PSTAT 174/274, Spring 2021: Homework # 3.

Note: $\{Z_t\} \sim WN(0, \sigma_Z^2)$ denotes white noise.

1. You are given the following time series model: $X_t = \frac{2}{3}X_{t-1} + \frac{1}{2}X_{t-2} + Z_t$.
Determine whether this time series is stationary and/or invertible.

invertible? AR(2) where p=2

AR(p) always invertible, by construction: $Z_t = \phi B X_t$

invertible and stationary

Stationary?

$B^k X_t = X_{t-k}$

$X_t - \frac{2}{3}X_{t-1} + \frac{1}{2}X_{t-2} = Z_t$

$\left(1 - \frac{2}{3}B + \frac{1}{2}B^2\right)X_t = Z_t$

When it has an MA(∞) rep.

$30 - 20B + 15B^2$

$= \dfrac{20 \pm \sqrt{(20)^2 - 4 \cdot 15 \cdot 30}}{2(15)}$

$B = \frac{2}{3} \pm i\frac{\sqrt{14}}{3}$

roots lie outside unit circle so it's stationary

2. You are given the following statements about a time series modeled as an AR(3) process:
I. Partial Autocorrelation for lag 3 is always equal to zero.
II. Partial Autocorrelation for lag 4 is always equal to zero.
III. Partial Autocorrelation for lag 4 is always greater than zero.
Determine which of the above statements are true.

II. b/c the acfs after lag 3 are equal to zero since this is an AR(3) process

3. For a stationary ARMA(1,1) model, you are given the following information: $\rho_X(1) = 0.7$, $\rho_X(2) = 0.3$. Calculate ϕ_1.
Hint: Formulas for ACF of ARMA (1,1) model $X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$ are given in §5.1 of lecture notes and slide 24 of week 3. Can you determine a recursive relation between $\rho_X(k)$ and $\rho_X(k-1)$? Use it!

ACF ARMA(1,1) $X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$

$$(X_{t-k}) X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1} (X_{t-k})$$

$$E\left[X_t X_{t-k} - \phi_1 X_{t-1} X_{t-k}\right] = E\left[X_{t-k} Z_t + \theta_1 X_{t-k} Z_{t-1}\right]$$

$$E[X_t X_{t-k}] - \phi_1 E[X_{t-1} X_{t-k}] = E[\cancel{X_{t-k} Z_t}]^{\to 0} + \theta_1 E[\cancel{X_{t-k} Z_{t-1}}]^{\to 0}$$

$$\frac{\gamma_x(k) - \phi_1 \gamma_x(k-1)}{\gamma_x(0)} = 0$$

$$\rho_x(k) = \phi_1 \rho_x(k-1)$$

$$\rho_x(2) = \phi_1 \rho_x(1)$$

* $\rho_x(1) = 0.7$
$\rho_x(2) = 0.3$

$$\frac{\rho_x(2)}{\rho_x(1)} = \phi_1$$

$$\frac{0.3}{0.7} = \phi_1$$

$$\phi_1 = \frac{3}{7}$$

4. You are given PACF for a stationary process: $\phi_{11} = -0.60$, $\phi_{22} = 0.36$, $\phi_{kk} = 0$ for $k \geq 3$. What time series model could have this PACF? Identify model's coefficients and write model equation.

Hint: • *§6 of Lecture Notes and slides 32-33 of Week 3 provide relationship between PACF and ACF, allowing to calculate ACFs of the model from given PACFs.*

• *Yule-Walker equations, in §4.3 of Lecture Notes and on slide 14, Week 3, provide relationship between ACF and model coefficients.*

$\phi_{11} = \rho_x(1) = -0.60$

$\alpha(1) = \phi_{11} = \rho_x(1)$

$$\rho_x(1) = \frac{\phi_1}{1-\phi_2}$$

$$-0.60 = \frac{\phi_1}{1-\phi_2} = \quad -0.60 = \frac{\phi_1}{1-.744}$$

$\phi_1 = -.1536$

$$\alpha(2) = \phi_{22} = \frac{\rho_x(2) - (\rho_x(1))^2}{1-(\rho_x(1))^2}$$

$$0.36 = \frac{\rho_x(2) - (0.60)^2}{1-(0.60)^2}$$

$$(.64)\,0.36 = \frac{\rho_x(2) - .36}{.64}\,(.64)$$

$.2304 = \rho_x(2) - .36$

$+.36 \qquad +.36$

$\rho_x(2) = .5904$

$$\rho_x(2) = \frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2}$$

$.5904 = \phi_1^2 + \phi_2$

$\phi_1 = -0.60(1-\phi_2)$

$.5904 = (-0.60 + 0.60\phi_2) + \phi_2$

$$\frac{1.1904}{1.6} = \frac{1.6\phi_2}{1.6}$$

$\phi_2 = .744$

AR(2) model becauss PACF stops after lag $k=2$, b/c $k \geq 3$

$X_t = -.1536X_{t-1} + .744X_{t-2} + Z_t$

5. You are given the following time-series model: $X_t = 0.8X_{t-1} + 2 + Z_t - 0.5Z_{t-1}$. Which of the following statements about this model is false?

A. $\rho_X(1) = 0.4$. B. $\rho_X(k) < \rho_X(1), k \geq 2$. C. The model is ARMA(1,1).
D. The model is stationary. E. The mean, μ_X, is 2.

B. is false, because for $\rho_X(k)$, $k \geq 1$ for an AR(2) model

C. is false, b/c $X_t = 0.8X_{t-1} + 2 + Z_t - 0.5Z_{t-1}$ is not an ARMA(1,1) model, it is not in the same form: $X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$

6. The Notion of *parameter redundancy* pertains to the situation when AR and MA characteristic polynomials $\phi(z)$ and share $\theta(z)$ share a common factor, in which case model may be simplified. Determine which of the following models are parameter redundant:
I. $X_t = \frac{1}{2}X_{t-1} + Z_t - \frac{1}{2}Z_{t-1}$;
II. $X_t = \frac{1}{2}X_{t-1} + Z_t - \frac{1}{9}Z_{t-2}$;
III. $X_t = -\frac{5}{6}X_{t-1} - \frac{1}{6}X_{t-2} + Z_t + \frac{8}{12}Z_{t-1} + \frac{1}{12}Z_{t-2}$.

I. $X_t = \frac{1}{2}X_{t-1} + Z_t - \frac{1}{2}Z_{t-1}$

$X_t - \frac{1}{2}BX_t = Z_t - \frac{1}{2}BZ_t$

$(1 - \frac{1}{2}B)X_t = (1 - \frac{1}{2}B)Z_t$

shared a common factor

$X_t = Z_t$

parameter redundant

II. $X_t = \frac{1}{2}X_{t-1} + Z_t - \frac{1}{9}Z_{t-2}$

$X_t - \frac{1}{2}BX_t = Z_t - \frac{1}{9}B^2 Z_t$

$(1 - \frac{1}{2}B)X_t = (1 - \frac{1}{9}B^2)Z_t$

not redundant

III. $X_t = -\frac{5}{6}X_{t-1} - \frac{1}{6}X_{t-2} + Z_t + \frac{8}{12}Z_{t-1} + \frac{1}{12}Z_{t-2}$

$X_t + \frac{5}{6}X_{t-1} + \frac{1}{6}X_{t-2} = Z_t + \frac{8}{12}Z_{t-1} + \frac{1}{12}Z_{t-2}$

$(1 + \frac{5}{6}B + \frac{1}{6}B^2)X_t = (1 + \frac{8}{12}B + \frac{1}{12}B^2)Z_t$

$X_t(1 + \frac{1}{3}B)\cancel{(1 + \frac{1}{2}B)} = (1 + \frac{1}{6}B)\cancel{(1 + \frac{1}{2}B)}$

$X_t(1 + \frac{1}{3}B) = (1 + \frac{1}{6}B)Z_t$ shared a common factor

parameter redundant