Linear Statistical Models — Quiz 03 — Full Name ______________________________

1. We learned the general result:

If $\underset{n\times 1}{\boldsymbol{Z}} \sim N(\boldsymbol{\mu}, \Sigma)$, and if $\underset{k\times 1}{\boldsymbol{a}}$ and $\underset{k\times n}{M}$ are nonrandom, then $\boldsymbol{a} + M\boldsymbol{Z} \sim N(\boldsymbol{a} + M\boldsymbol{\mu}, M\Sigma M^T)$

Apply the general result to specific cases by identifying $\boldsymbol{Z}$, $\boldsymbol{\mu}$, Σ, $\boldsymbol{a}$, and M for the specific cases.

(i) (2) $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 I_n)$, find distribution of $\boldsymbol{Y} = X\boldsymbol{b} + \boldsymbol{\varepsilon}$

$\boldsymbol{Z} = \boldsymbol{\varepsilon}$ $\quad \boldsymbol{\mu} = \underline{0}$ $\quad \Sigma = \sigma^2 I$ $\quad \boldsymbol{a} = X\underline{b}$ $\quad M = I$

$\boldsymbol{Y} \sim N(X\underline{b} + I\underline{0}, I\sigma^2 I I) = N(X\underline{b}, \sigma^2 I)$

(ii) (2) $\boldsymbol{Y} \sim N(?, ?)$, find distribution of $\widehat{\boldsymbol{B}} = (X^TX)^{-1}X^T\boldsymbol{Y}$

$\boldsymbol{Z} = \underline{Y}$ $\quad \boldsymbol{\mu} = X\underline{b}$ $\quad \Sigma = \sigma^2 I$ $\quad \boldsymbol{a} = \underline{0}$ $\quad M = (X^TX)^{-1}X^T$

$\widehat{\boldsymbol{B}} \sim N((X^TX)^{-1}X^TX\underline{b}, (X^TX)^{-1}X^T\sigma^2 I X(X^TX)^{-1})$

$N(\underline{b}, \sigma^2(X^TX)^{-1})$

(iii) (2) $\boldsymbol{Y} \sim N(?, ?)$, find distribution of $\widehat{\boldsymbol{Y}} = X(X^TX)^{-1}X^T\boldsymbol{Y} = P\boldsymbol{Y}$

$\boldsymbol{Z} = \underline{Y}$ $\quad \boldsymbol{\mu} = X\underline{b}$ $\quad \Sigma = \sigma^2 I$ $\quad \boldsymbol{a} = \underline{0}$ $\quad M = P$

$\widehat{\boldsymbol{Y}} \sim N(PX\underline{b}, P\sigma^2 I P^T)$

$N(X\underline{b}, \sigma^2 PP^T) = N(X\underline{b}, \sigma^2 P)$

(iv) (2) $\boldsymbol{Y} \sim N(?, ?)$, find distribution of $\widehat{\boldsymbol{e}} = (I - X(X^TX)^{-1}X^T)\boldsymbol{Y} = (I - P)\boldsymbol{Y}$

$\boldsymbol{Z} = \underline{Y}$ $\quad \boldsymbol{\mu} = X\underline{b}$ $\quad \Sigma = \sigma^2 I$ $\quad \boldsymbol{a} = \underline{0}$ $\quad M = I - P$

$\widehat{\boldsymbol{e}} \sim N((I-P)X\underline{b}, (I-P)\sigma^2 I(I-P)^T)$

$N(Xb - PXb, \sigma^2(I-P)) = N(\underline{0}, \sigma^2(I-P))$

2. Consider the model $\boldsymbol{Y} = X\boldsymbol{b} + \boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma^2 I)$, with design matrix: $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

(i) (3) Express the model by filling in the entries of the matrices below:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

(ii) (4) Do the matrix operations from (i) in order to write the model for each response:

$Y_1 = b_0 + \varepsilon_1$

$Y_2 = b_0 + \varepsilon_2$

$Y_3 = b_0 + b_1 + \varepsilon_3$

$Y_4 = b_0 + b_1 + \varepsilon_4$

(iii) (3) What is $E(Y_3) - E(Y_1)$ in terms of the elements of $\boldsymbol{b}$?

$$E(Y_3) - E(Y_1) = b_0 + b_1 - b_0 = b_1$$

(iv) (3) Given $X^TX = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ and $(X^TX)^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$, what is $\widehat{B}_0$ in terms of elements of $\boldsymbol{Y}$?

$$\underline{\hat{B}} = \begin{bmatrix} \hat{B}_0 \\ \hat{B}_1 \end{bmatrix} = (X^TX)^{-1}X^T\underline{Y} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}Y_1 + \frac{1}{2}Y_2 \\ -\frac{1}{2}Y_1 - \frac{1}{2}Y_2 + \frac{1}{2}Y_3 + \frac{1}{2}Y_4 \end{bmatrix}$$

$$\frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \hat{B}_0$$

(v) (3) What is the variance of $\widehat{B}_0$?

$$\text{Cov}\left(\begin{bmatrix}\hat{B}_0 \\ \hat{B}_1\end{bmatrix}\right) = \sigma^2 (X^T X)^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

so $\text{Var}(\hat{B}_0) = \sigma^2 \cdot \frac{1}{2}$

(vi) (3) What is the correlation between residuals $\widehat{e}_1$ and $\widehat{e}_2$ given that $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

$$\text{Cov}(\hat{e}) = \sigma^2 (I - P) = \sigma^2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\text{Corr}(\hat{e}_1, \hat{e}_2) = \frac{\text{Cov}(\hat{e}_1, \hat{e}_2)}{\sqrt{\text{Var}(\hat{e}_1)\text{Var}(\hat{e}_2)}}$$

$$= \frac{-\sigma^2/2}{\sqrt{\frac{\sigma^2}{2} \cdot \frac{\sigma^2}{2}}} = -1$$

3. (3) $T = Z/\sqrt{W/k}$ has a t distribution with k degrees of freedom if what 3 things are true?

(1) $Z \sim N(0,1)$

(2) $W \sim \chi^2_k$

(3) Z independent of W