PSTAT 174/274 Spring 2021 Homework 7

Note: $\{Z_t\} \sim WN(0, \sigma_Z^2)$ denotes white noise.

1. (Updating AR(2) forecast in presence of more information.) You are given the following AR(2) model: $X_t = 33 + 0.5X_{t-2} + Z_t$, where Z_t is a white noise process.
After observing $x_{76} = 64$ and $x_{77} = 68$, your forecast for X_{80} is a.
After observing, in addition, $x_{78} = 66$, your forecast for X_{80} becomes b. Calculate $b - a$.
all *Hint: Check Example 13.2 in lecture notes or on slide 11 of Week 7.*

$AR(p) = X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t$ $\quad p=2$

$AR(2) = X_t = 33(1) + 0.5X_{t-2} + Z_t$

$X_{76} = 64$

$X_{77} = 68$ $\quad X_{78} = 66$

$\vdots$ $\quad \vdots$

$X_{80} = a$ $\quad X_{80} = b$

One-step

$P_{80}X_{81} = E_{80}[X_{81}] = 33E_{80}(X_{80}) + 0.5E_{80}(X_{79}) + E_{80}(Z_{81})$

$= 33X_{80} + 0.5_2 X_{79}$

$33X_{79} + 0.5_{78}$ $\quad 33_{78} + 0.5_{77}$

$33(66) + 0.5(68)$

$= 2178 + 34$

$\simeq 2212$

two-step

$P_{80}X_{82} = E_{80}(X_{82}) = 33E_{80}(X_{81}) + 0.5E_{80}(X_{80}) + E_{80}(Z_{82})$

$= \emptyset_1 P_n X_{n+h-1}$

$= 33P_{80}X_{81} + 0.5X_{80}$ $\quad$ if $h=2$ $\quad P_n X_{n+2} = \phi_1 P_n X_{n+1} = \phi_1^2 X_n$

$= (33)^2 X_{80} + 0.5X_{80}$

$33_{77} + 0.5_{76}$

$33(68) + 0.5(64)$

$= 2244 + 32$

$\simeq 2276$ $\quad b-a$

$\simeq 64$

2. (Standard error of forecast error.) You are given the following fitted AR(1) model: $X_t = 5 + 0.85X_{t-1} + Z_t$. The estimated mean squared error (that is, variance of Z_t) is 13.645. Calculate the two-step ahead forecast standard error.

Hint: Check Example 13.1 in lecture notes or on slide 10 of Week 7.

$AR(1): X_t = 5 + \underset{\phi_1}{0.85} X_{t-1} + Z_t$ M.S.E. $\underset{(var)}{(Z_t)} = 13.645$

2-step ahead forecast standard error $h=2$

$h=2$

$P_n X_{n+2} = \phi_1^2 X_n$

$= 0.85^2 X_n$

$e_n(1) = X_{n+1} - 0.85^2 X_n = (0.85 X_{n+1-1} + Z_{n+1}) - .85 X_n = Z_{n+1}$

$e_n(2) = 0.85(Z_{n+1}) + Z_{n+2}$

$var(e_n(2)) \equiv 13.645\left(\frac{1-0.85^4}{1-0.85^2}\right)$

$= 23.5035125$

3. (Box-Pierce Statistics.) A modeler fitted 100 observations, using ARMA$(\overset{p}{1}, \overset{q}{2})$ model. For the 100 residuals, she have determined the first seven autocorrelation coefficients:

k	1	2	3	4	5	6	7
$\hat{\rho}_{\hat{W}}(k)$	.20	.15	-.18	.16	.08	.07	.09

Calculate the value of the Box-Pierce χ^2 statistics and determine the number of degrees of freedom for its distribution.

$$Q_W = n \sum_{j=1}^{h} \hat{\rho}_{\hat{W}}^{\,2}(j) \sim \chi^2(h-p-q)$$

$h=7$
$j=1$
$n=100$

$$Q_W = 100 \sum_{j=1}^{7} (.2)^2 + (.15)^2 + (-.18)^2 + (.16)^2 + (.08)^2 + (.07)^2 + (.09)^2$$

$$.04 + .0225 + .0324 + .0256 + .0064 + .0049 + .0081$$

$$100\,(.1399)$$

$$Q_W = 13.99 \sim \chi^2(7-1-2)$$

$$Q_W = 13.99 \sim \chi^2(4)$$

4 degrees of freedom

4. (Prediction intervals.) A Gaussian AR(1) model was fitted to a time series based on a sample of size n. You are given $\hat{\phi}_1 = 0.8$, $\hat{\mu} = 2$, $\hat{\sigma}_Z^2 = 9 \times 10^{-4}$, $x_n = 2.05$. Write the 95% prediction interval for the observation three periods ahead.

Hint: review Example 13.1 of Week 7; slide 10. Do not forget that the mean is not 0!

$$|\hat{\rho}(h)| < \frac{1.96}{\sqrt{n}} \text{ for all } h \geq 1$$

$$h=1 : P_n X_{n+1} = \phi_1 X_n$$
$$(0.8)(2.05)$$
$$= 1.64$$

$$e_n(1) = X_{n+1} - P_n X_{n+1} = (\phi_1 X_n + Z_{n+1}) - (\phi_1 X_n) = Z_{n+1}$$
$$= 0.8(2.05) + (Z_{n+1}) - 0.8(2.05)$$
$$= 1.64(Z_{n+1}) - 1.64$$
$$= Z_{n+1}$$

$$\text{var}(e_n(1)) \equiv \frac{(9 \cdot 10^{-4})\left((\cancel{0.8})^{\cancel{2(0)}}\right)}{1-(0.8)}$$

$$= (9 \times 10^{-4}) \left(\frac{1 - \cancel{0.8}}{1-(0.8)^2} \right)$$

$$= (9 \times 10^{-4}) \left(\frac{1}{.36} \right)$$

$$= .0025$$