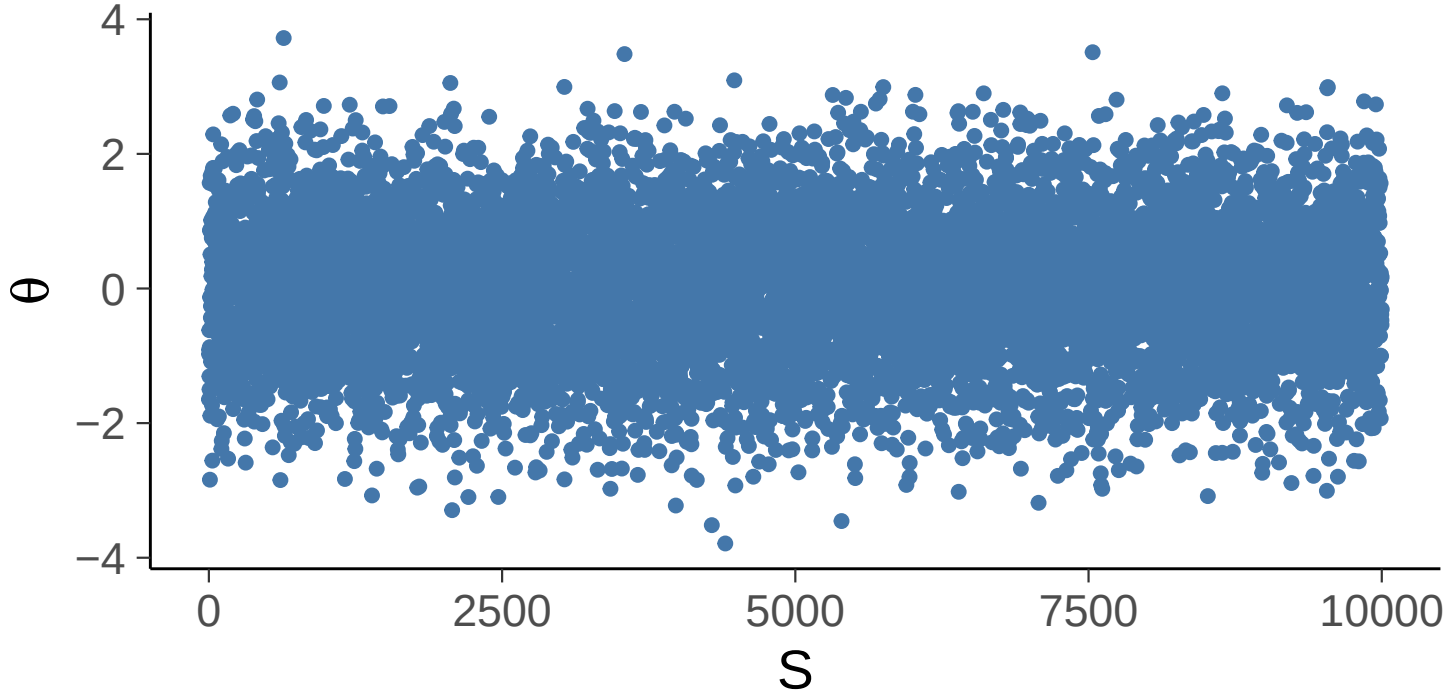


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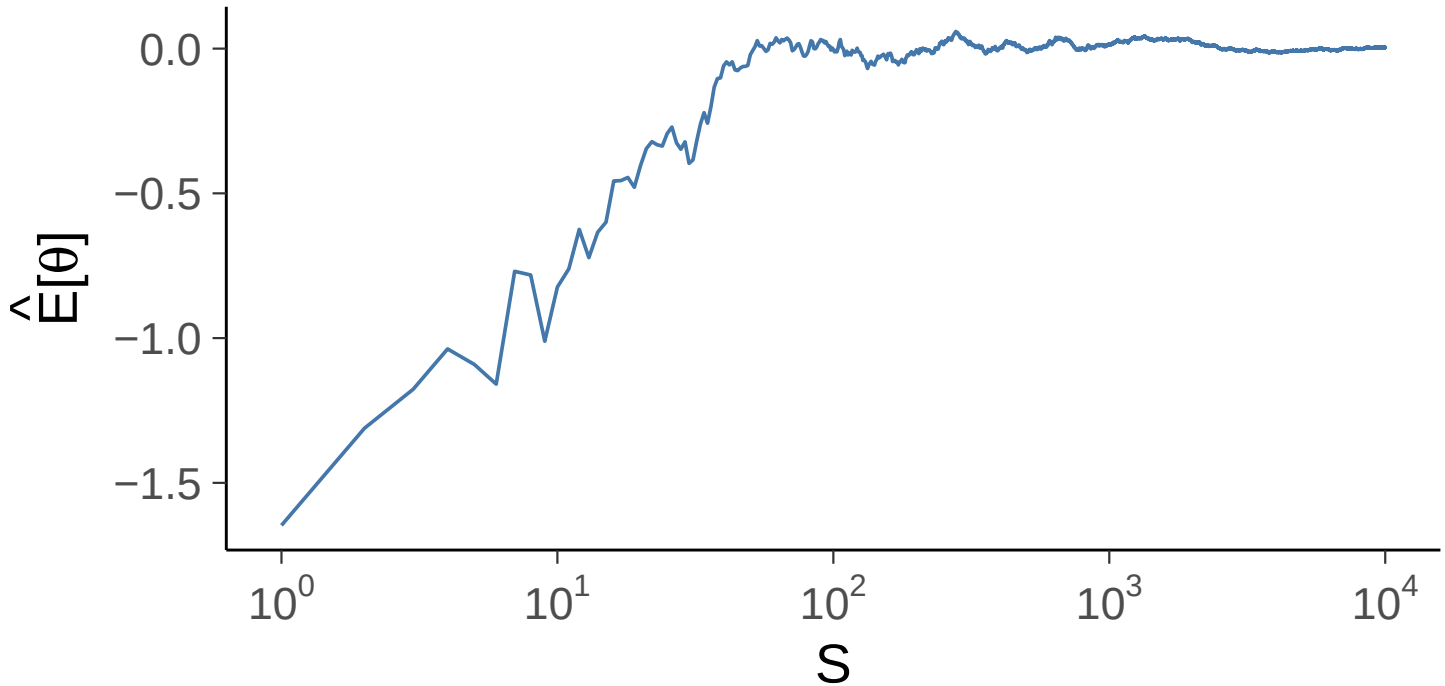


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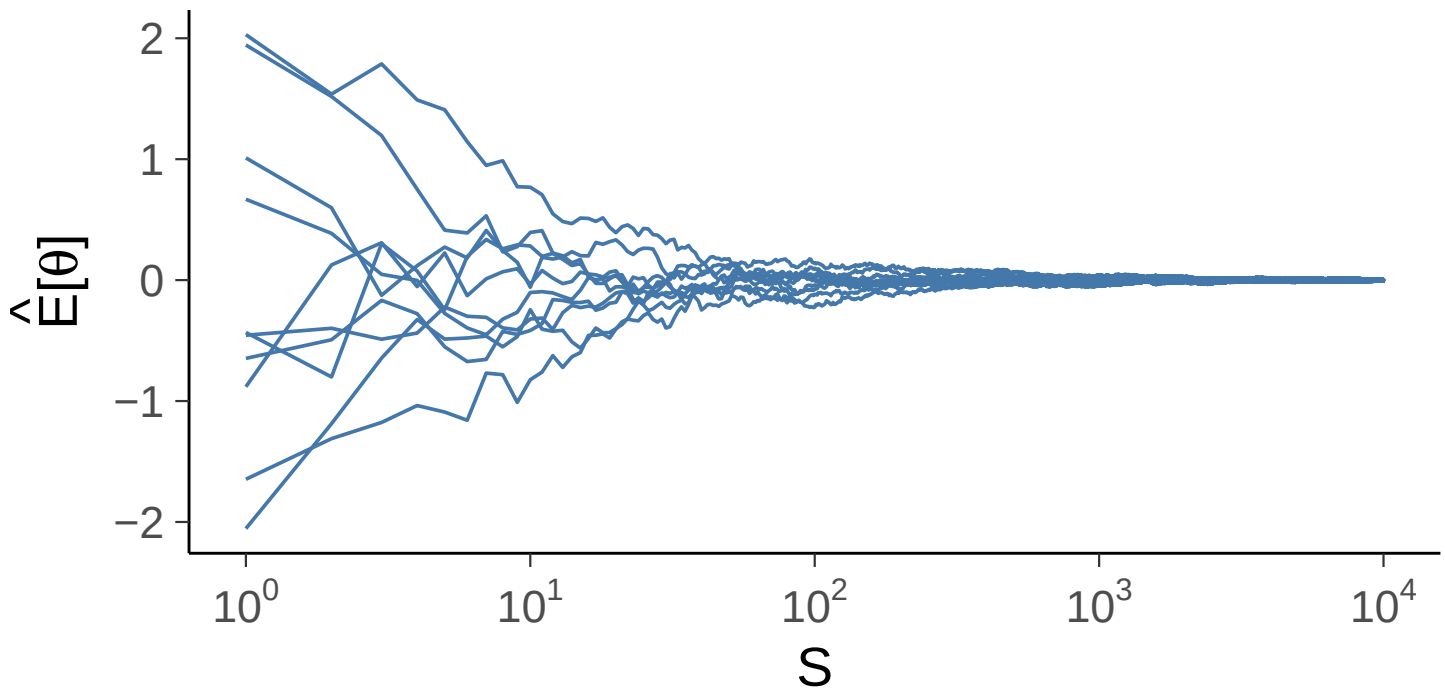

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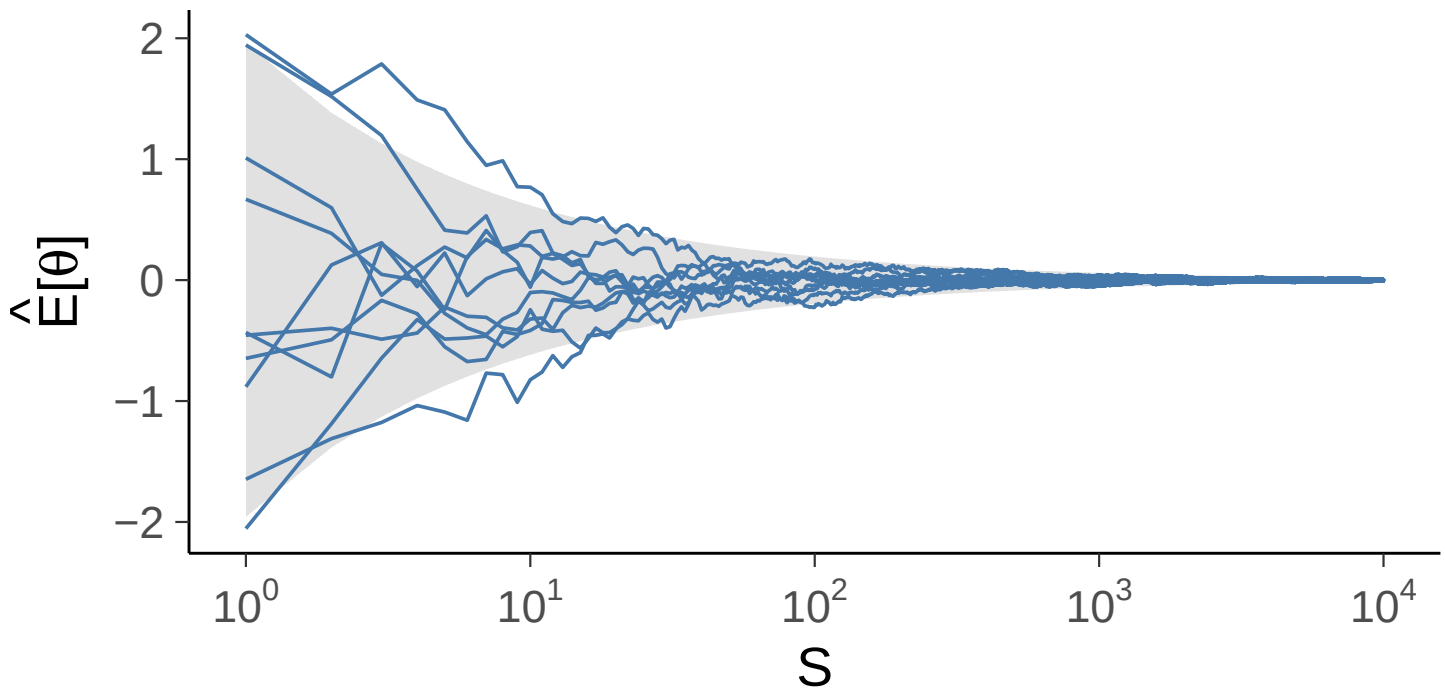

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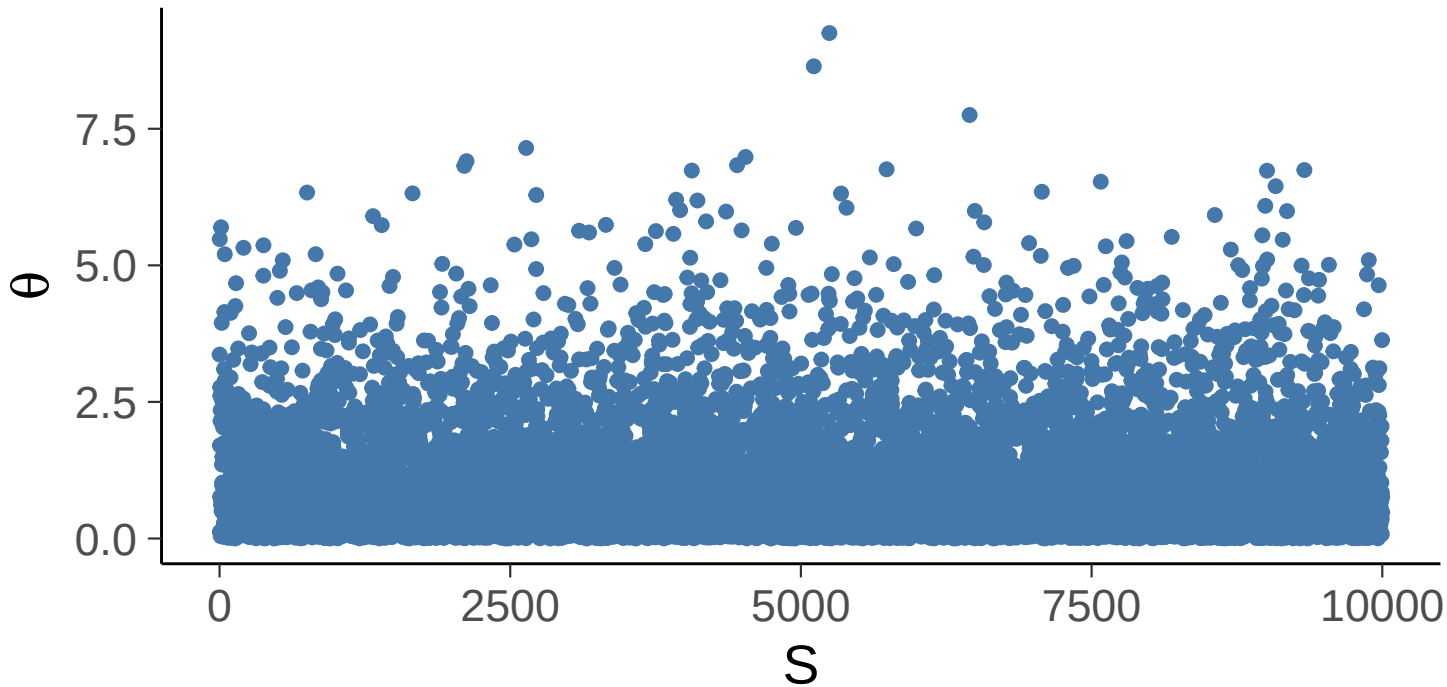


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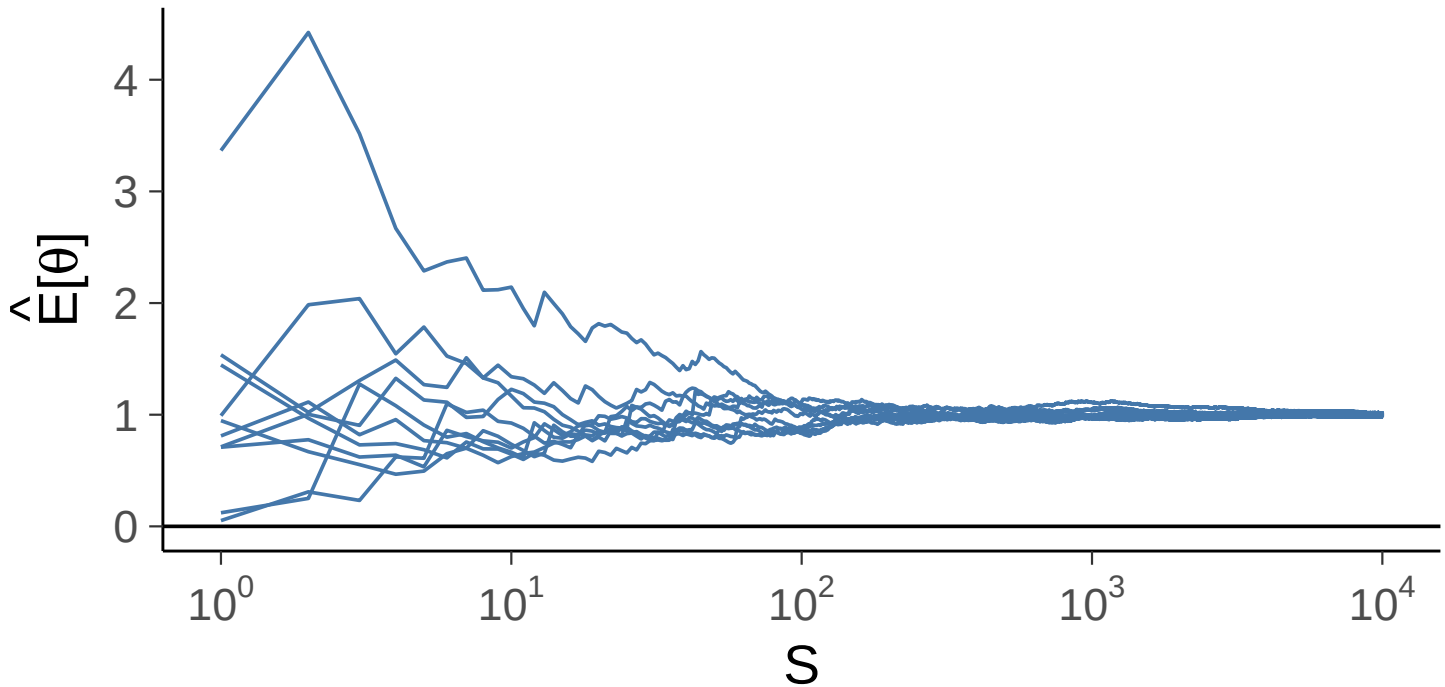


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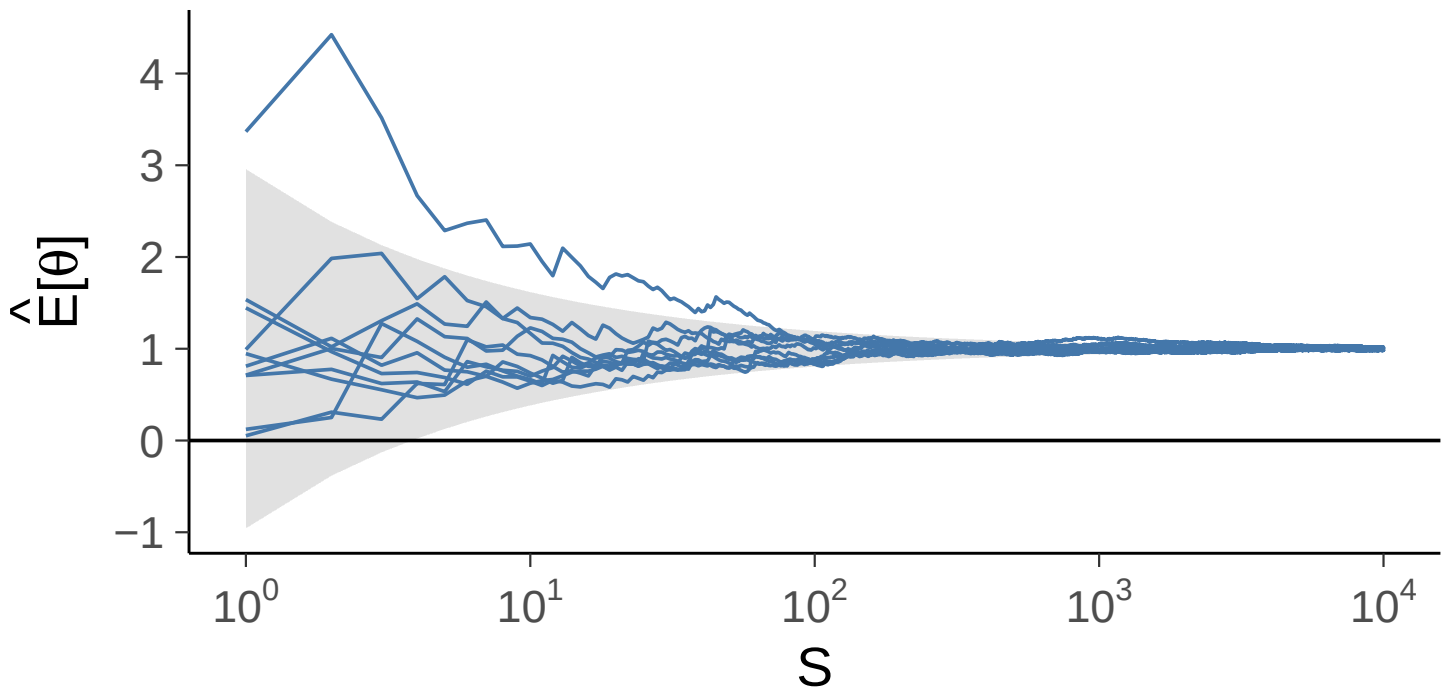

How many simulation draws are needed?

- Expectation of unknown quantity $\mathrm{E}(\theta) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$
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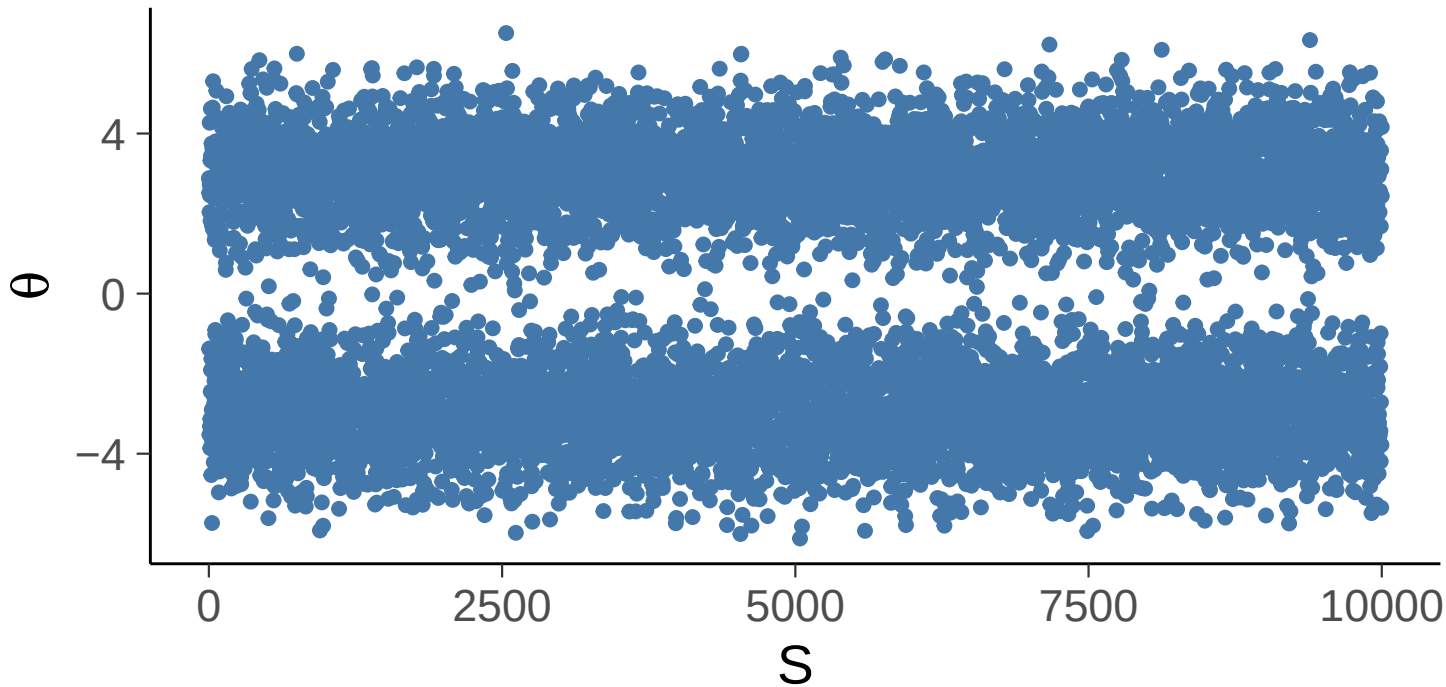


```
rnorm(n=10000, mean=sample(c(-3,3), 10000, replace=TRUE))
```

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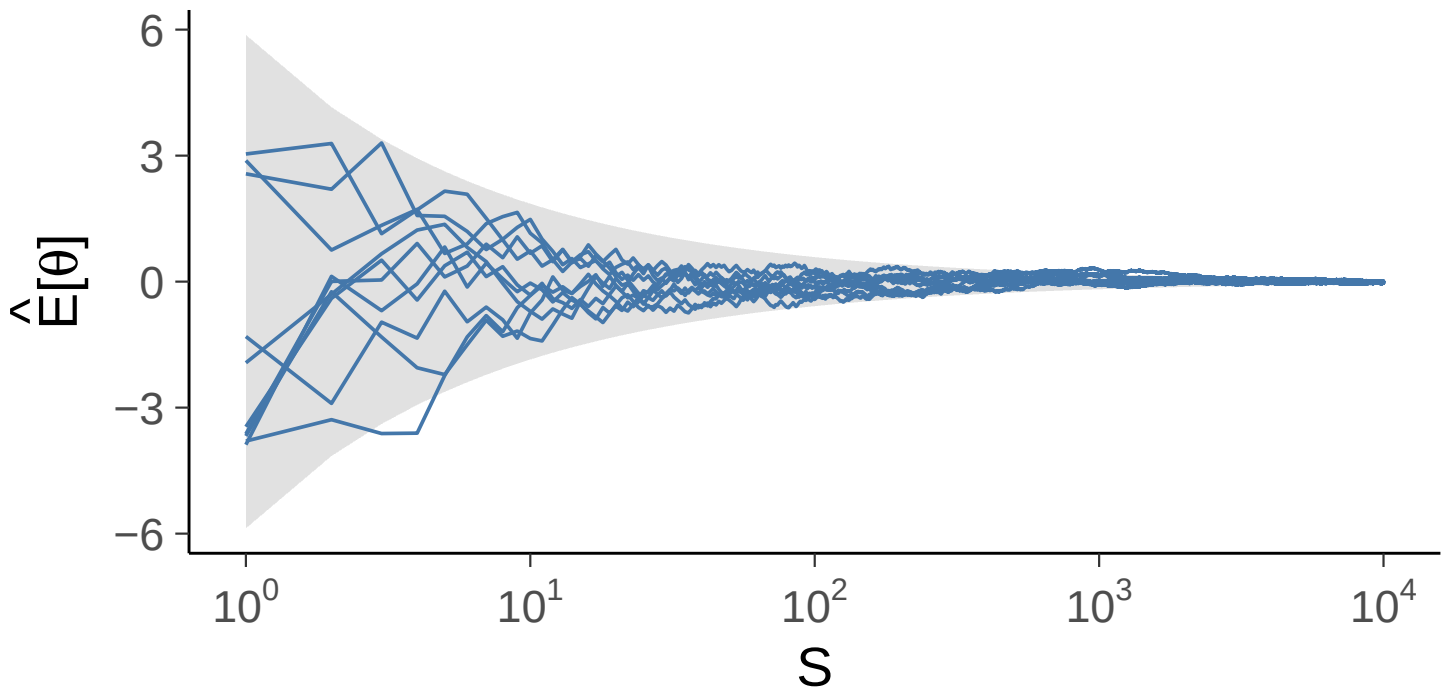

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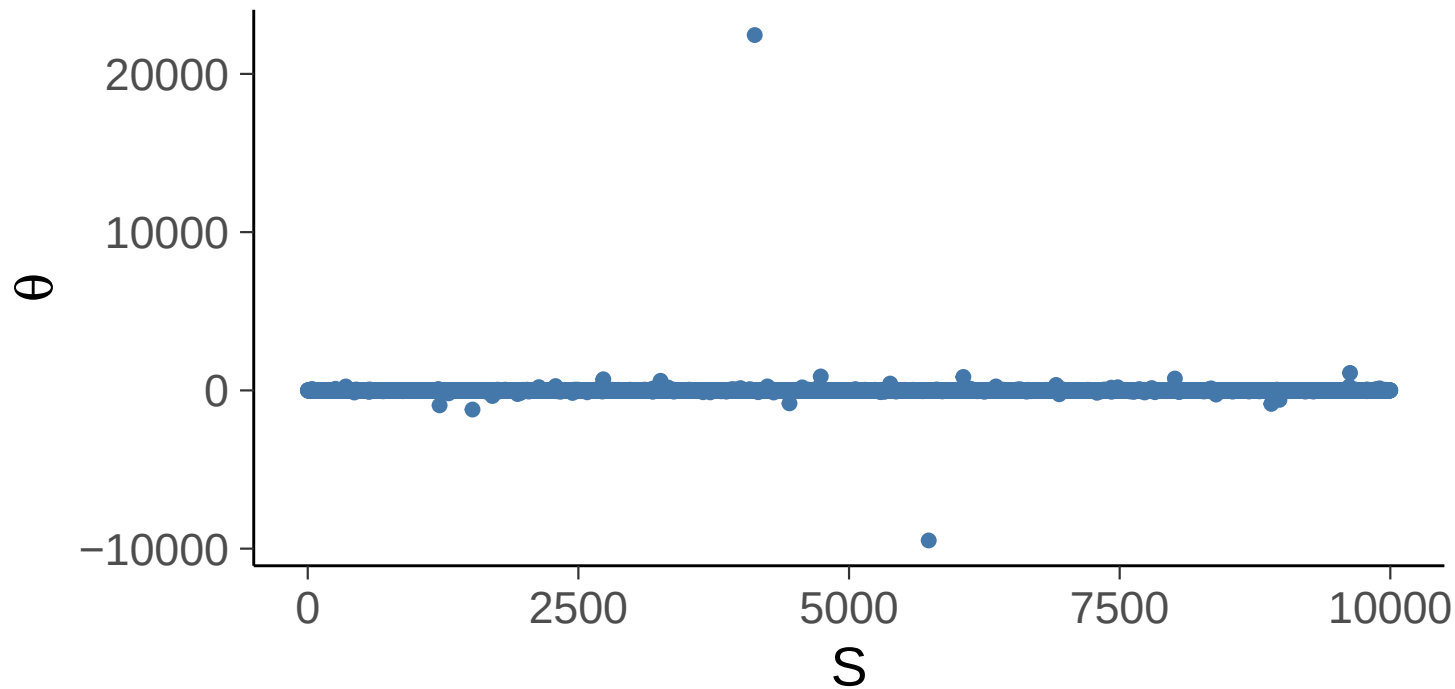

`rt(n=10000, df=1)`

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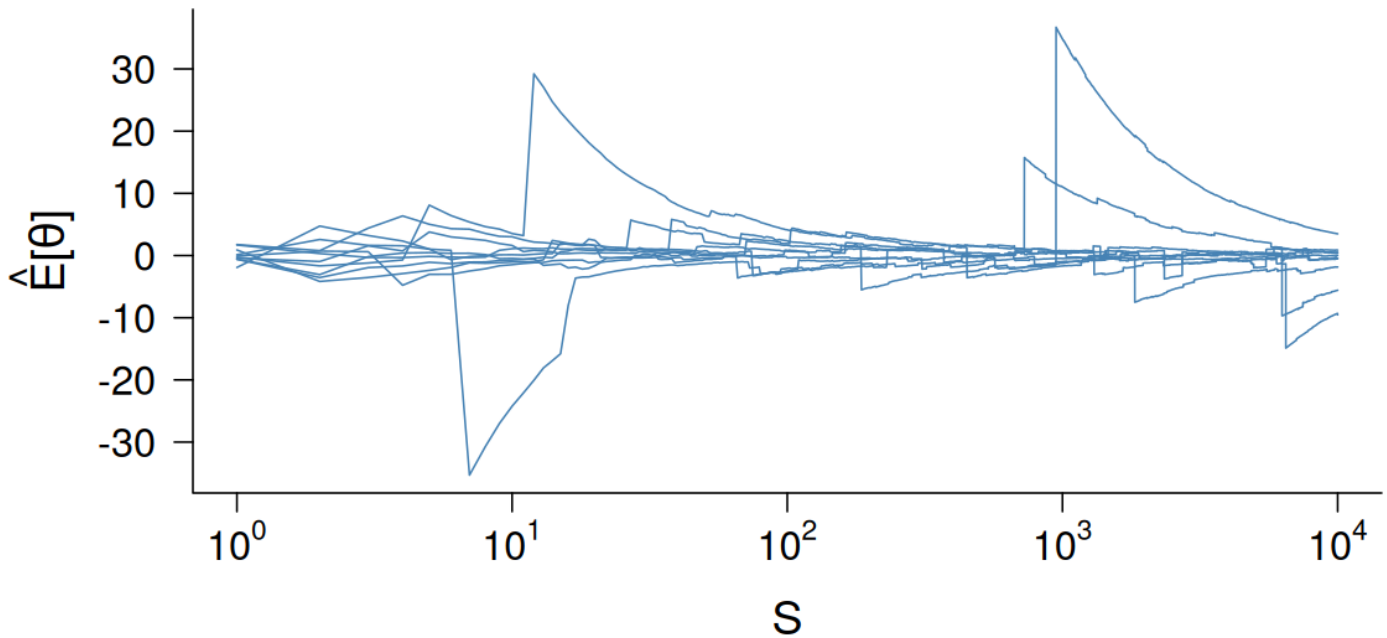


```
cummean(rt(n=10000, df=1))
```

How many simulation draws are needed?

- Expectation of unknown quantity $\mathrm{E}(\theta) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$
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  - $\sigma_\theta / \sqrt{S}$ is called Monte Carlo standard error (MCSE)
  - In practice, σ_θ will be estimated by $\sqrt{1/(S-1) \sum_{s=1}^{S} (\theta^{(s)} - \mathrm{E}(\theta))^2}$

Central limit theorem

- Valid also when $p(\theta)$ discrete
  - the distribution of mean is discrete, but the comparison to continuous normal is done using cumalative distribution functions

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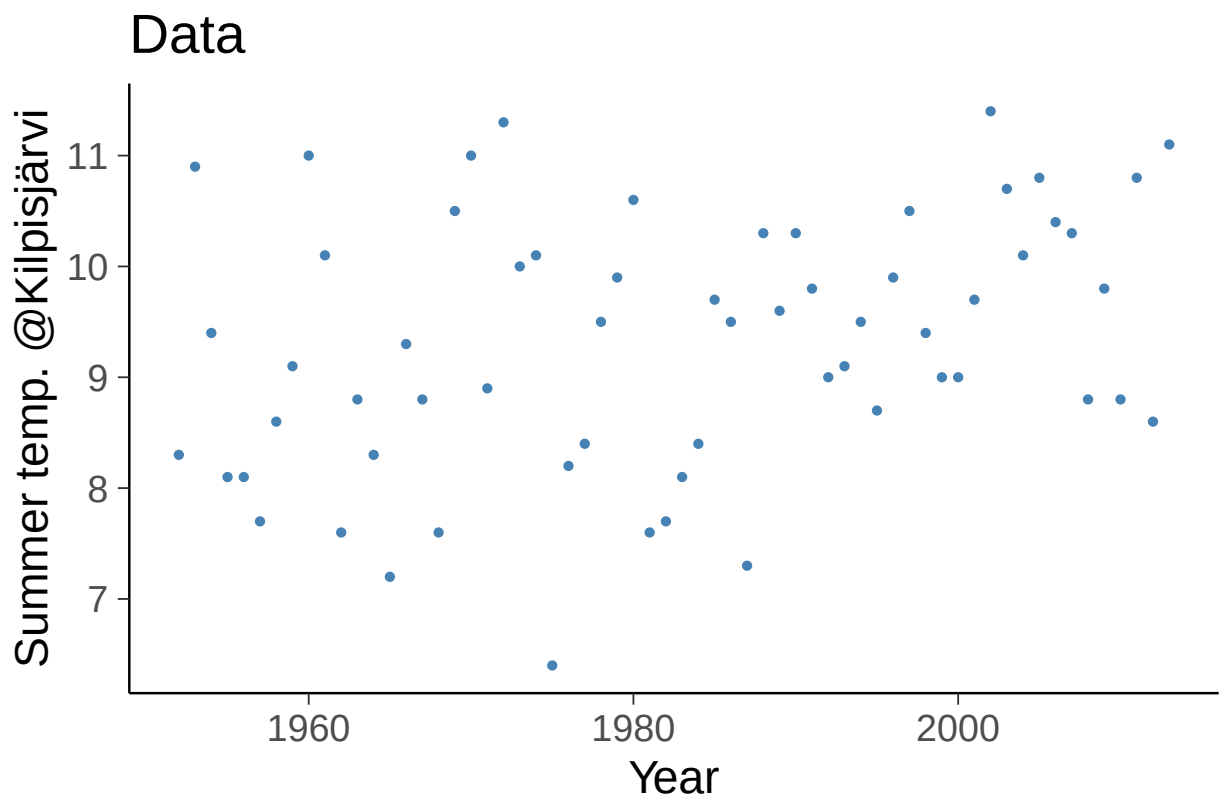
  - the distribution of mean is discrete, but the comparison to continuous normal is done using cumalative distribution functions
- 3Blue1Brown YouTube videos with nice visualisations
  - CLT with discrete distributions: *But what is the Central Limit Theorem?* https://www.youtube.com/watch?v=zeJD6dqJ5lo
  - CLT with continuous distributions: *Convolutions | Why X+Y in probability is a beautiful mess* https://www.youtube.com/watch?v=IaSGqQa5O-M

Example: Kilpisjärvi summer temperature

Average temperature in June, July, and August at Kilpisjärvi, Finland in 1952–2013



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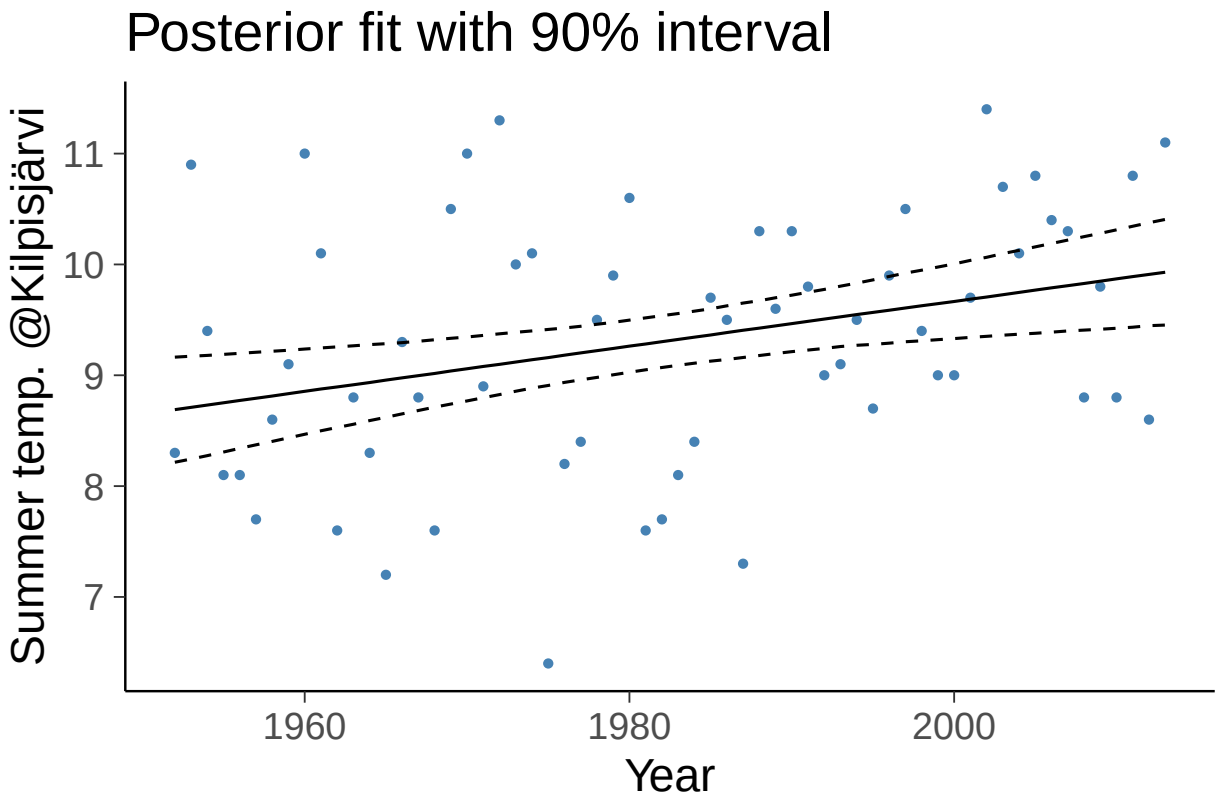

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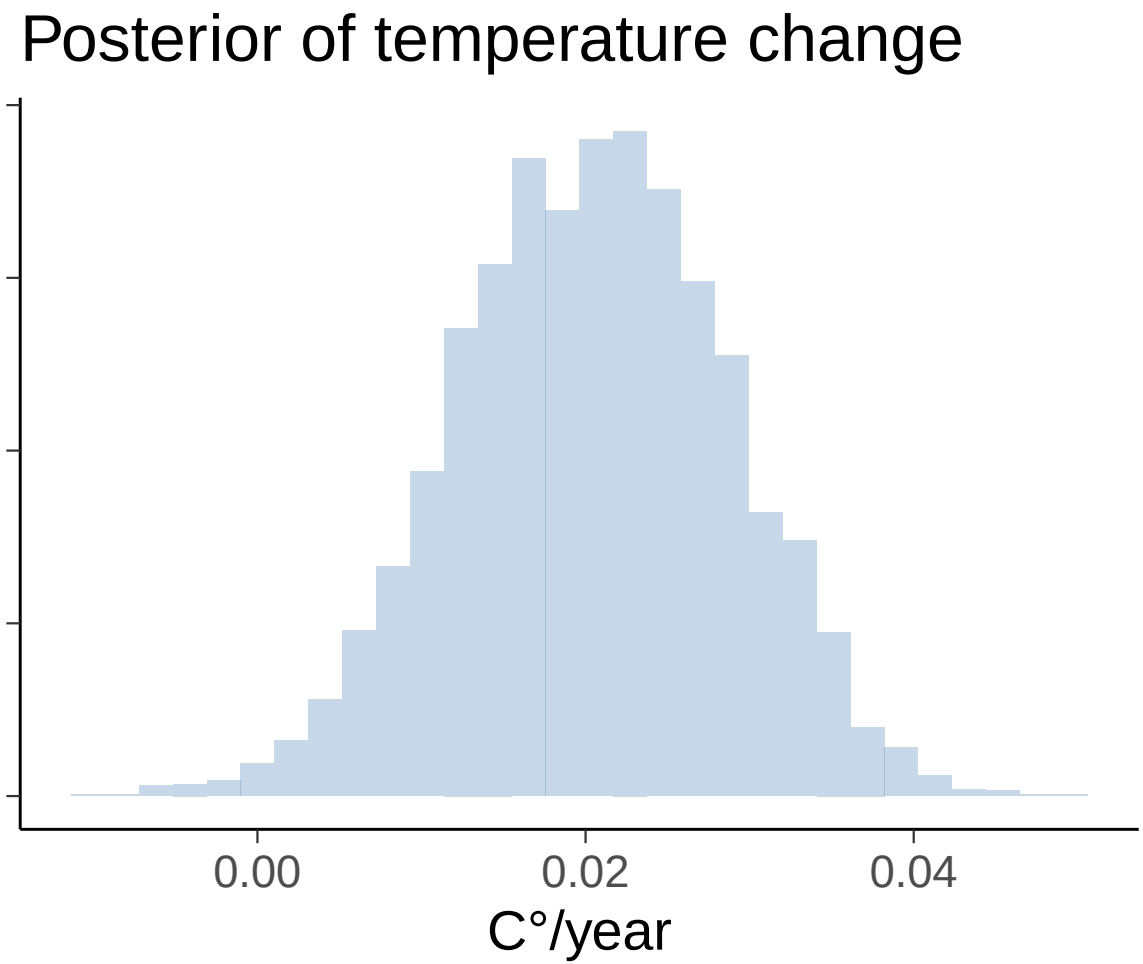

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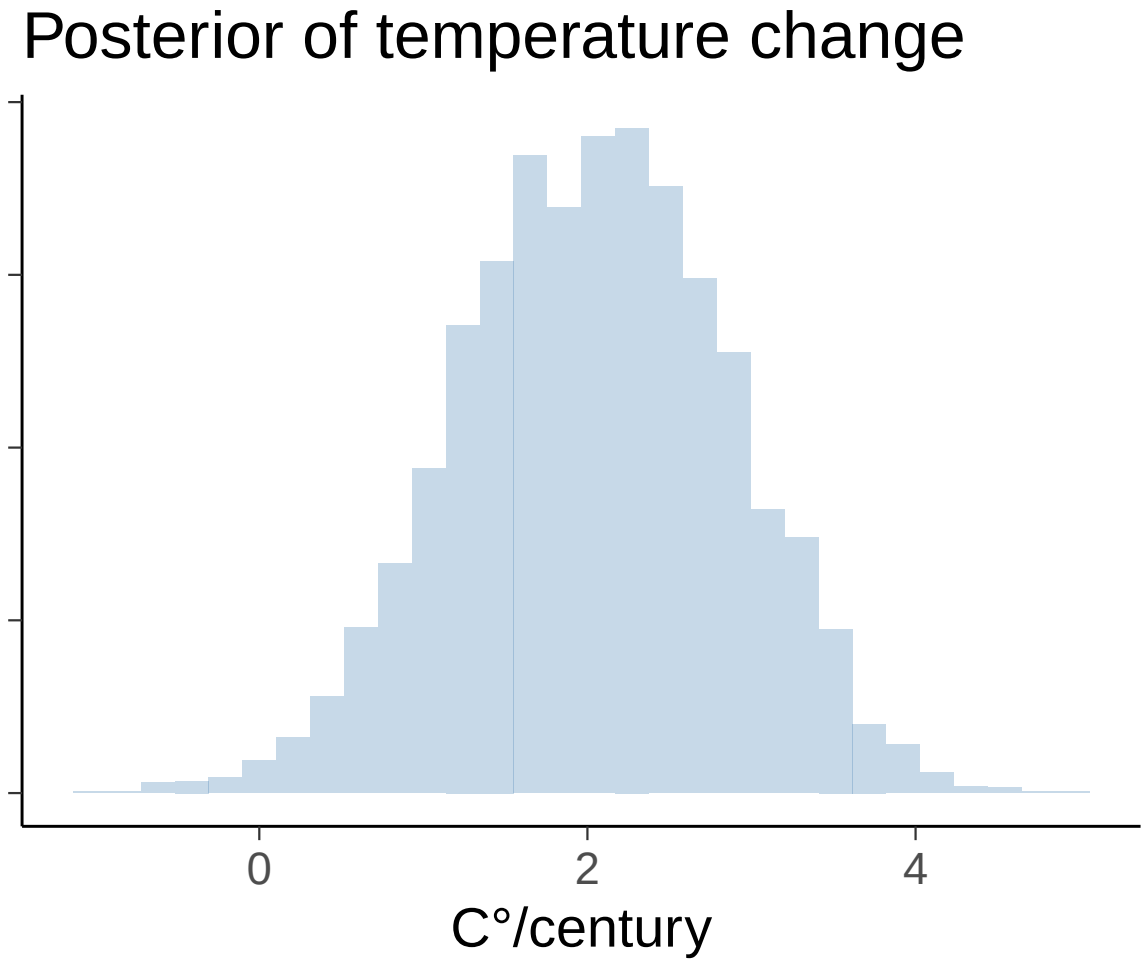

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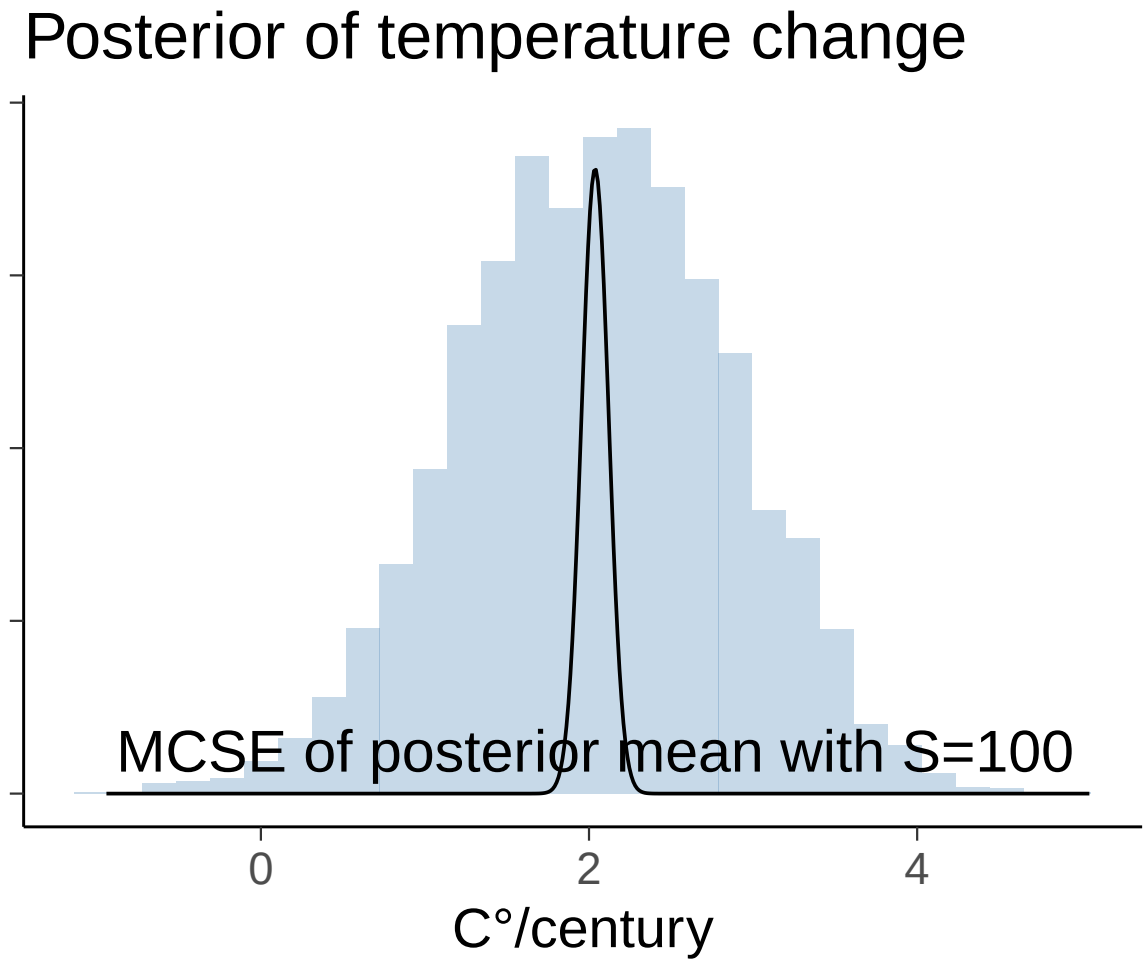

$\sigma_\theta \approx 0.83$, MCSE $= \sigma_\theta/\sqrt{S} \approx 0.083$,

in repeated sampling we may expect mean estimate to vary within (1.8, 2.1) (90% interval)

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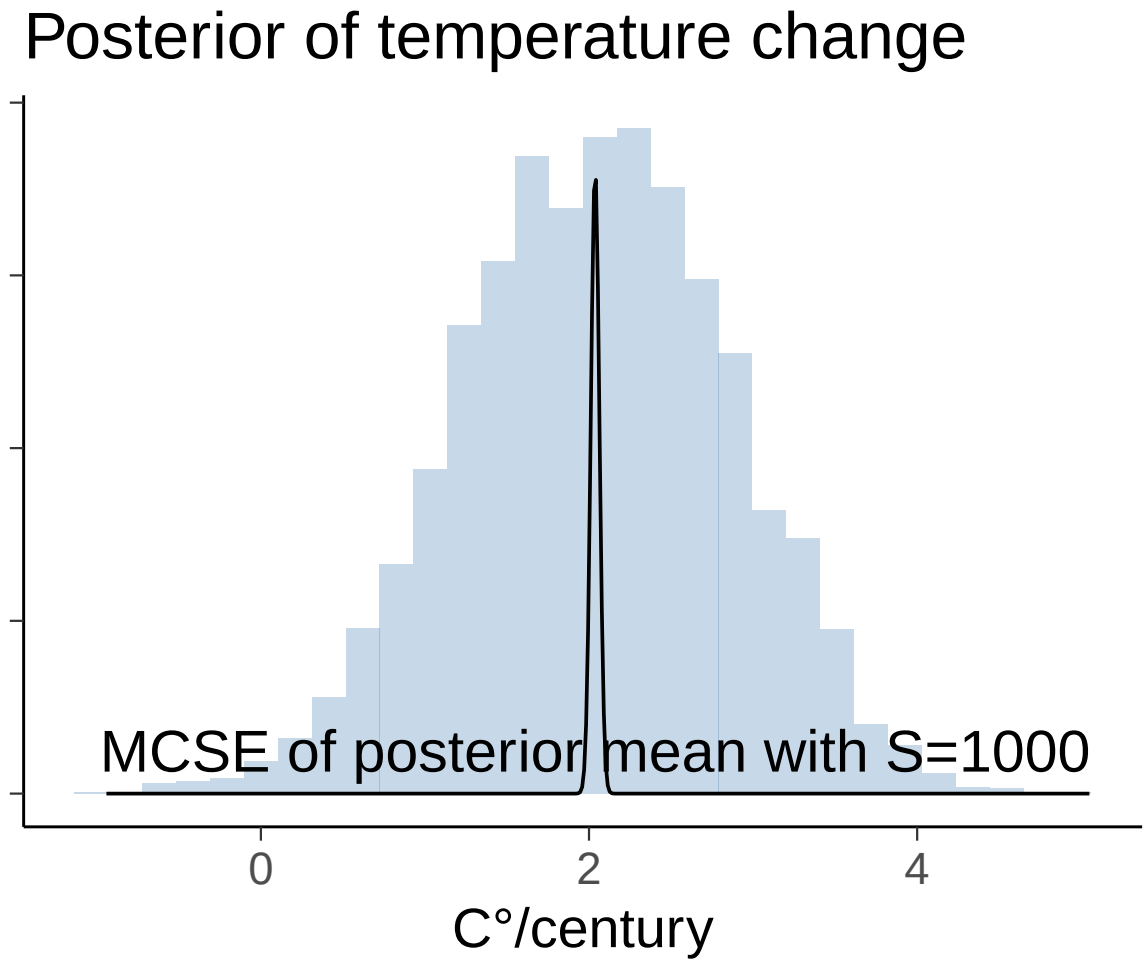
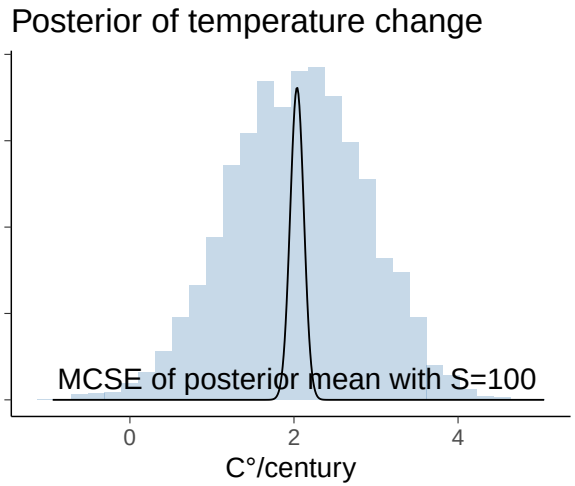
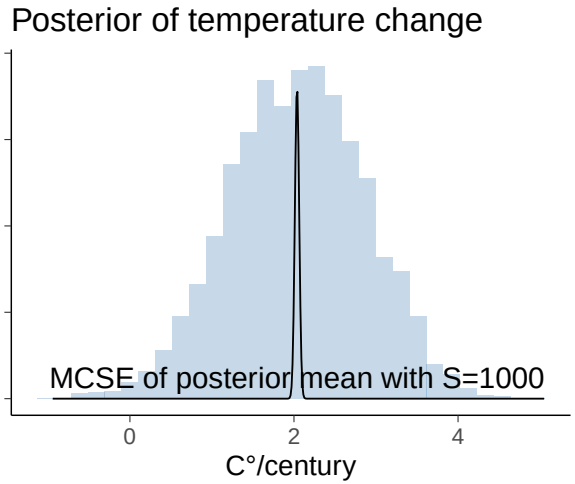


$\sigma_\theta \approx 0.83$, MCSE ≈ 0.026,

in repeated sampling we may expect mean estimate to vary within (1.9, 2.0) (90% interval)

Example: Kilpisjärvi summer temperature





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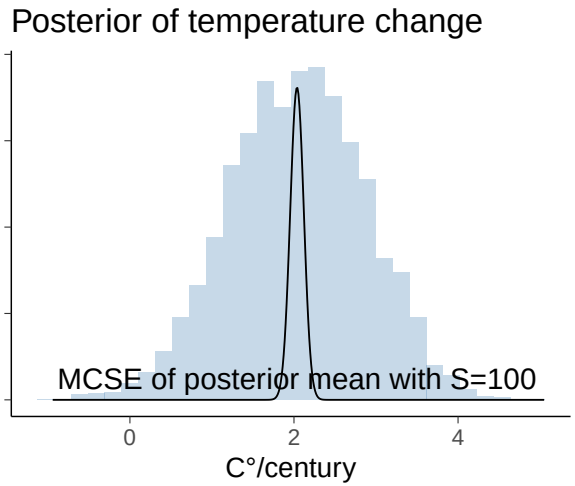

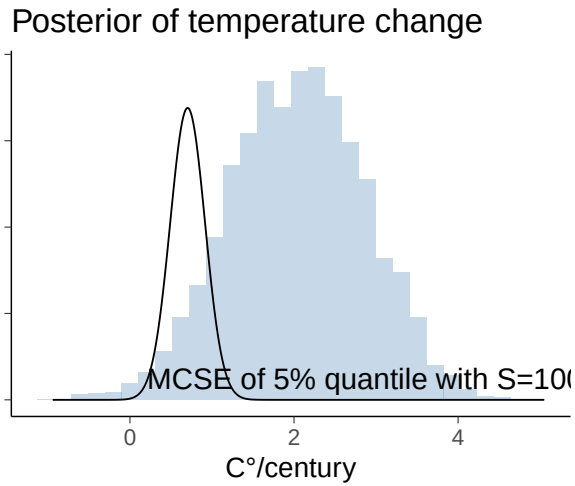

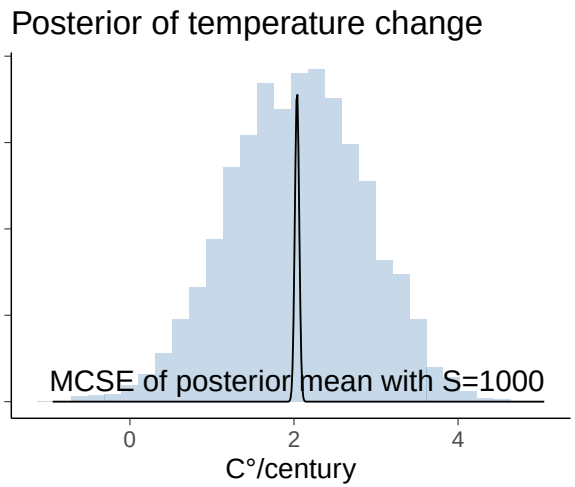

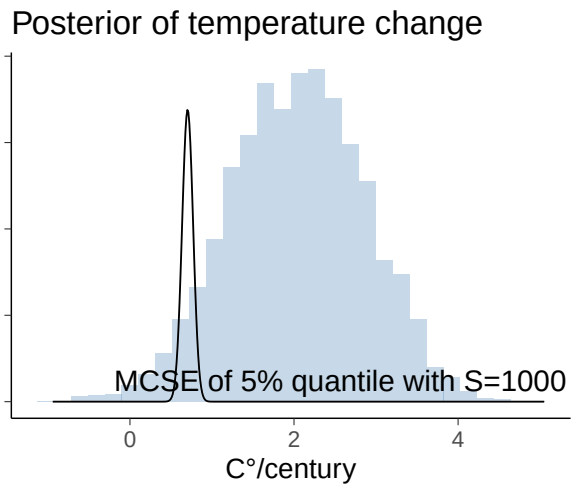

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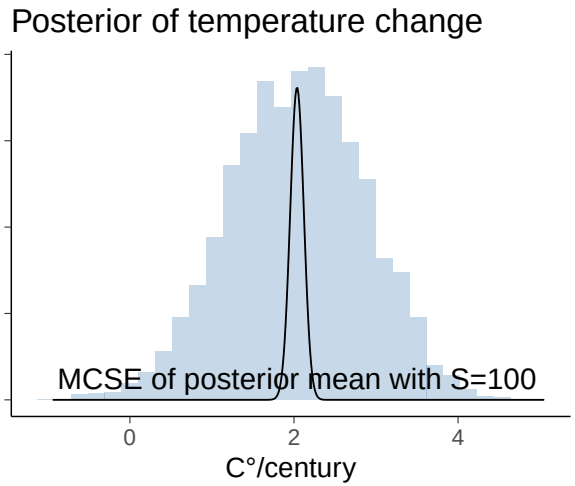

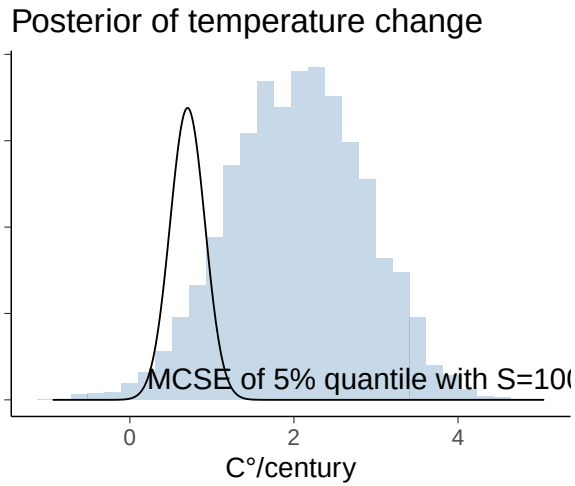

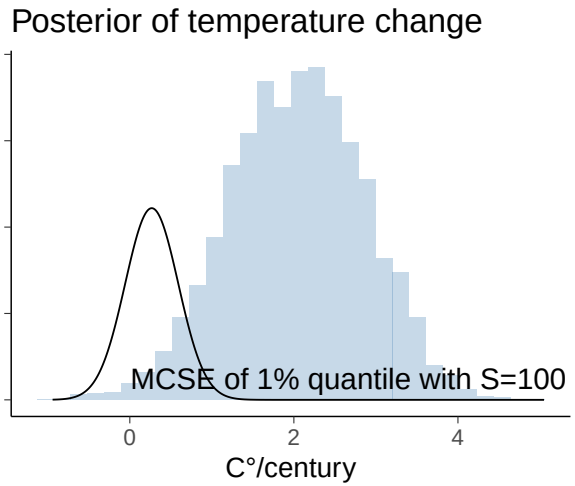

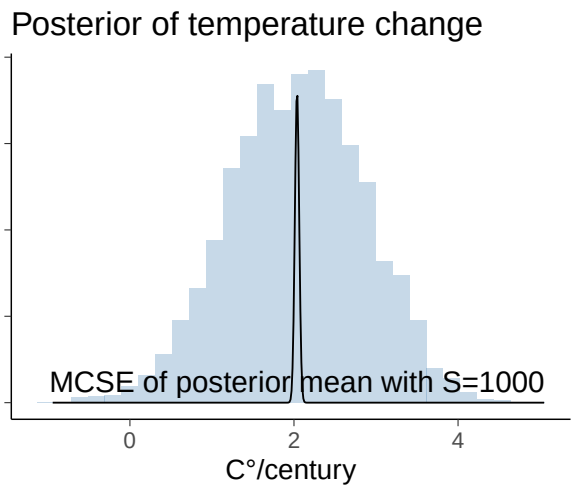

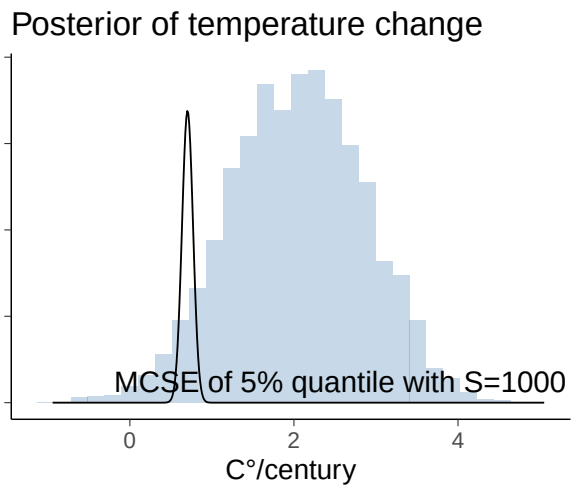

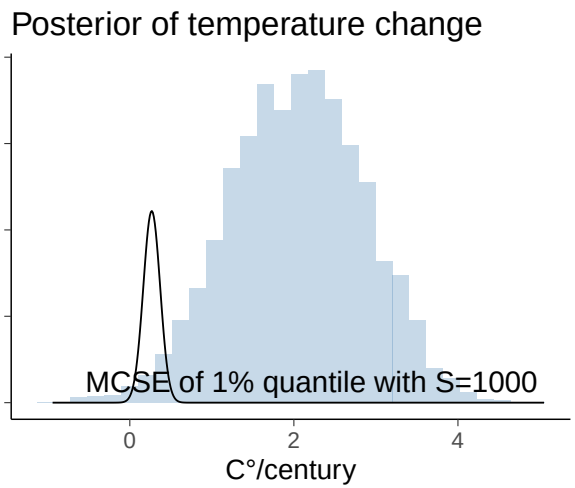

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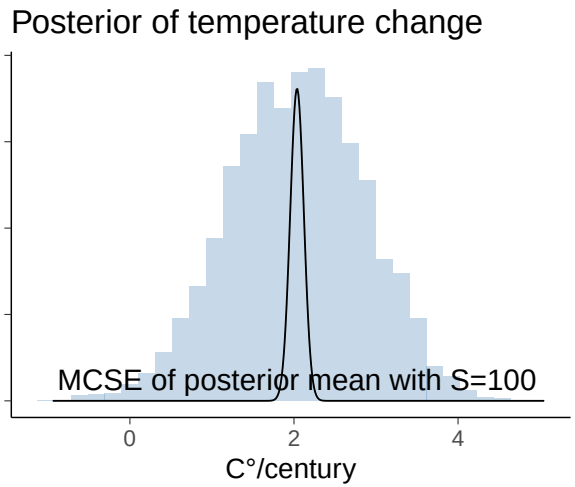

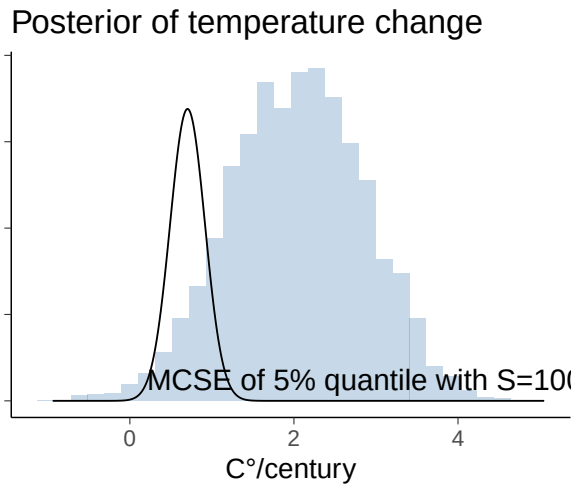

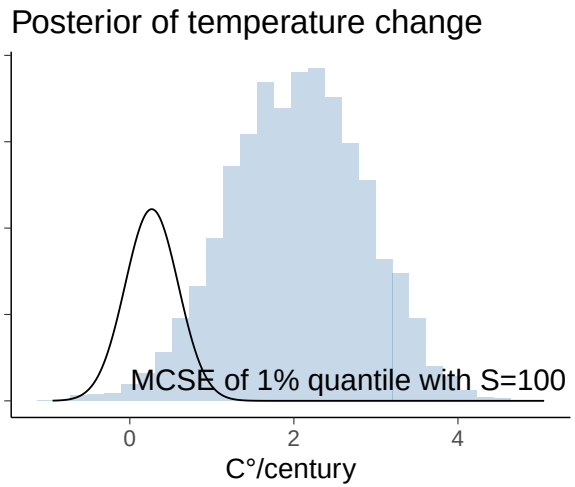

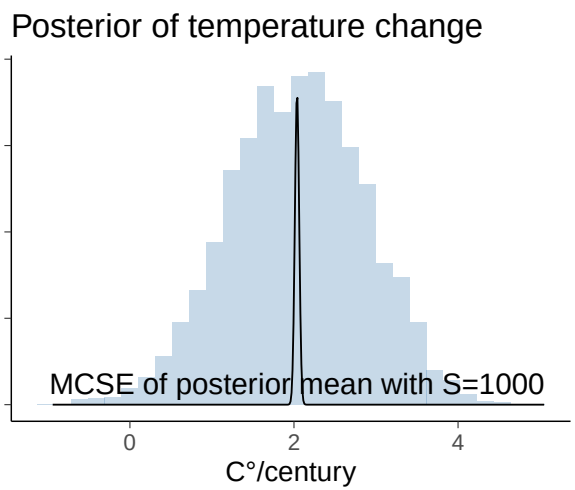

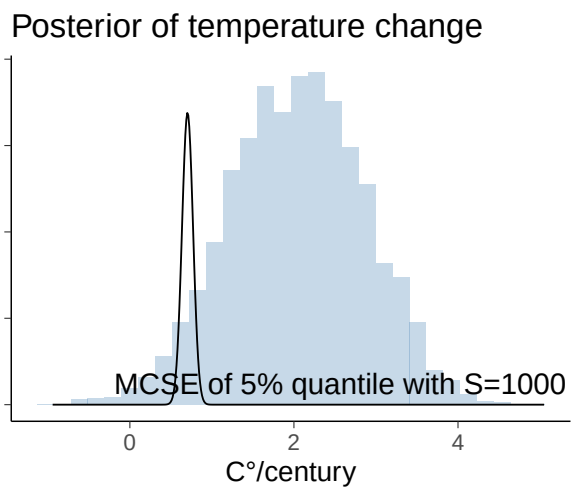

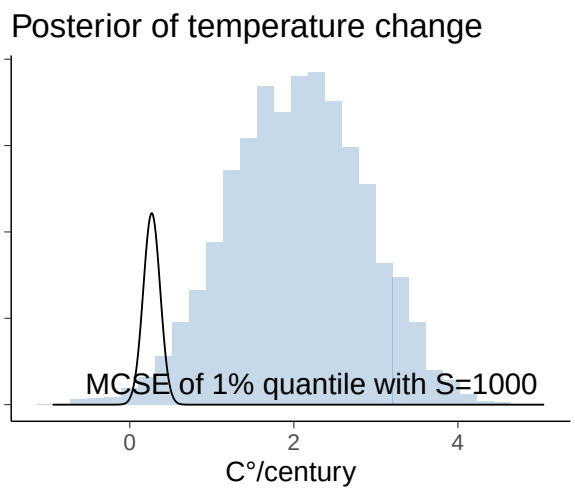

Tail quantiles are more difficult to estimate

See Vehtari, Gelman, Simpson, Carpenter, & Bürkner (2021) for quantile MCSE computation.

How many simulation draws are needed?

- Posterior probability

$$p(\theta \in A) \approx \frac{1}{S} \sum_{l} I(\theta^{(s)} \in A)$$

where $I(\theta^{(s)} \in A) = 1$ if $\theta^{(s)} \in A$

  - The sum of $I(\cdot)$ is binomially distributed as $p(\theta \in A)$
    - use beta CDF, or normal approximation

    $\rightarrow$ $\text{var}(I(\cdot)) = p(1-p)S$ (Appendix A, p. 579)

    $\rightarrow$ standard deviation of p is $\sqrt{p(1-p)/S}$

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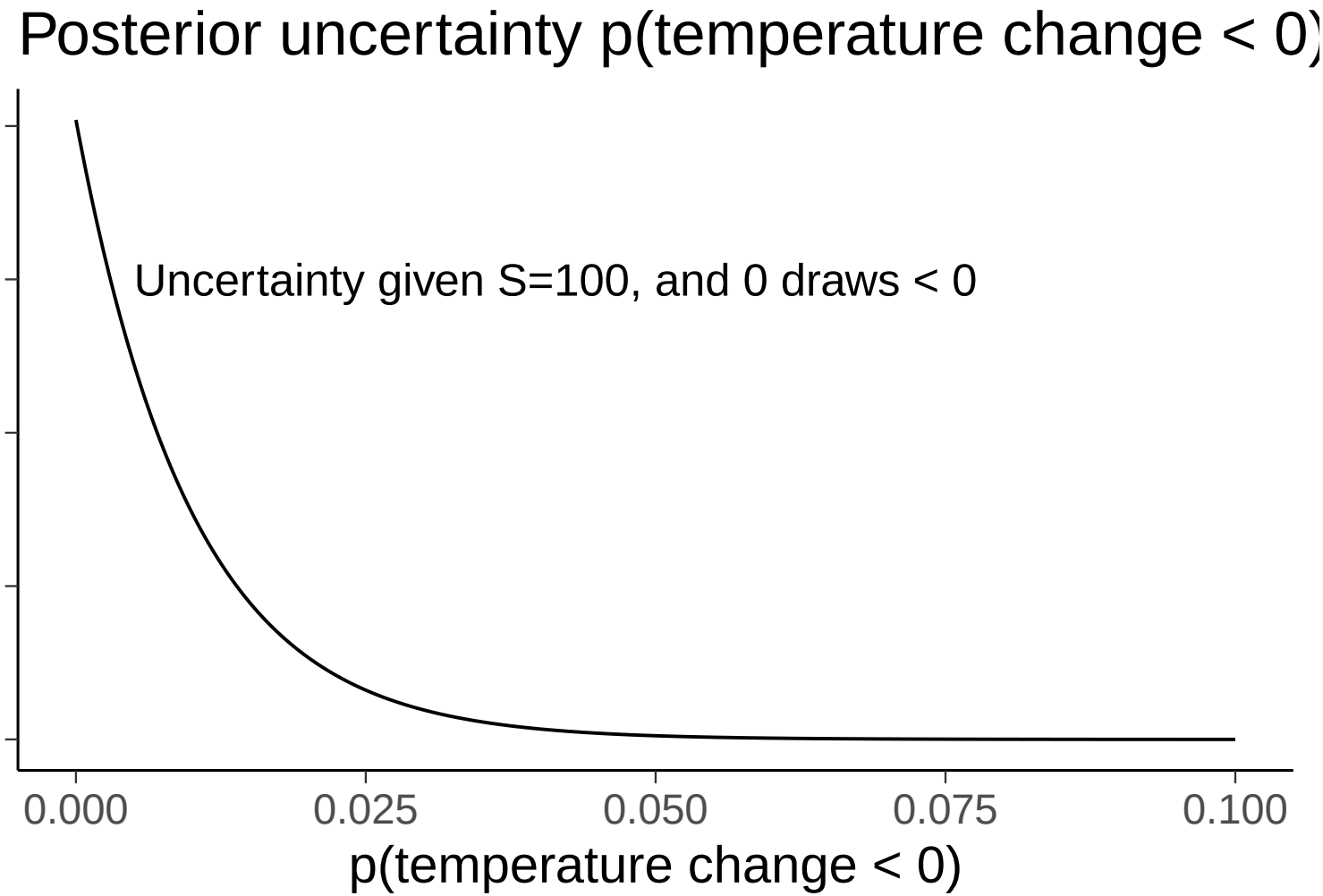
    or from quantiles of beta distribution the range is $(0.02, 0.1)$
  - $S = 2000$ draws needed for 1% unit accuracy
- To estimate small probabilities, a large number of draws is needed
  - to be able to estimate small p, need to get draws with $\theta^{(l)} \in A$, which in expectation requires $S \gg 1/p$

Example: Kilpisjärvi summer temperature



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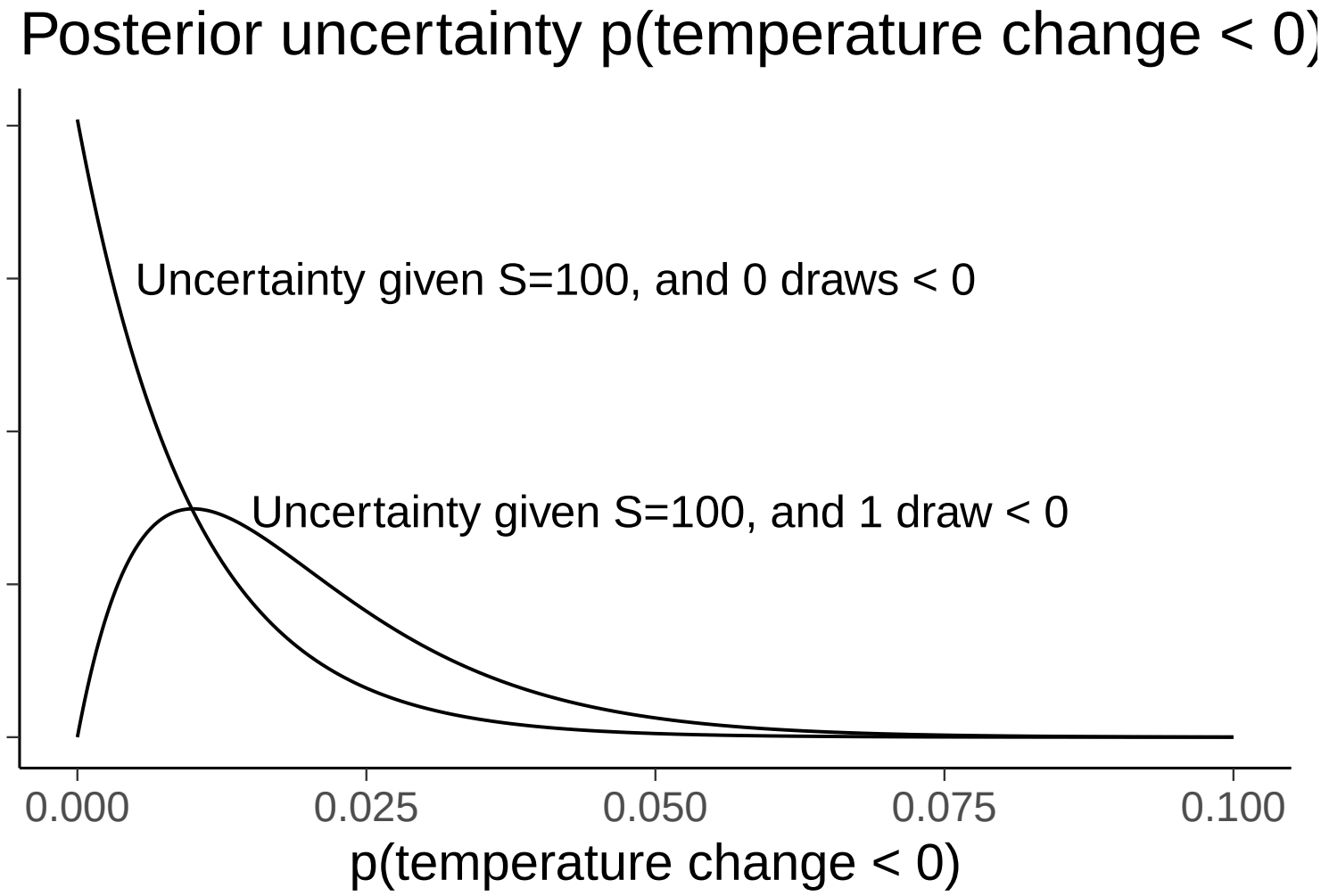

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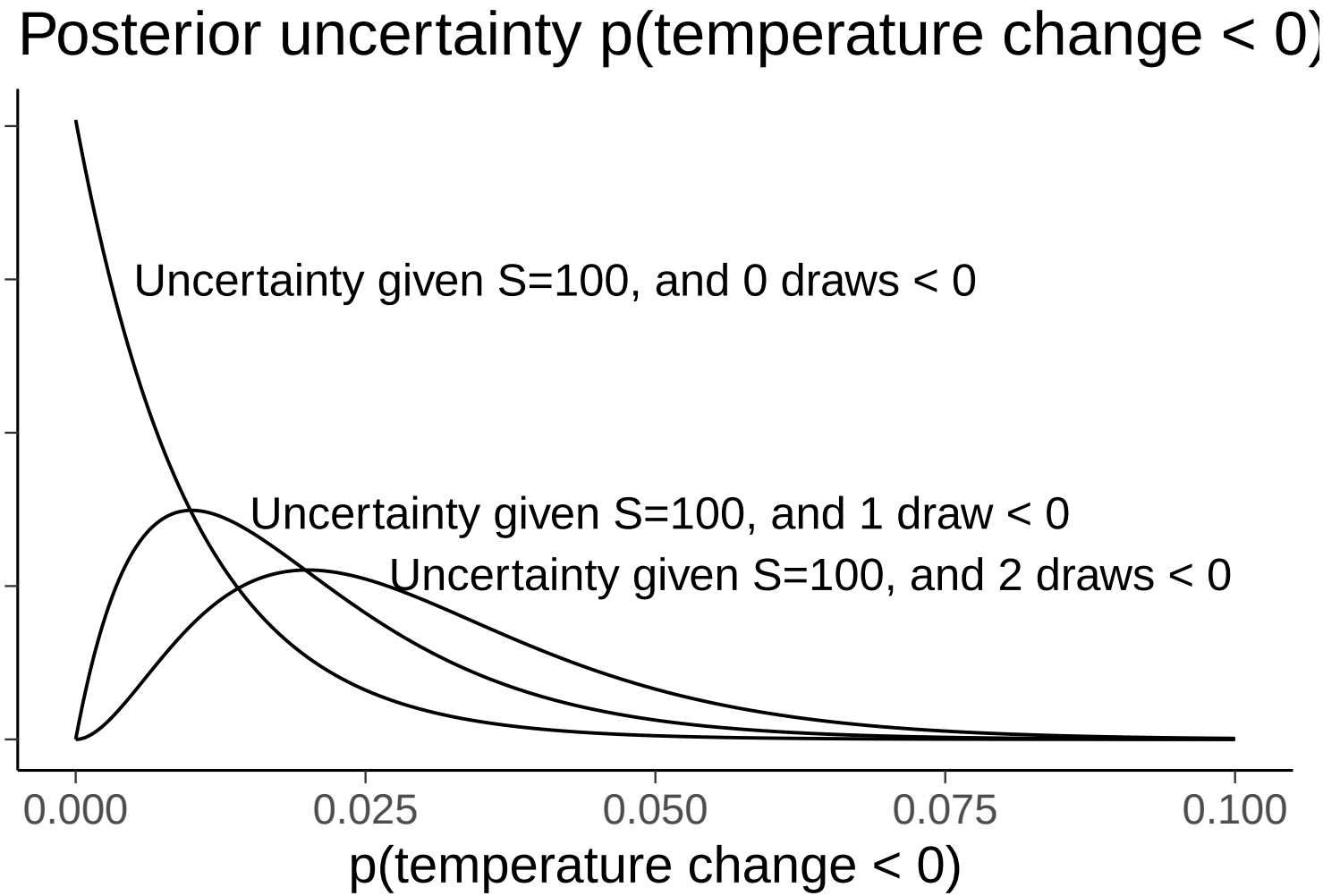

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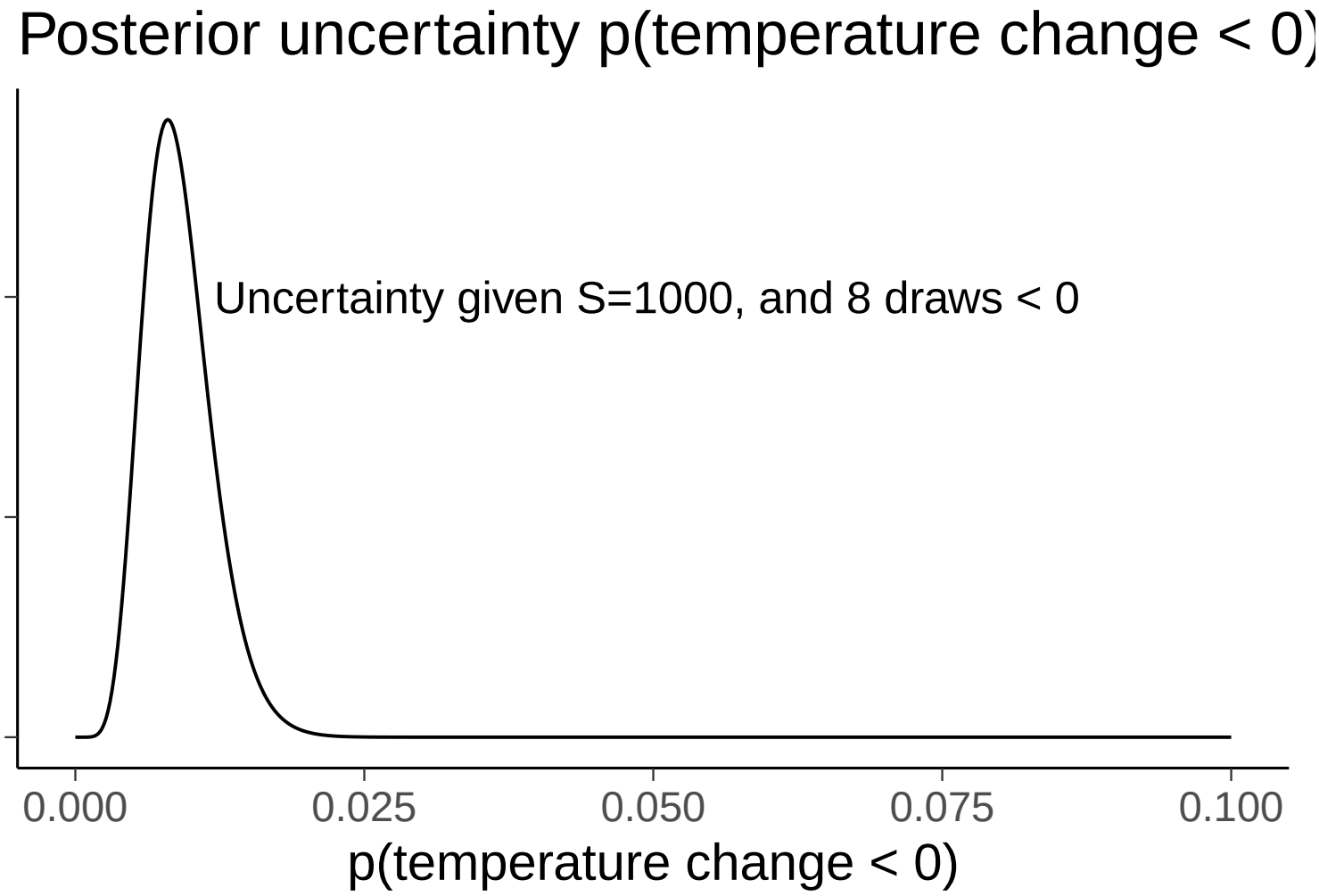

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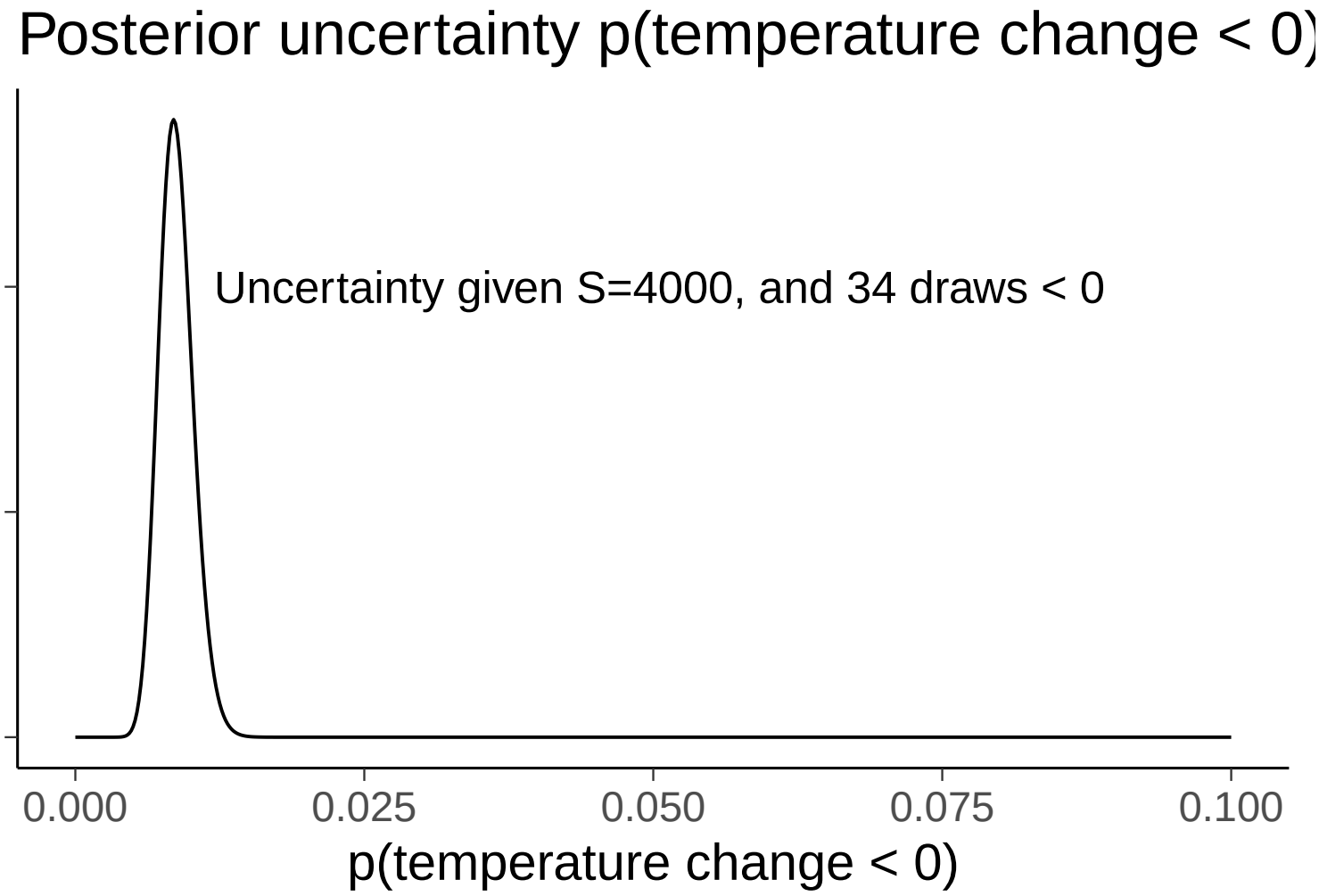

From probabilities to quantiles

- Probability: $p(\theta < A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} < A)$
- 5%-quantile: Find A so that $p(\theta < A) = 0.05$

From probabilities to quantiles

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- 5%-quantile: Find A so that $p(\theta < A) = 0.05$
- If $S = 1000$ and uncertainty interval for 5% probability is $(0.04, 0.06)$ (see earlier slide), we can find uncertainty interval (A^-, A^+), so that $p(\theta < A^-) = 0.04$, and $p(\theta < A^+) = 0.06$

From probabilities to quantiles

- Probability: $p(\theta < A) \approx \frac{1}{S} \sum_l I(\theta^{(s)} < A)$
- 5%-quantile: Find A so that $p(\theta < A) = 0.05$
- If $S = 1000$ and uncertainty interval for 5% probability is $(0.04, 0.06)$ (see earlier slide), we can find uncertainty interval (A^-, A^+), so that $p(\theta < A^-) = 0.04$, and $p(\theta < A^+) = 0.06$
  - we can summarise this interval by transforming it to MCSE
  - see examples in https://avehtari.github.io/casestudies/Digits/digits.html
  - if interested, see algorithm details in Vehtari, Gelman, Simpson, Carpenter, & Bürkner (2021), doi.org/10.1214/20-BA1221.

`posterior` package

Posterior mean and 5% and 95% quantiles:

```
draws |>
  subset_draws("beta100") |>
  summarize_draws(mean,
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These _mcse functions are for MCMC draws, but if the number of draws is big (≥ 1000), then these are accurate enough for independent MC draws, too

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Posterior probability and the corresponding MCSE estimate:

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draws |>
  mutate_variables(beta0p = beta100>0) |>
  subset_draws("beta0p") |>
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  - For probabilities close to 0 or 1, consider also when the model assumption justify certain accuracy

How many digits to report Summary

- Too many digits are difficult to read and can be misleading
- Check the Monte Carlo standard error to avoid showing random noise
- Show meaningful digits given the posterior uncertainty
- Take into account model assumptions and the context/audience

Check this case study
https://users.aalto.fi/~ave/casestudies/Digits/digits.html

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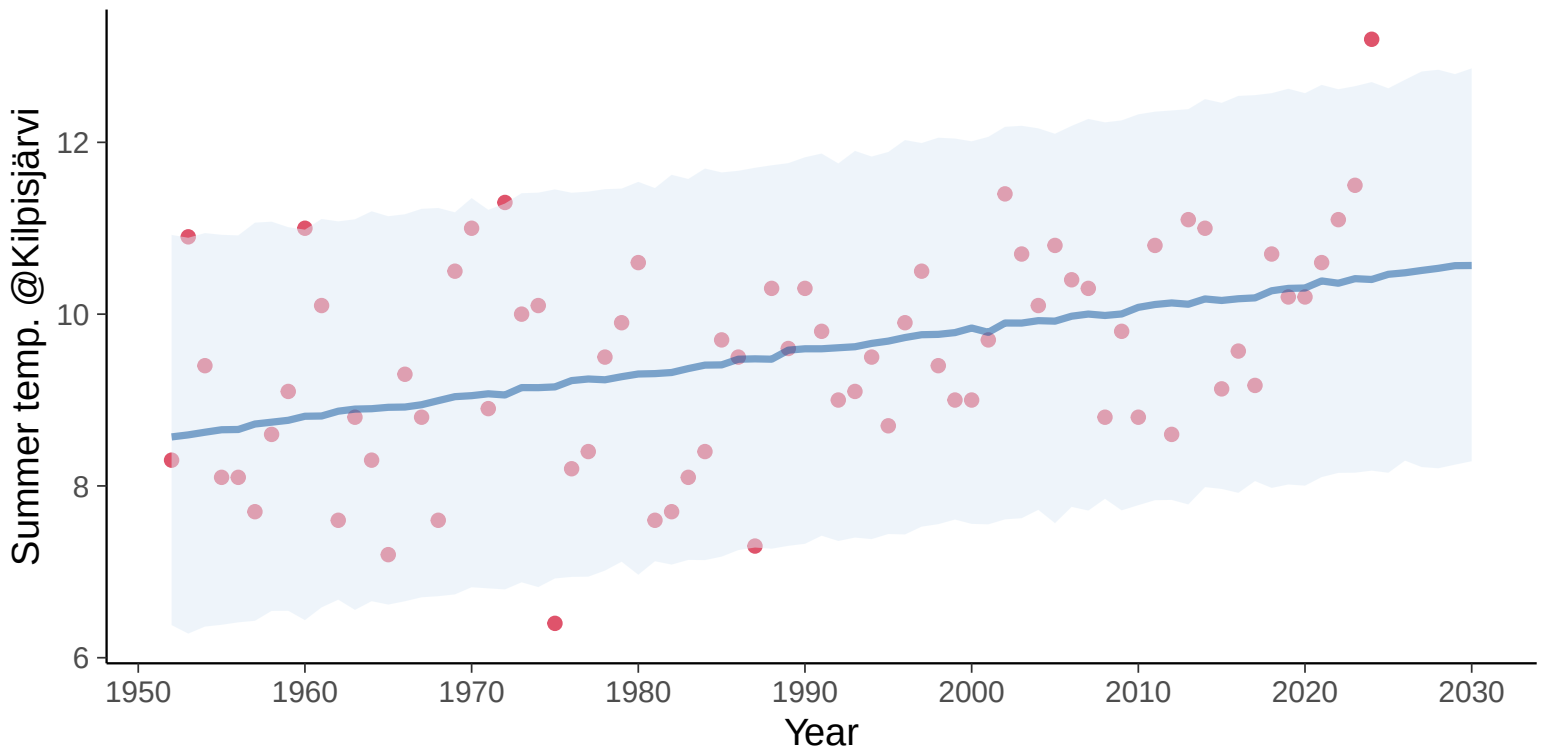
- Summer 2023 was the second hottest in the recorded history
- Summer 2024 was the hottest in the recorded history



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Simple example: $x \sim N, \quad t_4, \quad t_2, \quad t_1$

- N has all moments finite
- t_ν has less than ν fractional moments

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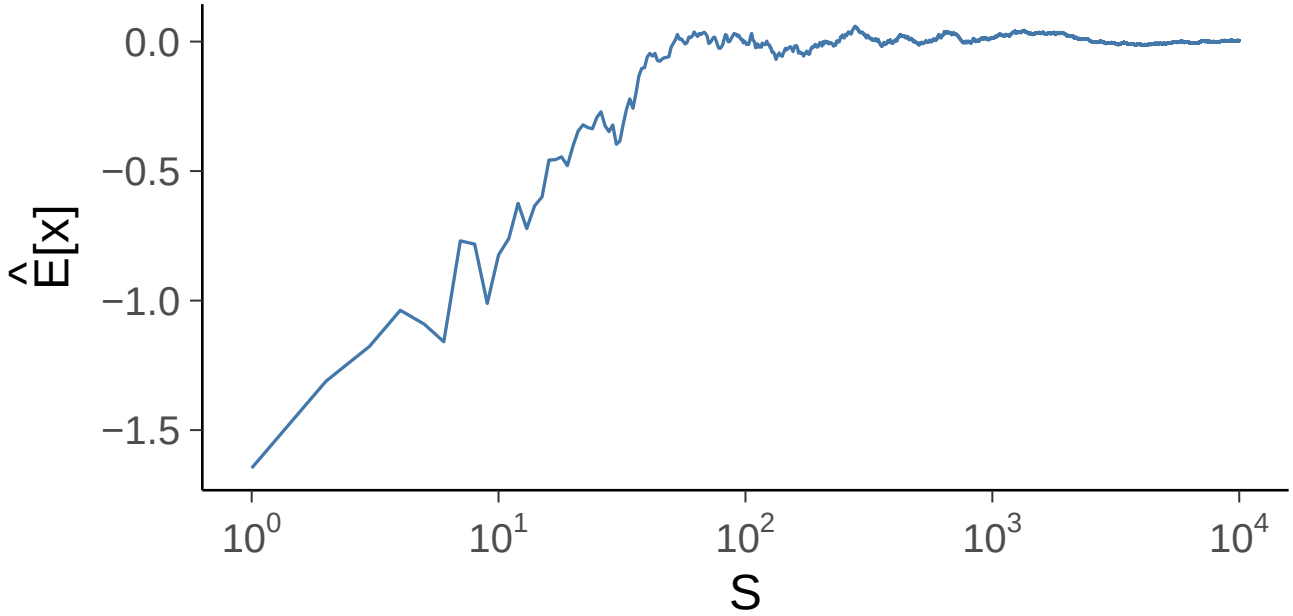

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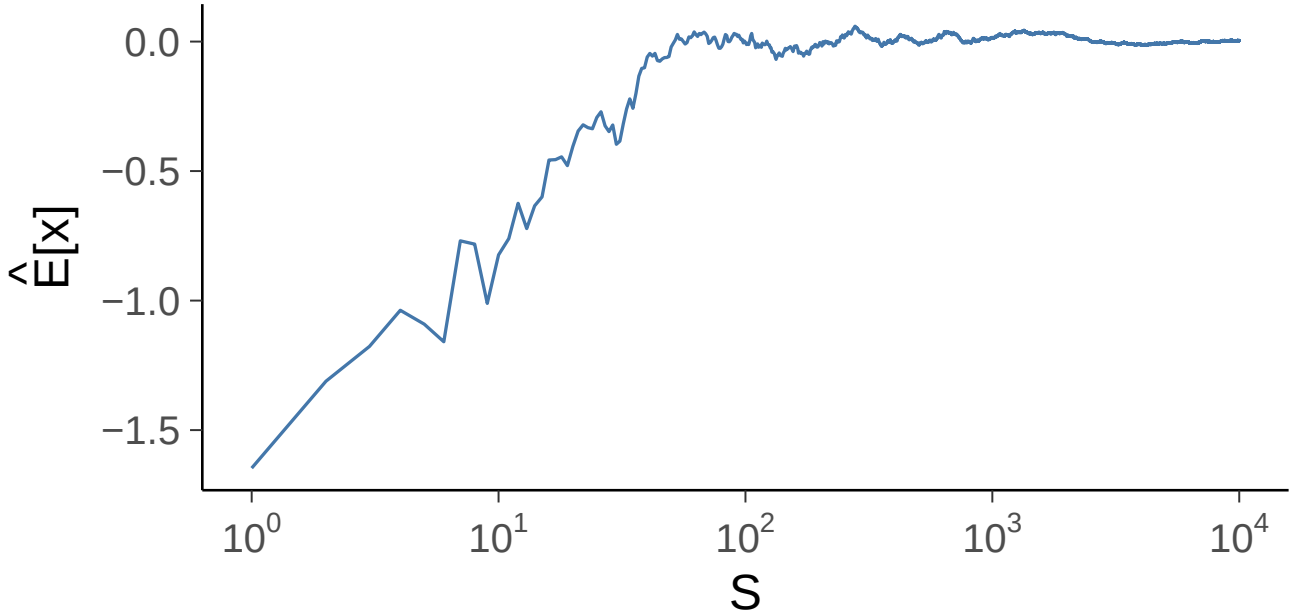

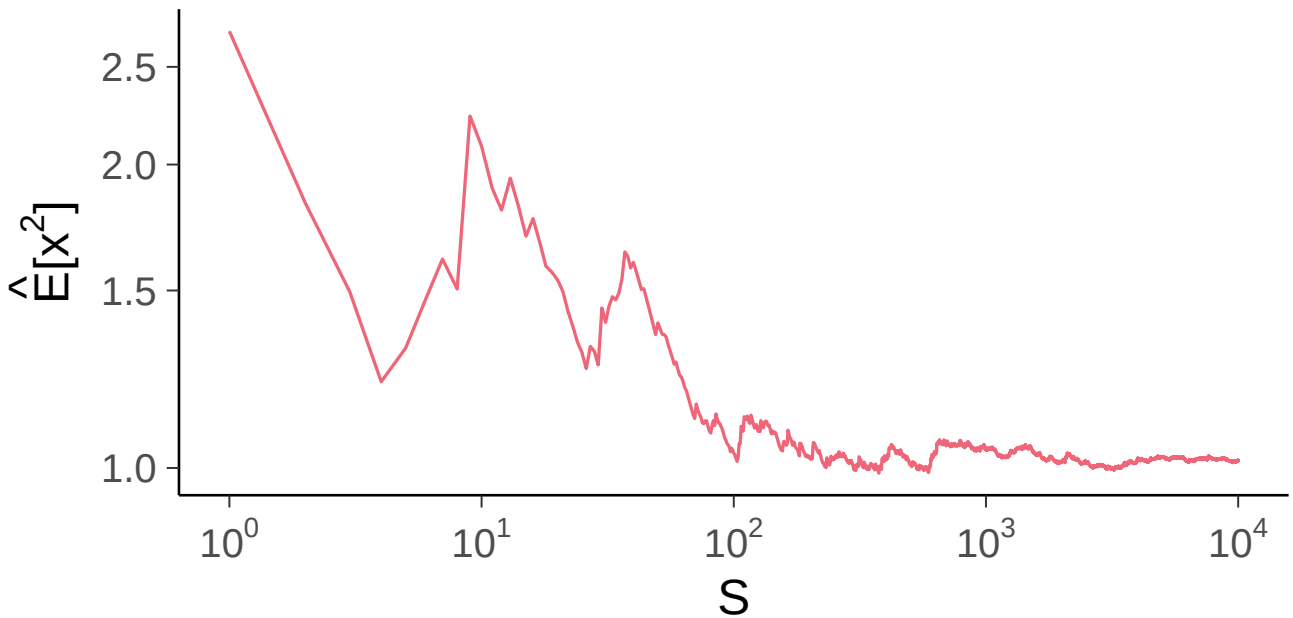

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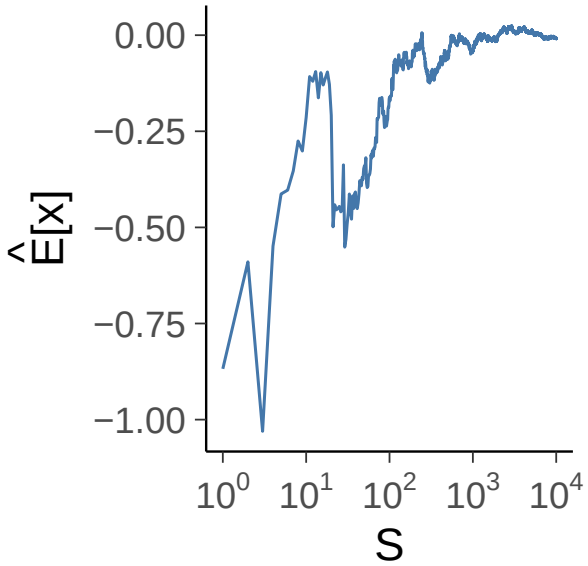

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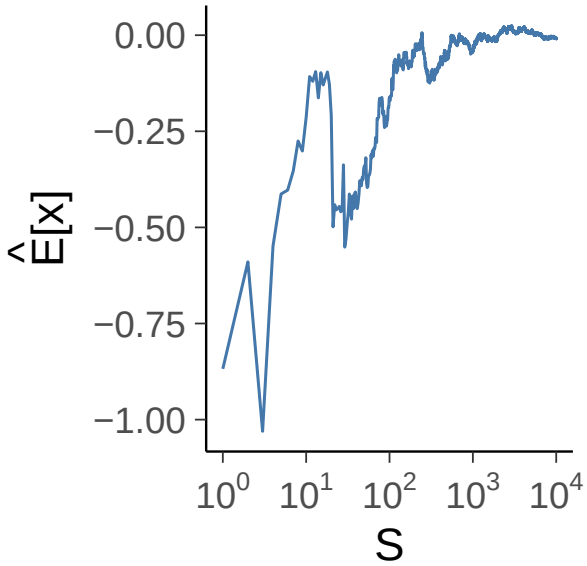

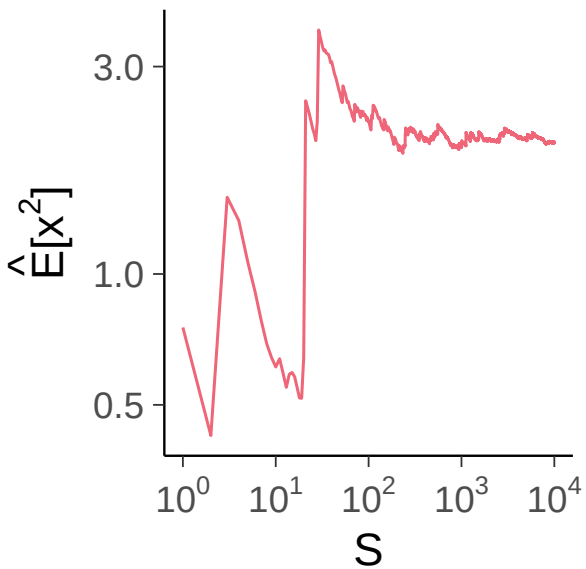

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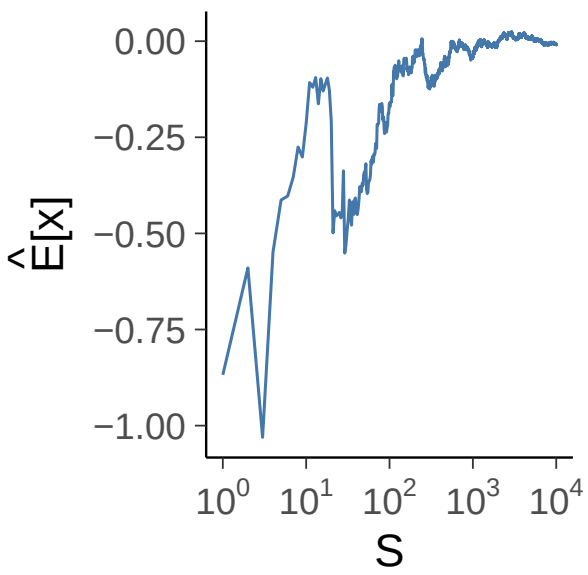

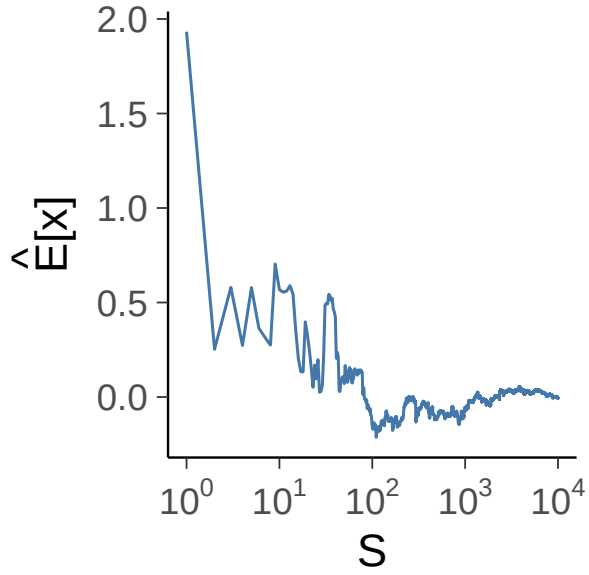

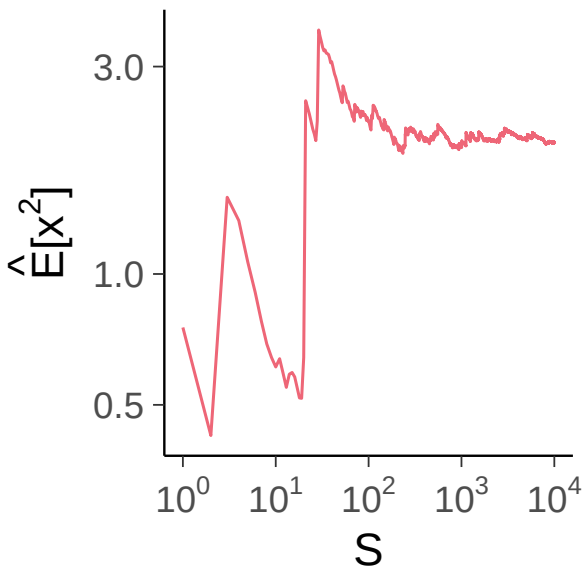

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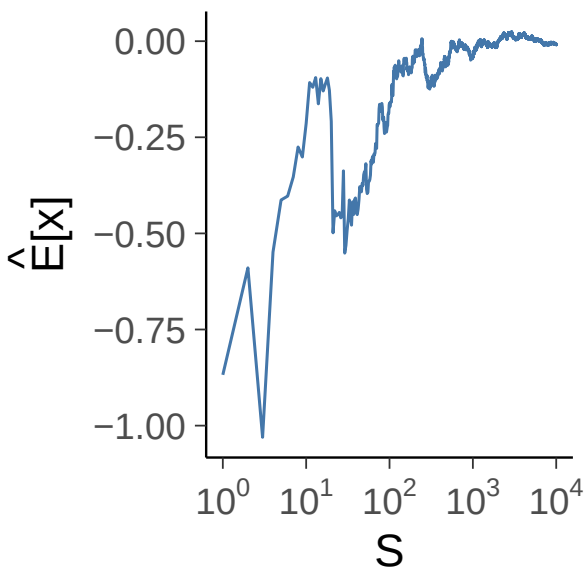

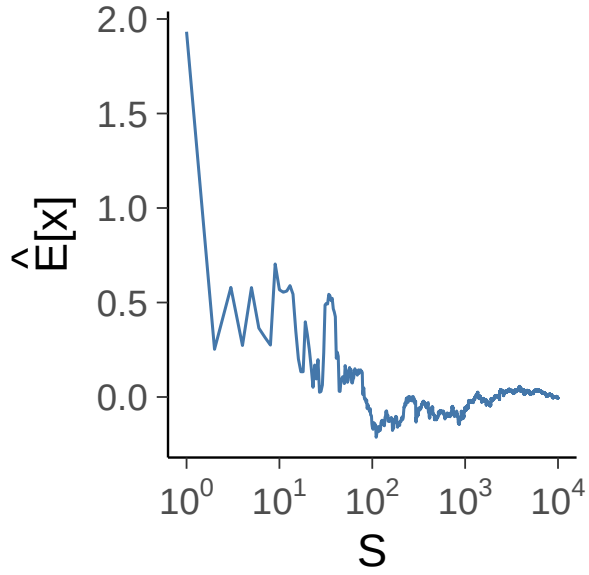

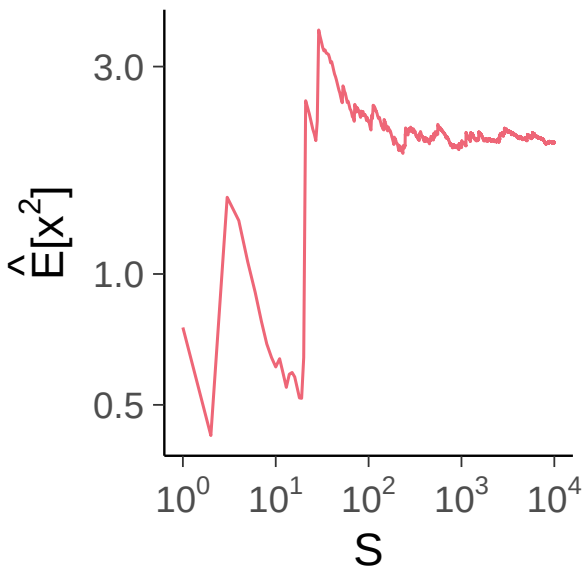

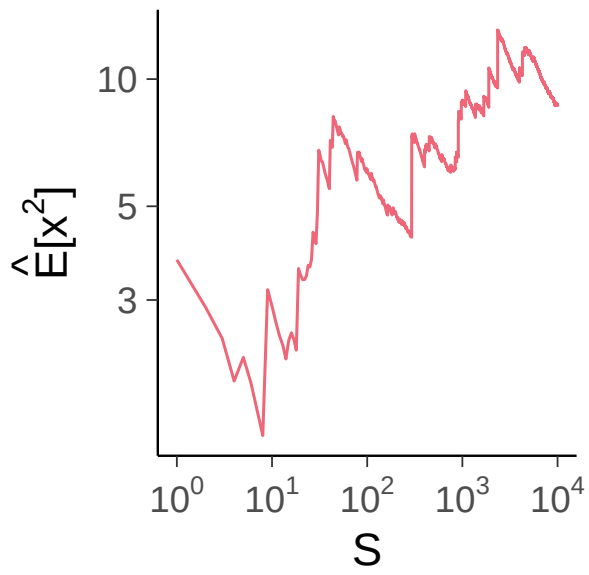

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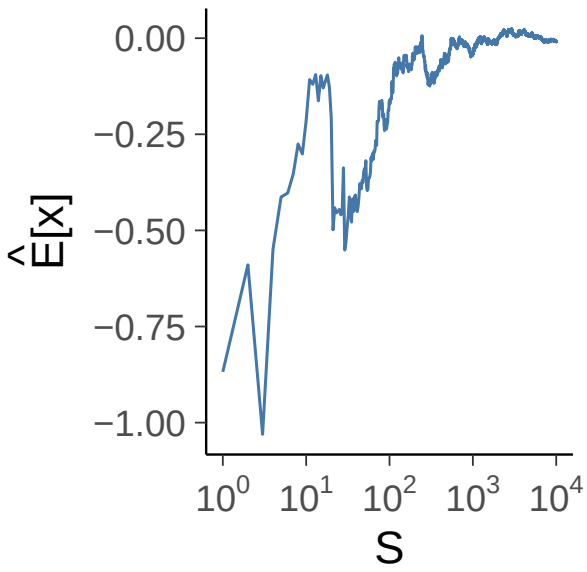

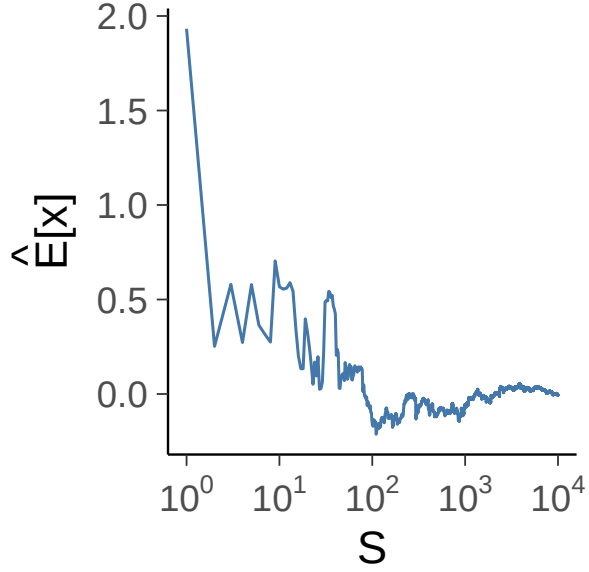

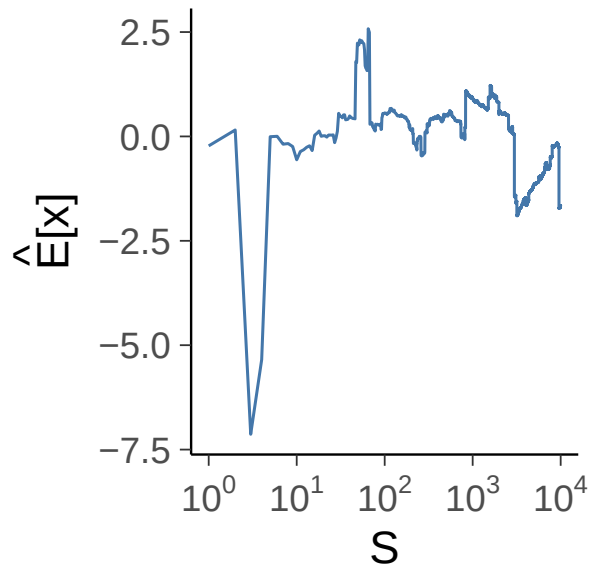

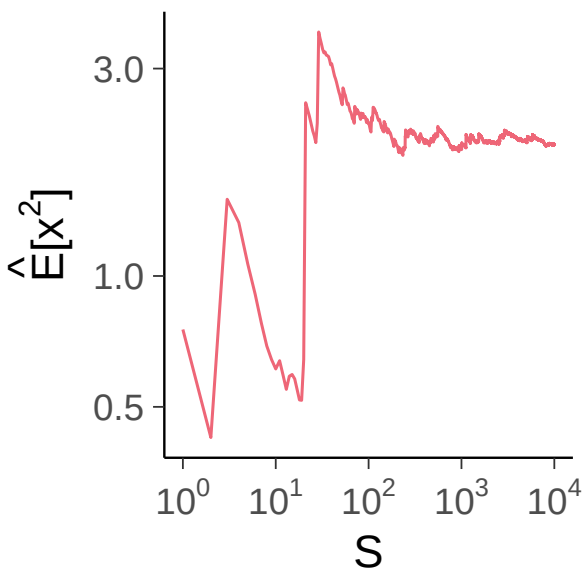

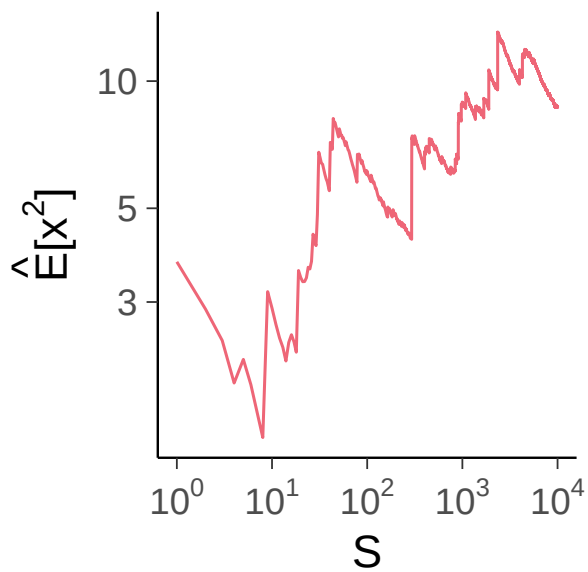

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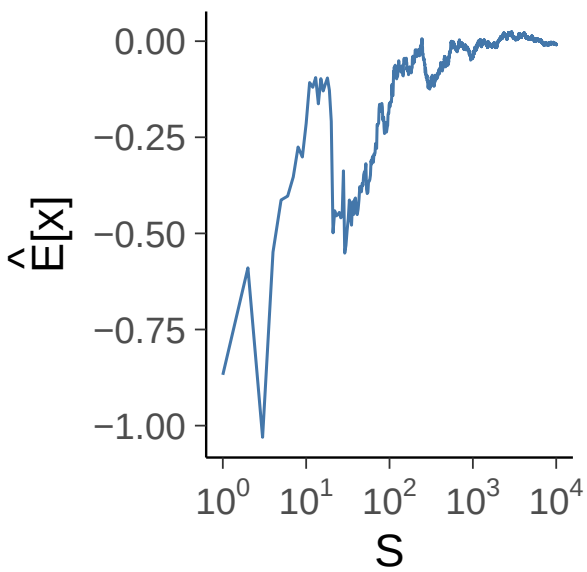
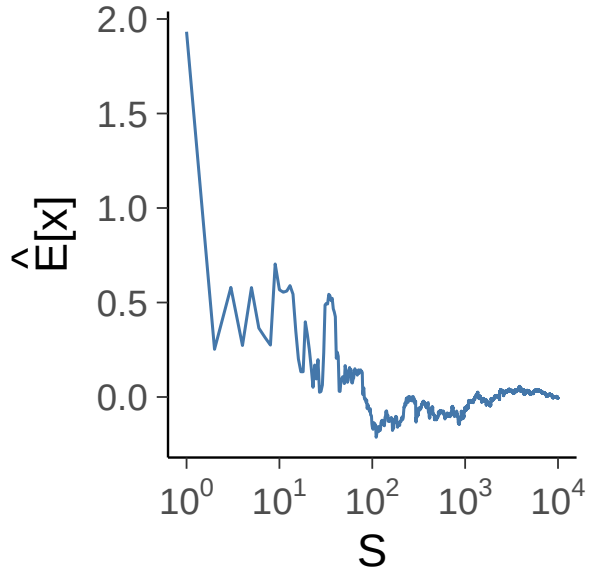
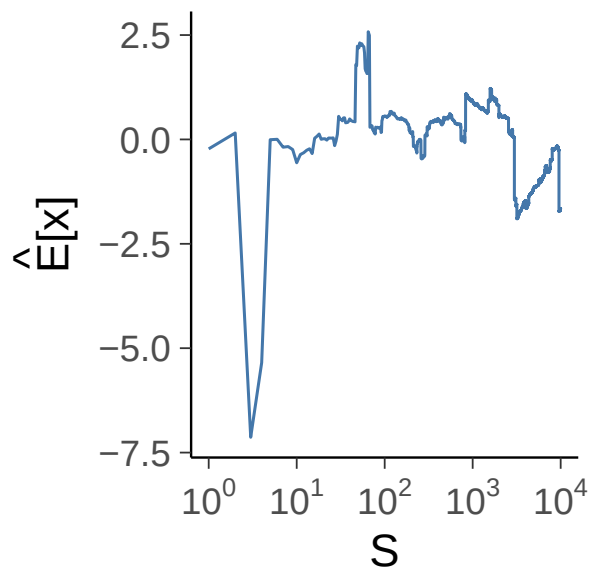
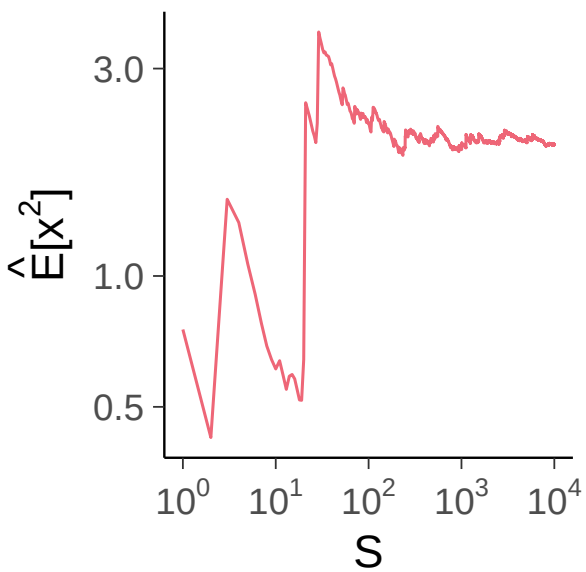
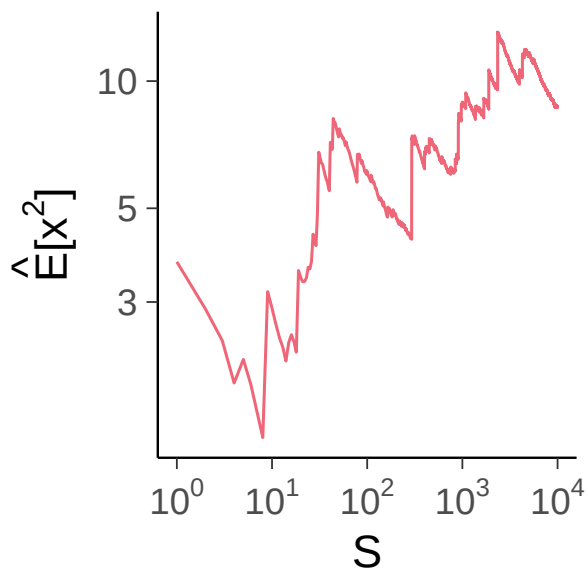
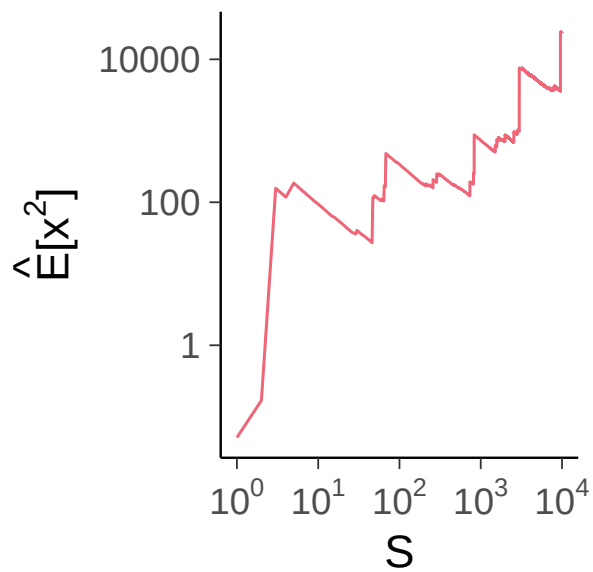






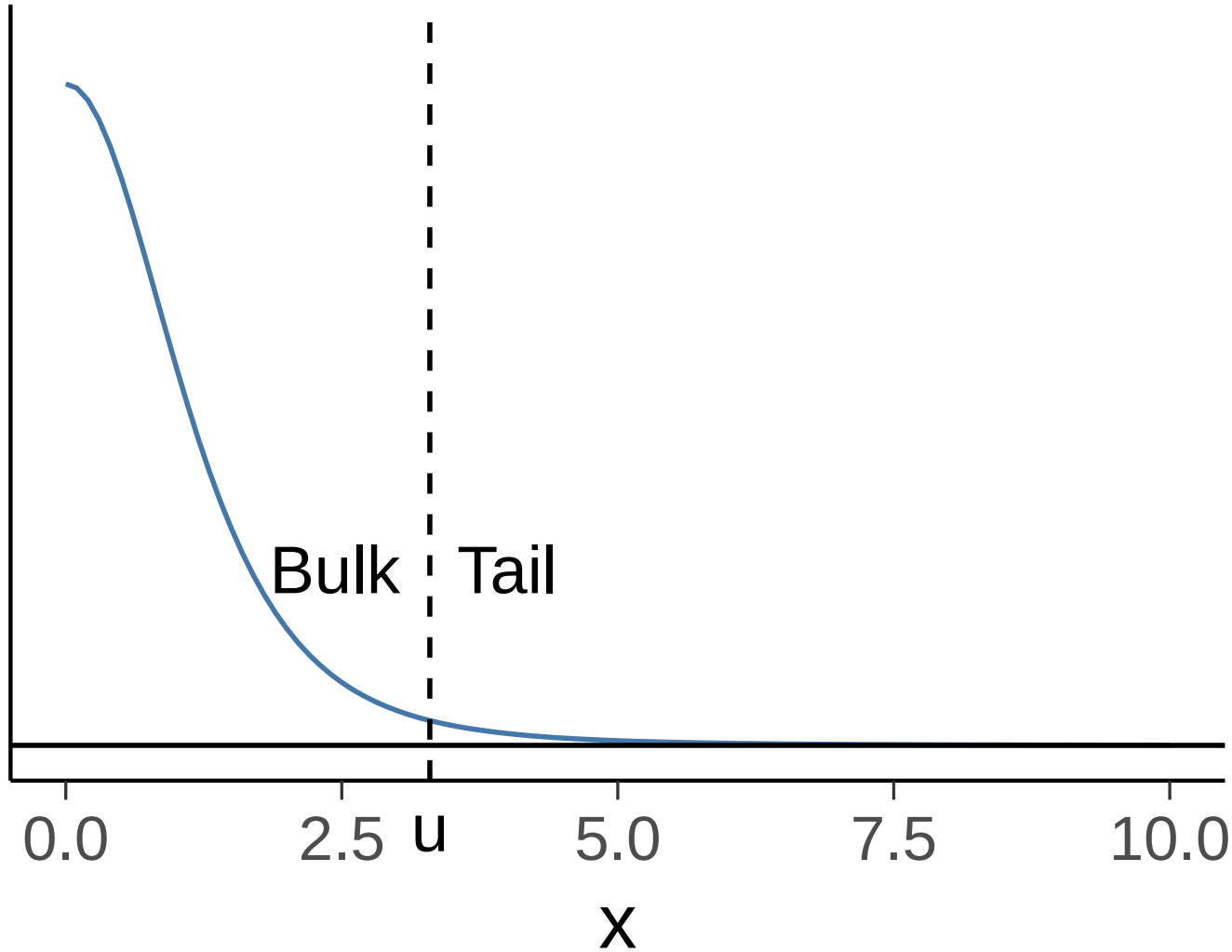






Pareto-$\hat{k}$ diagnostic

Pickands (1975): many distributions have tail ($x > u$) that is well approximated with Generalized Pareto distribution (GPD)



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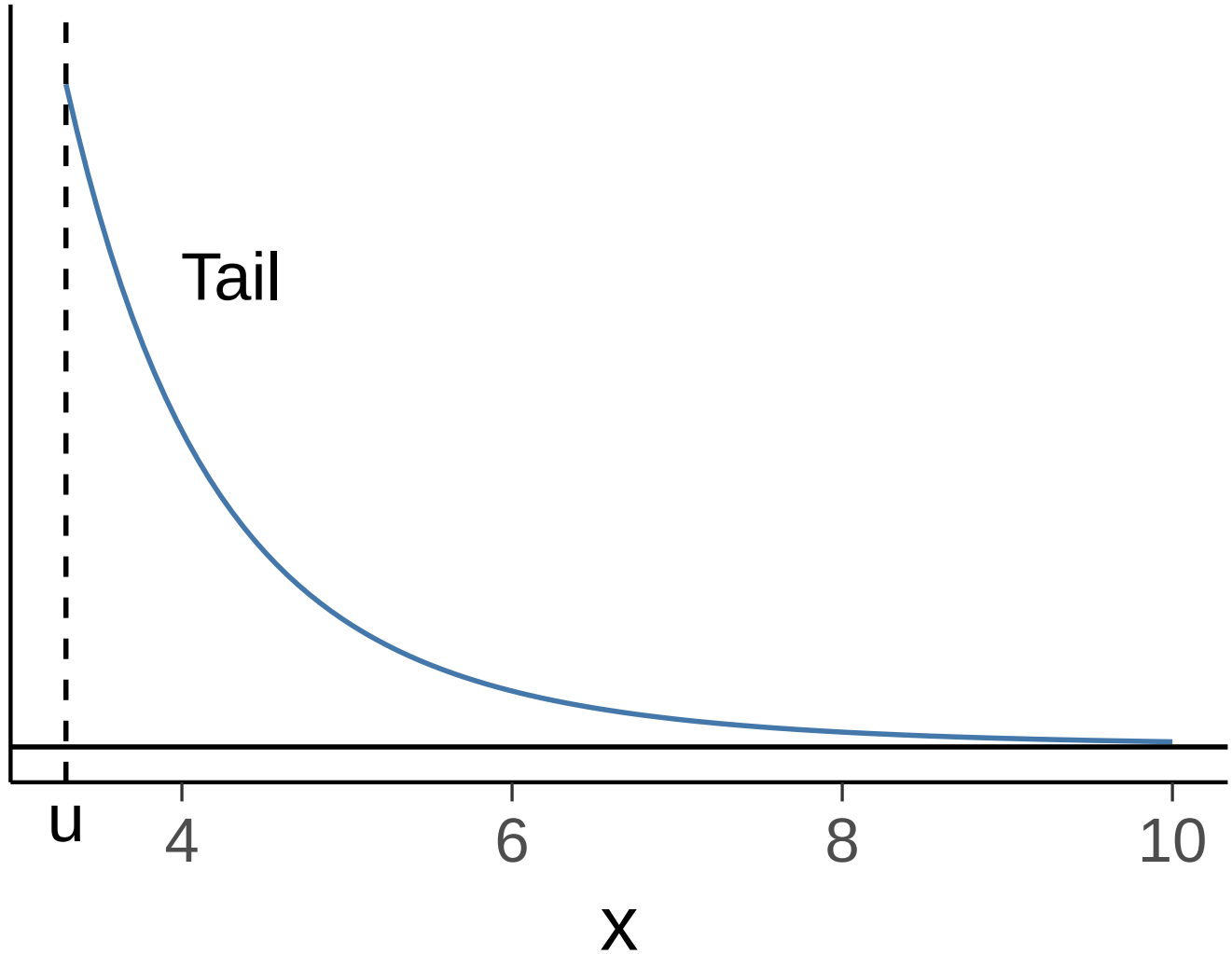

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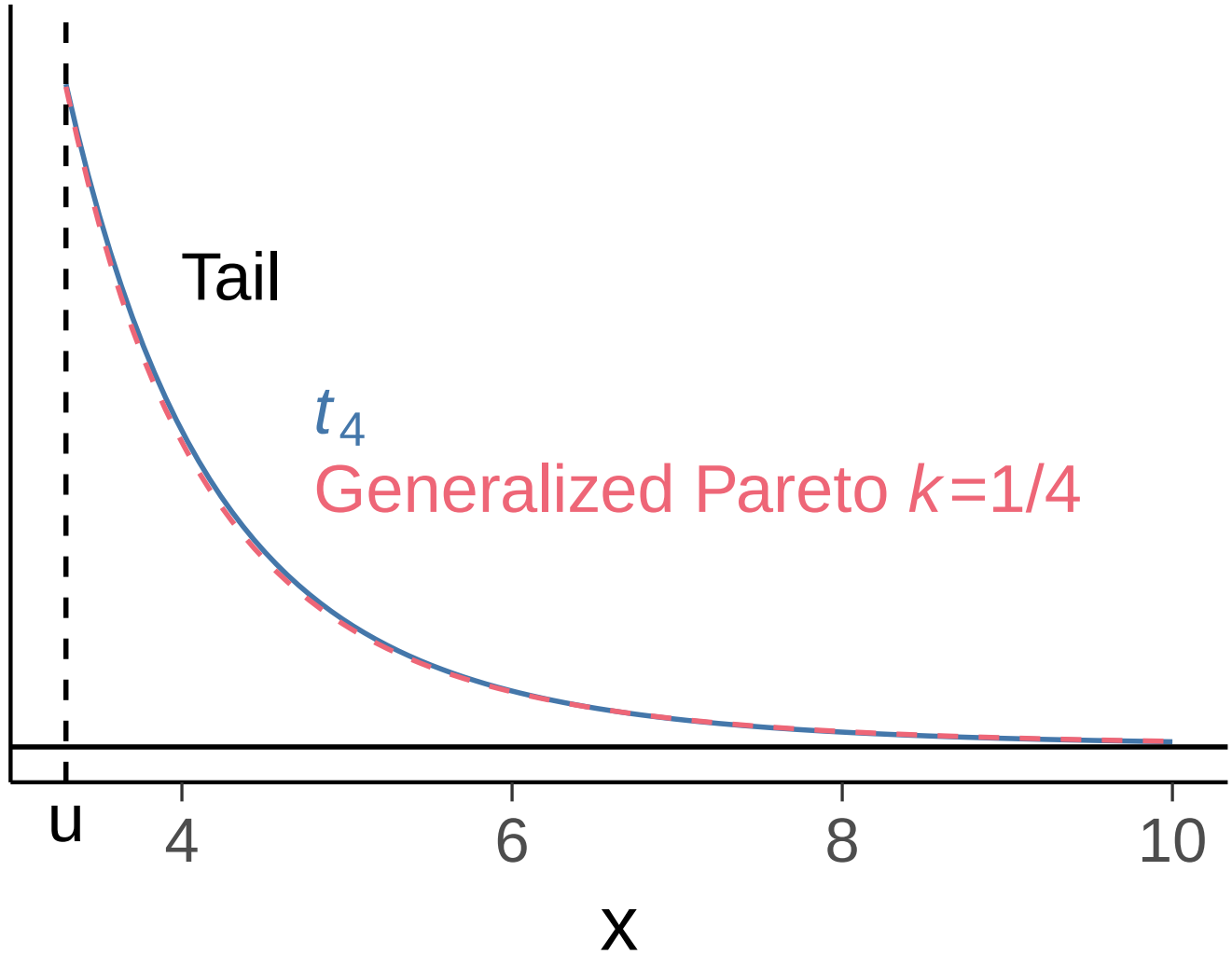

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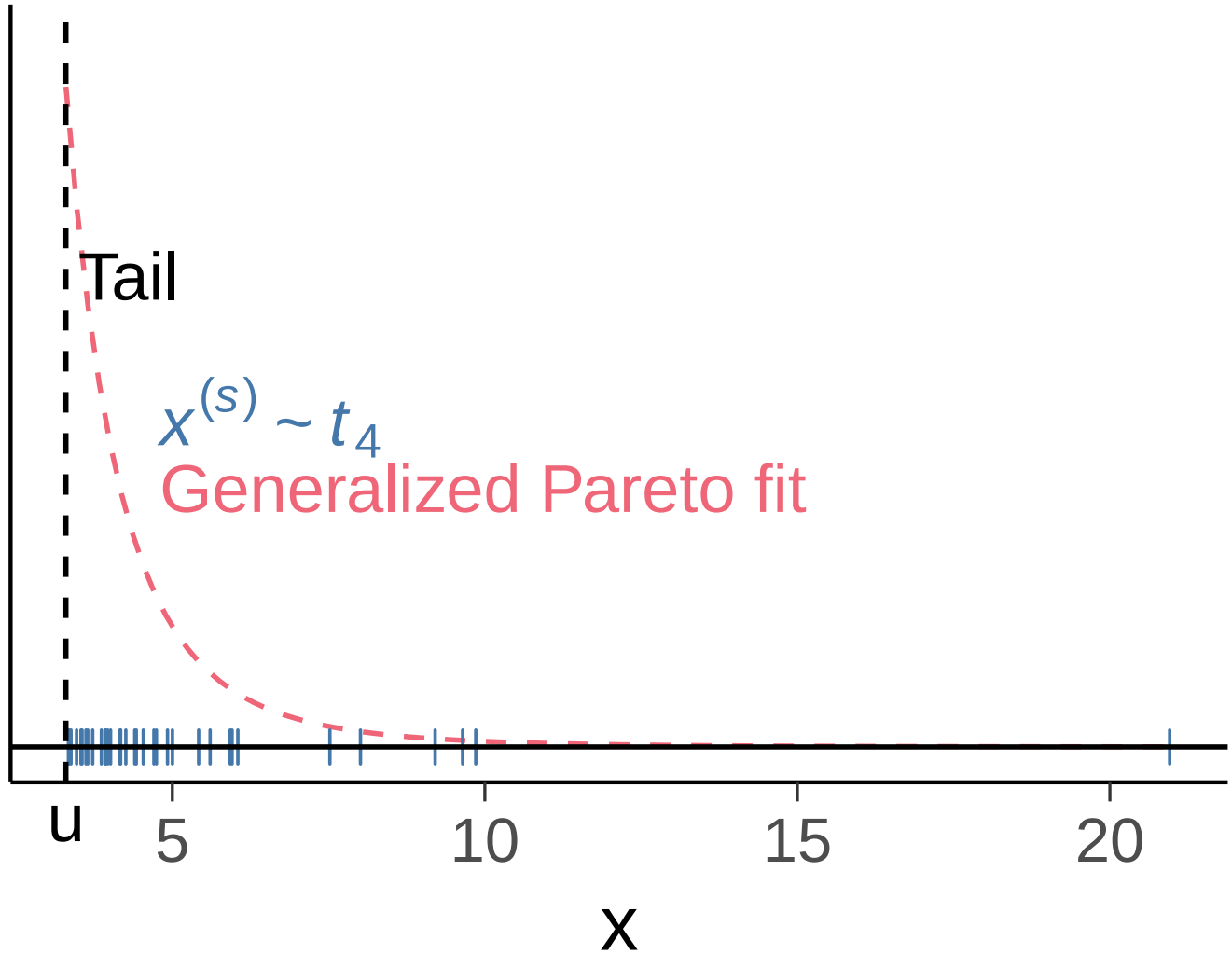
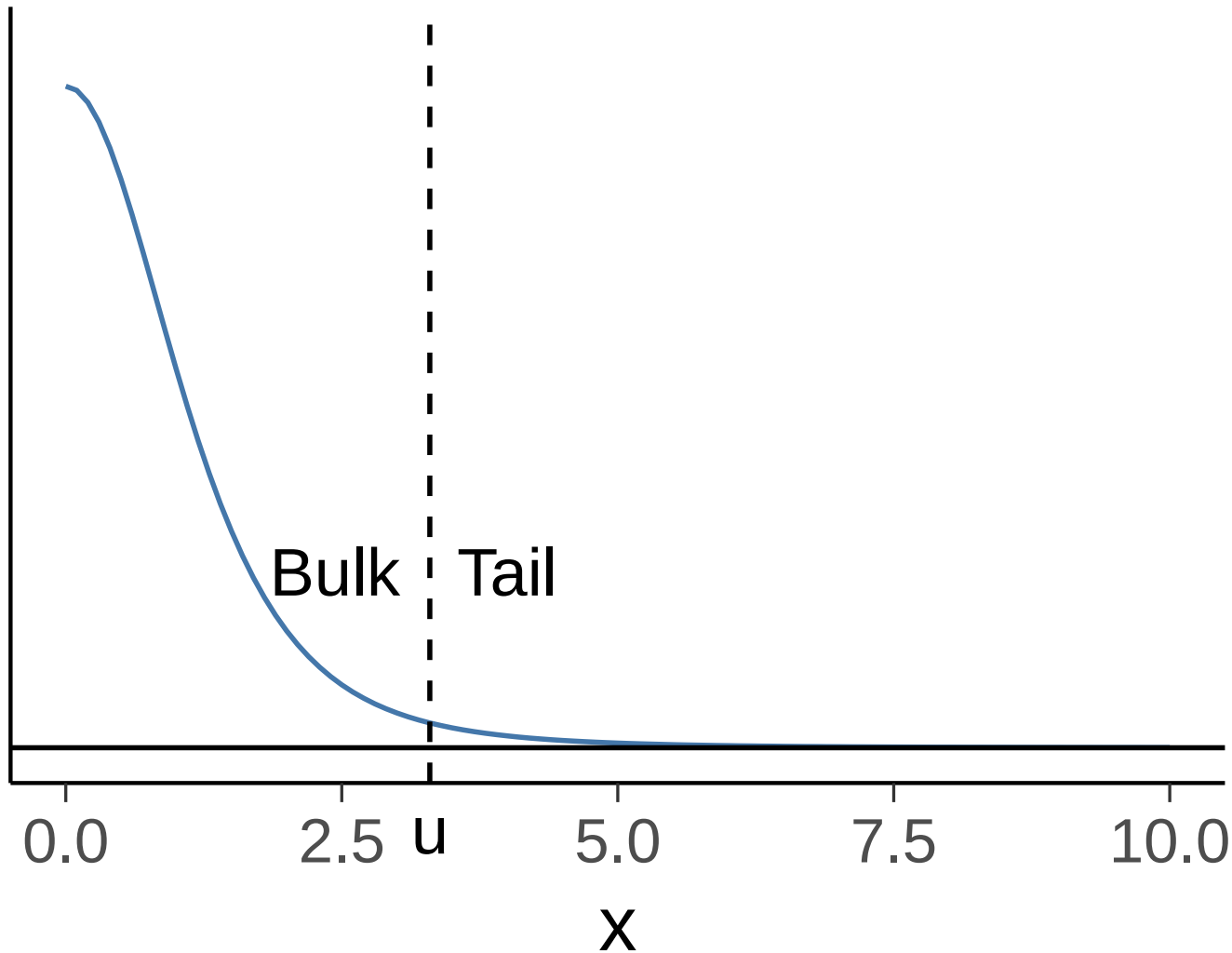


Pareto-$\hat{k}$ diagnostic

GPD has a shape parameter k,
and $1/k$ finite fractional moments



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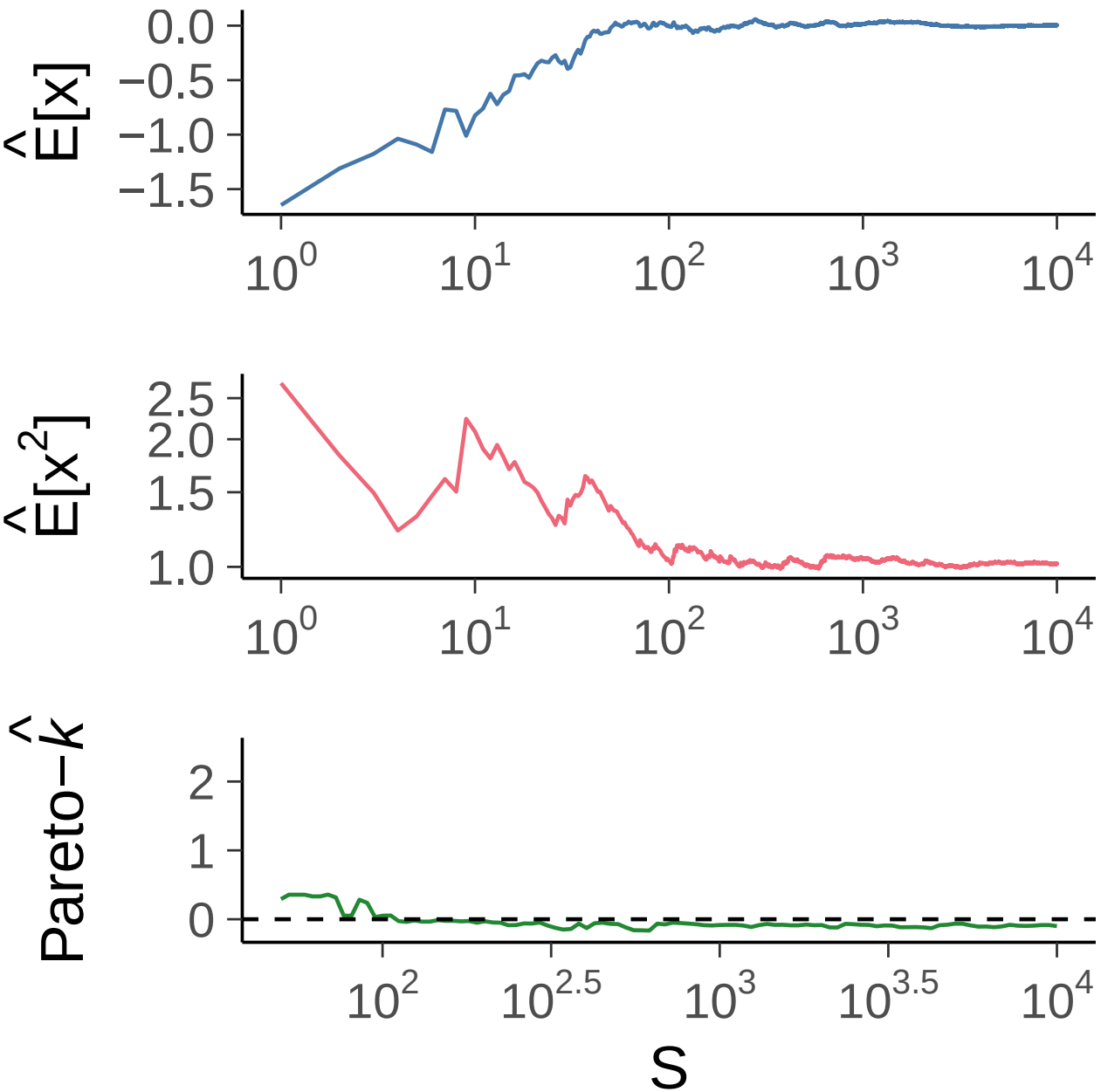
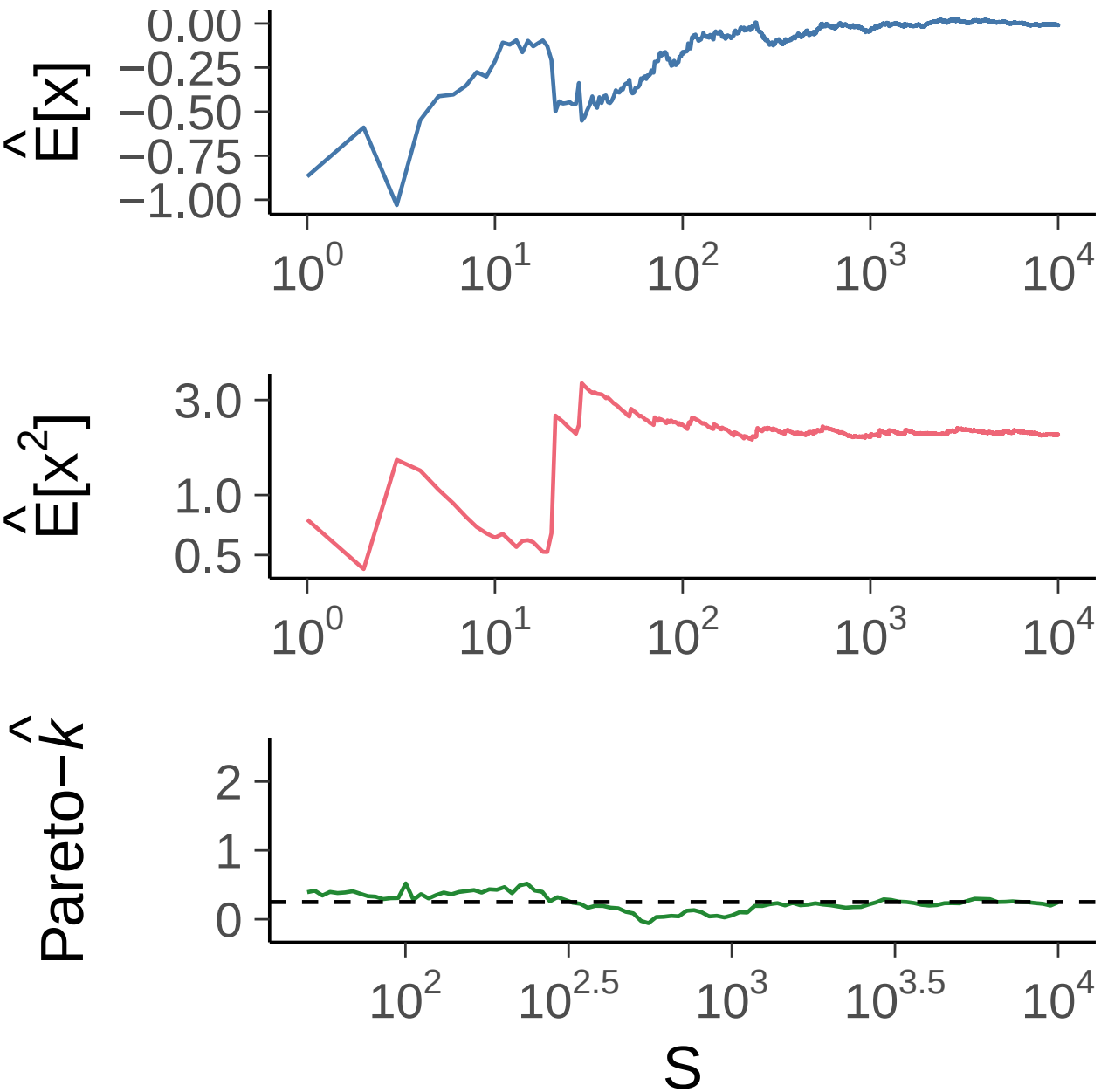
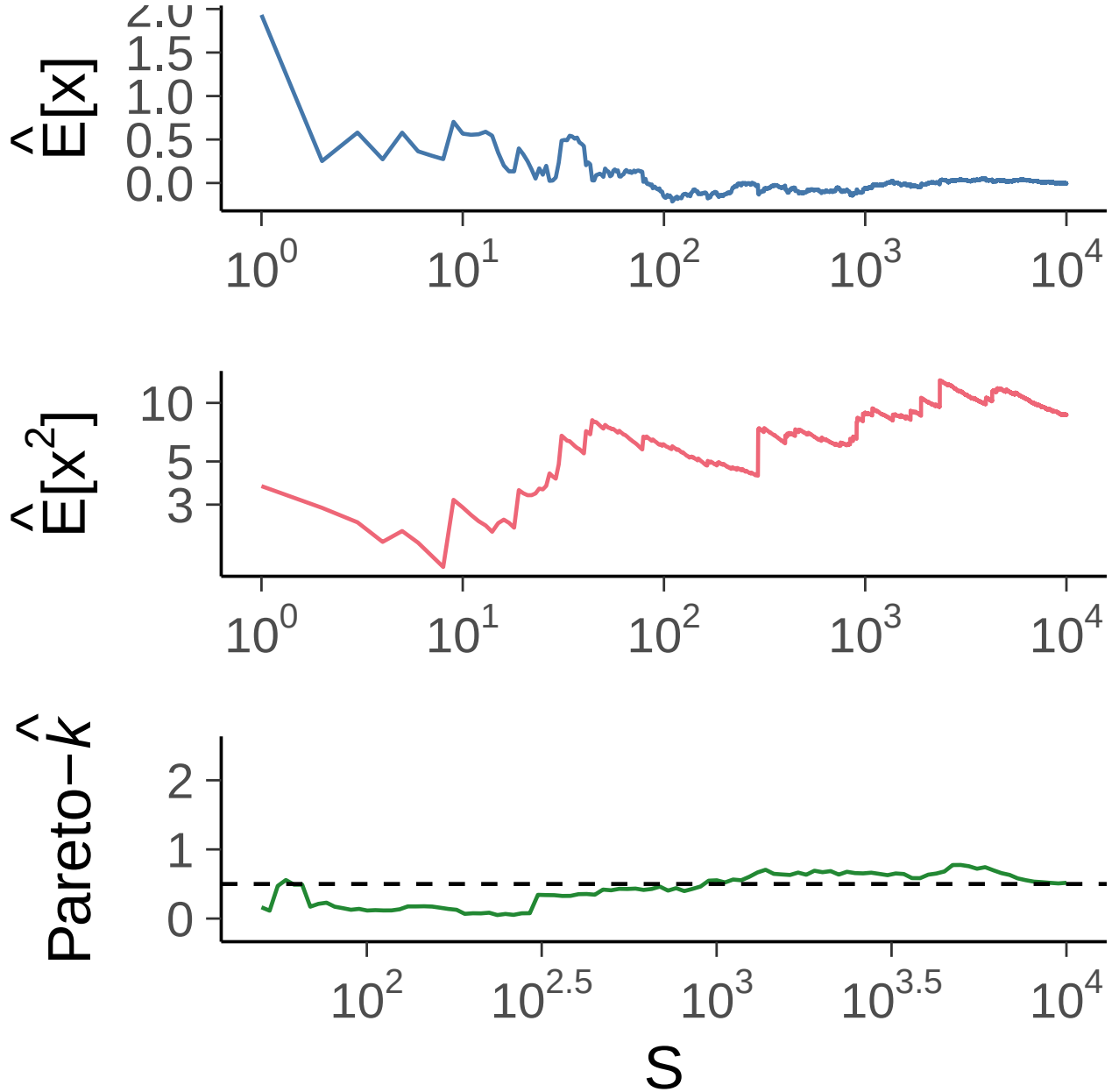


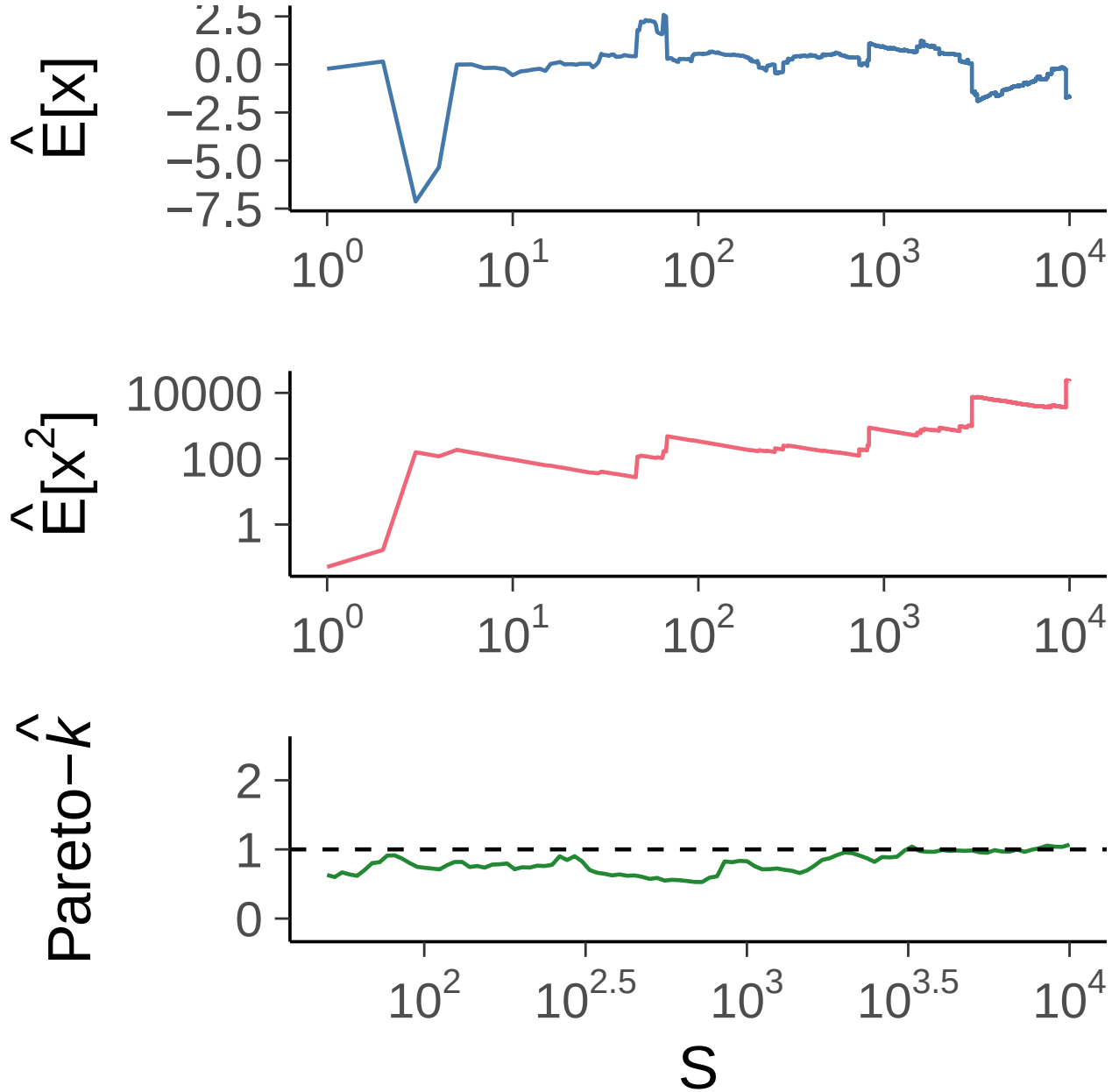
Pareto-$\hat{k}$ diagnostic: $x \sim t_4$



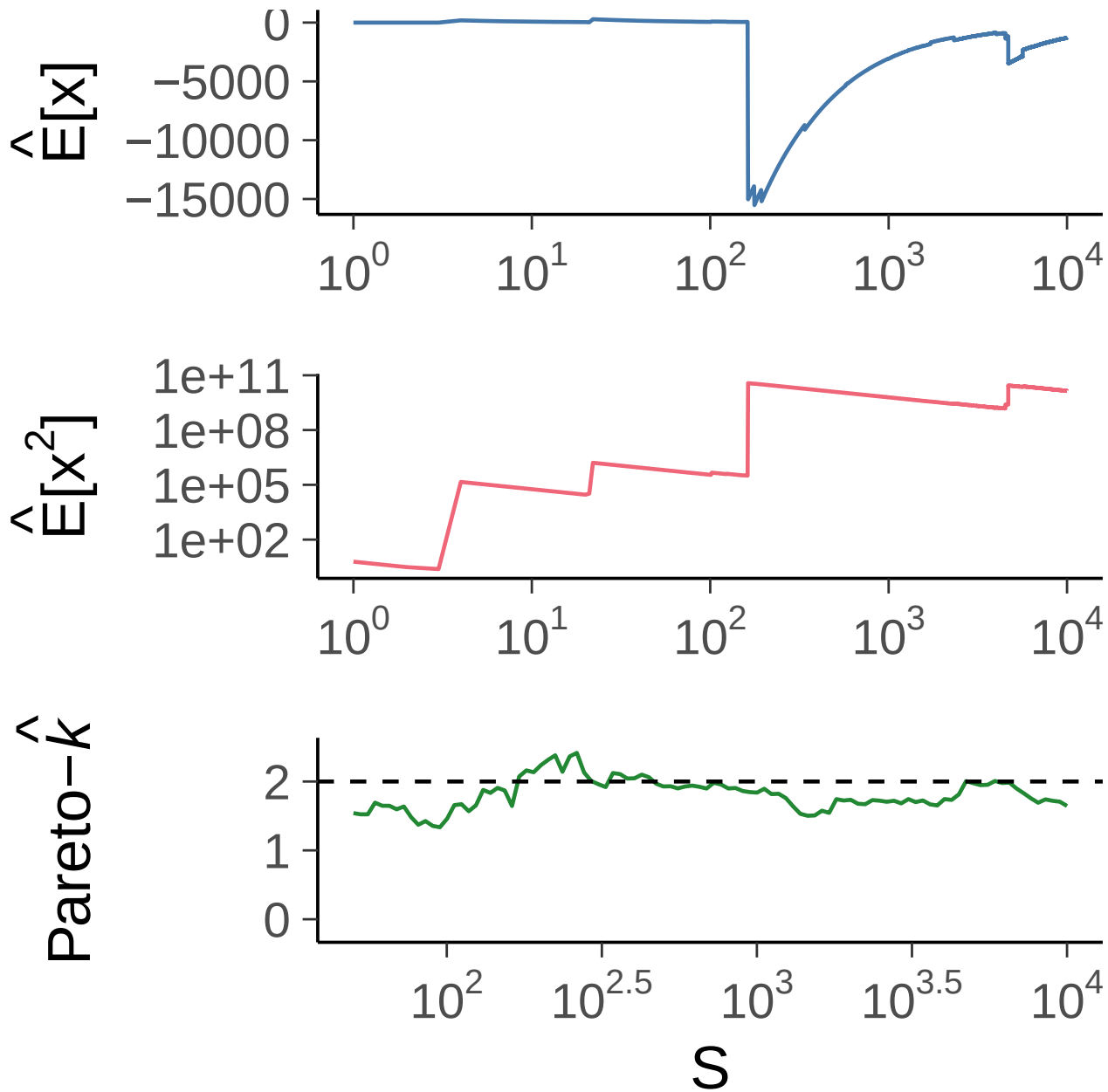
Pareto-$\hat{k}$ diagnostic: $x \sim t_2$



Pareto-$\hat{k}$ diagnostic: $x \sim t_1$



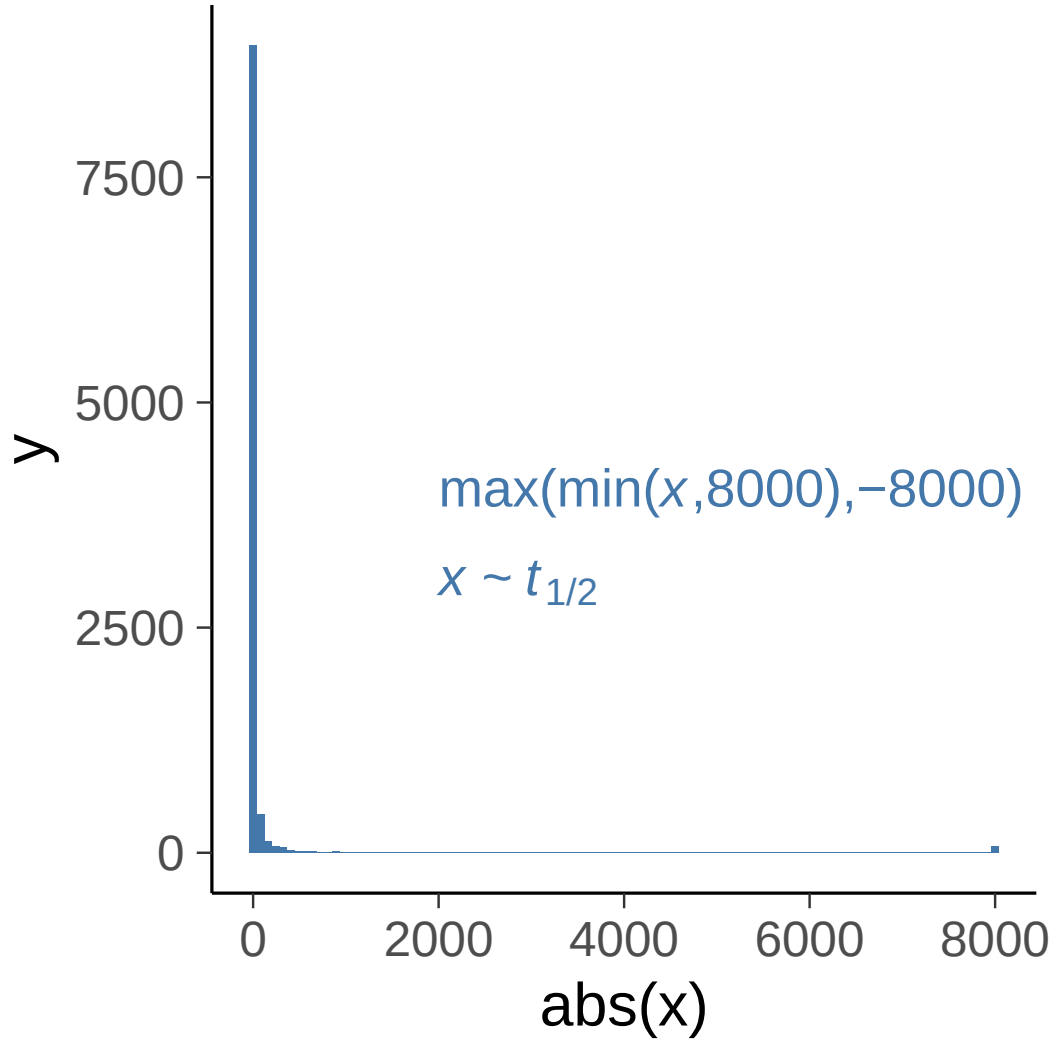
Pareto-$\hat{k}$ diagnostic: $x \sim t_{1/2}$



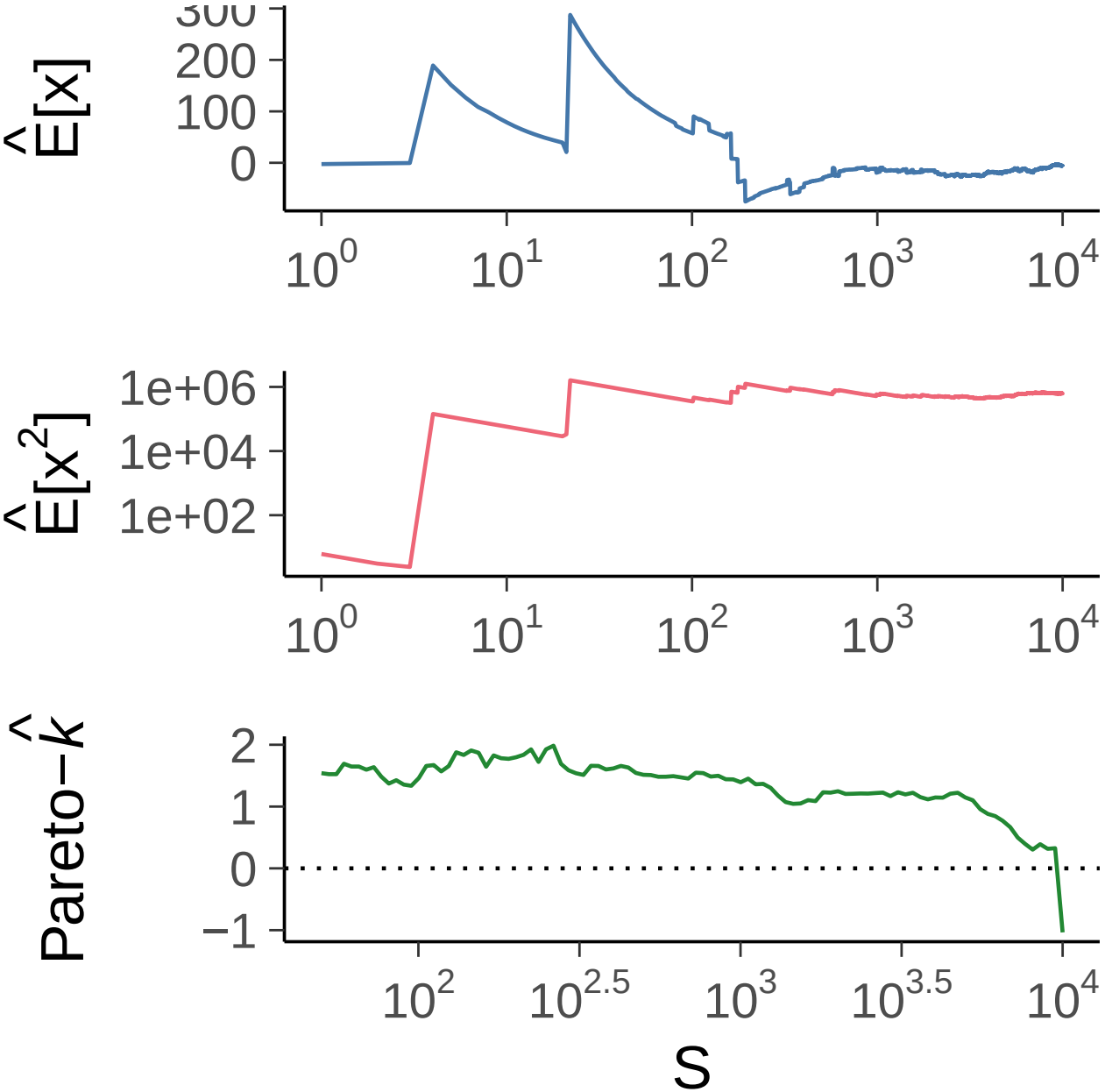
Pareto-$\hat{k}$ diagnostic is pre-asymptotic diagnostic

Thick tailed but truncated distribution

We can make estimates only based on what we have observed.



Pareto-$\hat{k}$ diagnostic: thick-tailed bounded distribution



Thick-tailed bounded distributions in practice

- Thick-tailed distributions are common in importance sampling and variational divergence estimation

Pareto-$\hat{k}$ in `posterior` package

```
> drt |> summarise_draws(mean, sd, mcse_mean)
```

variable	mean	sd	mcse_mean
xn	0.007	0.99	0.01
xt3	0.004	1.66	0.02
xt2_5	0.002	2.01	0.02
xt2	-0.008	3.00	0.03
xt1_5	-0.067	8.14	0.08
xt1	-1.57	122.	1.21

Pareto-$\hat{k}$ in `posterior` package

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> drt |> summarise_draws(mean, sd, mcse_mean, pareto_khat)
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variable	mean	sd	mcse_mean	pareto_khat
xn	0.007	0.99	0.01	-0.02
xt3	0.004	1.66	0.02	0.36
xt2_5	0.002	2.01	0.02	0.43
xt2	-0.008	3.00	0.03	0.53
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xt1	-1.57	122.	1.21	1.08

How to use Pareto-$\hat{k}$ diagnostic

- To check posterior of any quantity of interest
  - if high $\hat{k}$, maybe use some other summary than mean, e.g., quantiles

See more in Vehtari, Simpson, Gelman, Yao, and Gabry (2024). Pareto smoothed importance sampling. *JMLR*, 25(72):1-58.

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  - automated diagnostic as in PSIS-LOO (Lecture 9) and `priorsense` (Lecture ?)

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- Pareto-smoothing improves the mean estimate
  - reliable mean and MCSE estimates when Pareto-$k < 0.7$
  - required minimum sample size and convergence rate estimates for different values of k
  - more on lecture 9

See more in Vehtari, Simpson, Gelman, Yao, and Gabry (2024). Pareto smoothed importance sampling. *JMLR*, 25(72):1-58.

Direct simulation

- Produces independent draws
  - Using analytic transformations of uniform random numbers (e.g. appendix A)
  - factorization
  - numerical inverse-CDF
- Problem: restricted to limited set of models

Random number generators

- Good pseudo random number generators are sufficient for Bayesian inference
  - pseudo random generator uses deterministic algorithm to produce a sequence which is difficult to make difference from truly random sequence
  - modern software used for statistical analysis have good pseudo RNGs

Direct simulation: Example

- Box-Muller -method:
  If U_1 and U_2 are independent draws from distribution $\mathrm{U}(0,1)$, and

$$X_1 = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$
$$X_2 = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$$

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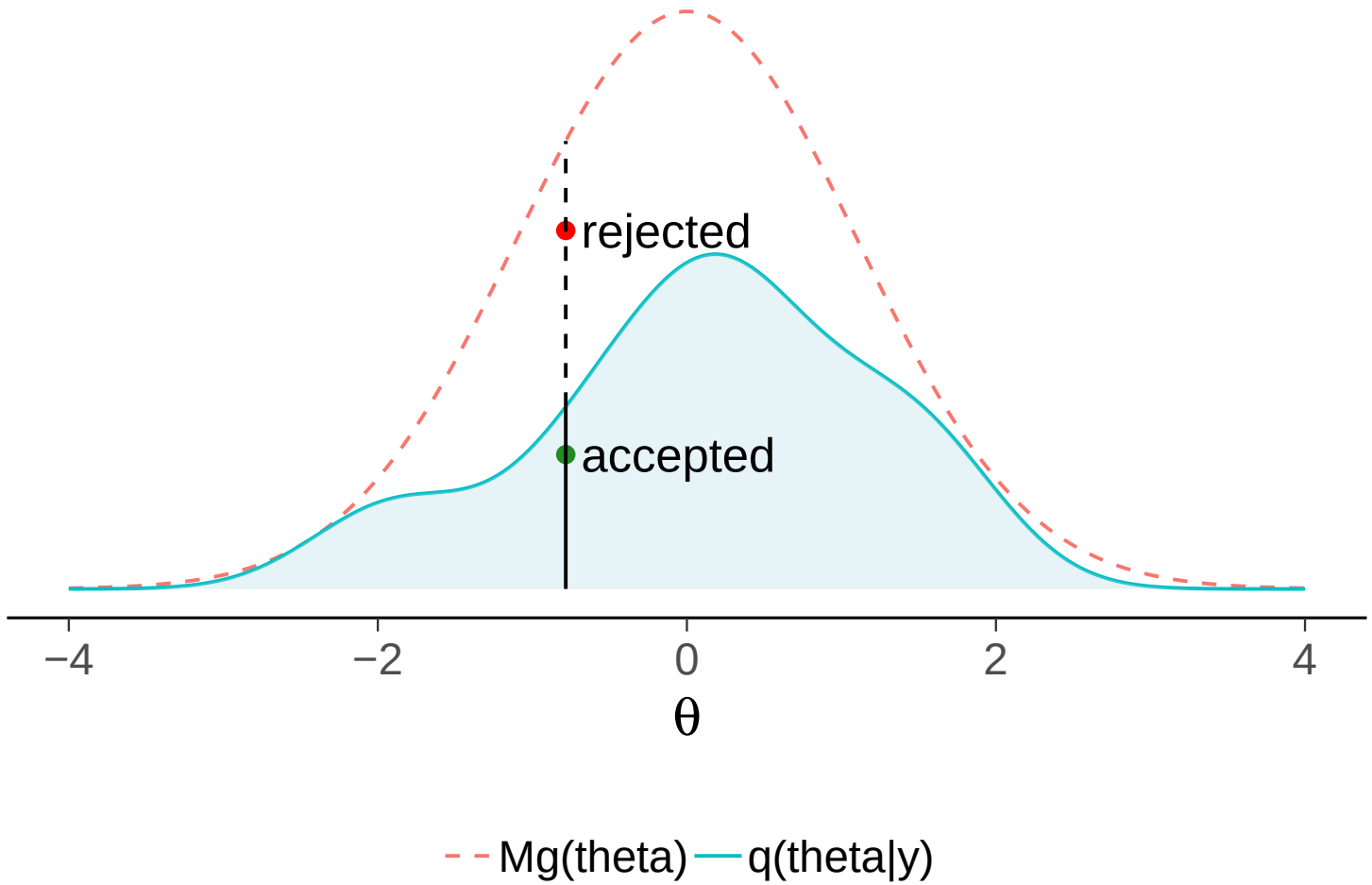
  then X_1 and X_2 are independent draws from the distribution $\mathrm{N}(0,1)$
  - not the fastest method due to trigonometric computations
  - for normal distribution more than ten different methods
  - e.g. R uses inverse-CDF

Indirect sampling

- Rejection sampling
- Importance sampling
- Markov chain Monte Carlo (next week)

Rejection sampling

- Proposal forms envelope over the target distribution $q(\theta|y)/Mg(\theta) \leq 1$
- Draw from the proposal and accept with probability $q(\theta|y)/Mg(\theta)$



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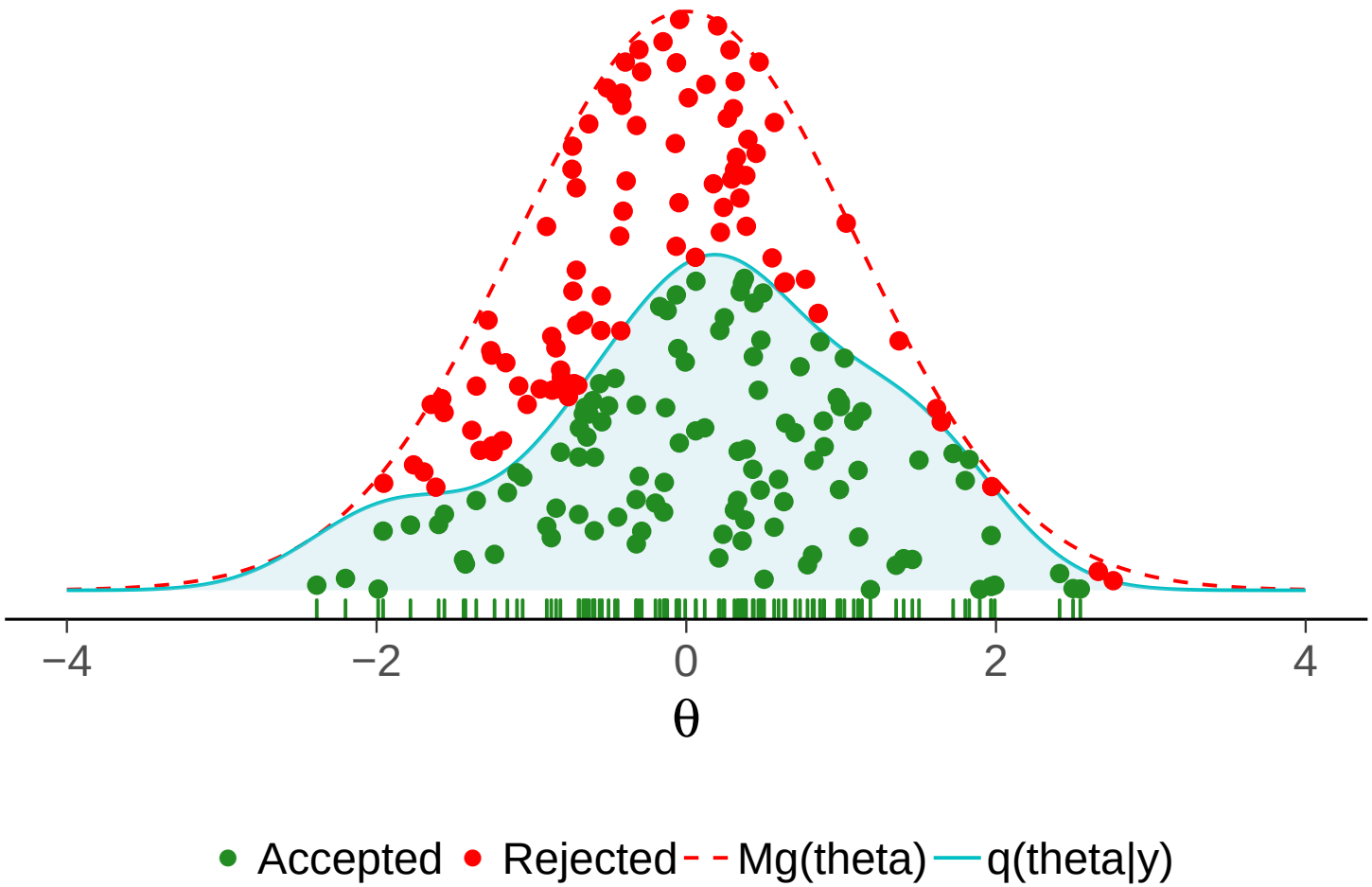
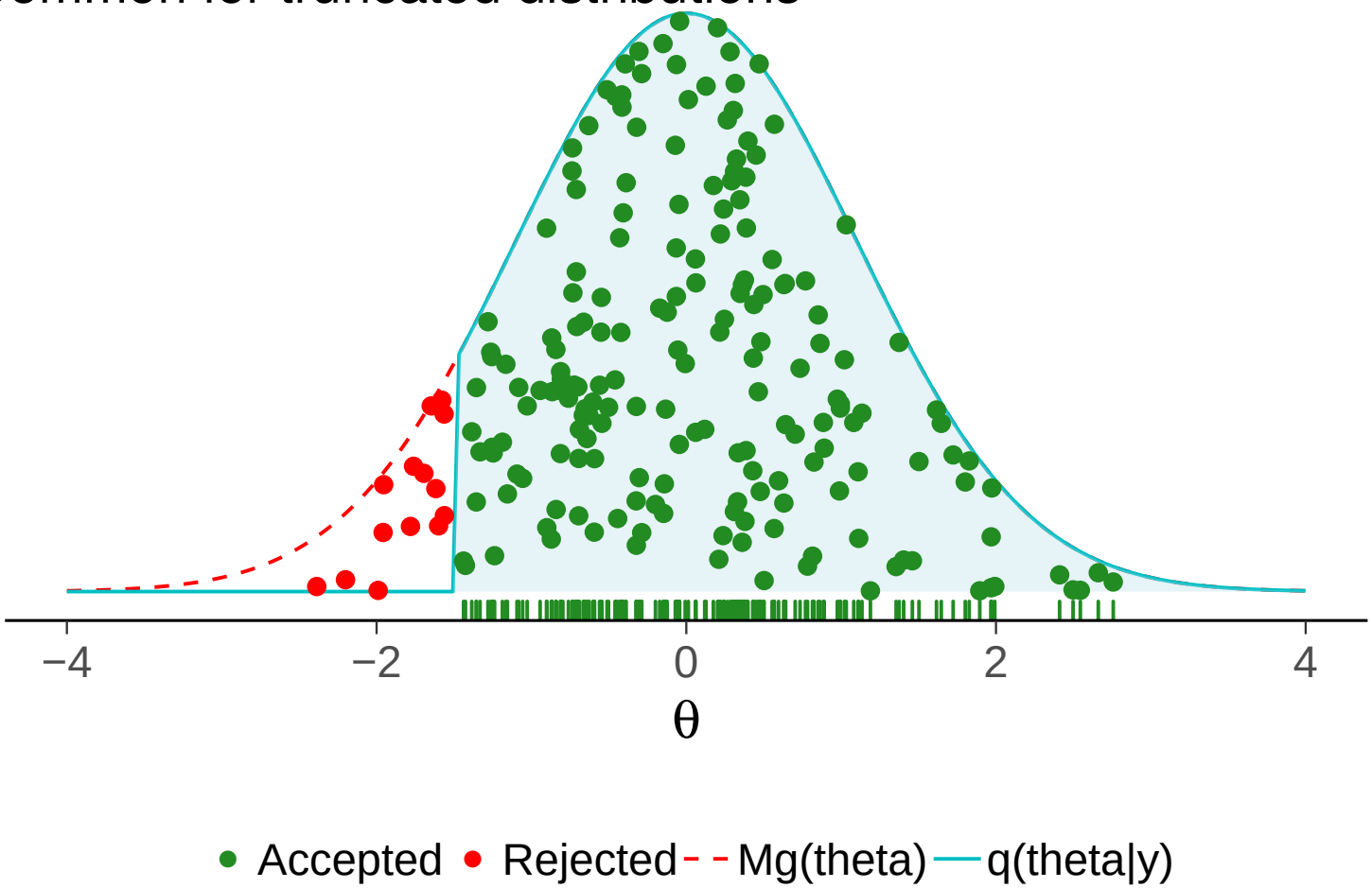


Rejection sampling

- Proposal forms envelope over the target distribution $q(\theta|y)/Mg(\theta) \leq 1$
- Draw from the proposal and accept with probability $q(\theta|y)/Mg(\theta)$
- Common for truncated distributions

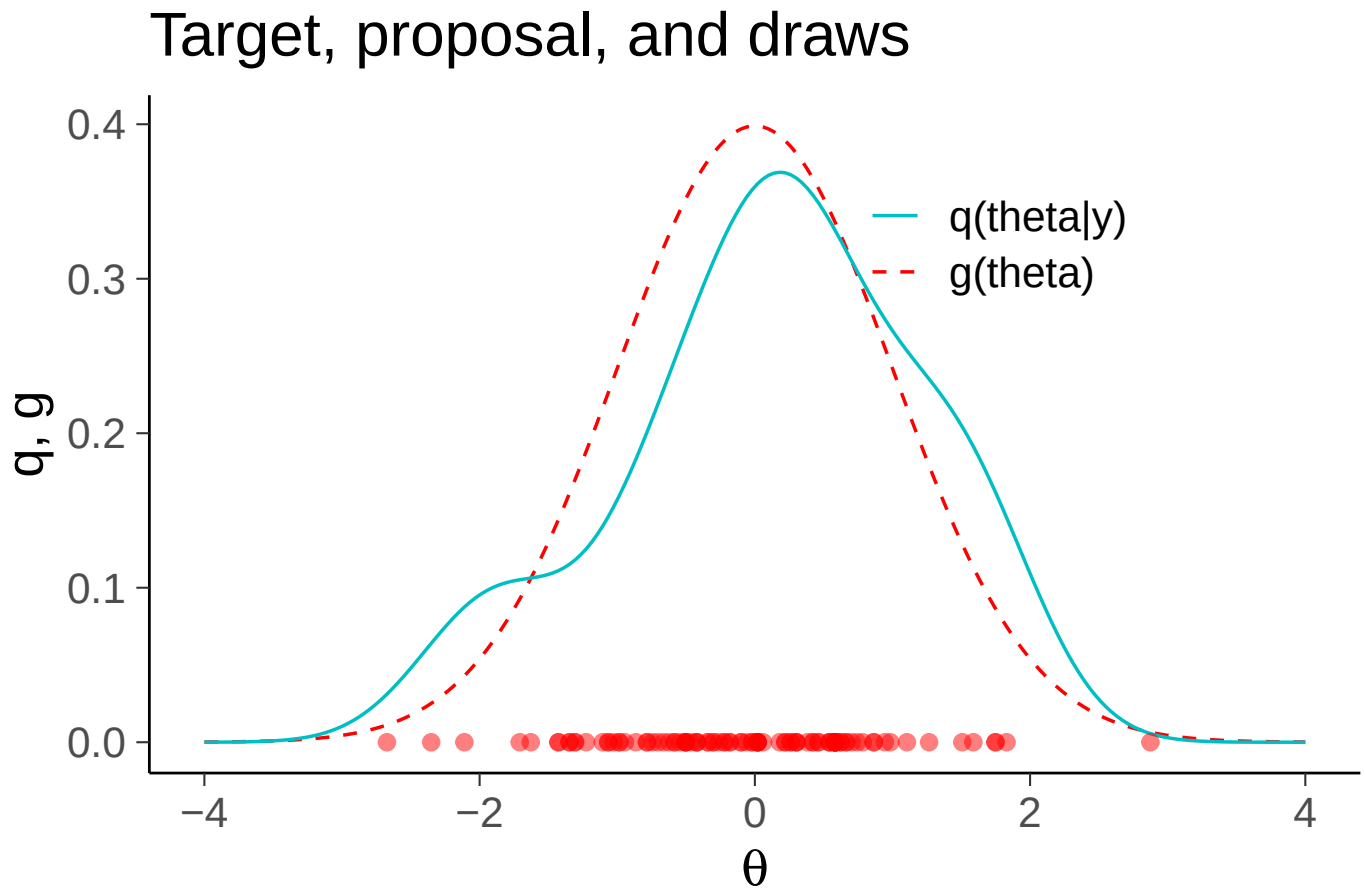


Rejection sampling

- The effective sample size (ESS) is the number of accepted draws
  - with bad proposal distribution may require a lot of trials
  - selection of good proposal gets very difficult when the number of dimensions increase
  - reliable diagnostics and thus can be a useful part

Importance sampling

- Proposal does not need to have a higher value everywhere



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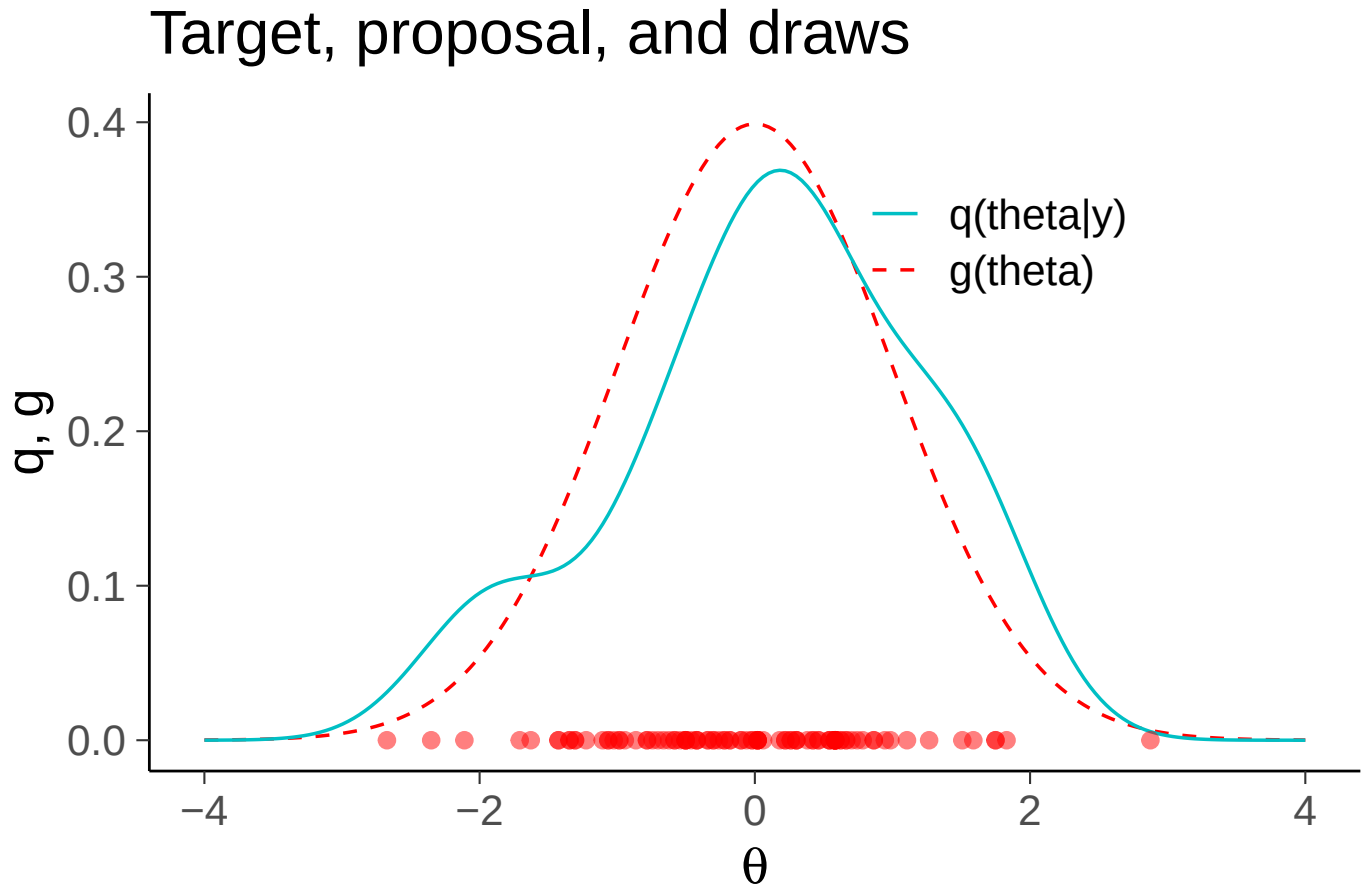

$$\mathrm{E}[h(\theta)] \approx \frac{\sum_s w_s h(\theta^{(s)})}{\sum_s w_s}, \qquad \text{where} \quad w_s = \frac{q(\theta^{(s)})}{g(\theta^{(s)})}$$

Importance sampling

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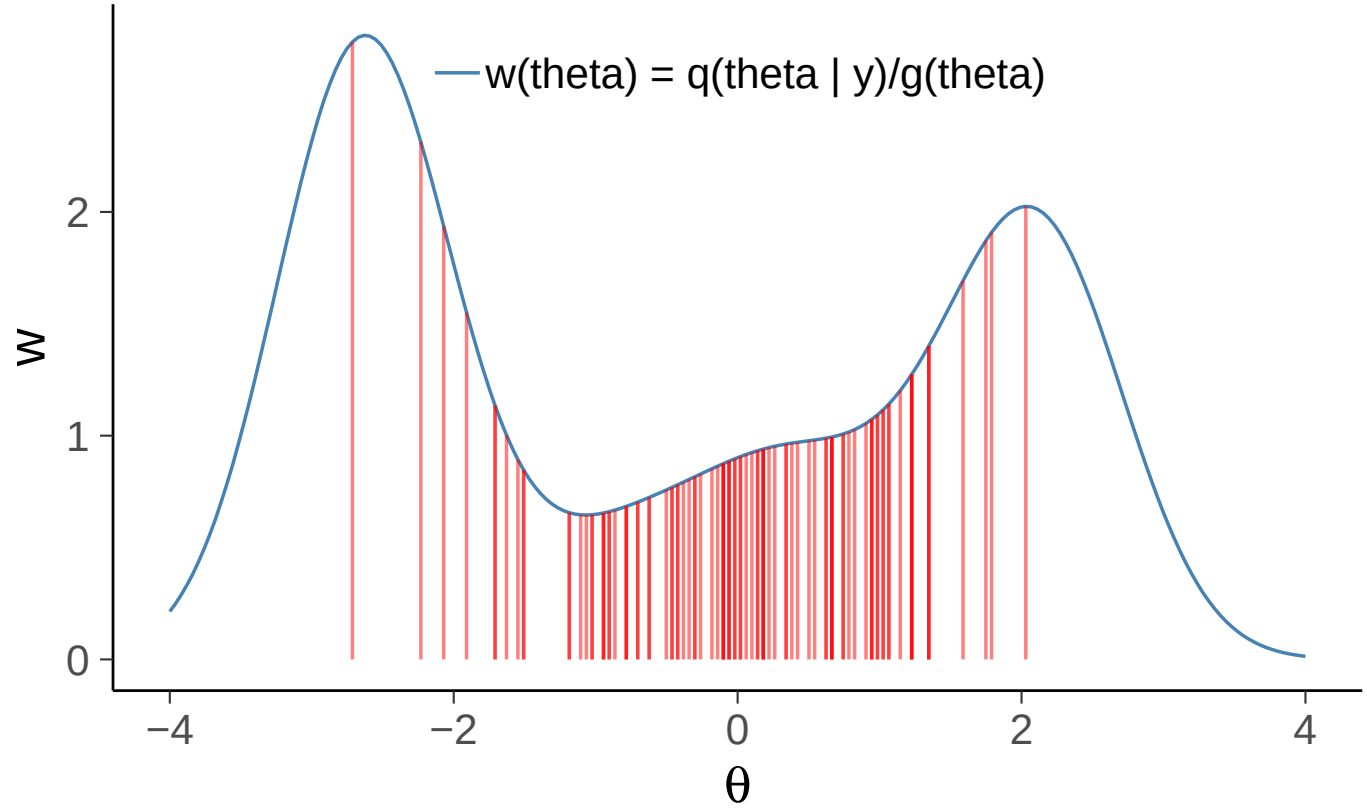
Draws and importance weights



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Some uses of importance sampling

In general selection of good proposal gets more difficult when the number of dimensions increase, but there are many special use case which scale well (e.g. I've used IS up to 10k dimensions)

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In general selection of good proposal gets more difficult when the number of dimensions increase, but there are many special use case which scale well (e.g. I've used IS up to 10k dimensions)

- Fast leave-one-out cross-validation (`loo`)
- Fast bootstrapping
- Fast prior and likelihood sensitivity analysis (`priorsense`)
- Conformal Bayesian computation
- Particle filtering
- Improving distributional approximations (e.g Laplace, Pathfinder, VI)

IS finite variance and central limit theorem

- If $h(\theta)w$ and w have finite variance $\rightarrow$ CLT
  - variance goes down as $1/S$
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  - generally not trivial
- Pre-asymptotic and asymptotic behavior can be really different!

Importance re-sampling

- Using the weighted draws is good

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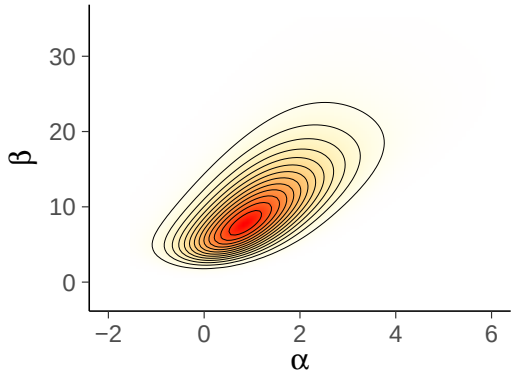
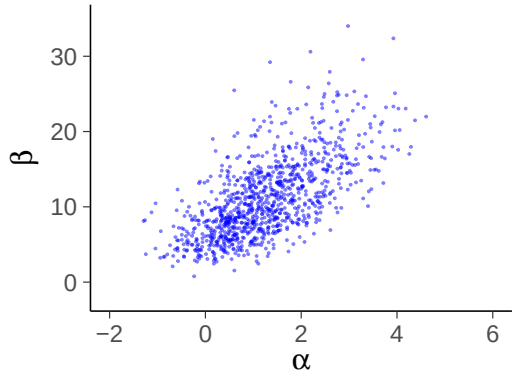
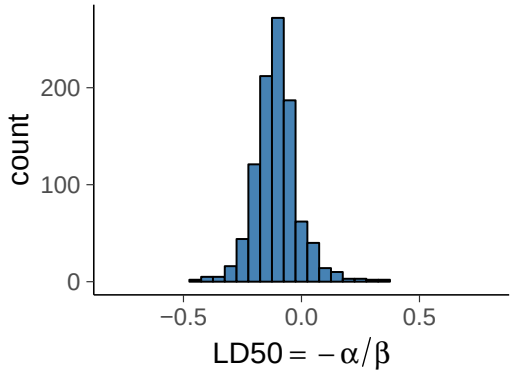
- But it can be convenient to obtain draws with equal weights
  - resample the draws according to the weights
  - some original draws may be included more than once
  - loses some information, but now the weights are equal

Example: Importance sampling in Bioassay

Grid







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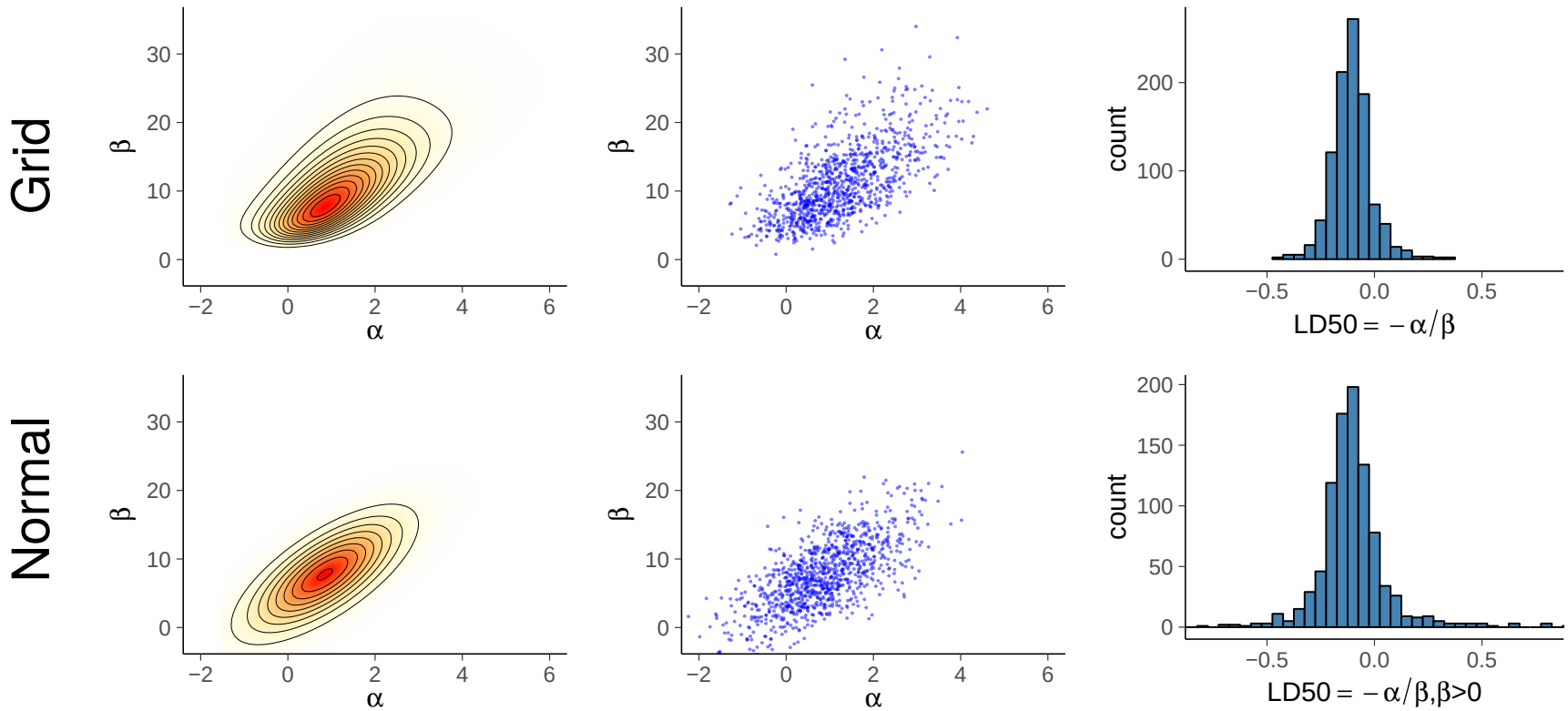

Normal approximation is discussed more in BDA3 Ch 4

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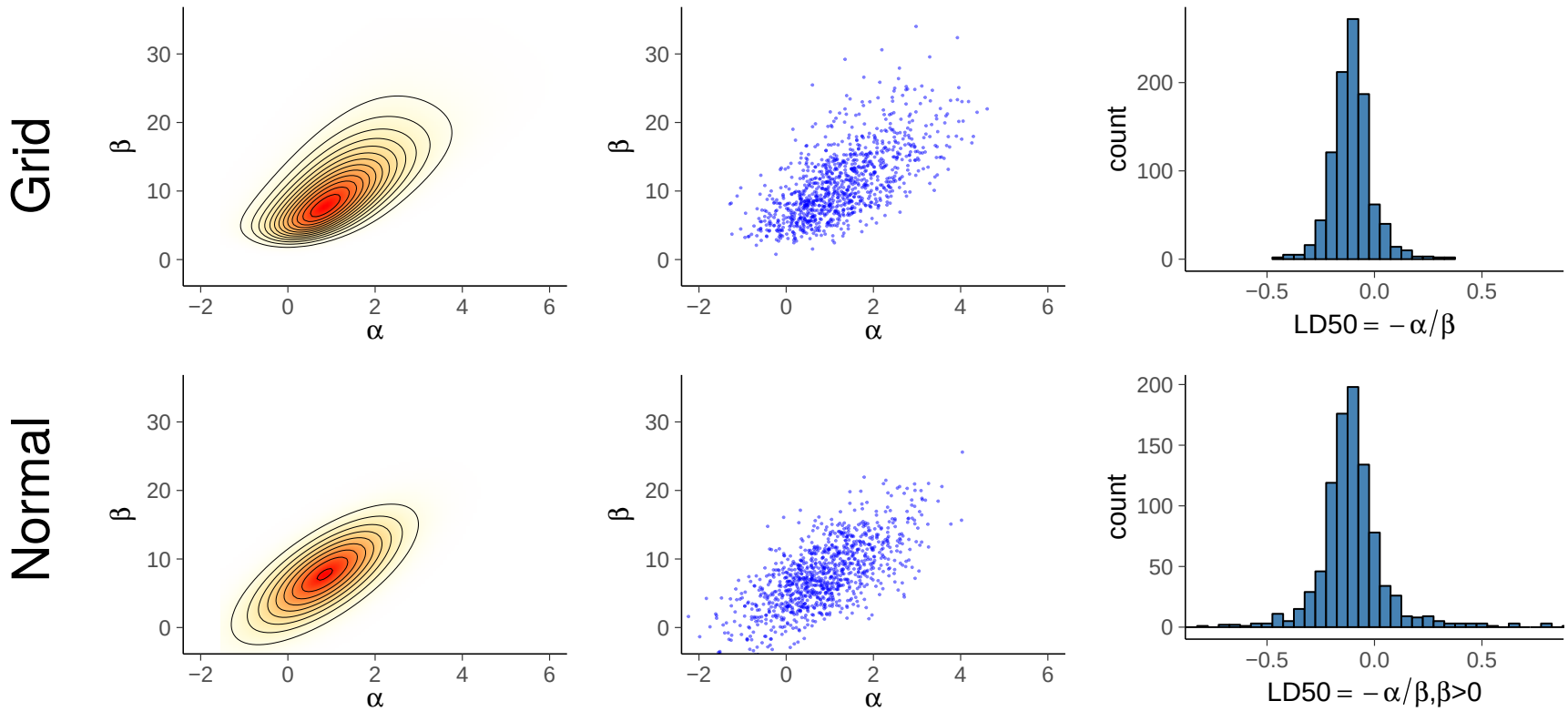

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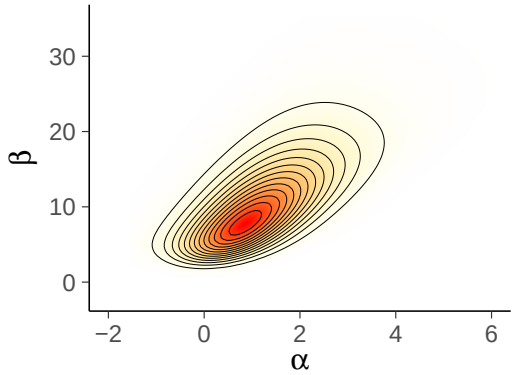
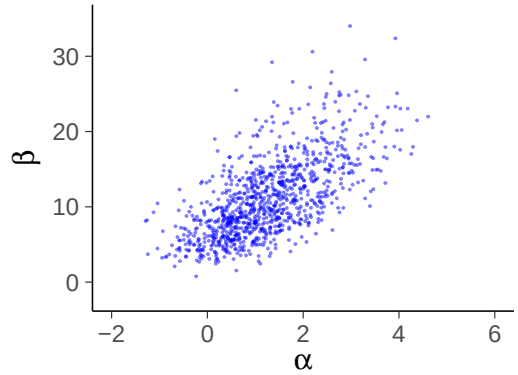
But the normal approximation is not that good here:

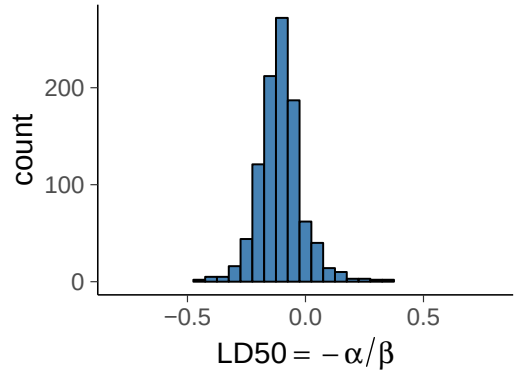
Grid sd(LD50) $\approx$ 0.1, Normal sd(LD50) $\approx$ .75!

Example: Importance sampling in Bioassay

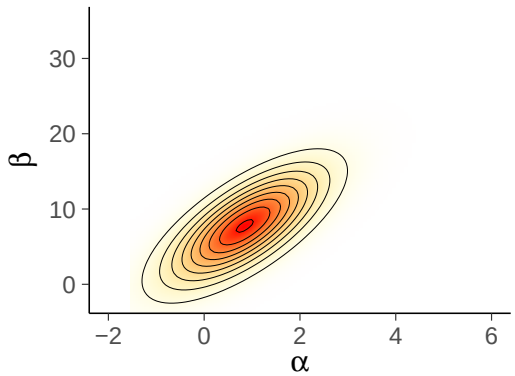
Grid

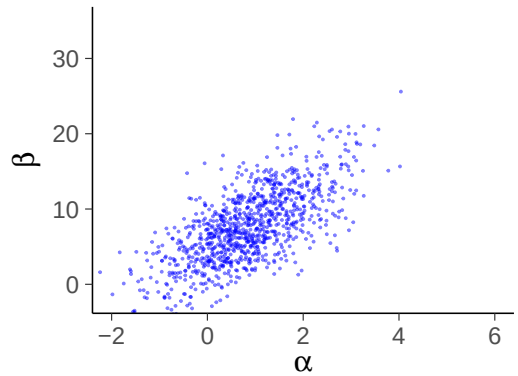


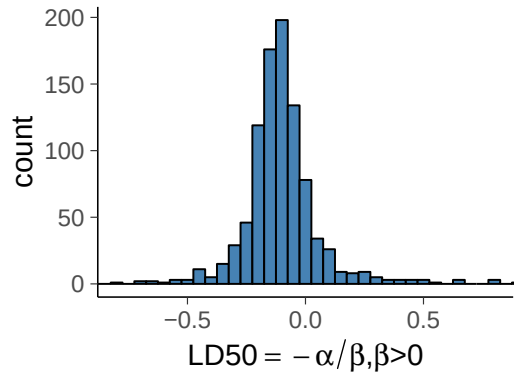




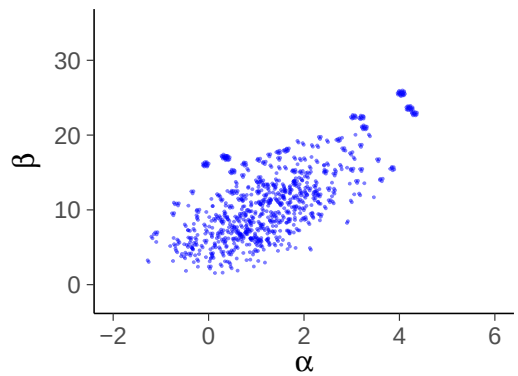
Normal

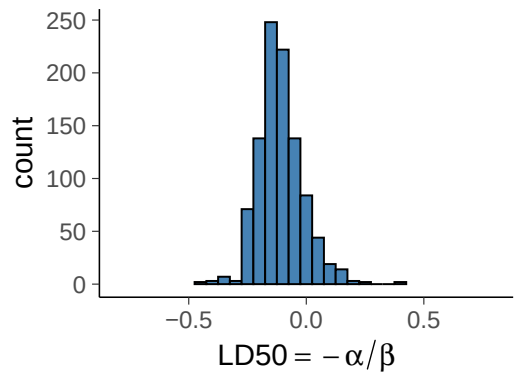






IR





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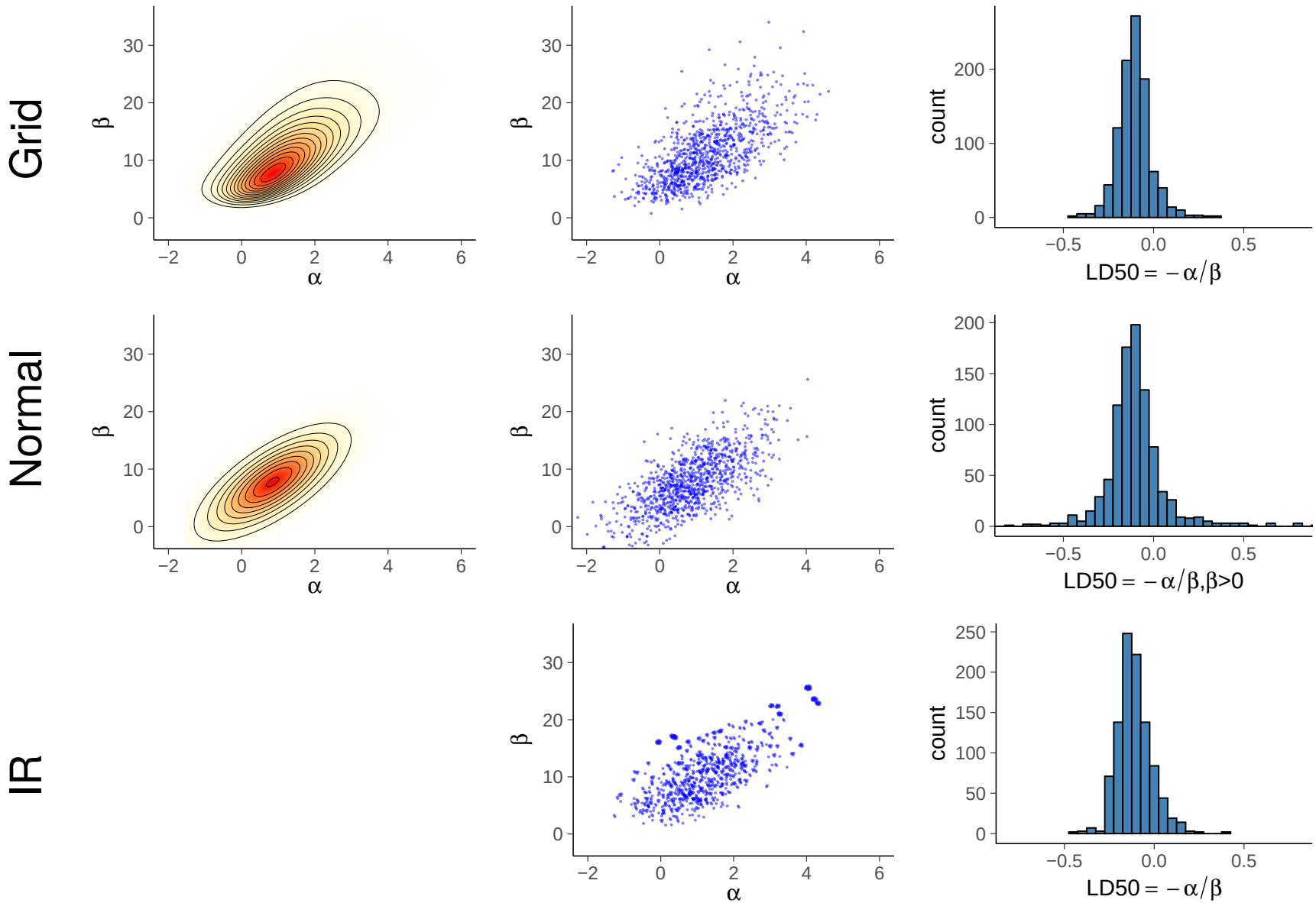

Grid sd(LD50) ≈ 0.1, IR sd(LD50) ≈ 0.1

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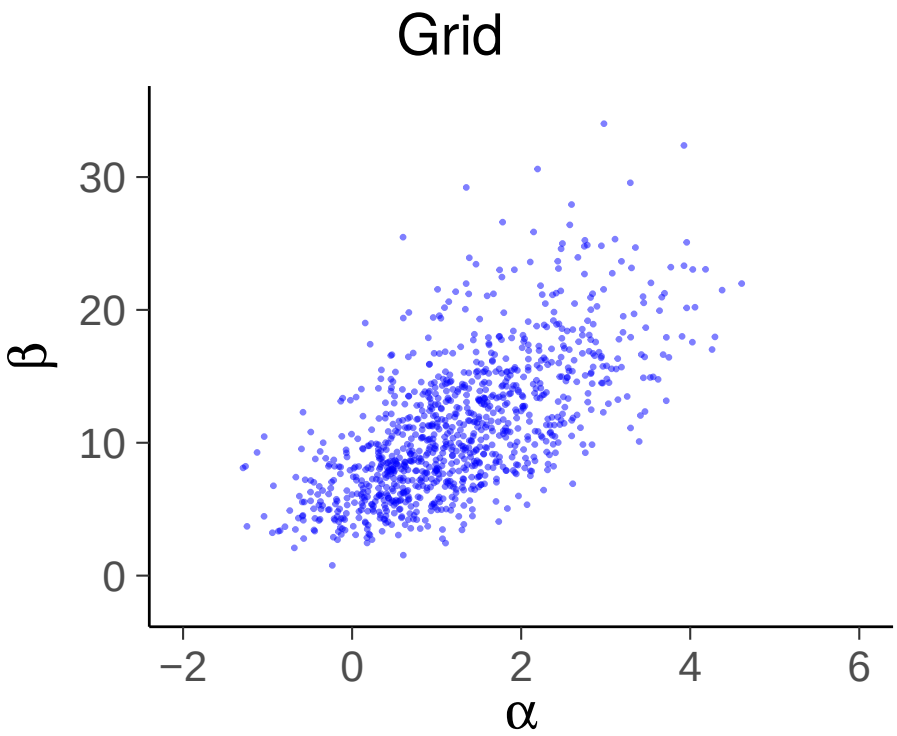

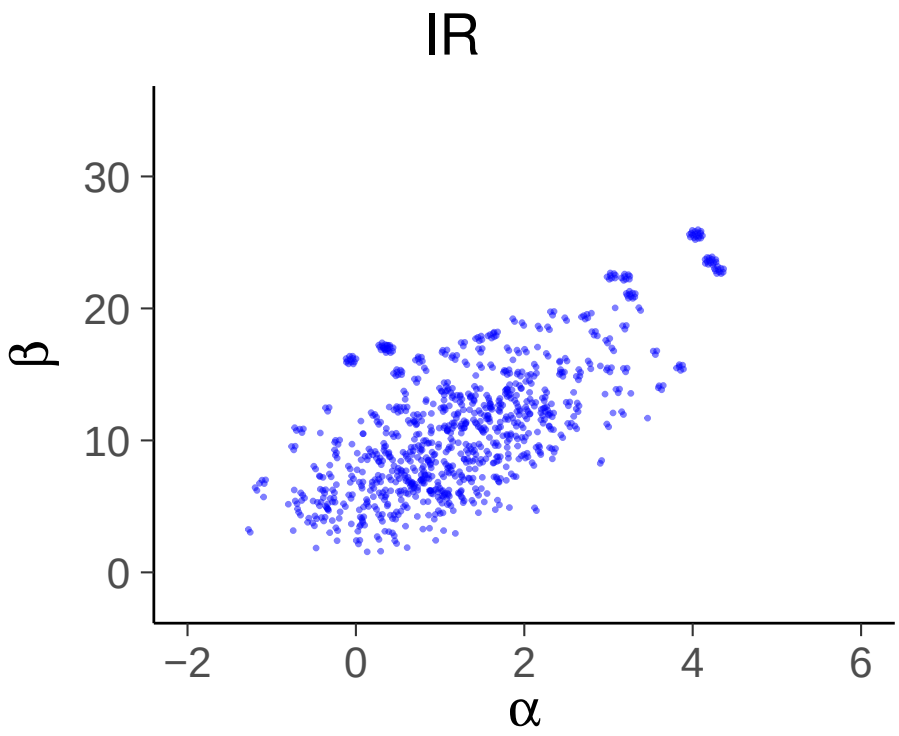

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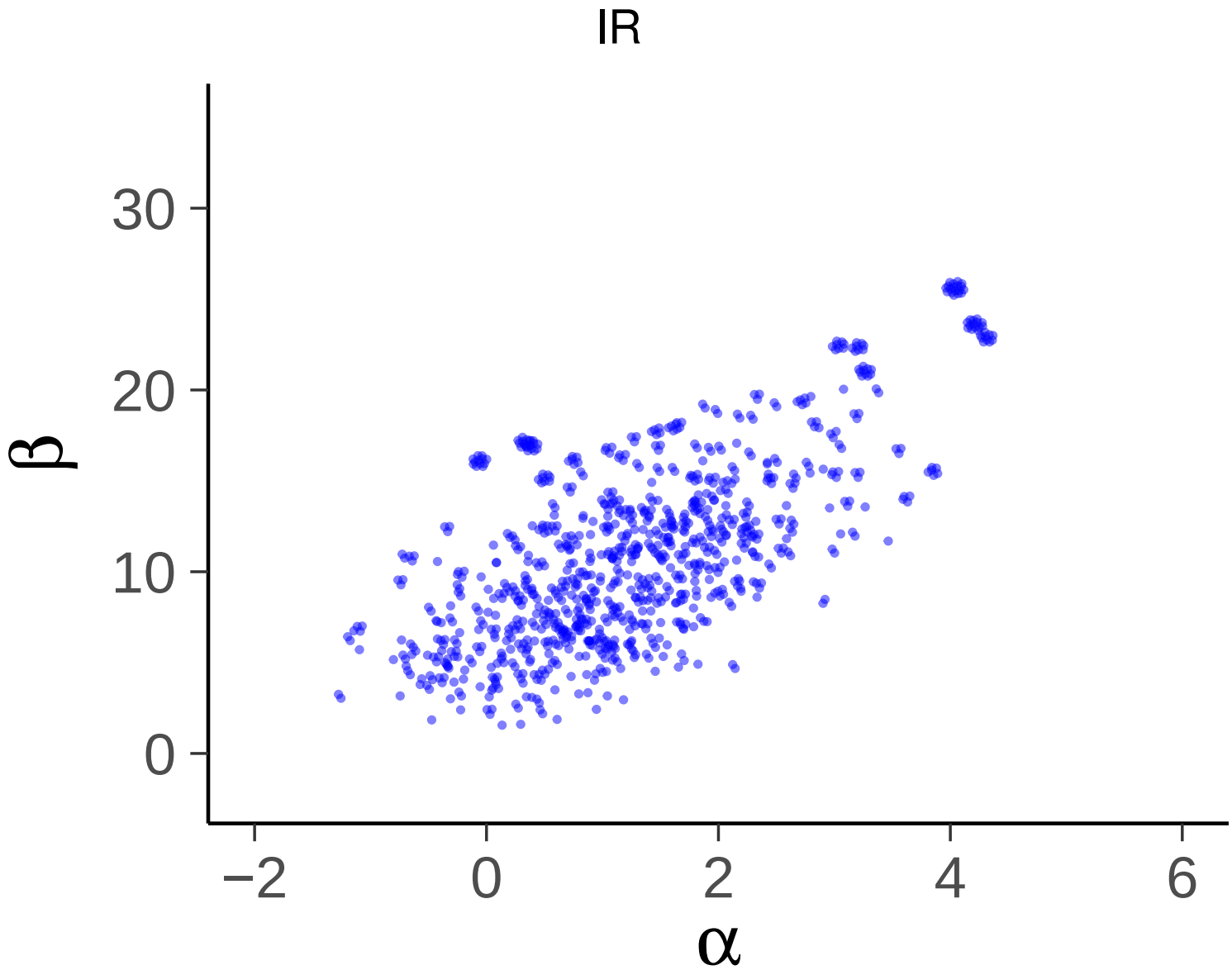

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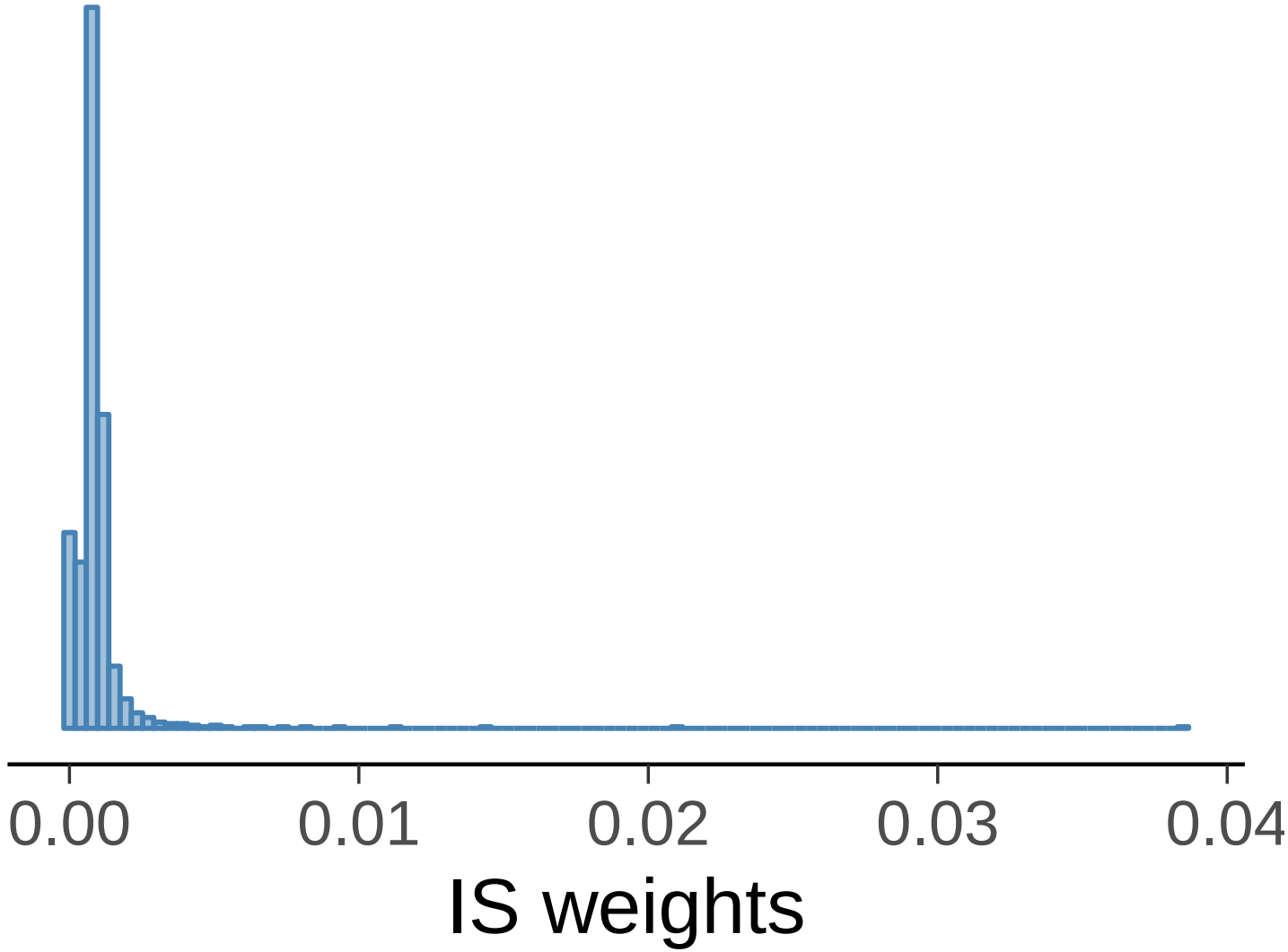

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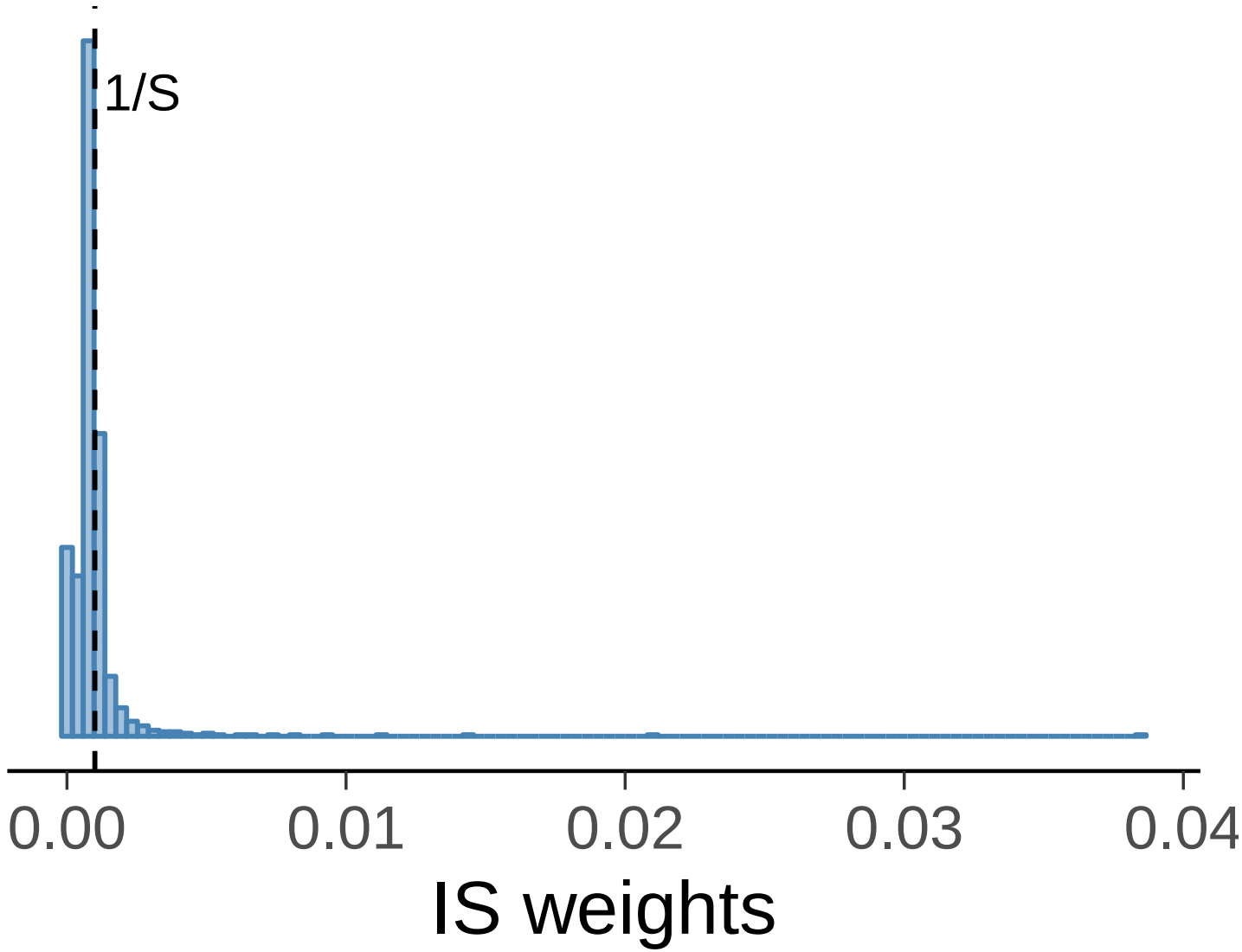

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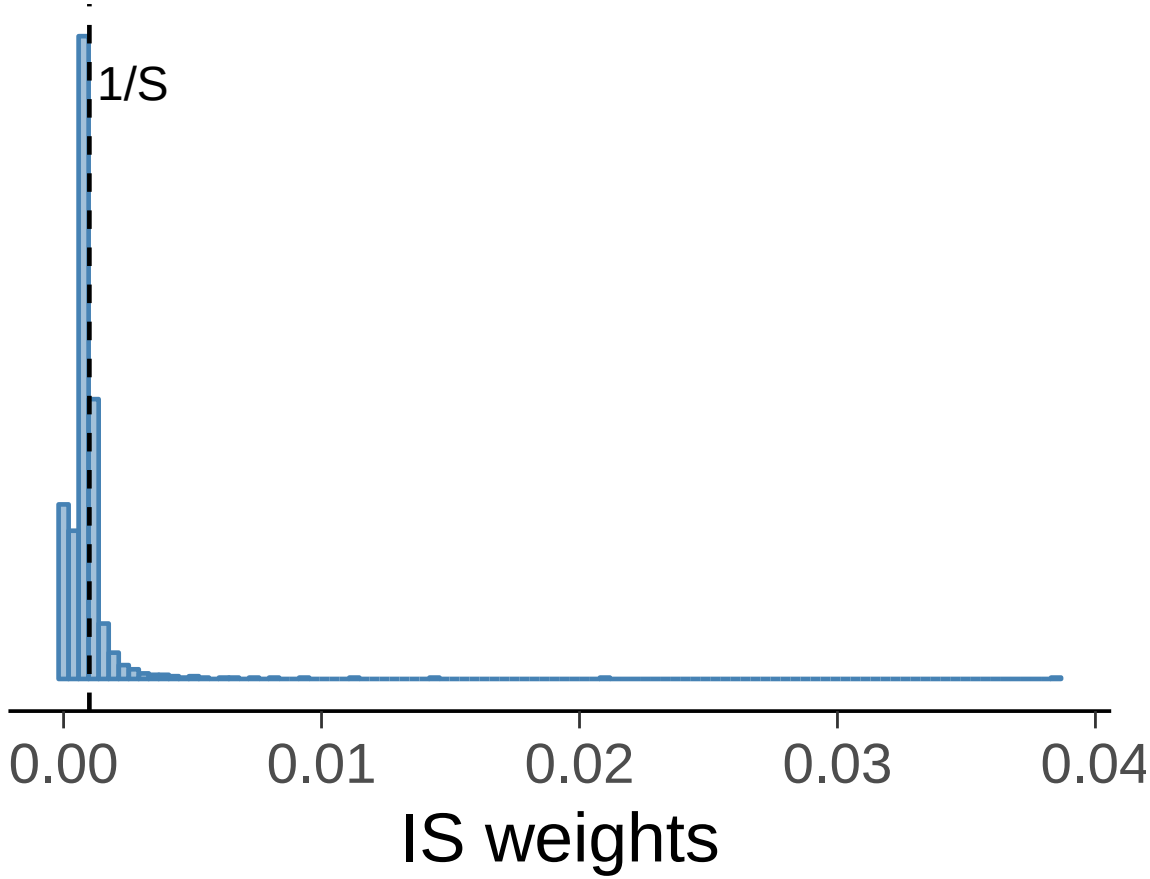

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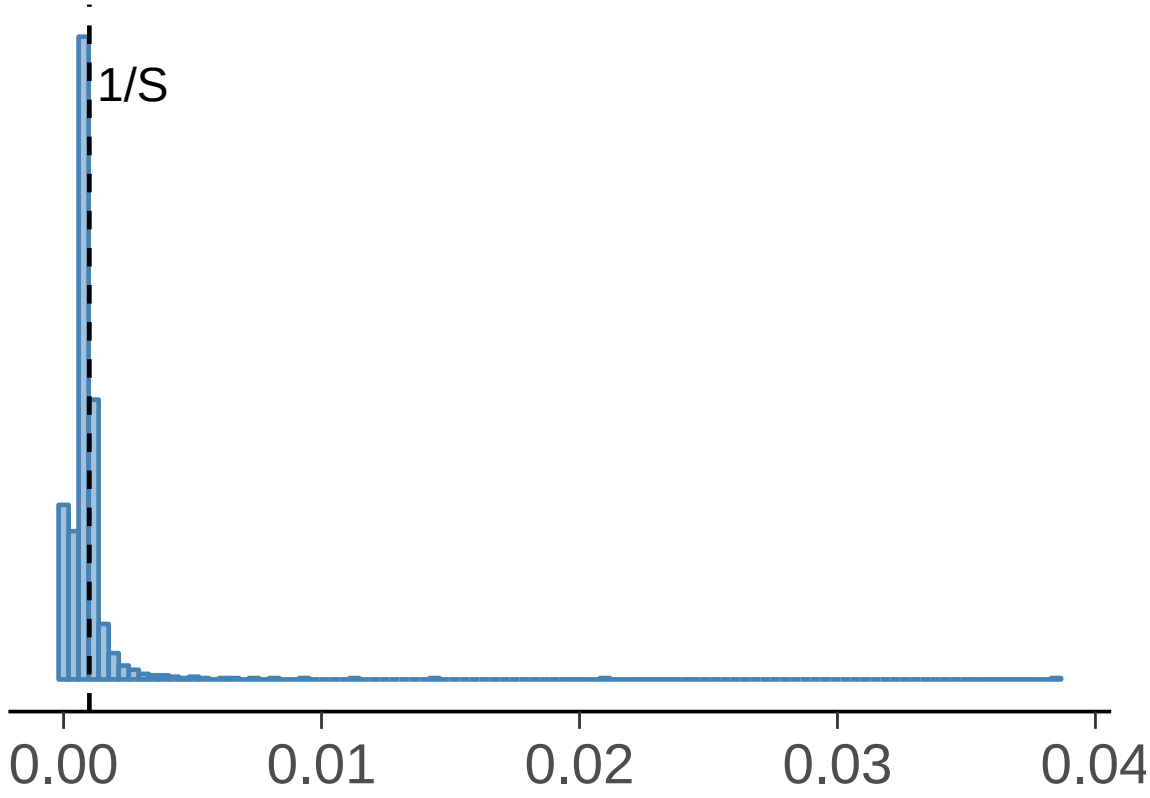

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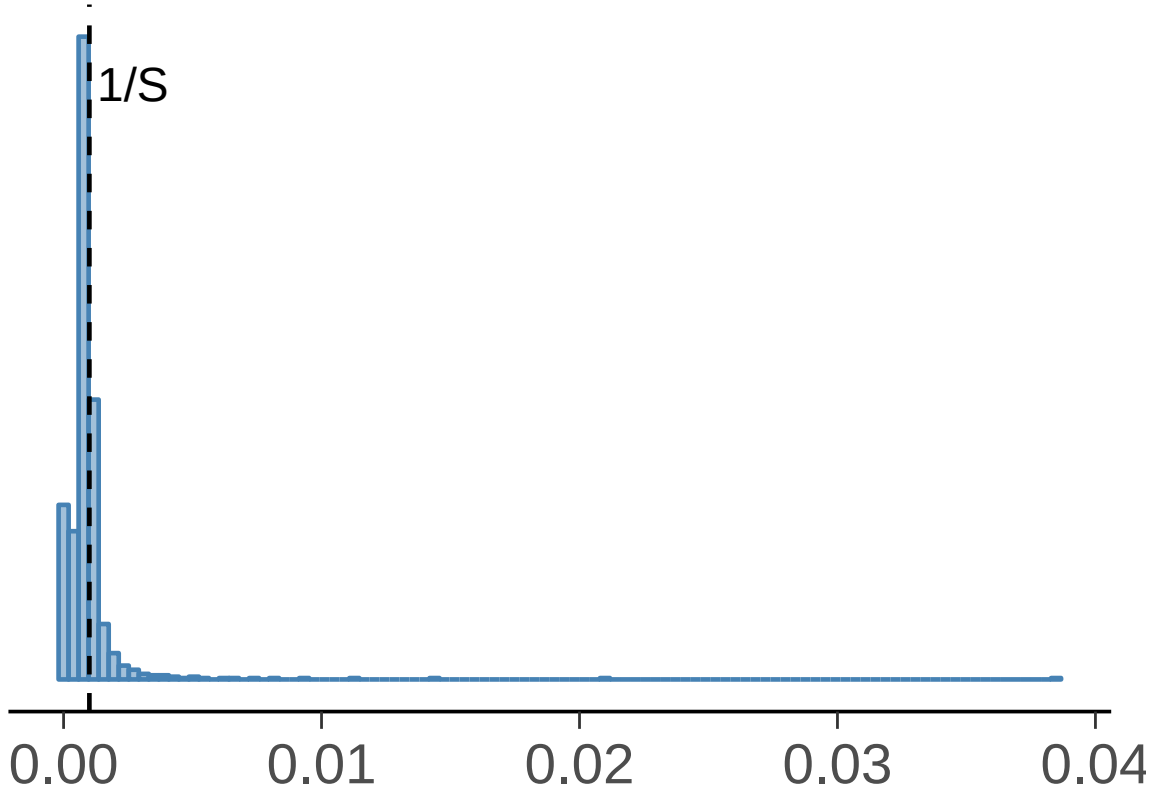
BDA3 1st (2013) and 2nd (2014) printing have an error for $\tilde{w}(\theta^s)$. The equation should not have the multiplier S (the normalized weights should sum to one). Online version is correct. Errata for the book http://www.stat.columbia.edu/~gelman/book/errata_bda3.txt

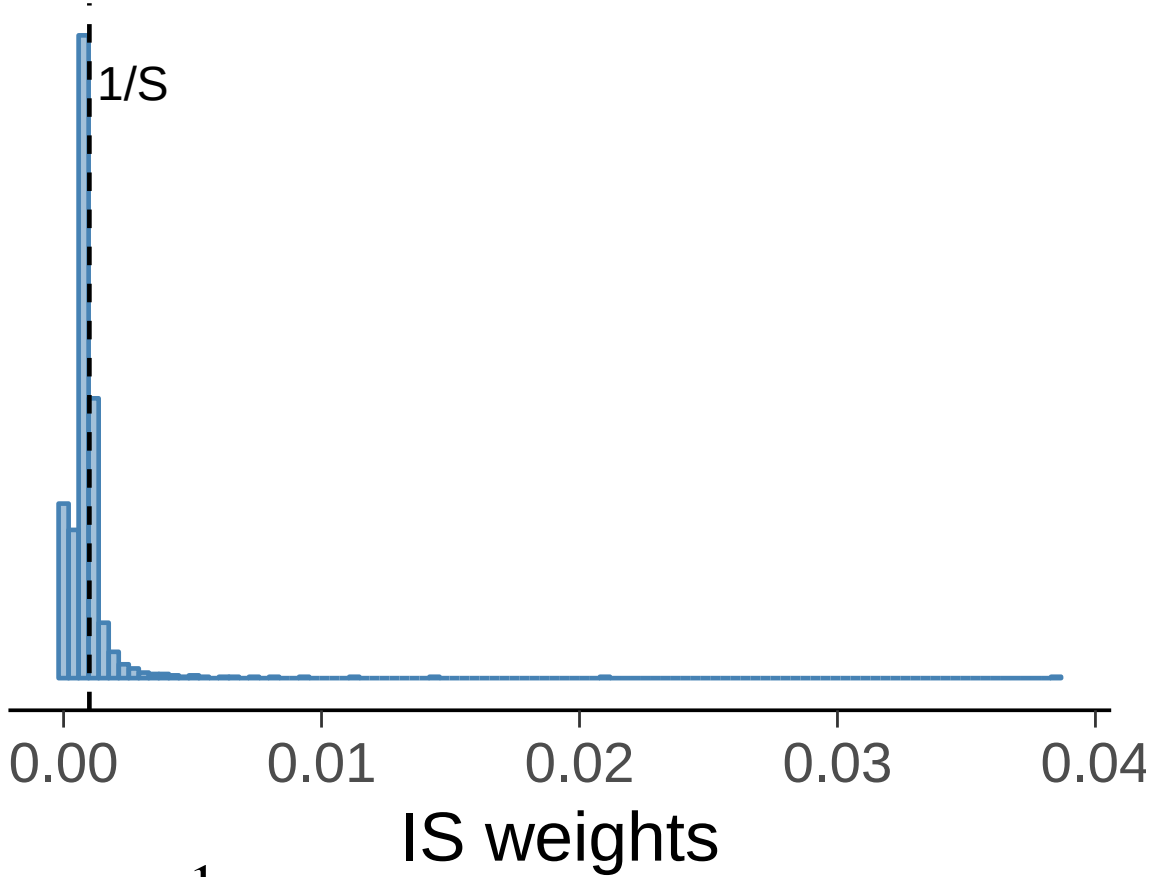
Example: Importance sampling in Bioassay



$$\text{ESS} = \frac{1}{\sum_{s=1}^{S} (\tilde{w}(\theta^s))^2}, \quad \text{where } \tilde{w}(\theta^s) = w(\theta^s) / \sum_{s'=1}^{S} w(\theta^{s'})$$

$$\text{ESS} \approx 396, \quad (\text{ESS} < S = 1000)$$

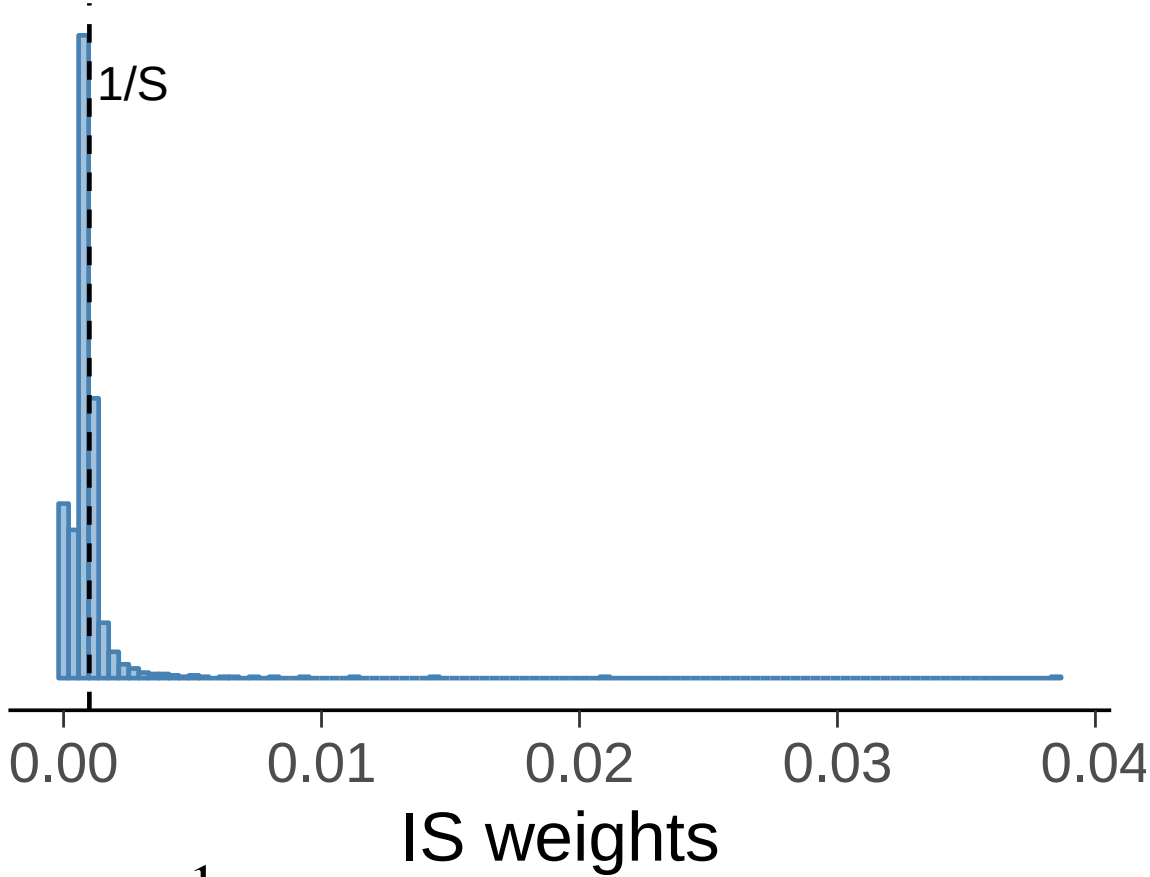
Example: Importance sampling in Bioassay



$$\text{ESS} = \frac{1}{\sum_{s=1}^{S} (\tilde{w}(\theta^s))^2},$$ is based on variance of $\tilde{w}(\theta^s)$

$\text{ESS} \approx 396$

Example: Importance sampling in Bioassay

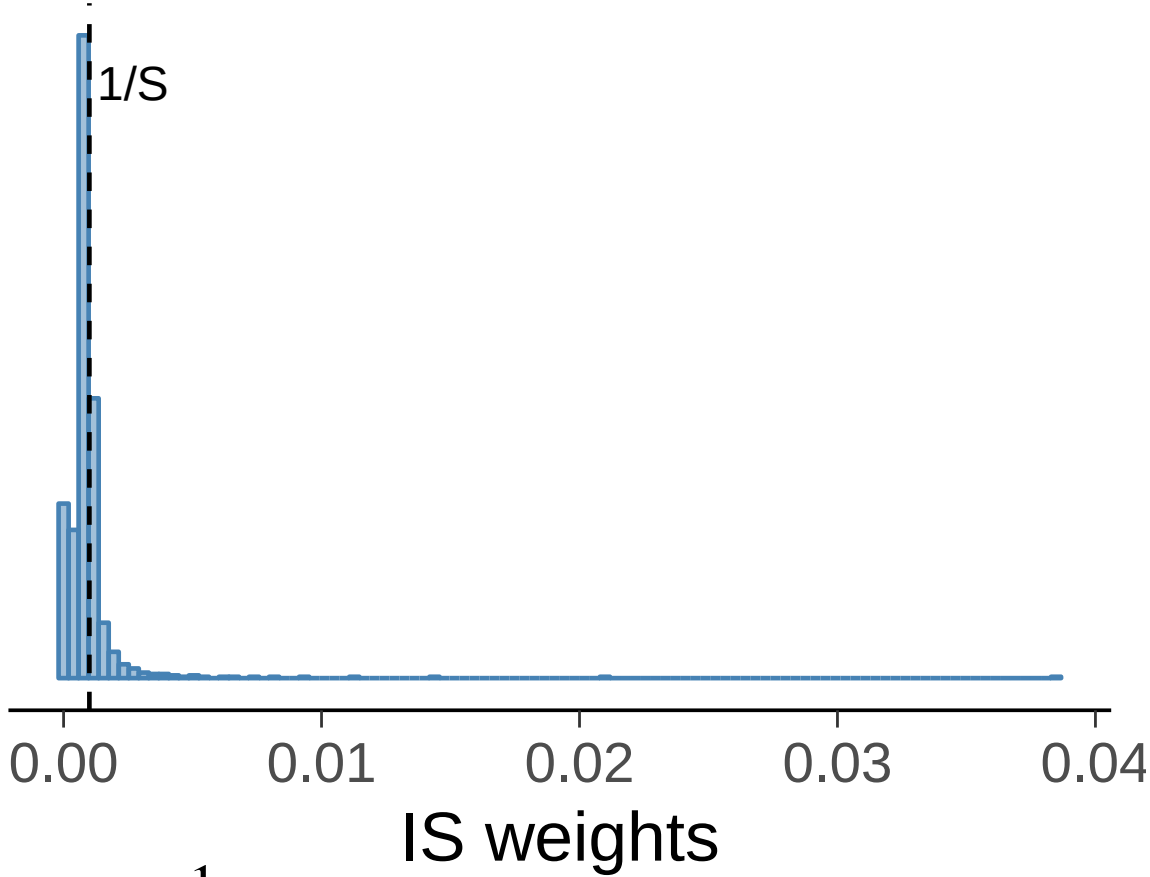


$$\text{ESS} = \frac{1}{\sum_{s=1}^{S} (\tilde{w}(\theta^s))^2}, \quad \text{is based on variance of } \tilde{w}(\theta^s)$$

$$\text{ESS} \approx 396$$

If all $\tilde{w}(\theta^s) = 1/S$, then $\text{ESS} = 1/(SS^{-2}) = S$

Example: Importance sampling in Bioassay



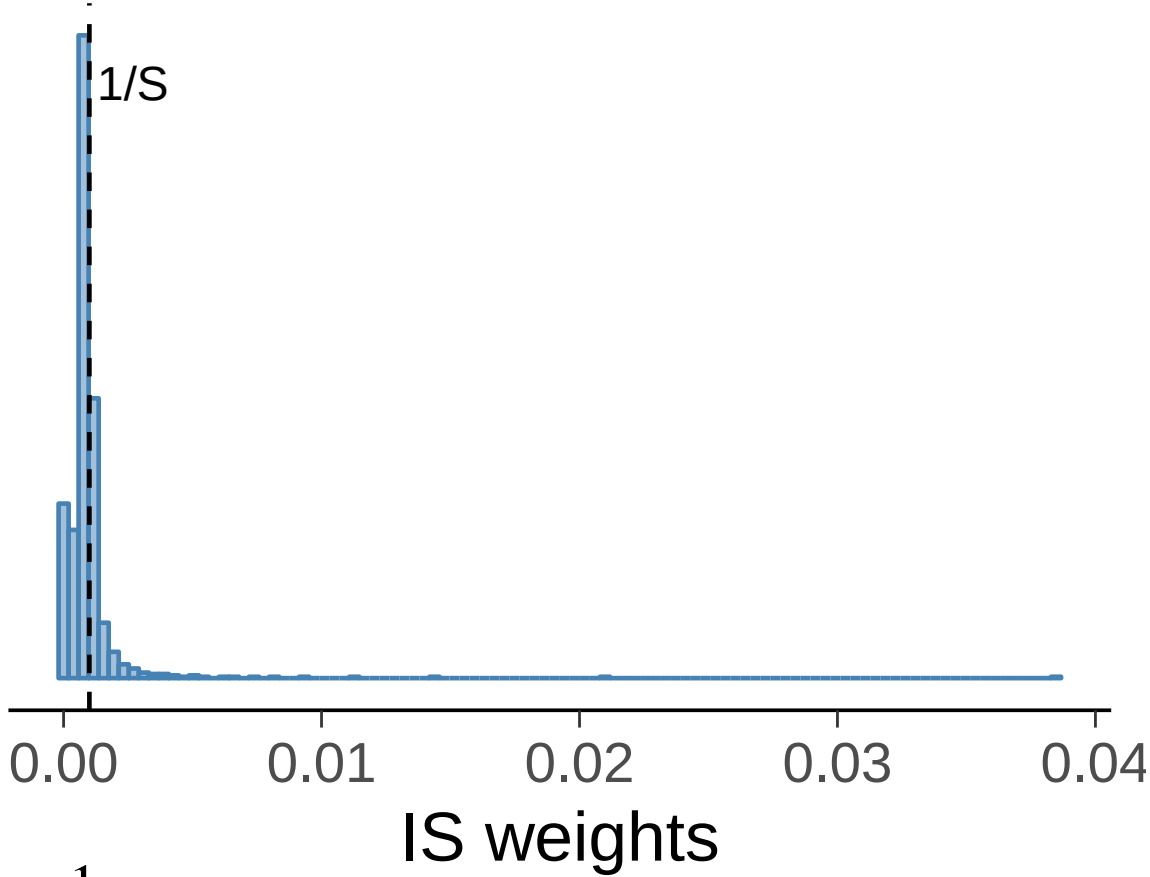
$$\text{ESS} = \frac{1}{\sum_{s=1}^{S} (\tilde{w}(\theta^s))^2}, \quad \text{is based on variance of } \tilde{w}(\theta^s)$$

$$\text{ESS} \approx 396$$

If all $\tilde{w}(\theta^s) = 1/S$, then ESS $= 1/(SS^{-2}) = S$

If one $\tilde{w}(\theta^s) = 1$, and others 0, then ESS $= 1/1 = 1$

Example: Importance sampling in Bioassay

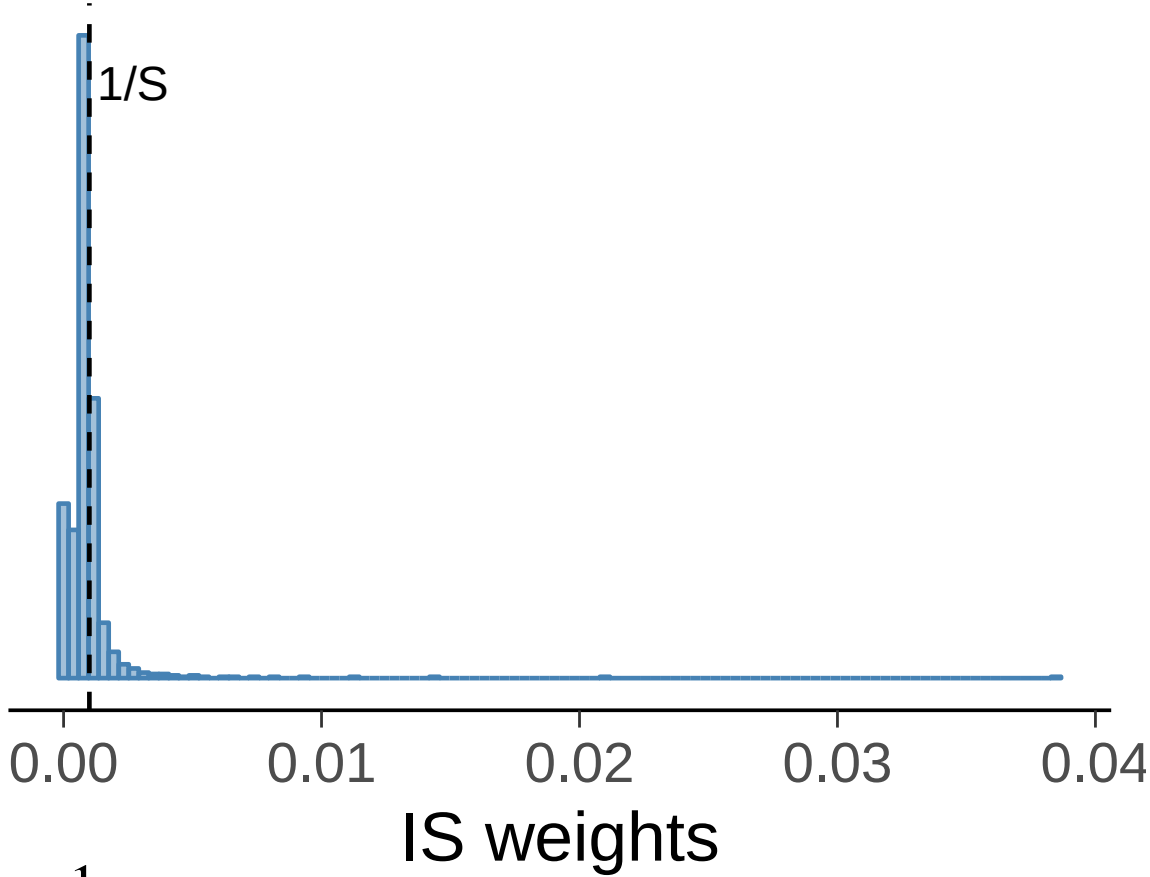


$$\text{ESS} = \frac{1}{\sum_{s=1}^{S} (\tilde{w}(\theta^s))^2},$$ is based on variance of $\tilde{w}(\theta^s)$

$\text{ESS} \approx 396$

Pareto-$\hat{k} \approx 0.65$, CLT does not hold

Example: Importance sampling in Bioassay



ESS $= \dfrac{1}{\sum_{s=1}^{S}(\tilde{w}(\theta^s))^2}$, is based on variance of $\tilde{w}(\theta^s)$

ESS ≈ 396

Pareto-$\hat{k} \approx 0.65$, CLT does not hold

with Pareto-smoothing the estimate would be fine if $\hat{k} < 0.7$

Importance sampling leave-one-out cross-validation

- Later in the course you will learn how $p(\theta|y)$ can be used as a proposal distribution for $p(\theta|y_{-i})$
  - which allows fast computation of leave-one-out cross-validation

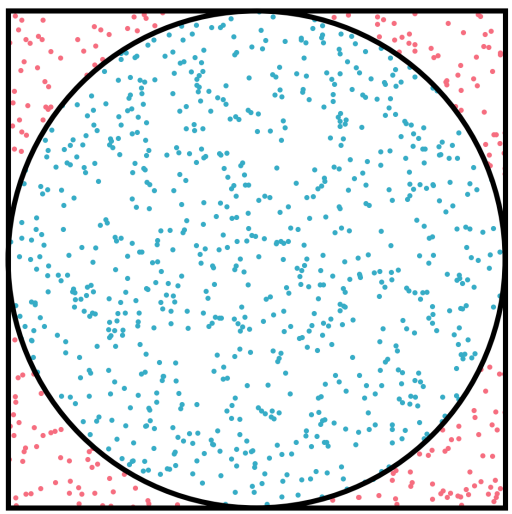
$$p(y_i|y_{-i}) = \int p(y_i|\theta)p(\theta|y_{-i})d\theta$$

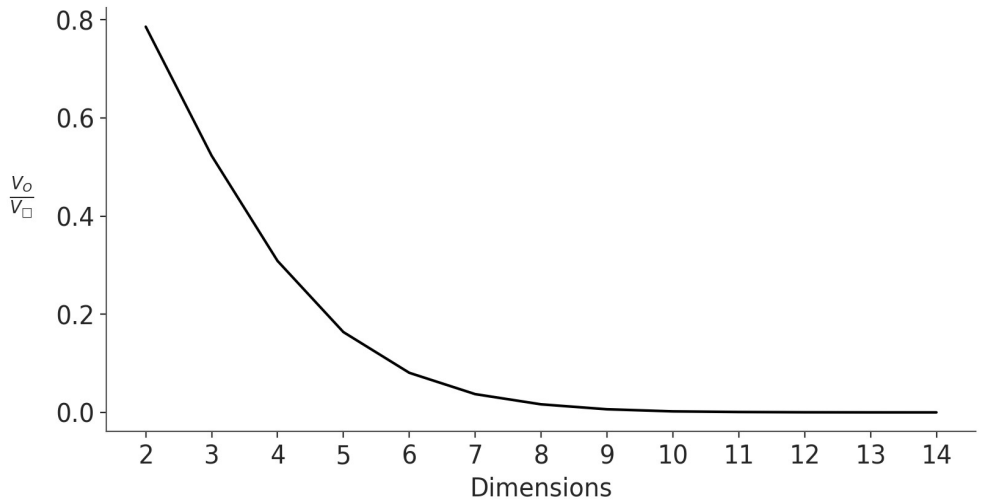
Pareto-$\hat{k}$ diagnostic use cases

- Importance sampling
  - leave-one-out cross-validation (Vehtari et al., 2016, 2017; Bürkner at al, 2020)
  - Bayesian stacking (Yao et al., 2018, 2021, 2022)
  - leave-future-out cross-validation (Bürkner et al., 2020)
  - Bayesian bootstrap (Paananen et al, 2021, online appendix)
  - prior and likelihood sensitivity analysis (Kallioinen et al., 2021)
  - improving distributional approximations (Yao et al., 2018; Zhang et al., 2021; Dhaka et al., 2021)
  - implicitly adaptive importance sampling (Paananen et al., 2021)
- Stochastic optimization (Dhaka et al., 2020)
- Divergences and gradients in VI (Dhaka et al., 2021)
- MCMC (Paananen et al., 2021)

Curse of dimensionality

- Number of grid points increases exponentially
- Concentration of the measure, that is, where is the most of the mass?





Markov chain Monte Carlo (MCMC)

- Pros
  - Markov chain goes where most of the posterior mass is
  - Certain MCMC methods scale well to high dimensions
- Cons
  - Draws are dependent (affects how many draws are needed)
  - Convergence in practical time is not guaranteed
- MCMC methods in this course
  - Gibbs: “iterative conditional sampling”
  - Metropolis: “random walk in joint distribution”
  - Dynamic Hamiltonian Monte Carlo: “state-of-the-art” used in Stan