Multivariable Control Problems

Coursework

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1 Topology and Mobility Analysis

Exercise 1

Serial Robot

The topology tree will contain seven (including the base) nodes depicting the links and six edges depicting the joints. The base link is depicted by B.

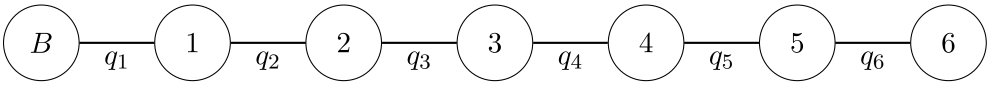

Figure 1.1: Graph Topology of Serial Robot

Parallel Robot

The topology tree will contain fourteen (including the top-platform and base) nodes depicting the links and eighteen edges depiciting the joints (six of which are the prismatic joints). The prismatic joint edges are colored green and the spherical joints are colored blue. The base link is depicted by B and the top platform is depicted by EE.

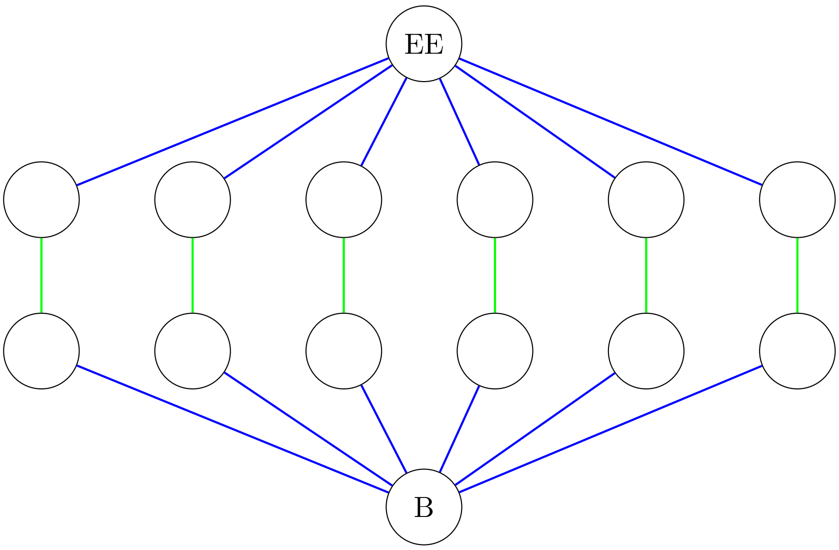

Figure 1.2: Graph Topology of Parallel Robot

Exercise 2

Both the serial and parallel robots are considered spatial mechanisms hence for both $s = 6$.

For the serial robot which has six revolute joints (one degree of freedom each),

$$N = 7, j = 6, f = 6$$
$$m_s = 6$$

The parallel robot has twelve spherical joints (three degrees of freedom each) and six prismatic joints (one degree of freedom each) with fourteen links.

$$N = 14, j = 18, f = 42$$
$$m_s = 12$$

Simplification for Serial Robots

The Chebychev-Grübler-Kutzbach formula can be simplified for serial manipulators which consist of no redundant degrees of freedom (eg. due to multiple closed kinematic chains). The simplification is given by

$$m_s = \sum_{i=1}^{j} f_i \tag{1.1}$$

where f_i is the degrees of freedom allowed by the i^{th} joint.

Modification for Parallel Robots

It is clear that (1.1) doesn't apply to parallel mechanisms which have multiple kinematic loops and hence redundant degrees of freedom. Moreover, the original Chebychev-Grübler-Kutzbach formula also sometimes doesn't work for all parallel mechanisms. For instance, the parallel mechanism given in this question is the *Stewart Platform*, which always provides six degrees of freedom to the top platform, hence the formula should ideally give 6 as an answer (which infact happens for the configuration where universal joints connect the legs and the base). But the redundant degrees of freedom introduced due to spherical joints on both sides of the legs are not considered. This redundant degree of freedom is the rotation of the leg about its own axis, which essentially has no influence on the resultant mechanism motion. Hence, this redundancy needs to be compensated. For six legs, the redundant degrees of freedom will be six. The modified Chebychev-Grübler-Kutzbach becomes

$$m_s = s(N - j - 1) + \sum_{i=1}^{j} f_i - C \tag{1.2}$$

where $C = 6$ and we get the correct answer of six for the given parallel mechanism.

Exercise 3

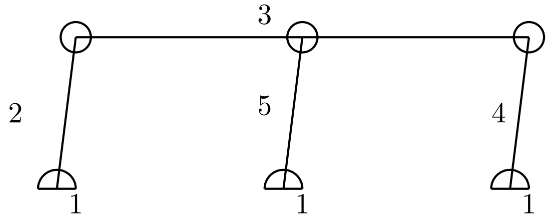

Figure 1.3: Four-bar linkage with one additional link and two additional revolute joints

Figure 1.3 depicts a two-dimensional four-bar linkage with an additional link (5) and two additional revolute joints connecting link 5 to link 3 and the base. According to the Chebychev-Grübler-Kutzbach formula

$$N = 5, j = 6, f = 6, s = 3$$
$$m_s = 0$$

But in reality, the described mechanism will have a single degree of freedom. Hence, in this case the formula fails. The reason is that in the above mechanism, the constraint introduced by the additional joints is redundant, and the instrinsic assumption of the Chebychev-Grübler-Kutzbach formula is that the constraints provided by the joints are independent.

2 Geometry and Kinematics

Exercise 4

The geometric object in which the end-effector lives is called the *Workspace*.

Forward Kinematics

The formulation of the forward kinematics is based on [?], which makes use of the *Denavit-Hartenberg* parameters.

For the given formulation, it is assumed that $q_1 \in [0, \pi]$ and $q_2 \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

For brevity, only the Z and X axes of the joint frames are drawn in 2D, axis coming out of the plane is denoted by $\oplus$ and axis going into the plane is denoted by $\otimes$.

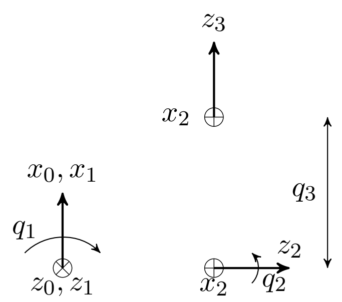

Figure 2.1: Joint Frame Assignment

Figure 2.1 illustrates the joint frame allocation along with the base frame $\{0\}$.

Frame (i)	α_{i-1} [deg]	a_{i-1} [m] d_i [m]	θ_i [deg]
1	0	0	q_1
2	-90	0	q_2
3	90	q_3	0

Table 2.1: Denavit-Hartenberg parameters

Table 2.1 summarizes the DH parameters according to the frame allocation of the manipulator. The homogenous transformation matrix is used to represent the pose. The transformation matrix between two arbitrary frames $\{A\}$ and $\{B\}$ is given by

$$\boldsymbol{T}_B^A = \left[\begin{array}{c|c} \boldsymbol{R}_B^A & \boldsymbol{r}_B^A \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]$$

where $\boldsymbol{R}_B^A$ describes the orientation of $\{B\}$ with respect to $\{A\}$ while $\boldsymbol{r}_B^A$ represents the position of origin of $\{B\}$ with respect to the origin of $\{A\}$. Assuming $E\,(x, y, z)$ directly lies on the origin of $\{3\}$, the forward kinematics are given by

$$\boldsymbol{T}_3^0 = \left[\begin{array}{c|c} \boldsymbol{R}_3^0 & \boldsymbol{r}_3^0 \\ \hline \boldsymbol{0}_{1\times 3} & 1 \end{array}\right]$$

where $\boldsymbol{r}_3^0 = (x\ y\ z)^\intercal$ and

$$\boldsymbol{T}_3^0 = \boldsymbol{T}_1^0 \boldsymbol{T}_2^1 \boldsymbol{T}_3^2$$

where

$$\boldsymbol{T}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ -\sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q_3\cos(q_1)\sin(q_2) \\ q_3\sin(q_1)\sin(q_2) \\ q_3\cos(q_2) \end{bmatrix} \tag{2.1}$$

Inverse Kinematics

Inverse kinematics can be computed with the help of the obtained forward kinematics expression. We know from (2.1)

$$\begin{aligned} x^2 + y^2 + z^2 &= q_3^2 \\ \implies q_3 &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

Moreover,

$$\begin{aligned} \cos(q_2) &= \frac{z}{q_3} = \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \sin(q_2) &= \sqrt{1-\cos(q_2)^2} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \\ \implies \tan(q_2) &= \frac{\sqrt{x^2+y^2}}{z} \\ \implies q_2 &= \arctan 2\left(\sqrt{x^2+y^2}, z\right) \end{aligned}$$

The negative version of $\sin(q_2)$ is not considered since it is assumed that $q_2 \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and hence $\arctan 2$ is used. Finally the expression for q_1 is obtained trivially

$$q_1 = \arctan\left(\frac{y}{x}\right)$$

Since $q_1 \in [0, \pi]$, hence arctan is used and zero values of x represents arbritary values of q_1. Hence, the inverse kinematics can be summarized as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \arctan\left(\frac{y}{x}\right) \\ \arctan 2\left(\sqrt{x^2+y^2}, z\right) \\ \sqrt{x^2+y^2+z^2} \end{bmatrix} \tag{2.2}$$

Program Verification

The verification of kinematics is done in the MATLAB® script *Kinematics.m*. The input DH parameters are read from the file *dhParam.txt*. The script first computes the forward kinematics according to (2.1) for the input DH parameters and then the inverse kinematics according to (2.2). The output of the inverse kinematics matches the inputs in the the file *dhParam.txt*.

Workspace

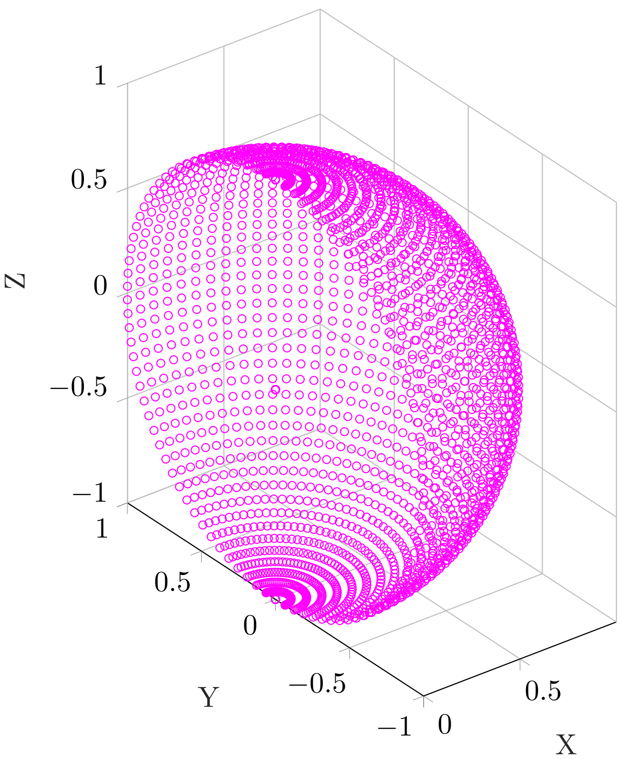

Figure 2.2: 3D Workspace of the manipulator

Figure 2.2 illustrates the three-dimensional workspace of the manipulator for the given joint constraints. It should be noted that the workspace is plotted with respect to the assumed base frame orientation and the present exposition of the workspace figure may not be optimal for illustration.

3 Dynamics

4 Control