1. In modeling the weekly sales of a certain commodity over the past six months, the time series model $X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$ was thought to be appropriate. Suppose the model was fitted and the autocorrelations of the residuals were:

k	1	2	3	4	5	6	7	8
$\hat{\rho}_{\hat{W}}(k)$	-.04	-.50	.03	-.01	.01	.02	.03	-.01
st. dev $\hat{\rho}_{\hat{W}}(k)$	.08	.10	.11	.11	.11	.11	.11	.11

Is the assumed model really appropriate? If not, how would you modify the model? Explain.

Hint: Check slides 8 - 9 of Lecture 11 for the 95% confidence intervals for autocorrelation function of the fitted residuals. You might also find slide 26 of Week 2 and slide 15 of Lecture 11 useful.

No trend, No seasonality, No change in variance

The fit is good if the residuals resembles Gaussian WN(0,1)

P/ACF of residuals

$\phi_Z(B) Z_t = \theta_Z B W_t \quad W_t \sim WN(0, \sigma_w^2)$

$\phi_Z(B)\phi(B) X_t = \theta(B)\theta_Z(B) W_t \quad W_t \sim WN(0, \sigma_w^2)$

95% C.I.:

$|\hat{\rho}(h)| < \frac{1.96}{\sqrt{n}}$ for all $h \geq 1$

$|\hat{\rho}(h)| < 0.69296$

$$\left.\begin{array}{l} \hat{\rho}(1) = -.04 < 0.69296 \\ \hat{\rho}(2) = -.50 < 0.69296 \\ \hat{\rho}(3) = .03 < 0.69296 \\ \hat{\rho}(4) = -.01 < 0.69296 \\ \hat{\rho}(5) = .01 < 0.69296 \\ \hat{\rho}(6) = .02 < 0.69296 \\ \hat{\rho}(7) = .03 < 0.69296 \\ \hat{\rho}(8) = -.01 < 0.69296 \end{array}\right\} \text{all within 95\% C.I.}$$

The fitted model is good if the residuals resembles WN, since the autocorrelations are within the 95% C.I. its WN

$X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$

$(1 - \phi_1 B) X_t = (1 + \theta_1 B) Z_t$ using backshift operator

$X_t = (1 - \phi_1 B)^{-1} (1 + \theta_1 B) Z_t$

2. Suppose that in a sample of size 100 from an AR(1) process with mean μ, $\phi = 0.6$, and $\sigma^2 = 2$, we obtain $\bar{x}_{100} = 0.271$. Construct an approximate 95% confidence interval for μ. Are the data compatible with the hypothesis that $\mu = 0$?

Help: (i) Here $\sigma^2 \equiv \sigma_Z^2$ is the variance of noise Z_t.
(ii) Large sample distribution of $\bar{X}_n$ is given as follows (See, for example, Slide 49 or §10.1 of Week 4): For n large, distribution of the sample mean $\bar{X}_n$ is approximately normal with mean $\mu \equiv EX_t$ and variance $n^{-1}v$, where $v \approx \gamma_X(0) + 2\sum_{h=1}^{\infty} \gamma_X(h)$.
(iii) For ACVF for AR(1) process see slide 43 of Week 2.

$\mu = 0$

$n = 100$ AR(1) $\phi = 0.6$ 95%

$\sigma^2 = 2$

$\bar{x}_{100} = 0.271$ $\quad X_t - \mu = \phi(X_{t-1} - \mu) + Z_t$, $WN(0, \sigma^2)$

$\bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$ $\quad \bar{X}_n \approx N(\mu_x, \frac{v}{n})$ with

$E(X_t) = \mu_x$

$\gamma_x(k) = \frac{\sigma^2}{1-\phi^2}$

var: $\frac{1}{n}\left[1 + 2\sum_{k=1}^{\infty} \phi^k \gamma_x(k)\right]$

$= \frac{1}{n}\left[1 + 2\left(\frac{1}{1-\phi} - 1\right)\right]\left(\frac{\sigma^2}{1-\phi^2}\right)$

$= \frac{1}{n}\left(\frac{2}{1-\phi} - 1\right)\left(\frac{\sigma^2}{1-\phi^2}\right)$

$= \frac{1}{n}\left(\frac{1+\phi}{1-\phi^2}\right)$

$= \frac{\sigma^2}{n(1-\phi^2)}$

95% CI μ_x: $\left(\bar{X}_n - 1.96\frac{\sqrt{v}}{\sqrt{n}}, \bar{X}_n + 1.96\frac{\sqrt{v}}{\sqrt{n}}\right)$

: $(-.419, .961)$ since $\mu = 0$ lies within C.I.

$\mu = 0$ is true