PSTAT 174/274: Homework # 1.

This homework is based on Lectures 1–2 and lab 1. Please review the lecture notes, lecture slides and your lab 1 work *before* starting working on this problems. *Good Luck!*

1. *Understanding deterministic and stochastic trends.* You are given the following statements about time series:
I. Stochastic trends are characterized by explainable changes in direction.
II. Deterministic trends are better suited to extrapolation than stochastic trends.
III. Deterministic trends are typically attributed to high serial correlation with random error.

Determine which statements are true. Explain.

A. I only	B. II only	C. III only
D. I, II, and III	E. The answer is not given by (A), (B), (C), or (D).	

B) II only

I. is incorrect because stochastic trends are caused by random variation

III. This is not true for the deterministic trend

2. *Random walk and stationarity.* A random walk is expressed as $X_1 = Z_1$, $X_t = X_{t-1} + Z_t$, $t = 2, 3, \ldots$, where $Z_t \sim WN(\mu_Z, \sigma_Z^2)$, that is, $E(Z_t) = \mu_Z$, $Var(Z_t) = \sigma_t^2$, and $Cov(Z_t, Z_s) = 0$ for $t \neq s$. Determine which statements are true with respect to a random walk model; show calculations and provide complete explanations.

I. If $\mu_Z \neq 0$, then the random walk is nonstationary in the mean.
(*Hint: Nonstationary in the mean means that the mean changes with time.*)

II. If $\sigma_Z^2 = 0$, then the random walk is nonstationary in the variance.
(*Hint: Nonstationary in the variance means that the variance changes with time.*)

III. If $\sigma_Z^2 > 0$, then the random walk is nonstationary in the variance.

I. $E[X_1] = E[Z_1] = \mu_Z$

$E[X_2] = E[X_1 + Z_2] = E[X_1] + E[Z_2] = \mu_Z + \mu_Z = 2\mu_Z$

$E[X_3] = E[X_2 + Z_3] = E[X_2] + E[Z_3] = 2\mu_Z + \mu_Z = 3\mu_Z$

when $t = 1, 2, \ldots n$

$E[X_t] = t\mu_Z$

dependent on t, non-stationary $\mu_Z \neq 0$ mean changes with time

True.

II. $\sigma_Z^2 = 0$

$V[X_1] = V[Z_1]$
$= \sigma_Z^2$

$V[X_2] = V[X_1 + Z_2]$
$= V[X_1] + V[Z_2] + 2\text{cov}[X_1, Z_2]$
$= \sigma_Z^2 + \sigma_Z^2 + 2 \cdot 0$
$= 2\sigma_Z^2$

$V[X_3] = V[X_2 + Z_3]$
$= V[X_2] + V[Z_3] + 2\text{cov}[X_2, Z_3]$
$= 2\mu_Z + \mu_Z + 2 \cdot 0$
$= 3\mu_Z$

where $t = 1, 2, \dots n$
$V[X_t] = t\sigma_Z^2$ dependent on t
$\sigma_Z^2 = 0$ stationary

False

III. From II
$V[X_t] = t\sigma_Z^2$
Variance is dependent on t, so nonstationary if $\sigma_Z^2 > 0$

True

3. *Calculation of sample acf.* You are given the following quarterly rainfall totals over a two-year span:

Quarter	Rainfall
2016 q1	25
2016 q2	19
2016 q3	10
2016 q4	32
2017 q1	26
2017 q2	38
2017 q3	22
2017 q4	20

Calculate the sample lag 4 autocorrelation. *Hints:*
(i) We are given a sample of size $n = 8$ *to estimate autocorrelation at lag 4:* $\rho(4) = Cor(X_1, X_5) = \frac{\gamma(4)}{\gamma(0)}$,
– for definition of autocorrelation at lag 4 see Week 1 slide 52 or (2.1.3) on p. 6 of Lecture Notes.
(ii) General formulas for calculating sample mean and covariance are given on slide 38 of week 1 and in §1.2 on p. 4 of Lecture notes for week 1. To estimate $\rho(4) = Cor(X_1, X_5)$ *we have:*

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t, \quad \hat{\rho}_4 = \frac{\hat{\gamma}(4)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{n-4}(x_t - \bar{x})(x_{t+4} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}.$$

	x_t	$x_t - \bar{x}$	$(x_t - \bar{x})^2$	$(x_{t+4} - \bar{x})$	$(x_t - \bar{x})(x_{t+4} - \bar{x})$
1	25	$25-24=1$	1		$(1)(2)=2$
2	19	$19-24=-5$	25		$(-5)(14)=-70$
3	10	$10-24=-14$	196		$(-14)(-2)=28$
4	32	$32-24=8$	64		$(8)(-4)=-32$
5	26	$26-24=2$	4	$26-24=2$	
6	38	$38-24=14$	196	$38-24=14$	
7	22	$22-24=-2$	4	$22-24=-2$	
8	20	$20-24=-4$	16	$20-4=-4$	

$\Sigma x_t = 192$ $\quad \Sigma(x_t - \bar{x})^2 = 506$ $\quad \sum_{t=1}^{n-4}(x_t - \bar{x})(x_{t+4} - \bar{x}) = 72$

$\bar{x} = \frac{1}{8}\sum_{t=1}^{8} x_t$
$= \frac{192}{8}$
$\bar{x} = 24$

$\hat{p}_4 = \frac{\sum_{t=1}^{n-4}(x_t - \bar{x})(x_{t+4} - \bar{x})}{\sum_{t=1}^{n=4}(x_t - \bar{x})^2}$
$= \frac{-72}{506}$
$\hat{p}_4 = -0.142$

4. *Polyroot command in R. Recall from algebra, that a function $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ is called a polynomial function of order n. Roots of a polynomial function f are solutions of the equation $f(z) = 0$. Roots of a quadratic equation $ax^2 + bx + c = 0$ are given by the formula $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.*

Let $f(z) = 1 - 0.4z$ and $g(z) = 1 + z - 6z^2$. Find their roots, show calculations. Check your answers using R command *polyroot*:

> *polyroot*$(c(1, -0.4))$ [1] 2.5+0i

> *polyroot*$(c(1, 1, -6))$. (*Do not forget to include your output!*)

[1] 0.5000000-0i - 0.3333333+0i

$f(z) = 1 - 0.4z$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 0$
$b = -0.4$
$c = 1$

$$x_{1,2} = \frac{+0.4 \pm \sqrt{\cancel{(0.4)^2 - 4(0)(1)}}}{2(0)}$$

$$x_{1,2} = \frac{0.4 \pm \sqrt{.16}}{2(0)}$$

$$x_1 = \frac{0.4 + \sqrt{.16}}{2(0)} \qquad x_2 = \frac{0.4 - \sqrt{.16}}{2(0)}$$

$$x_1 = \frac{0.8}{2(0)} \qquad x_2 = \frac{0.0}{2(0)}$$

$x_1 = \text{int}$ $\qquad x_2 = 2.5$

$g(z) = 1 + z - 6z^2 \qquad f(z) = 0$

$a = -6$
$b = 1$
$c = 1$

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(-6)(1)}}{2(-6)}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 24}}{-12}$$

$$x_{1,2} = \frac{-1 \pm 5}{-12}$$

$$x_1 = \frac{-1+5}{-12} \qquad x_2 = \frac{-1-5}{-12}$$

$$x_1 = \frac{4}{-12} \qquad x_2 = \frac{-6}{-12}$$

$x_1 = \frac{-1}{3}$ $\qquad x_2 = \frac{1}{2}$

5. *Gaussian White Noise and its square.* Let $\{Z_t\}$ be a Gaussian white noise, that is, a sequence of i.i.d. normal r.v.s each with mean zero and variance 1. Let $Y_t = Z_t^2$.

(a) Using R generate 350 observations of the Gaussian white noise Z. Plot the series and its acf.

(b) Using R, plot 350 observations of the series $Y = Z_t^2$. Plot its acf.

(c) Analyze graphs from (a) and (b).

– Can you see a difference between the plots of graphs of time series Z and Y? From the graphs, would you conclude that both series are stationary (or not)?

– Is there a noticeable difference in the plots of acf functions ρ_Z and ρ_Y? Would you describe Y as a non-Gaussian white noise sequence based on your plots?

Provide full analysis of your conclusions.

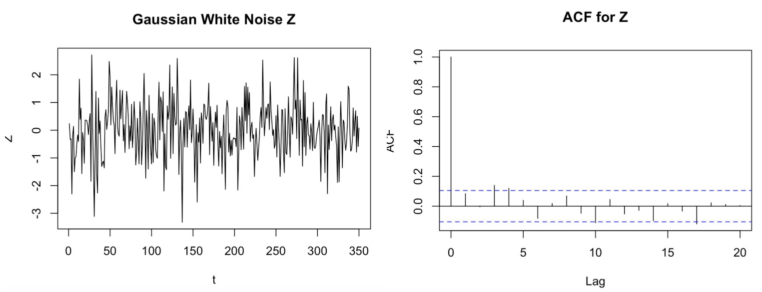
(d) Calculate the second-order moments of Y: $\mu_Y(t) = E(Y_t)$, $\sigma_Y^2(t) = Var(Y_t)$, and $\rho_Y(t, t+h) = Cor(Y_t, Y_{t+h})$. Do your calculations support your observations in (c)?

Hints: (i) Slides 65 and 68 of week 1 have R commands to generate MA(1) time series. White Noise is a MA(1) process with coefficient $\theta_1 = 0$. Here is a more direct code to generate WN $\{Z_t\} \sim N(0,1)$:

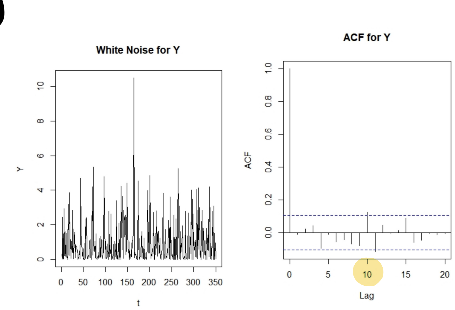
```
Z <- rnorm(350)
plot.ts(Z, xlab = "", ylab = "")
acf(Z, main = "ACF")
```

(ii) Useful for part (d): For $X \sim N(0, \sigma^2)$, $E(X^4) = 3(\sigma^2)^2$.

a)



b)



c) The (a) ACF graph lag is not sig. but the lag in ACF graph in (b) is sig. and correlated w/ serie. The (a) ACF graph has no corr., thus Z series is Stationary. While (b) has a non-zero corr., thus Y series is not Stationary. Since auto corr. f'n depends on lag and def of WN: Zt mean = 0, constant variance, uncorrelated and value 10 from lag values exceed expectation, Y is a non-Gassian WN sequence.

d) mean $= 0$, constant variance, uncorrelated

$\mu_{Y}(t) = E(Y_t)$ $\sigma^2_Y(t) = Var(Y_t)$ $\rho_Y(t, t+h) = Cor(Y_t, Y_{t+h})$

$X \sim N(0, \sigma^2)$, $E(X^4) = 3(\sigma^2)^2$