PSTAT 171. HW 4 (Winter 2021)

`Instruction: Review textbook chapters 3 and 4 first. Multiple reading might help. Then try to solve the homework problems quickly.`

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1. Find the value at $t = 0$ of a perpetuity that pays 1,000 at the end of each year starting at $t = 3$ assuming that $a(t) = (t+1)(t+3)/3$.

By definition, we have discount function $v(t) = 1/a(t) = 3/((t+1)(t+3))$ Since the perpetuity pays 1000 at time $t = 3, 4, \ldots$, the present value of the perpetuity is given by

$$\sum_{t=3}^{\infty} 1000v(t) = 3000 \sum_{t=3}^{\infty} \frac{1}{(t+1)(t+3)} = 3000 \cdot \frac{1}{2} \sum_{t=3}^{\infty} \Big(\frac{1}{t+1} - \frac{1}{t+3} \Big)$$

$$= 1500 \Big[\Big(\frac{1}{4} - \frac{1}{6} \Big) + \Big(\frac{1}{5} - \frac{1}{7} \Big) + \Big(\frac{1}{6} - \frac{1}{8} \Big) + \cdots \Big] = 1500 \Big(\frac{1}{4} + \frac{1}{5} \Big) = 675 .$$

Here the series is absolutely convergent (and hence, unconditionally convergent; Do we need a proof?), because it is nonnegative and the upper bound is given by

$$\sum_{t=3}^{\infty} v(t) \le \sum_{t=1}^{\infty} \frac{3}{t^2} = \frac{\pi^2}{2} < \infty .$$

2. Serena receives a fifty-year annuity-due that has payments that start at $ 2,000 and increase by 3% per year through the twenty-fourth payment, then stay level at $4,000. Find the accumulated value of this annuity at the end of fifty years if the annual effective rate of interest remains 4.2 % throughout the time of the annuity.

This is annuity-due. The payments are $2000, 2000(1.03), 2000(1.03)^2, \ldots, 2000(1.03)^{23}$ at times $0, 1, \ldots, 23$, and then 4000 at times $24, \ldots, 49$. With $v = 1/1.042$, the accumulation at time 50 is

$$2000(1.042)^{50}(1 + 1.03v + 1.03^2v^2 + \cdots + (1.03v)^{23}) + 4000(1.042)^{26}(1 + v + \cdots + v^{25})$$

$$= 2000(1.042)^{50} \cdot \frac{1 - (1.03v)^{24}}{1 - 1.03v} + 4000(1.042)^{26} \ddot{a}_{\overline{26}|0.042} = 519729.1 .$$

3. A perpetuity paying $1,000 at the beginning of each two years has the same present value as another perpetuity with level payments, this one having payments at the end of each three years. Express the level payment amount of the second annuity as a function of the annual effective interest rate i.

We can solve in two methods: (1) method from Chapter 3 and (2) method from Chapter 4. Let X be the level payment amount of the second annuity.

(1) method from Chapter 3. The first perpetuity-due has the present value $1000 + (1000/((1+i)^2 - 1))$, because the two year effective rate is $(1+i)^2 - 1$. The second perpetuity-immediate has the present value $X/((1+i)^3 - 1)$, because the three year effective rate is $(1+i)^3 - 1$. Recall the formulae of the perpetuities (perpetuity-due and perpetuity-immediate). Since they are the same, we have

$$1000 + \frac{1000}{(1+i)^2 - 1} = \frac{X}{(1+i)^3 - 1} ,$$

and hence, using $i + (s_{\overline{n}|i})^{-1} = 1/a_{\overline{n}|i}$, we obtain

$$X = 1000\Big(1 + \frac{1}{(1+i)^2 - 1}\Big)((1+i)^3 - 1) = 1000\Big(1 + \frac{1}{is_{\overline{2}|i}}\Big) \cdot is_{\overline{3}|i}$$

$$= 1000\Big(i + \frac{1}{s_{\overline{2}|}}\Big)s_{\overline{3}|i} = 1000 \cdot \frac{s_{\overline{3}|i}}{a_{\overline{2}|i}} .$$

(2) method from Chapter 4. The first perpetuity-due has the price $1000/(ia_{\overline{2}|i})$ and the second perpetuity-immediate has the price $X/(is_{\overline{3}|i})$. Thus we have

$$\frac{1000}{ia_{\overline{2}|i}} = \frac{X}{is_{\overline{3}|i}}$$

and hence,

$$X = 1000 \cdot \frac{s_{\overline{3}|i}}{a_{\overline{2}|i}}.$$

4. Jason purchases a deferred perpetuity for \$ 13,520. The perpetuity has quarterly payments of \$ 750. Express the waiting time until the first payment as a function of the annual effective interest rate i.

The quarterly effective interest rate is $j := (1+i)^{1/4} - 1$ and the quarterly effective discount rate is $d_j := j/(1+j) = ((1+i)^{1/4} - 1)\,/\,(1+i)^{1/4}$.
(1) Deferred perpetuity-immediate. If it is a deferred perpetuity-immediate with the first payment at time T (years), then since the value $750\,/\,j$ of perpetuity-immediate is calculated at time $T - (1/4)$, its time-0 value is

$$13520 = \frac{750}{j} \cdot (1+i)^{-(T-\frac{1}{4})},$$

and hence, the waiting time T is solved as

$$T = \frac{1}{4} + \frac{\log(750/(13520j))}{\log(1+i)} \approx \frac{1}{4} - \frac{\log(18.03\,j)}{\log(1+i)}.$$

(2) Deferred perpetuity-due. If it is a deferred perpetuity-due with the first payment at time T (years), then since the value $750\,/\,d_j$ of perpetuity is calculated at time T, its time-0 value is

$$13520 = \frac{750}{d_j} \cdot (1+i)^{-T},$$

and hence, the waiting time T is

$$T = \frac{-\log(13520d_j/750)}{\log(1+i)} \approx \frac{-\log(18.03\,d_j)}{\log(1+i)}.$$

5. Jasper is bequeathed a thirty-year deferred annuity that has a payment at the end of each third year. The first payment is for \$15,000 and is made five years after she receives the inheritance. There is always an increase of \$ 4,000 from one payment to the next. Find the value of her legacy at the time of the bequest, assuming that the annual effective interest rate remains level at 5 %.
- Do this problem twice, once using Chapter 3 methods and once using Section (4.4) techniques.

This is an increasing annuity starting from 15000 and increasing by 4000 every three years.
(Chapter 3 method) The effective three year interest rate is $j := (1.05)^3 - 1 \approx 0.15763$. The time-2 value of the annuity is given by the increasing annuity formula at rate j:

$$(I_{15000,4000}\,a)_{\overline{10}|j} = 15000a_{\overline{10}|j} + \frac{4000}{j}(a_{\overline{10}|j} - 10v_j^{10}) \approx 138171.8181,$$

and hence, the time-0 value of the annuity is about 125325.91 $(= 1.05^{-2} \cdot 138171.8181)$.

(Chapter 4 method) By the increasing annuity formula, the time-2 value is given by

$$15000 \cdot \frac{a_{\overline{30}|0.05}}{s_{\overline{3}|0.05}} + \frac{4000}{0.05s_{\overline{3}|0.05}}\left(\frac{a_{\overline{30}|0.05}}{s_{\overline{3}|0.05}} - \frac{10}{1.05^{30}}\right) \approx 138171.8181.$$

Thus, the time-0 value is again 125325.91 $(= 1.05^{-2} \cdot 138171.8181)$.

6. Bob deposits \$1,500 at the beginning of each quarter for sixteen years in a fund earning a nominal rate of interest of 6 % convertible monthly. The interest from this fund is paid out monthly and can only be reinvested at an effective annual rate of 5.2 %. Find the accumulated value of Bob's investments at the end of twentieth year.

Every interest $1500 \cdot 0.06/12 = 7.5$ earned from the fund is reinvested at 5.2 % for 64 quarters. It becomes the increasing annuity of 64 quarters with $m = 3$. The quarterly effective interest is $j := (1.052)^{1/4} - 1 \approx 1.2754\%$. After 64 quarters, the reinvestment account receives $480 (= 64 \cdot 7.5)$ for next 16 quarters. The balance of the 5.2 % reinvestment account at the end of 20 years is the sum of

$$7.5 \cdot 3(I\,s)^{(3)}_{\overline{64}|j} \cdot (1.052)^4 = 22.5 \cdot \frac{\ddot{s}_{\overline{64}|j} - 64}{3((1+j)^{1/3} - 1)} \cdot (1.052)^4 \approx 76570.3352$$

and

$$12 \cdot 480 s^{(12)}_{\overline{4}|} = 5760 \cdot \frac{(1.052)^4 - 1}{12(1.052^{1/12} - 1)} \approx 25488.25393\,.$$

The balance of the fund at the end of the twenty years will be $1500 \cdot 64 = 96000$.

Therefore, we obtain the total accumulation at the end of 20 years is

$$76570.3352 + 25488.25393 + 96000 = 198058.59\,.$$

7. An annuity lasting n interest periods has a payment at the end of each m-th of an interest period. The first payment is for an amount P. Payments are level within each interest period, and the the individual payments increases by an amount Q from one interest period to the next. Show that the accumulated value at the time of the last payment is

$$m\Big(P\,s^{(m)}_{\overline{n}|i} + \frac{Q}{i^{(m)}}(s_{\overline{n}|i} - n)\Big)\,.$$

Let $J := i^{(m)}/m = (1+i)^{1/m} - 1$ be the effective interest rate for the interest period. For $k = 1, 2, \ldots, n$, the payments in the k-th interest period is $P + (k-1)Q$ and the accumulated value is $(P + (k-1)Q)s_{\overline{m}|J} = Ps_{\overline{m}|J} + (k-1)Qs_{\overline{m}|J}$. Thus it becomes an increasing annuity with the first payment being $Ps_{\overline{m}|J}$ and increasing by $Qs_{\overline{m}|J}$:

$$(I_{Ps_{\overline{m}|J}, Qs_{\overline{m}|J}}\,s)_{\overline{n}|i} = Ps_{\overline{m}|J}s_{\overline{n}|i} + \frac{Qs_{\overline{m}|J}}{i}(s_{\overline{n}|i} - n)\,.$$

Note that as we learned, we have

$$s_{\overline{m}|J} \cdot s_{\overline{n}|i} = ms^{(m)}_{\overline{n}|i}$$

and

$$\frac{s_{\overline{m}|J}}{mi} = \frac{((1+J)^m - 1)/J}{mi} = \frac{1}{mJ} = \frac{1}{i^{(m)}}\,.$$

Therefore, we conclude

$$(I_{Ps_{\overline{m}|J}, Qs_{\overline{m}|J}}\,s)_{\overline{n}|i} = m\Big(P\,s^{(m)}_{\overline{n}|i} + \frac{Q}{i^{(m)}}(s_{\overline{n}|i} - n)\Big)\,.$$

8. This problem concerns the annuity underlying the annuity symbols $(I^{(m)}\,a)^{(m)}_{\overline{n}|i}$.
(a) The rate of payment over an interval is the amount paid normalized by dividing by the length of the interval. Show that this annuity pays at a rate of $j\,/\,m$ over the interval $[(j-1)\,/\,m\,, j\,/\,m]$.
(b) Show that the payments of this annuity total $(n^2/2) + (n/(2m))$.
(c) Show that the limit $\lim_{m\to\infty}(n^2/2) + (n/(2m))$ of the total payment amounts found in (b) is equal to $\int_0^n t dt$, the total payments of the continuously paying annuity underlying the symbol $(\overline{I}\,\overline{a})_{\overline{n}|}$.

(a) The payment in the interval is j/m^2. The rate of payment for a unit interval is $j/m^2/(1/m) = j/m$.

(b) This annuity pays k/m^2 at time k for mn payments. Thus, the total amount of payments is

$$\sum_{k=1}^{mn} \frac{k}{m^2} = \frac{mn(mn+1)}{2m^2} = \frac{n^2}{2} + \frac{n}{2m}\,.$$

(c) As $m \to \infty$, we obtain

$$\lim_{m\to\infty} \frac{n^2}{2} + \frac{n}{2m} = \frac{n^2}{2} = \int_0^n t dt\,.$$

Flipgrid Homework Submission

`Here is the instruction how to upload your solution.`

1. Find below the submission link to Flipgrid of your section (also listed in Week 1 in GauchoSpace). There are two Fligprid groups for different sections. Please choose the correct one and login with your UCSB google email.

- Group: Y. Feng sections F 11AM/F 1PM
`https://flipgrid.com/61b03eae`
- Group: J.K. Chen section M 11 AM + M. Becerra section T 3 PM
`https://flipgrid.com/fa4f6895`

- If it is your first time, then your account is automatically created.
- If you have login problems, please let the instructor know.

Upload your video HW 4 Problem 4. Please read the problem carefully, since the problem is slightly different.

That is it for HW 3! We will be having online quiz next week (Monday - Tuesday) based on the homework problems.