CS 103: Mathematical Foundations of Computing
Problem Set #2

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Due Friday, July 12 at 5:30 pm Pacific

Problems One through Six are to be answered by editing the appropriate files (see the Problem Set #2 instructions). You won't include your answers to those problems here.

Symbols Reference

Here are some symbols that may be useful for this PSet. If you are using LaTeX, view this section in the template file (the code in `cs103-ps2-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your Pset in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- Logical AND: $\wedge$
- Logical OR: $\vee$
- Logical NOT: $\neg$
- Logical implies: $\to$
- Logical biconditional: $\leftrightarrow$
- Logical TRUE: $\top$
- Logical FALSE: $\bot$
- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Seven: Yablo's Paradox

i.

Theorem: There does not exist a natural number n where the statement S_n is true.

Proof: We wish to show that there does not a natural number n where the statement S_n is true. Assume, for the sake of contradiction, that there exists a natural number n where the statement S_n is true.

Since S_n is true, we know that for any natural number m, such that $m > n$, the statement S_m is false. This allows us to conclude that the statement S_{n+1} is false.

Since statement S_{n+1} is false, we know that there exists some natural number l, such that $l > n + 1$, where the statement S_l is true.

However, this is impossible due to our prior assumption and the fact that $l > n$, which implies that the statement S_l must be false.

We have arrived at a contradiction and so our assumption must have been wrong. Therefore, there does not exist a natural number n where the statement S_n is true. ■

ii.

Theorem: There does not exist a natural number n where the statement S_n is false.

Proof: We want to show that there does not exist a natural number n where the statement S_n is false. Assume for the sake of contradiction that there exists a natural number n such that the statement S_n is false.

Since the statement S_n is false, we know that there must exist a natural number m, such that $m > n$, where the statement S_m is true.

Since S_m is true, we know that for any natural number l, such that $l > m$, the statement S_l is false.

Similar the the previous question, since S_l is false, there must exist a natural number o, such that $o > l$, where the statement S_o is true.

However, this is impossible due to the earlier fact that since $o > m$, the statement S_o must be false.

We have arrived at a contradiction, which means our inital assumption must have been wrong. Therefore, there does not exist a natural number n where the statement S_n is false. ■

iii.

Since there are no statements after $T_{9,999,999,999}$, this statement is vacuosly true.

Any statement T_n, where $0 <= n < 9,999,999,999$, is false, since if they were true, it must mean that $T_{9,999,999,999}$ is false, which is impossible since as mentioned earlier, it is vacuosly true.

Problem Eight: Hereditary Sets

i.

Theorem: There exists a hereditary set.

Proof: We wish to show that there exists a hereditary set. Let $S = \emptyset$. For S to be a hereditary set, we need to show that every element in S is also a hereditary set.

Since S is the empty set and has no elements, the statement that every element in S is an hereditary set is vacuosly true, which is what we wanted to show. ■

ii.

Theorem: If S is a hereditary set, then $\wp(S)$ is also a hereditary set.

Proof: Pick an arbitrary set S such that S is a hereditary set. We want to show that $\wp(S)$ is also a hereditary set. To do so, pick an arbitrary $y \in \wp(S)$. We need to show that y is also a hereditary set. To do so, pick an arbitrary $x \in y$. We need to show that x is a hereditary set.

Since $y \in \wp(S)$, we know that $y \subseteq S$. Since $x \in y$ and $y \subseteq S$, we know that $x \in S$. Since S is a hereditary set and $x \in S$, we know that x is also a hereditary set, which is what we wanted to show. ■

Problem Nine: Tournament Champions

i.

D is not a champion. E is a champion. ☻

ii.

Theorem: If player c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion.

Proof: Assume for the sake of contradiction that there exists a tournament T such that there exists a player c where c either won the most games in T or is tied for winning the most games and c is not a champion.

Let the number of games that c won be an arbitrary natural number x. Since c is not a tournament champion, there must exist another player p such that p won against c and for every player q other than p and c, if c won against q then p won against q. This implies that player p has $x + 1$ wins, since p beat c, on top of every player that c had beat.

This is impossible, due to earlier assumption that c must have either the most wins or is tied for the most wins. We have arrived at a contradiction and hence, our assumption must have been wrong. Therefore, if player c won more games or is tied for winning the most number of games, then c is a tournament champion. ■

Optional Fun Problem: Insufficient Connectives

Theorem: We cannot express every possible propositional logic formula using just $\leftrightarrow$ and $\bot$.

Proof: Assume for the sake of contradiction that we can express every propositional logic formula using just $\leftrightarrow$ and $\bot$.

Using just $\leftrightarrow$ and $\bot$, we can only generate $\neg$, with $\neg p$ as $p \leftrightarrow \bot$, and $\top$, with $\bot \leftrightarrow \bot$. We are however, unable to generate $\rightarrow$, which is required to express the remaining two other connectives $\wedge$ and $\vee$. Since we cannot generate all the logical connectives from just $\leftrightarrow$ and $\bot$, we cannot express every possible propositional logic formula with just these two connectives.

We have arrived at a contradiction and hence our initial assumption must have been wrong. Therefore, we cannot express every possible propositional logic formula using just $\leftrightarrow$ and $\bot$. ■