Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

Parametric Uncertainty Modeling: An Overview

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Abstract

Robust control system analysis and design is based on an uncertainty description, called a linear fractional transformation (LFT), which separates the uncertain (or varying) part of the system from the nominal system. This paper provides an overview of the parametric uncertainty modeling process for nonlinear parameter-dependent systems.

1.0 Introduction

Recent advances in robust control theory have made possible the analysis and design of multivariable control systems for robustness to explicitly defined system uncertainty. This theory provides a mechanism for achieving robust stability and performance of multiple-input multiple-output (MIMO) systems, and is based on a generalized system description that separates the nominal and uncertain system components.

The parametric uncertainty modeling problem is concerned with constructing the state-space model depicted in Figure 1.

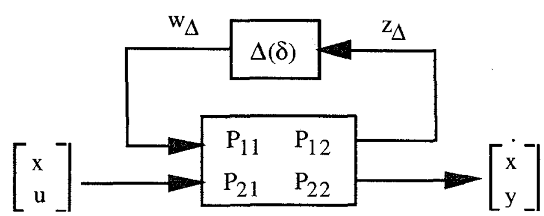

Figure 1. Parametric Uncertainty Model

The separated system description of the nominal part, P, and the uncertain part, Δ, is referred to as a P-Δ model, and falls within a broader class of system descriptions generally referred to as a linear fractional transformation (LFT). For direct implementation in current robust control algorithms, the LFT model must be obtained in a state-space model form and should be of low order. For practical control problems, the formulation of a P-Δ model which accurately characterizes realistic system uncertainties is very important, because the robustness results obtained using these algorithms depend directly on the uncertainty model used in the analysis or design.

Parametric uncertainty models can also be utilized in the application of linear parameter varying (LPV) control methods for nonlinear parameter-dependent systems (see Ref. [1]). LPV control system design provides a formal mathematical method for designing gain-scheduled control systems which can function over the operational envelope of the nonlinear system. The LPV control method can be performed whether or not the system model is represented in LFT form. However, when the parameterized model is expressed as an LFT, the solution is much simpler to compute and a larger number of varying parameters can be considered. In addition, the use of an LFT model allows LPV control to be combined with μ-Synthesis (see Ref. [2]) to design robust gain-scheduled control systems.

The process for forming LFT models can be quite complicated for nonlinear parameter variations. While some research has been performed and published on this subject (e.g., see Refs. [3] - [10] for some key results), there has not been a thorough comprehensive treatment of parametric uncertainty modeling for practical applications. This paper summarizes the process required in developing LPV and LFT models of nonlinear parameter-dependent systems for use in robust control system analysis and design and for applying LPV control methods to gain-scheduling problems.

2.0 Process Overview

The key steps in the parametric uncertainty modeling process include: the formulation of an LPV model, transformation of the LPV model into LFT form, and scaling of the LFT model to obtain the standard normalized P-Δ model form required for robust control system analysis and design. These steps are discussed in the following sub-sections.

2.1 LPV Model Formulation

A simulation and/or laboratory model of the nonlinear system can be used to generate data over the independent parameter space being considered in the development of an LPV model of the uncertain system. A method for parameterizing the data to obtain multivariate polynomial functions of the varying parameters is the subject of Ref. [11], and validation of the parameterized model in a stochastic sense is part of the process. This method can be used to obtain nonlinear parameter-dependent dynamical equations of the system based on either linear or nonlinear models.

An LPV model can be formulated using several approaches. One approach is to parameterize the matrix elements associated with a set of linear models generated over the uncertain parameter space. Another approach is to parameterize the underlying physical parameters of a linear model of the system. A third approach is to parameterize the physical parameters of the nonlinear system model, and then linearize the system at an equilibrium point while retaining the variable parameters of interest. Ref. [14] presents a method for formulating LPV state models from nonlinear parameter-dependent dynamical equations. This method allows the determination of equilibrium surfaces and the formulation of LPV models at or near bifurcation points. A symbolic modeling tool based on this result is presented in Ref. [15].

The resulting LPV model can be represented by the following equation.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\underline{\delta}) & \mathbf{B}(\underline{\delta}) \\ \mathbf{C}(\underline{\delta}) & \mathbf{D}(\underline{\delta}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{S}(\underline{\delta}) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{1}$$

where: $$\underline{\delta} = [\underline{\delta}_1, \underline{\delta}_2, \cdots, \underline{\delta}_m] \in \mathbf{R}^m \tag{2}$$

The $\underline{\delta}_i$ parameters represent the independent uncertain or varying parameters of the system. In standard P-Δ model form, each δ_i parameter is normalized. The underscore is used in the above equations to indicate that the $\underline{\delta}$ parameters are not in normalized form. Scaling associated with the parameterization process and for normalizing the uncertain parameters can be accomplished as a separate step once the LFT model structure has been obtained. This is discussed in greater detail in Section 2.3.

2.2 LFT Model Formulation

Transformation of the LPV model into LFT form is the most difficult step in the parametric uncertainty modeling process. It involves the formulation of the LFT problem to be solved, and the computation of an LFT model solution. Note that the LFT representation of the LPV model is exact, and should be of the lowest possible dimension. The general problem is to find a separated state-space uncertainty model in the LFT form, as defined by Figure 1. The LFT model equations prior to normalization are given below.

$$\mathbf{z}_\Delta = \underline{\mathbf{P}}_{11}\mathbf{w}_\Delta + \underline{\mathbf{P}}_{12}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{3a}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \underline{\mathbf{P}}_{21}\mathbf{w}_\Delta + \underline{\mathbf{P}}_{22}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{w}_\Delta = \underline{\Delta}\,\mathbf{z}_\Delta \tag{3b}$$

$$\underline{\Delta}(\underline{\delta}) = \mathrm{diag}\,[\underline{\delta}_1\mathbf{I}_{n_1}, \underline{\delta}_2\mathbf{I}_{n_2}, \ldots, \underline{\delta}_m\mathbf{I}_{n_m}] \tag{4a}$$

$$\dim(\underline{\Delta}) = n_\Delta = \sum_{i=1}^{m} n_i, \quad n_i = \dim(\mathbf{I}_i) \tag{4b}$$

Note that finding a solution for $\underline{\mathbf{P}}_{11}$, $\underline{\mathbf{P}}_{12}$, $\underline{\mathbf{P}}_{21}$, and $\underline{\mathbf{P}}_{22}$ such that the resulting uncertainty model is low-order means that n_Δ in equation (4b) should be as small as possible. This is equivalent to requiring that the number of repetitions of each parameter be minimized.

2.2.1 LFT Problem Formulation

In order to obtain an LFT model of the LPV system, an LFT problem must first be formed. Closing the Δ-loop in Figure 1 using equations (3) yields the following equation.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \left\{ \underline{\mathbf{P}}_{21}\underline{\Delta}(\mathbf{I} - \underline{\mathbf{P}}_{11}\underline{\Delta})^{-1}\underline{\mathbf{P}}_{12} + \underline{\mathbf{P}}_{22} \right\} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{5}$$

Comparing equations (1) and (5) yields the following LFT equation for $\mathrm{S}(\underline{\delta})$.

$$\begin{aligned} \mathbf{S}(\underline{\delta}) &= \underline{\mathbf{P}}_{21}\underline{\Delta}(\mathbf{I} - \underline{\mathbf{P}}_{11}\underline{\Delta})^{-1}\underline{\mathbf{P}}_{12} + \underline{\mathbf{P}}_{22} \\ &= \underline{\mathbf{P}}_{21}(\mathbf{I} - \underline{\Delta}\underline{\mathbf{P}}_{11})^{-1}\underline{\Delta}\underline{\mathbf{P}}_{12} + \underline{\mathbf{P}}_{22} \end{aligned} \tag{6}$$

$$\Rightarrow \quad \mathbf{S}(\underline{\delta}) = \mathbf{S}_\Delta(\underline{\delta}) + \mathbf{S}_o \tag{7}$$

Thus, $\mathrm{S}(\underline{\delta})$ contains a nominal component ($\underline{\mathbf{P}}_{22}$) and an uncertain component that depends on $\underline{\Delta}$. Comparing equations (6) and (7) yields the following equations for the nominal and uncertain components of $\mathrm{S}(\underline{\delta})$.

$$\mathbf{S}_o = \begin{bmatrix} \mathbf{A}_o & \mathbf{B}_o \\ \mathbf{C}_o & \mathbf{D}_o \end{bmatrix} = \underline{\mathbf{P}}_{22} \tag{8}$$

$$\begin{aligned} \mathbf{S}_\Delta(\underline{\delta}) &= \underline{\mathbf{P}}_{21}\underline{\Delta}(\mathbf{I} - \underline{\mathbf{P}}_{11}\underline{\Delta})^{-1}\underline{\mathbf{P}}_{12} \\ &= \underline{\mathbf{P}}_{21}(\mathbf{I} - \underline{\Delta}\underline{\mathbf{P}}_{11})^{-1}\underline{\Delta}\underline{\mathbf{P}}_{12} \end{aligned} \tag{9}$$

Since $\underline{P}_{22}$ is easily determined from the nominal system model, formulation of the LFT problem consists of forming equation (9) from the system model of equation (1). If the elements of the A, B, C, and D matrices of equation (1) are multivariate polynomial functions of $\underline{\delta}$, the nominal and uncertain parts of the system can be easily separated, as shown below.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \mathbf{S}(\underline{\delta})\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \{\mathbf{S_o} + \mathbf{S}_\Delta(\underline{\delta})\}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \qquad (10)$$

$$\mathbf{S}_\Delta(\underline{\delta}) = \begin{bmatrix} \mathbf{A}_\Delta(\underline{\delta}) & \mathbf{B}_\Delta(\underline{\delta}) \\ \mathbf{C}_\Delta(\underline{\delta}) & \mathbf{D}_\Delta(\underline{\delta}) \end{bmatrix} = \underline{\mathbf{P}}_{21}(\mathbf{I} - \Delta\underline{\mathbf{P}}_{11})^{-1}\Delta\underline{\mathbf{P}}_{12} \qquad (11)$$

Then equation (10) can be directly rewritten in the form of equations (3) by defining an input vector, w_Δ, and an output vector, z_Δ, associated with the uncertain part of the system.

If the elements of the system A, B, C, and D matrices given in equation (1) are rational functions of $\underline{\delta}$, the expansion of equation (10) cannot be performed directly. For this case, a matrix fraction description (MFD) of the system model, $S(\underline{\delta})$, must first be obtained in order to decompose the system into numerator and denominator matrices. This is represented by either of the following equations.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \mathbf{S_N}(\underline{\delta})\mathbf{S_D}^{-1}(\underline{\delta})\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \qquad (12a)$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \tilde{\mathbf{S}}_\mathbf{D}^{-1}(\underline{\delta})\tilde{\mathbf{S}}_\mathbf{N}(\underline{\delta})\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \qquad (12b)$$

As shown in equations (12), the MFD can be obtained in either its right (12a) or left (12b) form. This decomposition can always be performed, and the details associated with doing it are discussed in Ref. [7].

Once the MFD has been formed, the resulting numerator and denominator matrices, $S_N(\underline{\delta})$ and $S_D(\underline{\delta})$, contain multivariate polynomial functions of the uncertain parameters $\underline{\delta}$. Thus, $S_N(\underline{\delta})$ and $S_D(\underline{\delta})$ can be easily separated, and the associated uncertain components, $S_{N\Delta}(\underline{\delta})$ and $S_{D\Delta}(\underline{\delta})$, can be grouped together to form a single $S_\Delta(\underline{\delta})$ matrix. The result for the right MFD case is formalized in Theorem 2.1.

Theorem 2.1: Rational LFT Problem Formulation for a Right MFD

Given a right MFD description of an uncertain system as defined in equation (12a), then:

i.) the uncertain components from the numerator ($S_{N\Delta}(\underline{\delta})$) and denominator ($S_{D\Delta}(\underline{\delta})$) of the system can be separated from the nominal system components and combined into a single matrix to obtain the following LFT description:

$$\mathbf{S}_\Delta(\underline{\delta}) = \begin{bmatrix} \mathbf{S}_{\mathbf{N}_\Delta}(\underline{\delta}) \\ \mathbf{S}_{\mathbf{D}_\Delta}(\underline{\delta}) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{\mathbf{21}_\mathbf{N}} \\ \mathbf{Q}_{\mathbf{21}_\mathbf{D}} \end{bmatrix}(\mathbf{I} - \Delta\mathbf{Q}_{11})^{-1}\Delta\mathbf{Q}_{12} \qquad (13)$$

ii.) the state-space P-Δ model is given by equations (3) and (4), where:

$$\underline{\mathbf{P}}_{11} = \mathbf{Q}_{11} - \mathbf{Q}_{12}\mathbf{S}_{\mathrm{D}_0}^{-1}\mathbf{Q}_{21\mathrm{D}} \qquad (14a)$$

$$\underline{\mathbf{P}}_{12} = \mathbf{Q}_{12}\mathbf{S}_{\mathrm{D}_0}^{-1} \qquad (14b)$$

$$\underline{\mathbf{P}}_{21} = \mathbf{Q}_{21\mathrm{N}} - \mathbf{S}_{\mathrm{N}_0}\mathbf{S}_{\mathrm{D}_0}^{-1}\mathbf{Q}_{21\mathrm{D}} \qquad (14c)$$

$$\underline{\mathbf{P}}_{22} = \mathbf{S}_{\mathrm{N}_0}\mathbf{S}_{\mathrm{D}_0}^{-1} = \mathbf{S}_\mathrm{o} \qquad (14d)$$

Proof: See Ref. [7].

A similar formulation can be constructed for a left MFD system model, as presented in Theorem 2.2.

Theorem 2.2: Rational LFT Problem Formulation for a Left MFD

Given a left MFD description of an uncertain system as defined in equation (12b), then:

i.) the uncertain components from the numerator ($\tilde{S}_{\Delta_N}(\underline{\delta})$) and denominator ($\tilde{S}_{\Delta_D}(\underline{\delta})$) of the system can be separated from the nominal system components and combined into a single matrix to obtain the following LFT description:

$$\mathbf{S}_\Delta(\underline{\delta}) = \begin{bmatrix} \tilde{\mathbf{S}}_{\mathbf{N}_\Delta}(\underline{\delta}) & \tilde{\mathbf{S}}_{\mathbf{D}_\Delta}(\underline{\delta}) \end{bmatrix} = \tilde{\mathbf{Q}}_{\mathbf{21}}(\mathbf{I} - \Delta\tilde{\mathbf{Q}}_{11})^{-1}\Delta\begin{bmatrix} \tilde{\mathbf{Q}}_{\mathbf{12}_\mathbf{N}} & \tilde{\mathbf{Q}}_{\mathbf{12}_\mathbf{D}} \end{bmatrix} \qquad (15)$$

ii.) the state-space P-Δ model is given by equations (3) and (4), where:

$$\underline{\mathbf{P}}_{11} = \tilde{\mathbf{Q}}_{11} - \tilde{\mathbf{Q}}_{12_\mathrm{D}}\tilde{\mathbf{S}}_{\mathrm{D}_0}^{-1}\tilde{\mathbf{Q}}_{21} \qquad (16a)$$

$$\underline{\mathbf{P}}_{12} = \tilde{\mathbf{Q}}_{12_\mathrm{N}} - \tilde{\mathbf{Q}}_{12_\mathrm{D}}\tilde{\mathbf{S}}_{\mathrm{D}_0}^{-1}\tilde{\mathbf{S}}_{\mathrm{N}_0} \qquad (16b)$$

$$\underline{\mathbf{P}}_{21} = \tilde{\mathbf{S}}_{\mathrm{D}_0}^{-1}\tilde{\mathbf{Q}}_{21} \qquad (16c)$$

$$\underline{\mathbf{P}}_{22} = \tilde{\mathbf{S}}_{\mathrm{D}_0}^{-1}\tilde{\mathbf{S}}_{\mathrm{N}_0} = \mathbf{S}_\mathrm{o} \qquad (16d)$$

Proof: See Ref. [7].

Solution of the rational problem involves solving equation (13) (or (15)) for $\mathbf{Q}_{11}$, $\mathbf{Q}_{12}$, $\mathbf{Q}_{21}$, (or $\tilde{\mathbf{Q}}_{11}$, $\tilde{\mathbf{Q}}_{12}$, $\tilde{\mathbf{Q}}_{21}$) and Δ, and then computing $\underline{\mathbf{P}}_{11}$, $\underline{\mathbf{P}}_{12}$, $\underline{\mathbf{P}}_{21}$, and $\underline{\mathbf{P}}_{22}$ from equations (14) (or (16)).

The concept of decomposing the uncertain system into simpler matrix components that can then be used to obtain the P-Δ model (as presented above for the rational problem) is extendible to other difficult problems as a means of simplifying the solution process. Several extensions to the rational problem formulation for use in further decomposing difficult problems are discussed in Ref. [7], from which either a single compound $S_{\Delta}(\underline{\delta})$ matrix or individual $S_{\Delta_j}(\underline{\delta})$ matrices can be formed.

Note that the above decompositions are used to obtain multivariate polynomial matrix components. Other decomposition methods that can be used to obtain linear matrix components are discussed in Refs. [6], [10], and [13]. The approaches presented in [10] and [13] are designed to reduce unnecessary repetitions of the $\underline{\delta}$ paramters in the resulting LFT model.

2.2.2 LFT Model Construction

Once equation (9) (or its equivalent) and the corresponding $S_{\Delta}(\delta)$ matrix have been formed for the uncertain system, an LFT model can be obtained. Constructing the LFT model consists of solving equation (9) for $\underline{P}_{21}$, $\underline{P}_{12}$, and $\underline{P}_{11}$ such that the dimension of the $\underline{\Delta}$ matrix (as defined by equation (4)) is as small as possible, and determining $\underline{P}_{22}$ from the nominal system matrices as given in equation (8). This is equivalent to a multidimensional minimal realization problem for which no general theory has been developed. See Ref. [16] for a discussion of multidimensional systems theory, and Refs. [8], [9], and [17] for some recent theoretical extensions to parametric LFT modeling.

A numerical method to compute an LFT model for multivariate polynomial problems is presented in Ref. [12]. Note that this method can be directly applied to solve rational parameter problems using the LFT problem formulation of Theorems 2.1 and 2.2, as well as the other matrix decompositions presented in Ref. [7]. The decomposition approach of Refs. [10] and [13] can also be used in combination with the multivariate polynomial LFT solution of Ref. [12].

If the decomposition methods of Refs. [6], [10], and [13] are used to reduce the system to linear matrix components, then the linear LFT solution of Refs. [3] and [4] can be used to obtain a solution for each matrix component. This approach requires "post-processing" of the individual LFT models to obtain the standard LFT model of the full system. Reduction of the full model is also usually required to eliminate unnecessary repetitions of the uncertain parameters in Δ.

Ref. [18] presents a method to obtain uncertainty bounds for systems with parametric and nonparametric uncertainties given the parametric LFT model structure.

2.3 LFT Model Scaling

Once the LFT model has been formed, a simple matrix scaling operation can be performed to obtain the standard normalized P-Δ model shown in Figure 1. The standard P-Δ model equations are given by equations (3) and (4) with the underbar notation removed and with $|\delta_i| \le 1$. A general relationship between $\underline{\delta}_i$ and δ_i is given below to incorporate separate scaling and normalization operations.

$$\underline{\delta}_i = \frac{1}{\underline{s}_i}\bar{\delta}_i \ , \quad \bar{\delta}_{i\,min} \le \bar{\delta}_i \le \bar{\delta}_{i\,max} \qquad (17a)$$

$$\bar{\delta}_i = \bar{\delta}_{i_o} + \frac{k_{1i}\delta_i}{1 - k_{2i}\delta_i} \ , \quad |\delta_i| \le 1 \qquad (17b)$$

$$k_{1i} = \frac{2(\bar{\delta}_{i\,max} - \bar{\delta}_{i_o})(\bar{\delta}_{i_o} - \bar{\delta}_{i\,min})}{(\bar{\delta}_{i\,max} - \bar{\delta}_{i\,min})} \qquad (17c)$$

$$k_{2i} = \frac{(\bar{\delta}_{i\,max} + \bar{\delta}_{i\,min}) - 2\bar{\delta}_{i_o}}{(\bar{\delta}_{i\,max} - \bar{\delta}_{i\,min})} \qquad (17d)$$

The scaling term, $\underline{s}_i$, is used to reflect scaling that was performed in parameterizing the data prior to forming the LPV model. The overbar is used in the above equations to reflect the unscaled parameters, $\bar{\delta}_i$. In developing parameterized models using the method of Ref. [11], deviations from the nominal parameter value are usually modeled in order to obtain functions of the varying parameters, $\underline{\delta}_i$. Thus, the nominal term in equation (17b), i.e. $\bar{\delta}_{i_o}$, is often zero. Substitution of equations (17) into the $\underline{\Delta}$ matrix defined in equation (4) yields the following relationship between $\underline{\Delta}$ and the normalized Δ matrix of Figure 1.

$$\underline{\Delta}(\underline{\delta}) = K_o + (I - K_2\Delta)^{-1} K_1 \Delta$$

$$K_o = \mathrm{diag}[(\frac{\bar{\delta}_{1_o}}{\underline{s}_1} I_{n_1}), (\frac{\bar{\delta}_{2_o}}{\underline{s}_2} I_{n_2}), \cdots, (\frac{\bar{\delta}_{m_o}}{\underline{s}_m} I_{n_m})]$$

$$K_1 = \mathrm{diag}[(\frac{k_{11}}{\underline{s}_1} I_{n_1}), (\frac{k_{12}}{\underline{s}_2} I_{n_2}), \cdots, (\frac{k_{1m}}{\underline{s}_m} I_{n_m})]$$

$$K_2 = \mathrm{diag}[(k_{21} I_{n_1}), (k_{22} I_{n_2}), \cdots, (k_{2m} I_{n_m})]$$

Then, the following expressions can be derived relating the standard normalized P-Δ model matrices of Figure 1 to the LFT model of equations (3).

$$
\begin{aligned}
P_{11} &= K_2 + K_1 (I - \underline{P}_{11} K_o)^{-1} \underline{P}_{11} \\
P_{12} &= K_1 (I - \underline{P}_{11} K_o)^{-1} \underline{P}_{12} \\
P_{21} &= \underline{P}_{21} (I - K_o \underline{P}_{11})^{-1} \\
P_{22} &= \underline{P}_{22} + \underline{P}_{21} (I - K_o \underline{P}_{11})^{-1} K_o \underline{P}_{12}
\end{aligned}
$$

Note that P_{22} is unaffected by the parameterization process when the nominal parameter values, $\bar{\delta}_{i_0}$, are zero.

3.0 Example: LFT Problem Formulation

The rational system formulation presented in Section 2 is very useful for many systems whose equations naturally occur in a rational form. For example, the state space model associated with aircraft equations of motion can be formulated as shown below.

$$
\begin{aligned}
E\,\dot{x} &= A\,x + B\,u \quad \Rightarrow \quad \dot{x} = E^{-1} A\,x + E^{-1} B\,u \\
y &= C\,x + D\,u \qquad\qquad\; y = C\,x + D\,u
\end{aligned}
$$

This system can be represented using the rational formulation of Theorem 2.2 for a left MFD, as indicated by the following equations.

$$
\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \tilde{S}_D^{-1} \tilde{S}_N \begin{bmatrix} x \\ u \end{bmatrix}
$$

$$
\Rightarrow \quad S_\Delta(\delta) = \begin{bmatrix} \tilde{S}_{N_\Delta} & \tilde{S}_{D_\Delta} \end{bmatrix} = \begin{bmatrix} A_\Delta & B_\Delta & E_\Delta \\ C_\Delta & D_\Delta & 0 \end{bmatrix}
$$

Note that a block zero column was omitted from the $S_\Delta(\delta)$ matrix. Using this formulation, uncertainties appearing in E can be modeled directly without having to perform the indicated inversion and multiplication. This clearly simplifies the uncertainty modeling process. Other examples are given in Ref. [7].

4.0 Concluding Remarks

This paper has presented a comprehensive overview of the parametric uncertainty modeling process, including: the development of parameterized models to characterize parameter-dependent nonlinear data associated with a given system; the development of an LPV model of the system; representation of the LPV model in LFT form; and scaling of the LFT model to obtain the standard normalized P-Δ model form. Details associated with each of these steps were presented or referenced in this paper.

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