PSTAT 171. HW 8 Solution (Winter 2021)

`Instruction: Review textbook Chapter 8 for Problems 1-4, review Lecture 16 and then Problems 5-10 from Chapter 9. Multiple reading might help. Then try to solve the homework problems quickly.`

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1. On January 1, 2005, Aliza secured a ten-year \$600,000 loan with annual payments from Trinity National Bank. At the end of each year, she will pay interest due together with \$ 60,000 of principal. The interest rate is reset annually to the one-year prime rate of Trinity National Bank plus 175 basis points. The one-year prime rates for select dates are given in the table below. Calculate the amount of Aliza's third payment.

Date	1-year prime rate
1/1/2005	3.52%
1/1/2006	4.75%
1/1/2007	3.94%
1/1/2008	5.21%

This is a problem of floating rate bond. The principal payment is 60000 each year for ten years. The third payment occurs at the end of 2007. The interest rate is 5.69% (3.94 % plus 1.75 % (175 basis points)). The outstanding balance is $600000 \cdot (1 - 0.2) = 480000$ at the beginning of January 1, 2007. Thus, the payment is

$$480000 \cdot 0.0569 + 60000 = 87312 .$$

2. On April 1, 1994, Great Savings Bank had issued many fixed-rate CDs, and had loaned out much of the money it obtained from these at variable rates. The board of directors is concerned that the Bank faces too much interest-rate risk and decides to take the fixed-leg in a fixed-for-floating interest rate swap — that is, lock in a fixed interest rate. The notional balance is \$35,000,000, the fixed rate is $i(4) = 5.25\%$, and the floating rate is the three-month LIBOR. The swap is for one year and the LIBOR rates, given on an "Actual/360 basis," turn out to be 4.250% for the period beginning 4/01/94, 4.875% for the period beginning 7/01/94, 5.688% for the period beginning 10/01/94, and 6.328% for the period beginning 1/01/95. Determine the amount and time of all payments.

With the fixed nominal quarterly interest rate $i(4) = 5.25\%$, the fixed income is $35000000 \cdot i(4)/4 = 459375$. In the calculation with the floating rate, we count the actual dates because "Actual/360 basis", i.e.,

Term	# Dates	Interest payments (Liabilities)
4/01/94 - 6/30/94	91	$35000000 \cdot 0.0425 \cdot 91/360 = 376006.94$
7/01/94 - 9/30/94	92	$35000000 \cdot 0.04875 \cdot 92/360 = 436041.67$
10/01/94 - 12/31/94	92	$35000000 \cdot 0.05688 \cdot 92/360 = 508760$
1/01/95 - 3/31/95	90	$35000000 \cdot 0.06328 \cdot 90/360 = 553700$

Thus, the net cash flows are
on July 1, 1994, $459375 - 376006.94 = +83368.06$ (inflow);
on October 1, 1994, $459375 - 436041.67 = +23333.33$ (inflow);
on January 1, 1995, $459375 - 508760 = -49385$ (outflow);
on April 1, 1995, $459375 - 553700 = -94325$ (outflow).

3. The annual yields of zero-coupon bonds are as given below. What is the swap rate for a four-year interest rate swap with level notional amount and annual settlement?

Length of term in years	1	2	3	4	5
Annual yield	1.5%	2.5%	3.3%	4.2%	4.9%

With the zero coupon bond prices $P_0 = 1$, $P_1 = 1/1.015$, $P_2 = 1/1.025^2$, $P_3 = 1/1.033^3$, $P_4 = 1/1.042^4$, the swap rate is given by

$$R = \frac{P_0 - P_4}{P_1 + P_2 + P_3 + P_4} \approx 0.0411 .$$

4. Leif Corporation borrowed \$3,000,000. The interest rate for this loan is reset each year to the one-year spot rate. At the end of each year for three years, Leif will pay the interest due, along with \$1,000,000 in principal. To avoid the risk of interest rate fluctuation, Leif Corporation entered a three-year amortizing swap that mirrors the terms of the loan with Perma Inc. Under the swap, Leif has agreed to pay at the fixed rate and receive payments based on the one-year spot rate. Assuming that the spot rates are as below, determine Perma's market value at the beginning of the second year. Spot interest rates at initiation of the swap were:

t (years)	1	2	3
r_t	1.28%	2.03%	2.84%

Spot interest rates at the beginning of the second year were:

t (years)	1	2	3
r_t	1.54%	2.61%	3.25%

With the zero coupon bond prices $P_0 = 1$, $P_1 = 1/1.0428$, $P_2 = 1/1.0203^2$, $P_3 = 1/1.0284^3$, the swap rate is

$$R = \frac{3 \cdot (P_0 - P_1) + 2 \cdot (P_1 - P_2) + 1 \cdot (P_2 - P_3)}{3P_1 + 2P_2 + P_3} \approx 0.02285 .$$

At the beginning of year 2, we shall evaluate the market value: the remaining fixed rate payments are $2000000R$ and $1000000R$, and the floating rate payments are $2000000 \cdot 0.0154$ and $1000000 \cdot (1.025^2/1.0154 - 1)$ because of the forward rate. Thus the market value is

$$2000000 \cdot (R - 0.0154) \cdot v_{0.0154} + 1000000 \cdot \Big(R - \Big(\frac{1.0261^2}{1.0154} - 1\Big)\Big) \cdot v_{0.0261}^2 \approx 1329.11 .$$

5. Compute the Macaulay duration of a ten-year 6% \$1,000 bond having annual coupons and a redemption of \$ 1,200 if the yield to maturity is 8%.

The coupon is $1000 \cdot 0.06 = 60$. The price of the bond is

$$P(0.08) = 60 a_{\overline{10}|0.08} + 1200 v_{0.08}^{10} \approx 958.437069 .$$

The Macaulay duration is

$$D(0.08, \infty) = \frac{1}{P}\Big(\sum_{k=1}^{9} 60\, k\, v_{0.08}^k + 10 \cdot 1260 v_{0.08}^{10}\Big)$$

$$= \frac{60}{P} \cdot (Ia)_{\overline{9}|0.08} + \frac{12600\, v_{0.08}^{10}}{P} \approx 7.846 .$$

6. Calculate the Macaulay duration $D(.05, \infty)$ and the modified duration $D(.05, 2)$ of a preferred stock that pays dividends forever of \$ 50 each six months, with the next dividend in exactly six months.

With the effective 6 month interest $j = (1+i)^{1/2} - 1$, the price is $P(i) = 50/j = 50/((1 + i)^{1/2} - 1)$. Its derivative is

$$P'(i) = -50((1+i)^{1/2} - 1)^{-2} \cdot \frac{1}{2}(1+i)^{-1/2} .$$

Thus, we obtain the modified duration and the Macaulay duration

$$D(i,1) \;=\; -\frac{P'(i)}{P(i)} \;=\; \frac{1}{2((1+i)-(1+i)^{1/2})}\,, \quad D(i,\infty) \;=\; D(i,1)\cdot(1+i) \;=\; \frac{(1+i)^{1/2}}{2((1+i)^{1/2}-1)}\,.$$

Hence,

$$D(0.05,\infty) \;=\; 20.75\,, \quad D(0.05,2) \;=\; D(0.05,\infty)\cdot(1.05)^{-1/2} \;=\; 20.25\,.$$

7. A bond has Macaulay duration $D(i,\infty)=5.8$ and Macaulay convexity $C(i,\infty)=1.2$. Determine $C(i,4)$ as a function of i.

Note that $1+i \;=\; (1+i^{(m)}/m)^m = e^\delta$. Taking derivatives, we obtain

$$\frac{\partial\delta}{\partial i^{(m)}} \;=\; \Big(1+\frac{i^{(m)}}{m}\Big)^{-1}, \quad \frac{\partial^2\delta}{\partial {i^{(m)}}^2} \;=\; -\frac{1}{m}\Big(1+\frac{i^{(m)}}{m}\Big)^{-2}.$$

Then the second partial derivative of P with respect to $i^{(m)}$ is

$$\frac{\partial^2 P}{\partial {i^{(m)}}^2} \;=\; \frac{\partial}{\partial i^{(m)}}\Big(\frac{\partial P}{\partial\delta}\cdot\frac{\partial\delta}{\partial i^{(m)}}\Big) \;=\; \frac{\partial^2 P}{\partial\delta^2}\cdot\Big(\frac{\partial\delta}{\partial i^{(m)}}\Big)^2 + \frac{\partial P}{\partial\delta}\cdot\frac{\partial^2\delta}{\partial {i^{(m)}}^2}\,.$$

Thus, substituting the derivatives of δ with respect to $i^{(m)}$, we obtain

$$\begin{aligned} C(i,m) \;&=\; \frac{1}{P}\frac{\partial^2 P}{\partial {i^{(m)}}^2} \;=\; \frac{1}{P}\Big(\frac{\partial^2 P}{\partial\delta^2}\cdot\Big(\frac{\partial\delta}{\partial i^{(m)}}\Big)^2 + \frac{\partial P}{\partial\delta}\cdot\frac{\partial^2\delta}{\partial {i^{(m)}}^2}\Big) \\ &=\; \Big(1+\frac{i^{(m)}}{m}\Big)^{-2}\Big(C(i,\infty)+\frac{1}{m}D(i,\infty)\Big)\,. \end{aligned}$$

Particularly, we have

$$C(i,4) \;=\; \Big(1+\frac{i^{(4)}}{4}\Big)^{-2}\Big(1.2+\frac{5.8}{4}\Big) \;=\; \frac{2.65}{(1+i)^{1/2}}\,.$$

8. Providence Health Care is obligated to make a payment of \$300,000 in exactly three years. In order to provide for this obligation, their financial officer decides to purchase a combination of one-year zero-coupon bonds and four-year zero-coupon bonds. Each of these is sold to yield an annual effective yield of 4%. How much of each type of bond should be purchased so that the present value and duration conditions of Redington immunization are satisfied? Is the convexity condition also satisfied at i = 4%?

Let us denote the amount of the one-year zero coupon bond and the amount of the four-year zero coupon bond invested by x_1 and x_2, respectively. Then for the Redington immunization, the present value should be matched with the present value of the obligation (liability)

$$x_1 + x_2 \;=\; \frac{300000}{1.04^3}\,,$$

and the Macaulay duration of the portfolio should be matched with the duration of the obligation:

$$\frac{x_1}{x_1+x_2} + \frac{4x_2}{x_1+x_2} \;=\; 3\,.$$

Solving for x_1 and x_2, we obtain

$$x_1 \;=\; 88899.64\,, \quad x_2 \;=\; 177799.27\,.$$

The convexity of the portfolio is

$$\frac{x_1}{x_1+x_2}\cdot 1^2 + \frac{x_2}{x_1+x_2}\cdot 4^2 \;=\; 11$$

greater than the convexity $3^2 \;=\; 9$ of the liability. Therefore, the convexity condition is satisfied.

9. The price of Ada's bond is \$1416.89 when calculated using an annual yield of 4.8%. Using a first-order modified approximation, Ada calculated that the change in price of her bond would be \$32.01 if the annual yield increased from 4.80% to 4.95%. Calculate the first-order Macaulay approximation of the price of this bond for an interest rate of 4.95%.

> This problem is about the first order Macaulay approximation but we did not have time to cover it in the lecture. It will not be in the final exam.

Let us denote by $D(0.048, 1)$ the modified duration. Then

$$1416.89 - 32.01 \;=\; 1416.89 \cdot (1 - D(0.048, 1) \cdot (0.0495 - 0.048))\,.$$

Solving for the modified duration, we obtain

$$D(0.048, 1) \;=\; 15.061$$

and hence, $D(0.048, \infty) \;=\; D(0.048, 1) \cdot 1.048 \;=\; 15.784$. The Macaulay approximation is

$$1416.89 \cdot \left(\frac{1.048}{1.0495}\right)^{D(0.048,\infty)} \approx 1385.26\,.$$

10. A bond without a call provision has Macaulay duration of 4.5381, Macaulay convexity of X, and price of \$35,328.70 when calculated using an annual interest rate of 5.7%. Using a second-order Macaulay approximation, Leta estimated that the change in price of this bond would be \$305.03 if the interest rate were to decrease to 5.5%. Find the value of X.

> This problem is about the second order Macaulay approximation but we did not have time to cover it in the lecture. It will not be in the final exam.

The second-order Macaulay approximation is

$$35328.7 + 305.03 \;=\; 35328.7 \cdot \left(\frac{1.057}{1.055}\right)^{4.5381} \cdot \left(1 + \frac{1}{2}\left(\frac{0.055 - 0.077}{1.057}\right)^2 \left(X - 4.5381\right)^2\right).$$

Solving for X, we obtain $X \;=\; 21.772$.

Flipgrid Homework Submission

`Here is the instruction how to upload your solution.`

1. Find below the submission link to Flipgrid of your section (also listed in Week 1 in GauchoSpace). There are two Fligprid groups for different sections. Please choose the correct one and login with your UCSB google email.

- Group: Y. Feng sections F 11AM/F 1PM
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- If it is your first time, then your account is automatically created.
- If you have login problems, please let the instructor know.

Upload your video HW 8 Problem 5.

♯♯ **Please read the problem carefully, since the problem is slightly different**.

That is it for HW 8 and all the homework! You did a lot this quarter!