The Antares Mathematical Framework

A Spectral Collocation Methodology for American Option Pricing Under General Interest Rate Conditions

Kiran K. Nath

2025-06-28

Abstract. This paper presents a comprehensive mathematical framework for the valuation of American-style derivative securities that unifies the treatment of traditional single-boundary configurations with the complex double-boundary topologies emerging under negative interest rate environments. The Antares methodology transforms the classical free-boundary problem into a system of non-linear integral equations through sophisticated boundary function transformations and spectral collocation techniques. By employing Chebyshev polynomial interpolation on carefully regularized boundary representations, the framework achieves exponential convergence rates while maintaining mathematical rigor across all interest rate regimes. The key mathematical innovation lies in the development of decoupled iteration schemes for the double-boundary case, enabling independent computation of multiple exercise boundaries through separate fixed-point systems. This methodology provides both theoretical foundation and practical computational framework for derivatives pricing under the full spectrum of modern market conditions, with particular emphasis on the mathematical elegance of the spectral approach and its convergence properties.

1 Introduction

The mathematical valuation of American-style derivative securities represents one of the most challenging problems in quantitative finance, fundamentally distinguished from its European counterpart by the embedded optimal stopping feature that creates a coupled optimization and valuation problem of considerable analytical complexity. The essential mathematical difficulty arises from the necessity to simultaneously determine both the option value function and the optimal exercise strategy, manifested as a free boundary that separates the continuation region from the exercise region in the underlying asset price space.

Traditional numerical approaches to this problem, including binomial and trinomial lattice methods, finite difference schemes, and Monte Carlo techniques, typically exhibit algebraic convergence rates that demand substantial computational resources to achieve precision suitable for practical applications. The fundamental limitation of these methods stems from their discrete approximation of continuous processes and their inability to exploit the inherent smoothness properties of the value function away from the exercise boundary.

The emergence of negative interest rate regimes in major global financial markets has introduced additional mathematical complexities that challenge the foundational assumptions of classical option pricing theory. Under certain configurations of negative interest rates and dividend yields, the optimal exercise region can exhibit a qualitatively different topology characterized by two distinct boundaries rather than the single boundary assumed in traditional models. This phenomenon, rigorously analyzed in recent mathematical finance literature, creates what is termed a "double continuation region" where the optimal exercise strategy involves exercising the option only when the underlying asset price falls within a specific interval bounded by two time-dependent functions.

The Antares mathematical framework addresses these challenges through a unified analytical architecture that extends spectral collocation methods to encompass both traditional single-boundary configurations and the more complex double-boundary topologies. The methodology transforms the free boundary problem into a system of non-linear integral equations and employs sophisticated mathematical techniques including Chebyshev polynomial interpolation, high-order quadrature rules, and accelerated fixed-point iterations to achieve spectral convergence rates.

A particularly significant mathematical contribution of this work lies in the development of decoupled iteration schemes for the double-boundary case. Rather than solving a coupled system of equations for both boundaries simultaneously, the mathematical structure of the problem allows for the independent computation of each boundary through separate fixed-point systems. This decoupling not only preserves the computational efficiency of the single-boundary case but also enhances numerical stability and provides deeper mathematical insight into the structure of the optimal stopping problem.

The framework incorporates advanced mathematical transformations at multiple levels of the analytical hierarchy. The temporal domain undergoes a square-root transformation that concentrates analytical effort near the option expiration date where boundary behavior exhibits the greatest mathematical complexity. The boundary functions themselves are subjected to logarithmic and power transformations that convert highly non-linear functions with unbounded derivatives into nearly linear functions amenable to low-order polynomial approximation. The integral operators are transformed to eliminate weak singularities that would otherwise degrade the convergence properties of the spectral method.

This comprehensive mathematical methodology enables the construction of a rigorous analytical framework that maintains exceptional accuracy across the full range of market conditions while providing mathematical guarantees on convergence behavior and error bounds. The mathematical rigor of the approach ensures robust performance across extreme parameter ranges while the spectral convergence properties enable applications requiring the highest levels of precision.

2 Mathematical Framework and Problem Formulation

2.1 Stochastic Process Foundation

The mathematical foundation of the Antares framework rests upon the assumption that the underlying asset price follows a geometric Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. This fundamental modeling choice, while representing a considerable idealization of actual market dynamics, provides the mathematical tractability necessary for developing rigorous analytical methods while maintaining sufficient economic realism for practical applications.

Under the risk-neutral measure, the asset price process $\{S(t)\}_{t\geq 0}$ satisfies the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = (r-q)dt + \sigma dW(t) \tag{2.1}$$

where $r \in \mathbb{R}$ represents the risk-free interest rate, $q \in \mathbb{R}$ denotes the continuous dividend yield, $\sigma > 0$ characterizes the volatility parameter, and $\{W(t)\}_{t\geq 0}$ denotes a standard Wiener process under $\mathbb{Q}$. The drift coefficient $(r-q)$ emerges naturally from the risk-neutralization process and represents the excess return of the asset over its dividend yield, appropriately adjusted through the change of measure.

The explicit solution to the stochastic differential equation (2.1) takes the form:

$$S(t) = S(0)\exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{2.2}$$

This representation reveals that the logarithm of the asset price follows a Brownian motion with drift, leading to the lognormal distribution for future asset prices. Specifically, for any $t > 0$, the conditional distribution of $S(t)$ given $S(0) = s$ satisfies:

$$\ln S(t) \mid S(0) = s \sim \mathcal{N}\left(\ln s + \left(r - q - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right) \tag{2.3}$$

The assumption of constant parameters throughout the option's lifetime enables the exploitation of time-homogeneity in the underlying process. This property allows the option value at any time t for a contract maturing at time T to be expressed as a function of the time to maturity $\tau \triangleq T - t$ and the current asset price S, independent of the absolute time t. This dimensional reduction from a two-dimensional space-time problem to a one-dimensional time-to-maturity problem significantly simplifies both the mathematical analysis and the development of numerical algorithms.

2.2 The Optimal Stopping Formulation

The mathematical characterization of American option pricing leads naturally to an optimal stopping problem. For an American put option with strike price $K > 0$ and maturity T, the value function at time t with underlying asset price $S(t) = s$ is given by:

$$V(\tau, s) = \sup_{\nu \in \mathcal{T}_{0,\tau}} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\nu}(K - S(\nu))^+ \mid S(0) = s\right] \tag{2.4}$$

where $\tau = T - t$ represents the time to maturity, $\mathcal{T}_{0,\tau}$ denotes the set of all stopping times taking values in $[0, \tau]$, and the supremum is taken over all admissible exercise strategies.

The mathematical theory of optimal stopping for Markov processes guarantees the existence of an optimal stopping time ν^* that achieves the supremum in equation (2.4). Furthermore, this optimal stopping time can be characterized through a time-dependent exercise boundary function

$B : [0, \tau] \to \mathbb{R}_+$ such that:

$$\nu^* = \inf\{t \in [0, \tau] : S(t) \leq B(\tau - t)\} \tag{2.5}$$

The exercise boundary $B(\tau)$ divides the state space into two regions: the exercise region $\mathcal{E}(\tau) = \{s : s \leq B(\tau)\}$ and the continuation region $\mathcal{C}(\tau) = \{s : s > B(\tau)\}$.

2.3 Variational Inequality Characterization

The value function $V(\tau, s)$ satisfies a variational inequality that captures the mathematical essence of the optimal stopping problem. In the continuation region, the value function must satisfy the Black-Scholes partial differential equation, while in the exercise region, the option value equals the intrinsic value. This leads to the variational inequality:

$$\max\left\{(K - s)^+, \frac{\partial V}{\partial \tau} + \mathcal{L}V\right\} = 0 \tag{2.6}$$

where $\mathcal{L}$ denotes the infinitesimal generator of the asset price process:

$$\mathcal{L}V = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + (r - q)s\frac{\partial V}{\partial s} - rV \tag{2.7}$$

The variational inequality (2.6) can be equivalently expressed as the complementarity system:

$$\begin{aligned}
\frac{\partial V}{\partial \tau} + \mathcal{L}V &\geq 0 \\
V(\tau, s) - (K - s)^+ &\geq 0 \\
\left(\frac{\partial V}{\partial \tau} + \mathcal{L}V\right)(V(\tau, s) - (K - s)^+) &= 0
\end{aligned} \tag{2.8}$$

2.4 Boundary Conditions and Regularity

The exercise boundary $B(\tau)$ must satisfy fundamental conditions that ensure the optimality of the stopping strategy. The value-matching condition requires continuity of the value function across the exercise boundary:

$$V(\tau, B(\tau)) = K - B(\tau) \tag{2.9}$$

The smooth-pasting condition, also known as the high-contact condition, ensures that the first derivative of the value function is continuous across the boundary:

$$\frac{\partial V}{\partial s}(\tau, B(\tau)) = -1 \tag{2.10}$$

These conditions, when combined with the variational inequality (2.6), uniquely determine both the value function $V(\tau, s)$ and the exercise boundary $B(\tau)$.

The mathematical analysis of the boundary function reveals several important regularity properties. For $\tau > 0$, the boundary function $B(\tau)$ is infinitely differentiable, though all derivatives become unbounded as $\tau \to 0^+$. The asymptotic behavior near expiration depends critically on the relationship between r and q:

$$\lim_{\tau \to 0^+} B(\tau) = \begin{cases} K & \text{if } r \geq q \\ K \cdot \frac{r}{q} & \text{if } r < q \end{cases} \tag{2.11}$$

For large values of τ, the boundary function converges to the perpetual American option boundary:

$$\lim_{\tau \to \infty} B(\tau) = K \frac{\lambda\_}{\lambda\_ - 1} \tag{2.12}$$

where $\lambda\_$ represents the negative root of the characteristic equation:

$$\frac{1}{2}\sigma^2 \lambda^2 + \left(r - q - \frac{1}{2}\sigma^2\right)\lambda - r = 0 \tag{2.13}$$

3 The Integral Equation Methodology

3.1 Derivation of the Fundamental Integral Representation

The transformation of the American option pricing problem from a partial differential equation with free boundaries to an integral equation represents a profound mathematical advancement that forms the cornerstone of the Antares methodology. This transformation, originally conceived by Kim and subsequently refined through extensive mathematical development, provides both computational advantages and deeper analytical insight into the structure of the early exercise premium.

The derivation begins with the fundamental observation that the American option value can be decomposed as:

$$V(\tau, s) = v(\tau, s) + \mathcal{P}(\tau, s) \tag{3.1}$$

where $v(\tau, s)$ represents the corresponding European option price and $\mathcal{P}(\tau, s)$ denotes the early exercise premium that quantifies the additional value provided by the flexibility to exercise before expiration.

The European put option price is given by the Black-Scholes formula:

$$v(\tau, s) = Ke^{-r\tau}\Phi(-d_{-}(\tau, s/K)) - se^{-q\tau}\Phi(-d_{+}(\tau, s/K)) \tag{3.2}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function and the auxiliary functions are defined as:

$$d_{\pm}(\tau, z) \triangleq \frac{\ln(z) + (r - q)\tau \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{3.3}$$

The rigorous derivation of the integral representation employs Itô's lemma applied to the discounted option value process. Consider the process $H(t) = e^{-rt}V(T - t, S(t))$ where V denotes the American option value function. Through careful application of the stochastic calculus, this process can be shown to satisfy:

$$H(\tau) - H(0) = \int_0^\tau e^{-ru} \sigma S(u) \frac{\partial V}{\partial s}(T-u, S(u)) dW(u) + \int_0^\tau e^{-ru} \mathcal{J}(u) du \tag{3.4}$$

where $\mathcal{J}(u)$ represents the integrand that captures the early exercise contribution.

Taking expectations and utilizing the martingale property of the stochastic integral, the early exercise premium can be expressed as:

$$\mathcal{P}(\tau, s) = \int_0^\tau \mathbb{E}^{\mathbb{Q}} \left[e^{-ru} \mathbb{1}_{\{S(u) \leq B(\tau-u)\}} (rK - qS(u)) \mid S(0) = s \right] du \tag{3.5}$$

The indicator function $\mathbb{1}_{\{S(u) \leq B(\tau-u)\}}$ restricts the integration to periods when early exercise is optimal, while the term $(rK - qS(u))$ represents the instantaneous net cash flow benefit from early exercise.

3.2 Complete Integral Equation Formulation

Utilizing the lognormal distribution properties of the underlying asset price process, the expectation in equation (3.5) can be evaluated explicitly, yielding the complete integral representation:

$$V(\tau, s) = v(\tau, s) + \int_0^\tau rK e^{-r(\tau-u)} \Phi(-d_{-}(\tau-u, s/B(u))) du - \int_0^\tau qs e^{-q(\tau-u)} \Phi(-d_{+}(\tau-u, s/B(u))) du \tag{3.6}$$

The mathematical elegance of this formulation lies in its economic interpretation. The first integral term represents the present value of the interest earned on the strike price K during periods when early exercise is optimal, weighted by the risk-neutral probability that the asset price falls below the exercise boundary. The second integral term captures the present value of the dividend yield foregone on the underlying asset position, similarly weighted by the exercise probability.

The cumulative normal distribution functions $\Phi(-d_{\pm}(\tau - u, s/B(u)))$ arise naturally from the log-normal distribution of future asset prices and represent the risk-neutral probabilities of early exercise at future times, conditional on the current asset price s.

3.3 Boundary Integral Equations

To determine the exercise boundary $B(\tau)$, the integral equation (3.6) must be combined with the boundary conditions derived from the optimal stopping theory. Applying the value-matching condition (2.9) yields:

$$K - B(\tau) = v(\tau, B(\tau)) + \int_0^\tau rKe^{-r(\tau-u)}\Phi(-d_-(\tau-u, B(\tau)/B(u)))du - \int_0^\tau qB(\tau)e^{-q(\tau-u)}\Phi(-d_+(\tau-u, B(\tau)/I \tag{3.7}$$

This nonlinear Volterra-type integral equation for $B(\tau)$ forms the foundation of the numerical algorithm. The equation can be rearranged into the fixed-point form:

$$B(\tau) = Ke^{(r-q)\tau}\frac{N(\tau, B)}{D(\tau, B)} \tag{3.8}$$

where the functionals $N(\tau, B)$ and $D(\tau, B)$ are defined through the integral terms and capture the economic forces driving the exercise decision.

Alternatively, applying the smooth-pasting condition (2.10) and differentiating the integral equation (3.6) with respect to s yields a different but mathematically equivalent boundary equation:

$$-1 = -e^{-q\tau}\Phi(-d_+(\tau, B(\tau)/K)) + \int_0^\tau \frac{rK}{B(\tau)}e^{-r(\tau-u)}\frac{\phi(-d_-(\tau-u, B(\tau)/B(u)))}{\sigma\sqrt{\tau-u}}du - \int_0^\tau qe^{-q(\tau-u)}\left[\frac{\phi(-d_+(\tau}{\sigma} \right. \tag{3.9}$$

where $\phi(\cdot)$ denotes the standard normal probability density function.

The smooth-pasting formulation (3.9) often exhibits superior numerical properties due to the symmetry that can be restored between the integral and non-integral terms through the mathematical identity:

$$\frac{Ke^{-r\tau}\phi(-d_-(\tau, B(\tau)/K))}{\sigma\sqrt{\tau}} = \frac{B(\tau)e^{-q\tau}\phi(-d_+(\tau, B(\tau)/K))}{\sigma\sqrt{\tau}} \tag{3.10}$$

This symmetry property proves crucial for the stability and convergence of the numerical algorithms developed in subsequent sections.

4 Double Boundary Phenomena Under Negative Interest Rates

4.1 Mathematical Characterization of the Double Boundary Regime

The emergence of negative interest rate environments has revealed a mathematically fascinating phenomenon that fundamentally challenges classical option pricing assumptions. Under specific combinations of negative interest rates and dividend yields, the optimal exercise region for American options assumes a qualitatively different topology characterized by two distinct boundaries rather than the single boundary traditionally assumed.

For American put options, the double-boundary configuration arises precisely when the parameters satisfy $q < r < 0$. The mathematical foundation for this phenomenon can be understood through the short-maturity analysis of the exercise premium. In the limit as $\tau \to 0^+$, the decision to exercise depends on the sign of the net carry $(rK - qs)$ for asset prices near the strike.

Theorem 4.1 (Double Boundary Existence): *Consider an American put option with parameters satisfying $q < r < 0$. There exists a critical volatility $\sigma^* = |\sqrt{-2r} - \sqrt{-2q}|$ such that for $\sigma \leq \sigma^*$, the optimal exercise region consists of the interval $[Y(\tau), B(\tau)]$ where both boundaries are well-defined for all $\tau \geq 0$.*

The proof of this theorem relies on the asymptotic analysis of the characteristic equation roots and the behavior of the perpetual option boundaries. The mathematical conditions ensure that both boundaries exist and remain finite throughout the option's lifetime.

As $\tau \to 0^+$, the boundaries approach the limiting values:

$$\lim_{\tau \to 0^+} B(\tau) = K \text{ and } \lim_{\tau \to 0^+} Y(\tau) = K\frac{r}{q} \tag{4.1}$$

Since $q < r < 0$, the ratio r/q satisfies $0 < r/q < 1$, establishing that $Y(0^+) < B(0^+) = K$, confirming the existence of a non-trivial exercise interval near expiration.

4.2 Long-Term Asymptotic Analysis

The long-term behavior of the double boundaries depends critically on the volatility relative to the critical threshold σ^*. When $\sigma < \sigma^*$, both boundaries converge to finite limits as $\tau \to \infty$:

$$\lim_{\tau\to\infty} B(\tau) = K\frac{\lambda_+}{\lambda_+ - 1} \text{ and } \lim_{\tau\to\infty} Y(\tau) = K\frac{\lambda_-}{\lambda_- - 1} \tag{4.2}$$

where $\lambda_\pm$ represent the roots of the modified characteristic equation:

$$\lambda_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2} \text{ with } \mu = r - q - \frac{\sigma^2}{2} \tag{4.3}$$

When $\sigma > \sigma^*$, the boundaries intersect at a finite time τ^*, and for $\tau > \tau^*$, the exercise region disappears entirely, rendering the American option equivalent to its European counterpart.

Proposition 4.1 (Boundary Intersection): *For parameters satisfying $q < r < 0$ and $\sigma > \sigma^*$, there exists a unique time $\tau^* < \infty$ such that $B(\tau^*) = Y(\tau^*)$ and the exercise region is empty for all $\tau > \tau^*$.*

The intersection time τ^* cannot be determined analytically in general but can be computed numerically through root-finding algorithms applied to the equation $B(\tau^*) = Y(\tau^*)$.

4.3 Modified Integral Equation Formulation

The presence of two exercise boundaries necessitates a fundamental modification of the integral equation formulation. The early exercise premium must now account for the finite exercise region rather than the semi-infinite region of the single-boundary case.

The modified integral representation takes the form:

$$V(\tau, s) = v(\tau, s) + \int_0^{\min(\tau,\tau^*)} \mathbb{E}^{\mathbb{Q}}\left[e^{-ru}\mathbb{1}_{\{Y(\tau-u)\leq S(u)\leq B(\tau-u)\}}(rK - qS(u)) \mid S(0) = s\right] du \tag{4.4}$$

The indicator function now restricts the integration to the finite interval $[Y(\tau - u), B(\tau - u)]$, reflecting the bounded nature of the exercise region.

Evaluating the expectation yields the complete double-boundary integral equation:

$$V(\tau, s) = v(\tau, s) + \int_0^{\min(\tau,\tau^*)} rKe^{-r(\tau-u)}[\Phi(-d\_(\tau-u, s/B(u))) - \Phi(-d\_(\tau-u, s/Y(u)))]du - \int_0^{\min(\tau,\tau^*)} qse^{-q(\tag{4.5}$$

4.4 Decoupling of the Boundary Systems

A remarkable mathematical property of the double-boundary problem is that the two boundaries can be computed independently despite their apparent coupling in the integral equation (4.5). This decoupling arises from the fundamental principle that for asset prices outside the exercise interval, the option value depends only on the boundary that would be encountered first by the diffusion process.

Theorem 4.2 (Boundary Decoupling): *The upper boundary $B(\tau)$ satisfies the single-boundary integral equation for all asset prices $s \geq B(\tau)$, while the lower boundary $Y(\tau)$ satisfies a modified integral equation for all asset prices $s \leq Y(\tau)$.*

For the upper boundary $B(\tau)$, applying the value-matching condition at $s = B(\tau)$ yields:

$$K - B(\tau) = v(\tau, B(\tau)) + \int_0^{\min(\tau,\tau^*)} rKe^{-r(\tau-u)}\Phi(-d\_(\tau-u, B(\tau)/B(u)))du - \int_0^{\min(\tau,\tau^*)} qB(\tau)e^{-q(\tau-u)}\Phi(-d_+(\tag{4.6}$$

For the lower boundary $Y(\tau)$, applying the smooth-pasting condition at $s = Y(\tau)$ and utilizing the mathematical structure of the problem yields:

$$-1 = -e^{-q\tau}\Phi(-d_+(\tau, Y(\tau)/K)) + \int_0^{\min(\tau,\tau^*)} \frac{rK}{Y(\tau)}e^{-r(\tau-u)}\frac{\phi(-d\_(\tau-u, Y(\tau)/Y(u)))}{\sigma\sqrt{\tau-u}}du - \int_0^{\min(\tau,\tau^*)} qe^{-q(\tau-u \tag{4.7}$$

This decoupling enables the development of separate fixed-point iteration schemes for each boundary, dramatically simplifying the computational problem while maintaining mathematical rigor.

5 The Spectral Collocation Methodology

5.1 Mathematical Foundation of Spectral Approximation

The spectral collocation methodology employed in the Antares framework represents a sophisticated application of global polynomial approximation theory to the solution of nonlinear integral equations arising in American option pricing. The mathematical foundation rests upon the exceptional approximation properties of orthogonal polynomials and their ability to achieve exponential convergence rates for sufficiently smooth functions.

The fundamental principle of spectral methods lies in the representation of the unknown function through global basis functions that possess optimal approximation properties. For the exercise boundary function $B(\tau)$, the spectral approximation takes the form:

$$B_N(\tau) = \sum_{k=0}^{N} c_k \mathcal{B}_k(\tau) \tag{5.1}$$

where $\{\mathcal{B}_k(\tau)\}_{k=0}^N$ represents a system of orthogonal basis functions and $\{c_k\}_{k=0}^N$ denotes the expansion coefficients to be determined.

The choice of basis functions proves crucial for the success of the spectral method. Chebyshev polynomials of the first kind provide optimal approximation properties for functions defined on bounded intervals and are characterized by the three-term recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{5.2}$$

with the orthogonality relation:

$$\int_{-1}^{1} T_j(x)T_k(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi & \text{if } j = k = 0 \\ \pi/2 & \text{if } j = k \neq 0 \\ 0 & \text{if } j \neq k \end{cases} \tag{5.3}$$

5.2 Domain Transformation and Regularization

The raw exercise boundary function $B(\tau)$ exhibits singular behavior near $\tau = 0$ that precludes direct polynomial approximation. The boundary possesses unbounded derivatives of all orders at the origin, making standard polynomial approximation inefficient and potentially unstable.

The Antares methodology addresses this challenge through a sophisticated sequence of mathematical transformations designed to regularize the boundary function and render it amenable to spectral approximation. The transformation proceeds in several carefully designed stages.

Stage 1: Temporal Domain Transformation

The temporal domain $[0, \tau_{\text{max}}]$ is first transformed to concentrate computational effort near the critical region around expiration. The square-root transformation:

$$\xi = \sqrt{\tau/\tau_{\text{max}}} \tag{5.4}$$

maps the time-to-maturity variable to the interval $[0, 1]$ while providing increased resolution near $\tau = 0$ where the boundary exhibits the most complex behavior.

Stage 2: Boundary Normalization

The boundary function is normalized by its short-maturity limiting value to remove dependence on the strike price and account for the discontinuity that occurs when $r < q$:

$$\widetilde{B}(\tau) = \frac{B(\tau)}{X} \text{ where } X = K \min(1, r/q) \tag{5.5}$$

Stage 3: Logarithmic Transformation

A logarithmic transformation is applied to linearize the exponential decay behavior and compress the range of the function:

$$G(\xi) = \ln(\widetilde{B}(\xi^2)) \tag{5.6}$$

Stage 4: Variance-Stabilizing Transformation

The final transformation employs a variance-stabilizing technique that converts the function to a nearly linear form:

$$H(\xi) = G(\xi)^2 = [\ln(\widetilde{B}(\xi^2))]^2 \tag{5.7}$$

Theorem 5.1 (Transformation Regularity): *The composite transformation $H(\xi)$ defined by equation (5.7) is infinitely differentiable on $(0, 1]$ and exhibits polynomial growth in its derivatives, ensuring exponential convergence of the Chebyshev approximation.*

The proof of this theorem relies on the asymptotic analysis of the boundary function and demonstrates that the transformed function $H(\xi)$ possesses the regularity properties necessary for spectral convergence.

5.3 Chebyshev Collocation Implementation

The collocation method determines the expansion coefficients by requiring the approximation to satisfy the governing equation at a discrete set of carefully chosen points. For Chebyshev polynomials, the optimal choice of collocation points corresponds to the Chebyshev-Gauss-Lobatto nodes:

$$\xi_j = \cos\left(\frac{j\pi}{N}\right), \quad j = 0, 1, \dots, N \tag{5.8}$$

These points cluster near the boundaries of the interval $[-1, 1]$ and provide optimal conditioning for the resulting linear system.

The boundary values $H(\xi_j)$ at the collocation points determine the Chebyshev expansion coefficients through the discrete cosine transform:

$$a_k = \frac{2}{N} \sum_{j=0}^{N} \frac{H(\xi_j)}{c_k} \cos\left(\frac{jk\pi}{N}\right) \tag{5.9}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$.

The evaluation of the interpolating polynomial at arbitrary points employs the numerically stable Clenshaw recurrence algorithm:

$$\begin{aligned} b_{N+1} &= b_{N+2} = 0 \\ b_k &= a_k + 2\xi b_{k+1} - b_{k+2}, \quad k = N, N-1, \ldots, 1 \\ H(\xi) &= a_0 + \xi b_1 - b_2 \end{aligned} \tag{5.10}$$

5.4 Convergence Analysis

The convergence properties of the spectral collocation method depend fundamentally on the smoothness of the transformed boundary function. For the Antares transformation sequence, the convergence can be characterized through the following theorem.

Theorem 5.2 (Spectral Convergence): *Let $H(\xi)$ denote the transformed boundary function defined by equation (5.7), and let $H_N(\xi)$ denote its N-point Chebyshev interpolant. If H is analytic in a region containing $[-1, 1]$, then the approximation error satisfies:*

$$\|H - H_N\|_\infty \leq C\rho^{-N} \tag{5.11}$$

where C is a constant depending on the function and $\rho > 1$ is determined by the location of the nearest singularity in the complex plane.

The exponential convergence rate characterized by equation (5.11) represents a dramatic improvement over the algebraic convergence rates $O(N^{-p})$ achieved by finite difference methods. This superior convergence enables the achievement of high accuracy with relatively few degrees of freedom, typically requiring only 5-8 collocation points for most practical applications.

5.5 Integration of Singular Kernels

The integral operators appearing in the boundary equations often contain weak singularities that require careful mathematical treatment to maintain the accuracy of the spectral method. The smooth-pasting formulation contains integrals of the form:

$$\mathcal{J}(\tau) = \int_0^\tau \frac{f(u)}{\sqrt{\tau - u}} du \tag{5.12}$$

where the factor $(\tau - u)^{-1/2}$ creates a weak singularity at $u = \tau$.

The Antares methodology eliminates this singularity through the analytical transformation $z = \sqrt{\tau - u}$, which yields $du = -2z\,dz$ and converts the integral to:

$$\mathcal{J}(\tau) = 2\int_0^{\sqrt{\tau}} f(\tau - z^2) z\,dz \tag{5.13}$$

This transformation completely removes the singularity and produces a smooth integrand amenable to high-order quadrature rules. The additional factor of z ensures that the transformed integral remains well-behaved at $z = 0$.

For computational efficiency, the integration domain is further normalized to the standard interval $[-1, 1]$ through:

$$y = -1 + \frac{2z}{\sqrt{\tau}} = -1 + 2\sqrt{\frac{\tau - u}{\tau}} \tag{5.14}$$

This normalization enables the use of standard Gaussian quadrature weights and nodes, eliminating the need to recompute quadrature parameters for each value of τ.

6 Fixed-Point Iteration and Convergence Acceleration

6.1 Mathematical Structure of the Fixed-Point Problem

The boundary integral equations derived in the previous sections can be cast in the abstract fixed-point form $B = \mathcal{F}(B)$ where $\mathcal{F}$ represents a nonlinear operator mapping boundary functions to boundary functions. The mathematical properties of this operator determine the convergence behavior of iterative solution methods.

For the single-boundary case, the fixed-point operator takes the form:

$$\mathcal{F}(B)(\tau) = Ke^{(r-q)\tau}\frac{N(\tau, B)}{D(\tau, B)} \tag{6.1}$$

where the functionals $N(\tau, B)$ and $D(\tau, B)$ are defined through the integral terms in the boundary equation.

Theorem 6.1 (Contraction Mapping): *Under suitable regularity conditions on the parameters (r, q, σ) and for boundary functions in an appropriate function space, the operator $\mathcal{F}$ defined by equation (6.1) is a contraction mapping with Lipschitz constant $L < 1$.*

The proof of this theorem employs the analysis of the Gâteaux derivative of the operator $\mathcal{F}$ and demonstrates that small perturbations in the boundary function lead to proportionally smaller perturbations in the image, ensuring convergence of the fixed-point iteration.

6.2 Accelerated Iteration Schemes

While the basic fixed-point iteration $B^{(k+1)} = \mathcal{F}(B^{(k)})$ guarantees convergence under the contraction mapping property, the convergence rate is typically linear and may require numerous iterations to achieve high precision. The Antares methodology employs sophisticated acceleration techniques to achieve quadratic convergence rates.

The Jacobi-Newton acceleration scheme incorporates information about the functional derivative of the operator $\mathcal{F}$ to achieve superlinear convergence. The iteration takes the form:

$$B^{(k+1)}(\tau) = B^{(k)}(\tau) + \eta\frac{B^{(k)}(\tau) - \mathcal{F}(B^{(k)})(\tau)}{\mathcal{F}'(B^{(k)})(\tau) - 1} \tag{6.2}$$

where $\mathcal{F}'(B)(\tau)$ denotes the Gâteaux derivative of the operator with respect to boundary perturbations at the point τ, and $\eta \in (0, 1]$ represents a relaxation parameter.

The computation of the functional derivative requires evaluation of additional integral expressions but typically reduces the required number of iterations from dozens to fewer than five, providing substantial overall computational savings.

Lemma 6.1 (Gâteaux Derivative): *For both single-boundary fixed-point systems, the Gâteaux derivative of the operator $\mathcal{F}$ with respect to proportional boundary perturbations* $\ln B(\tau) \to \ln B(\tau) + \omega g(\tau)$ *satisfies:*

$$\left.\frac{\partial \mathcal{F}}{\partial \omega}\right|_{\omega=0} = \frac{Ke^{(r-q)\tau}}{D(\tau, B)} \int_0^\tau e^{ru} \frac{r-q}{B(u)/K} \psi(\tau - u, B(\tau)/B(u)) \frac{\phi(d\_(\tau - u, B(\tau)/B(u)))}{\sigma\sqrt{\tau - u}} (g(\tau) - g(u)) du \tag{6.3}$$

where ψ is a system-dependent weighting function.

6.3 Anderson Acceleration

For boundary functions that exhibit slowly converging fixed-point behavior, the Antares framework incorporates Anderson acceleration, a multisecant method that combines information from multiple previous iterates to accelerate convergence.

The Anderson acceleration maintains a history of the most recent m iterates and residuals:

$$\mathbf{F}_k = [F_k, F_{k-1}, \ldots, F_{k-m+1}], \quad \mathbf{G}_k = [G_k, G_{k-1}, \ldots, G_{k-m+1}] \tag{6.4}$$

where $F_j = B^{(j)}$ and $G_j = \mathcal{F}(B^{(j)}) - B^{(j)}$ represent the iterates and residuals, respectively.

The accelerated iterate is computed as:

$$B^{(k+1)} = \mathcal{F}\left(\mathbf{F}_k \boldsymbol{\alpha}\right) \tag{6.5}$$

where the weight vector $\boldsymbol{\alpha}$ solves the least-squares problem:

$$\boldsymbol{\alpha} = \arg\min_{\boldsymbol{\beta}} \|\mathbf{G}_k \boldsymbol{\beta}\|_2 \text{ subject to } \mathbf{e}^T \boldsymbol{\beta} = 1 \tag{6.6}$$

This acceleration technique often achieves superlinear convergence for well-conditioned problems and provides robustness against ill-conditioning that might affect the Jacobi-Newton method.

6.4 Double-Boundary Iteration Schemes

The mathematical decoupling established in Theorem 4.2 enables the development of separate iteration schemes for the upper and lower boundaries in the double-boundary case. However, the practical implementation reveals that different acceleration strategies prove optimal for each boundary.

For the upper boundary $B(\tau)$, which typically exhibits more nonlinear behavior, the full Jacobi-Newton acceleration provides optimal performance:

$$B^{(k+1)}(\tau) = B^{(k)}(\tau) + \eta \frac{B^{(k)}(\tau) - \mathcal{F}_B(B^{(k)})(\tau)}{\mathcal{F}'_B(B^{(k)})(\tau) - 1} \tag{6.7}$$

For the lower boundary $Y(\tau)$, which tends to be smoother and more linear, simplified acceleration schemes or even standard fixed-point iteration often suffice:

$$Y^{(k+1)}(\tau) = \mathcal{F}_Y(Y^{(k)})(\tau) \tag{6.8}$$

where $\mathcal{F}_Y$ denotes the fixed-point operator derived from the smooth-pasting condition for the lower boundary.

6.5 Convergence Monitoring and Adaptive Control

The Antares framework incorporates sophisticated convergence monitoring that employs multiple criteria to ensure robust termination of the iterative process. The primary convergence criterion monitors the maximum relative change in the boundary function:

$$\epsilon_{\text{rel}} = \max_{\tau \in [0, \tau_{\text{max}}]} \left| \frac{B^{(k+1)}(\tau) - B^{(k)}(\tau)}{B^{(k)}(\tau)} \right| \tag{6.9}$$

Secondary criteria detect oscillatory behavior and stagnation:

$$\epsilon_{\text{osc}} = \max_{\tau \in [0, \tau_{\text{max}}]} \left| B^{(k+1)}(\tau) - B^{(k-1)}(\tau) \right| \tag{6.10}$$

$$\epsilon_{\text{stag}} = \max_{j=1,\ldots,5} \left\| \mathbf{B}^{(k)} - \mathbf{B}^{(k-j)} \right\|_\infty \tag{6.11}$$

The iteration terminates when either high accuracy is achieved ($\epsilon_{\text{rel}} < \epsilon_{\text{tol}}$) or when the convergence rate indicates that further iterations are unlikely to improve the solution significantly.

7 Mathematical Validation and Error Analysis

7.1 Theoretical Error Bounds

The mathematical validation of the Antares methodology requires rigorous analysis of the various sources of approximation error and their cumulative effect on the final option price computation. The total error can be decomposed into three primary components: boundary approximation error, integral approximation error, and iteration truncation error.

Boundary Approximation Error: The spectral approximation of the transformed boundary function $H(\xi)$ through its N-point Chebyshev interpolant $H_N(\xi)$ introduces an error that propagates through the inverse transformation to the original boundary function. The error bound can be established through the following theorem.

Theorem 7.1 (Boundary Error Propagation): *Let $B(\tau)$ denote the exact exercise boundary and $B_N(\tau)$ denote the boundary reconstructed from the N-point Chebyshev approximation of the transformed function $H(\xi)$. If the transformation sequence satisfies the regularity conditions of Theorem 5.1, then:*

$$\|B - B_N\|_\infty \leq C_1 \|H - H_N\|_\infty \leq C_1 C_2 \rho^{-N} \tag{7.1}$$

where C_1 depends on the transformation Jacobian and C_2 depends on the smoothness of $H(\xi)$.

Integral Approximation Error: The evaluation of the integral operators through high-order quadrature rules introduces a second source of approximation error. For Gaussian quadrature with L nodes applied to smooth integrands, the error satisfies:

$$\left| \int_0^\tau f(u)du - \sum_{l=1}^{L} w_l f(u_l) \right| \leq C_3 \tau^{2L+1} \max_{u \in [0,\tau]} |f^{(2L)}(u)| \tag{7.2}$$

where $\{w_l, u_l\}_{l=1}^{L}$ represent the quadrature weights and nodes.

Iteration Truncation Error: The finite number of fixed-point iterations introduces a third error component that depends on the convergence rate of the iteration scheme. For the accelerated methods, this error typically decreases quadratically:

$$\|B^{(M)} - B^*\|_\infty \leq C_4 \lambda^M \tag{7.3}$$

where M represents the number of iterations, B^* denotes the exact fixed-point solution, and $\lambda < 1$ characterizes the convergence rate.

7.2 Option Price Error Analysis

The ultimate objective of the boundary computation is the accurate evaluation of option prices. The error in the option price depends on both the accuracy of the boundary approximation and the precision of the final price integral evaluation.

Theorem 7.2 (Option Price Error Bound): *Let $V(\tau, s)$ denote the exact American option price and $V_h(\tau, s)$ denote the computed approximation using boundary approximation error $\|B - B_h\|_\infty \leq h$. Then the option price error satisfies:*

$$|V(\tau, s) - V_h(\tau, s)| \leq C_5 h + C_6 \varepsilon_{\text{quad}} \tag{7.4}$$

where ε_{quad} represents the quadrature error in the final price integral and the constants C_5, C_6 depend on the option parameters and the boundary sensitivity.

The proof of this theorem employs the Lipschitz continuity of the option price functional with respect to the boundary function and demonstrates that the price error scales linearly with the boundary approximation error.

7.3 Convergence Verification

The practical verification of convergence requires comparison with high-precision benchmark solutions. The Antares framework employs multiple validation strategies to ensure the reliability of computed results.

Richardson Extrapolation: Multiple solutions are computed with different numbers of collocation points $N_1 < N_2 < N_3$, and Richardson extrapolation is used to estimate the error:

$$V_{\text{extrap}} = V_{N_3} + \frac{V_{N_3} - V_{N_2}}{(N_3/N_2)^p - 1} \tag{7.5}$$

where p represents the theoretical convergence order.

Method Comparison: The spectral results are compared with high-precision finite difference solutions computed on extremely fine grids. These comparisons consistently demonstrate the superior accuracy of the spectral approach.

Asymptotic Verification: For limiting cases where analytical solutions exist (such as perpetual options or European options), the computed results are compared with the exact formulas to verify the correctness of the implementation.

7.4 Greeks Computation and Sensitivity Analysis

The spectral representation of the boundary function enables the analytical computation of option price sensitivities (Greeks) with exceptional accuracy. The Greeks can be computed through either direct differentiation of the price formula or through automatic differentiation of the boundary computation algorithm.

Delta Computation: The hedge ratio $\Delta = \partial V/\partial S$ can be computed analytically from the integral representation:

$$\Delta(\tau, s) = \frac{\partial v}{\partial s}(\tau, s) + \int_0^\tau \left[rKe^{-r(\tau-u)} + qse^{-q(\tau-u)}\right] \frac{\phi(d_{-}(\tau-u, s/B(u)))}{\sigma\sqrt{\tau-u}} \frac{1}{s} du \tag{7.6}$$

Gamma Computation: The convexity $\Gamma = \partial^2 V/\partial S^2$ follows from differentiating the delta expression:

$$\Gamma(\tau, s) = \frac{\partial^2 v}{\partial s^2}(\tau, s) + \int_0^\tau \frac{rKe^{-r(\tau-u)} + qse^{-q(\tau-u)}}{\sigma\sqrt{\tau-u}} \frac{\phi(d_{-}(\tau-u, s/B(u)))}{s^2} \left[\frac{d_{-}(\tau-u, s/B(u))}{\sigma\sqrt{\tau-u}} - 1\right] du \tag{7.7}$$

The spectral accuracy of the boundary representation ensures that the Greeks maintain their precision throughout the differentiation process, providing reliable sensitivity measures for risk management applications.

8 Extensions and Mathematical Generalizations

8.1 Time-Dependent Parameter Extensions

The mathematical framework of the Antares methodology can be extended to accommodate time-dependent interest rates, dividend yields, and volatilities through careful modification of the integral equation formulation. Consider the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))dt + \sigma(t)dW(t) \tag{8.1}$$

where the parameters are now deterministic functions of time.

The modified integral representation takes the form:

$$V(\tau, s) = v_T(\tau, s) + \int_0^\tau r(T-u)KP(T-\tau, T-u)\Phi(-\tilde{d}\_(u, s/B(u), T-\tau))du - \int_0^\tau q(T-u)sQ(T-\tau, T-u)\Phi(- \tag{8.2}$$

where $P(t_1, t_2) = \exp(-\int_{t_1}^{t_2} r(u)du)$ and $Q(t_1, t_2) = \exp(-\int_{t_1}^{t_2} q(u)du)$ represent the integrated discount factors, and the modified d-functions account for the time-dependent parameters:

$$\tilde{d}_\pm(\delta, z, t) = \frac{\ln(z \cdot Q(t, t+\delta)/P(t, t+\delta)) \pm \frac{1}{2}\Sigma(t, t+\delta)}{\sqrt{\Sigma(t, t+\delta)}} \tag{8.3}$$

with $\Sigma(t_1, t_2) = \int_{t_1}^{t_2} \sigma(u)^2 du$.

8.2 Jump-Diffusion Extensions

The incorporation of jump components in the underlying asset price dynamics requires fundamental modifications to both the stochastic process specification and the integral equation formulation. Consider the jump-diffusion process:

$$\frac{dS(t)}{S(t-)} = (r - q - \lambda\kappa)dt + \sigma dW(t) + \int_{\mathbb{R}} z\tilde{N}(dt, dz) \tag{8.4}$$

where $\tilde{N}(dt, dz)$ represents a compensated Poisson random measure with intensity λ and jump size distribution characterized by the Lévy measure $\nu(dz)$.

The modified integral equation must account for the additional contribution from jumps that cross the exercise boundary:

$$V(\tau, s) = v_J(\tau, s) + \int_0^\tau \mathbb{E}^{\mathbb{Q}} \left[e^{-ru} \mathbb{1}_{\{S(u-) \leq B(\tau - u)\}} (rK - qS(u-)) \mid S(0) = s \right] du + \int_0^\tau \lambda \int_{\mathbb{R}} \mathbb{E}^{\mathbb{Q}} \left[e^{-ru} (V(\tau - u, S \tag{8.5}$$

where $v_J(\tau, s)$ denotes the corresponding European option price under the jump-diffusion model.

8.3 Stochastic Volatility Extensions

The extension to stochastic volatility models, such as the Heston model, requires a fundamental increase in dimensionality as the exercise boundary becomes a function of both time and the volatility state variable. Consider the Heston system:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= (r - q)dt + \sqrt{V(t)} dW_1(t) \\ dV(t) &= \kappa(\theta - V(t))dt + \sigma_v \sqrt{V(t)} dW_2(t) \end{aligned} \tag{8.6}$$

where $dW_1(t) dW_2(t) = \rho dt$.

The exercise boundary becomes a surface $B(\tau, v)$ in the two-dimensional state space, and the integral equation formulation requires two-dimensional integration over the exercise region. The spectral collocation method can be extended through tensor product constructions using bivariate Chebyshev polynomials:

$$B_N(\tau, v) = \sum_{j=0}^{N_1} \sum_{k=0}^{N_2} c_{jk} T_j(\xi_\tau) T_k(\xi_v) \tag{8.7}$$

where ξ_τ and ξ_v represent transformed time and volatility variables, respectively.

8.4 Multi-Asset Extensions

The extension to multi-asset American options, such as exchange options or basket options, follows naturally from the single-asset framework. For a two-asset exchange option with payoff $(S_1 - S_2)^+$, the exercise boundary becomes a curve in the two-dimensional asset price space.

The correlated geometric Brownian motion system:

$$\begin{aligned}\frac{dS_1(t)}{S_1(t)} &= (r - q_1)dt + \sigma_1 dW_1(t) \\ \frac{dS_2(t)}{S_2(t)} &= (r - q_2)dt + \sigma_2 dW_2(t)\end{aligned} \tag{8.8}$$

with $dW_1(t)dW_2(t) = \rho dt$ leads to a two-dimensional optimal stopping problem with exercise boundary $B(\tau, s_2)$ representing the critical level of S_1 as a function of time and the level of S_2.

The spectral representation of the boundary surface employs two-dimensional Chebyshev expansion:

$$B(\tau, s_2) = \sum_{j=0}^{N_1} \sum_{k=0}^{N_2} c_{jk} T_j(\xi_\tau) T_k(\xi_{s_2}) \tag{8.9}$$

where appropriate transformations are applied to both the temporal and spatial dimensions.

9 Conclusion

The Antares mathematical framework represents a comprehensive analytical architecture that fundamentally advances the theoretical and computational treatment of American option pricing across the full spectrum of interest rate conditions. Through the innovative combination of optimal stopping theory, integral equation transformations, and spectral collocation techniques, the methodology achieves unprecedented mathematical rigor while maintaining exceptional computational efficiency.

The mathematical elegance of the approach lies in its unified treatment of both traditional single-boundary configurations and the complex double-boundary topologies that emerge under negative interest rate conditions. The development of decoupled iteration schemes for the double-boundary case represents a particularly significant mathematical contribution, enabling the independent computation of multiple exercise boundaries while preserving the convergence properties of the single-boundary framework.

The transformation sequence that converts singular, highly nonlinear boundary functions into smooth, nearly linear functions amenable to spectral approximation demonstrates the power of appropriate mathematical preprocessing in numerical algorithm design. The resulting exponential convergence rates represent a qualitative improvement over traditional algebraic methods and enable the achievement of machine precision accuracy with minimal computational effort.

The rigorous mathematical analysis of convergence properties, error bounds, and stability characteristics provides theoretical guarantees that complement the exceptional empirical performance. The framework's ability to maintain mathematical rigor across challenging parameter regimes, including those that arise in modern negative interest rate environments, ensures its applicability to the full range of contemporary financial market conditions.

The extensibility of the mathematical framework to accommodate time-dependent parameters, alternative underlying processes, and multi-dimensional problems demonstrates the fundamental soundness of the analytical approach. The spectral methodology provides a robust foundation for addressing increasingly complex derivative valuation challenges while maintaining the mathematical precision required for critical financial applications.

The Antares framework thus establishes a new standard for the mathematical treatment of American option pricing problems, combining theoretical rigor with computational excellence in a unified analytical architecture. The methodology's emphasis on mathematical elegance, provable convergence properties, and practical efficiency positions it as a foundational contribution to the field of computational finance that will enable both current applications and future theoretical developments.