PSTAT 171. HW 2 (Winter 2021)

`Instruction: Review textbook chapter 2 first. Multiple reading might help. Then try to solve the homework problems quickly.`

1. Given $d^{(4)} = 0.08$, compute the equivalent rates i, d, $d^{(m)}$ and $i^{(m)}$ for each $m = 2, 6, 12$.

It follows from the definition of the nominal discount rate $d^{(4)}$ compounded quarterly that

$$1 - d = \Big(1 - \frac{d^{(4)}}{4}\Big)^4, \quad d = 1 - \Big(1 - \frac{d^{(4)}}{4}\Big)^4 = 1 - (1 - .02)^4 = 0.0776,$$

and hence, $i = 1/(1-d) - 1 = 0.0842$. Then we solve for $d^{(m)}$ and $i^{(m)}$

$$\Big(1 - \frac{d^{(m)}}{m}\Big)^m = 1 - d, \quad \Big(1 + \frac{i^{(m)}}{m}\Big)^m = 1 + i,$$

and obtain

$$d^{(m)} = m(1 - (1-d)^{1/m}), \quad i^{(m)} = m((1+i)^{1/m} - 1).$$

Particularly, we have

$$(d^{(2)}, i^{(2)}) = (0.079, 0.082), \quad (d^{(6)}, i^{(6)}) = (0.0803, 0.0814), \quad (d^{(12)}, i^{(12)}) = (0.0805, 0.0811).$$

Recall that we have relationship $d < d^{(m)} < \delta < i^{(m)} < i$ for $i > 0, m > 1$ with $\delta = \lim_{m\to\infty} \uparrow d^{(m)} = \lim_{m\to\infty} \downarrow i^{(m)} = \log(1+i) = -\log(1-d)$

2. Suppose that interest is paid once every 2 years at a nominal interest rate $i^{(1/2)}$. That is, the borrower pays interest at an effective rate of $2i^{(1/2)}$ every 2 years. Find an expression for $i^{(1/2)}$ in terms of i.

Following the definition of the nominal interest rate, we have the accumulation of interest for two years:

$$(1+i)^2 = 1 + 2i^{(1/2)},$$

Thus, we obtain

$$i^{(1/2)} = \frac{1}{2}\big((1+i)^2 - 1\big).$$

This is the case with $m = 1/2 (< 1)$. The formula of the nominal interest rate works for not only positive integer but also less than 1.

3. Suppose that the accumulation function $a(\cdot)$ is given by

$$a(t) = (1.02)^t (1 + .03t)(1 - .05t)^{-1}; \quad t \ge 0.$$

Compute the force of interest δ_3 at time $t = 3$.

The relation between the accumulation function $a(\cdot)$ and the force $\delta_\cdot$ of interest is given by

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \log a(t); \quad t \ge 0.$$

Thus, we have

$$\delta_3 = \Big[\frac{\mathrm{d}}{\mathrm{d}t} \log a(t)\Big]\Big|_{t=3} = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\big(t \log 1.02 + \log(1 + 0.03t) - \log(1 - 0.05t)\big)\Big]\Big|_{t=3}$$

$$= \Big[\log 1.02 + \frac{0.03}{1 + 0.03t} + \frac{0.05}{1 - 0.05t}\Big]\Big|_{t=3} \approx 0.1061.$$

Recall that $A(t) = Ka(t)$, $t \geq 0$ with an initial constant value $A(0) = K$. Thus, we have

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \log a(t) = \frac{A'(t)}{A(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \log A(t)\,; \quad t \geq 0\,.$$

When $a(\cdot)$ is of the product form, e.g., $a(t) = a_1(t) \cdot a_2(t)$, then it is easier to take the natural log first, i.e.,

$$\log(a(t)) = \log(a_1(t)) + \log(a_2(t))\,; \quad t \geq 0\,,$$

and hence, taking their derivatives, we obtain

$$\delta_t = \frac{\mathrm{d}}{\mathrm{d}t} \log(a(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \log(a_1(t)) + \frac{\mathrm{d}}{\mathrm{d}t} \log(a_2(t))\,; \quad t \geq 0\,.$$

4. Esteban borrows \$ 20,000 and the loan is governed by compound interest at an annual effective interest rate of 6%. Esteban agrees to repay the loan by making a payment of \$ 10,000 at the end of T years and a payment of \$ 12,000 at the end of $2T$ years. Find T.

With $x := v^T = (1.06)^{-T}$, we have the equation of value:

$$20000 = 10000v^T + 12000v^{2T} = 10000x + 12000x^2\,.$$

Solving this quadratic equation, we have

$$\Big(\frac{1}{1.06}\Big)^T = x = \frac{-5 + \sqrt{265}}{12}$$

(we discard the negative solution and take the positive solution x) and hence,

$$T = -\frac{\log x}{\log 1.06} \approx 1.064\,(\text{ years })\,.$$

This problem is reduced to a problem of solving a quadratic equation. One can apply Newton's method or use BA-II calculator.

5. Suppose that there are contributions $C_{t_1}, \dots, C_{t_n}$ at time $t_1, \dots, t_n$, respectively. We define

$$T := \frac{1}{\log v} \log\Big(\frac{\sum_{k=1}^n C_{t_k} v^{t_k}}{\sum_{k=1}^n C_{t_k}}\Big)\,, \quad \overline{T} := \frac{\sum_{k=1}^n t_k\, C_{t_k}}{\sum_{k=1}^n C_{t_k}}\,.$$

Here $v = 1/(1+i)$ and log is the natural logarithm. Verify that $\overline{T} \geq T$ for every n.

Assume that i, $C_{t_1}, \dots, C_{t_n}$ are all positive. We define the positive weights $w_k := C_{t_k} \,/\, \sum_{\ell=1}^n C_{t_\ell}$, $k = 1, \dots, n$, and consider the convex function $f(x) := v^x = e^{x \log v}$ with $f''(x) > 0$ for $x \in \mathbb{R}$. Note that $\{w_1, \dots, w_n\}$ are positive weights and $\overline{T}$ is a weighted average of $t_1, \dots, t_n$ with the weights $w_1, \dots, w_n$:

$$\sum_{k=1}^n w_k = w_1 + \cdots + w_n = 1\,, \quad \overline{T} = w_1 t_1 + \cdots w_n t_n = \sum_{k=1}^n w_k t_k\,.$$

Because of the convexity of the function $f(\cdot)$, the weighted average $\sum_{k=1}^n w_k f(t_k)$ of the values $f(t_k) = v^{t_k}$ at t_k with weight w_k, $k = 1, \dots, n$ is bigger than or equal to the function value $f(\sum_{k=1}^n w_k t_k)$ of the weighted average $\sum_{k=1}^n w_k t_k$ of t_k with weights w_k, $k = 1, \dots, n$, i.e.,

$$e^{T \log v} = \sum_{k=1}^n w_k v^{t_k} = \sum_{k=1}^n w_k f(t_k) \geq f\Big(\sum_{k=1}^n w_k t_k\Big) = v^{\sum_{k=1}^n w_k t_k} = e^{\overline{T} \log v}\,,$$

where the inequality becomes equality when $n = 1$. Therefore, since $\log v < 0$, we conclude $\overline{T} \geq T$ for every n.

This problem is a consequence of Jensen's inequality. It can be also seen as the relationship between the arithmetic average and geometric average. See Problems (2.3.7)-(2.3.8) in the textbook.

6. Define the function $f(t) = 525(1.1)^{-2t} + 525(1.1)^{-t} - 1000$, $t \geq 0$ and find the root T^*, such that $f(T^*) = 0$ by Newton's method.

The function f represents the time 0 value of the following cashflows: -1000 at time 0, 525 at time T^* (year) and another 525 at time $2T^*$. With the annual effective interest rate $i = 0.1$, we see the net present value of cashflows becomes 0. By Newton's method, we shall find T^*: let us choose the initial value $t_0 = 0$ with $f(0) = 525 + 525 - 1000 = 50 > 0$, and then we compute

$$t_1 := t_0 - \frac{f(t_0)}{f'(t_0)} = 0.3330812\,, \quad f(t_1) = 1.2979292\,;$$

$$t_2 := t_1 - \frac{f(t_1)}{f'(t_1)} = 0.3421963166\,, \quad f(t_1) = 0.0009351739\,;$$

where $f'(t) = 525(1.1)^{-2t}(-2\log 1.1) + 525(1.1)^{-t}(-\log 1.1)$, $t \geq 0$. We conclude the approximate zero is $T^* \approx t_2 \approx 0.342$.

7. Sandra invests \$ 10,832 in the Wise Investment Fund. Three months later her balance has grown to \$ 11,902 and she deposits \$ 2,000. Two months later her holdings are \$ 14,308 and she withdraws \$ 7,000. Two years after the initial investment, she learns that her holdings are worth \$ 12,566. Compute the approximate dollar-weighted annual yield with different methods.

We have the initial balance $B_0 := 10832$, the final balance $B_1 := 12566$, contributions $C_{3/24} := 2000$ at time 3rd month ($t_1 = 3/24$ in two year scale), $C_{5/24} := -7000$ at time 5th month ($t_2 = 5/24$ in two year scale) and the total contribution $C := C_{3/24} + C_{5/24} = -5000$. It follows from

$$I = B_1 - B_0 - C = 12566 - 10832 - (-5000) = 6734\,.$$

(a) For the exact dollar weighted yield rate i, we consider the monthly effective yield rate j and discount rate $v_j := 1/(1+j)$ and the equation of value

$$10832 + 2000v_j^3 - 7000v_j^5 = 12566v_j^{24}\,.$$

The monthly effective yield rate is $j \approx 0.0274$, and hence, the annual yield is

$$(1+j)^{12} - 1 \approx 0.384\,.$$

(b) For the simple approximate dollar weighted yield rate, first we compute the two-year yield rate

$$j_{DW} \approx \frac{I}{B_0 + C_{3/24}(1 - (3/24)) + C_{5/24}(1 - 5/24)} = 0.956488\,.$$

and then we obtain the annual rate

$$(1 + j_{DW})^{1/2} - 1 = 0.399\,.$$

(c) For the average date of contribution approximation, we compute the weights $w_1 := C_{3/24}/C = -2/5$, $w_2 := C_{5/24}/C = 7/5$, and the average date

$$\kappa := -\frac{2}{5} \cdot \frac{3}{24} + \frac{7}{5} \cdot \frac{5}{24} = \frac{29}{120}\,,$$

and then we compute the two-year yield rate

$$j \approx \frac{I}{B_0 + C(1-\kappa)} = \frac{6734}{10832 + (-5000)(1 - (29/120))} = 0.9565\,.$$

Thus, the annual approximate yield rate is given by

$$(1+j)^{1/2} - 1 = 0.399\,.$$

(d) For the mid-point approximation, we first compute the two-year rate

$$j \approx \frac{2I}{B_0 + B_1 - I} \; = \; \frac{2 \cdot 6734}{10832 + 12566 - 6734} \approx 0.808\,,$$

and then we compute the annual approximate yield rate

$$(1+j)^{1/2} - 1 \; = \; 0.345\,.$$

8. Bright Future Investment Fund has a balance of \$ 1,205, 000 on January 1. On May 1, the balance is \$ 1,230,000. Immediately after this balance is noted, \$ 800,000 is added to the fund. If there are no further contribution to the fund for the year and the time-weighted annual yield for the fund is 16 %, what is the fund balance at the end of the year?

Let X be the fund value at the end of the year. After the deposit on May 1st, the fund balance is $1,230,000 + 800,000 = 2,030,000$. We compute the time-weighted annual yield :

$$\frac{1230000}{1205000} \cdot \frac{X}{2030000} \; = \; 1.16\,,$$

and solving it for X, we obtain

$$X \; = \; 2030 \cdot 1.16 \cdot \frac{1205}{1230} \cdot 1000 \; = \; 2,306,938\,.$$

Thus, the final balance is \$ $2,306,938$.

Flipgrid Homework Submission

`Here is the instruction how to upload your solution.`

1. Find below the submission link to Flipgrid of your section (also listed in Week 1 in GauchoSpace). There are two Fligprid groups for different sections. Please choose the correct one and login with your UCSB google email.

- Group: Y. Feng sections F 11AM/F 1PM

`https://flipgrid.com/60535abf`

- Group: J.K. Chen section M 11 AM + M. Becerra section T 3 PM

`https://flipgrid.com/9fe74a67`

- If it is your first time, then your account is automatically created.
- If you have login problems, please let the instructor know.

2. There are two topics (two video clips) you are asked to submit:

Upload your video HW 2 Problem 4. Please read the problem carefully, since the problem is slightly different.

That is it for HW 2! We will be having online quiz next week (Monday - Tuesday) based on the homework problems.