① $Var(aZ+bZ') = a^2 Var(Z) + b^2 Var(Z') + 2ab\,Cov(Z, Z')$

Z, Z'

$a, b \in \mathbb{R}$

$* \; var(X) = E[X-E(X)]^2$

$* \; cov(X,Y) = E[(X-E(X))(Y-E(Y))]$

$$
\begin{aligned}
& Var(aZ+bZ') \\
&= E[aZ+bZ']^2 - [E[aZ+bZ']]^2 \\
&= E[a^2Z^2+b^2Z'^2+2abZZ'] - [E[aZ']+E[bZ']]^2 \\
&= E[a^2Z^2]+E[b^2Z'^2]+E[2abZZ'] - (aE[Z]+bE[Z'])^2 \\
&= a^2E[Z^2]+b^2E[Z']^2+2ab\,E[2Z'] - a^2E[Z^2]-a^2[E(Z)]^2+b^2E[Z']^2-b^2[E[Z']]^2+2abE[ZZ']-2abE[Z]E[Z'] \\
&= a^2E[Z^2]+b^2E[Z']^2+2abE[Z]\cdot E[Z'] - a^2E[Z^2]-a^2[E(Z)]^2+b^2E[Z']^2-b^2[E[Z']]^2+2abE[ZZ']-2abE[Z]E[Z'] \\
&= a^2(E[Z^2]-[E[Z]]^2)+b^2(E[Z'^2]-[E[Z']]^2)+2ab(E[ZZ']-E[Z]E[Z']) \\
&= a^2 var(Z) + b^2 var(Z'] + 2ab\,cov[Z,Z']
\end{aligned}
$$

②

a) Does pairwise indep. imply mutual indep.

pairwise indep.:

$P(A_i \cap A_j) = P(A_i)P(A_j)$ $\quad i,j = 1,2,3,\dots,n$, $i \neq j$

$P(A_1 \cap A_2) = P(A_1)P(A_2)$

$P(A_1 \cap A_3) = P(A_1)P(A_3)$

$P(A_2 \cap A_3) = P(A_2)P(A_3)$

mutually indep.

$P(A_1 \cap A_2) = P(A_1)P(A_2)$

$P(A_2 \cap A_3) = P(A_2)P(A_3)$

$P(A_1 \cap A_3) = P(A_1)P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

No, pairwise indep. doesn't imply mutual indep.

The probabilities of A, B, C

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$ $P(C) = \frac{1}{2}$

$P(A) = P(B) = P(C) = \frac{1}{2}$

Pairwise:

$P(A \cap B) = \frac{1}{4}$ $P(A \cap C) = \frac{1}{4}$ $P(B \cap C) = \frac{1}{4}$

$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$

so

$P(A \cap B) = P(A)P(B)$
$P(A \cap C) = P(A)P(C)$
$P(B \cap C) = P(B)P(C)$

Although, $P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$

$\frac{1}{4} \neq \frac{1}{8}$

Thus, A,B,C are not mutually independent and pairwise indep. doesn't imply so

$$Y_n = B_0 + B_1 x_n + \epsilon_n, \; n = 1, \dots, N$$

(3) yes, linear regression can be used to model dynamics for which the regression f'n isn't described as a straight line because the paramaters $(B_0, B_1, \dots B_n)$ are linearly related, the independent variables $(X_1, X_2 \dots X_n)$ however can have non-linear transformations and can be transformed linearly. So, linear regression can be used to model nonlinear data.

(4) $\sum_{n=1}^{N} e_n = 0$

$Y_n = B_0 + B_1 x_n + \epsilon_n, \; n = 1, \dots, N$

$\hat{y} = \hat{B}_0 + \hat{B}_1 x$

$\bar{y} = \bar{B}_0 + B_1 x$

$* \, e_n = y - \hat{y}$

$\sum_{n=1}^{N} e_n = 0$

$= \sum_{n=1}^{N} (y - \hat{y})$

$$= \sum_{n=1}^{N} (y - \hat{B}_0 - \hat{B}_1 x) \quad * \hat{B}_0 = \bar{y} - \hat{B}_1 \bar{x}$$

$$= \sum_{n=1}^{N} (y - (\bar{y} - \hat{B}_1 \bar{x}) - \hat{B}_1 x)$$

$$= \sum_{n=1}^{N} y - \bar{y} \sum_{n=1}^{N} 1 + \hat{B}_1 \sum_{n=1}^{N} x - \hat{B}_1 \sum_{n=1}^{N} x_n$$

$$= N \frac{\sum_{n=1}^{N} y}{N} - \bar{y} N + \hat{B}_1 \bar{x} N - \hat{B}_1 N \frac{\sum_{n=1}^{N} x}{N}$$

$$= N \cancel{\bar{y} - N\bar{y}} + N\bar{x}\cancel{\hat{B}_1 - N\bar{x}}\hat{B}_1$$

$$= 0$$

$$\therefore \sum_{n=1}^{N} e_n = 0$$

5 $E\left[(f(X) - Y)^2\right] \quad = E\left[f(X) - E[Y|X]\right]^2 + E\left[(Y - E[Y|X])^2\right]$

$$= E\left[(f(X) - Y + E[Y|X] - E[Y|X])^2\right]$$

$$= E\left[(f(X) - E[Y|X])^2\right] + E\left[Y - E[Y|X]\right]^2 + \underbrace{E\left[f(X) - E(Y|X)(Y - E[Y|X])\right]}_{=0}$$

$$* E[Y - E(Y|X)] = 0$$

$$= E\left[(f(X) - E[Y|X])\right]^2 + E\left[Y - E[Y|X]\right]^2$$

6) $(\hat{B}_0, \hat{B}_1)$ $\text{argmin}_{(\alpha_0, \alpha_1) \in \mathbb{R}^2} \sum_{n=1}^{N} (y_n - (\alpha_0 + \alpha_1 x_n))^2$

(0,1) (1,0) (1,1) $N=3$

$$S(\alpha_0, \alpha_1) = \sum_{i=1}^{N} (y_i - \overline{\alpha_0 + \alpha_1 x_i})^2$$

$\frac{ds}{d\alpha_0} = \sum_1^N (y_i - \alpha_0 - \alpha_1 x_i)(-2)$
$= 0$

$\alpha_0 + \alpha_1 \bar{x} = \bar{y}$
$\alpha_0 = \bar{y} - \alpha_1 \bar{x}$

$\frac{\partial^2 s}{\partial \alpha_0^2} = (-2)(-N)$ $\quad \frac{\partial^2 s}{\partial \alpha_0 \partial \alpha_1} = (-2)(\sum x_i)$

$\frac{\partial^2 s}{\partial \alpha_1^2} = (-2)(-\sum x_i^2)$

$\frac{ds}{d\alpha_1} = -2\sum_1^N (y_i - \alpha_0 - \alpha_1 x_i)(x_i)$
$= 0$

$\sum(y_i - \bar{y} - \alpha_1 \bar{x} - \alpha_1 x_i)^{x_i = 0}$

$= \alpha_1 \sum (x_i - \bar{x}) x_i = \sum (y_i - \bar{y}) x_i$

$\alpha_1 = \frac{\sum (y_i - \bar{y})(x_i - \bar{x})}{\sum (x_i - \bar{x})^2}$

$\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} = N\sum x_i^2 - (\sum x_i)^2$
$= N[\sum x_i^2 - N\bar{x}^2]$
$= N\sum(x_i - \bar{x})^2 > 0$

So it's pos.

$\therefore (\hat{B}_0, \hat{B}_1)$ is min of $S(\alpha_0, \alpha_1)$

$\sum (x_i - \bar{x})(y_i - \bar{y})$
$= \sum x_i y_i - N\bar{x}\bar{y}$
$= 1 - 3(\frac{2}{3})(\frac{2}{3})$
$= -\frac{1}{3}$

$\sum (x_i - \bar{x})^2$
$= \sum x_i^2 - N\bar{x}^2$
$= 2 - 3(\frac{2}{3})^2$
$= 2 - \frac{4}{3}$
$= \frac{2}{3}$

$\hat{B}_1 = \frac{-\frac{1}{3}}{\frac{2}{3}} = -\frac{1}{2}$ $\quad (\hat{B}_0, \hat{B}_1) = (1, -\frac{1}{2})$

$\hat{B}_0 = \frac{2}{3} + \frac{1}{2}(\frac{2}{3}) = 1$

7)

```
x=data.frame(USJudgeRatings)$FAMI
y=data.frame(USJudgeRatings)$WRIT
fit= lm(y~x)
Coeff=fit$coefficient
Y=Coeff[1]+Coeff[2]*x
Coeff
abline(fit,col=3,lwd=2)
```

(Intercept)	Slope x
-0.1299279	1.0033755
$\hat{B}_0$	$\hat{B}_1$