PSTAT 174/274, Spring 2021: Homework # 4.

Note: In all problems, $\{Z_t\} \sim WN(0, \sigma_Z^2)$ denotes white noise, and B denotes the backshift operator $BX_t = X_{t-1}$.

1. Determine which of the following indicates that a nonstationary time series can be represented as a random walk: $X_t = Z_1 + \dots + Z_t$ or $X_t = X_{t-1} + Z_t$

I. A plot of the series detects a linear trend and increasing variability.
II. The differenced series follows a white noise model.
III. The standard deviation of the original series is greater than the standard deviation of the differenced series.

II. $\nabla X_t \sim WN(0, \sigma_Z^2)$

$\nabla X_t = Z_t$

$X_t - X_{t-1} = Z_t$

$X_t = X_{t-1} + Z_t$: RW

= nonstationary

II.) The differenced series follows a WN model so, a non-stationary time series can be represented as a RW

2. Identify the following time series model as a specific ARIMA or SARIMA model:
2.a $X_t = 1.5X_{t-1} - 0.5X_{t-2} + Z_t - 0.1Z_{t-1}$
2.b $X_t = 0.5X_{t-1} + X_{t-4} - 0.5X_{t-5} + Z_t - 0.3Z_{t-1}$

2a) ARIMA(p, d, q)

$X_t - 1.5X_{t-1} - 0.5X_{t-2} = Z_t - 0.1Z_{t-1}$

$(1 - 1.5B + 0.5B^2)X_t = (1 - 0.1B)Z_t$

$(1-B)(1-0.5B)X_t = (1-0.1B)Z_t$ ← MA(1)

$d=1$ $p=1$ $q=1$

ARIMA(1,1,1)

2b) SARIMA(p,d,q) × (P,D,Q)$_S$

SARIMA(1,0,1)×(0,1,0)

$X_t = 0.5X_{t-1} + X_{t-4} - 0.5X_{t-5} + Z_t - 0.3Z_{t-1}$

$X_t - 0.5X_{t-1} - X_{t-4} + 0.5X_{t-5} = Z_t - 0.3Z_{t-1}$

$(1 - 0.5B - B^4 + 0.5B^5)X_t = (1-0.3B)Z_t$

$(1-B^4)(1-0.5B)X_t = \underbrace{(1-0.3B)Z_t}_{\text{MA}(1)}$

$d=0$ $D=1$ $S=4$ $p=1$ $q=1$

$Q=0$

3. (a) For the following SARIMA$(p,d,q)\times(P,D,Q)_s$ models, specify parameters p,d,q,P,D,Q and s, and write corresponding equations: (i) SARIMA $(2,1,1)\times(0,1,1)_6$, (ii) SARIMA$(1,1,2)\times(2,0,1)_4$.

Hint: It is helpful to first write equations in the form $\phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D X_t = \theta(B)\Theta(B^s)Z_t$, $Z_t \sim WN(0,\sigma_Z^2)$, *then proceed to rewrite the equations in the form that eliminates use of the backshift operator* B.

(b) For the following processes $\{X_t\}$, identify SARIMA $(p,d,q)\times(P,D,Q)_s$ model: (i) $(1-B^6)^2X_t = (1-0.3B)Z_t$; (ii) $X_t = 0.3X_{t-12} + Z_t$;

(c) You are given a time series model where PACF is zero except for lags 12 and 24. Which model will have this pattern?

a) SARIMA$(p,d,q)\times(P,D,Q)$ $Z_t \sim WN(0,\sigma_Z^2)$

i.) SARIMA$(2,1,1)\times(0,1,1)_6$

AR(2) ↑ MA(1)
differenced @ lag 1

$\phi(B)\Phi(B^s)(1-B)^d(1-B^s)^D X_t = \theta(B)\Theta(B^s)Z_t$

$p=2, d=1, q=1 \quad P=0, D=1, Q=1, S=6$

AR(2) $p=2$ $d=1$ $D=1$ $q=1$ MA(1) $Q=$ SMA(1)

$(1-\phi_1 B-\phi_2 B^2)(1-B)(1-B^6)X_t = (1+\theta_1 B)(1+\Theta_1 B^6)Z_t$

ii.) SARIMA$(1,1,2)\times(2,0,1)_4$

$p=1, d=1, q=2 \quad P=2, D=0, Q=1, S=4$

$p=1$ $d=1$ $P=2$ $S=4$ $q=2$

$(1-\phi_1 B)(1-B)(1-\Phi_1 B^4-\Phi_2 B^8) = (1+\theta_1 B+\theta_2 B^2)(1+\Theta_1 B^4)Z_t$

b) $\{X_t\}$ SARIMA$(p,d,q)\times(P,D,Q)_s$

i) $(1-B^6)^2 X_t = (1-0.3B)Z_t$

$p=0, d=0, q=1$

$P=0, D=2, Q=0$

$S=6$

SARIMA$(0,0,1)\times(0,2,0)_6$

ii.) $X_t = 0.3X_{t-12} + Z_t$ $\quad X_t - 0.3X_{t-12} = Z_t$

$(1-0.3B^{11})X_t = Z_t$

$p=0, d=0, q=0$

$P=1, D=0, Q=0$

$S=11$

SARIMA$(0,0,0)\times(1,0,0)_{11}$

c) PACF $=0$
lag(12) $\neq 0$
lag(24) $\neq 0$

$S=12$ $\quad (1-B)^{12}X_t = Z_t$

a SARMA model w/seasonality $=12$

SARIMA$(0,0,0)\times(0,1,0)_{12}$

$MA(1)$ $SMA(1)$

4. Write the form of the model equation for a SARIMA $(0,0,1)(0,0,1)_{12}$ model.

$s=12$

A. $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$.
B. $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_1\theta_2 Z_{t-3}$.
C. $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_1\theta_2 Z_{t-12}$.
D. $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-11} + \theta_1\theta_2 Z_{t-12}$.
E. $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-12} + \theta_1\theta_2 Z_{t-13}$.

$X_t = (1+\theta_1 B)(1+\theta_2 B^{12})Z_t$

E.) $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-12}\theta_2 Z_{t-13}$

AR(2)
lag $k=2$

5. You are given PACF for two stationary processes:
(a) For time series $\{X_t, t=1,2,\dots\}$: $\phi_{11}=0$, $\phi_{22}=0.36$, $\phi_{kk}=0$ for $k \geq 3$.
(b) For time series $\{Y_t, t=1,2,\dots\}$: $\phi_{11}=0.7$, $\phi_{kk}=0$ for $k \geq 2$.
In each case, write an appropriate equation for the corresponding stationary process.

a) $\phi_{11} = \rho_x(1) = 0$

$\phi_{22} = \dfrac{\rho_x(2) - [\rho_x(1)]^2}{1-(\rho_x(1))^2}$

$0.36 = \dfrac{\rho_x(2) - 0}{1-0^2}$

$0.36 = \rho_x(2)$

$\rho_x(2) = \dfrac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2}$

$0.36 = \dfrac{\cancel{\phi^2} + \phi_2\cancel{(1-\phi_2)}}{\cancel{(1-\phi_2)}}$

$\phi_2 = 0.36$

$\rho_x(1) = \dfrac{\phi_1}{1-\phi_2}$

$(1-\phi_2)0 = \dfrac{\phi_1}{1-\phi_2}(1-\phi_2)$

$\phi_1 = 0$

b) $\phi_{11} = \rho_x(1) = 0.7$

$\rho_x(1) = \dfrac{\phi_1}{1-\phi_2}$

$(1-\phi_2)0.7 = \dfrac{\phi_1}{1-\phi_2}(1-\phi_2)$

$\phi_1 = 0.7(1-\phi_2)$