Nonlinear vibration testing: peculiarities and challenges

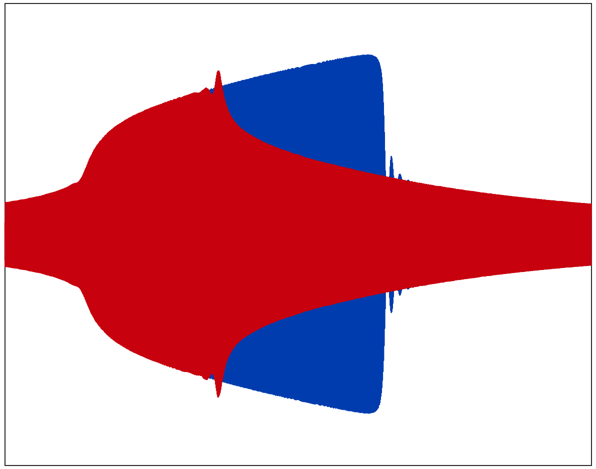
Ghislain Raze

Space Structures and Systems Lab

Aerospace and Mechanical Eng. Dept.

University of Liège, Belgium

g.raze@uliege.be



ADVANCED STRUCTURAL DYNAMICS – 17 Jun 2025

Nonlinear vibration testing?

Share your thoughts!

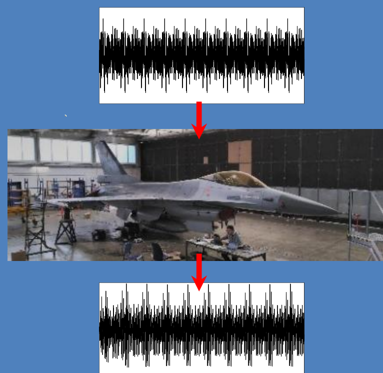
How familiar are you with vibration testing, nonlinear dynamics, and control?

https://app.wooclap.com/BQGYHZ

Experimental modal analysis: an industrial standard

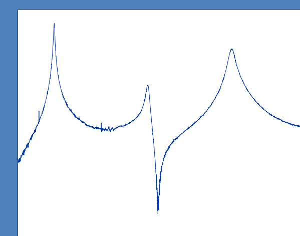
From raw data…

1. Testing



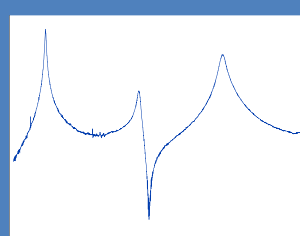
2. FRF estimation

$$H(\omega) = \frac{S_1(\omega)}{S_2(\omega)}$$



3. FRF fitting

$$\hat{\theta} = \arg\min \left| H(\omega) - \hat{H}(\omega;\theta) \right|_2$$



4. Modal characteristics

$$\hat{\theta}$$

$$\downarrow$$

$$\omega_n, \zeta_n, \phi_n$$

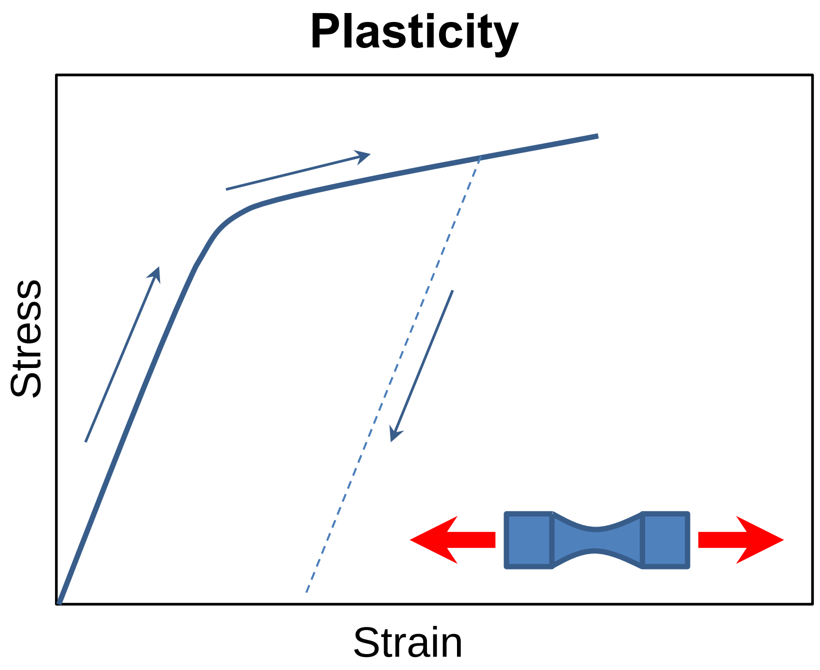
… to useful information.

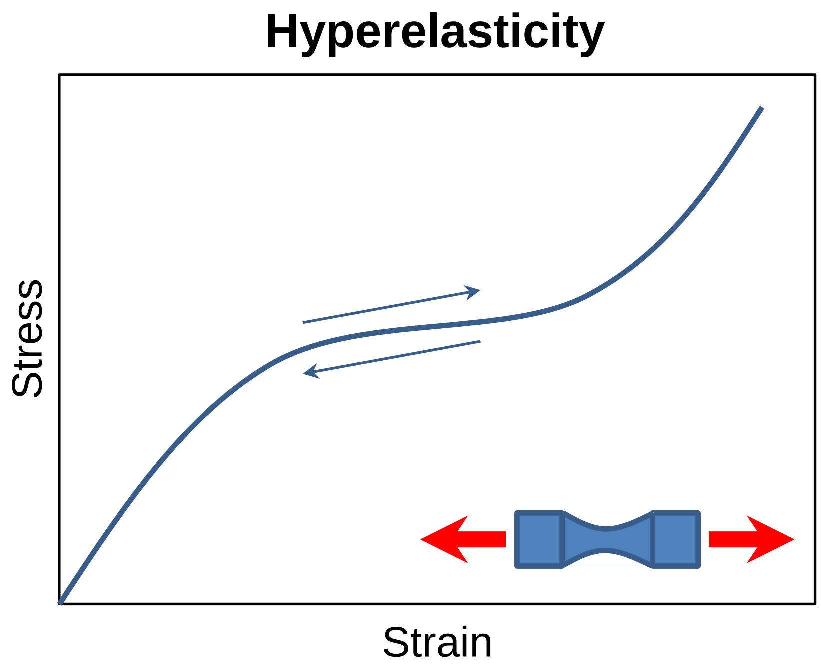
Main assumptions in linear structural dynamics

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t)$$

1. Linear elasticity	$\rightarrow$	Nonlinear materials
2. Small displacements and rotations	$\rightarrow$	Geometric nonlinearity Boundary conditions
3. Viscous damping	$\rightarrow$	Nonlinear damping
4. Time invariance	$\rightarrow$	Wear, environmental variations (temperature,…).

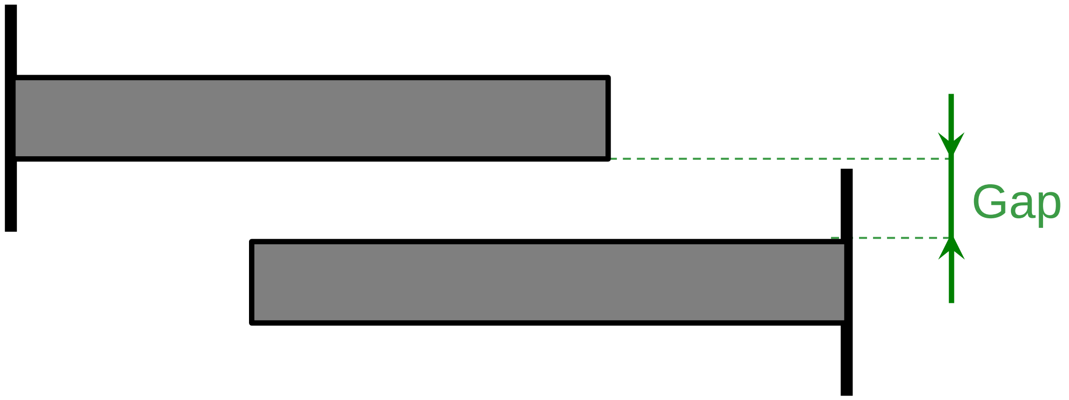
Nonlinear material behavior

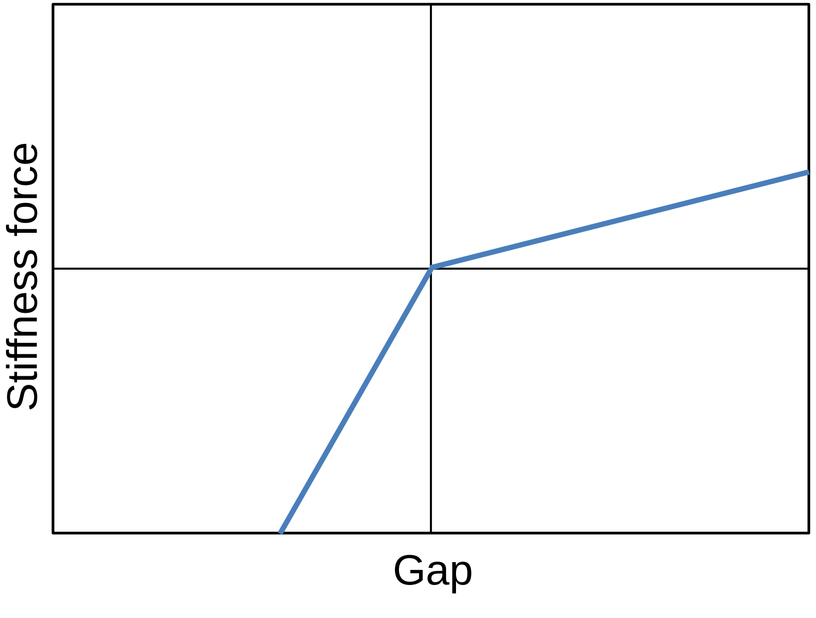




Materials behave nonlinearly when strained enough.

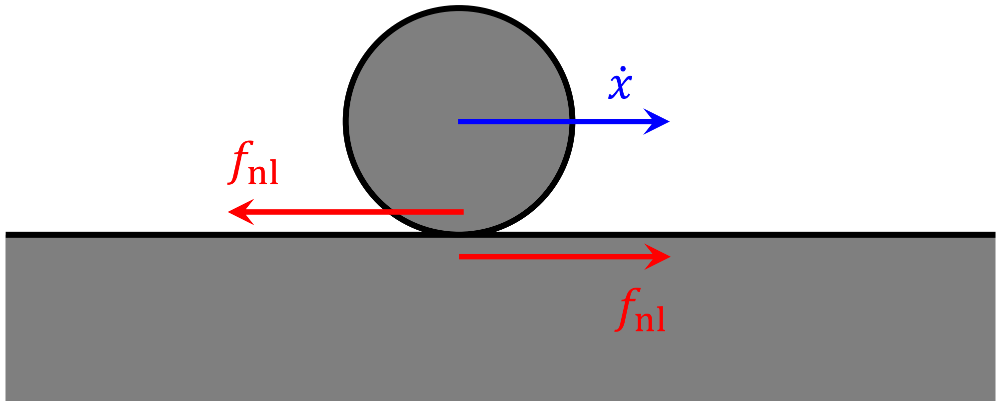
Nonlinear boundary conditions





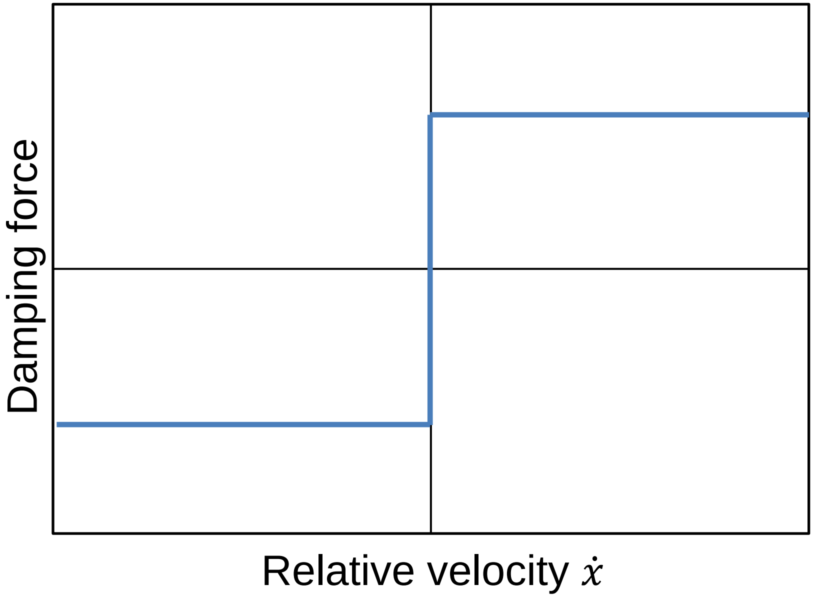
Anything with gaps or clearances can exhibit contact and impacts.

Nonlinear damping

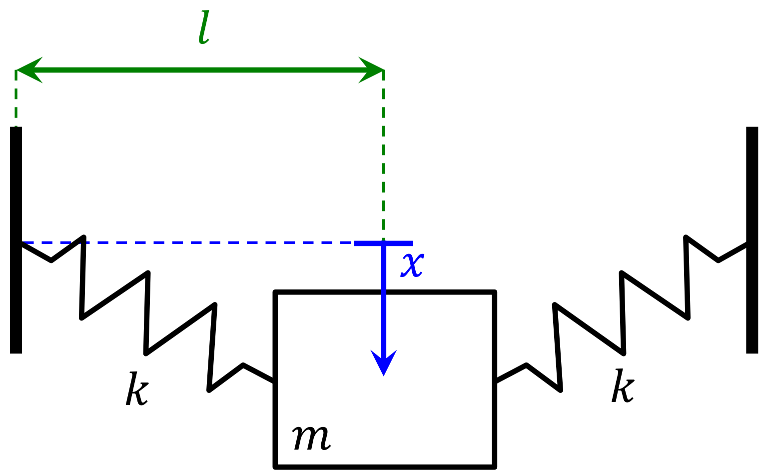


Solids rubbing against each other will exhibit friction, e.g., Coulomb damping.

Any bolted or jointed connection can exhibit friction.



A mass with two linear springs



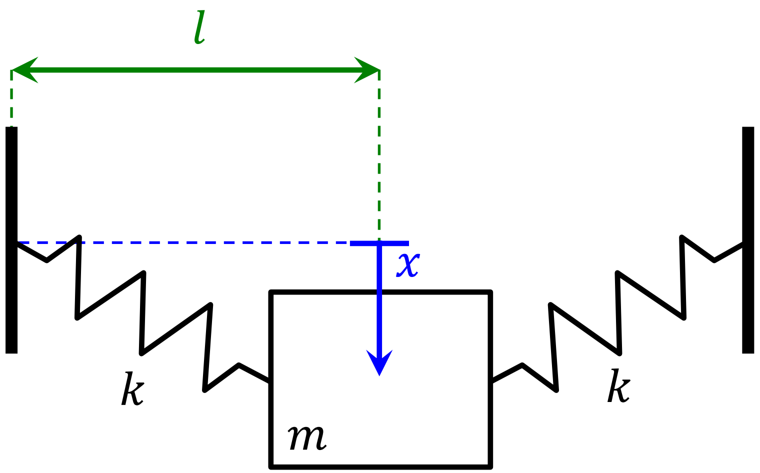
Let's consider a mass m connected to two linear springs of stiffness k of rest length l_0 moving vertically along x. The kinetic and potential energies are

$$\mathcal{T} = \frac{1}{2} m\dot{x}^2, \qquad \mathcal{V} = k\left(\sqrt{x^2 + l^2} - l_0\right)^2, \qquad \mathcal{L} = \mathcal{T} - \mathcal{V}$$

and the equations of motion are derived using Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + 2k\left(1 - \frac{l_0}{\sqrt{x^2 + l^2}}\right)x = 0$$

Geometric nonlinearities



$$m\ddot{x} + 2k\left(1 - \frac{l_0}{\sqrt{x^2 + l^2}}\right)x = 0$$

We can use a Taylor development of the stiffness force:

$$2k\left(1 - \frac{l_0}{\sqrt{x^2 + l^2}}\right)x = 2k\left(1 - \frac{l_0}{l}\right)x + \frac{k l_0}{l^3}x^3 + O(x^5)$$

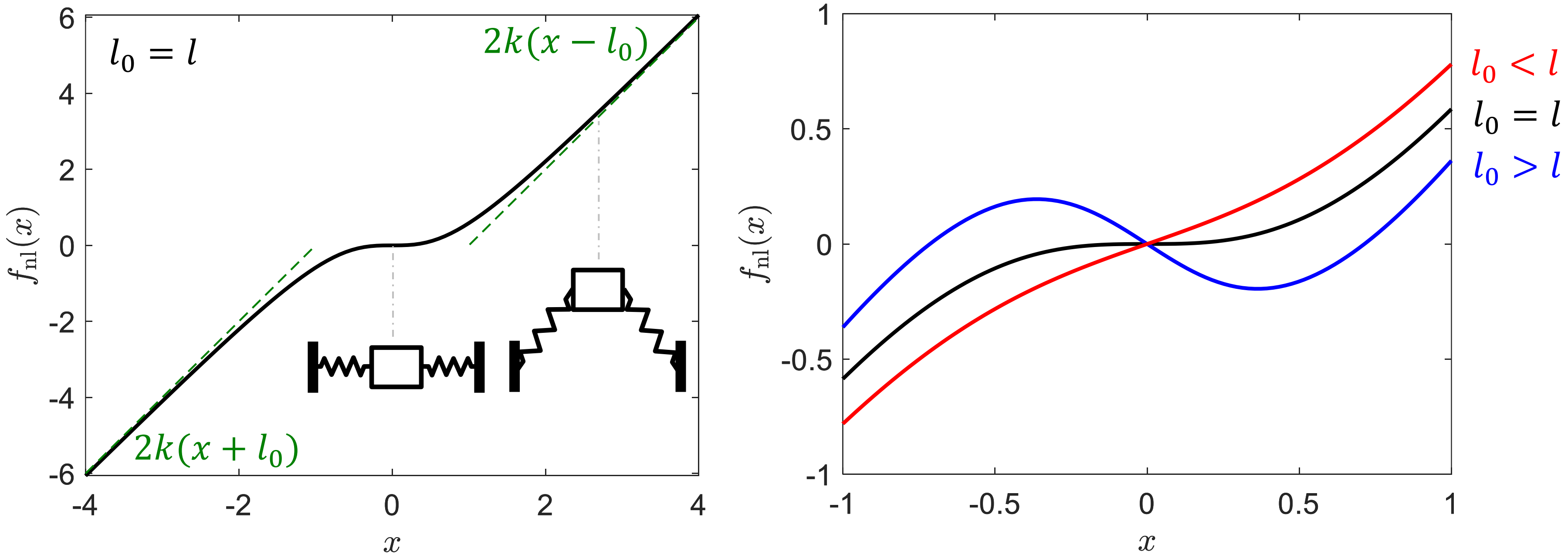
Consider $l = l_0$. If we linearize this equation, we get

$$m\ddot{x} = 0$$

i.e., a free mass!

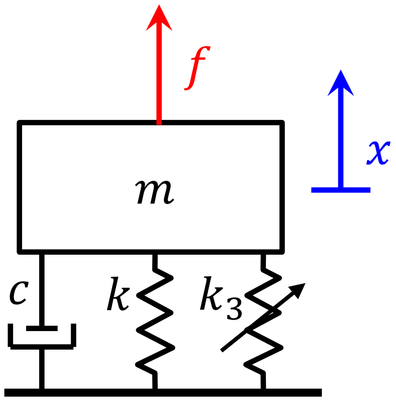
Geometrically nonlinear stiffness

$$f_{\mathrm{nl}}(x) = 2k\left(1 - \frac{l_0}{\sqrt{x^2 + l^2}}\right)x$$



The dynamic effects of this nonlinear stiffness are complicated to analyze, but we can capture them with the next term in the Taylor series (i.e., cubic).

A toy model: the Duffing oscillator



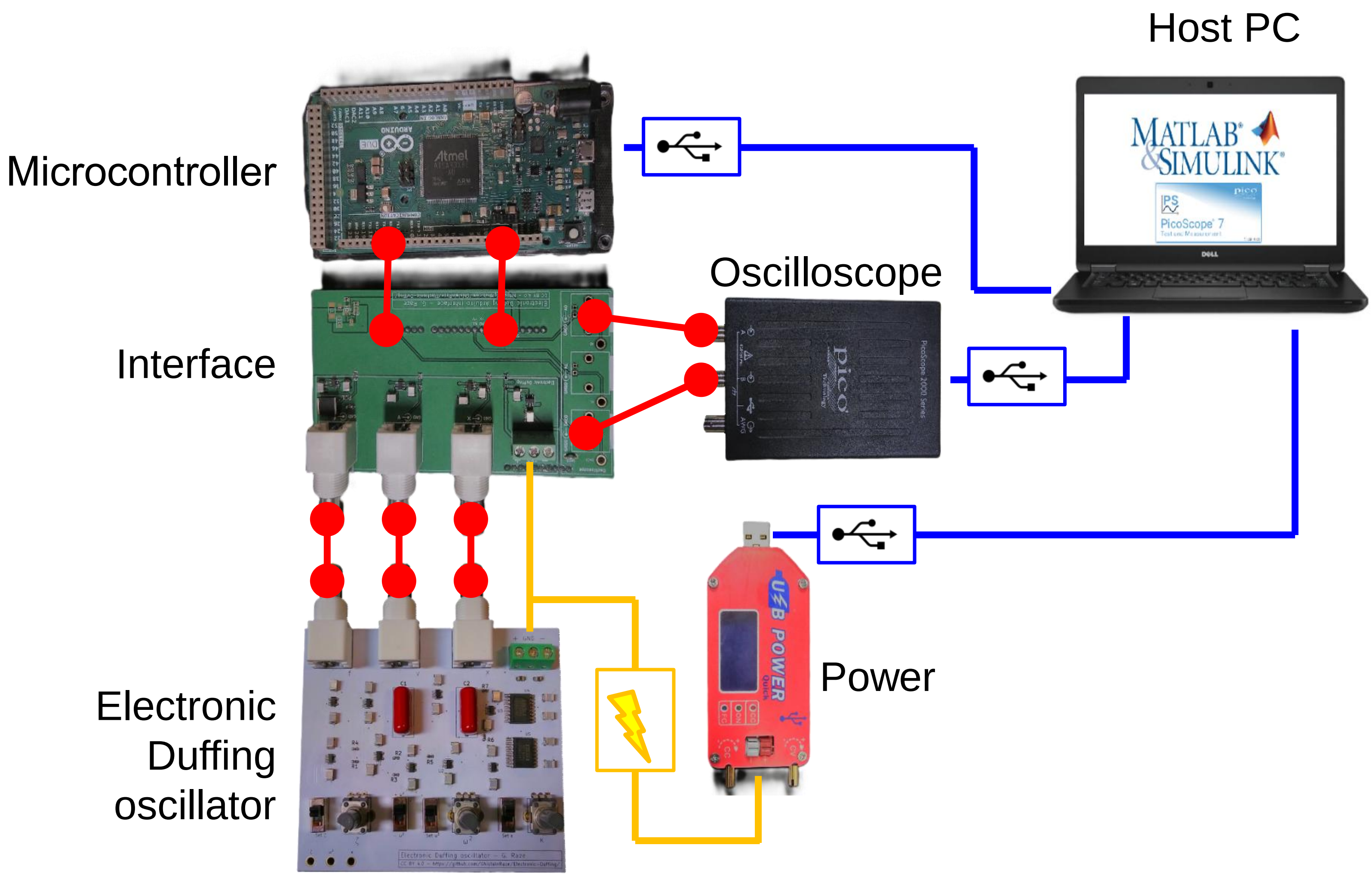
We can capture the essential features of a geometric nonlinearity by adding a cubic stiffness to the classical harmonic oscillator. The resulting oscillator is often known as the Duffing oscillator. It is governed by

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_3x^3(t) = f(t)$$

where m, c and k are the mass, damping coefficient and stiffness coefficient of the oscillator, k_3 its nonlinear stiffness coefficient, x its displacement and f the external force applied to it.

What is the impact of this cubic stiffness term on the dynamics? Let's find out together!

An electronic Duffing setup

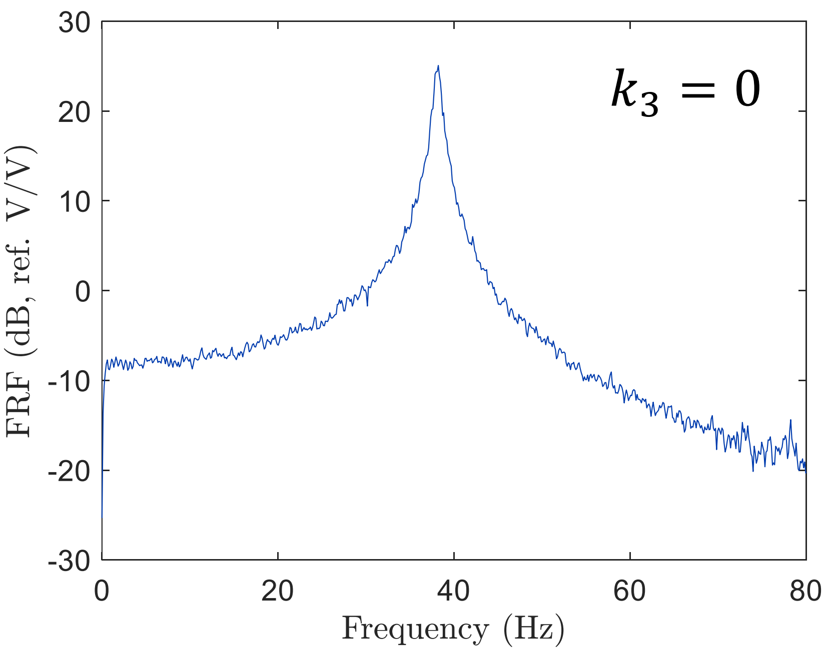


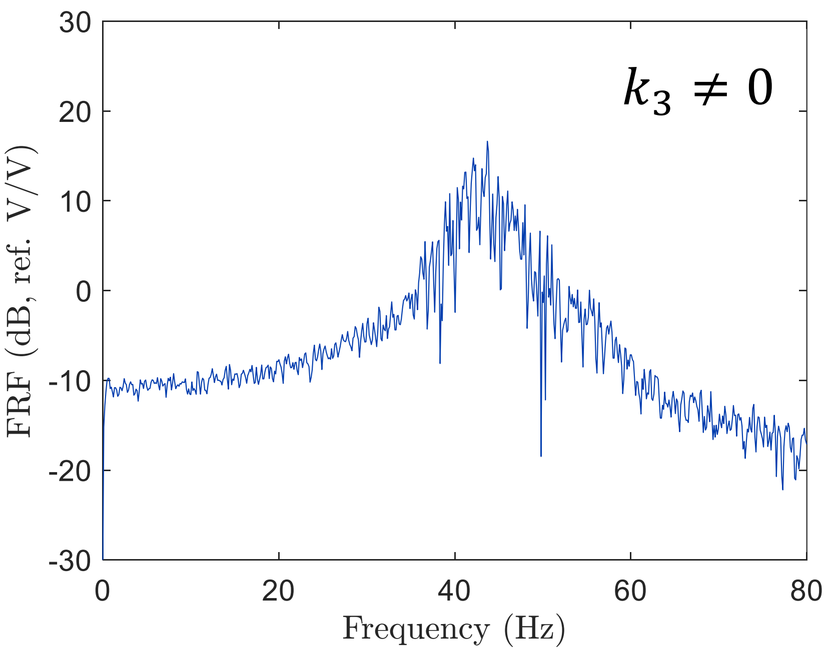
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_3x^3(t) = f(t)$$

https://github.com/GhislainRaze/Electronic-Duffing/

How does experimental modal analysis fare?

We can compute the FRF under broadband excitation conditions.





Experimental modal analysis falls short

Applying experimental modal analysis to the Duffing oscillator does not work!

Why is that?

Because the system is nonlinear!

Let’s now investigate

- the general characteristics of nonlinear systems
- how we can recognize nonlinear behavior
- and what we can do about that

What is the problem?

Challenges associated with experimental nonlinear dynamics

Reminder: how do we compute forced responses?

We wish to find the steady-state motion of the linear oscillator

$$m\ddot{x} + c\dot{x} + kx = f\,\sin(\omega t)$$

Let us assume that the response is periodic and single-harmonic

$$x(t) = a\,\sin(\omega t + \phi)$$

we then get

$$(k - \omega^2 m)a\sin(\omega t + \phi) + c\omega a\cos(\omega t + \phi) = f\sin(\omega t)$$

Balancing the sine and cosine terms yields

$$(k - \omega^2 m)a\cos(\phi) - c\omega a\sin(\phi) = f$$

$$(k - \omega^2 m)a\sin(\phi) + c\omega a\cos(\phi) = 0$$

Frequency response function

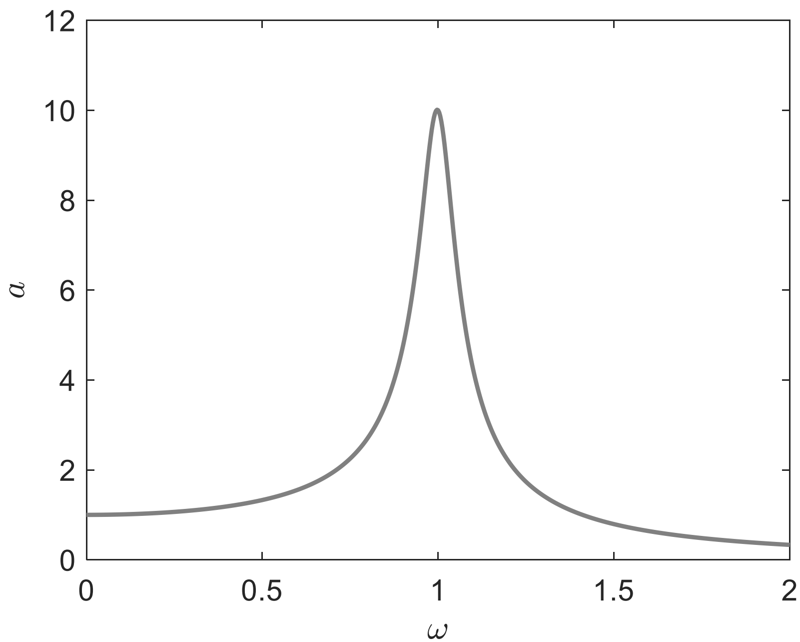
These two equations are equivalent to

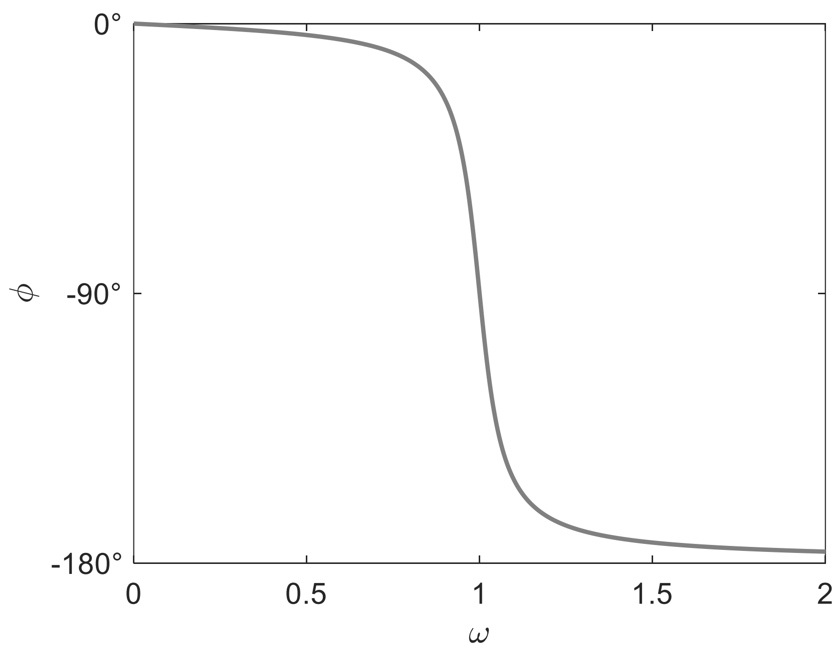
$$(k - \omega^2 m)a = f\cos(\phi)$$

$$c\omega a = -f\sin(\phi)$$

Adding their squares,

$$\left((k - \omega^2 m)^2 + c^2\omega^2\right)a^2 = f^2$$





How to solve the Duffing equation?

We wish to find or approximate the steady-state solutions of

$$m\ddot{x} + c\dot{x} + kx + k_3x^3 = f\,\sin(\omega t)$$

Let us assume that the response is periodic and single-harmonic

$$x(t) = a\,\sin(\omega t + \phi)$$

If we operate a time shift $t = \tau - \phi/\omega$,

$$(k - \omega^2 m)a\,\sin(\omega\tau) + c\omega a\,\cos(\omega\tau) + \frac{k_3}{4}a^3(3\,\sin(\omega\tau) + \sin(3\omega\tau))$$
$$= f\,(\sin(\omega\tau)\cos(\phi) - \cos(\omega\tau)\sin(\phi))$$

Balancing the harmonics and neglecting the third one yields

$$(k - \omega^2 m)a + \frac{3k_3}{4}a^3 = f\,\cos(\phi)$$

$$c\omega a = -f\,\sin(\phi)$$

One-term harmonic balance solution

Taking the sum of squares of the two resulting equations, we get cubic equation in a^2.

$$c_3 a^6 + c_2(\omega) a^4 + c_1(\omega) a^2 - f^2 = 0$$

with

$$c_3 = \frac{9k_3^2}{16} > 0$$

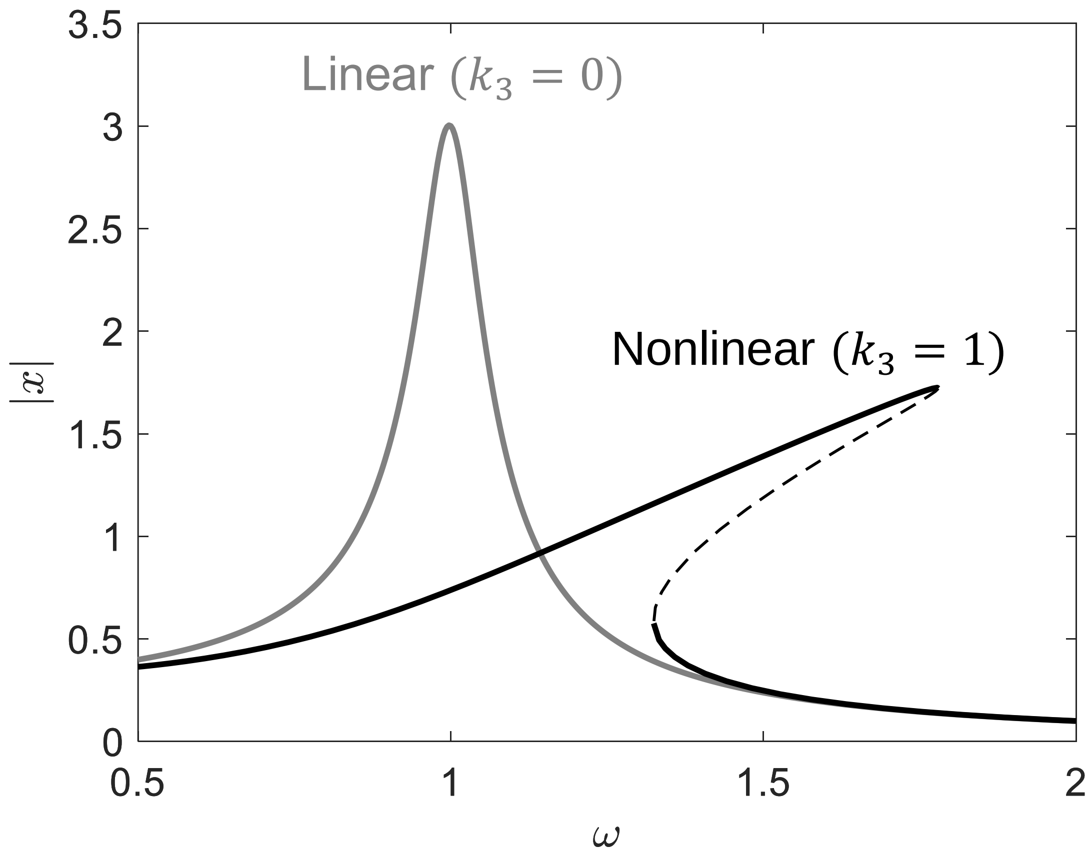
$$c_2(\omega) = \frac{3k_3}{2}(k - \omega^2 m)$$

$$c_1(\omega) = (k - \omega^2 m)^2 + c^2\omega^2 > 0$$

Hence this equation only has either one or three real positive roots in a^2, depending on the value of ω.

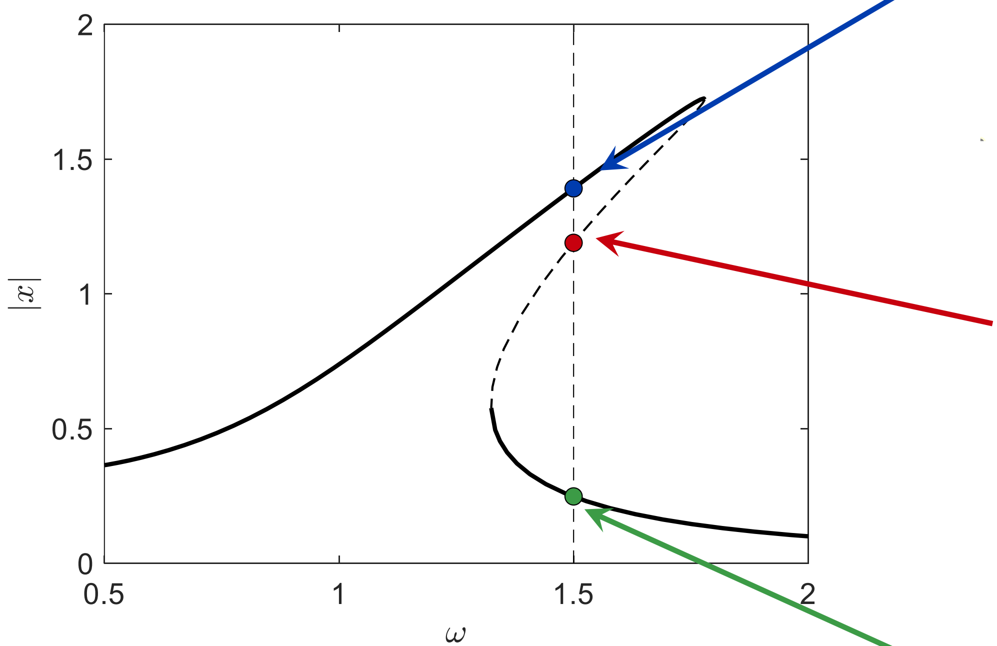
Nonlinear frequency response (NFR)

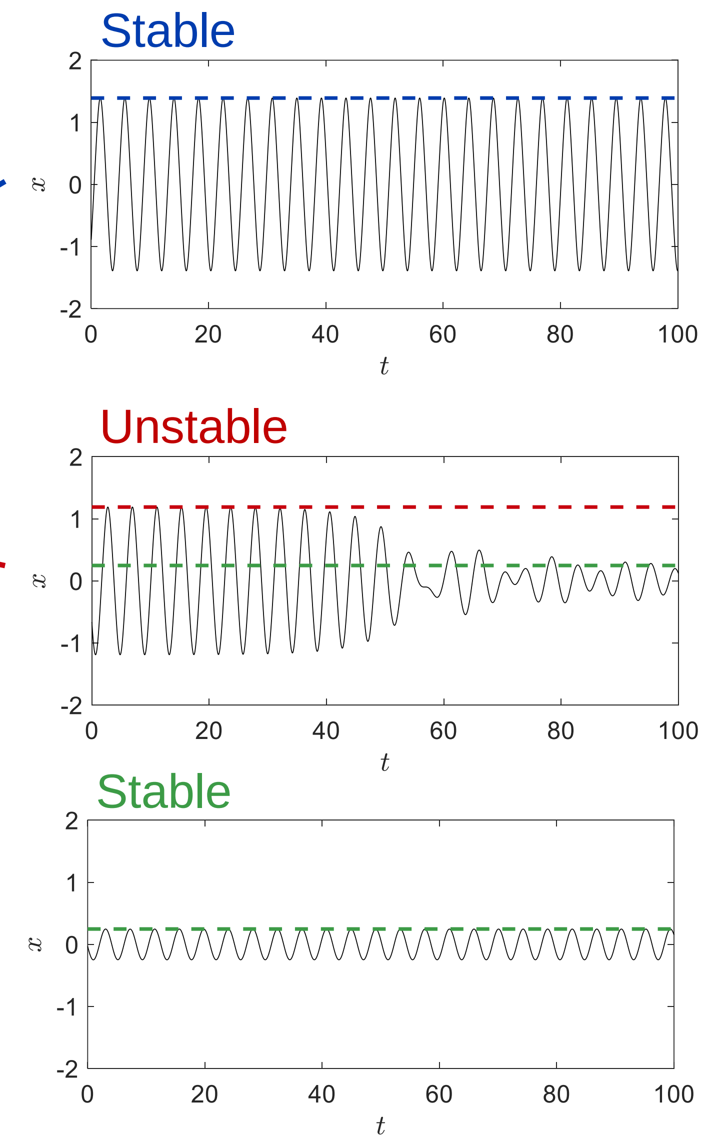
$$\ddot{x} + 0.1\dot{x} + x + k_3 x^3 = 0.3\sin(\omega t)$$



Multistability in the Duffing oscillator

$$\ddot{x} + 0.1\dot{x} + x + k_3 x^3 = 0.3\sin(\omega t)$$

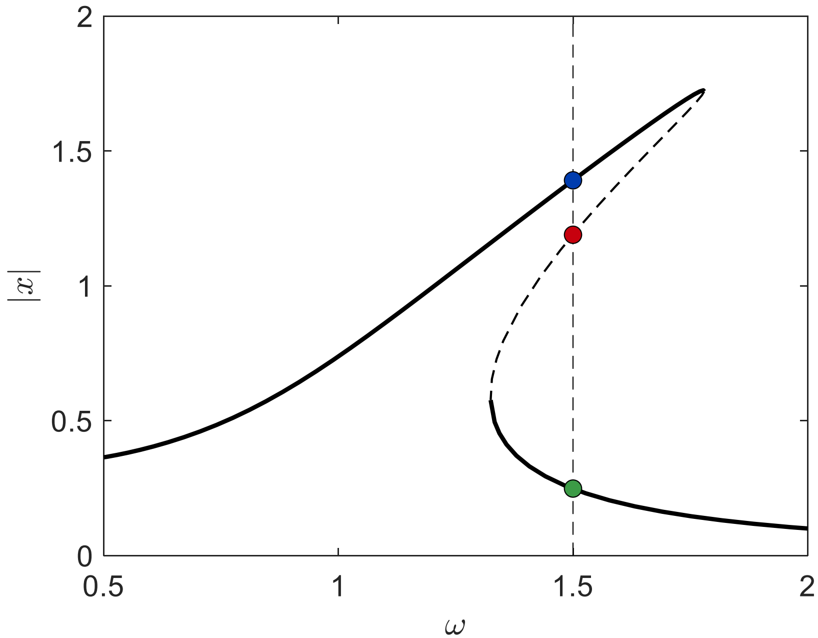


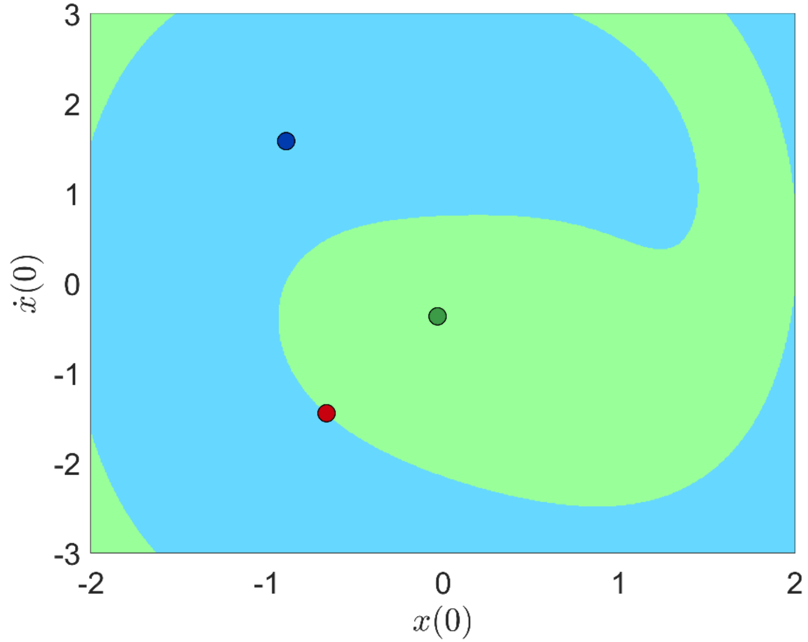


The steady state is determined by initial conditions

$$\ddot{x} + 0.1\dot{x} + x + x^3 = 0.35\sin(\omega t)$$

The initial conditions dictate which attractor is attained…



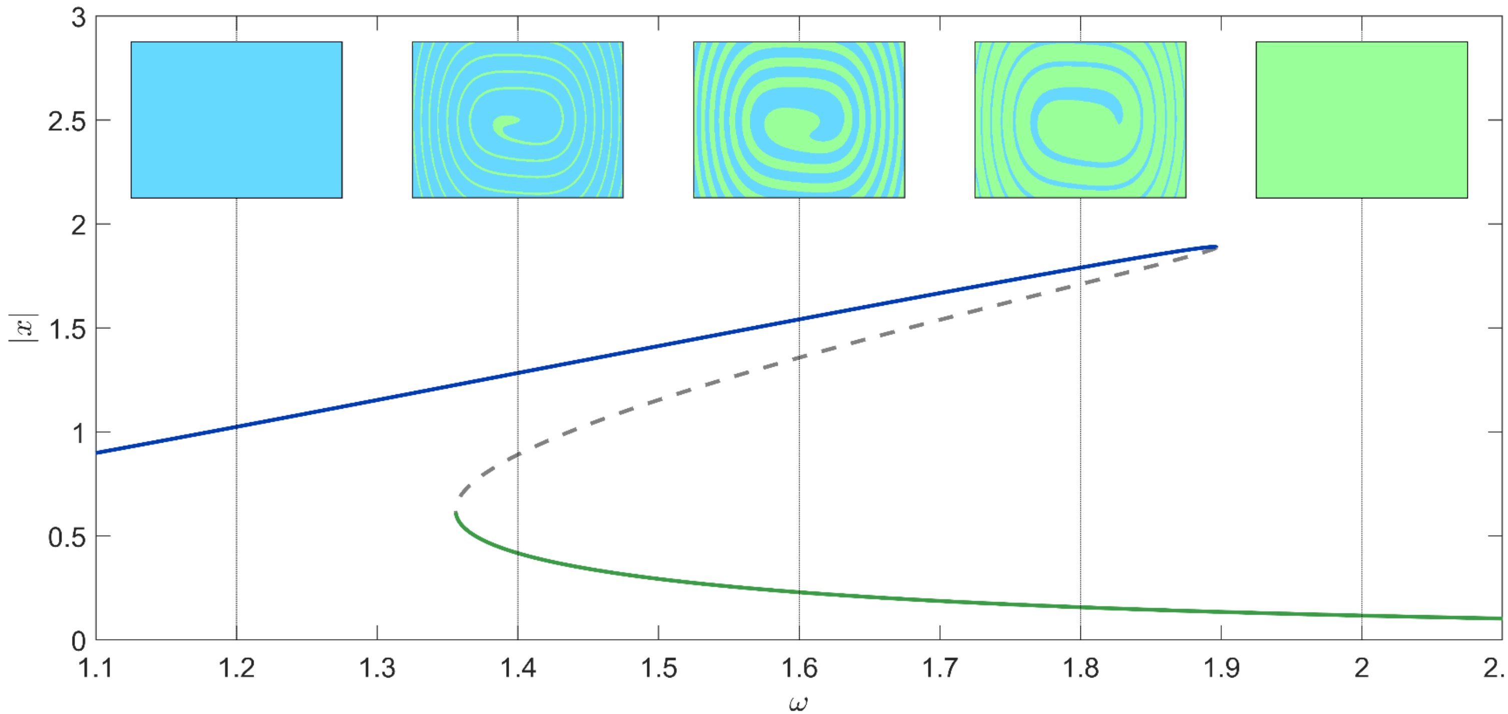


… but initial conditions are very complicated to control in an experimental set-up!

Nonlinear frequency response of the Duffing oscillator

$$\ddot{x} + 0.1\dot{x} + x + x^3 = 0.35\sin(\omega t)$$

The initial conditions dictate which attractor is attained…

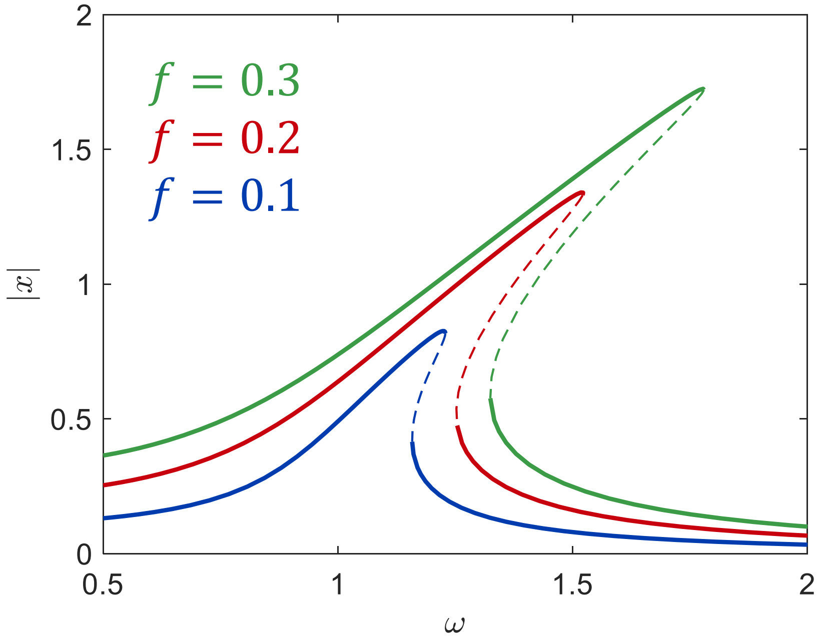


… but initial conditions are very complicated to control in an experimental set-up!

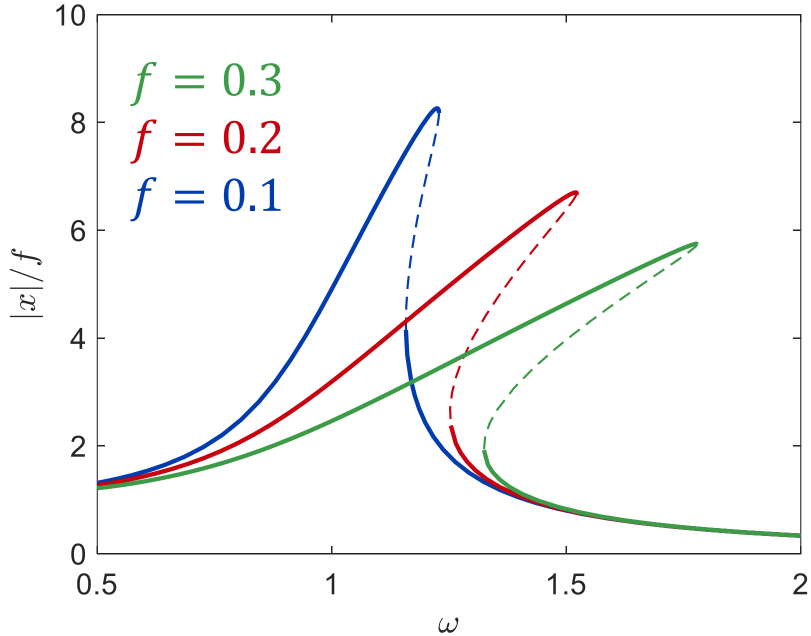
Amplitude-dependent response characteristics

$$\ddot{x} + 0.1\dot{x} + x + x^3 = f\,\sin(\omega t)$$

NFR



Nonlinear "FRF"

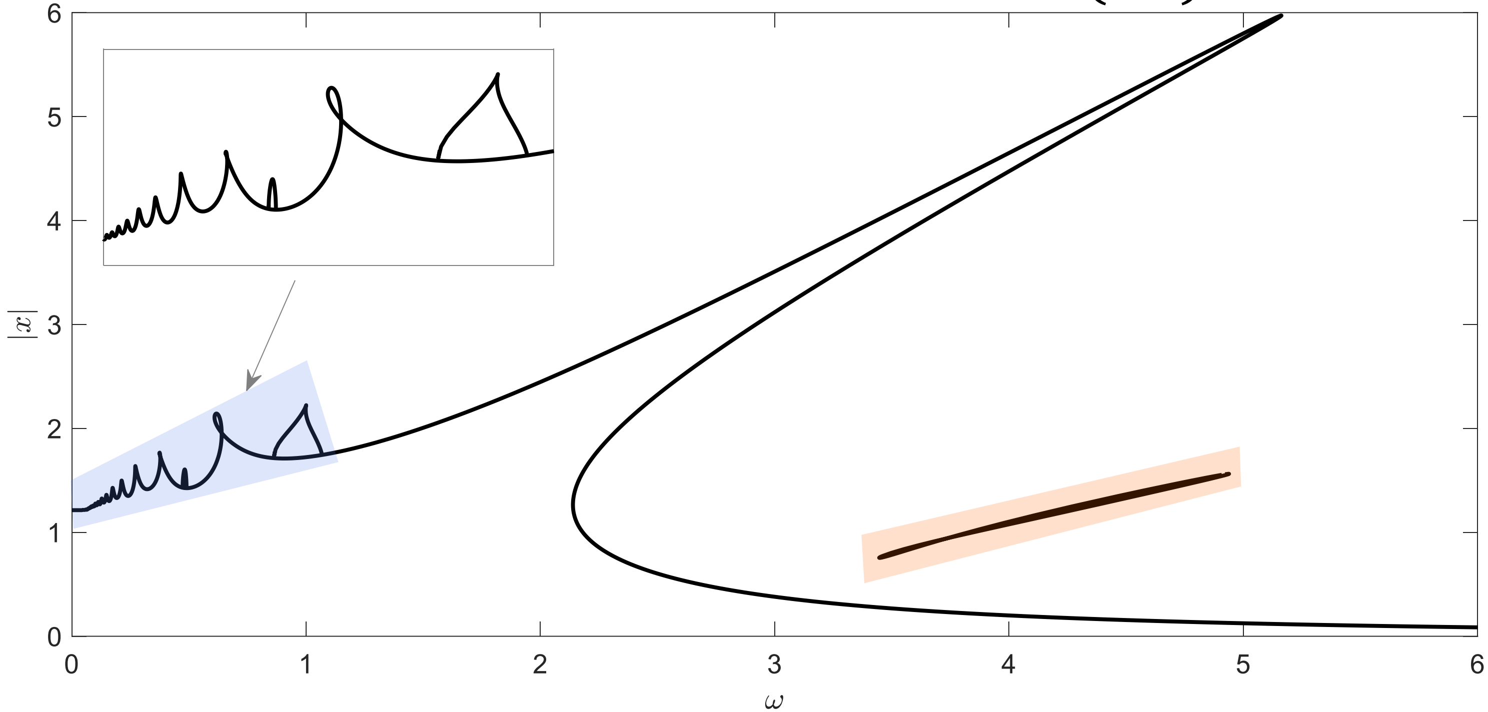


Nonlinear systems feature amplitude-dependent NFRs, resonance frequencies, modal damping and mode shapes.

Measurements must be made at multiple forcing amplitudes.

And a lot of other strange phenomena can happen!

$$\ddot{x} + 0.1\dot{x} + x + x^3 = 3\sin(\omega t)$$



Superharmonic and subharmonic resonances can appear (resonances excited by harmonics). In harmonically-forced multiple-degree-of-freedom systems, even more resonances (of internal and combination types) can appear.

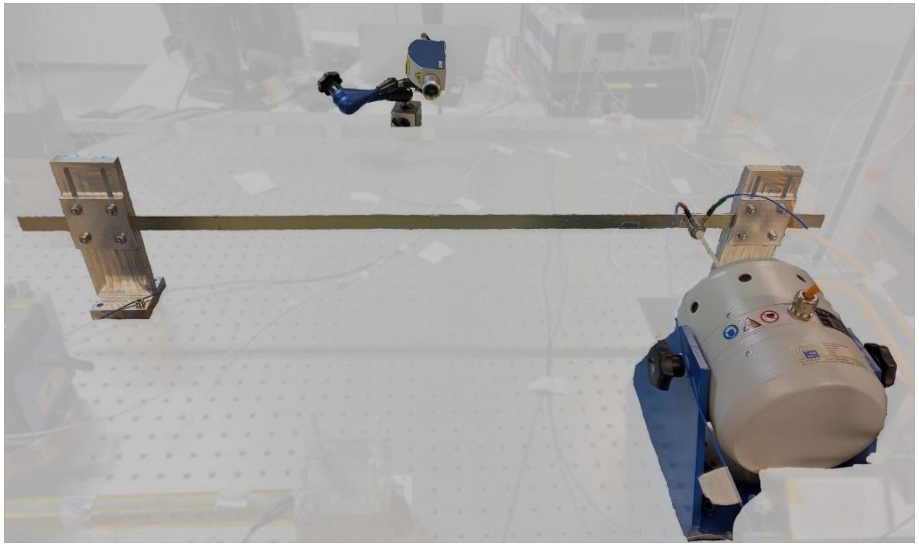
Some parts of the NFR are isolated (“isolas”).

Nonlinear systems can exhibit nonperiodic (quasiperiodic or chaotic) responses to a periodic excitation.

How can I recognize this problem?

Symptoms of nonlinearity in experiments

A thin clamped-clamped beam



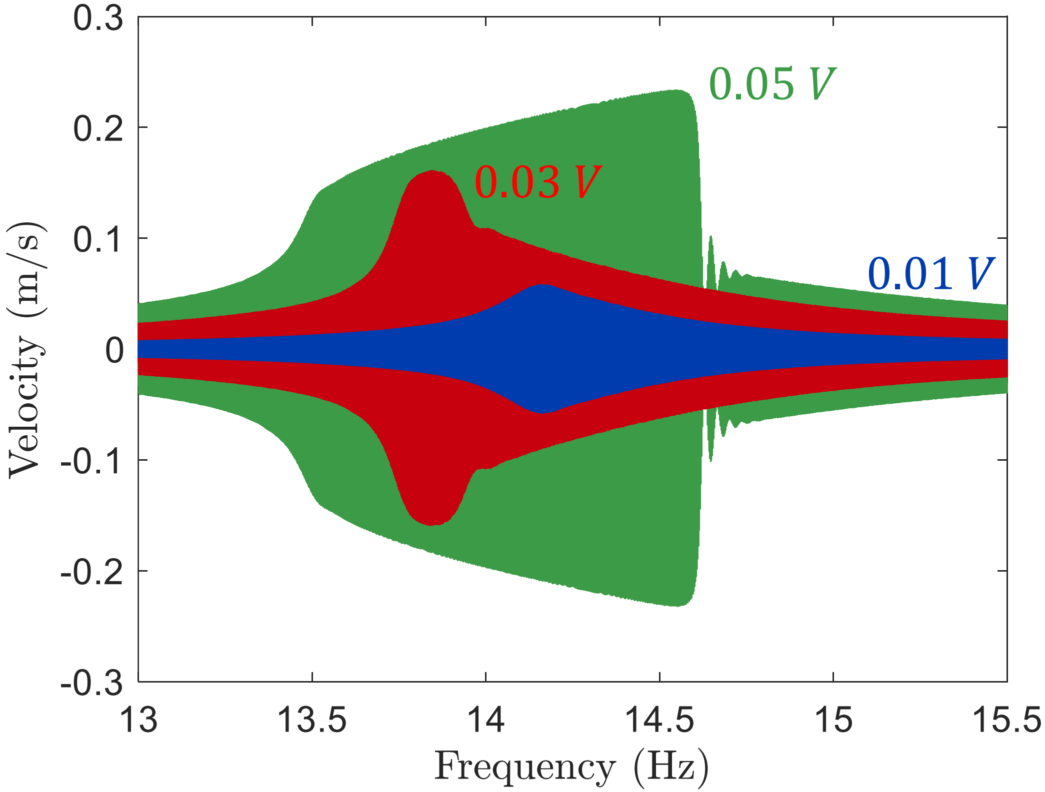
What type of nonlinearity would you expect to be dominant with this clamped-clamped beam?

https://app.wooclap.com/BQGYHZ

Symptom 1: non-homogeneity

Swept-sine excitations are most often used in nonlinear vibration testing.

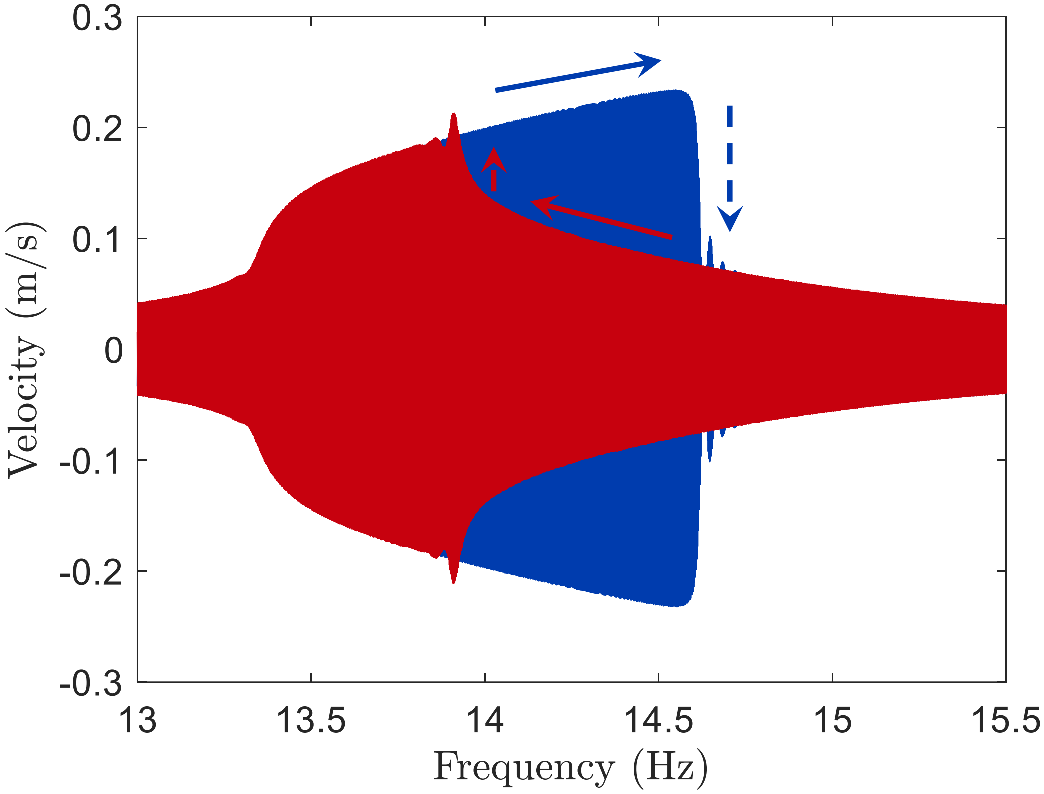
The response depends on the excitation amplitude and does not merely scale with it.



The principle of superposition does not apply.

Symptom 2: jumps and hysteresis

The response depends on initial conditions.

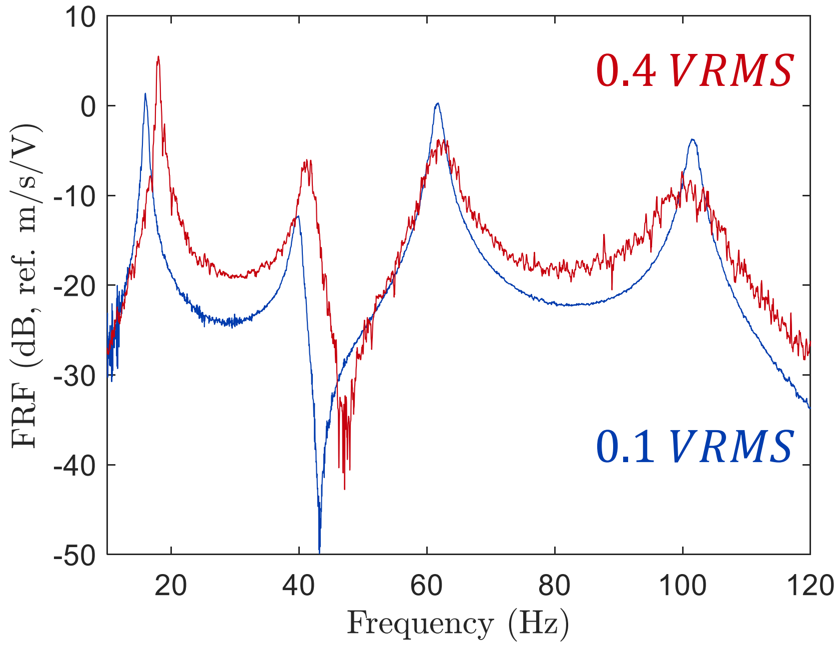


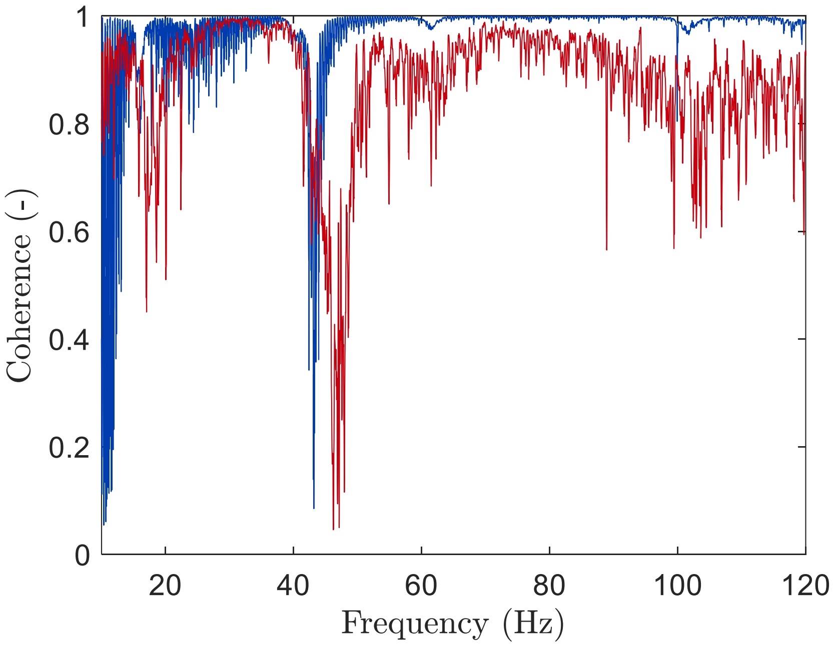
At $0.05\,V$, a sweep up leads to a jump down, and a sweep down leads to a jump up.

These jumps result in a nonlinear hysteresis phenomenon, which is a signature of nonlinearity.

Symptom 3: distorted FRF, low coherence

What happens if we use a broadband excitation signal and compute the FRF?

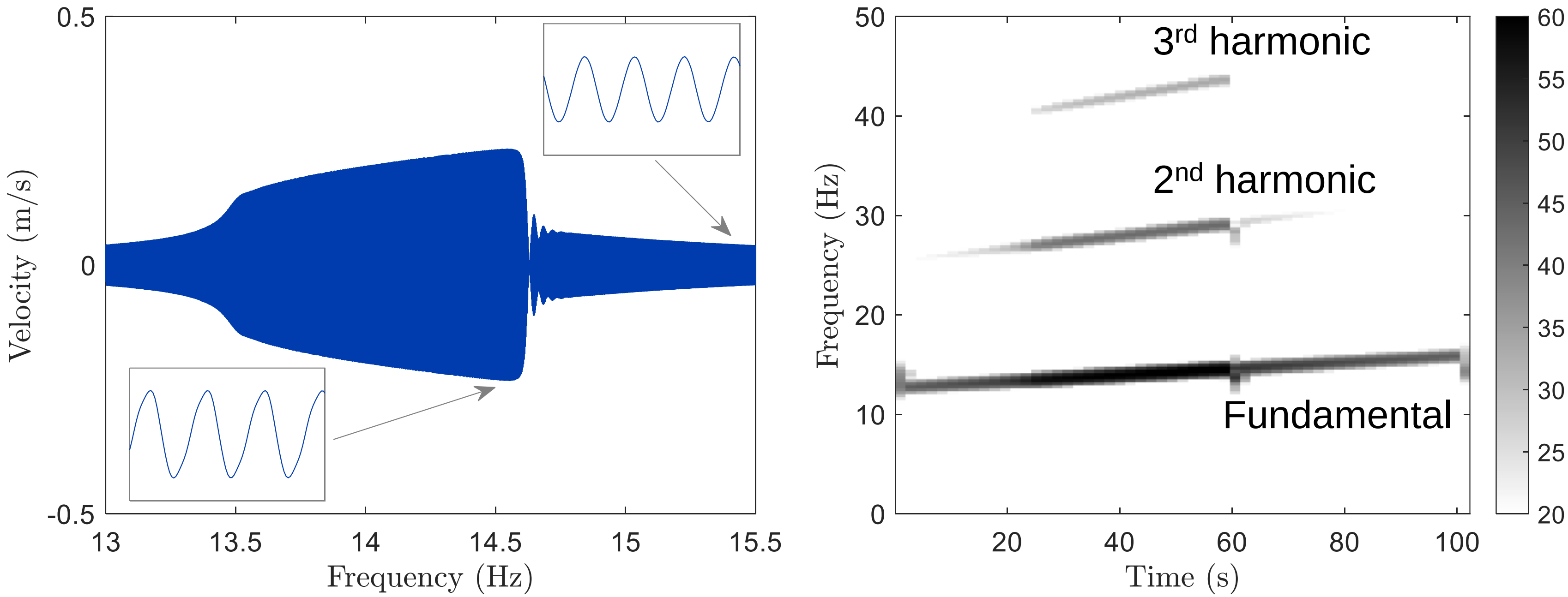




Nonlinearities may result in noisy FRFs and low coherence values.

(NB: these tests were made on a different day than the others, hence the different resonance frequency of the first mode)

Symptom 4: presence of harmonics

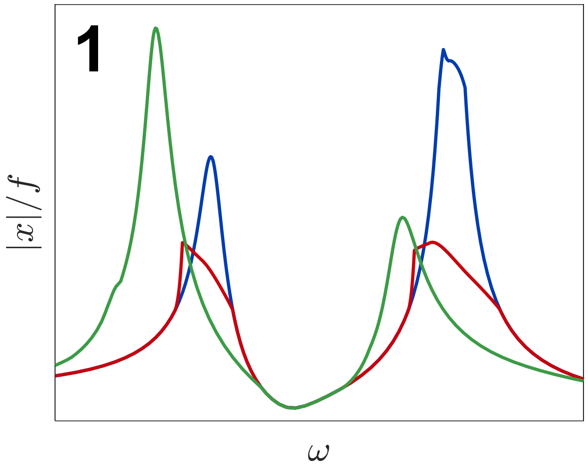


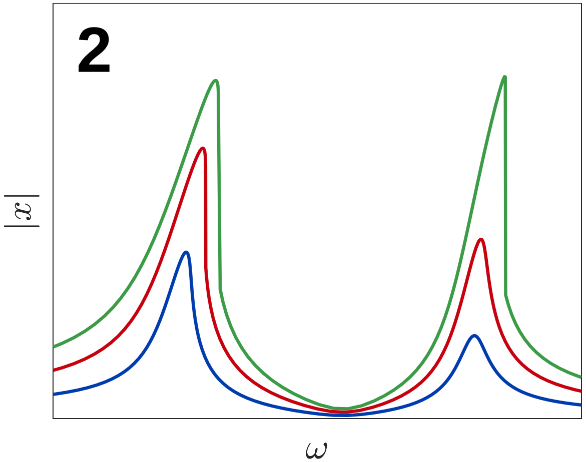
Nonlinear regimes of motion generate harmonics.

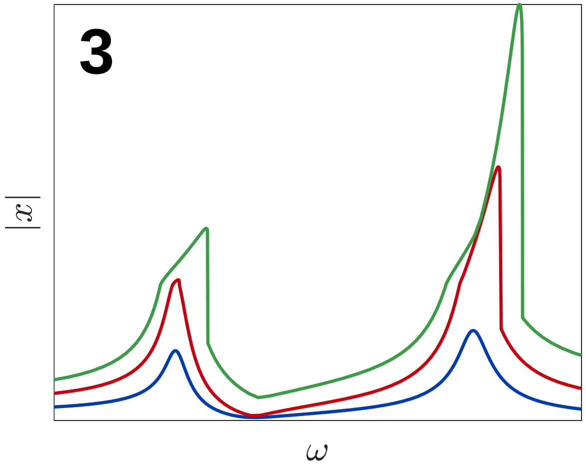
They can be detected with a time-frequency analysis (e.g., a spectrogram or a wavelet transform).

Recognizing nonlinear behavior

To which behavior can we associate these swept-sine responses? $f_1 < f_2 < f_3$







https://app.wooclap.com/BQGYHZ

What can I do about this problem?

Model-based and model-free methods

First solution: model-based approaches

We can extend the FRF fitting procedure to nonlinear systems. The goal is to fit our measured results to equations of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{\mathrm{nl}}(\mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\theta}_{\mathrm{nl}}) = \mathbf{f}(t)$$

Goal: estimate $\boldsymbol{\theta}_{\mathrm{nl}}$, the nonlinear parameters, together with $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ (or, in most cases, quantities similar to these matrices).

Main three steps:

1. **Detection**
   Is my structure nonlinear?
2. **Characterization**
   What kind of nonlinearity is there?
3. **Estimation**
   What is the “best” mathematical model for this nonlinearity?

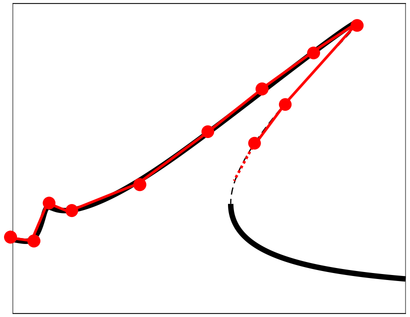
Characterization is usually the most challenging step, while estimation boils down to a (linear or nonlinear) least-squares fitting problem.

Second solution: model-free approaches

Modeling can be very challenging if one has little prior insight on the nonlinear structural behavior.

Experimental continuation comes as a model-free alternative to system identification approaches.

The main idea is to combine
- A **continuation** algorithm to wander along the NFR curve
- **Control** to stabilize its unstable parts



It requires more sophisticated means than traditional open-loop testing.

Experimental nonlinear dynamics

Model-based approaches

- Nonlinear system identification
- Nonlinear model updating

+ Yields a reusable model
Insight gained from the model

- Model-dependent
Risk of overfitting

Model-free approaches

- Experimental continuation

+ Robust and fast testing
Insight gained during the test

- More complex set-up
Stability must be ensured

NB: these two approaches are not incompatible!

Additional matters for nonlinear vibration testing

Sampling frequency
It must be selected high enough to capture the harmonics present in the response. As a rule of thumb,

$$f_{\mathrm{s}} \geq 20 f_{\mathrm{typ}}$$

where f_{typ} is a typical frequency of interest (e.g. that of the mode which is tested).

Actuator-structure interactions
It is typically very hard to impose the desired force profile on a structure. Commercial force control solutions are usually not robust enough to handle that problem. This must be kept in mind when analyzing test results.

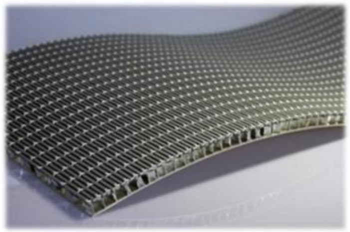
Nonlinearity type and location
Identifying the type of nonlinearity may not be easy, especially if several types are combined. There is almost no method to locate the nonlinearities. One needs to rely on one's understanding and physical insight into the problem.

So what?

Summary

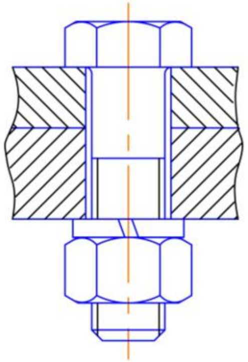
Summary: common sources of nonlinearity

Nonlinear materials



Geometric nonlinearity

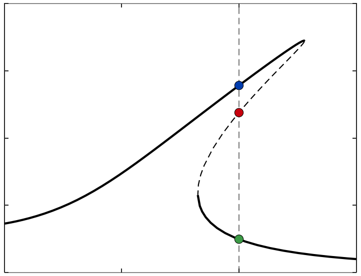
Boundary conditions

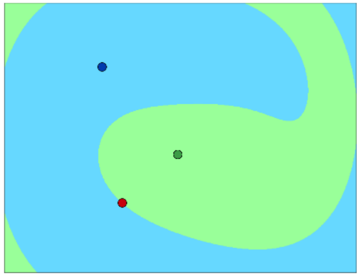


Nonlinear damping

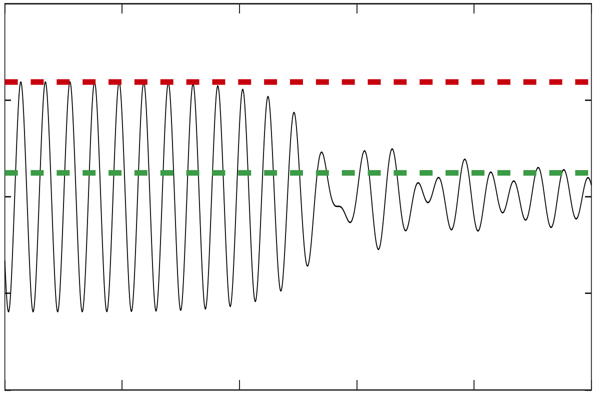
Summary: challenges in nonlinear vibration testing

Multistability



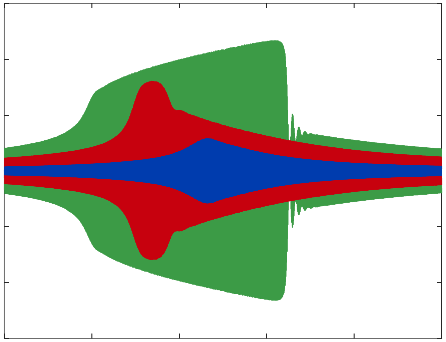


Instabilities

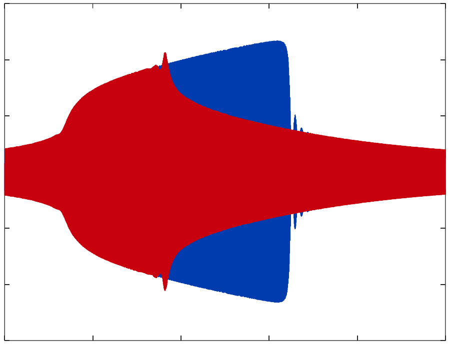


Summary: symptoms of nonlinearity

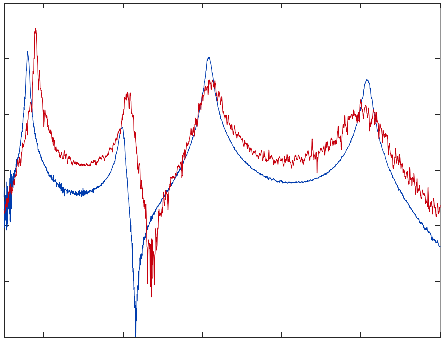
Lack of homogeneity



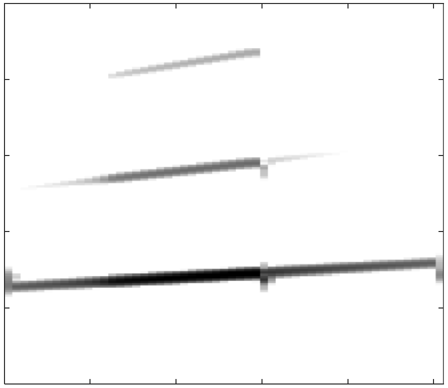
Jumps and hysteresis



Distorted spectral quantities



Harmonics



References (1/2)

D. J. Ewins, “Modal Testing: Theory, Practice and Application,” 2001.

M. Géradin and D. J. Rixen, *Mechanical vibrations: theory and application to structural dynamics*. John Wiley & Sons, 2014.

K. Worden and G. R. Tomlinson, *Nonlinearity in Structural Dynamics*. CRC Press, 2019. doi: 10.1201/9780429138331.

A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. Wiley, 1995. doi: 10.1002/9783527617586.

G. Genta, *Vibration Dynamics and Control*, vol. 57, no. 2. in Mechanical Engineering Series, vol. 57. Boston, MA: Springer US, 2009. doi: 10.1007/978-0-387-79580-5.

T. Detroux, L. Renson, L. Masset, and G. Kerschen, “The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems,” *Computer Methods in Applied Mechanics and Engineering*, vol. 296, pp. 18–38, Nov. 2015, doi: 10.1016/j.cma.2015.07.017.

References (2/2)

G. Kerschen, K. Worden, A. F. Vakakis, and J.-C. Golinval, “Past, present and future of nonlinear system identification in structural dynamics,” *Mechanical Systems and Signal Processing*, vol. 20, no. 3, pp. 505–592, Apr. 2006, doi: 10.1016/j.ymssp.2005.04.008.

J. P. Noël and G. Kerschen, “Nonlinear system identification in structural dynamics: 10 more years of progress,” *Mechanical Systems and Signal Processing*, vol. 83, pp. 2–35, Jan. 2017, doi: 10.1016/j.ymssp.2016.07.020.

D. J. Ewins, B. Weekes, and A. delli Carri, “Modal testing for model validation of structures with discrete nonlinearities,” *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 373, no. 2051, p. 20140410, Sep. 2015, doi: 10.1098/rsta.2014.0410.

U. Parlitz and W. Lauterborn, “Superstructure in the bifurcation set of the Duffing equation,” *Physics Letters A*, vol. 107, no. 8, pp. 351–355, Feb. 1985, doi: 10.1016/0375-9601(85)90687-5.

A. Marchionne, P. Ditlevsen, and S. Wieczorek, “Synchronisation vs. resonance: Isolated resonances in damped nonlinear oscillators,” *Physica D: Nonlinear Phenomena*, vol. 380–381, pp. 8–16, Oct. 2018, doi: 10.1016/j.physd.2018.05.004.