CHAPTER 4

STABILITY THEORY

In this chapter it is shown how the standard problem can be reduced to the model-matching problem. The procedure is to parametrize, via a parameter matrix Q in $\mathbf{RH}_\infty$, all K's which stabilize G.

4.1 Coprime Factorization Over $\mathbf{RH}_\infty$

Recall that two polynomials $f(s)$ and $g(s)$, with, say, real coefficients, are said to be *coprime* if their greatest common divisor is 1 (equivalently, they have no common zeros). It follows from Euclid's algorithm that f and g are coprime iff there exist polynomials $x(s)$ and $y(s)$ such that

$$fx + gy = 1\,. \tag{1}$$

Such an equation is called a Bezout identity.

We are going to take the practical route and define two functions f and g in $\mathbf{RH}_\infty$ to be *coprime* (over $\mathbf{RH}_\infty$) if there exist x, y in $\mathbf{RH}_\infty$ such that (1) holds. (The more primitive, but equivalent, definition is that f and g are coprime if every common divisor of f and g is invertible in $\mathbf{RH}_\infty$, i.e.

$$h,\ fh^{-1},\ gh^{-1} \in \mathbf{RH}_\infty \Rightarrow h^{-1} \in \mathbf{RH}_\infty.)$$

More generally, two matrices F and G in $\mathbf{RH}_\infty$ are *right-coprime* (over $\mathbf{RH}_\infty$) if they have equal number of columns and there exist matrices X and Y in $\mathbf{RH}_\infty$ such that

$$[X \;\; Y]\begin{bmatrix} F \\ G \end{bmatrix} = XF + YG = I\,.$$

This is equivalent to saying that the matrix $\begin{bmatrix} F \\ G \end{bmatrix}$ is left-invertible in $\mathbf{RH}_\infty$.

Similarly, two matrices F and G in $\mathbf{RH}_\infty$ are *left-coprime* (over $\mathbf{RH}_\infty$) if they have equal number of rows and there exist X and Y in $\mathbf{RH}_\infty$ such that

$$[F \;\; G]\begin{bmatrix} X \\ Y \end{bmatrix} = FX + GY = I\,;$$

equivalently, $[F \;\; G]$ is right-invertible in $\mathbf{RH}_\infty$.

Now let G be a proper real-rational matrix. A *right-coprime factorization* of G is a factorization $G = NM^{-1}$ where N and M are right-coprime $\mathbf{RH}_\infty$-matrices. Similarly, a *left-coprime factorization* has the form $G = \tilde{M}^{-1}\tilde{N}$ where $\tilde{N}$ and $\tilde{M}$ are left-coprime. Of course implicit in these definitions is the requirement that M and $\tilde{M}$ be square and non-singular. We shall require special coprime factorizations, as described in the next lemma.

Lemma 1. For each proper real-rational matrix G there exist eight $\mathbf{RH}_\infty$-matrices satisfying the equations

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{2}$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I\,. \tag{3}$$

Equations (2) and (3) together constitute a *doubly-coprime factorization* of G. It should be apparent that N and M are right-coprime and $\tilde{N}$ and $\tilde{M}$ are left-coprime; for example, (3) implies

$$[\tilde{X} \;\; -\tilde{Y}]\begin{bmatrix} M \\ N \end{bmatrix} = I\,,$$

proving right-coprimeness.

It's useful to prove Lemma 1 constructively by deriving explicit formulas for the eight matrices. The formulas use state-space realizations, and hence are readily amenable to computer implementation.

We start with a state-space realization of G,

$$G(s) = D + C(s-A)^{-1}B \tag{4}$$

$$A,\ B,\ C,\ D \text{ real matrices,}$$

with (A,B) stabilizable and (C,A) detectable. It's convenient to introduce a new data structure: let

$$[A, B, C, D]$$

stand for the transfer matrix

$$D + C(s-A)^{-1}B .$$

Now introduce state, input, and output vectors x, u, and y respectively so that $y = Gu$ and

$$\dot{x} = Ax + Bu \tag{5a}$$

$$y = Cx + Du . \tag{5b}$$

Next, choose a real matrix F such that $A_F := A + BF$ is stable (all eigenvalues in Re $s < 0$) and define the vector $v := u - Fx$ and the matrix $C_F := C + DF$. Then from (5) we get

$$\dot{x} = A_F x + Bv$$

$$u = Fx + v$$

$$y = C_F x + Dv .$$

Evidently from these equations the transfer matrix from v to u is

$$M(s) := [A_F, B, F, I] \tag{6a}$$

and that from v to y is

$$N(s) := [A_F, B, C_F, D] . \tag{6b}$$

Therefore

$$u = Mv , \quad y = Nv$$

so that $y = NM^{-1}u$, i.e. $G = NM^{-1}$.

Similarly, by choosing a real matrix H so that $A_H := A + HC$ is stable and defining

$$B_H := B + HD$$

$$\tilde{M}(s) := [A_H, H, C, I] \tag{6c}$$

$$\tilde{N}(s) := [A_H, B_H, C, D] , \tag{6d}$$

we get $G = \tilde{M}^{-1}\tilde{N}$. (This can be derived as above by starting with $G(s)^T$ instead of $G(s)$.) Thus we've obtained four matrices in $\mathbf{RH}_\infty$ satisfying (2).

Formulas for the other four matrices to satisfy (3) are as follows:

$$X(s) := [A_F, -H, C_F, I] \tag{7a}$$

$$Y(s) := [A_F, -H, F, 0] \tag{7b}$$

$$\tilde{X}(s) := [A_H, -B_H, F, I] \tag{7c}$$

$$\tilde{Y}(s) := [A_H, -H, F, 0]\,. \tag{7d}$$

The explanation of where these latter four formulas come from is deferred to Section 4.

Exercise 1. Verify that the matrices in (6) and (7) satisfy (3).

Example 1.

As an illustration of the use of these formulas, consider the scalar-valued example

$$G(s) = \frac{s-1}{s(s-2)}\,.$$

A minimal realization is

$$G(s) = [A, B, C, D]$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [-1 \ \ 1], \quad D = 0.$$

Choosing F to place the eigenvalues of A_F (arbitrarily) at $\{-1, -1\}$, we get

$$F = [-1 \ \ -4]$$

$$A_F = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

Then

$$N(s) = [A_F, B, C, 0]$$

$$= \frac{s-1}{(s+1)^2}$$

and

$$M(s) = [A_F, B, F, 1]$$

$$= \frac{s(s-2)}{(s+1)^2}\,.$$

Similarly, the assignment

$$H = \begin{bmatrix} -5 \\ -9 \end{bmatrix}$$

yields

$$A_H = \begin{bmatrix} 5 & -4 \\ 9 & -7 \end{bmatrix}$$

$$\begin{aligned} X(s) &= [A_F, -H, C, 1] \\ &= \frac{s^2+6s-23}{(s+1)^2} \end{aligned}$$

$$\begin{aligned} Y(s) &= [A_F, -H, F, 0] \\ &= \frac{-41s+1}{(s+1)^2} . \end{aligned}$$

Finally, in this example we have

$$\tilde{N} = N, \ \tilde{M} = M, \ \tilde{X} = X, \ \tilde{Y} = Y .$$

4.2 Stability

This section provides a test for when a proper real-rational K stabilizes G. Introduce left- and right-coprime factorizations

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{1a}$$

$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U} . \tag{1b}$$

Theorem 1. The following are equivalent statements about K:

(i) K stabilizes G,

(ii) $$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0 \ I]N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty ,$$

(iii) $$\begin{bmatrix} \tilde{M} & \tilde{N}\begin{bmatrix} 0 \\ I \end{bmatrix} \\ \tilde{U}[0\ I] & \tilde{V} \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

The idea underlying the equivalence of (i) and (ii) is simply that the determinant of the matrix in (ii) is the least common denominator (in $\mathbf{RH}_\infty$) of all the transfer functions from w, v_1, v_2 to z, u, y; hence the determinant must be invertible for all these transfer functions to belong to $\mathbf{RH}_\infty$, and conversely.

The proof of Theorem 1 requires a preliminary result. Insert the factorizations (1) into Figure 1, split apart the factors, and introduce two new signals ξ and η to get Figure 2.

Lemma 1. The nine transfer matrices in Figure 1 from w, v_1, v_2 to z, u, y belong to $\mathbf{RH}_\infty$ iff the six transfer matrices in Figure 2 from w, v_1, v_2 to ξ, η belong to $\mathbf{RH}_\infty$.

Proof. (If) This direction follows immediately from the equations

$$\begin{bmatrix} z \\ y \end{bmatrix} = N\xi + \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

$$u = U\eta + v_1 ,$$

which in turn follow from Figure 2.

(Only if) By right-coprimeness there exist $\mathbf{RH}_\infty$-matrices X and Y such that

$$XM + YN = I .$$

Hence

$$\xi = XM\xi + YN\xi . \tag{2}$$

But from Figure 2

$$M\xi = \begin{bmatrix} w \\ u \end{bmatrix} , \ N\xi = \begin{bmatrix} z \\ y - v_2 \end{bmatrix} .$$

Substitution into (2) gives

$$\xi = X\begin{bmatrix} 0 \\ u \end{bmatrix} + Y\begin{bmatrix} z \\ y \end{bmatrix} + X\begin{bmatrix} w \\ 0 \end{bmatrix} - Y\begin{bmatrix} 0 \\ v_2 \end{bmatrix} .$$

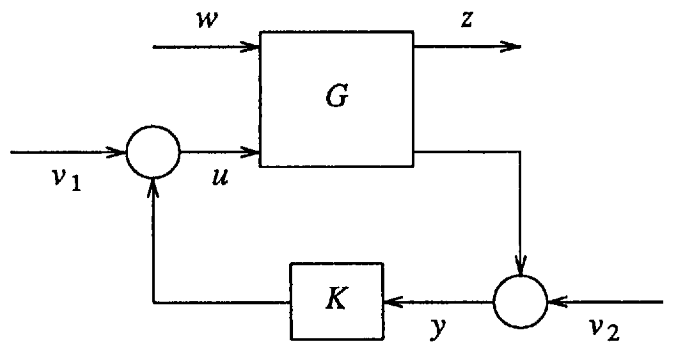

Figure 4.2.1. Diagram for stability definition

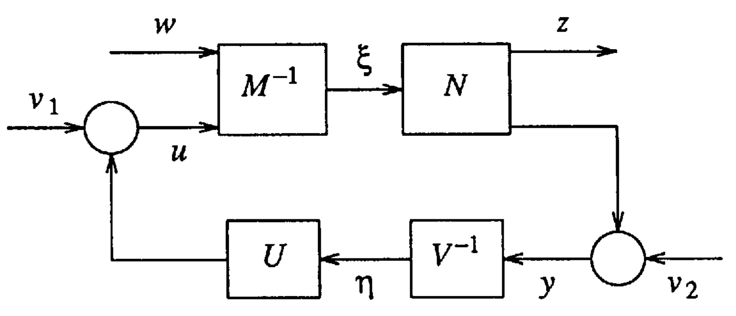

Figure 4.2.2. With internal signals

Hence the three transfer matrices from w, v_1, v_2 to ξ belong to $\mathbf{RH}_\infty$.

A similar argument works for the remaining three transfer matrices to η. □

Proof of Theorem 1. We shall prove the equivalence of (i) and (ii). First, let's see that the matrix displayed in (ii) is indeed nonsingular, i.e. its inverse exists as a rational matrix. We have

$$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0\ I]N & V \end{bmatrix} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ I \end{bmatrix} K \\ [0\ I]G & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & K \\ G_{21} & G_{22} & I \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}. \tag{3}$$

Now

$$\begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}$$

is nonsingular because both M and V are. Also, since G_{22} is strictly proper, we have that

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & K \\ G_{21} & G_{22} & I \end{bmatrix}$$

is nonsingular when evaluated at $s=\infty$: its determinant equals 1 at $s=\infty$. Thus both matrices on the right-hand side of (3) are nonsingular.

The equations corresponding to Figure 2 are

$$\begin{bmatrix} M & -\begin{bmatrix} 0 \\ I \end{bmatrix} U \\ -[0\ I]N & V \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} w \\ v_1 \end{bmatrix} \\ v_2 \end{bmatrix}.$$

Thus by Lemma 1 K stabilizes G iff

$$\begin{bmatrix} M & -\begin{bmatrix} 0 \\ I \end{bmatrix} U \\ -[0\ I]N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

But this is equivalent to (ii). □

Exercise 1. Prove equivalence of (i) and (iii) in Theorem 1.

4.3 Stabilizability

Let's say that G is *stabilizable* if there exists a (proper real-rational) K which stabilizes it. Not every G is stabilizable; an obvious non-stabilizable G is $G_{12}=0$, $G_{21}=0$, $G_{22}=0$, G_{11} unstable. In this example, the unstable part of G is disconnected from u and y. In terms of a state-space model G is stabilizable iff its unstable modes are controllable from u (stabilizability) and observable from y (detectability). The next result is a stabilizability test in terms of left- and right-coprime factorizations

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

Theorem 1. The following conditions are equivalent:

(i) G is stabilizable,

(ii) M, $[0\ I]N$ are right-coprime and

M, $\begin{bmatrix} 0 \\ I \end{bmatrix}$ are left-coprime,

(iii) $\tilde{M}, \tilde{N}\begin{bmatrix} 0 \\ I \end{bmatrix}$ are left-coprime and

$\tilde{M}$, $[0\ I]$ are right-coprime.

The proof requires some preliminaries. The reader will recall the following fact. For each real matrix F there exist real matrices G and H such that

$$F = G\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} H.$$

The matrices G and H may be obtained by elementary row and column operations, and the size of the identity matrix equals the rank of F. The following analogous result for $\mathbf{RH}_\infty$-matrices is stated without proof.

Lemma 1. For each matrix F in $\mathbf{RH}_\infty$ there exist matrices G, H, and F_1 in $\mathbf{RH}_\infty$ satisfying the equation

$$F = G \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix} H$$

and having the properties that G and H are invertible in $\mathbf{RH}_\infty$ and F_1 is nonsingular.

This result is now used to prove the following useful fact that if M and N are right-coprime, then the matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ can be filled out to yield a square matrix which is invertible in $\mathbf{RH}_\infty$.

Lemma 2. Let M and N be $\mathbf{RH}_\infty$-matrices with equal number of columns. Then M and N are right-coprime iff there exist matrices U and V in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

Proof. (If) Define

$$\begin{bmatrix} X & Y \\ ? & ? \end{bmatrix} := \begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} ,$$

where a question mark denotes an irrelevant block. Then

$$[X \ Y] \begin{bmatrix} M \\ N \end{bmatrix} = I ,$$

so M and N are right-coprime.

(Only if) Define

$$F := \begin{bmatrix} M \\ N \end{bmatrix}$$

and bring in matrices G, H, and F_1 as per Lemma 1. Since F is left-invertible in $\mathbf{RH}_\infty$ (by right-coprimeness), it follows that

$$\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$$

is left-invertible in $\mathbf{RH}_\infty$ too. But then it must have the form

$$\begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

with $F_1^{-1} \in \mathbf{RH}_\infty$. Defining

$$K := G \begin{bmatrix} F_1 H & 0 \\ 0 & I \end{bmatrix},$$

we get

$$\begin{bmatrix} M \\ N \end{bmatrix} = K \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Thus the definition

$$\begin{bmatrix} U \\ V \end{bmatrix} := K \begin{bmatrix} 0 \\ I \end{bmatrix}$$

gives the desired result, that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = K$$

is invertible in $\mathbf{RH}_\infty$. □

The obvious dual of Lemma 2 is that M and N are left-coprime iff there exist U and V such that

$$\begin{bmatrix} M & N \\ U & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

Proof of Theorem 1. We shall prove equivalence of (i) and (ii).

(i) $\Rightarrow$ (ii): If G is stabilizable, then by Theorem 2.1 there exist U and V in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0\ I]N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

This implies by Lemma 2 and its dual that

$$M,\ [0\ I]N \text{ are right-coprime}$$

and

$$M,\ \begin{bmatrix} 0 \\ I \end{bmatrix} U \text{ are left-coprime.}$$

But the latter condition implies left-coprimeness of

$$M,\ \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

(ii) $\Rightarrow$ (i): Choose, by right-coprimeness and Lemma 2, matrices X and Y in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & X \\ [0\ I]N & Y \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

Also, choose, by left-coprimeness, matrices R and T in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} \begin{bmatrix} R \\ T \end{bmatrix} = I . \tag{1}$$

Now define

$$U := TX \tag{2a}$$

$$V := Y - [0\ I]NRX . \tag{2b}$$

Then we have from (1) and (2) that

$$\begin{bmatrix} M & X \\ [0\ I]N & Y \end{bmatrix} \begin{bmatrix} I & -RX \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0\ I]N & V \end{bmatrix} . \tag{3}$$

The two matrices on the left in (3) have inverses in $\mathbf{RH}_\infty$, hence so does the matrix on the right in (3).

The next step is to show that V is nonsingular. We have

$$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0\ I]N & V \end{bmatrix} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ I \end{bmatrix} U \\ [0\ I]G & V \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & U \\ G_{21} & G_{22} & V \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} . \tag{4}$$

Evaluate all the matrices in (4) at $s=\infty$; then take determinants of both sides noting that G_{22} is strictly proper and the matrix on the left-hand side of (4) is invertible in $\mathbf{RH}_\infty$. This gives

$$0 \neq \det V(\infty) \det M(\infty) .$$

Thus $\det V(\infty) \neq 0$, i.e. V^{-1} exists. Hence we can define $K := UV^{-1}$.

Next, note that U and V are right-coprime (this follows from invertibility in $\mathbf{RH}_\infty$ of the matrix on the right-hand side of (3)). We conclude from Theorem 2.1 that K stabilizes G. □

Exercise 1. Prove equivalence of (i) and (iii) in Theorem 1.

Hereafter, G will be assumed to be stabilizable. Intuitively, this implies that G and G_{22} share the same unstable poles (counting multiplicities), so to stabilize G it is enough to stabilize G_{22}. Let's define the latter concept explicitly: K *stabilizes* G_{22} if in Figure 2.1 the four transfer matrices from v_1 and v_2 to u and y belong to $\mathbf{RH}_\infty$.

Theorem 2. K stabilizes G iff K stabilizes G_{22}.

The necessity part of the theorem follows from the definitions. To prove sufficiency we need a result analogous to Lemma 2.1.

Lemma 3. The four transfer matrices in Figure 2.1 from v_1, v_2 to u, y belong to $\mathbf{RH}_\infty$ iff the four transfer matrices in Figure 2.2 from v_1, v_2 to ξ, η belong to $\mathbf{RH}_\infty$.

The proof is omitted, it being entirely analogous to that of Lemma 2.1.

Proof of Theorem 2. Suppose K stabilizes G_{22}. To prove that K stabilizes G it suffices to show, by Lemma 2.1, that the six transfer matrices in Figure 2.2 from w, v_1, v_2 to ξ, η belong to $\mathbf{RH}_\infty$. But by Lemma 3 we know that those from v_1, v_2 to ξ, η do. So it remains to show that the two from w to ξ, η belong to $\mathbf{RH}_\infty$.

Set v_1=0 and v_2=0 in Figure 2.2 and write the corresponding equations:

$$M\xi = \begin{bmatrix} I \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ I \end{bmatrix} U\eta \tag{5}$$

$$V\eta = [0 \;\; I]N\xi \,. \tag{6}$$

By left-coprimeness there exist matrices R and T in $\mathbf{RH}_\infty$ such that

$$\begin{bmatrix} M & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{bmatrix} \begin{bmatrix} R \\ T \end{bmatrix} = I \,. \tag{7}$$

Post-multiply (7) by $\begin{bmatrix} I \\ 0 \end{bmatrix} w$ to get

$$MR\begin{bmatrix} I \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ I \end{bmatrix} T \begin{bmatrix} I \\ 0 \end{bmatrix} w = \begin{bmatrix} I \\ 0 \end{bmatrix} w \,. \tag{8}$$

Now subtract (8) from (5), rearrange, and define

$$\xi_1 := \xi - R\begin{bmatrix} I \\ 0 \end{bmatrix} w \tag{9}$$

$$v_1 := T\begin{bmatrix} I \\ 0 \end{bmatrix} w \tag{10}$$

to get

$$M\xi_1 = \begin{bmatrix} 0 \\ I \end{bmatrix} (v_1 + U\eta) \,. \tag{11}$$

Also, rearrange (6) and define

$$v_2 := [0 \; I]NR\begin{bmatrix} I \\ 0 \end{bmatrix} w \tag{12}$$

to get

$$V\eta = [0 \; I]N\xi_1 + v_2 \, . \tag{13}$$

The block diagram corresponding to (11) and (13) is Figure 1. By Lemma 3 and the fact that K stabilizes G_{22} we know that the transfer matrices in Figure 1 from v_1, v_2 to ξ_1, η belong to $\mathbf{RH}_\infty$. But by (10) and (12) those from w to v_1, v_2 belong to $\mathbf{RH}_\infty$. Hence those from w to ξ_1, η belong to $\mathbf{RH}_\infty$. Finally, we conclude from (9) that the transfer matrix from w to ξ belongs to $\mathbf{RH}_\infty$. □

Exercise 2. Suppose $G_{11}=G_{12}=G_{21}=G_{22}$. Prove that G is stabilizable.

4.4 Parametrization

This section contains a parametrization of all K's which stabilize G_{22}. To simplify notation slightly, in this section the subscripts 22 on G_{22} are dropped. The relevant block diagram is Figure 1.

Bring in a doubly-coprime factorization of G,

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \, , \tag{1}$$

and coprime factorizations (not necessarily doubly-coprime) of K,

$$K = UV^{-1} = \tilde{V}^{-1}\tilde{U} \, .$$

The first result is analogous to Theorem 2.1; the proof is omitted. **Lemma 1.** The following are equivalent statements about K:

(i) K stabilizes G,

(ii) $\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty$,

(iii) $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \in \mathbf{RH}_\infty$.

The main result of this chapter is the following.

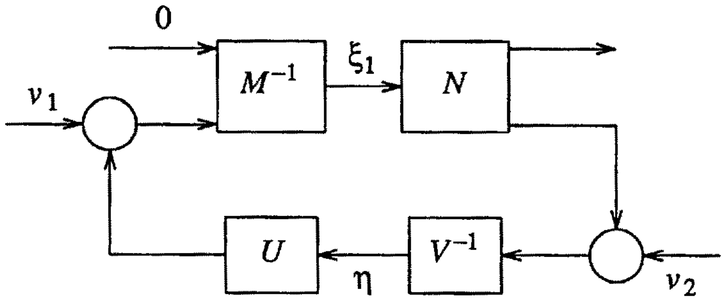

Figure 4.3.1. For proof of Theorem 4.3.2

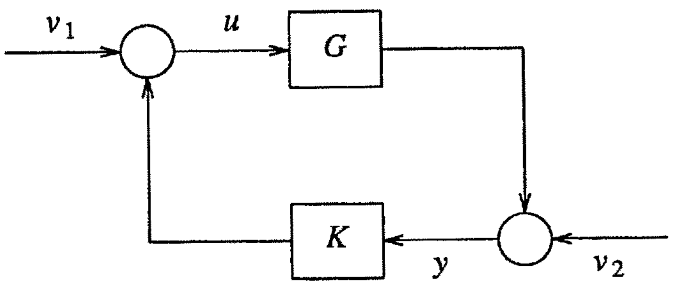

Figure 4.4.1. Diagram for controller parametrization

Theorem 1. The set of all (proper real-rational) K's stabilizing G is parametrized by the formulas

$$K = (Y - MQ)(X - NQ)^{-1} \tag{2}$$

$$= (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}) \tag{3}$$

$$Q \in \mathbf{RH}_\infty .$$

Proof. Let's first prove equality (3). Let $Q \in \mathbf{RH}_\infty$. From (1) we have

$$\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} = I$$

so that

$$\begin{bmatrix} \tilde{X} - Q\tilde{N} & -(\tilde{Y} - Q\tilde{M}) \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y - MQ \\ N & X - NQ \end{bmatrix} = I . \tag{4}$$

Equating the (1,2)-blocks on each side in (4) gives

$$(\tilde{X} - Q\tilde{N})(Y - MQ) = (\tilde{Y} - Q\tilde{M})(X - NQ) ,$$

which is equivalent to (3).

Next, we show that if K is given by (2), it stabilizes G. Define

$$U := Y - MQ , \quad V := X - NQ$$

$$\tilde{U} := \tilde{Y} - Q\tilde{M} , \quad \tilde{V} := \tilde{X} - Q\tilde{N}$$

to get from (4) that

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = I . \tag{5}$$

It follows from (5) that U, V are right-coprime and $\tilde{U}$, $\tilde{V}$ are left-coprime (Lemma 3.2). Also from (5)

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

So from Lemma 1 K stabilizes G.

Finally, suppose K stabilizes G. We must show K satisfies (2) for some Q in $\mathbf{RH}_\infty$. Let $K=UV^{-1}$ be a right-coprime factorization. From (1) and defining $D:=\tilde{M}V-\tilde{N}U$ we have

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & \tilde{X}U-\tilde{Y}V \\ 0 & D \end{bmatrix} . \tag{6}$$

The two matrices on the left in (6) have inverses in $\mathbf{RH}_\infty$, the second by Lemma 1. Hence $D^{-1}\in\mathbf{RH}_\infty$. Define

$$Q:=-(\tilde{X}U-\tilde{Y}V)D^{-1} ,$$

so that (6) becomes

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix} . \tag{7}$$

Pre-multiply (7) by

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix}$$

and use (1) to get

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix} .$$

Therefore

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} (Y-MQ)D \\ (X-NQ)D \end{bmatrix} .$$

Substitute this into $K=UV^{-1}$ to get (2). □

As a special case suppose G is already stable, i.e. $G\in\mathbf{RH}_\infty$. Then in (1) we may take

$$N=\tilde{N}=G$$

$$\tilde{X}=M=I , \quad X=\tilde{M}=I$$

$$Y=0, \quad \tilde{Y}=0,$$

in which case the formulas in the theorem become simply

$$K = -Q(I-GQ)^{-1}$$

$$= -(I-QG)^{-1}Q .$$

There is an interpretation of Q in this case: $-Q$ equals the transfer matrix from v_2 to u in Figure 1 (check this).

We can now explain the idea behind the choice (1.7) of X, Y, $\tilde{X}$, $\tilde{Y}$ in Section 1. Recall that the state-space equations for G were

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du ,$$

that

$$A_F := A + BF , \quad A_H := A + HC$$

were stable, and that we defined

$$B_H := B + HD , \quad C_F := C + DF .$$

Let's find a stabilizing K by observer theory. The familiar state-space equations for K are

$$\dot{\hat{x}} = A\hat{x} + Bu + H(C\hat{x} + Du - y)$$

$$u = F\hat{x} ,$$

or equivalently

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y$$

$$u = \hat{C}\hat{x} ,$$

where

$$\hat{A} := A + BF + HC + HDF = A_F + HC_F$$

$$\hat{B} := -H$$

$$\hat{C} := F .$$

Thus in terms of our data structure

$$K(s) = [\hat{A}, \hat{B}, \hat{C}, 0] .$$

By observer theory K stabilizes G.

Now find coprime factorizations of K in the same way as we found coprime factorizations of G in Section 1. To get a right-coprime factorization $K = YX^{-1}$ we first choose $\hat{F}$ so that $\hat{A}_F := \hat{A} + \hat{B}\hat{F}$ is stable. It is convenient to take $\hat{F} := C_F$, so that $\hat{A}_F = A_F$. By analogy with (1.6) we get $K = YX^{-1}$, where

$$X(s) := [\hat{A}_F, \hat{B}, \hat{F}, I]$$

$$= [A_F, -H, C_F, I]$$

$$Y(s) := [\hat{A}_F, \hat{B}, \hat{C}, 0]$$

$$= [A_F, -H, F, 0] .$$

A similar derivation leads to a left-coprime factorization $K = \tilde{X}^{-1}\tilde{Y}$, where

$$\tilde{X}(s) := [A_H, -B_H, F, I]$$

$$\tilde{Y}(s) := [A_H, -H, F, 0] .$$

These formulas coincide with (1.7).

By Lemma 1 we know that

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1} \in \mathbf{RH}_\infty$$

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \in \mathbf{RH}_\infty .$$

Hence the product

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix}$$

must be invertible in $\mathbf{RH}_\infty$. The only surprise is that the product equals the identity matrix, as is verified by algebraic manipulation.

Exercise 1. In Figure 4 suppose $G(s) = \dfrac{1}{s(s-1)}$. Consider a controller of the form

$$K = \frac{-Q}{1-GQ}$$

where Q is real-rational. Find necessary and sufficient conditions on Q in order that K

stabilize G.

4.5 Closed-Loop Transfer Matrices

Now we return to the standard set-up of Figure 1, Chapter 3. Theorem 4.1 gives every stabilizing K as a transformation of a free parameter Q in $\mathbf{RH}_\infty$. The objective in this section is to find the transfer matrix from w to z in terms of Q.

In the previous section we dropped the subscripts on G_{22}; now we must restore them. Bring in a doubly-coprime factorization of G_{22}:

$$G_{22} = N_2 M_2^{-1} = \tilde{M}_2^{-1} \tilde{N}_2$$

$$\begin{bmatrix} \tilde{X}_2 & -\tilde{Y}_2 \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix} \begin{bmatrix} M_2 & Y_2 \\ N_2 & X_2 \end{bmatrix} = I . \tag{1}$$

Then the formula for K is

$$K = (Y_2 - M_2 Q)(X_2 - N_2 Q)^{-1} \tag{2a}$$

$$= (\tilde{X}_2 - Q\tilde{N}_2)^{-1} (\tilde{Y}_2 - Q\tilde{M}_2) . \tag{2b}$$

Now define

$$T_1 := G_{11} + G_{12} M_2 \tilde{Y}_2 G_{21} \tag{3a}$$

$$T_2 := G_{12} M_2 \tag{3b}$$

$$T_3 := \tilde{M}_2 G_{21} . \tag{3c}$$

Theorem 1. The matrices T_i (i=1–3) belong to $\mathbf{RH}_\infty$. With K given by (2) the transfer matrix from w to z equals $T_1 - T_2 Q T_3$.

Proof. The first statement follows from the realizations to be given below. For the second statement we have

$$z = [G_{11} + G_{12}(I - KG_{22})^{-1} K G_{21}] w . \tag{4}$$

Substitute $G_{22} = N_2 M_2^{-1}$ and (2b) into $(I - KG_{22})^{-1}$ and use (1) to get

$$(I - KG_{22})^{-1} = M_2 (\tilde{X}_2 - Q\tilde{N}_2) .$$

Thus from (2b) again

$$(I-KG_{22})^{-1}K=M_2(\tilde{Y}_2-Q\tilde{M}_2).$$

Substitute this into (4) and use the definitions of T_i to get

$$z=(T_1-T_2QT_3)w. \ \square$$

For computations it is useful to have explicit realizations of the transfer matrices T_i (i=1–3). Start with a minimal realization of G:

$$G(s)=[A,\ B,\ C,\ D].$$

Since the input and output of G are partitioned as

$$\begin{bmatrix} w \\ u \end{bmatrix}, \ \begin{bmatrix} z \\ y \end{bmatrix},$$

the matrices B, C, and D have corresponding partitions:

$$B=[B_1\ \ B_2]$$

$$C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

Then

$$G_{ij}(s)=[A,\ B_j,\ C_i,\ D_{ij}], \quad i,j=1,2.$$

Note that D_{22}=0 because G_{22} is strictly proper. It can be proved that stabilizability of G (an assumption from Section 3) implies that (A,B_2) is stabilizable and (C_2,A) is detectable.

Next, find a doubly-coprime factorization of G_{22} as developed in Section 1. For this choose F and H so that

$$A_F:=A+B_2F, \ A_H:=A+HC_2$$

are stable. Then the formulas are as follows:

$$M_2(s)=[A_F, B_2, F, I]$$

$$N_2(s)=[A_F, B_2, C_2, 0]$$

$$\tilde{M}_2(s)=[A_H, H, C_2, I]$$

$$\tilde{N}_2(s)=[A_H, B_2, C_2, 0]$$

$$X_2(s)=[A_F, -H, C_2, I]$$

$$Y_2(s)=[A_F, -H, F, 0]$$

$$\tilde{X}_2(s)=[A_H, -B_2, F, I]$$

$$\tilde{Y}_2(s)=[A_H, -H, F, 0]\,.$$

Finally, substitution into (3) yields the following realizations:

$$T_1(s)=[\underline{A}, \underline{B}, \underline{C}, D_{11}]$$

$$\underline{A}=\begin{bmatrix} A_F & -B_2F \\ 0 & A_H \end{bmatrix}$$

$$\underline{B}=\begin{bmatrix} B_1 \\ B_1+HD_{21} \end{bmatrix}$$

$$\underline{C}=[C_1+D_{12}F \;\; -D_{12}F]$$

$$T_2(s)=[A_F, B_2, C_1+D_{12}F, D_{12}]$$

$$T_3(s)=[A_H, B_1+HD_{21}, C_2, D_{21}]\,.$$

It can be observed that $T_i \in \mathbf{RH}_\infty$ ($i=1$–3), as claimed in Theorem 1. For example, this is how the realization of T_2 is obtained:

$$\begin{aligned} T_2(s)&=G_{12}(s)M_2(s) \\ &=[A, B_2, C_1, D_{12}]\times[A_F, B_2, F, I] \\ &=\left[\begin{bmatrix} A & B_2F \\ 0 & A_F \end{bmatrix}, \begin{bmatrix} B_2 \\ B_2 \end{bmatrix}, [C_1 \;\; D_{12}F], D_{12}\right] \\ &=\left[\begin{bmatrix} A & 0 \\ 0 & A_F \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, [C_1 \;\; C_1+D_{12}F], D_{12}\right] \qquad (5) \\ &=[A_F, B_2, C_1+D_{12}F, D_{12}]\,. \end{aligned}$$

Similarity transformation by $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ was used in (5).

Example 1.

Consider the tracking example of Chapter 3 with

$$P(s)=\frac{s-1}{s(s-2)}$$

$$W(s)=\frac{s+1}{10s+1}$$

and ρ=1. We have

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$G_{11}(s)=\begin{bmatrix} \frac{s+1}{10s+1} \\ 0 \end{bmatrix},\ G_{12}(s)=\begin{bmatrix} -\frac{s-1}{s(s-2)} \\ 1 \end{bmatrix}$$

$$G_{21}(s)=G_{11}(s),\ G_{22}(s)=\begin{bmatrix} 0 \\ \frac{s-1}{s(s-2)} \end{bmatrix}.$$

A minimal realization of G is

$$G(s)=[A,B,C,D]$$

$$A=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C=\begin{bmatrix} .09 & -1 & 1 \\ 0 & 0 & 0 \\ .09 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ D=\begin{bmatrix} .1 & 0 \\ 0 & 1 \\ .1 & 0 \\ 0 & 0 \end{bmatrix}.$$

For F and H we may take

$$F=[0\ -3\ -1]$$

$$H=\begin{bmatrix} 0 & 0 \\ 0 & -9 \\ 0 & -5 \end{bmatrix}.$$

Then

$$A_F=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ A_H=\begin{bmatrix} -.1 & 0 & 0 \\ 0 & -7 & 9 \\ 0 & -4 & 5 \end{bmatrix}$$

$$\underline{A}=\begin{bmatrix} -.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & 9 \\ 0 & 0 & 0 & 0 & -4 & 5 \end{bmatrix}, \ \underline{B}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{C}=\begin{bmatrix} .09 & -1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 3 & 1 \end{bmatrix}.$$

Finally,

$$T_1(s)=\begin{bmatrix} \dfrac{s+1}{10s+1} \\ 0 \end{bmatrix}$$

$$T_2(s)=\begin{bmatrix} -\dfrac{s-1}{s^2+s+1} \\ \dfrac{s(s-2)}{s^2+s+1} \end{bmatrix}$$

$$T_3(s)=T_1(s).$$

Exercise 1. It's desired to find a K which stabilizes G and makes the dc gain from w to z equal to zero (asymptotic rejection of steps). Give an example of a stabilizable G for which no such K exists. Find necessary and sufficient conditions (in terms of T_i, i=1–3) for such K to exist.

The results of this chapter can be summarized as follows. The matrix G is assumed to be proper, with G_{22} strictly proper. Also, G is assumed to be stabilizable. The formula (2) parametrizes all K's which stabilize G. In terms of the parameter Q the transfer matrix from w to z equals $T_1-T_2QT_3$. Such a function of Q is called *affine*.

In view of these results the standard problem can be solved as follows: First, find a Q in $\mathbf{RH}_\infty$ to minimize $\|T_1-T_2QT_3\|_\infty$, i.e. solve the model-matching problem of Chapter 3. Then obtain a controller K by substituting Q into (2).

Notes and References

The material of this chapter is based on Doyle (1984). Earlier relevant references are Chang and Pearson (1978) and Pernebo (1981); a more general treatment is given in Nett (1985).

As a general reference for the material of this chapter see Vidyasagar (1985a). The idea of doing coprime factorization over $\mathbf{RH}_\infty$ is due to Vidyasagar (1972), but the idea was first fully exploited by Desoer *et al.* (1980). The state-space formulas in Section 1 are from Nett *et al.* (1984). The important parametrization of Theorem 4.1 is due to Youla *et al.* (1976) as modified by Desoer *et al.* (1980). Finally, see Minto (1985) for a comprehensive treatment of stability theory by state-space methods.