Lesson 8 Red-Black Trees

Wholeness of the Lesson

Red-black trees provide a solution to the problem of unacceptably slow worst case performance of binary search trees. This is accomplished by introducing a new element: nodes of the tree are colored red or black, adhering to the balance condition for red-black trees. The balance condition is maintained during insertions and deletions and doing so introduces only slight overhead.

Science of Consciousness: Red-black trees, as an example of BSTs with a balance condition, exhibit the Principle of the Second Element for solving the problem of skewed BSTs.

The Goal

- Improve worst-case performance of BSTs by introducing a balance condition that
  - can be maintained after each insertion and deletion
  - guarantees the height of the tree is always O(log n), where n is the number of nodes in the tree.
- Achieving this objective implies that all insertions, deletions and searches have a worst-case running time of O(log n).

Overview of the Lesson

- Specify the balance condition that defines a red-black tree.
- Verify that the height of any red-black tree is $O(\log n)$.
- Work through the insertion algorithm for a red-black tree.

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Tree Terminology And Conventions

1. We always think of our trees as terminating with a NULL reference. This convention will be important when we compute the height of red-black trees. Sometimes the NULL nodes are not shown.
2. External nodes are NULL references; internal nodes are non-NULL nodes; if we refer to a “node” without specifying external or internal, we mean “internal node”.
3. In the present context, a “leaf node” is an internal node that has only NULL children.

4. The height of an empty tree is -1.
5. A path from a node x_0 to a node x_k is a sequence x_0; x_1; … ; x_k so that x_{i+1} is a child of x_i (for i = 0; 1; 2; … ; k - 1). The length of such a path is the number of predecessors of the last element x_k (so in this case, = k).
6. The height of a node x is the length of the longest path from x to a leaf node. (Note: the height of NULL is -1.)

Definition of Red-Black Trees

A BST is red-black if it has the following 4 properties:

A. Every node is colored either red or black

B. The root is colored black.

C. If a node is red, its children are black.

D. For each node x, every path from x to a NULL reference has the same number of black nodes. This number is called the black height of x and is denoted bh(x).

Note: The black height of NULL is 0.

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Height of a Red-Black Tree

- Idea: Note that if we have a binary tree with $n \geq 2^h$, we can take logs on both sides to obtain $h \leq \log(n)$. The proof below shows that replacing h with bh(r) in this calculation is good enough to show h is O(log n).

- Suppose T is red-black and has n nodes, height h, and root r.

1. Fact 1: $bh(r) \geq h/2$
2. Fact 2: $n \geq 2^{bh(r)} - 1$
3. Theorem: $h \leq 2\log(n+1)$

Proof of Theorem:

$$
\begin{aligned}
n &\geq 2^{bh(r)} - 1 \\
&\geq 2^{h/2} - 1 \\
n + 1 &\geq 2^{h/2} \\
\log(n+1) &\geq h/2 \\
2\log(n+1) &\geq h
\end{aligned}
$$

Height of a Red-Black Tree: Fact 1

Lemma 1. For any red-black tree having height h,

$$\mathrm{bh}(r) \geq \frac{h}{2}$$

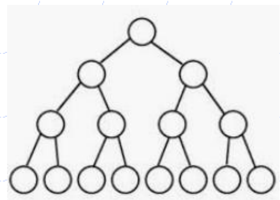
where r is the root.

Intuition. Though the black height of a red-black tree is typically less than the height of the tree, it is not "too much" smaller — it's a reasonable approximation.

Proof. If the tree is null, this is obvious. Otherwise, follow any path from the root to a deepest leaf. The root is black, and one (or more) of the next two nodes is black, then one (or more) of the two nodes after that are black, and so forth. Thus, at least half the nodes along such a path are black. ■

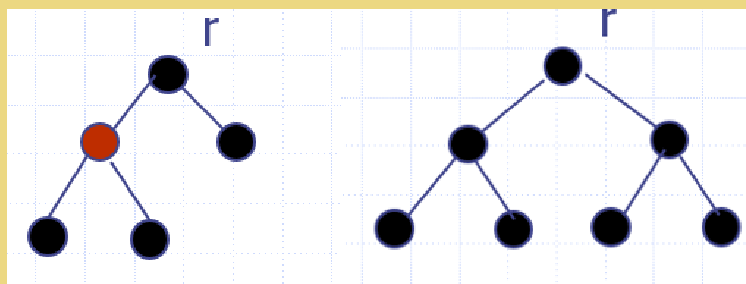
Fact 2: $n \geq 2^{bh(r)} - 1$

We can see why Fact 2 holds in the following way: In a binary tree in which every level is filled



we have $n \geq 2^h$; this continues to be true if we color every node in this tree black (as in the second example on right) and in this case bh(r) is approximately equal to h.

In the worst case though, approximately half of the nodes are colored red. Imagine moving black nodes up in a red-black tree to replace all the red nodes. This would reduce the height to approximately h/2, so we would have $n \geq 2^{h/2}$. Since bh(r) is approximately h/2, one can show that $n \geq 2^{bh(r)} - 1$.



$n = 5$	$n = 7$
$bh(r) = 2$	$bh(r) = 3$
$h = 2$	$h = 2$
$5 \geq 2^2 - 1$	$7 \geq 2^3 - 1$

Two Examples

Height of a Red-Black Tree: Fact 2 Formal Proof

Lemma 2. Suppose x is any node in T (internal or external). Let $n(T_x)$ denote the number of *internal* nodes in the subtree of T at x. Then

$$n(T_x) \geq 2^{\mathrm{bh}(x)} - 1.$$

Intuition. In a *completely filled* binary tree, we know that $n = 2^{h+1} - 1$. The idea is that when a tree is balanced, it must branch a lot and its number of nodes must be exponentially bigger than its height. Since black height is an approximation to height, therefore, the lemma says that "since red-black trees are highly balanced, the number of nodes is always exponentially larger than the height."

Proof. Proceed by induction on the height $h(x)$ of x in T.

- If $h(x) = -1$, then x must be null. It follows $\mathrm{bh}(x) = 0 = n(T_x)$ and the result follows.
- If $h(x) = 0$, then x is a leaf node, so $\mathrm{bh}(x) = 0$ or 1. Again

$$2^{\mathrm{bh}(x)} - 1 \leq 1 = n(T_x)$$

- For the induction step, assume the result holds true for heights $< h$, where $h \geq 1$. Let x_L and x_R be left and right children of x.

  Claim. $h(x_L) < h(x)$ and $h(x_R) < h(x)$

  Proof of Claim. We show it for x_L; the other case is similar. If x_L is null, then $h(x_L) = -1 < 1 \leq h(x)$. If x_L is not null, then by definition of height, $h(x_L) = h(x) - 1 \leq h(x)$.

Continuation of proof of theorem. By the claim, the induction hypothesis can be applied to T_{x_L} and T_{x_R}:

$$(*)\qquad \begin{aligned} n(T_{x_L}) &\geq 2^{\mathrm{bh}(x_L)} - 1 \\ n(T_{x_R}) &\geq 2^{\mathrm{bh}(x_R)} - 1 \end{aligned}$$

Notice also that if x is red, $\mathrm{bh}(x_L) = \mathrm{bh}(x)$ (likewise for x_R), whereas if x is black, $\mathrm{bh}(x_L) = \mathrm{bh}(x) - 1$ (likewise for x_R). Therefore

$$(**)\qquad \begin{aligned} \mathrm{bh}(x_L) &\geq \mathrm{bh}(x) - 1 \\ \mathrm{bh}(x_R) &\geq \mathrm{bh}(x) - 1 \end{aligned}$$

Putting $(*)$ and $(**)$ together, we have

$$\begin{aligned} n(T_x) &= n(T_{x_L}) + n(T_{x_R}) + 1 \\ &\geq 2^{\mathrm{bh}(x_L)} - 1 + 2^{\mathrm{bh}(x_R)} - 1 + 1 \quad (\text{by } (*)) \\ &\geq 2^{\mathrm{bh}(x)-1} - 1 + 2^{\mathrm{bh}(x)-1} - 1 + 1 \quad (\text{by } (**)) \\ &\geq 2 \cdot 2^{\mathrm{bh}(x)-1} - 1 \\ &= 2^{\mathrm{bh}(x)} - 1. \end{aligned}$$

Main Point

Because of their balance condition, red-black trees always have height O(log n), so their primary operations all have running times that are O(log n). Techniques for maintaining the red-black properties after insertions and deletions are techniques that maintain balance in the midst of change.

Science of Consciousness: “Far away indeed from the balanced intellect is action devoid of greatness.” (Gita, II.49) The “balanced intellect” is a state of life in perfect balance in which each area of life from most expressed to most subtle and refined is spontaneously given due attention. Such a life is filled with great accomplishment and success. The technique to maintain balance is regular contact with the Self, the field of pure consciousness; having contacted this field, awareness is able to maintain a profound state of balance even in the context of active life.

Overview of the Lesson

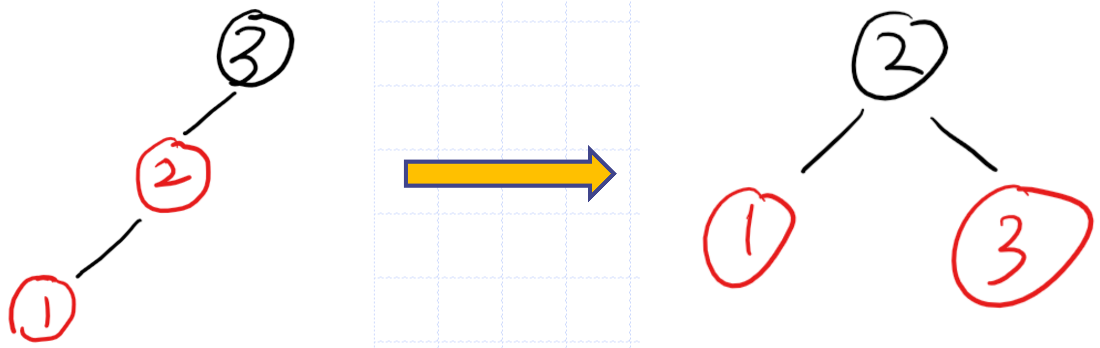
- Specify the balance condition that defines a red-black tree.
- Verify that the height of any red-black tree is O(log n).
- Work through the insertion algorithm for a red-black tree.

Preparing to Insert into a Red Black Tree: *Rotations*

- Introducing the primary re-balancing technique: *rotations*
  - Used to keep a red-black tree balanced
  - It has the effect of raising the height of some nodes and lowering others, but preserving the BST property

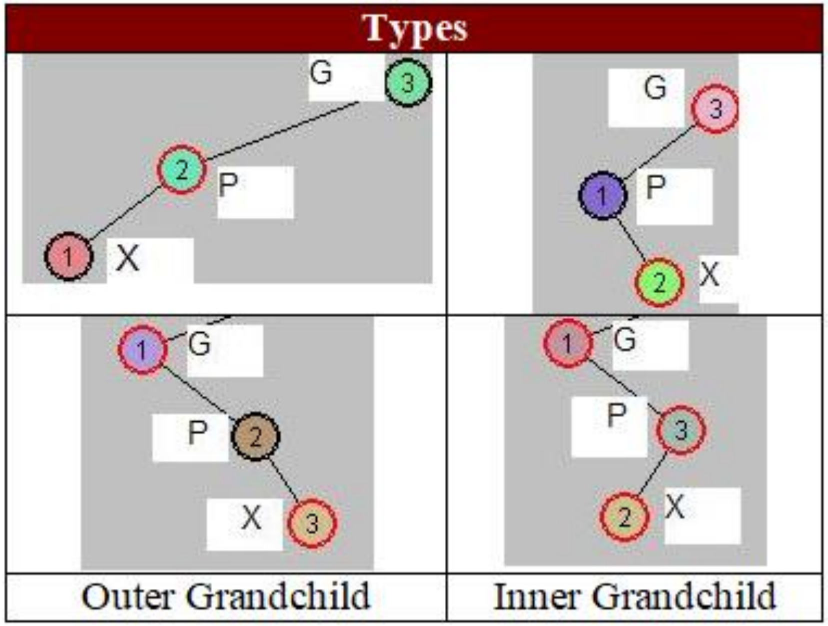
Examples of Rotations

a. Perform a right rotation of tree obtained from 3,2,1 insertion. At first, the node 3 is the *top*. After rotation, tree is balanced and still a BST.



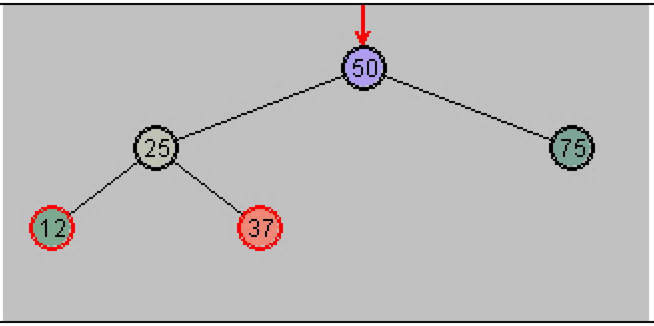
Examples of Rotations

b. *Rotations involving a grandchild.* Often need to move a node X and the parent P and grandparent G need to be examined. Relative to the grandparent, X can be an *outer grandchild* or an *inner grandchild.*

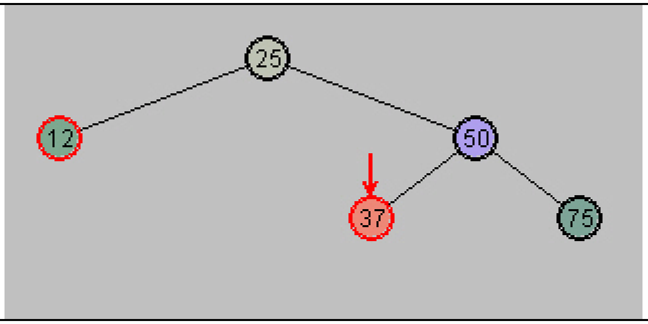


Examples of Rotations

c) In complicated rotations (for instance, in a right rotation where the child being lifted already has a right child), there may be a *crossover node.*



Perform a right rotation of 25, 50, with top 50



37 is a *crossover node*

Algorithm: **Top-Down Insertion in a Red-Black Tree**

Input: a Red-Black Tree t and a value x

Output: t with x inserted

if t is empty then create a black node with value x and return it

perform search (BST style) on t to locate proper location of x, perform color flips as needed along the way

if color flip introduces a red-red violation, use rebalancing technique to fix it

place at the correct insertion point in t a red node with value x

if insertion causes red-red violation, again use rebalancing technique to fix it

<u>Case 1:</u> Outer grandchild causes a red-red violation.

- Change color of G
- Change color of P
- Rotate P, G in the direction that lifts X

<u>Case 2:</u> Inner grandchild causes a red-red violation.

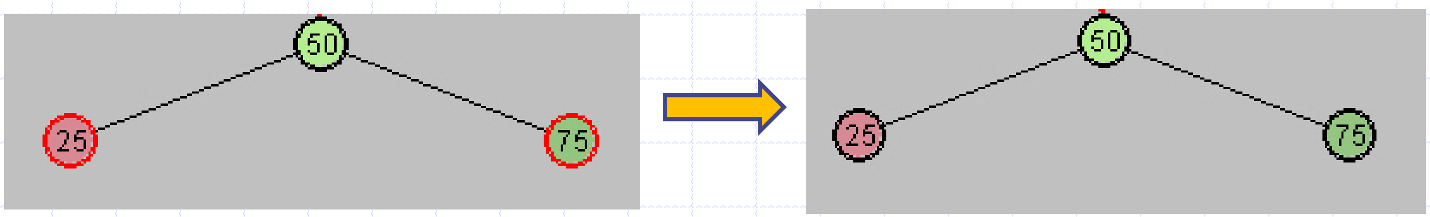
- Change color of G
- Change color of X
- Perform double rotation:
  - P, X, lifting X
  - G, X, lifting X

Rebalancing technique

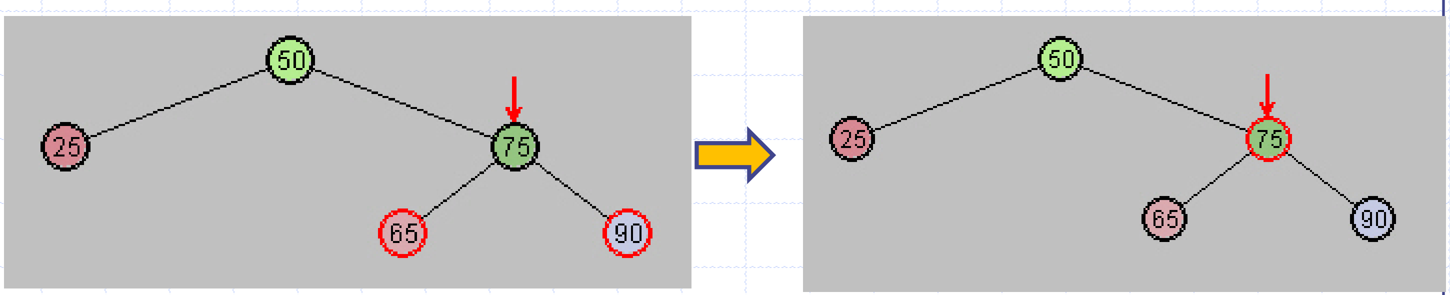
Color Flips On The Way Down

Color Flip Strategy: During the search for the insertion point, when a black node having two red children is encountered, a color flip is done. A color flip changes colors of all three nodes unless the top node is the root, in which case only the children's nodes change color.

top is root



top is not root



Color Flips On The Way Down

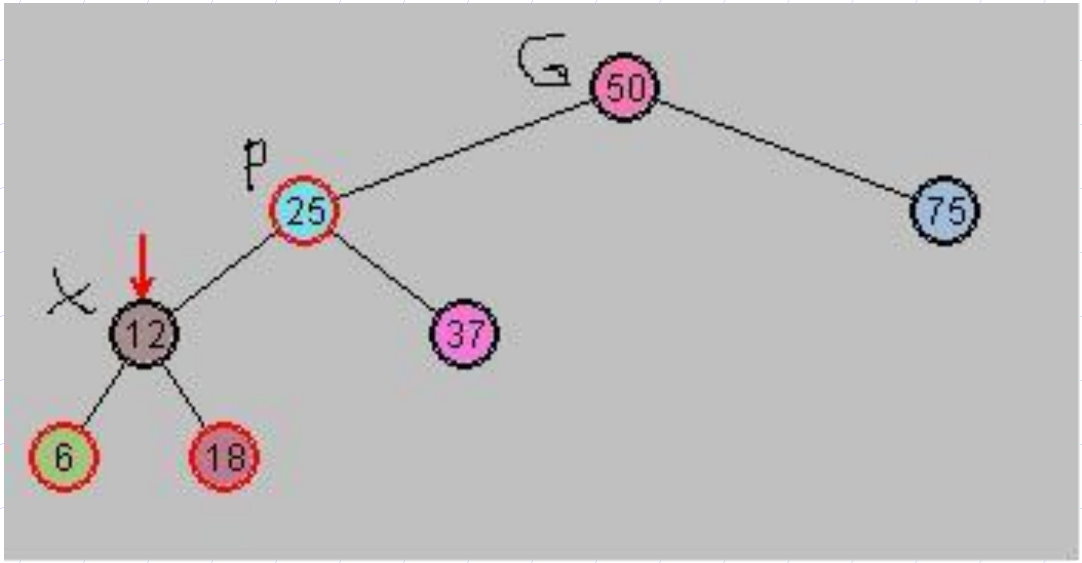
- A color flip can introduce a *red-red violation*, which must be corrected to maintain red-black property. Before moving further to the insertion point, the violation must be corrected, using one or two *rotations*.

Rotations On The Way Down

- Case 1: Outer grandchild causes a red-red violation after color flip. The Rule:
  - Change color of G
  - Change color of P
  - Rotate P, G in the direction that lifts X
- Case 2: Inner grandchild causes a red-red violation. The Rule:
  - Change color of G
  - Change color of X
  - Perform double rotation:
    - P, X, lifting X
    - G, X, lifting X

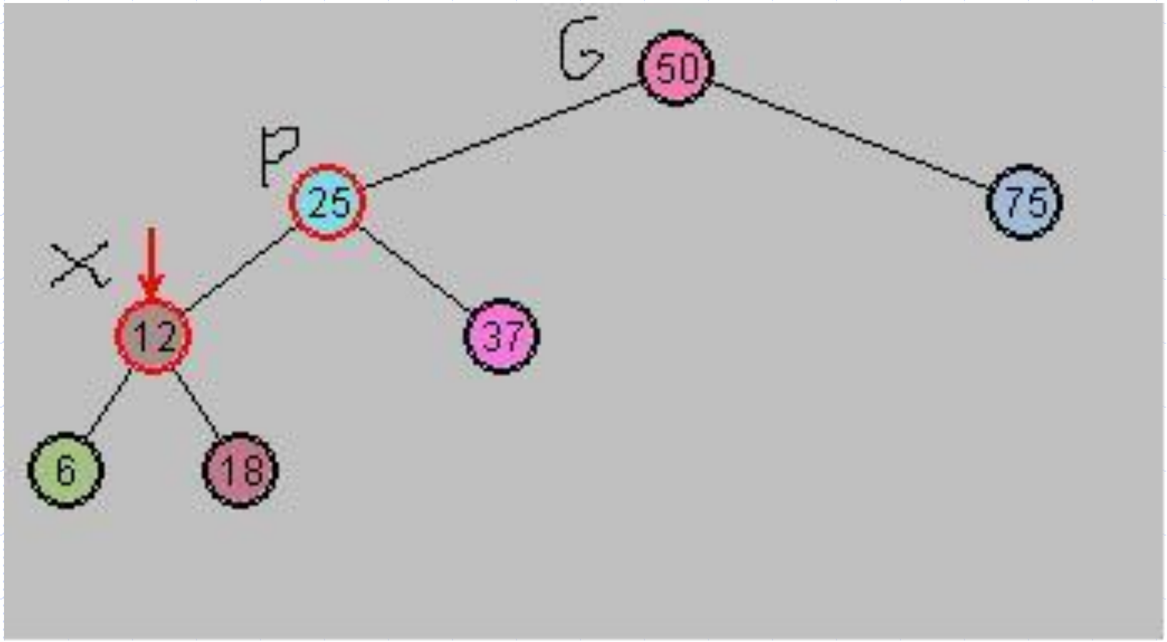
Example for Case 1

- We wish to insert the node 3, but we first notice need for color flip



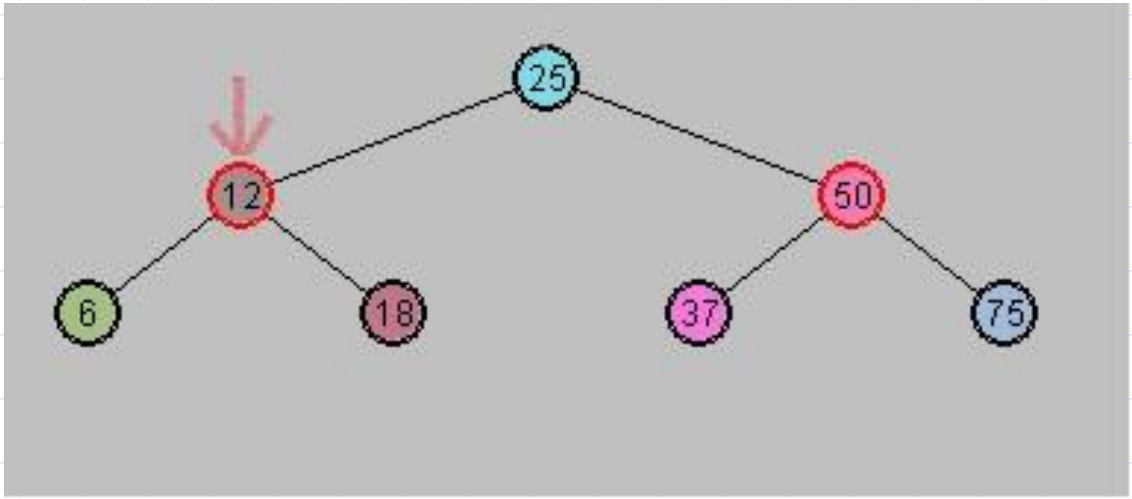
Example for Case 1 - Continued

- After color flip, we encounter red-red violation caused by outer grandchild



Example for Case 1 - Continued

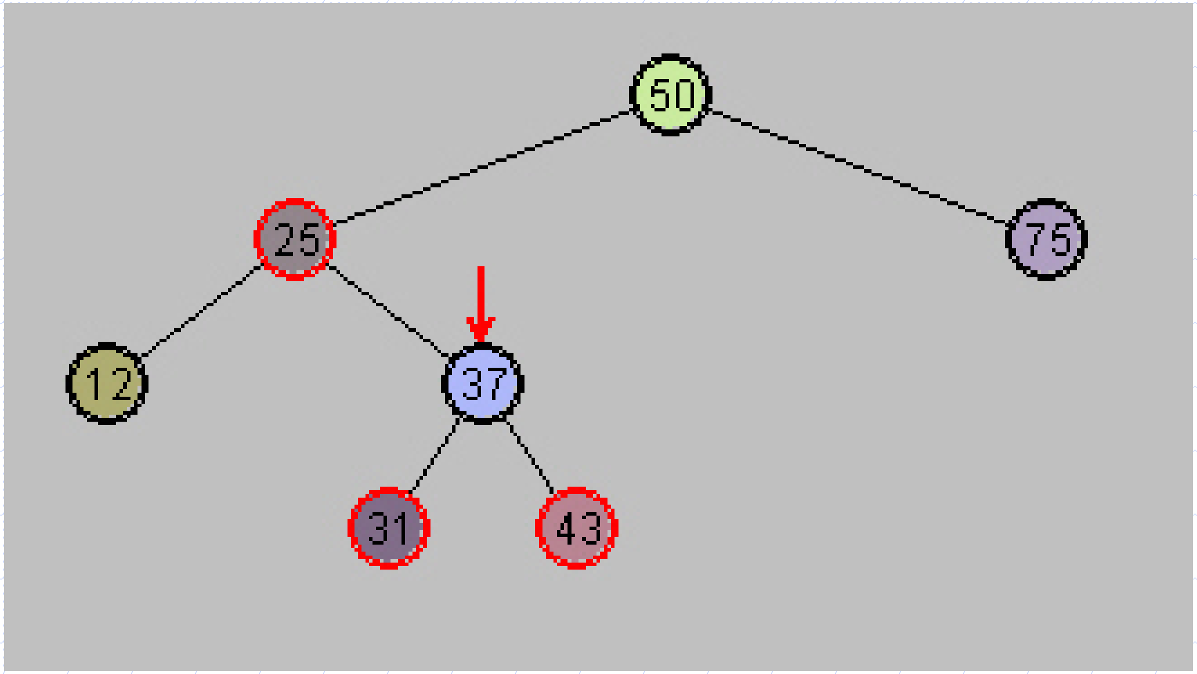
- We apply the rule for this case:



- Red-red violation has been eliminated and tree is now balanced.

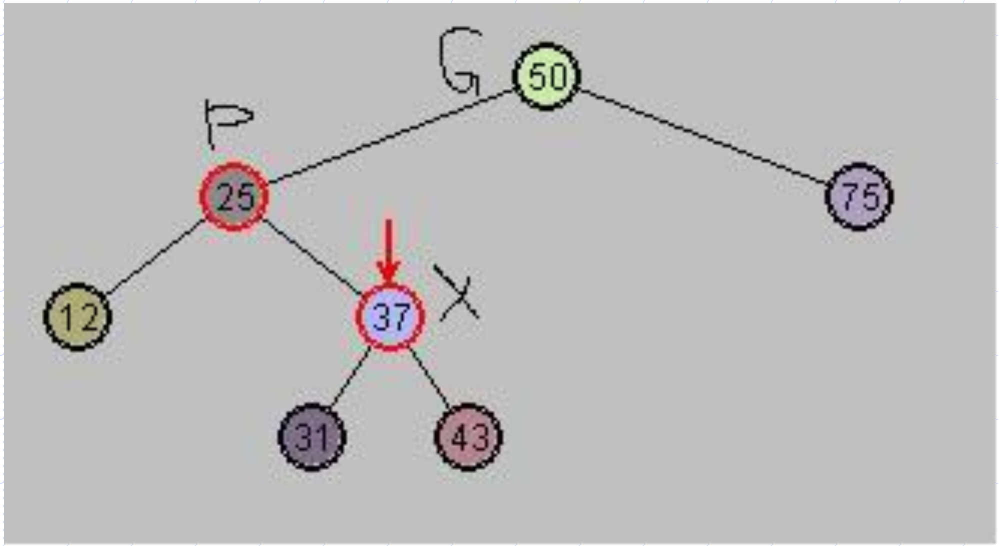
Example for Case 2

- Suppose we are trying to insert the node 28.

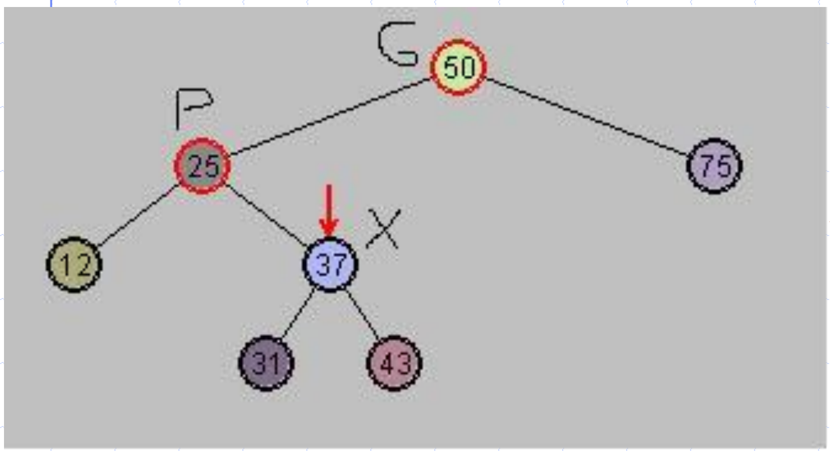


Example for Case 2 - Continued

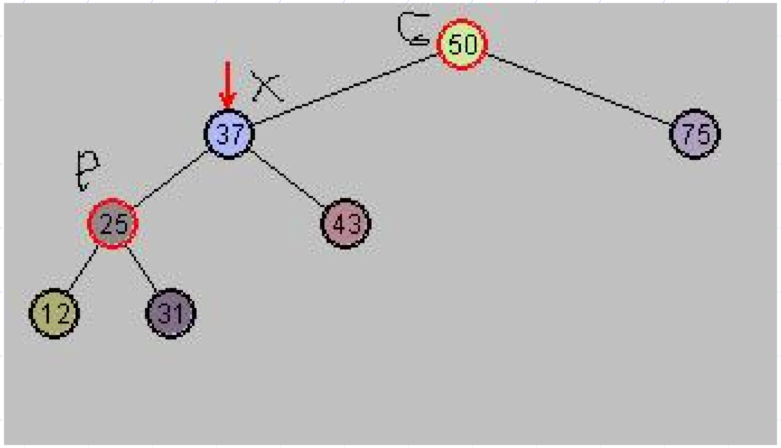
- Need to do a color flip which leads to a red-red violation caused by inner grandchild.



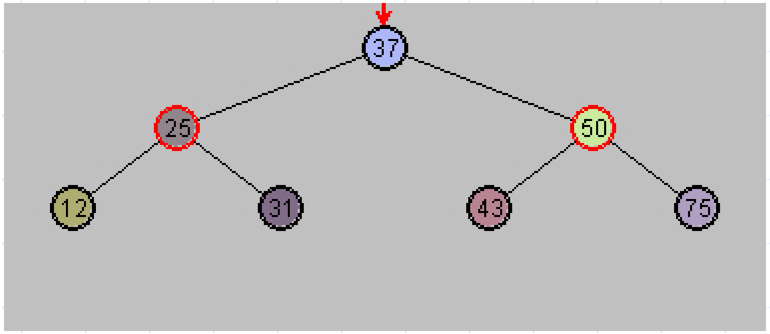
Example for Case 2 - Continued



2. Rotate X,P in the direction that lifts X.



1. First change colors of G, X



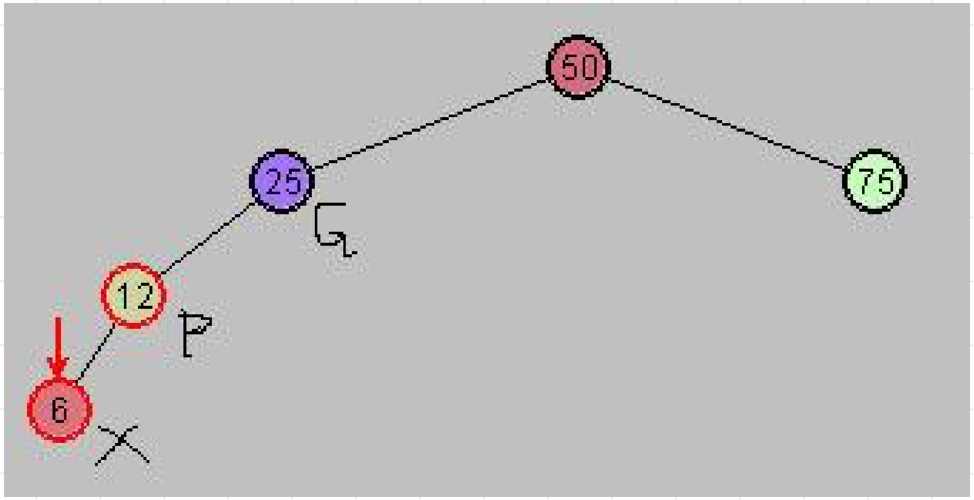
3. rotate X, G in the direction that lifts X. Tree is now balanced. Can now insert 28.

Insertion And Corrections

- Nodes are always inserted as red nodes. This can lead to further red-red violations.
- Three cases when new node X is inserted.
  - P is black, so no adjustment necessary (since X is red)
  - P is red and X is an outside grandchild. The Rule:
    - Change color of G
    - Change color of P
    - Rotate P, G in direction that lifts X
  - P is red and X is an inside grandchild. The Rule:
    - Change color of G
    - Change color of X
    - Double rotation:
      - P, X, lifting X
      - G, X, lifting X

Exercise - 1

- Insert 6 to the tree. Insertion results in outer grandchild red-red violation.



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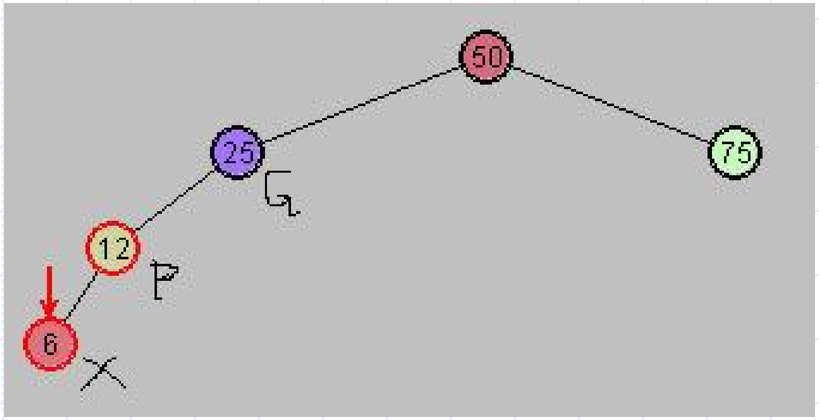
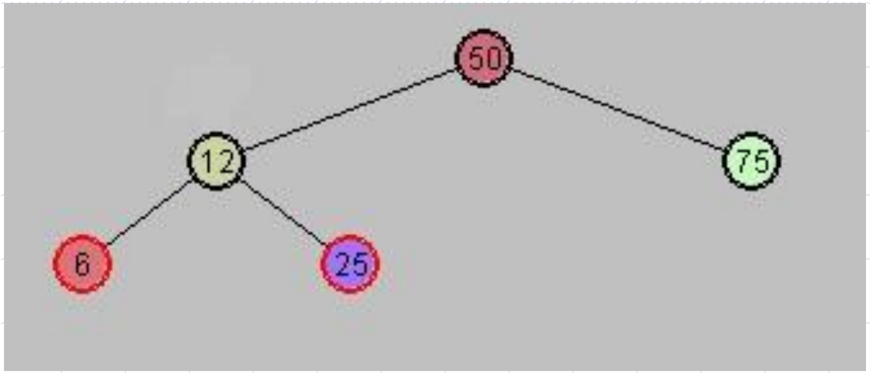
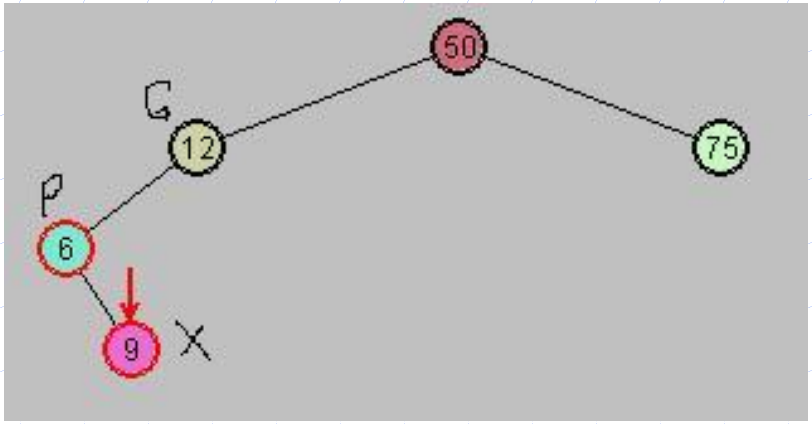




Exercise - 2

- Insert 9 to the tree. Insertion results in inner grandchild red-red violation.



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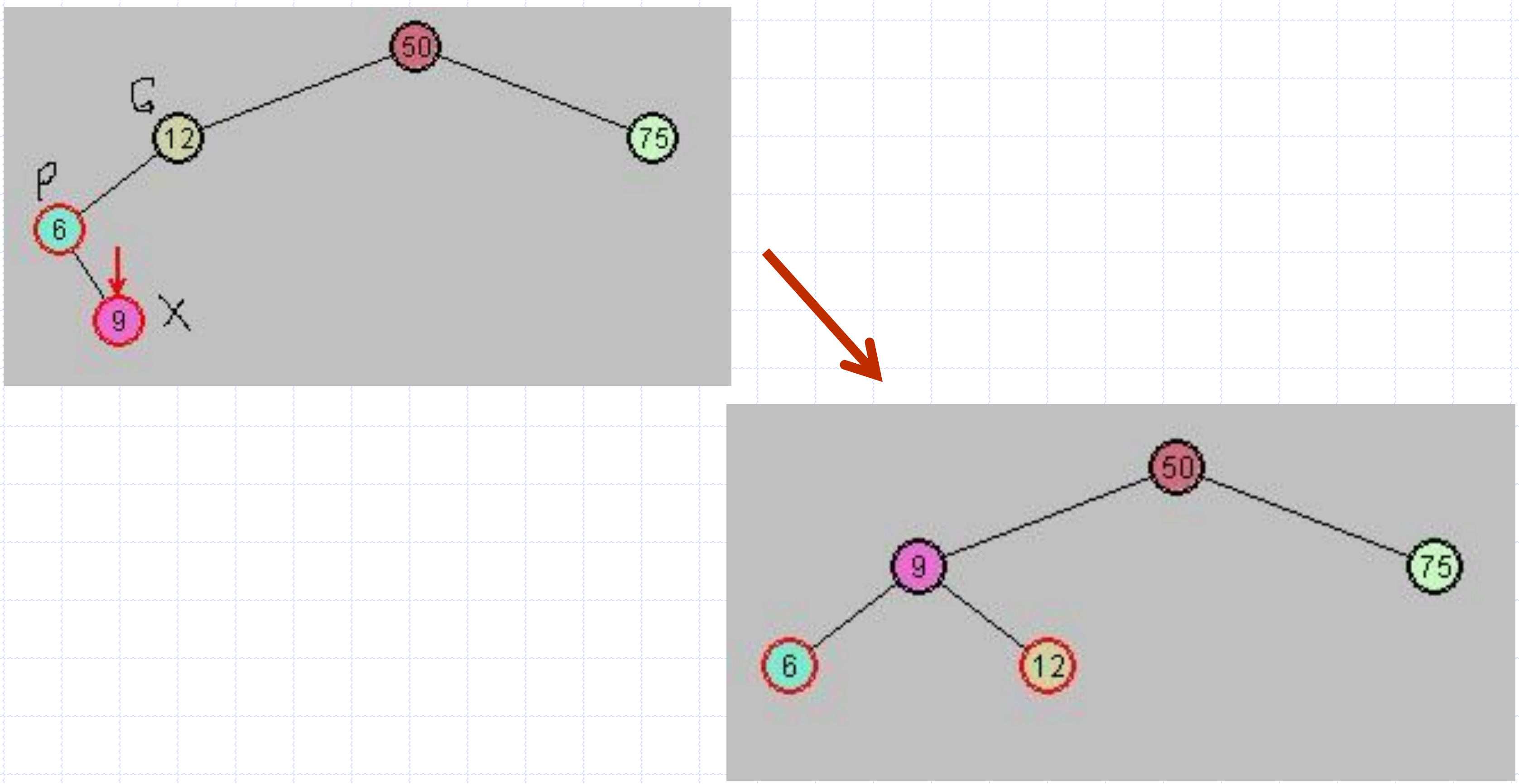

Building Red-black Trees

- Use the insertion algorithm for red-black trees to successively insert the following nodes, starting with an empty tree.

  1, 2, 3, 4, 5, 6

Show work on board.

Conclusions

- Insertions of a node at depth d require
  - O(d) to make flips and adjustments on the way to insertion point
  - O(1) to make final color changes and adjustments
- Therefore, worst case running time for insertions is O(log n)
- Searches are done as in BSTs – as before, these require O(log n)
- Deletions can be shown also to take O(log n). Since the algorithm requires several cases and is a bit complicated, we do not go through it here.

Main Point

The integrity of red-black trees is preserved after tree operations (insertions and deletions) are performed by maintaining the balance condition after execution of each operation. This maintenance does not increase the cost of operations because it requires only constant time, involving local color changes, color flips, and rotations.

Science of Consciousness: The ability to maintain its fundamental character in the face of change is the expression of the *invincible* quality of pure consciousness. Pure consciousness, in giving rise to diversity, maintains its unbounded and immortal status. In society, this invincible quality is seen when a small percentage of a population engages in group practice of the TM and TM-Sidhi Programs – the inherent harmony of the society is enlivened to the extent that it "averts the birth of an enemy."

Connecting the Parts of Knowledge With the Wholeness of Knowledge

Balanced BSTs

1. A Binary Search Tree can be used to maintain data in sorted order more efficiently than is possible using any kind of list. Average case running time for insertions and searchers is O(log n).
2. In a Binary Search Tree that does not incorporate procedures to maintain balance, insertions, deletions and searches all have a worst-case running time of $\Omega(n)$. By incorporating balance conditions, the worst case can be improved to O(log n).
3. *Transcendental Consciousness* is the field of perfect balance. All differences have Transcendental Consciousness as their common source.
4. Impulses Within the Transcendental Field. The sequential unfoldment that occurs within pure consciousness and that lies at the basis of creation proceeds in such a way that each new expression remains fully connected to its source. In this way, the balance between the competing emerging forces is maintained.
5. Wholeness Moving Within Itself. In Unity Consciousness, balance between inner and outer has reached such a state of completion that the two are recognized as alternative viewpoints of a single unified wholeness.