CS 103: Mathematical Foundations of Computing
Problem Set #2

[TODO: Replace this with your name(s)]

July 10, 2024

Due Friday, July 12 at 5:30 pm Pacific

Problems One through Six are to be answered by editing the appropriate files (see the Problem Set #2 instructions). You won't include your answers to those problems here.

Symbols Reference

Here are some symbols that may be useful for this PSet. If you are using LaTeX, view this section in the template file (the code in `cs103-ps2-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your Pset in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- Logical AND: $\wedge$
- Logical OR: $\vee$
- Logical NOT: $\neg$
- Logical implies: $\to$
- Logical biconditional: $\leftrightarrow$
- Logical TRUE: $\top$
- Logical FALSE: $\bot$
- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Seven: Yablo's Paradox

i.

- Theorem: There does not exist a natural number n where the statement S_n is true.
- Proof: We wish to show that there does not a natural number n where the statement S_n is true. Assume, for the sake of contradiction, that there exists a natural number n where the statement S_n is true.
- Since S_n is true, we know that for any natural number m, such that $m > n$, the statement S_m is false. This allows us to conclude that the statement S_{n+1} is false. Since statement S_{n+1} is false, we know that there exists some natural number l, such that $l > n + 1$, where the statement S_l is true.
- However, this is impossible due to our prior assumption and the fact that $l > n$, which implies that the statement S_l must be false. We have arrived at a contradiction and so our assumption must have been wrong. Therefore, there does not exist a natural number n where the statement S_n is true. ■

ii.

- Theorem: There does not exist a natural number n where the statement S_n is false.
- Proof: We want to show that there does not exist a natural number n where the statement S_n is false. Assume for the sake of contradiction that there exists a natural number n such that the statement S_n is false.
- Since the statement S_n is false, we know that there must exist a natural number m, such that $m > n$, where the statement S_m is true. Since S_m is true, we know that for any natural number l, such that $l > m$, the statement S_l is false. Similar the the previous question, since S_l is false, there must exist a natural number o, such that $o > l$, where the statement S_o is true. However, this is impossible due to the earlier fact that since $o > m$, the statement S_o must be false. We have arrived at a contradiction, which means our inital assumption must have been wrong. Therefore, there does not exist a natural number n where the statement S_n is false. ■

iii.

- Evaluating T_0:
  - Since there are no statements after $T_{9,999,999,999}$, this statement is vacuosly true.
  - Any statement T_n, where $0 <= n < 9,999,999,999$, is false, since if they were true, it must mean that $T_{9,999,999,999}$ is false, which is impossible since as mentioned earlier, it is vacuosly true.

Problem Eight: Hereditary Sets

i.

- Theorem: There exists a hereditary set.
- Proof: We wish to show that there exists a hereditary set. Let $S = \emptyset$. For S to be a hereditary set, we need to show that every element in S is also a hereditary set.
- Since S is the empty set and has no elements, the proposition that every element in S is an hereditary set is vacuosly true. Therefore, S itself is a hereditary set. ■

ii.

- Theorem: If S is a hereditary set, then $\wp(S)$ is also a hereditary set.
- Proof: Assume that S is a hereditary set. We wish to prove that $\wp(S)$ is a hereditary set. To do so, we need to show that any $y \in \wp(S)$ is a hereditary set.
- Since S is a hereditary set, any $x \in S$ is also a hereditary set. Since any $x \in S$ is a hereditary set, any subset of S is also a hereditary set. And since $\wp(S)$ produces the set whose elements are all possible subsets of S, which are themselves hereditary sets, any $y \in \wp(S)$ is also a hereditary set, which is what we wanted to show. ■ Check for repeating definition.

Problem Nine: Tournament Champions

i.

- D is not a champion. E is a champion. ☻

ii.

- Theorem: If player c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion.
- Proof: Assume for the sake of contradiction that there exists a player c such that c won more games or is tied for winning the greatest number of games and c is not a tournament champion.
- Since c is not a champion, there must exist a player p such that p beat c and for any other player q, q beat c or p beat q. Assume that the tournament now only consists of the two players p and c. Since c is not a champion, p beat c. This must mean that p has more wins than c, which is impossible since earlier we assumed that c must have either won the most games or is tied for the greatest number of wins. We have arrived at a contradiction, which means our inital assumption must have been wrong.
- Therefore, if player c won the most games or is tied for the most number of wins, then c is a tournament champion. ■

Optional Fun Problem: Insufficient Connectives

- Theorem: We cannot express every possible propositional logic formula using just $\leftrightarrow$ and $\perp$.