CS 103: Mathematical Foundations of Computing
Problem Set #6

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Due Friday, August 11 at 4:00 pm Pacific

Some symbols you may want to use here:

- The empty string is denoted ε.
- Alphabets are denoted Σ.
- The language of an automaton is denoted $\mathscr{L}(D)$.
- You can say two strings are distinguishable relative to L by writing $x \not\equiv_L y$.
- You can write a character in monospace `in text mode` and $\texttt{inmathmode}$.
- The Optional Fun Problem uses the notation $\mathscr{F}$.

Problems One and Two are to be answered by editing the appropriate files (`RegularExpressions.regexes` and `Grammars.cfgs`, respectively). Do not put your answers to Problems One and Two in this file.

Problem Three: Finite Languages

Pick an arbitrary finite language L. Let n be the number of strings in L, that is $|L| = n$. We can view L as containing n strings, that is, $L = \{s_1, s_2, \ldots, s_n\}$. A trivial regular expression for L would then be $s_1|s_2|\ldots|s_n$, which ensures that only the n strings in L are accepted.

Consider the edge case where $|L| = 0$. The regular expression for L would then be $\emptyset$, implying that no strings are accepted, as required.

Problem Four: Distinguishability

i.

1. $w = a$
2. $w = \varepsilon$
3. $w = \varepsilon$
4. No such w.

ii.

1. aa and aaa. Both already contain the substring aa.
2. Yes.
3. aa and aaa.
4. Yes.
5. Not a distinguishing set. The two strings $abba$ and $abbababba$.
6. Yes
7. Not a distinguising set. The two strings are ε and b.

iii.

Let $S = \{ab, a, b, baa\}$.

ab and a: $abba \in L$ while $aba \notin L$. $w = ba$
ab and b: $aba \notin L$ while $ba \in L$. $w = a$
ab and baa: $aba \notin L$ while $baaa \in L$. $w = a$

a and b: $aa \notin L$ while $ba \in L$. $w = a$.
a and baa: $aa \notin L$ while $baaa \in L$. $w = a$.

b and baa: $bb \notin L$ while $baab \in L$. $w = b$.

iv.

Theorem: L is not regular.

Proof: Let $S = \{a^n b^n | n \in \mathbb{N}\}$. We will prove that S is infinite and that S is a distinguishing set for L.

To see that S is infinite, observe that S contains one string for each natural number.

To see that S is a distinguishing set for L, pick arbitrary strings $a^n b^n, a^m b^m \in S$ such that $n \neq m$. Let

$w = a^n b^n$. It is trivial that $w \in \Sigma*$. Observe that $a^n b^n a^n b^n \notin L$ and $a^m b^m a^n b^n \in L$. Therefore, we see that $a^n a^n \not\equiv_L a^m b^m$, implying that S is a distinguishing set, as required.

Since S is infinite and a distinguishing set for L, by the Myhill-Nerode Theorem, we see that L is not regular. ■

Problem Five: Brzozowksi's Theorem

i.

1. aab
2. $bbbaaa$
3. bba

ii.

Proof: We want to show that L is not regular. It is given that S is infinite, hence, we only need to show that S is a distinguishing set for L. To do so, pick arbitrary strings $x, y \in S$ such that $x \neq y$. Since $x \neq y$, we know that $\partial_x L \neq \partial_y L$. Since $\partial_x L \neq \partial_y L$, we know that, without loss of generality, there exists a string w such that $w \in \partial_x L$ and $w \notin \partial_y L$. This implies that $xw \in L$ and $yw \notin L$, which tells us that $x \not\equiv_L y$. Hence, we see that S is a distinguishing set for L as required.

Since S is infinite and a distinguishing set for L, by the Myhill-Nerode Theorem, we see that L is not regular. ■

Optional Fun Problem: Birget's Theorem

Write your answer to the Optional Fun Problem here.