

Exercises for Linear Statistical Models

- Suppose we have data x_1, \dots, x_n and y_1, \dots, y_n , which we model as

$$Y_i = b_0 + b_1 x_i + \varepsilon_i, \quad \varepsilon_i \stackrel{\text{ind}}{\sim} N(0, \sigma^2).$$

- Write down the formula for the sum of squares criterion.
- Take the partial derivatives of the sum of squares criterion to derive the normal equations.
- Solve the normal equations to show that the least squares estimates are

$$\begin{aligned} \hat{b}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ \hat{b}_0 &= \bar{y} - \hat{b}_1 \bar{x} \end{aligned}$$

- In terms of the estimates \hat{b}_0 and \hat{b}_1 , write down the formulas for the fitted values, the residuals, and $\hat{\sigma}^2$ (the variance estimate).
- Explain the difference between an estimate and an estimator.
- Write down how you would explain all of the assumptions of the simple linear model for y_1, \dots, y_n given x_1, \dots, x_n to your grandmother (who was a chemical engineer).
- Suppose we have data x_1, \dots, x_n and y_1, \dots, y_n . The quantity sxy is given to the left of the equals sign and is always equal to **exactly one** of the three expressions to the right of the equals sign. Circle the correct expression and show why sxy is equal to it.

$$\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \quad = \quad \sum_{i=1}^n (x_i - \bar{x})y_i \quad \sum_{i=1}^n (x_i - \bar{x})\bar{y} \quad \sum_{i=1}^n \bar{x}(y_i - x_i)$$

- Derive a formula for $\text{Cov}(\hat{B}_0, \hat{B}_1)$
- Use the formulas for the variances and covariances of \hat{B}_0 and \hat{B}_1 to derive a simplified expression for the prediction variance $\text{Var}(\hat{B}_0 + \hat{B}_1 x_i)$.
- Suppose we have data x_1, \dots, x_n and y_1, \dots, y_n . Which of the following equations qualify as statistical models for y_1, \dots, y_n ?

$$Y_i = b_0 + b_1 x_i + \varepsilon_i, \quad \varepsilon_i \stackrel{\text{ind}}{\sim} N(0, \sigma^2)$$

$$Y_i = b_0 + b_1 x_i$$

$$Y_i = e^{\cos(x_i) + \varepsilon_i}, \quad \varepsilon_i \stackrel{\text{ind}}{\sim} N(0, \sigma^2)$$

$$0 \leq Y_i \leq 10$$

- Under the simple linear model, there is a formula for the least squares estimators of the regression coefficients. Which property of the estimators allows us to conclude that the estimators are normally distributed?

10. Select all that are correct: A confidence interval for the slope b_1 . . .
 - (a) contains the slope values that we rejected using a hypothesis test
 - (b) contains the slope values that we failed to reject using a hypothesis test
 - (c) is a set of plausible values of the slope given the data
 - (d) has endpoints that are random variables
11. Name two desirable properties of estimators, and explain why they are desirable.
12. Let

$$\mathbf{Y} = X\mathbf{b} + \boldsymbol{\varepsilon}, \quad \varepsilon_i \stackrel{ind}{\sim} N(0, \sigma^2 I)$$

where \mathbf{Y} is $n \times 1$ and X is $n \times p + 1$.

What are the dimensions of \mathbf{b} and $\boldsymbol{\varepsilon}$?

What is the expectation of \mathbf{Y} ?

What is the expectation of $X^T \mathbf{Y}$?

What is the expectation of $\hat{\mathbf{B}} = (X^T X)^{-1} X^T \mathbf{Y}$?

What is the covariance matrix for $\boldsymbol{\varepsilon}$?

What is the covariance matrix for $\hat{\mathbf{B}}$?

13. Recall the simple linear model

$$Y_i = b_0 + b_1 x_i + \varepsilon_i, \quad \varepsilon_i \stackrel{ind}{\sim} N(0, \sigma^2).$$

Write down the design matrix X .

Calculate the entries of $X^T X$

Calculate the entries of $(X^T X)^{-1}$

Calculate $X^T \mathbf{y}$.

Show that $\hat{\mathbf{b}} = (X^T X)^{-1} X^T \mathbf{y}$ produces the same least squares estimates that we originally derived. You'll have to use the formula for the inverse of a 2×2 matrix.

14. The residual sum of squares criterion for the multiple linear model is

$$rss(\mathbf{b}^*) = \sum_{i=1}^n (y_i - \sum_{j=0}^p b_j^* x_{ij})^2$$

Derive the k th normal equation by differentiating the residual sum of squares.

15. Show that the set of $p + 1$ normal equations can be written as $X^T \mathbf{y} = X^T X \hat{\mathbf{b}}$

16. Let

$$X = [\mathbf{x}_0 \quad \cdots \quad \mathbf{x}_p]$$

And suppose for this problem that $\mathbf{x}_0, \dots, \mathbf{x}_p$ are orthonormal, which means that $\mathbf{x}_j^T \mathbf{x}_k = 0$ if $j \neq k$ and 1 if $j = k$.

Show that the least squares estimates of \mathbf{b} are $\hat{\mathbf{b}}_j = \mathbf{x}_j^T \mathbf{y}$.

17. What are the two defining properties of projection matrices?

18. Show that $X(X^T X)^{-1} X^T$ is a projection matrix.

19. Suppose that

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix} \text{ and } M = UU^T$$

Calculate the entries of M and show that M is a projection matrix.

20. Consider

$$P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Show that P is a projection matrix.

Describe the space that P projects onto.

21. Show that the fitted values are in the column space of X . In other words, show that the fitted values can be written as a linear combination of the columns of X .

22. Let $\mathbf{b} = (b_0, b_1, b_2)$ and $\hat{\mathbf{B}}$ be the least squares estimator for \mathbf{b} . Further, let $M = \sigma^2(X^T X)^{-1}$, and M_{ij} be the (i, j) entry of M .

What is the expected value of $[0 \quad 1 \quad -1] \hat{\mathbf{B}}$?

What is the variance of $[0 \quad 1 \quad -1] \hat{\mathbf{B}}$?

23. Let

$$\mathbf{Y} = X\mathbf{b} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(0, \sigma^2 I)$$

Recall that if $\mathbf{Z} \sim N(\boldsymbol{\mu}, \Sigma)$, then $\mathbf{a} + M\mathbf{Z} \sim N(\mathbf{a} + M\boldsymbol{\mu}, M\Sigma M^T)$.

What is the distribution of \mathbf{Y} ?

What is the distribution of $X^T \mathbf{Y}$?

What is the distribution of $\hat{\mathbf{B}} = (X^T X)^{-1} X^T \mathbf{Y}$?

Does $\mathbf{Y}^T \mathbf{Y}$ follow a normal distribution? Why or why not?

24. The following is from a longitudinal study measuring heights, weights, etc. of a group of girls. Height2 = height at age 2, Height9 = height at age 9, Height18 = height at age 18, and LegCirc9 = leg circumference at age 9.

Below is output from a regression of Height18 on Height2, Height9, and LegCirc9

Call: `lm(formula = Height18 ~ Height2 + Height9 + LegCirc9)`

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	35.6880	11.2669	3.167	0.00233
Height2	0.2945	0.1827	1.612	0.11163
Height9	0.9016	0.1195	7.547	1.71e-10
LegCirc9	-0.5986	0.2089	-2.865	0.00559

Residual standard error: 3.404 on 66 degrees of freedom

Multiple R-squared: 0.6997, Adjusted R-squared: 0.6861

F-statistic: 51.27 on 3 and 66 DF, p-value: < 2.2e-16

(a) How many individual girls were in this dataset?

(b) The model fit to the height data was

$$Y_i = b_0 + b_1x_{i1} + b_2x_{i2} + b_3x_{i3} + \varepsilon_i, \quad \varepsilon_i \stackrel{ind}{\sim} N(0, \sigma^2),$$

where the covariates are listed in the same order as in the output above. What is the estimate of σ ?

(c) Given an interpretation for the result of the t -test for the Height2 regression coefficient. Why does this make sense for height data?

(d) Can we conclude that height at age 2 is not important for predicting height at age 18?

(e) What is the residual sum of squares for this model?

(f) What is *syx* for this dataset, and what is the standard deviation of the response?

25. Show that $X^T \hat{\mathbf{e}} = \mathbf{0}$, that is, the observed residual vector is orthogonal to every column of the design matrix.

26. Let $T = Z/\sqrt{W/m}$. T has a t distribution with m degrees of freedom if what 3 things are true?

27. Suppose we have a hypothesis for the multiple linear model that can be written as

$$H_0 : \mathbf{c}^T \mathbf{b} = a$$

where \mathbf{c} is a $p + 1$ by 1 vector, a is a known (hypothesized) scalar, and, as usual, \mathbf{b} is the vector of regression coefficients. For example, the hypothesis $H_0 : b_2 - b_1 = 0$ can be written in this form.

(a) Write down the t statistic for this test in terms of $\hat{\mathbf{b}}$, X , and $\hat{\sigma}^2$.

(b) Show that the random version of the t statistic has a T distribution.

28. Here is a dataset with results from the pinewood derby:

i	racer	lane	heat	time
1	Mary	1	1	3.123
2	Suzy	2	1	3.147
3	June	3	1	3.201
4	Mary	3	2	3.133
5	Suzy	1	2	3.168
6	June	2	2	3.192
7	Mary	2	3	3.118
8	Suzy	3	3	3.146
9	June	1	3	3.225

(a) Write down the design matrix for the one-factor model with **racer** as the factor.

(b) Write down the design matrix for the one-factor model with **name** as the factor.

(c) Write down the design matrix for the one-factor model with **heat** as the factor.

29. Suppose we have the factor model:

$$Y_i = b_0 + b_{j(i)} + \varepsilon_i \quad \varepsilon_i \stackrel{ind}{\sim} N(0, \sigma^2)$$

where $j(i)$ is the factor level of the i th observation. Our dataset has the following design matrix:

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) How many levels of the factor are in our dataset?

(b) What is the rank of the design matrix?

(c) Compute $X^T X$.

(d) State which of the following are estimable functions. For those that are, give a linear combination of observations whose expected value is the estimable function.
 $b_0 + b_1$, $b_1 + b_2$, $b_2 - b_3$, $b_1 - 0.5(b_2 + b_3)$, $2b_3 - b_1 - b_2$

(e) Suppose we pick a solution to the normal equations for which $\hat{b}_0 = 0$. Then how do we interpret \hat{b}_2 ?

(f) Suppose we pick a solution to the normal equations for which $\hat{b}_1 = 0$. Then how do we interpret \hat{b}_2 ?

30. Suppose that we have six dogs. Dogs 1 and 2 are greyhounds, dogs 3 and 4 are whippets, and dogs 5 and 6 are Italian greyhounds. We model their weights as:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I_6).$$

Dog 3 is a whippet that weights 26.0 pounds and is the cutest dog you'll ever meet.

- (a) Write these quantities in terms of the notation above:

$$E(Y_1) =$$

$$E(Y_3) =$$

$$E(Y_5) =$$

- (b) Give interpretations in words of the following quantities. If no interpretation exists, write "no interpretation". Hint: use your answers from the previous part.

$$b_0$$

$$b_0 + b_1$$

$$b_3 - b_2$$

- (c) Using mathematical notation, write down the hypothesis that all three breeds of the dogs have the same expected weight.

Should we use a t test or an F test? Give the degrees of freedom for the test.

- (d) Using mathematical notation, write down the hypothesis that we expect a whippet to weigh 15 pounds more than an Italian greyhound.

Should we use a t test or an F test? Give the degrees of freedom for the test.

31. The SAT data contains average SAT scores from each state. Each state was grouped into 1 of 9 regions: ENC, ESC, MA, MTN, NE, PAC, SA, WNC, WSC. Consider the following model for the average SAT math score from the i th state:

$$Y_i = b_0 + b_{j(i)} + \epsilon_i, \quad \epsilon_i \stackrel{ind}{\sim} N(0, \sigma^2),$$

where $j(i)$ is the region number (1-9) of the i th state. Below is R output from fitting the model:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	551.000	8.093	68.088	< 2e-16 ***
regionESC	1.750	12.139	0.144	0.886059
regionMA	-52.333	13.215	-3.960	0.000284 ***

regionMTN	-11.500	10.316	-1.115	0.271285	
regionNE	-49.167	10.957	-4.487	5.52e-05	***
regionPAC	-36.200	11.445	-3.163	0.002899	**
regionSA	-61.333	10.093	-6.077	3.08e-07	***
regionWNC	29.857	10.596	2.818	0.007339	**
regionWSC	-11.750	12.139	-0.968	0.338599	

Residual standard error: 18.1 on 42 degrees of freedom
Multiple R-squared: 0.7733, Adjusted R-squared: 0.7302
F-statistic: 17.91 on 8 and 42 DF, p-value: 2.832e-11

- Which region's coefficient did R set to zero (the reference region)?
- What is the estimated expected SAT score in the reference region?
- What is the estimated expected SAT score in the NE region?
- Which region has the highest estimated expected SAT score, and what is the estimate?
- What is the t statistic for testing whether MA has the same expected SAT score as the reference region?
- How many degrees of freedom in the t statistic from the previous part?
- In terms of b_j 's, write down the hypothesis tested by the F test in the output.
- In the F test, what is the rss for the full model?
- What is the rss for the reduced model? You can use knowledge of the F statistic or the R^2 statistic to solve this. Try it both ways and make sure you get the same answer.
- Think carefully: how many "states" are in this dataset?