Resource Allocation for MIT Fraternity Rush

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1 Introduction

Fraternity Rush at MIT is the process by which new freshmen interact with various fraternities and ultimately decide which, if any, they will join. Rush consists of a series of mixers and events held over six days, after which fraternities issue invitations to freshmen (bids). A freshman can receive zero, one, or multiple bids. After receiving bids, freshmen have approximately a week to select one, or none, of the bids and ultimately join (pledge) that fraternity, which marks the end of the year's Rush.

In Fall 2017, there were 25 different fraternities that participated in MIT Rush. Out of 595 men in the incoming freshman class (Class of 2021), 297 eventually pledged a fraternity, and we estimate that around 400 attended at least one Rush event. Due to the large number of independent fraternities at MIT, competition between fraternities during Rush is fierce. Rush events typically include trips to Six Flags, lobster dinners, river boat cruises, indoor skydiving, whitewater rafting, and more. Fraternities each spend anywhere from $15,000 to $25,000 on Rush week each year, with the goal of recruiting around 10-20 new members. As a result, the amortized "cost per pledge" for fraternities can easily average around $1,000!

We view these excessive costs as an inefficiency of the Rush market. Anecdotally, fraternities often end up spending a large portion of their Rush money on freshmen who will ultimately not join, which suggests that the money is poorly allocated. During a typical Rush, perhaps 100-150 freshmen will visit a fraternity, around 50 will be present at multiple events, but only 10-20 will ultimately receive+accept a bid. This imbalance leads us to believe there may be a better allocation of fraternity resources that targets the freshmen most desired/most likely to pledge. Such an allocation strategy would reduce waste and save fraternities money during Rush.

In this paper, we analyze MIT Rush as a modified version of a college-student matching market. We show that the process of issuing bids + pledging is equivalent to the college-proposing deferred acceptance (DA^C) mechanism, with a caveat: **Each fraternity's capacity is unlimited.** As a result, fraternities simply extend bids to all acceptable candidates, and DA^C always terminates after one round. This is designed to mirror the real Rush process where bids are only issued once, and frats never issue a second round of bids, even though they could in theory. We attribute this behavior to the fact that pledge class size rarely places a hard limit on the # of bids alloted; fraternities simply aim for a reasonable range (10-20), and extend bids to all acceptable students.

Due to this capacity relaxation, we show that the college-proposing deferred acceptance outcome (μ^C) is the same as the student-proposing deferred acceptance outcome (μ^S). This tells us that MIT Rush is strategy-proof for students. We go further and show that the capacity relaxation also makes MIT Rush strategy-proof for the fraternities: it is always optimal to extend bids to all acceptable candidates.

To introduce the notion of spending money, we allow fraternities a limited number of "swaps", which they can use to adjust student's preference lists prior to extending bids. Up to one swap can be spent per student per fraternity, and each fraternity can spend up to N total swaps. Spending a swap on a student moves a fraternity up one position on that student's preference list, and when breaking ties between fraternities at equal positions, the fraternity that spent a swap is valued higher than the fraternity that did not.

Under this formulation, when N is sufficiently large, we show that allocating swaps according to the Gale-Shapley outcome of the original preference rankings provides a Nash Equilibrium for the Rush market. We compare this swap strategy to other wasteful ones such as $top - students$, which corresponds to spending all N swaps on top-ranked candidates. Via simulation, we show that the $Gale - Shapley$ swap strategy consistently leads to better outcomes for fraternities, and often leads to fewer swaps being spent per fraternity. This corresponds to greater efficiency and fewer dollars being spent during MIT Rush.

2 Rush Market Formulation

The market for Rush has many similarities to a traditional college-student matching market, where colleges and students have preference lists over each other, and colleges have strict capacities. However there are a few important differences. We describe MIT fraternity Rush as:

1. Fraternities each have a (strict) ranking over the students. These rankings are **fixed**, and publicly known.

2. Fraternities' capacities are unlimited.

3. Students each begin with a (strict) ranking of fraternities, and these rankings are publicly known.

4. Each fraternity can spend up to N swaps to modify students' rankings, where N is larger than all fraternities' capacities. Note that a fraternity is limited to one swap per student, but multiple fraternities can use swaps on the same student. These swaps represent money/attention spent on a student, improving their perception of a fraternity. **For example:**

   Fraternity F1 starts with preferences: $[S1 > S2 > S3 > S4 > None > ... > S100]$

   Student S1 starts with preferences: $[F3 > F2 > F1 > F4 > None > ... > F25]$

   F1 spends 1 of their swaps on S1, resulting in

   Student S1 updated preferences: $[F3 > F1 > F2 > F4 > None > ... > F25]$

5. Fraternities simultaneously perform their swaps.
6. Fraternities extend bids to all acceptable students.
7. The students choose their most preferred fraternity (or none) out of their received bids.

At the end of this process, some students will be matched to fraternities, and the Rush process is completed.

2.1 Unlimited Capacity

The most important difference between our formulation of Rush, and a traditional college-student matching market, is that we assume fraternities have unlimited capacities. Our justification for this is that in practice, fraternities rarely have hard limits on pledge class size. More often than not, the limiting factor is the number of acceptable students. This happens because the acceptance criteria is very strict — for most fraternities, extending a bid requires unanimous agreement between the brothers.

As a result, fraternities rarely set a fixed capacity before Rush begins. Instead they aim for a pledge class size range (e.g. 10-20 pledges) and after meeting/judging students, fraternities set their capacity to the # of students they find acceptable. For the sake of the model, we interpret this behavior as unlimited capacities for fraternities.

Note that the unlimited capacity assumption also aligns with the fact that bids are only extended **once** during Rush: If a fraternity had an acceptable list larger than its capacity, there would be a strong incentive to continue handing out bids if some of the first round bids were rejected. In reality this never happens, which implies that fraternities extend bids to **all** acceptable candidates in the very first round. This process can be interpreted as deferred acceptance

(DA^C), and as we will show in Section 3.1, this mechanism terminates in one round if the original capacities are unlimited (or at least larger than each fraternity's acceptable list).

2.2 N swap limit

We assume that N is larger than the capacity of each frat, because fraternities should be able to spend at least enough swaps (dollars) to entice the # of students they wish to receive. This is a basic assumption that we find to be consistent across all MIT fraternities (money is not the limiting factor).

2.3 Swap Tie-Breaking

We allow fraternities to apply swaps to student preference lists — however, we must be careful to ensure that swaps are commutative, and can be expressed with a utility function. To this end, we assume that preference lists assign a *score* to each fraternity. For example, the student preference list,

$$S1 : [F1 > F2 > F3 > F4 > None]$$

corresponds to scores:

$$F1 = 0$$

$$F2 = -1$$

$$F3 = -2$$

$$F4 = -3$$

If a fraternity applies a swap to a student, it increases their score by $1 + \epsilon$, where $\epsilon << 1$. This corresponds to the idea that a lower fraternity applying a swap (spending money on you) is more valuable than a higher fraternity not swapping on you (not spending on you). For examples, if $F2$ applies a swap on the student above, the new preference list is:

$$S1' : [F2 > F1 > F3 > F4 > None]$$

If instead, **both** $F2$ and $F3$ apply swaps, the original preference list is unchanged:

$$S1'' : [F1 > F2 > F3 > F4 > None]$$

Since fraternities are limited to one swap per student, we know this utility function is valid (fraternity scores are always either of the form s or $s + \epsilon$) and any set of swaps corresponds to a unique output preference list, regardless of the order in which they are performed.

3 Theoretical Observations

Based on our formulation, we can make the following claims about Rush:

1. The bids + pledging process is equivalent to the *college-proposing deferred-acceptance* (DA^C) mechanism.

2. Under the unlimited capacity assumption, DA^C terminates after one round.

3. Under the unlimited capacity assumption, the matching output of DA^C is the same as the matching output of DA^S, the *student-proposing deferred acceptance* mechanism.

4. Under the unlimited capacity assumption, DA^C is strategy-proof for fraternities.

5. Since the bids + pledging process can be replaced by DA^S, we claim that the bids+pledging process is also strategy-proof for students, with respect to their final (possibly swapped) preference lists.

We prove the above claims in the following sections.

3.1 Bids + Pledging Mechanism

At the conclusion of Rush week, fraternities hand out bids to selected students, and students then select one or none of their received bids to join a fraternity. At first glance, this appears to be a truncated version of DA^C, with only one round of proposals. However, we claim that due to the unlimited capacities of fraternities, standard DA^C would terminate after one round anyways:

Lemma 3.1. *If colleges have unlimited capacities, DA^C terminates after one round.*

Proof. Given unlimited capacities, all colleges extend offers to all acceptable candidates in the first round. After students accept their top offer, colleges with unfilled seats have no more acceptable candidates to consider, so they are finished. No college extends a second round of offers. $\square$

In other words, the bids + pledging process is exactly equivalent to DA^C, and simply happens to terminate after one round due to the unlimited capacities of fraternities. This matches real-life Rush, where multiple rounds of bids are allowed, but *never actually occur*, due to the unlimited capacities.

One other observation we make is that the matching output of DA^C under this constraint is identical to the matching output under DA^S:

Lemma 3.2. *If college capacities are unlimited, the college-proposing deferred-acceptance matching, μ^C, and the student-proposing deferred-acceptance matching, μ^S, are equal.*

Proof. Since college capacities are unlimited, under DA^C, all acceptable students for a college C receive an offer from C in the first round. Thus, every student receives an offer from every college for which they are acceptable. Since students select their favorite offer among those received, and no further rounds occur (Lemma 3.1), every student is matched to their favorite college for which they are acceptable.

Now, under DA^S, students will propose to their favorite colleges in order, and will be matched to the first college for which they are acceptable. This is again due to the unlimited capacity constraint. Just as before, students are matched to their favorite college for which they are acceptable.

Therefore, both DA^C and DA^S produce the same matching. □

3.2 Strategy-proofedness

Under our unlimited capacity assumption, we claim that DA^C, and therefore Rush, is strategy-proof for the fraternities:

Theorem 3.3. *If colleges have unlimited capacities, DA^C is strategy-proof for all colleges.*

Proof. We know that given unlimited capacity, every college extends offers to all acceptable candidates in the first round. Clearly, there is no added benefit for a college C to extend an offer to an unacceptable student. On the other hand, if C chooses to NOT give an offer to an acceptable student s, that student will either:

1. Accept an offer at a different college C'.
2. Receive no offers, and remain unmatched.

In the first case, college C' accepts s, and this has no effect on the other matched students (all capacities are unlimited, no students are shifted/dropped).

In the second case, we again have no effect on the other students, and simply end up with an inefficiency (both s and C would prefer to be matched but are not). There is no way for college C to manipulate its preferences to receive a better match. $\square$

Furthermore, due to the equivalence shown in Lemma 3.2, we claim that Rush is also strategy-proof for the students. From 14.19, we know that the student-proposing deferred-acceptance algorithm is strategy-proof for students:

Theorem 3.4. *DA^S is strategy-proof for all students. (14.19 notes)*

Since the Rush matching is always guaranteed to be equal to the DA^S matching, we can replace the Rush mechanism with DA^S, and therefore Rush is strategy-proof for students as well.

4 Swap Allocation Strategies

We now explore the various swap allocation strategies that the proposed framework allows for. Recall that each fraternity is allocated N swaps, and no more than 1 swap may be used on a given student per fraternity. Since swaps are a proxy for money spent, the goal is to achieve a favorable outcome while minimizing the number of swaps used.

4.1 Top-Students

The top students strategy (top-swap) is a rather simple one - fraternities use all of their swaps on their highest ranked candidates. We will show that this is not an optimal strategy by a simple example:

Consider three students, $S1, S2, S3$, who all have the following preferences over three fraternities: $[F1 > F2 > F3 > None]$. The three fraternities, $F1, F2, F3$, all have a capacity of 2, and the following preferences over three students: $[S1 > S2 > S3 > None]$. Assume each fraternity is allocated 2 swaps. If each fraternity uses the *top-students strategy*, and thus allocates their swaps to $S1$ and $S2$, then the outcome will be the following:

$$F1 : [S1, S2], F2 : [S3], F3 : [None]$$

However, consider the following potential action - $F3$ instead decides to spend their swaps on $S2$ and $S3$. Then, the outcome will instead be the following:

$$F1 : [S1, S2], F2 : [None], F3 : [S3].$$

Even though F3 prefers S3 the least, they are incentivized to misrepresent their preferences in the swaps and thus **top-students strategy is not strategy proof**. We find a blocking pair in the matching, as $F2$ would prefer $S3$ over $None$ and $S3$ would prefer $F2$ over $F3$. Note the downside of the top-students strategy - each fraternity spends all of their possible swaps. In our

4.2 Gale-Shapley

The next strategy we consider is assigning swaps to the students allocated to a frat in their original Gale-Shapley outcome — before swaps occur. We call this strategy *GS-swap*. This can be understood as essentially a *defensive* policy: spend swaps to protect the students you would have originally received. Interestingly, we can also prove that this strategy leads to a Nash Equilibrium, given a few basic assumptions.

Lemma 4.1. *For a given fraternity, F_i, assuming that the number of swaps, N, is $\geq$ the size of their Gale-Shapley outcome from the original game, $|F_{i_S}|$, then assigning swaps to the students who were the outcome of the original game,*

F_{i_S}, *will result in a Nash-Equilibrium.*

Proof. We start by recalling that the bids + pledging process, which is analogous to DA^S, is strategy-proof for both fraternities and students, given our unlimited capacity assumption. Assume, for the sake of contradiction, that there does not exist a Nash-Equilibrium after the GS-swap strategy is applied and bids + pledging is run. Note that playing the GS-swap strategy should result in the same outcome as running bids + pledging without swaps. This is because for a given student, to have received them in the original game you must have had the highest score for them amongst available frats. Then after swaps, your score for the student must have weakly increased, so the list of students a fraternity receives will be unchanged. Thus, if there doesn't exist a Nash-Equilibrium after the GS-swap strategy is applied and bids + pledging is run then this means that the outcome of DA^S is not a Nash-Equilibrium, which is a contradiction. $\square$

4.3 Smart-Gale-Shapley

One additional improvement we can make to the Gale-Shapley strategy is to recognize that we don't always have to spend swaps on our GS-outcome students. If s is a student in fraternity F's original GS-outcome, and the fraternity directly below F on s' preference list finds s unacceptable, then F does not have to spend a swap on s. This is true because the only fraternity that can "steal" s from F is the fraternity directly below F on s' preference list, but if they are not a threat, there is no need to "protect" s with a swap.

We call this strategy *smart-GS-swap*, and since it produces the same matching as GS-swap, it also constitutes a Nash Equilibrium. We also note that smart-GS-swap always results in weakly fewer swaps spent per fraternity.

4.4 Overlapping Strategies

Now we must consider the case where fraternities use varying strategies and introduce the following lemma:

Lemma 4.2. *No fraternity using top-swap can use it to beat a fraternity using GS-swap.*

Proof. Assume for the sake of contradiction that a fraternity, F_1 uses the *top-swap* strategy and uses it to 'steal' a student, S_1 from a fraternity, F_2, using *GS-swap*. Without swaps, F_2 would be assigned S_1. Thus, since F_2 is using the *GS-swap*, they must use a swap on S_1. If F_1 does not use a swap on S_1, they can not get S_1. However, even if they do use a swap on S_1, the fact that F_2 used a swap on S_1 as well means that F_2 will be higher in preference order than F_1. Thus, a contradiction arises and F_1 can not actually steal S_1 from F_2. $\square$

Importantly, note that playing the *GS-swap* strategy will guarantee each fraternity spends a weakly lesser amount of swaps than they would have in the *top-swap* strategy. This leads to an important outcome. **If a fraternity decides to use *GS-swap* or *smart-GS-swap*, they will only spend swaps (money) on students that they are guaranteed to receive**.

5 Simulations

In order to validate our theoretical frameworks and results, we built a simulation of the Rush process. The code utilized for these simulations is available here: `https://github.com/aveni/mit_rush`

5.1 Simulation Methods

The simulation we built has easily modifiable parameters for the number of students, fraternities, swaps, capacity, and more. For the capacity of each fraternity we sampled from a Normal distribution, as in the real world fraternities rarely have exactly the same capacity, but do stay within a fairly limited range. Preference lists are randomized Next, after randomizing, preference lists are truncated. This is similar to the real world as most students are not actually willing to join every fraternity. In a similar vein, fraternities do not preference more students than they have the capacity for.

5.2 Simulation Results

We explain three simulation results to shed light on our process.

5.2.1 Simulation 1 - Small Example

This simulation had 2 fraternities, $F0, F1$, each with a capacity of 2, and 3 students, $S0, S1, S2$. Preferences were generated completely randomly as there were very few actors in this game, and were as follows:

$$\mathbf{F0} : [S0 > S2 > S1], \mathbf{F1} : [S2 > S1 > S0]$$

$$\mathbf{S0} : [F1 > F0], \mathbf{S1} : [F1 > F0], \mathbf{S2} : [F0 > F1]$$

If there were no swaps, this would be the Gale-Shapley outcome:

$$\mathbf{F0} : [S2], \mathbf{F1}[S0, S1]$$

Now, we run the simulation with varied strategies and observe the outcomes:

Strategy	F1: Top	F1: Gale
F0: Top	**F0**:[S0, S2], **F1**[S1]	**F0**:[S2], **F1**[S0,S1]
F0: Gale	**F0**:[S2], **F1**[S0,S1]	**F0**:[S2], **F1**[S0,S1]

As expected, and proved by Lemma 4.1, the outcome of both fraternities using GS-swap is the same as if they did not use the swaps at all. The outcome of when one fraternity uses top-swap and one uses GS-swap is also the same. What's interesting though is the output when both fraternities use top-swap - the outcome is different from the other three. F0 will receive two students while F1 will only receive one, a clearly worse outcome for F1. This clearly shows that the top-swap strategy is not a Nash Equilibrium among players.

5.2.2 Simulation 2 - Large Example

In this simulation we attempted to model the full MIT fraternity rush, using the following parameter set: *400 students, 25 frats, 20 swaps, capacity drawn from* $\mathcal{N}(20, 2)$*, acceptable # frats drawn from* $\mathcal{N}(6, 1)$. Running the simulation multiple times, due to the stochasticity involved in the methodology, we see on average only 93 students are matched to fraternities, leaving 307 unmatched. This suggests a major flaw inherent in our methodology, and reveals that our simulation may not be ideal for this sort of situation. Since each student is randomly drawing a limited (mean size 6) fraternity preference list, and vice-verse, there is likely to be lots of situations where a student only likes fraternities who do not have him on their acceptable list. In the real world, the problem is more of a two-way street - students that are liked by certain fraternities tend to like the same fraternities back, as the fit has to be mutual. This leads us to our third simulation, which more accurately depicts the strengths of our methods.

5.2.3 Simulation 3 - Realistic Example

In this simulation we use the following parameter set: *80 students, 5 frats, 20 swaps, capacity drawn from* $\mathcal{N}(20, 2)$. We choose these parameters based on the observation that there are strong correlations between student and fraternity preference lists. In reality, Rush is not a 25-frat free-for-all, instead there are "cliques" of similar frats that compete for smaller, similar pools of students. We approximate this by zooming in one such “clique”, consisting of 5 fraternities and a smaller pool of students, who find all 5 frats acceptable. This turns out to be a very realistic model of the Rush process, where similar fraternities are competing over a similar pool of students.

We find that in one typical realization, the total number of students matched, and total number of swaps spent by fraternities is as follows:

Strategy	Total # students matched	Total # swaps spent
top-students	65	95
GS-swap	65	65
smart-GS-swap	65	33

We see that 65/80 students are matched to fraternities, which is consistent with real Rush statistics. We see that using the GS-swap, and smart-GS-swap strategies result in **far fewer total swaps spent**, without affecting the number of students matched. We achieve a nearly 66% reduction in total swaps spent. Furthermore, across all simulations we ran with these settings, we found the output matchings are identical! This is because if N is sufficiently large, using the top-students strategy must spend swaps across all students, including the GS-outcome students, and other students for whom spending swaps is wasteful. This proves that using our strategies does not negatively affect the matching outcomes for fraternities, and only saves them swaps (money).

A full text output of our matching is provided on our Github.

6 Conclusions / Future Work

In this work we have developed a framework for modeling fraternity Rush at MIT. We aimed to help alleviate one of Rush’s largest problems - the large amount of money spent inefficiently by fraternities. By modeling money as swaps on student preference lists, we were able to show that there is a strategy, *GS-swap*, for fraternities, that results in fraternities only spending money on students they are guaranteed to receive. Further gains can be achieved by only spending money on students who are being competed for, through the *smart-GS-swap* strategy.

An immediate direction for future work is better simulation of preference lists, to match the real world. As we showed in our simulations, the market model we formulated, where preference lists are randomly generated and truncated, does not always accurately describe Rush. Instead, preference lists tend to have strong two-way correlations between fraternities and students.

An important reformulation of the problem would be using money for preference discovery. In some sense, the reason fraternities spend money on Rush is not just to convince students to join, but also to understand students' preferences better. The first four to five days of Rush can be considered a preference discovery game, while the last day is an exploitation game (the focus of this project) where resources are spent convincing students to join one fraternity vs. the other.

The preference revelation setting is a much harder problem to solve, as there appear to be opportunities for manipulation from the student's side: A popular student could provide false preferences to multiple fraternities extort as many money (through events/meals) out of them as possible. We hypothesize that there may be no Nash Equilibrium for fraternities in this market; costs will keep increasing without bound as fraternities compete to uncover student preferences. This setting is also applicable to many other problems, such as sports recruiting and executive recruiting.

In conclusion, we recommend fraternities spend their swaps (money) defensively, only on students that they would have expected to receive in the absence of swaps. There is no benefit to spending money on top candidates if they are out-of-reach. By following smart-GS-swap, fraternities can save money while still protecting the students they deserve.