Linear Statistical Models — Quiz 07 — Full Name ______________________

1. Consider a drug trial where the first two subjects are sisters, and the third and fourth subjects are unrelated to each other and to the two sisters. Consider the following model:

$$Y_i = b_0 + b_2 x_{i2} + \varepsilon_i$$

where $x_{i2} = 1$ if subject i got the drug and 0 otherwise, and $\varepsilon_i \sim N(0, \sigma^2)$, that is, all variances are σ^2. However, the covariance between the two sisters is $0.5\sigma^2$, and all other covariances are zero.

(i) (3) Write down the covariance matrix Σ for $(Y_1, Y_2, Y_3, Y_4)^T$.

$$\Sigma = \begin{bmatrix} \sigma^2 & \sigma^2/2 & 0 & 0 \\ \sigma^2/2 & \sigma^2 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

(ii) (3) Subjects 1 and 3 got a placebo. Subjects 2 and 4 got the drug. What is the design matrix X?

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(iii) (4) Show that the following estimator is unbiased for b_2:

$$\widehat{B}_2 = \frac{2}{3}(Y_2 - Y_1) + \frac{1}{3}(Y_4 - Y_3)$$

$$E(\hat{B}_2) = \frac{2}{3}\left(E(Y_2) - E(Y_1)\right) + \frac{1}{3}\left(E(Y_4) - E(Y_3)\right)$$

$$= \frac{2}{3}\left(b_0 + b_2 - b_0\right) + \frac{1}{3}\left(b_0 + b_2 - b_0\right)$$

$$= \frac{2}{3} b_2 + \frac{1}{3} b_2 = b_2 \checkmark$$

(iv) (4) Calculate the following variances. Hint: $\text{Cov}(Y_2 - Y_1, Y_4 - Y_3) = 0$

$$\text{Var}(Y_2 - Y_1) = Var(Y_2) + Var(Y_1) - 2\,Cov(Y_2, Y_1)$$

$$= \sigma^2 + \sigma^2 - 2\frac{\sigma^2}{2} = \sigma^2$$

$$\text{Var}(Y_4 - Y_3) = Var(Y_4) + Var(Y_3) - 2\,Cov(Y_4, Y_3)$$

$$= \sigma^2 + \sigma^2 + 0 = 2\sigma^2$$

$$\text{Var}(\tfrac{2}{3}(Y_2 - Y_1) + \tfrac{1}{3}(Y_4 - Y_3)) = \frac{4}{9} Var(Y_2 - Y_1) + \frac{1}{9} Var(Y_4 - Y_3) + 0$$

$$= \frac{4}{9}\sigma^2 + \frac{1}{9}2\sigma^2 = \frac{6}{9}\sigma^2 = \frac{2}{3}\sigma^2$$

2. Consider this dataset and model for y_i:

i	$j(i)$	y_i
1	1	6.2
2	1	9.7
3	2	4.6
4	3	10.2

$$Y_i = b_0 + B_{j(i)} + \varepsilon_i, \quad B_1, \ldots, B_3 \overset{ind}{\sim} N(0, \sigma_1^2), \quad \varepsilon_1, \ldots, \varepsilon_4 \overset{ind}{\sim} N(0, \sigma_2^2)$$

(i) (3) What is the design matrix X?

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(ii) (3) What is the covariance matrix Σ

$$\Sigma = \begin{bmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 & 0 & 0 \\ \sigma_1^2 & \sigma_1^2 + \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_1^2 + \sigma_2^2 & 0 \\ 0 & 0 & 0 & \sigma_1^2 + \sigma_2^2 \end{bmatrix}$$

3. In the moving average model, we have

$$Y_i = b_0 + \rho\varepsilon_{i-1} + \varepsilon_i, \quad \varepsilon_i \overset{ind}{\sim} N(0, \sigma^2)$$

Let $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, and $\boldsymbol{Y} = (Y_1, Y_2, Y_3)^T$.

(i) (3) Find $\boldsymbol{\mu} = E(\boldsymbol{Y})$

$$E(\underline{Y}) = \begin{bmatrix} b_0 \\ b_0 \\ b_0 \\ \cancel{b_0} \end{bmatrix}$$

b.c. $E(Y_i) = b_0 + \rho E(\varepsilon_{i-1}) + E(\varepsilon_i)$

$= b_0 + 0 + 0$

(ii) (4) Find a matrix M such that $\boldsymbol{Y} = \boldsymbol{\mu} + M\boldsymbol{\varepsilon}$. Pay attention to the dimensions.

$Y_1 = b_0 + \rho\varepsilon_0 + \varepsilon_1$

$Y_2 = b_0 + \rho\varepsilon_1 + \varepsilon_2$

$Y_3 = b_0 + \rho\varepsilon_2 + \varepsilon_3$

~~$Y_4 = b_0 + \rho\varepsilon_3 + \varepsilon_4$~~.

$$\underline{M\varepsilon} = \begin{bmatrix} \rho & 1 & 0 & 0 \\ 0 & \rho & 1 & 0 \\ 0 & 0 & \rho & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$

(iii) (3) What is the covariance matrix for $\boldsymbol{Y}$? (Hint, you can use your previous answer).

$$\mathrm{Cov}(\underline{Y}) = M\,\mathrm{Cov}(\underline{\varepsilon})M^T = M\sigma^2 I M^T = \sigma^2 M M^T$$

$$= \sigma^2 \begin{bmatrix} \rho & 1 & 0 & 0 \\ 0 & \rho & 1 & 0 \\ 0 & 0 & \rho & 1 \end{bmatrix} \begin{bmatrix} \rho & 0 & 0 \\ 1 & \rho & 0 \\ 0 & 1 & \rho \\ 0 & 0 & 1 \end{bmatrix} = \sigma^2 \begin{bmatrix} \rho^2+1 & \rho & 0 \\ \rho & \rho^2+1 & \rho \\ 0 & \rho & \rho^2+1 \end{bmatrix}$$

4. (3) Bonus: In the autoregressive model, we have (in simplified form):

$$Y_1 = \frac{\sigma}{\sqrt{1-\rho^2}}\varepsilon_1, \qquad Y_i = \rho Y_{i-1} + \varepsilon_i \quad \text{for } i = 2,3,4, \qquad \boldsymbol{\varepsilon} \sim N(0, \sigma^2 I_4)$$

Find a matrix M such that $\boldsymbol{Y} = (Y_1, Y_2, Y_3, Y_4)^T = M^{-1}\boldsymbol{\varepsilon}$.

$$Y_1 = \frac{\sigma}{\sqrt{1-\rho^2}} \varepsilon_1$$

$$Y_1 \frac{\sqrt{1-\rho^2}}{\sigma} = \varepsilon_1$$

$$Y_2 - \rho Y_1 = \varepsilon_2$$

$$Y_3 - \rho Y_2 = \varepsilon_3$$

$$Y_4 - \rho Y_3 = \varepsilon_4$$

$$\underline{M}\,\underline{Y} = \underline{\varepsilon}$$

$$\begin{bmatrix} \frac{\sigma}{\sqrt{1-\rho^2}} & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 \\ 0 & -\rho & 1 & 0 \\ 0 & 0 & -\rho & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

$$\Downarrow$$

$$\underline{Y} = M^{-1} \underline{\varepsilon}$$