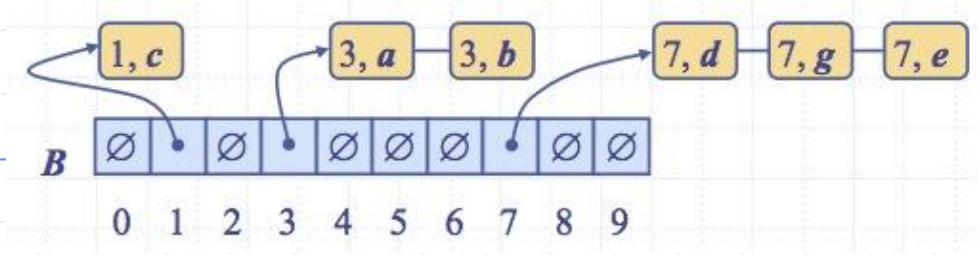
Lesson 6

Lower Bound on Comparison-Based Algorithms: *Discovering the Range of Natural Law*



Wholeness of the Lesson

Using the technique of decision trees, one establishes the following lower bound on comparison-based sorting algorithms: Every comparison-based sorting algorithm has at least one worst case for which running time is $\Omega(n \log n)$. Bucket Sort and its relatives, under suitable conditions, run in linear time in the worst case, but are not comparison-based algorithms. Each level of existence has its own laws of nature. The laws of nature that operate at one level of existence may not apply to other levels of existence.

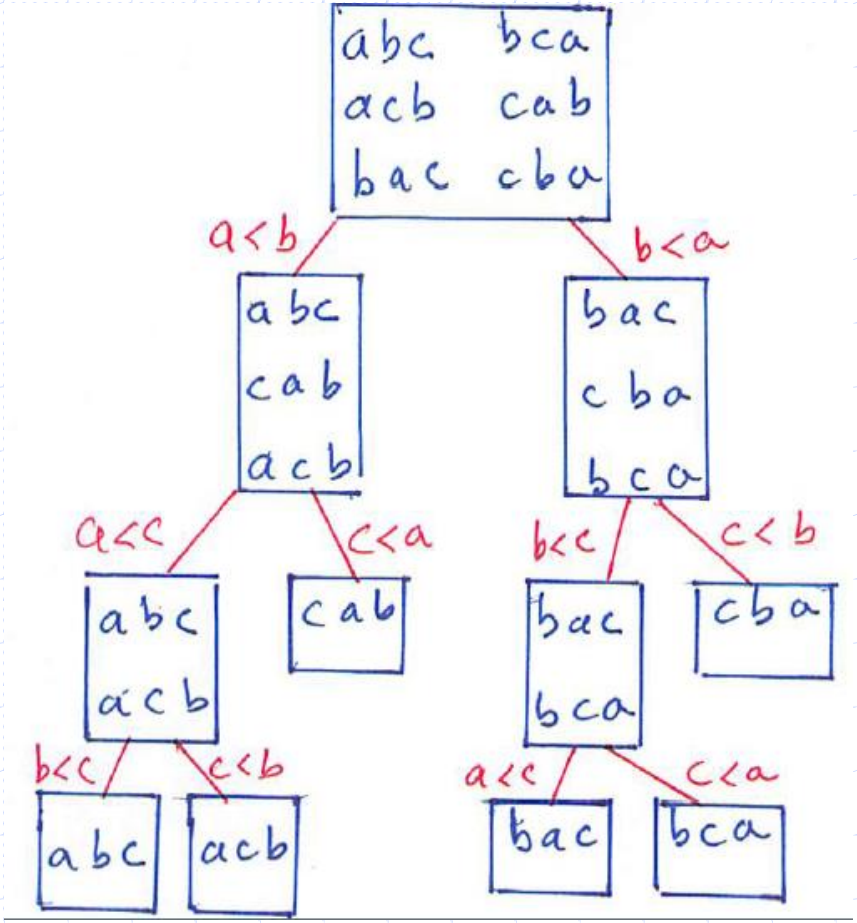
A Lower Bound on Comparison-Based Sorting Algorithms

- All sorting algorithms discussed so far have used comparison as a central operation
- So far, all sorting algorithms discussed have a worst-case running time of $\Omega(n \log n)$
- Using the technique of *decision trees* we can show that $n \log n$ is the best that can be done for comparison-based sorting algorithms

Sample Decision Tree

- Every comparison-based sorting algorithm, applied to a given array, can be represented by a decision tree.
- To illustrate the technique, we use a simple but typical example: sorting a 3-element array of distinct values *a,b,c.*

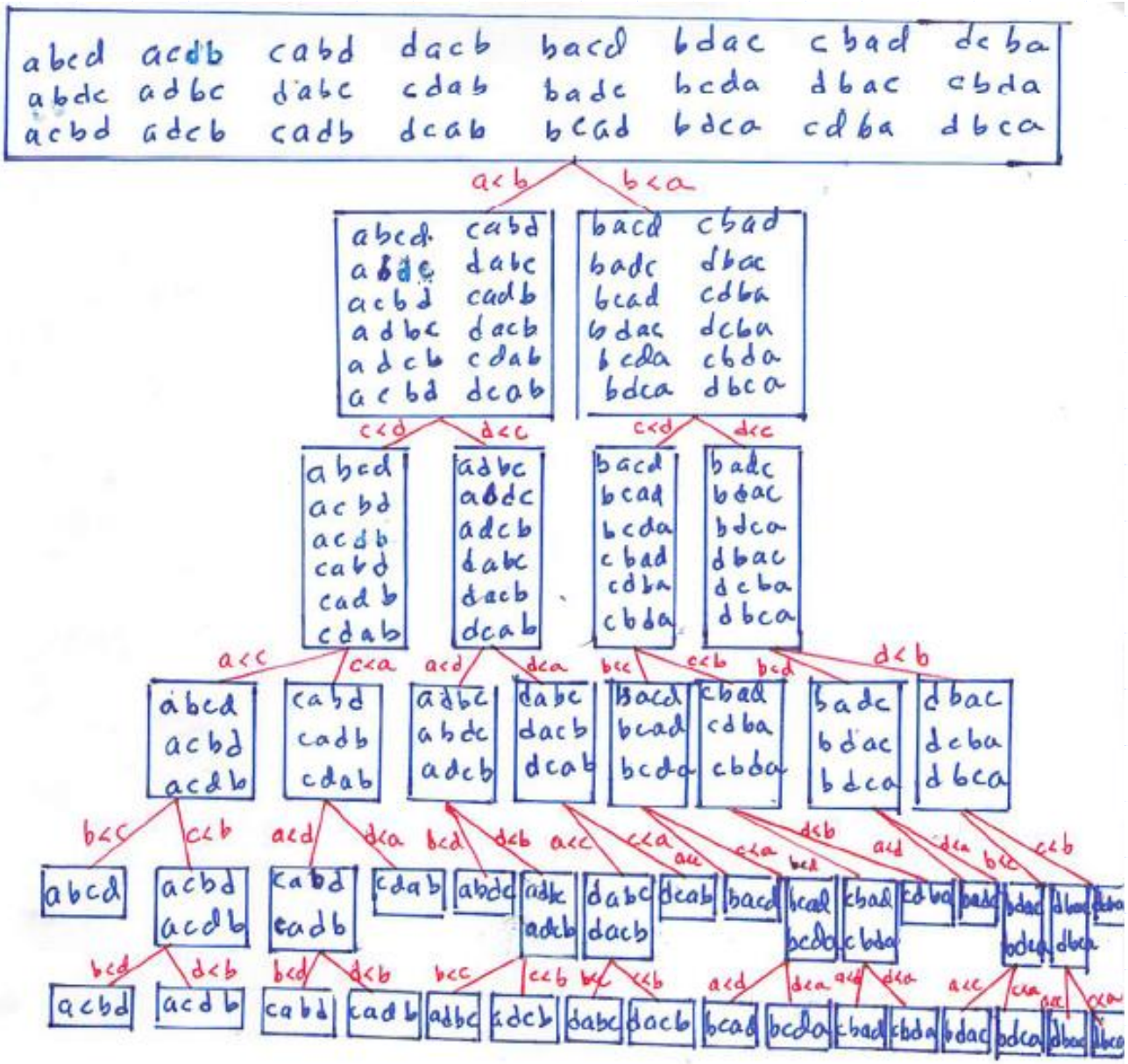
Decision Tree for Sorting 3 Elements



Anatomy of the Decision Tree

- Each node of the tree represents all possible sorting outcomes that have not been eliminated by the comparisons done so far.
- The labels on the links of the tree represent comparison steps as the algorithm runs. Different algorithms will perform some of the comparison steps in a different sequence.
- A leaf in the decision tree represents a possible sorting outcome. The different paths to the leaves represent all possible ways n distinct items could be put in sorted order; therefore, the leaves represent all possible arrangements of n distinct elements.

Decision Tree for MergeSort on 4 Elements



Strategy for Computing # Comparisons

- *Observation:* The number of comparisons performed in order to arrive at an arrangement found in a leaf node equals the depth of that leaf node in the decision tree.

- Therefore, to count # comparisons performed in the worst case, we determine the depth of the deepest node in the decision tree which is equal to height of the tree.

  (# comparisons performed in the worst case
  = the depth of the deepest node
  = height of the decision tree)

Mathematical Observations

Suppose T is a binary tree with L leaves and height h. Then

1. $L \leq 2^h$. (Lab 4 exercise)

Taking logs on both sides:

2. $h \geq \log L$.

It follows that

3. $h \geq \lceil \log L \rceil$

Establishing the Lower Bound

Summary: Suppose we have a decision tree T for a sorting algorithm running on input of size n.

a. T has n! leaves. (Follows because leaves represent all possible permutations of the n elements of the array)

b. Height of T is at least $\lceil \log n! \rceil$.

c. Therefore, in the worst case, at least $\lceil \log n! \rceil$ comparisons are performed by the algorithm, so running time of the algorithm is $\Omega(\log n!)$.

d. log n! is in $\Omega(n \log n)$ (Shown in the next slide)

e. So in the worst case, the running time for the sorting algorithms is $\Omega(n \log n)$.

The Computation of log n!

$$
\begin{aligned}
\log n! &= \log(1 \cdot 2 \cdot 3 \cdot \ldots \cdot n) \\
&= \log n + \log(n-1) + \log(n-2) + \ldots + \log 2 + \log 1 \\
&\geq \log n + \log(n-1) + \ldots + \log \frac{n}{2} \\
&\geq \sum_{i=1}^{n/2} \log \frac{n}{2} \\
&= \frac{n}{2} \cdot \log \frac{n}{2} \\
&= \frac{n}{2} \cdot (\log n - 1) \\
&= \frac{n}{2} \cdot \log n - \frac{n}{2} \\
&= \Omega(n \log n)
\end{aligned}
$$

Specialized Sorting: Bucket Sort

- The Context: Suppose we have an array A of n distinct integers a_1, a_2, ..., a_n, all lying in the range 0..m-1.
- Sorting strategy:
  1. Create an array bucket[] of size m, entries initialized to 0. Create an output array B of size n
  2. Scan A. When a_i is encountered, increment the value bucket a_i
  3. Scan bucket[]. For each j < m for which bucket[j] > 0, copy j into next available slot in B.

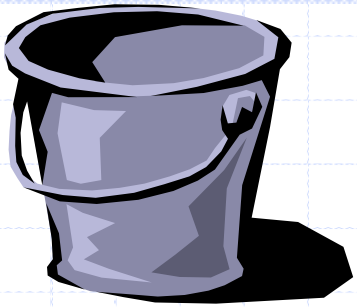
Analysis of Bucket Sort

- Step 1 requires O(m+n)
- Step 2 requires O(n)
- Step 3 requires O(m)+O(n)

** Therefore, BucketSort runs in O(m+n). When m is O(n), BucketSort runs in O(n)

- *Handling Duplicates (simple case):* Same algorithm, except in loading array B at the end, if bucket[j] = k, insert k copies of a_j into B.

Stability of Bucket-Sort



- Given n integers $a_1, a_2, \ldots, a_n$ in the range [0, m-1], possibly with duplicates, and n matched objects o1, o2, …, on. We sort array A whose elements are pairs $(a_1,o_1), (a_2,o_2),\ldots,(a_n,o_n)$.
- To handle duplicates in this case, array bucket[] now consist of m *lists* rather than integers. Output array B will consist of pairs (a,o).
  - Phase 1: Scan A. When (a_i,o_i) is encountered, copy it to the end of the list in bucket $B[a_i]$
  - Phase 2: For $j = 0, \ldots, m-1$, copy the items of bucket $bucket[j]$ to the end of sequence B.
- As before, running time is O(n+m). Notice the algorithm is stable because we always add newly scanned pairs to end of the list in the appropriate bucket

Algorithm *bucketSort*(*A*, *m*)

Input array *A* of (key, element) items with keys in the range [0, *m* – 1]

Output array *B* sorted by increasing keys

bucket ← array of *m* empty lists

for *i* ← 0 **to** *n* – 1

  (*k*, *o*) ← *A[i]*

  bucket[*k*].*insertLast*((*k*, *o*))

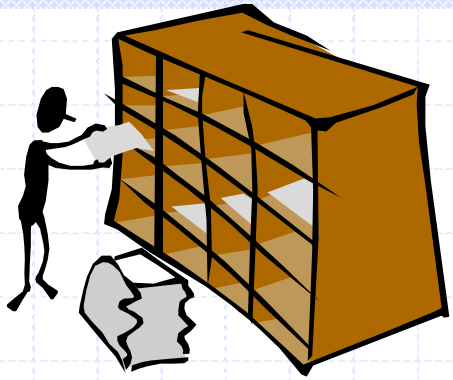
for *j* ← 0 **to** *m* – 1

  while ¬*bucket*[*j*].*isEmpty*()

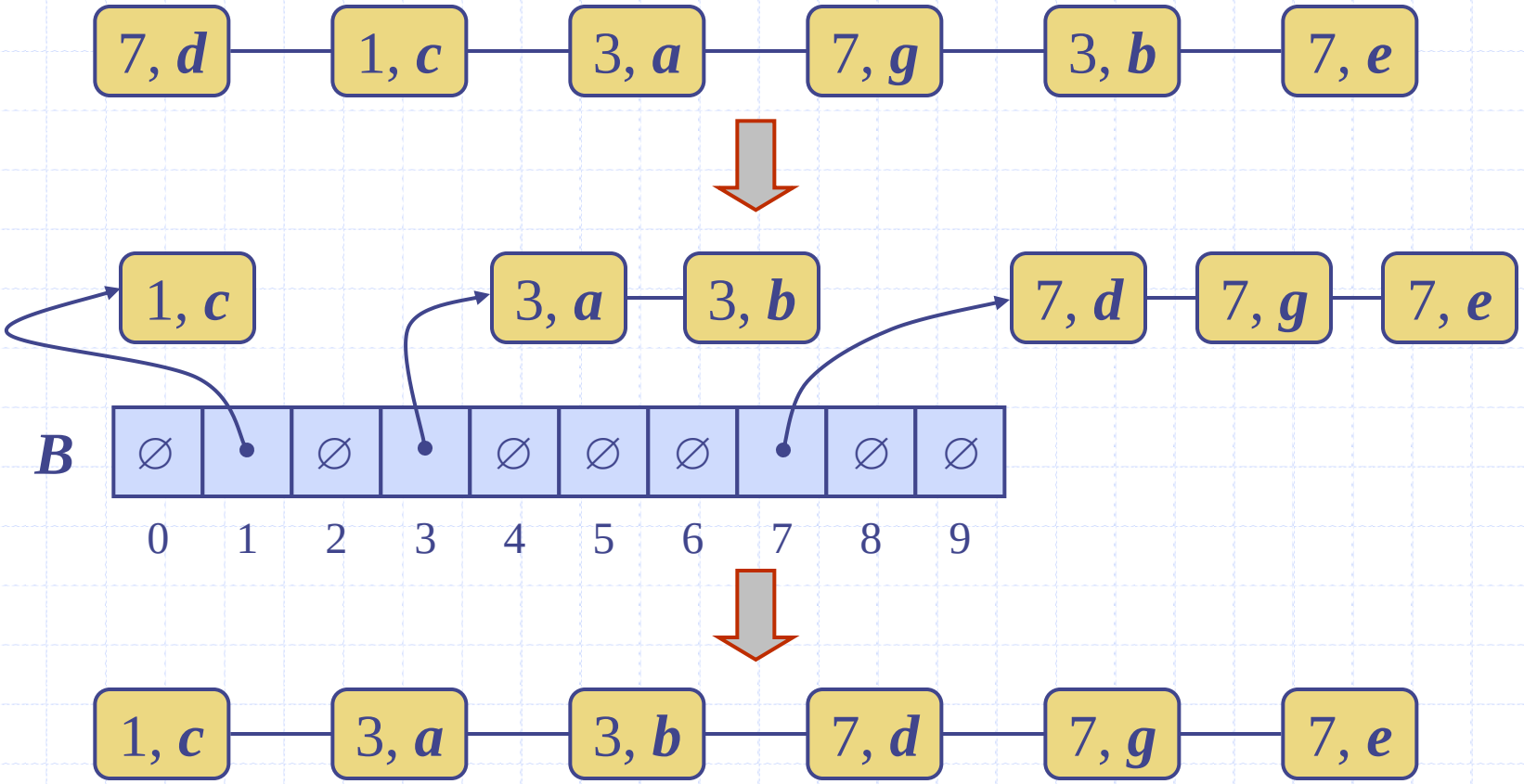
    (*k*, *o*) ← *bucket*[*j*].*removeFirst*()

    B.*insertLast*((*k*, *o*))

Example



- Key range $[0, 9]$

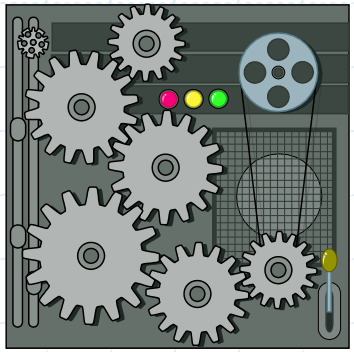


Expanding the Context

- Negative integers can be included if lower bound on these integers is known in advance. Similarly, integers in any range [a,b] can be sorted; worthwhile if b-a is O(n)

- If m is too big (i.e., not O(n)), BucketSort is not useful because scanning the bucket array is too costly

Radix-Sort



- BucketSort doesn't work because range is too big – would run in $\Omega(n^2)$
- Radix-sort is a generalization of BucketSort that uses multiple bucket arrays
- Example: Sort 48, 1, 6, 23, 37, 19 ,21

Radix Sort Example

- Strategy is to use 2 bucket arrays, each of size 7 (7 is the *radix*)
- Based on observation that every k in [0,48] can be written:
  k = 7q + r where 0 <= q < 7, 0 <= r < 7
- Procedure:
  - Pass #1: Scan initial array and place values the "remainders" bucket r[] – put x in r[i] if x % 7 = i. (Need to assume bucket array consists of lists)
  - Pass #2: Scan r[], reading from front of each list to back, and place values in the "quotients" bucket q[] – put x in q[i] if x/7 = i.
  - Output: Scan q[], again reading lists front to back

Analysis of RadixSort

- Running time is O(d(n+r)) where
  - d = # bucket arrays (in example d=2)
  - n = size of initial array (example: 7)
  - r = size of each bucket array (ex: 7)
- RadixSort runs in O(n) whenever the number of bucket arrays is O(1), and size of each bucket array is O(n).

Optional: Using 3 Bucket Arrays

- Generalizing previous example, we observe that whenever k belongs to [0, 342] (note $342 = 7^3 - 1$), then we can write

  $k = 49q_1 + 7q_2 + r$

  where $0 \le q_1 < 7$, $0 \le q_2 < 7$, $0 \le r < 7$.

  Example: $340 = 49*6 + 7*6 + 4$

Therefore to handle sorting approx 7 elements in the range [0, 342], create bucket arrays r[], q_1[], q_2[], and compute,for each x in input array, values x%7, (x/7)%7, x/49, respectively

What Was That Lower Bound Again?

- We established a lower bound of $\Omega(n \log n)$ on comparison-based sorting algorithms. Do BucketSort and RadixSort disprove our earlier findings?

What Was That Lower Bound Again?

- We established a lower bound of Ω(nlog n) on comparison-based sorting algorithms. Do BucketSort and RadixSort disprove our earlier findings?
- *Resolution:* Neither of these is comparison-based.

Main Point

A decision-tree argument shows that comparison-based sorting algorithms can perform no better than Θ(nlog n). However, BucketSort is an example of a sorting algorithm that runs in O(n). This is possible only because BucketSort does not rely primarily on comparisons in order to perform sorting. This phenomenon illustrates two points from SCI. First, to solve a problem, often the best approach is to bring a new element to the situation (in this case, bucket arrays); this is the Principle of the Second Element. The second point is that different laws of nature are applicable at different levels of creation. Deeper levels are governed by more comprehensive and unified laws of nature.

Connecting the Parts of Knowledge
With The Wholeness of Knowledge

Transcending The Lower Bound On Comparison-Based Algorithms

1. Comparison-based sorting algorithms can achieve a worst-case running time of $\Theta(n \log n)$, but can do no better.

2. Under certain conditions on the input, Bucket Sort and Radix Sort can sort in $O(n)$ steps, even in the worst case. The $n \log n$ bound does not apply because these algorithms are not comparison-based.

***3. Transcendental Consciousness* is the field of all possibilities and of pure orderliness. Contact with this field brings to light new possibilities and leads to spontaneous orderliness in all aspects of life.**

4. *Impulses Within The Transcendental Field*. The organizing power of pure knowledge is the lively expression of the Transcendent, giving rise to all expressions of intelligence.

5. *Wholeness Moving Within Itself*. In Unity Consciousness, the organizing dynamics at the source of creation are appreciated as an expression of one's own Self.