PSTAT 171. HW 5 Solution (Winter 2021)

Instruction: Review textbook chapter 5 first. Multiple reading might help. Then try to solve the homework problems quickly.

Instructor: Tomoyuki Ichiba

1. An amortized loan is repaid with annual payments which start at \$ 400 at the end of the first year and increase by \$ 45 each year until a payment of \$1,480 is made, after which they cease. If interest is 4% effective, find the amount of principal in the fourteenth payment.

The loan payment K_t at time t is $K_t = 400 + 45(t-1)$ for $t = 1, 2, \ldots$, e.g., $K_1 = 400$, $K_2 = 445$, $K_{14} = 985$, $K_{15} = 1030$, and $K_{25} = 1480$. Thus the last payment is made at time 25. The amount of principal payment at the 14th payment is the difference $B_{13} - B_{14}$ between the outstanding loan balances B_{13} at time 13 and B_{14} at time 14, where by the prospective formula

$$B_{13} = K_{14}v + K_{15}v^2 + \cdots + K_{25}v^{12} = (I_{985,45}a)_{\overline{25-13}|0.04} = 985a_{\overline{12}|0.04} + \frac{45}{0.04}(a_{\overline{12}|0.04} - 12v^{12})$$

$$\approx 11370.44547\,,$$

$$B_{14} = K_{15}v + K_{16}v^2 + \cdots + K_{25}v^{11} = (I_{1030,45}a)_{\overline{25-13}|0.04} = 1030a_{\overline{11}|0.04} + \frac{45}{0.04}(a_{\overline{11}|0.04} - 11v^{11})$$

$$\approx 10840.26328\,,$$

with $v = 1/1.04$. Therefore, the amount of principal in the 14th payment is

$$B_{13} - B_{14} \approx 11370.44547 - 10840.26328 \approx 530.18\,.$$

2. Let r and k denote positive integers, and set $n = rk$. An amortized loan lasting n interest periods has a payment of P at the end of each k interest periods. The effective interest rate per interest period is i.
(a) Explain why the outstanding loan balance at time jk, just after the payment of P is equal to $P \cdot a_{\overline{n-jk}|i} \,/\, s_{\overline{k}|i}$ for $j = 0, 1, 2, \ldots$ [HINT: Look at Section (4.2).]
(b) Use the result of (a) to establish that the interest in the payment at time $(j+1)k$ is $P(1-v^{n-jk})$.

(a) The level payments of amount P are made at time $k, 2k, \ldots, jk, \ldots, rk = n$. There are $r-j$ payments at time $(j+1)k, \ldots, n$ after the time-jk payment. The effective interest rate over the k interest period is $J := (1+i)^k - 1$, and its discount factor is $v_J := 1\,/\,(1+J) = (1+i)^{-k} = v^k$. Thus the outstanding loan balance at time jk after the payment is given by the prospective formula

$$B_{jk} = P \cdot a_{\overline{r-j}|J} = P \cdot \frac{1 - v_J^{r-j}}{J} = P \cdot \frac{1 - v^{(r-j)k}}{(1+i)^k - 1}$$

$$= P \cdot \frac{1 - v^{(r-j)k}}{i} \cdot \frac{i}{(1+i)^k - 1} = P \cdot \frac{a_{\overline{n-jk}|i}}{s_{\overline{k}|i}}\,.$$

(b) Using the result from part (a), the interest in the payment P at time $(j+1)k$ is

$$JB_{jk} = ((1+i)^k - 1) \cdot P \cdot \frac{a_{\overline{n-jk}|i}}{s_{\overline{k}|i}} = ((1+i)^k - 1) \cdot P \cdot \frac{1 - v^{(r-j)k}}{(1+i)^k - 1} = P(1 - v^{n-jk})\,.$$

3. An amortized loan lasting n interest periods has a level payment of P at the end of each m-th of an interest period. The effective interest rate per interest period is i.
(a) For $k = 1, 2, \ldots, mn$ define $a^{(m)}_{\overline{n-(k/m)}|i} := (1 - v^{(n-(k/m))})\,/\,i^{(m)}$. Explain why the outstanding loan balance at time k/m, just after the payment P, is $m\,P\,a^{(m)}_{\overline{n-(k/m)}|i}$. [HINT: You may find it helpful to look at Section (3.11).]
(b) Use the result of (a) to show that the interest at time $(k+1)\,/\,m$ is $P(1 - v^{n-m/k})$ for $k = 0, 1, \ldots, nm-1$.

(a) The "monthly" (m-thly) effective interest rate is $J := (1+i)^{1/m} - 1 = i^{(m)}/m$ and its discounting factor $v_J := 1/(1+J) = (1+i)^{-1/m} = v^{1/m}$. At time k/m after the payment, there are $nm-k$ payments remaining. The outstanding loan balance at time k/m is the time k/m value of these future payments, that is, $Pa_{\overline{nm-k}|J}$ evaluated at rate J. By direct calculation we have

$$a_{\overline{nm-k}|J} = \frac{1-v_J^{nm-k}}{J} = \frac{1-(1+i)^{-(nm-k)/m}}{i^{(m)}/m} = m \cdot \frac{1-v^{n-(k/m)}}{i^{(m)}} = m\, a^{(m)}_{\overline{n-(k/m)}|i}.$$

Therefore, the outstanding loan balance $B_{k/m}$ at time k/m, just after the payment P is $B_{k/m} = m\,P\,a^{(m)}_{\overline{n-(k/m)}|i}$.

(b) The interest payment at time $(k+1)/m$ is $J \cdot B_{k/m}$, where $B_{k/m}$ is the outstanding loan balance at time k/m. We use the "monthly" effective interest rate J. Thus, we have the interest amount

$$J \cdot B_{k/m} = J\,m\,Pa^{(m)}_{\overline{n-(k/m}|i} = J\,P \cdot \frac{1-v_J^{nm-k}}{J} = P(1-v_J^{nm-k}) = P(1-v^{n-m/k}),$$

thanks to part (a), where $v_J = 1/(1+J) = (1+i)^{-1/m} = v^{1/m}$.

4. Alan borrows \$18,000 for eight years and agrees to make quarterly payments of \$770. Each of these payments consists of interest for the just completed quarter and a deposit to a sinking fund that has a nominal interest rate of 6% convertible quarterly. For the first six years, each year the lender receives 8% nominal interest convertible quarterly. For the remaining two years, the lender receives 12% nominal interest convertible quarterly. Find the amount by which the sinking fund is short of repaying the loan at the end of the eight years.

Alan pays 770 every quarter. Among each 770, he pays interest to the lender and deposits to the sinking fund. In the first 24 payments (6 years) the interest is $18000 \cdot 0.08/4 = 360$ and the remaining is $770 - 360 = 410$, and in the next 8 payments (2 years) the interest is $18000 \cdot 0.12/4 = 540$ and the remaining is $770 - 540 = 230$. Thus, the the accumulation in the sinking fund at the effective quarterly interest $0.06/4 = 0.015$ is

$$410\, s_{\overline{24}|0.015} \cdot 1.015^8 + 230\, s_{\overline{8}|0.015} = 15164.29\,.$$

The difference between the accumulation of the sinking fund and the original loan amount 18000 to be paid at time 8 is

$$18000 - 15164.29 = 2835.713\,.$$

Thus, the sinking fund is short by 2835.71 .

5. Bob and Barbara are friends. Bob takes out a \$10,000 loan and agrees to repay it over twelve years by making annual level payments at an effective rate of 5.62499%. At the same time, Barbara takes out a \$10,000 loan and agrees to repay it by making annual interest payments at an annual effective interest rate of i. She also agrees to make annual level deposits into a sinking fund that earns 4% annual effective interest so as to accumulate \$10,000 at the end of the twelve years. Bob and Barbara discover they have the same total annual expenditures resulting from their loans. Find the rate i.

Bob pays the usual loan of level amount (say K_*) to repay 10000 at rate $i_* := 5.62499\%$, that is,

$$K_* \cdot a_{\overline{12}|i_*} = 10000\,, \qquad \text{and hence} \qquad K_* = \frac{10000}{a_{\overline{12}|i_*}}\,.$$

Barbara pays the interest $10000\,i$ and deposit (say $K_\circ$) to the sinking fund to make the loan of amount 10000 for 12 years, that is, $K_\circ\, s_{\overline{12}|0.04} = 10000$. Then her payment amount is

$$10000\,i + K_\circ = 10000\,i + \frac{10000}{s_{\overline{12}|0.04}}\,.$$

Since their payments are the same, we have

$$\frac{10000}{a_{\overline{12}|i_*}} = 10000\,i + \frac{10000}{s_{\overline{12}|0.04}}\,.$$

Solving this equation for i, we obtain

$$i = \frac{1}{10000}\Big(\frac{10000}{a_{\overline{12}|i_*}} - \frac{10000}{s_{\overline{12}|0.04}}\Big) = \frac{1}{a_{\overline{12}|i_*}} - \frac{1}{s_{\overline{12}|0.04}} = 0.05028485 = 5.028485\%\,.$$

6. A loan of \$39,999.85 is to be repaid by payments at the end of each quarter for eight years. Each payment is 2% higher than its predecessor. The loan is made at a nominal rate of discount of 4% payable quarterly. Find the balance just after the 20th payment, the amount of interest in the twentieth payment, and the amount of principal in the twentieth payment.

Let x be the first payment. Then the k-th payment is $x(1.02)^{k-1}$. Since the nominal rate of discount of 4 % payable quarterly means $d = 1\%$ effective quarterly discount rate, the discount factor is $v = 1 - d = 0.99$. The time-0 value of all the 32 payments of the 8-year loan paid quarterly is

$$39999.85 = \sum_{k=1}^{32} x(1.02)^{k-1}\, v^k = x \cdot 0.99 \sum_{k=1}^{32} (1.02 \cdot 0.99)^{k-1}\,.$$

Thus solving it for x, we obtain

$$x = \Big(0.99 \sum_{k=1}^{32} (1.02 \cdot 0.99)^{k-1}\Big)^{-1} \cdot 39999.85 = \frac{39999.85}{0.99} \cdot \frac{1.02 \cdot 0.99 - 1}{(1.02 \cdot 0.99)^{32} - 1} = 1081.1\,.$$

The outstanding loan balances B_{19} and B_{20} are

$$B_{19} = \sum_{k=20}^{23} x(1.02)^{k-1} v^{k-19} = 1081.1 \cdot 1.02^{19} \cdot 0.99 \cdot \frac{(1.02 \cdot 0.99)^{13} - 1}{1.02 \cdot 0.99 - 1} \approx 21505.47149\,,$$

$$B_{20} = \sum_{k=21}^{32} x(1.02)^{k-1} v^{k-20} = 1081.1 \cdot 1.02^{20} \cdot 0.99 \cdot \frac{(1.02 \cdot 0.99)^{12} - 1}{1.02 \cdot 0.99 - 1} \approx 20147.73992\,.$$

The principal in the 20th payment is

$$B_{19} - B_{20} \approx 1357.73\,.$$

With the interest rate $i = d/(1-d) = 0.01/0.99 = 1\,/\,99$, the interest in the 20th payment is

$$i\, B_{19} \approx \frac{1}{99} \cdot 21505.47149 \approx 217.23\,.$$

7. Mr. Beltram takes out a \$100,000 loan for twelve years. The applicable annual effective interest rate is a promotional rate of 2% for the first two years and 6% for the remainder of the loan term. Mr. Beltram's payments increase by 10% each year. Find the balance on the loan immediately following his fifth payment.

Let x denote the amount of his first payment. The payment at time k is $x(1.1)^{k-1}$ for $k = 1, \ldots, 12$. In the promotion period the discount factor is $v_* := 1/(1 + 0.02)$ and in the remainder period the discount factor is $v := 1/(1+0.06)$. The cumulative discount factor at time k is v_* and v_*^2 for $k = 1, 2$ respectively and $v_*^2 v^{k-2}$ for $k = 3, \ldots, 12$. Then the present value of the payments is

$$100000 = \sum_{k=1}^{2} x(1.1)^{k-1} v_*^k + \sum_{k=3}^{12} x(1.1)^{k-1} v_*^2 v^{k-2} = x\Big(v_* + 1.1 v_*^2 + v_*^2 \sum_{k=3}^{12} (1.1)^{k-1} v^{k-2}\Big)$$

$$= x\Big(v_* + 1.1 v_*^2 + v_*^2 (1.1)^2 v \sum_{\ell=0}^{9} (1.1 v)^{\ell}\Big) = x\Big(v_* + 1.1 v_*^2 + v_*^2 (1.1)^2 v \cdot \frac{(1.1v)^{10} - 1}{1.1v - 1}\Big)\,.$$

Solving this equation for x, we obtain

$$x = 100000\Big(v_* + 1.1 v_*^2 + v_*^2 (1.1)^2 v \cdot \frac{(1.1v)^{10} - 1}{1.1v - 1}\Big)^{-1} \approx 6634.34\,.$$

The outstanding loan balance B_5 at time 5 after the 5th payment is given by the prospective formula

$$B_5 = x(1.1)^5 v + x(1.1)^6 v^2 + \cdots + x(1.1)^{11} v^7 = x(1.1)^5 v \cdot \frac{(1.1v)^7 - 1}{1.1v - 1} \approx 79068.75 .$$

8. A bank makes a package of three loans to a small business.

(a) \$120,000 amortized monthly for ten years at a nominal discount rate of 6.8% convertible monthly.

(b) \$100,000 to be repaid by monthly sinking fund payments for ten years where interest is assessed at a rate of 5.4% nominal convertible monthly and the sinking fund earns 4% nominal interest convertible monthly. The bank receives the sinking fund deposits.

(c) \$200,000 to be repaid with interest at the end of ten years with an effective rate of discount of 8.2% throughout the ten years.

Find the bank's annual effective yield on each of these loans individually and on the package of loans over the ten-year period.

We analyze each loan (a)-(c) first and then consider the package of loans for 10-year period.
(a) This is the simple loan. The annual effective yield rate is the same as the annual effective interest rate of the loan. Since the nominal discount rate $d^{(12)} = 6.8\%$, the equivalent interest rate is

$$i = \left(1 - \frac{d^{(12)}}{12}\right)^{-1} - 1 = 0.07057233461 = 7.05723\% .$$

The effective monthly interest is $I := (d^{12}/12)\,/(1 - (d^{12}/12)) \approx 0.5698961\%$, and hence the bank receives the monthly payment $P := 120000\,/\,a_{\overline{120}|I} \approx 1383.349663 \approx 1383.35$ for up to the 119th month and then the last payment is 1383.29, because

$$1383.35 + (120000 - 1383.35 a_{\overline{120}|I})(1 + I)^{120} \approx 1383.29 .$$

(b) The bank receives the 5.4 % interest and the 4%-sinking fund deposit each month for 10 years. The monthly interest payment is $100000 \cdot 0.054/12 = 450$ The monthly sinking fund deposit amount is 679.12 up to the 119th month, because $100000/s_{\overline{120}|0.04/12} = 679.1180$. The 120th month deposit is $678.83 (= 679.12 - 0.29)$, that is, it is slightly less because $679.12 s_{\overline{12}|0.04/12} = 100000.29$. Thus, with the monthly yield rate j the equation of value is

$$100000 = (450 + 679.12) a_{\overline{120}|j} - 0.29(1 + j)^{-120} .$$

Solving this equation, we obtain $j \approx 0.005131230$ and hence, the annual effective yield $i = (1 + j)^{12} - 1 \approx 0.0656436 = 6.56436\%$.

(c) Similar to the loan in (a), the annual yield for this loan is $i = d\,/\,(1 - d) = 0.082/(1 - 0.082) = 0.089324619 \approx 8.93246\%$. The payment is $200000(1 - d)^{10} = 470547.1893$ at time 10 (year).

Combining the loans (a)-(c) together, the bank pays $120000 + 100000 + 200000 = 420000$ at time 0, receives

$$1383.35 + 1129.12 = 2512.47$$

for the $1, \ldots, 119$th month, and then at time 120 (month), the bank receives

$$1383.29 + 1128.83 + 470547.19 = 473059.31 .$$

Thus, with the monthly effective yield R, we have the equation of value

$$420000 = 2512.47 a_{\overline{120}|R} - 2512.47(1 + R)^{120} + 473059.31(1 + R)^{120} .$$

Solving it for R, we obtain $R = 0.0664104278$. The annual yield of the package loan is $(1 + R)^{12} - 1 \approx 0.08266875052 \approx 8.2668\%$.

9. A loan of \$10,000 is negotiated, with the borrower agreeing to repay the principal over ten years as well as to make annual end-of-year payments of interest at 4% effective per annum. A \$1,100 total payment will be due at the end of each year during the first five years, and a higher level end-of-year payment will be required during the second five years.

The lender will replace his capital by means of a sinking fund earning 5% per annum. Each time he receives a payment from the borrower, he will deposit that portion representing principal into the sinking fund.

What will be the lender's yield on the whole transaction, assuming all payments are made as scheduled?

Suppose that the loan of 10000 is repaid by 1100 in the first five years and by $1100 + x$ in the second five years. Since the annual effective interest rate is $i = 4\%$ with $v := 1/(1+i)$, we have the equation of value:

$$10000 = 1100 a_{\overline{10}|0.04} + x a_{\overline{5}|0.04} v^5 .$$

Solving for x, we obtain

$$x = \frac{10000 - 1100 a_{\overline{10}|0.04}}{a_{\overline{5}|0.04}} \cdot (1+i)^5 \approx 294.61 .$$

The lender uses the 5%-sinking fund for the replacement of the capital of 10000. The level amount to be deposited is the portion of principal payment, i.e., the payment from the loan minus the payment of interest with respect to the annual yield (say j) for the replacement of the capital. In the first five years the principal payments are $1100 - 10000\,j$ and the next five years the principal payments are $1100 + x - 10000\,j$ for each year. Thus the sinking fund accumulates 10000 in 10 years. Thus, we have the equation of value

$$10000 = (1100 - 10000\,j) s_{\overline{10}|0.05} + x s_{\overline{5}|0.05} ,$$

and solving it for the annual yield j, we obtain

$$j = \frac{1100 - s_{\overline{10}|0.05} - 10000 + x s_{\overline{5}|0.05}}{10000 s_{\overline{10}|0.05}} \approx 4.343802427 \approx 4.344\% .$$

Flipgrid Homework Submission

`Here is the instruction how to upload your solution.`

1. Find below the submission link to Flipgrid of your section (also listed in Week 1 in GauchoSpace). There are two Fligprid groups for different sections. Please choose the correct one and login with your UCSB google email.

- Group: Y. Feng sections F 11AM/F 1PM
`https://flipgrid.com/ace05ed8`
- Group: J.K. Chen section M 11 AM + M. Becerra section T 3 PM
`https://flipgrid.com/e39d73e9`

- If it is your first time, then your account is automatically created.
- If you have login problems, please let the instructor know.

Upload your video HW 5 Problem 2. Please read the problem carefully, since the problem is slightly different.

That is it for HW 5! We will be having online quiz next week (Monday - Tuesday) based on the homework problems.