Alaris: A Rigorous Mathematical Specification

Formal Derivation of the Earnings Volatility Calendar Spread System with Complete Fault Monitoring

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Abstract

This document provides a complete mathematical specification of the Alaris quantitative trading system. Beginning from first principles, we derive the option pricing methodology for American-style securities under both positive and negative interest rate regimes, formalise the earnings-based volatility trading strategy with mathematically precise entry criteria, and specify the complete fault monitoring framework as a system of inequalities and logical predicates. Every decision rule in the system corresponds to a mathematically specified condition, ensuring deterministic behaviour. Where stochasticity is inherent to the underlying processes, we explicitly characterise the probabilistic assumptions and derive appropriate estimators with known statistical properties.

Introduction

Purpose and Scope

The Alaris system implements a systematic approach to capturing the implied volatility premium that manifests around corporate earnings announcements. This document serves three purposes: to provide a rigorous mathematical foundation for all computational methods employed, to specify formally the decision rules governing strategy execution, and to derive the complete set of fault conditions that the system monitors.

A distinguishing characteristic of this specification is its emphasis on mathematical completeness. Every algorithmic decision in Alaris corresponds to an evaluable predicate derived from the mathematical framework. Where the underlying phenomena are inherently stochastic, we explicitly state the probabilistic model assumptions and derive estimators with known convergence properties.

Document Organisation

The document proceeds as follows. Section 2 establishes the mathematical framework for option pricing, deriving the free boundary formulation and its integral equation transformation. Section 3 extends this framework to the double boundary regime arising under negative interest rates. Section 4 develops the volatility trading strategy, specifying the signal generation predicates with their academic provenance. Section 5 formalises the complete fault detection and monitoring system as a set of mathematical inequalities. Section 6 addresses the inherent stochasticity in certain components and specifies how this is accounted for.

The Option Pricing Framework

Stochastic Foundation

The Risk-Neutral Measure

The Alaris pricing methodology operates under the risk-neutral probability measure $\mathbb{Q}$. Under this measure, the asset price process $S(t)$ satisfies the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = (r - q)\,dt + \sigma\,dW^{\mathbb{Q}}(t) \tag{1}$$

where:

- $r \in \mathbb{R}$ denotes the continuously compounded risk-free interest rate
- $q \in \mathbb{R}$ denotes the continuous dividend yield

- $\sigma > 0$ denotes the instantaneous volatility
- $W^{\mathbb{Q}}(t)$ is a standard Brownian motion under $\mathbb{Q}$

i Parameter Domain Extension

Unlike classical treatments that assume $r \geq 0$, Alaris explicitly permits $r < 0$ to accommodate negative interest rate environments. This extension requires careful treatment of the exercise region topology, addressed in Section 3.

The solution to Equation **??** is the geometric Brownian motion:

$$S(t) = S(0)\exp\left[\left(r - q - \frac{\sigma^2}{2}\right)t + \sigma W^{\mathbb{Q}}(t)\right] \tag{2}$$

Filtration and Information Structure

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space equipped with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Brownian motion $W^{\mathbb{Q}}(t)$, augmented by null sets. All stopping times referenced in this document are with respect to this filtration.

The Free Boundary Problem

Value Function Characterisation

For an American put option with strike K and maturity T, define the value function $V : [0, T] \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ as:

$$V(t, s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau - t)}(K - S(\tau))^+ \,\middle|\, S(t) = s\right] \tag{3}$$

where $\mathcal{T}_{t,T}$ denotes the set of stopping times taking values in $[t, T]$.

The supremum in Equation **??** is attained by the optimal stopping time:

$$\tau^* = \inf\{u \in [t, T] : S(u) \leq B(T - u)\} \tag{4}$$

where $B(\cdot)$ is the optimal exercise boundary function.

The Free Boundary

Definition 0.1 (Exercise Boundary)**.** The optimal exercise boundary $B : [0, T] \to \mathbb{R}_{>0}$ is the unique continuous function satisfying:

1. **Value Matching**: $V(t, B(T-t)) = K - B(T-t)$ for all $t \in [0, T]$
2. **Smooth Pasting**: $\frac{\partial V}{\partial s}(t, B(T-t)) = -1$ for all $t \in (0, T)$

The value function satisfies the variational inequality:

$$\max\{\mathcal{L}V - rV,\ (K-s) - V\} = 0 \tag{5}$$

where the infinitesimal generator is:

$$\mathcal{L}V = \frac{\partial V}{\partial t} + (r-q)s\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2}$$

The Integral Equation Formulation

Early Exercise Premium Decomposition

The American option value decomposes into European option value plus early exercise premium:

$$V(\tau, s) = v(\tau, s) + \mathcal{P}(\tau, s) \tag{6}$$

where $\tau = T - t$ denotes time to maturity, $v(\tau, s)$ is the European put price, and $\mathcal{P}(\tau, s)$ is the early exercise premium.

Theorem 0.1 (Kim Integral Representation)**.** *The early exercise premium satisfies:*

$$\mathcal{P}(\tau, s) = \int_0^\tau e^{-ru}\left[rK\Phi(-d_-(u, s/B(u))) - qse^{(r-q)u}\Phi(-d_+(u, s/B(u)))\right] du \tag{7}$$

where:

$$d_\pm(u, z) = \frac{\ln(z) + (r-q)u \pm \frac{1}{2}\sigma^2 u}{\sigma\sqrt{u}}$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function.

The Boundary Integral Equation

Applying the value-matching condition at the boundary yields the nonlinear integral equation for $B(\tau)$:

$$K - B(\tau) = v(\tau, B(\tau)) + \int_0^\tau \mathcal{K}(\tau, u, B(\tau), B(u))\, du \tag{8}$$

where the kernel $\mathcal{K}$ is:

$$\mathcal{K}(\tau, u, b, b') = e^{-ru}\left[rK\Phi(-d_-(u, b/b')) - qb \cdot e^{(r-q)u}\Phi(-d_+(u, b/b'))\right]$$

Asymptotic Behaviour

The boundary function exhibits the following asymptotic behaviour:

Near Expiration ($\tau \to 0^+$):

$$\lim_{\tau \to 0^+} B(\tau) = \begin{cases} K & \text{if } r \geq q \\ K \cdot \frac{r}{q} & \text{if } r < q \text{ and } r > 0 \end{cases} \tag{9}$$

Long Maturity ($\tau \to \infty$):

$$\lim_{\tau \to \infty} B(\tau) = B_\infty = K \cdot \frac{\lambda\_}{\lambda\_ - 1} \tag{10}$$

where $\lambda\_$ is the negative root of the characteristic equation:

$$\frac{1}{2}\sigma^2\lambda^2 + \left(r - q - \frac{\sigma^2}{2}\right)\lambda - r = 0 \tag{11}$$

The QD+ Approximation Method

Quasi-Analytic Framework

The Alaris system employs the QD+ approximation method of Andersen, Lake, and Offengenden (2016) as the first stage of boundary estimation. This method provides an accurate quasi-analytic approximation that serves as initialisation for subsequent refinement.

Definition 0.2 (QD+ Boundary Approximation)**.** The QD+ approximation expresses the boundary as:

$$B^{\text{QD}}(\tau) = K \cdot \frac{\lambda(\tau)}{\lambda(\tau) - 1} \tag{12}$$

where $\lambda(\tau)$ is obtained from a modified characteristic equation with time-dependent correction:

$$\frac{1}{2}\sigma^2\lambda^2 + \left(r - q - \frac{\sigma^2}{2}\right)\lambda - r \cdot h(\tau) = 0$$

with $h(\tau) = 1 - e^{-r\tau}$.

Super Halley's Method

The QD+ boundary equation is solved using Super Halley's method, a third-order root-finding algorithm:

$$S_{n+1} = S_n - \frac{f(S_n)}{f'(S_n)} \cdot \frac{1}{1 - \frac{f(S_n) \cdot f''(S_n)}{2[f'(S_n)]^2}} \tag{13}$$

Theorem 0.2 (Super Halley Convergence)**.** *For a function f with simple root S^* and continuous third derivative in a neighbourhood of S^*, Super Halley's method converges cubically:*

$$|S_{n+1} - S^*| = O(|S_n - S^*|^3)$$

The implementation specifies convergence tolerance $\epsilon = 10^{-10}$ and maximum iterations $N_{\max} = 50$.

Double Boundary Regime Under Negative Rates

Regime Characterisation

Mathematical Conditions for Double Boundaries

The emergence of double boundaries occurs under specific parameter configurations involving negative interest rates.

Definition 0.3 (Double Boundary Regime)**.** The double boundary regime is characterised by the predicate:

$$\mathcal{R}_{\text{double}} \equiv (q < r < 0) \tag{14}$$

When $\mathcal{R}_{\text{double}}$ holds, the optimal exercise region for an American put is the set:

$$\mathcal{E}(\tau) = \{s > 0 : Y(\tau) \leq s \leq B(\tau)\}$$

where $B(\tau)$ is the upper boundary and $Y(\tau)$ is the lower boundary.

Physical Interpretation

In the double boundary regime, immediate exercise is optimal only when the asset price falls within a bounded interval. This arises because:

1. When $s > B(\tau)$: Continuation is optimal due to potential for deeper in-the-money scenarios
2. When $Y(\tau) \leq s \leq B(\tau)$: Immediate exercise is optimal
3. When $s < Y(\tau)$: Continuation is optimal because the negative dividend yield implies expected price appreciation

Critical Volatility Threshold

The behaviour of the double boundary system depends on whether volatility exceeds a critical threshold.

Definition 0.4 (Critical Volatility)**.** Define the critical volatility as:

$$\sigma^* = \sqrt{\frac{2|r|(r-q)}{q}} \tag{15}$$

When $\sigma > \sigma^*$, the boundaries intersect at finite time τ^*.

Modified Integral Equation System

Two-Boundary Integral Representation

Under the double boundary regime, the American put value satisfies:

$$V(\tau, s) = v(\tau, s) + \mathcal{P}_B(\tau, s) - \mathcal{P}_Y(\tau, s) \tag{16}$$

where:

$$\mathcal{P}_B(\tau, s) = \int_0^\tau e^{-ru} \left[rK\Phi(-d_-(u, s/B)) - qse^{(r-q)u}\Phi(-d_+(u, s/B)) \right] du$$

$$\mathcal{P}_Y(\tau, s) = \int_0^\tau e^{-ru} \left[rK\Phi(-d_-(u, s/Y)) - qse^{(r-q)u}\Phi(-d_+(u, s/Y)) \right] du$$

Decoupled Fixed-Point Systems

A key mathematical result enabling efficient computation is the decoupling of the boundary equations.

Theorem 0.3 (Boundary Decoupling)**.** *The upper boundary $B(\tau)$ can be computed independently of $Y(\tau)$ using:*

$$K - B(\tau) = v(\tau, B(\tau)) + \int_0^\tau \mathcal{K}_B(\tau, u, B(\tau), B(u))\, du \tag{17}$$

Given $B(\tau)$, the lower boundary $Y(\tau)$ is then determined by:

$$K - Y(\tau) = v(\tau, Y(\tau)) + \int_0^\tau \mathcal{K}_B(\tau, u, Y(\tau), B(u)) - \mathcal{K}_Y(\tau, u, Y(\tau), Y(u))\, du \tag{18}$$

FP-B' Stabilisation

The Alaris system employs the FP-B' stabilised iteration scheme from Healy (2021) to prevent oscillatory behaviour in long-maturity options.

Definition 0.5 (FP-B' Iteration)**.** At iteration n, the stabilised update for the lower boundary is:

$$Y^{(n+1)}(\tau) = \mathcal{T}_Y \left[Y^{(n)}; B^{(n+1)} \right]$$

where the operator $\mathcal{T}_Y$ incorporates the just-computed upper boundary $B^{(n+1)}$ rather than the previous iteration $B^{(n)}$.

This stabilisation ensures convergence for maturities up to $T = 15$ years under the Healy benchmark parameters.

Boundary Validation Predicates

The computed boundaries must satisfy mathematical constraints from the free boundary theory.

Definition 0.6 (Boundary Validity Conditions)**.** A computed boundary pair $(B(\tau), Y(\tau))$ is valid if and only if:

$$\begin{aligned}
\mathcal{V}_1 &\equiv B(\tau) > 0 \quad \forall \tau \in [0, T] && \text{(Positivity - Upper)} \\
\mathcal{V}_2 &\equiv Y(\tau) > 0 \quad \forall \tau \in [0, T] && \text{(Positivity - Lower)} \\
\mathcal{V}_3 &\equiv B(\tau) > Y(\tau) \quad \forall \tau \in [0, T] && \text{(Ordering)} \\
\mathcal{V}_4 &\equiv B(\tau) < K \quad \forall \tau \in [0, T] && \text{(Put Constraint - Upper)} \\
\mathcal{V}_5 &\equiv Y(\tau) < K \quad \forall \tau \in [0, T] && \text{(Put Constraint - Lower)}
\end{aligned}$$

The aggregate validity predicate is:

$$\mathcal{V}_{\text{boundary}} \equiv \bigwedge_{i=1}^{5} \mathcal{V}_i$$

The Earnings Volatility Trading Strategy

Theoretical Foundation

The Volatility Risk Premium

The earnings volatility strategy exploits the documented phenomenon that implied volatility systematically exceeds realised volatility around corporate earnings announcements. Let:

- $\sigma_I(t)$ denote the market-implied volatility at time t
- $\sigma_R(t)$ denote the realised volatility over the same horizon

The volatility risk premium is:

$$\text{VRP}(t) = \sigma_I(t) - \sigma_R(t) \tag{19}$$

Earnings VRP Assumption

For securities with scheduled earnings announcements at time T_E, the volatility risk premium exhibits the pattern:

$$\mathbb{E}[\text{VRP}(t)] > 0 \quad \text{for } t \in [T_E - \Delta, T_E)$$

where $\Delta \approx$ 5-7 trading days, with mean reversion following the announcement.

Signal Generation Predicates

The Atilgan Criteria

The signal generation system implements the criteria from Atilgan et al. (2014), expressed as formal predicates.

Definition 0.7 (Trading Signal Predicates)**.** For a security with symbol ξ evaluated at time t with earnings date T_E, define:

IV/RV Ratio Criterion:

$$\mathcal{S}_1(\xi, t) \equiv \frac{\sigma_I^{30}(\xi, t)}{\sigma_R^{30}(\xi, t)} \geq 1.25 \tag{20}$$

Term Structure Criterion:

$$\mathcal{S}_2(\xi, t) \equiv \nabla_\tau \sigma_I(\xi, t) \leq -0.00406 \tag{21}$$

where $\nabla_\tau \sigma_I$ denotes the slope of the implied volatility term structure.

Liquidity Criterion:

$$\mathcal{S}_3(\xi, t) \equiv \bar{V}^{30}(\xi, t) \geq 1{,}500{,}000 \tag{22}$$

where $\bar{V}^{30}$ is the 30-day average daily trading volume.

Signal Strength Classification

Definition 0.8 (Signal Strength Function)**.** The signal strength function $\Sigma : \{\text{true}, \text{false}\}^3 \to \{\text{Recommended}, \text{Consider}, \text{Avoid}\}$ is:

$$\Sigma(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \begin{cases} \text{Recommended} & \text{if } \mathcal{S}_1 \wedge \mathcal{S}_2 \wedge \mathcal{S}_3 \\ \text{Consider} & \text{if exactly two of } \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\} \text{ hold} \\ \text{Avoid} & \text{otherwise} \end{cases} \tag{23}$$

Realised Volatility Estimation

The Yang-Zhang Estimator

The Alaris system employs the Yang-Zhang (2000) estimator for realised volatility, which is efficient for OHLC (open-high-low-close) data and robust to opening gaps.

Definition 0.9 (Yang-Zhang Estimator)**.** Given n trading days with price data (O_i, H_i, L_i, C_i), define:

Overnight Variance:

$$\sigma_o^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\ln\frac{O_i}{C_{i-1}} - \bar{o}\right)^2$$

Open-to-Close Variance:

$$\sigma_c^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\ln\frac{C_i}{O_i} - \bar{c}\right)^2$$

Rogers-Satchell Variance:

$$\sigma_{RS}^2 = \frac{1}{n}\sum_{i=1}^{n}\left[\ln\frac{H_i}{C_i}\cdot\ln\frac{H_i}{O_i} + \ln\frac{L_i}{C_i}\cdot\ln\frac{L_i}{O_i}\right]$$

Combined Estimator:

$$\sigma_{YZ}^2 = \sigma_o^2 + k\sigma_c^2 + (1-k)\sigma_{RS}^2 \tag{24}$$

where $k = 0.34/(1.34 + \frac{n+1}{n-1})$.

Theorem 0.4 (Yang-Zhang Efficiency)**.** *The Yang-Zhang estimator achieves efficiency factor $\eta \approx 8$ relative to close-to-close volatility estimation, meaning it requires approximately $1/8$ as many observations for equivalent precision.*

Term Structure Analysis

Linear Term Structure Model

The implied volatility term structure is modelled as:

$$\sigma_I(\tau) = \alpha + \beta\cdot\tau + \epsilon(\tau) \tag{25}$$

where τ is days to expiration.

The slope parameter β is estimated via ordinary least squares:

$$\hat{\beta} = \frac{\sum_{i=1}^{m}(\tau_i - \bar{\tau})(\sigma_i - \bar{\sigma})}{\sum_{i=1}^{m}(\tau_i - \bar{\tau})^2} \tag{26}$$

Definition 0.10 (Term Structure Trading Signal)**.** A negative slope $\hat{\beta} < \beta^* = -0.00406$ indicates elevated short-dated implied volatility relative to longer tenors, consistent with the pre-earnings IV premium.

Position Sizing: The Kelly Criterion

Optimal Growth Framework

Position sizing employs the Kelly criterion to maximise long-run geometric growth rate.

Theorem 0.5 (Kelly Criterion)**.** *For a bet with probability p of winning return b and probability $(1-p)$ of losing stake, the growth-optimal fraction is:*

$$f^* = \frac{p(b+1)-1}{b} = \frac{pb-(1-p)}{b} \tag{27}$$

For small expected returns, this simplifies to:

$$f^* \approx \frac{\mu}{\sigma^2}$$

where μ is expected return and σ^2 is variance.

Fractional Kelly Implementation

The Alaris system employs fractional Kelly sizing to reduce variance:

$$f_{\text{actual}} = \kappa \cdot f^* \quad \text{with } \kappa \in \{0.01, 0.02\} \tag{28}$$

The fraction κ depends on signal strength:

$$\kappa = \begin{cases} 0.02 & \text{if } \Sigma = \text{Recommended} \\ 0.01 & \text{if } \Sigma = \text{Consider} \\ 0 & \text{if } \Sigma = \text{Avoid} \end{cases}$$

Fault Detection and Monitoring System

Overview of the Fault Framework

The Alaris fault detection system is designed to identify conditions where model assumptions may be violated or execution may be compromised. Each fault condition is specified as a mathematically precise inequality or predicate.

Fault Classification

Faults are classified into four categories:

1. **Data Quality Faults**: Violations of input data validity assumptions
2. **Model Validity Faults**: Conditions where pricing models may be unreliable
3. **Execution Risk Faults**: Conditions threatening profitable execution
4. **Position Risk Faults**: Conditions requiring position adjustment or exit

Data Quality Validation

Price Data Validation

Definition 0.11 (Price Data Validity Predicates)**.** For price data point $P_i = (O_i, H_i, L_i, C_i, V_i)$:

$$\begin{aligned}
\mathcal{D}_1(P_i) &\equiv O_i > 0 \wedge H_i > 0 \wedge L_i > 0 \wedge C_i > 0 && \text{(Positivity)} \\
\mathcal{D}_2(P_i) &\equiv L_i \leq O_i \leq H_i \wedge L_i \leq C_i \leq H_i && \text{(OHLC Consistency)} \\
\mathcal{D}_3(P_i) &\equiv V_i \geq 0 && \text{(Volume Non-negativity)} \\
\mathcal{D}_4(P_i, P_{i-1}) &\equiv \left| \frac{O_i - C_{i-1}}{C_{i-1}} \right| < 0.50 && \text{(Gap Reasonableness)}
\end{aligned}$$

Aggregate price validity:

$$\mathcal{D}_{\text{price}}(P_i) \equiv \bigwedge_{j=1}^{4} \mathcal{D}_j$$

Implied Volatility Validation

Definition 0.12 (IV Data Validity Predicates)**.** For implied volatility observation σ_I:

$$\begin{aligned}
\mathcal{D}_5(\sigma_I) &\equiv 0.01 \leq \sigma_I \leq 5.00 && \text{(Range Constraint)} \\
\mathcal{D}_6(\sigma_I, \sigma_I') &\equiv |\sigma_I - \sigma_I'| < 0.50 && \text{(Temporal Continuity)}
\end{aligned}$$

Volume and Open Interest Validation

Definition 0.13 (Liquidity Data Validity)**.** For option contract with volume V and open interest OI:

$$\begin{aligned}
\mathcal{D}_7(V, OI) &\equiv V \geq 0 \wedge OI \geq 0 && \text{(Non-negativity)} \\
\mathcal{D}_8(V, OI) &\equiv V \leq 10 \cdot OI && \text{(Volume/OI Ratio)}
\end{aligned}$$

Execution Cost Validation

Cost Model Specification

The execution cost model computes total transaction cost as:

$$C_{\text{exec}}(n) = C_{\text{spread}}(n) + C_{\text{commission}}(n) + C_{\text{slippage}}(n) \tag{29}$$

where n is the number of contracts.

Spread Cost:

$$C_{\text{spread}}(n) = \frac{(A - B)}{2} \cdot n \cdot 100$$

where A is the ask price and B is the bid price.

Commission Cost:

$$C_{\text{commission}}(n) = (\phi_{\text{broker}} + \phi_{\text{exchange}} + \phi_{\text{regulatory}}) \cdot n$$

with $\phi_{\text{broker}} = 0.65$, $\phi_{\text{exchange}} = 0.30$, $\phi_{\text{regulatory}} = 0.02$.