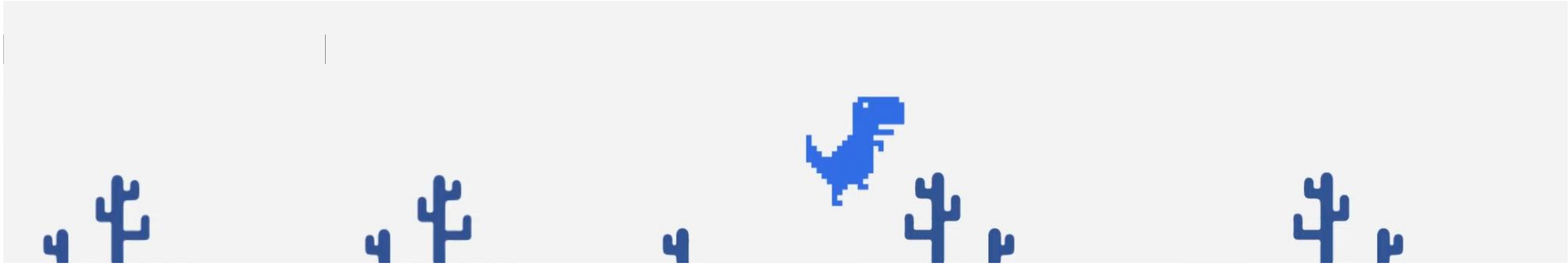
Tutorial on Nonlinear Reduced Order Modelling

A. Vizzaccaro

Advanced Structural Dynamics

Why ROMs?



High fidelity Models:
High number of DOFs

Run **full** dynamic simulations:
Hours-days

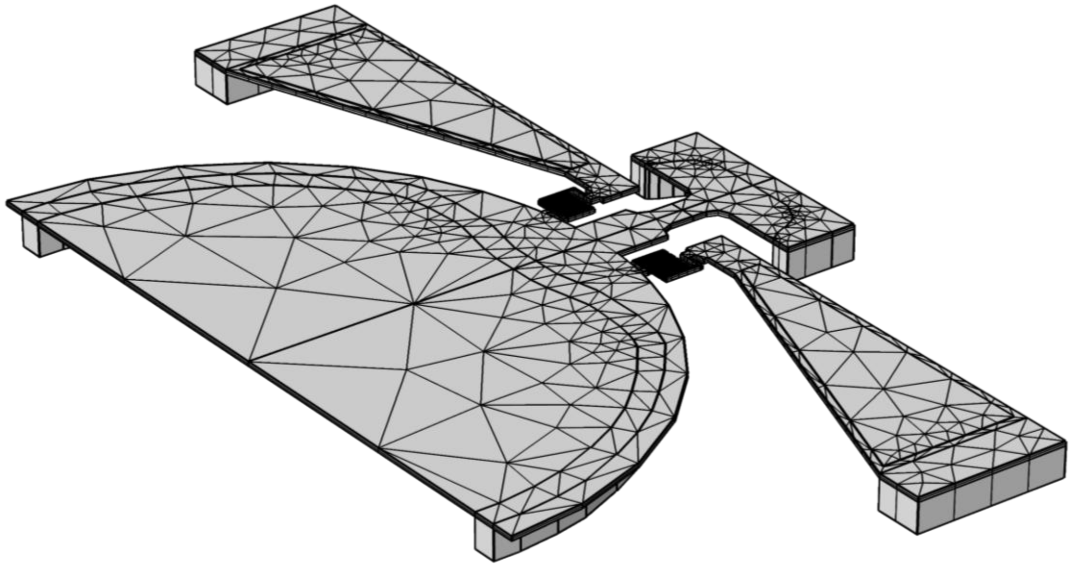
Hard to accelerate computationally

The dynamics live in a much lower dimensional space

With **ROM**:
Seconds-minutes

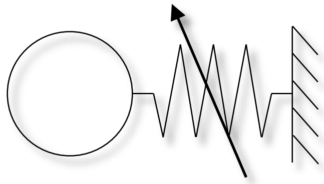
Reduced Order Modelling (ROM)

Large scale **3D** finite elements model



Nonlinear Dynamical system
with up to **Millions** of degrees of freedom

Capture the **characteristic** dynamics
with **low dimensional** models



Desired Properties:

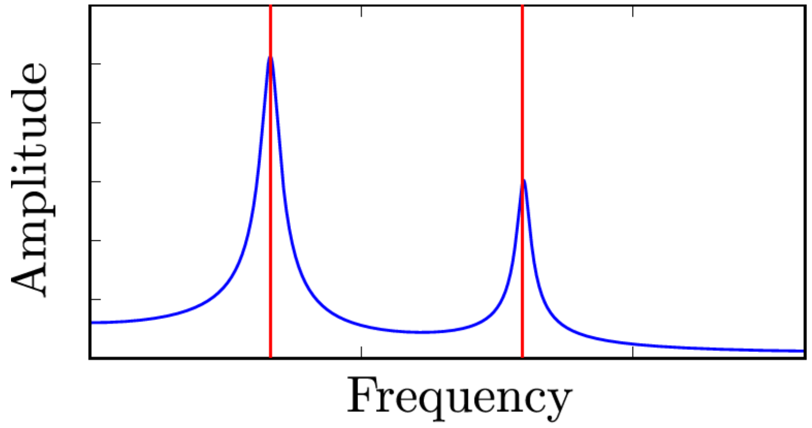
- **Small** number of parameters
- Physically **interpretable**
- From FE model
- **Fast** yet **accurate**

The simplest ROM

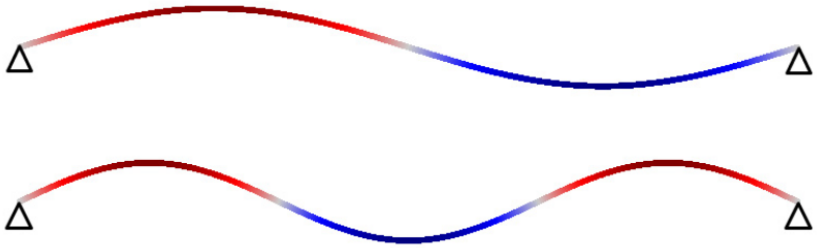
Large scale **3D** finite elements model with up to **millions** of degrees of freedom:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0}$$

ω_i Natural frequency



ϕ_i Mode shape



Change of Coordinates	$\mathbf{U} = \mathbf{\Phi}\mathbf{u}$
Reduced Dynamics	$\ddot{\mathbf{u}} + \mathbf{\Omega}^2\mathbf{u} = \mathbf{0}$

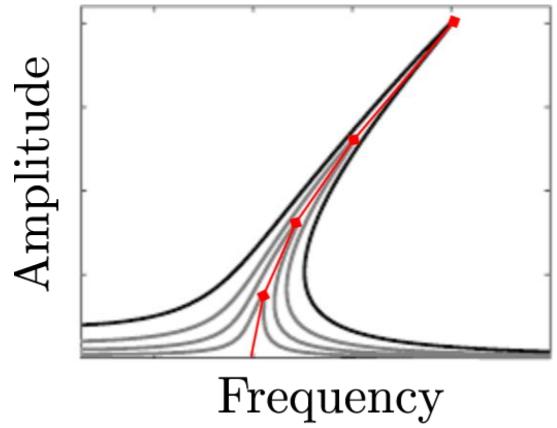
Linear Modal Analysis is the most trivial of ROMs

What happens when the system is Nonlinear?

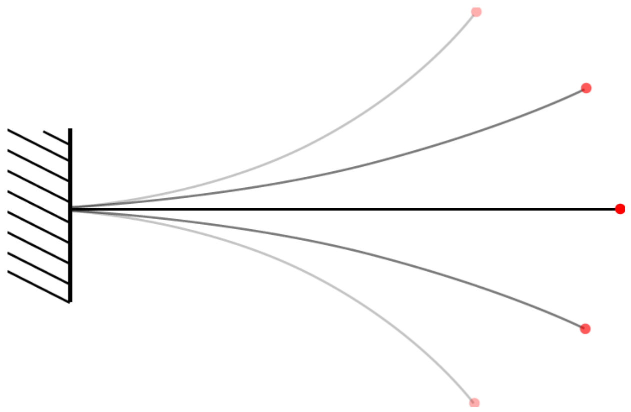
Large scale **3D** finite elements model with up to **millions** of degrees of freedom:

$$\mathbf{M\ddot{U}} + \mathbf{KU} + \mathbf{GUU} + \mathbf{HUUU} = \mathbf{0}$$

Natural frequencies $\omega_{\mathrm{nl}} = \omega_{\mathrm{nl}}(a)$



Mode shapes $\phi_{\mathrm{nl}} = \phi_{\mathrm{nl}}(a)$



Change of Coordinates	$\mathrm{U} = \mathrm{U}(\mathrm{u}, \dot{\mathrm{u}})$
Reduced Dynamics	$\ddot{\mathrm{u}} + \omega_{\mathrm{n}}{}^2\, \mathrm{u} + \mathrm{f}(\mathrm{u}, \dot{\mathrm{u}}) = 0$

The two main ingredients of every Nonlinear ROM

Naïve approach

Large scale **FE** model:

$$\mathbf{M\ddot{U}} + \mathbf{KU} + \mathbf{GUU} + \mathbf{HUUU} = \mathbf{0}$$

$$\mathbf{U} = \boldsymbol{\Phi}\mathbf{u} \quad \rightarrow \quad \ddot{\mathbf{u}} + \boldsymbol{\Omega}^2\mathbf{u} + \mathbf{guu} + \mathbf{huuu} = \mathbf{0}$$

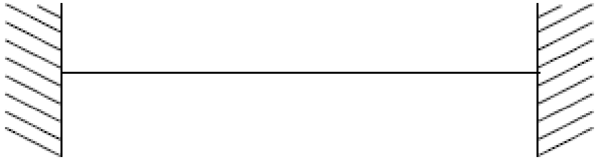
$\boxed{\text{Reduce on the first mode:}}$

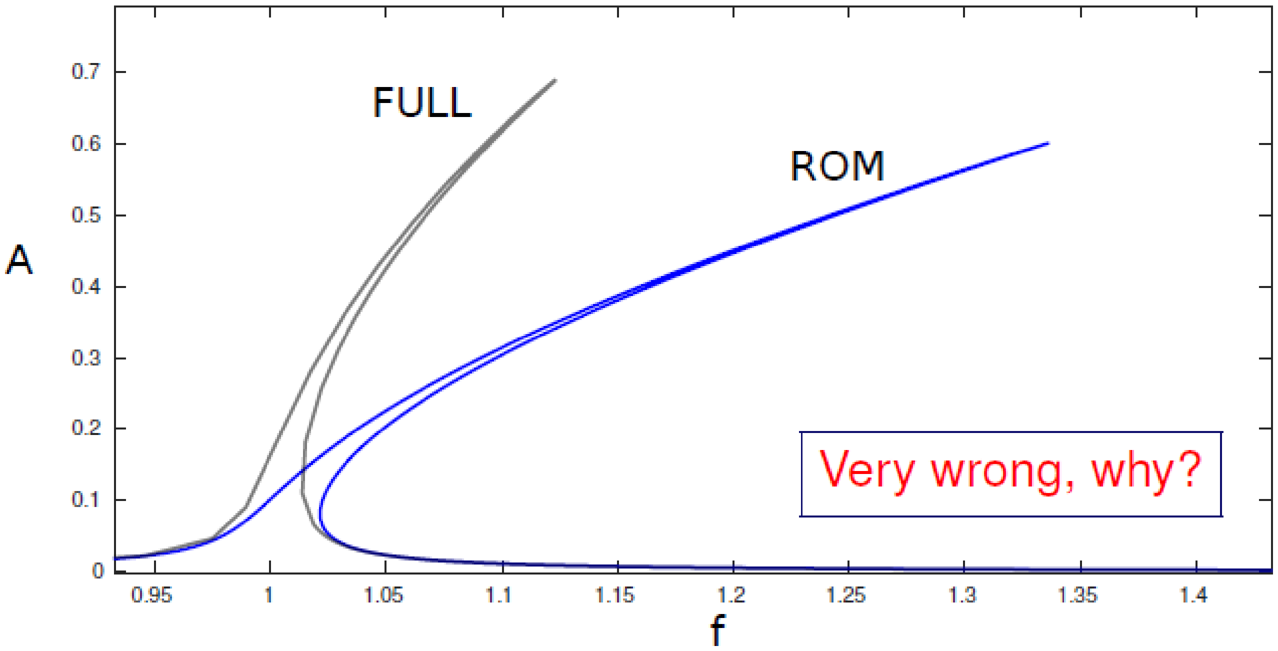
all modes zero, $u_1 \neq 0$

Reduced Dynamics

$$\ddot{u}_1 + \omega_1^2 u_1 + g_{11}^1 u_1^2 + h_{111}^1 u_1^3 = 0$$

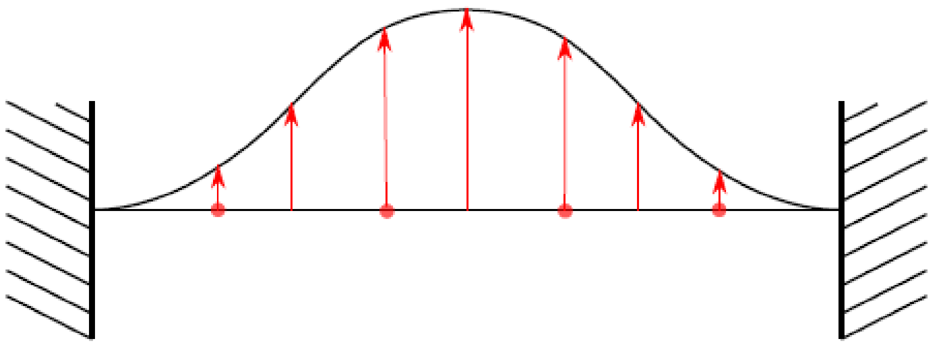
Classic testcase:
Clamped-Clamped beam



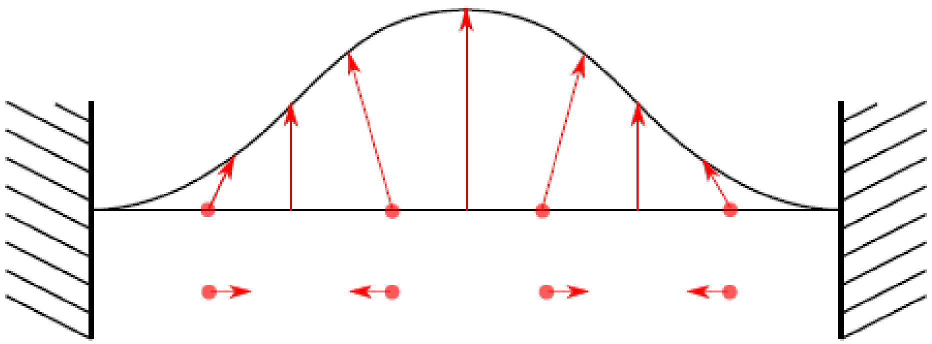


Membrane bending couplings

Linear modeshape



Nonlinear modeshape



Constrains the nonlinear structure on its linear modeshape

4th axial mode has to be included

Using the Linear mode as a basis:
cannot account for flexural-axial couplings!

$$\ddot{u}_1 + \omega_1^2 u_1 + g_{14}^1 u_1 p_4 + h_{111}^1 u_1^3 = 0$$

$$\ddot{u}_4 + \omega_4^2 u_4 + g_{11}^4 u_1^2 = 0$$

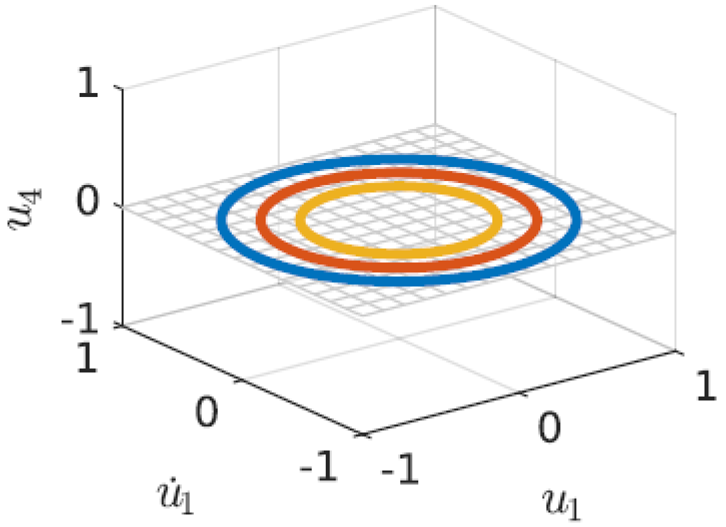
Invariant breaking terms

$$\ddot{u}_4 + \omega_4^2 u_4 + g_{11}^4 u_1^2 = 0$$

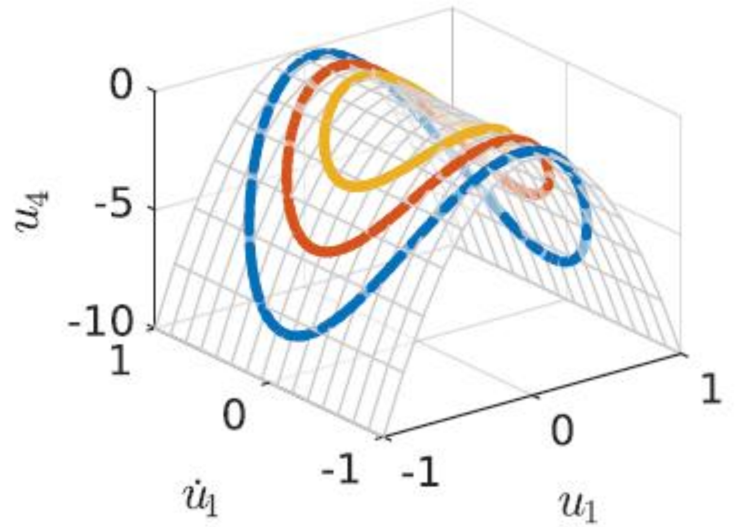
g_{11}^4 is the quadratic invariant breaking term

First modal amplitude nonzero $\Rightarrow$ the coupled mode: forcing like term in u_1^2

Linear Invariance



Nonlinear Invariant manifold



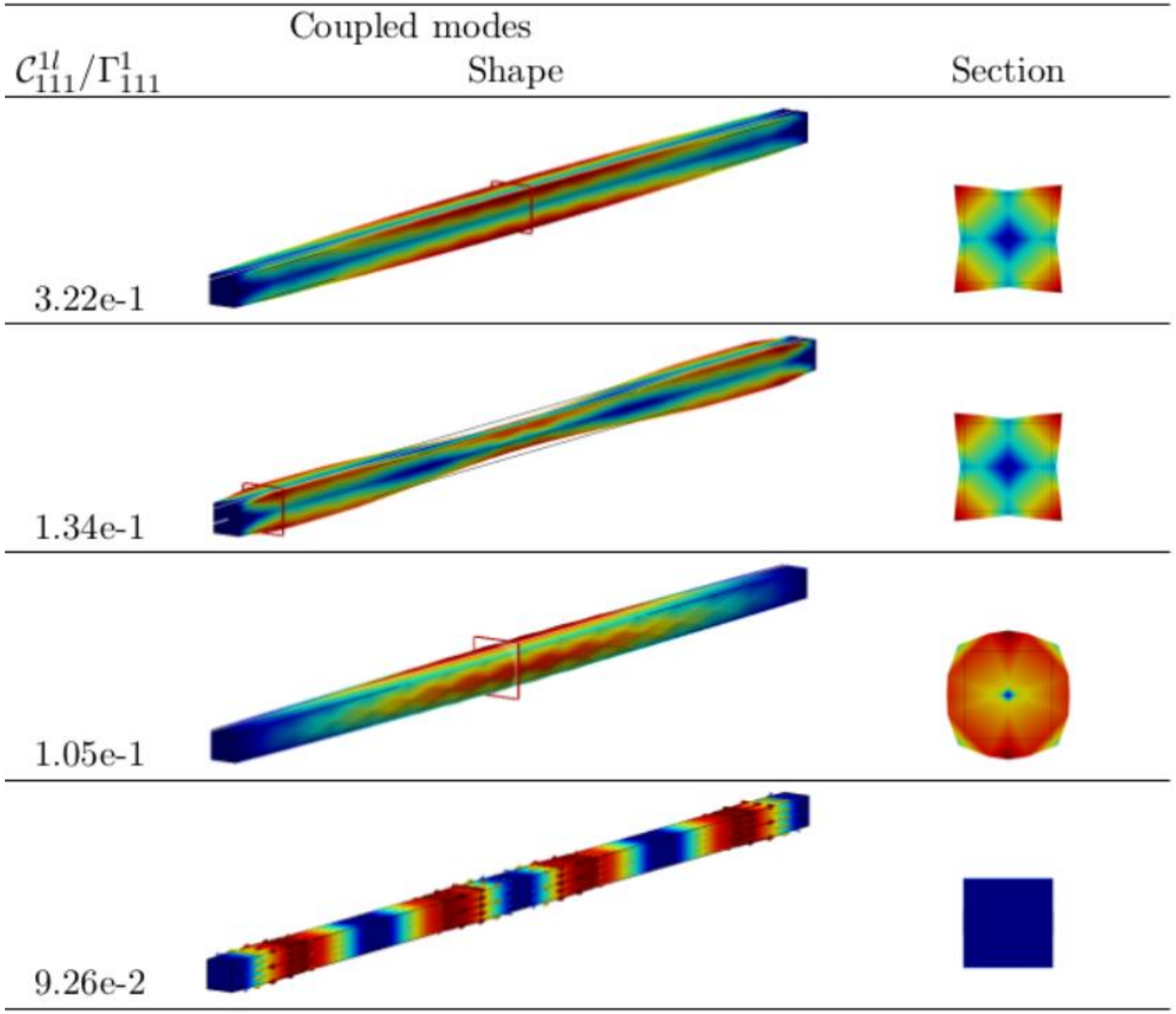
4th axial mode has to be included

So what's the problem?

- Many modes coupled to a single master mode
- Distributed all over the full spectrum

(Q1) How can we find them?

(Q2) How can we 'enslave' them?



Outline

- ROM by condensation
- Linear VS <u>Nonlinear</u> ROMs
- ROM by projection
- ROM by invariance

(A2) Static condensation (1D beam elements)

$$\ddot{u}_1 + \omega_1^2 u_1 + g_{14}^1 u_1 p_4 + h_{111}^1 u_1^3 = 0$$

$$\ddot{u}_4 + \omega_4^2 u_4 + g_{11}^4 u_1^2 = 0$$

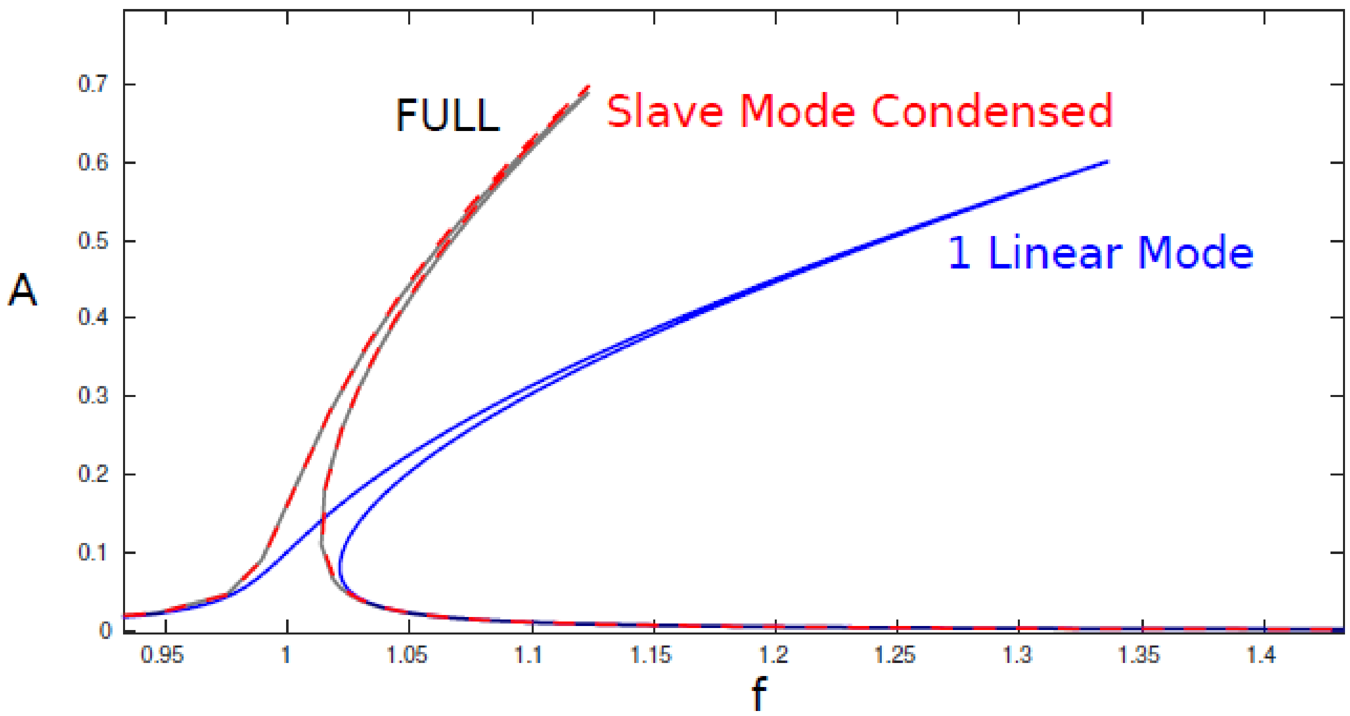
Now, perfect!

Slow-Fast Assumption: $\boxed{\omega_4 \gg \omega_1}$

$$u_4 = -\frac{g_{11}^4}{\omega_4^2} u_1^2$$

$$\ddot{u}_1 + \omega_1^2 u_1 + \left(h_{111}^1 - \frac{g_{14}^1 g_{11}^4}{\omega_4^2}\right) u_1^3 = 0$$

Correction Factor: $\frac{g_{14}^1 g_{11}^4}{\omega_4^2}$



(A2) Static condensation (3D beam elements)

$$\ddot{u}_1 + \omega_1^2 u_1 + g^1_{1s} u_1 u_s + h^1_{111} u_1^3 = 0$$

$$\ddot{u}_s + \omega_s^2 u_s + g^s_{11} u_1^2 = 0$$

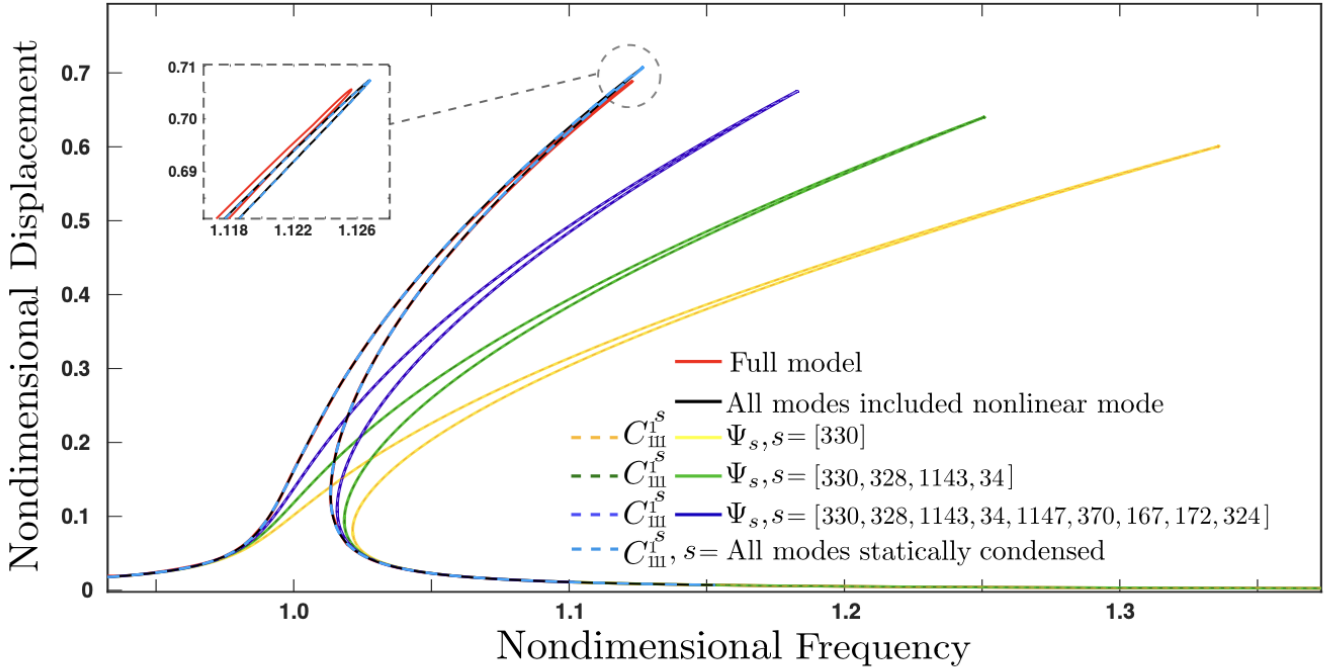
(Q1) How can we find them?

Slow-Fast Assumption: $\boxed{\omega_s \gg \omega_1}$

$$u_s = -\frac{g^s_{11}}{\omega_s^2} u_1^2$$

$$\ddot{u}_1 + \omega_1^2 u_1 + \Big(h^1_{111} - \sum \frac{g^1_{1s} g^s_{11}}{\omega_s^2}\Big) u_1^3 = 0$$

Correction Factor: $\sum \frac{g^1_{1s} g^s_{11}}{\omega_s^2}$



(A1) EITHER.. Implicit condensation (ICE)

$$\mathbf{KU} + \mathbf{GUU} + \mathbf{HUUU} = \mathbf{M}\phi_1\alpha$$

$$\boxed{\alpha = \alpha(u_1)}$$

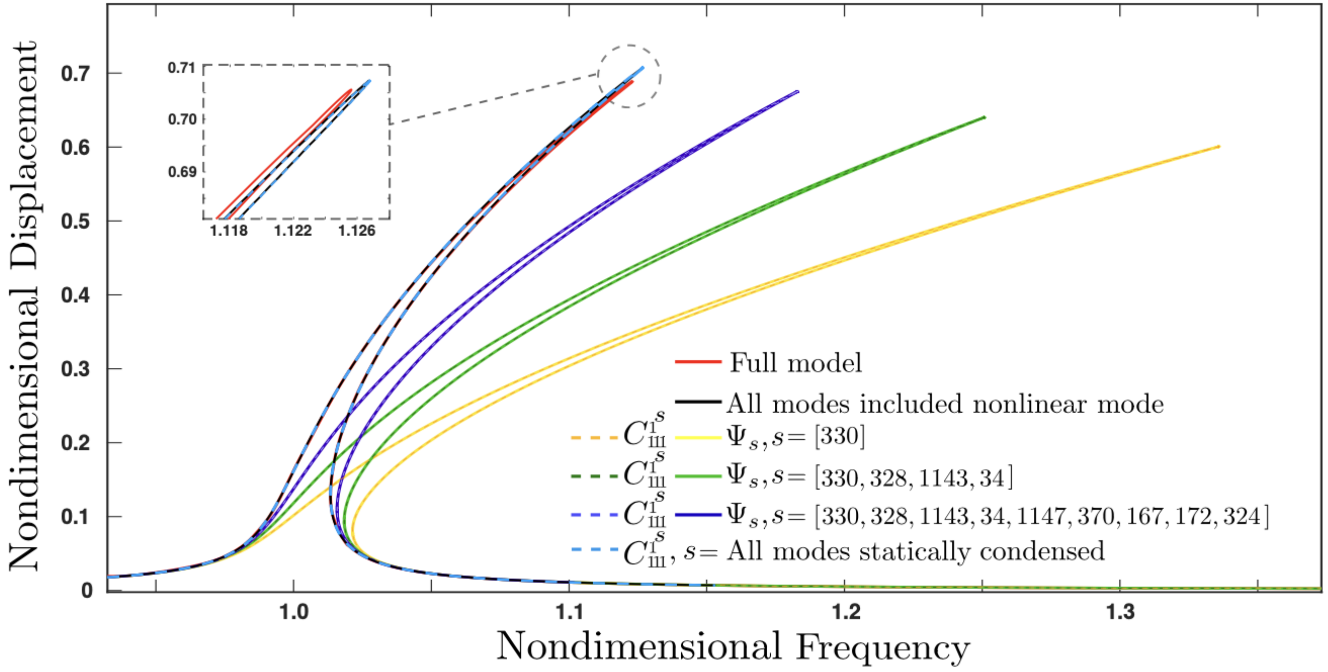
$$\ddot{u}_1 + \alpha(u_1) = 0$$

$$\omega_1^2 u_1 + g_{1s}^1 u_1 u_s + h_{111}^1 u_1^3 = \alpha$$

$$\omega_s^2 u_s + g_{11}^s u_1^2 = 0$$

$$u_s = -\frac{g_{11}^s}{\omega_s^2} u_1^2$$

$$\omega_1^2 u_1 + (h_{111}^1 - \sum \frac{g_{1s}^1 g_{11}^s}{\omega_s^2}) u_1^3 = \alpha$$



(A1) OR.. Static Modal Derivative (SMD)

Perturbed eigenproblem:

$$\left(-\tilde{\omega}_1^2(u_1)\mathbf{M}+\tilde{\mathbf{K}}(u_1)\right)\tilde{\phi}_1(u_1)=\mathbf{0}$$

Neglecting inertial contribution:

$$\mathbf{K}\frac{\partial\tilde{\phi}_1(u_1)}{\partial u_1}+\frac{\partial\tilde{\mathbf{K}}(u_1)}{\partial u_1}\phi_1=\mathbf{0}$$

$$\frac{\partial\tilde{\mathbf{K}}}{\partial u_1}\phi_1=\mathbf{G}\phi_1\phi_1\ 2$$

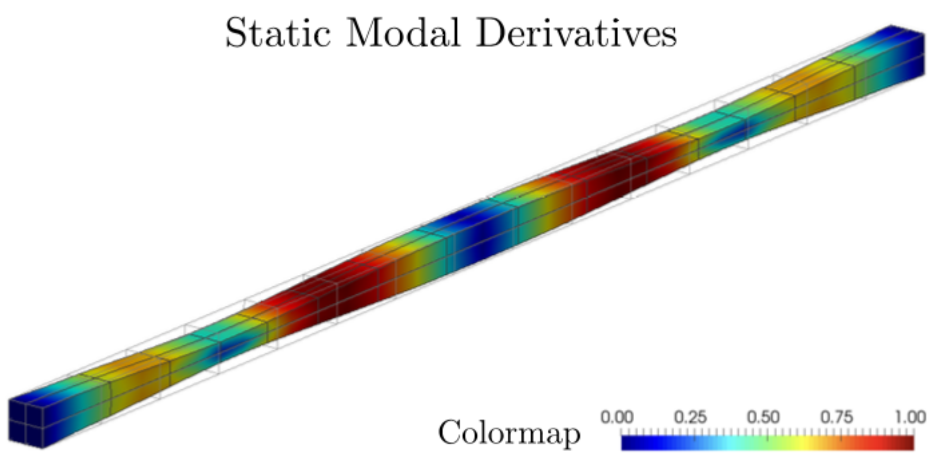
$$\theta_{11}=\frac{\partial\tilde{\phi}_1(u_1)}{\partial u_1}$$

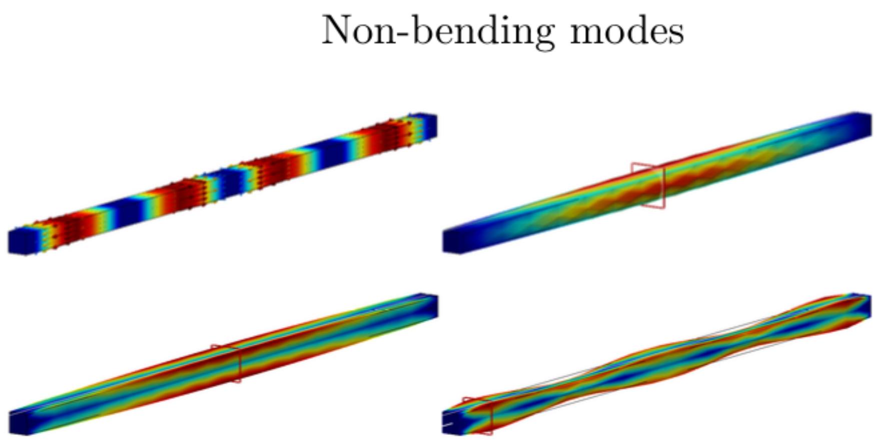
Static modal derivative:

$$\theta_{11}=-\mathbf{K}^{-1}\mathbf{G}\phi_1\phi_1\ 2$$

(A1) OR.. Static Modal Derivative (SMD)

$$\theta_{11} = -\mathbf{K}^{-1}\mathbf{G}\phi_1\phi_1$$





Inclusion of all slave modes under the slow fast assumption:

$$\theta_{11} = -\sum 2\phi_s \frac{g_{11}^s}{\omega_s^2}$$

Recap so far

- Finding the coupling (A1)
  - either by nonlinear ***static*** analysis: ICE/POD (***inverse*** problem)
  - or by ***direct*** force computation: SMD (direct problem + ***linear solve***)

- Including the found couplings (A2)
  - either keeping the found modes in as an additional DOF (linear ROM)
  - or ***keeping*** the ***aggregated*** vector in as an additional DOF: dual modes (linear ROM)
  - Or ***condensing*** the couplings and only keeping master: (nonlinear ROM)

- Coming up other nonlinear ROMs (A2)
  - ROM by ***projection***: QM/IC-ICE
  - ROM by ***invariance***: SP/NF/SSM/DNF/DPIM

(A2) Intuition on quadratic manifold (QM)

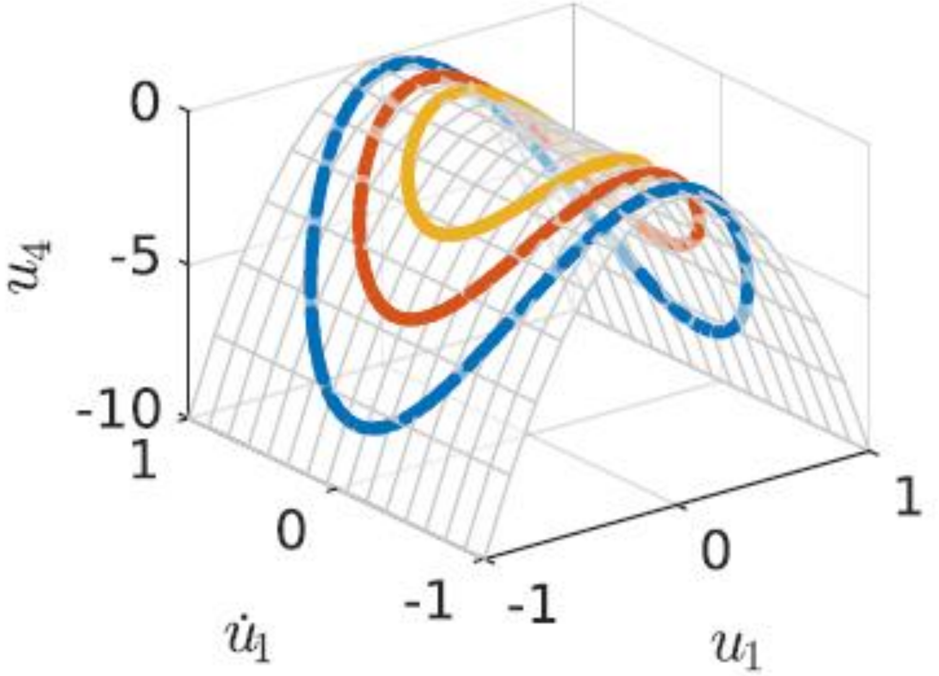
$$\theta_{11} = \frac{\partial \tilde{\phi}_1(u_1)}{\partial u_1}$$

$$\theta_{11} = -\sum 2\phi_s \frac{g_{11}^s}{\omega_s^2}$$

$$\frac{1}{2}\theta_{11} u_1^2 = -\sum \phi_s \frac{g_{11}^s}{\omega_s^2} u_1^2$$

Nonlinear Change of coordinates:

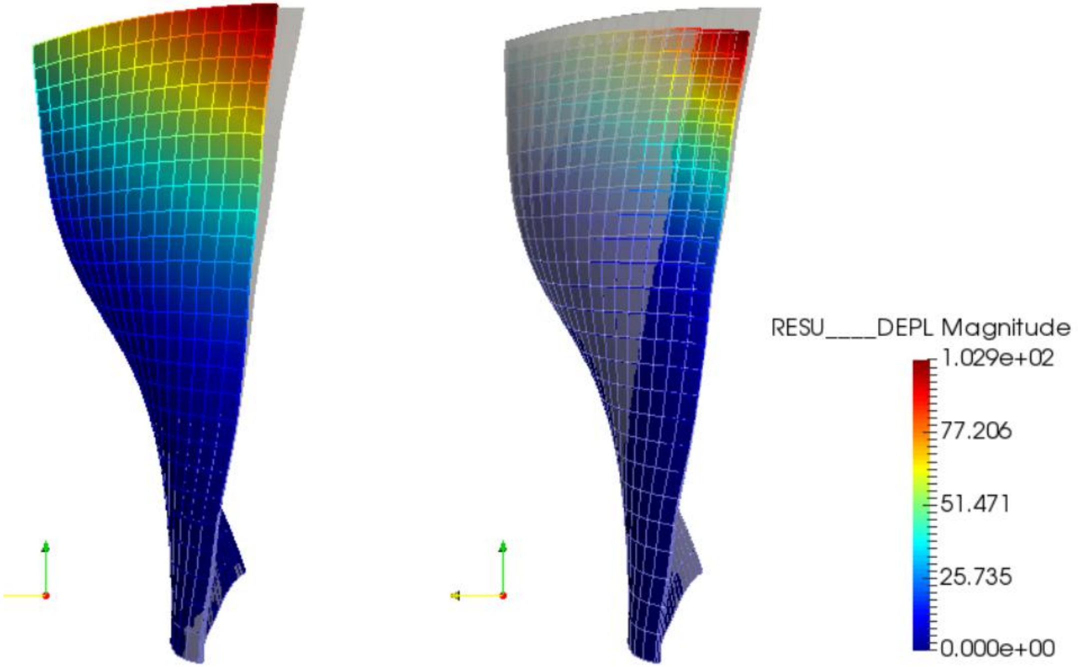
$$\boxed{\mathbf{U}(u_1) = \phi\, u_1 + \frac{1}{2}\,\theta\, u_1^2}$$



(A2) Intuition on quadratic manifold (QM)

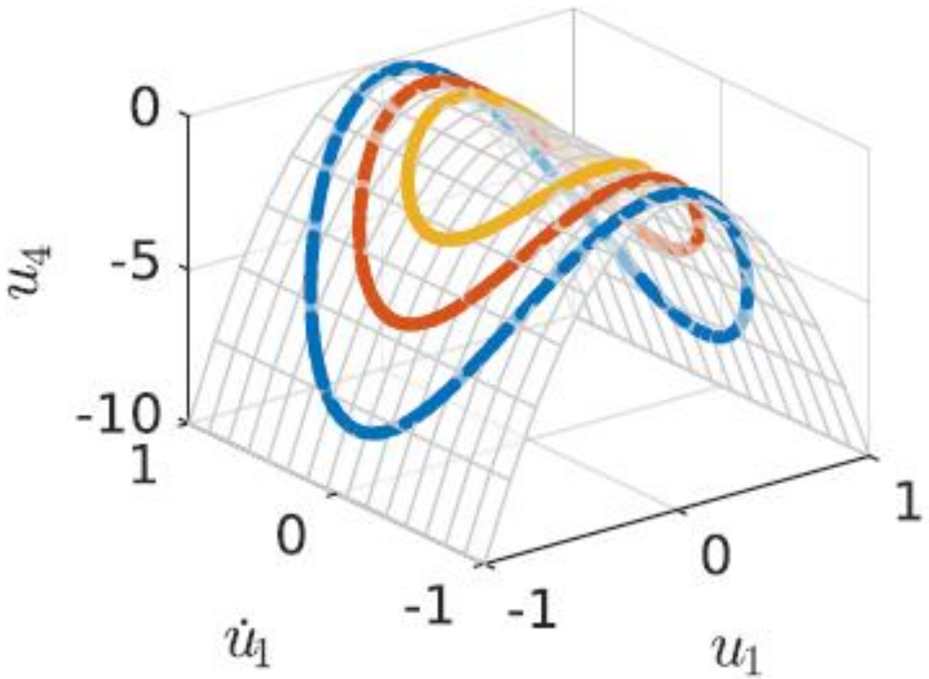
Mode: ϕ

Modal derivative: θ



Nonlinear Change of coordinates:

$$\boxed{\mathbf{U}(u_1) = \phi\, u_1 + \frac{1}{2}\,\theta\, u_1^2}$$



ROM by projection (QM / IC-ICE)

Nonlinear change of coordinates

$$U(u_1) = \phi \; u_1 + \frac{1}{2}\theta \; u_1^2$$

$$\ddot{U}(u_1) = \phi \; \ddot{u}_1 + \frac{1}{2}\theta \; (\ddot{u}_1 u_1 + \dot{u}_1^2)$$

Projection on tangent space!

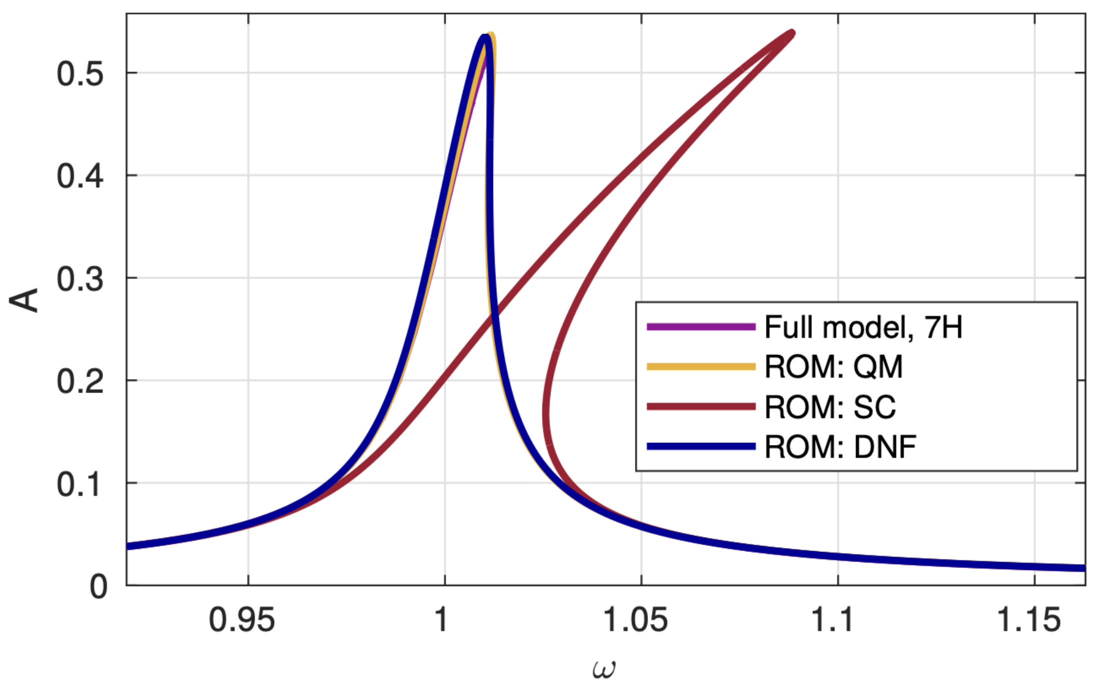
$$\nabla U(u_1) = \phi \; + \theta \; u_1$$

Reduced dynamics equation:

$$(\nabla U(u_1))^T \left(\mathbf{M}\ddot{U}(u_1) + \mathbf{K}\mathbf{U}(\mathbf{u_1}) + \mathbf{G}U(u_1)U(u_1) + \mathbf{H}U(u_1)U(u_1)U(u_1)\right) = \mathbf{0}$$

(A2) Inertial nonlinearity in QM

Cantilever beam



Nonlinear Change of coordinates:

$$\boxed{\mathbf{U}(u_1) = \phi\, u_1 + \frac{1}{2}\,\theta\, u_1^2}$$

nonlinear inertia component

$$\ddot{U}(u_1) = \phi\,\ddot{u}_1 + \frac{1}{2}\theta\,(\ddot{u}_1 u_1 + \dot{u}_1^2)$$

SC: $\ddot{u}_1 + \omega_1^2 u_1 + (h_{111}^1 - \sum \frac{g_{1s}^1 g_{11}^s}{\omega_s^2}) u_1^3 = 0$

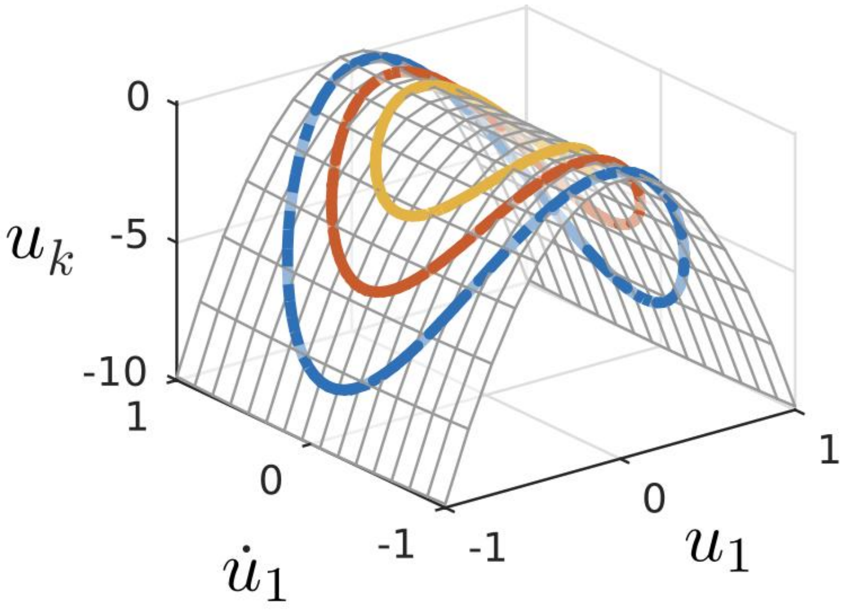
QM: $\ddot{u}_1 + \omega_1^2 u_1 + (h_{111}^1 - \sum \frac{g_{1s}^1 g_{11}^s}{\omega_s^2}) u_1^3 + \sum \frac{2 g_{1s}^1 g_{11}^s}{\omega_s^4} (\ddot{u}_1 u_1^2 + u_1 \dot{u}_1^2) = 0$

Displacement only change of coordinates

Nonlinear change of coordinates

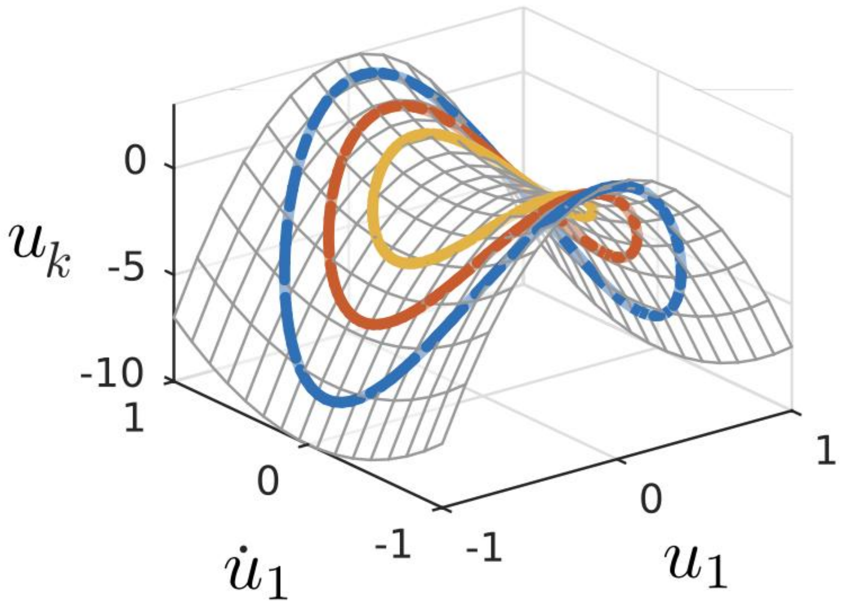
$$U(u_1) = \phi\, u_1 + \frac{1}{2}\theta\, u_1^2$$

Velocity independent manifold



Velocity dependent manifold

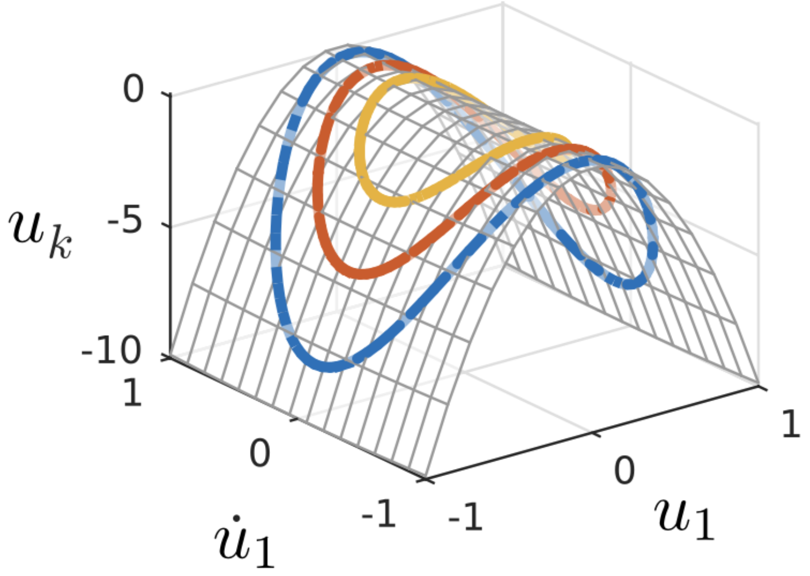
$$\mathbf{U} = \mathbf{U}(u_1, \dot{u}_1)$$



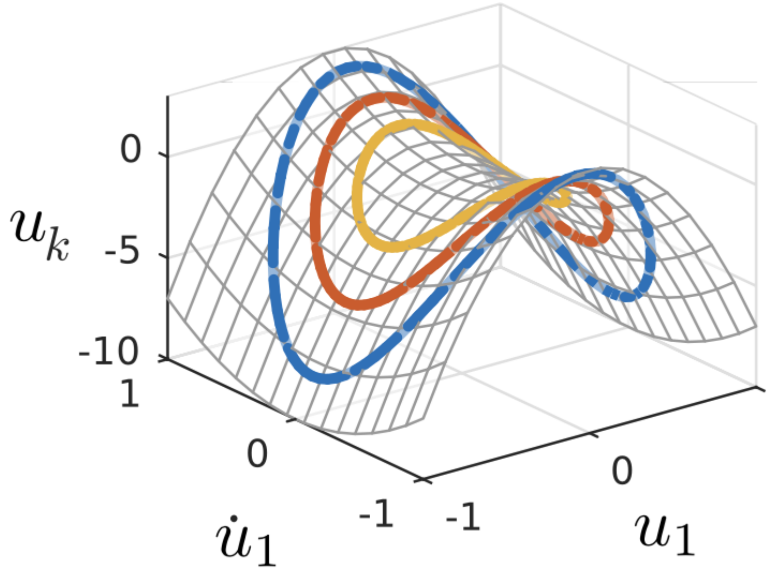
When is it needed?

When is the velocity dependence needed?

Velocity Independent Manifold



Velocity Dependent Manifold

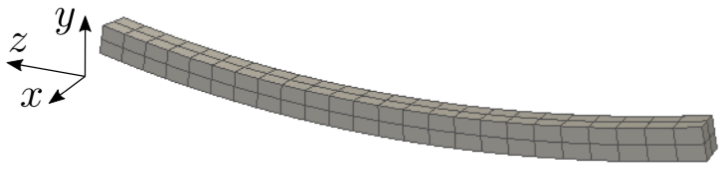


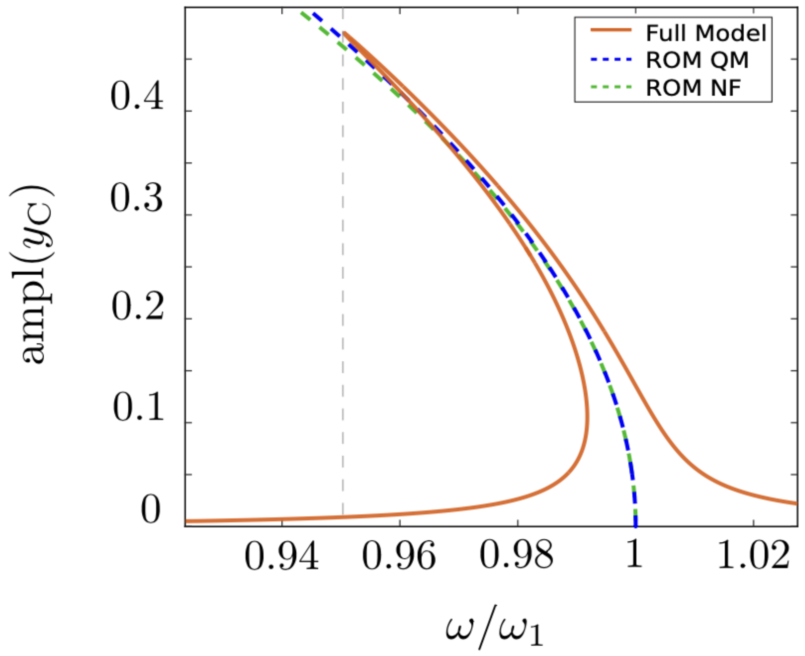
Velocity Independence: Frequency Coupled Mode $\gg$ Frequency Master Mode

$$\boxed{\omega_C > 3\omega_M}$$

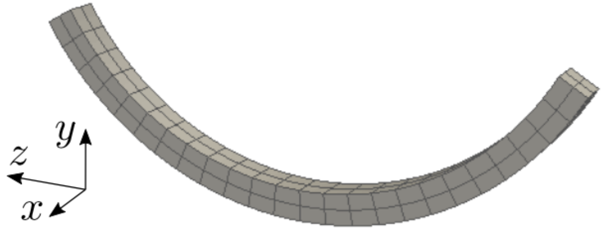
Example shallow VS non-shallow arch

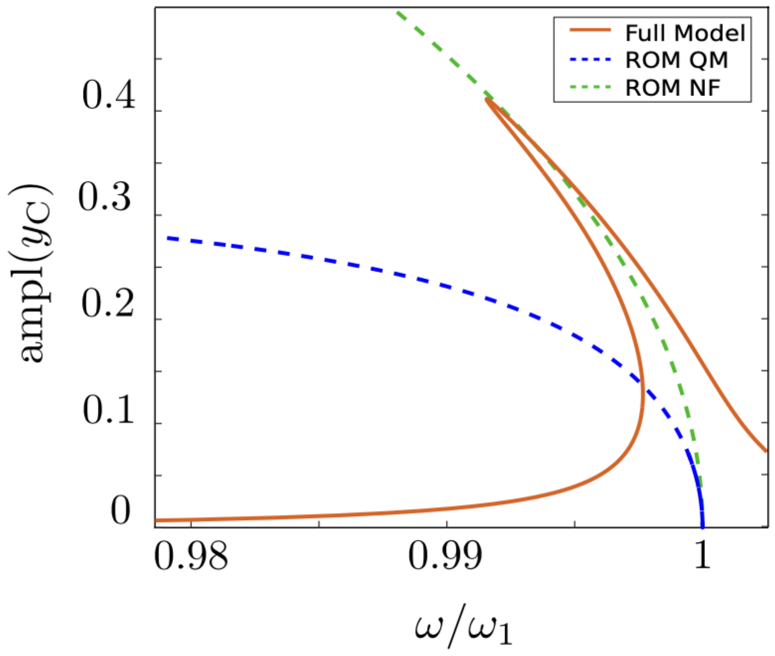
Shallow: $\boxed{\omega_C \approx 3.4\omega_M}$





Non-shallow: $\boxed{\omega_C \approx 1.7\omega_M}$





$$\mathbf{U} = \mathbf{U}(u_1, \dot{u}_1)$$

(A2) ROM by invariance (Shaw-Pierre SP)

Start by rewriting the system in first order:

$$\dot{\mathbf{U}} = \mathbf{V}$$

$$\mathbf{M}\dot{\mathbf{V}} + \mathbf{K}\mathbf{U} + \mathbf{G}\mathbf{U}\mathbf{U} + \mathbf{H}\mathbf{U}\mathbf{U}\mathbf{U} = \mathbf{0}$$

Change of coordinates in displacement and velocity:

$$\mathbf{U} = \mathbf{U}(u_1, v_1)$$

$$\mathbf{V} = \mathbf{V}(u_1, v_1)$$

Ansatz for reduced model (no projection needed):

$$\dot{u}_1 = v_1$$

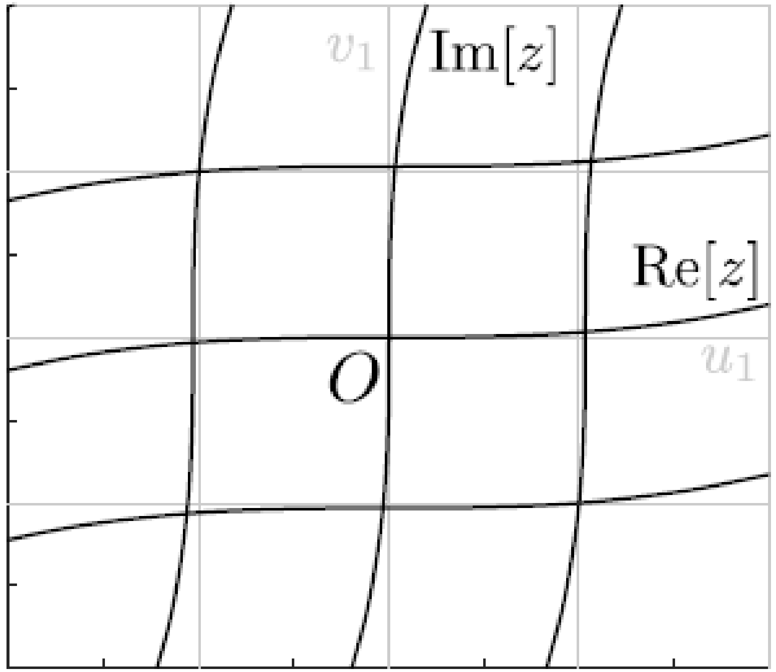
$$\dot{v}_1 + f(u_1, v_1) = 0$$

Time derivatives:

$$\dot{\mathbf{U}} = \partial_{u_1}\mathbf{U}\ v_1 - \partial_{v_1}\mathbf{U}\ f(u_1, v_1)$$

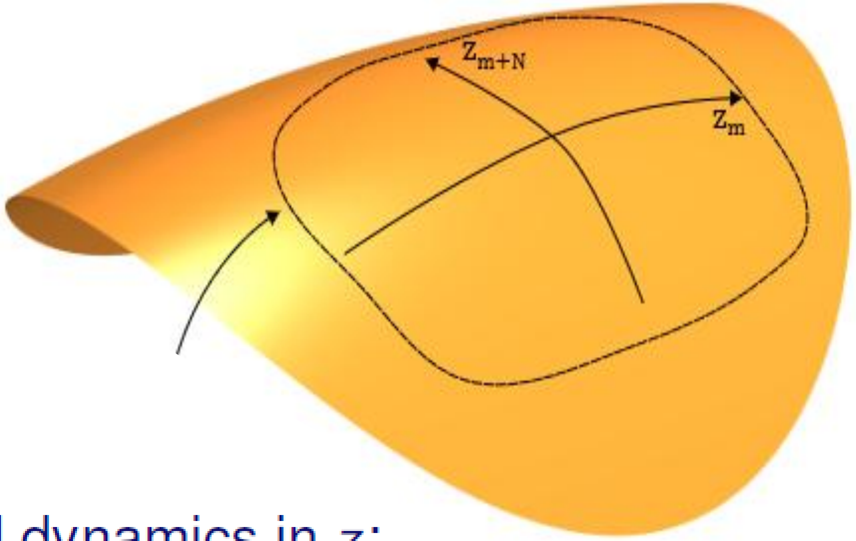
$$\dot{\mathbf{V}} = \partial_{u_1}\mathbf{V}\ v_1 - \partial_{v_1}\mathbf{V}\ f(u_1, v_1)$$

(A2) ROM by invariance (Normal Form Style)



Change of coordinates:
$\mathbf{U} = \mathbf{U}(z)$

Parametrisation of manifold as an embedding in the complex normal coordinates z



Reduced dynamics in z:
$\dot{z} = f(z)$

(A2) Invariance equation (SP/SSM/DNF/DPIM)

Change of coordinates:

$$\mathbf{U} = \mathbf{U}(z)$$

Reduced dynamics in z:

$$\dot{z} = f(z)$$

Time derivatives:

$$\dot{\mathbf{U}} = \nabla_z \mathbf{U} \cdot f;$$
$$\dot{\mathbf{V}} = \nabla_z \mathbf{V} \cdot f$$

Invariance Equation:

$$\mathbf{M}\nabla_z \mathbf{U} \cdot f = \mathbf{M}\mathbf{V}$$
$$\mathbf{M}\nabla_z \mathbf{V} \cdot f + \mathbf{K}\mathbf{U} + \mathbf{G}\mathbf{U}\mathbf{U} + \mathbf{H}\mathbf{U}\mathbf{U}\mathbf{U} = \mathbf{0}$$

Homological order p:

$$(\sigma^2 \mathbf{M} + \mathbf{K})\Psi^{(p)} = F^{(p)}$$

Homological order 2

$$\theta = (-\mathbf{K})^{-1}\mathbf{G}_{11}$$
$$\chi = ((2\omega_1)^2 \mathbf{M} - \mathbf{K})^{-1}\mathbf{G}_{11}$$

(A2) Velocity dependent manifold

SP:

$$\mathbf{U} = \phi u_1 + \mathbf{a} u_1^2 + \mathbf{b} v_1^2$$

QM:

$$\mathbf{U} = \phi u_1 + \theta u_1^2$$

Non-intrusive FE computation of **a**, **b**:

$$\theta = (-\mathbf{K})^{-1}\mathbf{G}_{11}$$ Standard Static Analysis

$$\chi = ((2\omega_1)^2\mathbf{M} - \mathbf{K})^{-1}\mathbf{G}_{11}$$ Non-Standard Analysis

$$\mathbf{a} = \frac{\theta + \chi}{2} \qquad \mathbf{b} = \frac{\theta - \chi}{2\omega_1^2}$$

Slow-fast: $\chi \approx \theta, \quad \mathbf{a} \approx \theta \quad \mathbf{b} \approx \mathbf{0}$

Recap

Starting from the equation of motion:

$$\mathbf{M\ddot{U}} + \mathbf{KU} + \mathbf{GUU} + \mathbf{HUUU} = \mathbf{0} \tag{4}$$

$\boxed{\text{Change of coordinates}}$

Modal ROM:

$$u_s = a_s^{(2)} u_1^2 + \dots$$

Direct ROM:

$$\mathbf{U} = \phi_1 u_1 + \Psi^{(2)} u_1^2 + \dots$$

Sought **geometry** of the approximated manifold is given by the unknown $a_s^{(2)}$ or $\Psi^{(2)}$

$\boxed{\text{Reduced dynamics}}$

$$\ddot{u}_1 + \omega_1^2 u_1 + f^{(2)} u_1^2 + \dots = 0$$

Sought **dynamics** on the approximated manifold is given by the unknown $f^{(2)}$

SC/ICE

Assume polynomial form in u_1:

$$\mathbf{U} = \phi_1 u_1 + \Psi^{(2)} u_1^2 + \dots$$

Neglect inertia and balance high order terms:

$$\mathbf{K}\Psi^{(2)} u_1^2 + \mathbf{G}\phi_1\phi_1 u_1^2 = \mathbf{0}$$

$$\vdots$$

Obtain reduced dynamics by projection of static forces:

$$\ddot{u}_1 + \phi_1^T(\mathbf{KU} + \mathbf{GUU} + \mathbf{HUUU}) =$$
$$\ddot{u}_1 + \omega_1^2 u_1 + f^{(2)} u_1^2 + f^{(3)} u_1^3 + \dots = 0$$

ICE method works similarly but in an implicit manner: easy to go to higher orders.

QM/IC-ICE

Assume polynomial form in u_1:

$$\mathbf{U} = \phi_1 u_1 + \Psi^{(2)} u_1^2 + \dots$$

Neglect inertia and balance high order terms:

$$\mathbf{K}\Psi^{(2)} u_1^2 + \mathbf{G}\phi_1\phi_1 u_1^2 = \mathbf{0}$$

$$\vdots$$

Obtain reduced dynamics by projection of whole e.o.m. on a tangent space:

$$(\phi_1^{\mathsf{T}} + u_1 \Psi^{(2)\mathsf{T}})(\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} + \mathbf{G}\mathbf{U}\mathbf{U} + \mathbf{H}\mathbf{U}\mathbf{U}\mathbf{U}) =$$
$$\ddot{u}_1 + \omega_1^2 u_1 + f_{NL}(u_1, \dot{u}_1, \ddot{u}_1) = 0$$

Appearance of inertial terms in acceleration and velocity

SP/SSM/DNF/DPIM

Assume polynomial form in u_1 and $\dot{u}_1$:

$$\mathbf{U} = \phi_1 u_1 + \Psi^{(2,0)} u_1^2 + \Psi^{(1,1)} u_1 \dot{u}_1 + \Psi^{(0,2)} \dot{u}_1^2 + \ldots$$

Assume reduced dynamics too: polynomial in u_1 and $\dot{u}_1$:

$$\ddot{u}_1 + \omega_1^2 u_1 + f^{(2,0)} u_1^2 + f^{(1,1)} u_1 \dot{u}_1 + f^{(0,2)} \dot{u}_1^2 + \ldots = 0$$

Balance same monomials in invariance equation:

$$\begin{aligned} &\mathbf{M}\nabla_z \mathbf{U} \cdot f = \mathbf{M}\mathbf{V} \\ &\mathbf{M}\nabla_z \mathbf{V} \cdot f + \mathbf{K}\mathbf{U} + \mathbf{G}\mathbf{U}\mathbf{U} + \mathbf{H}\mathbf{U}\mathbf{U}\mathbf{U} = 0 \end{aligned}$$

No need to enforce projection!

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Thank you for your attention