The Integral of $1/(x^n + 1)$

A Complete Treatment

December 2025

1 Preliminaries

Let $n \geq 2$ be a positive integer. The polynomial $P(x) = x^n + 1$ has n distinct roots in $\mathbb{C}$, obtained by solving $x^n = -1 = e^{i\pi(2m+1)}$ for any integer m:

$$\omega_j := e^{i\pi(2j+1)/n}, \qquad j = 0, 1, \ldots, n-1$$

These roots lie on the unit circle, equally spaced at angular intervals of $2\pi/n$, with the first root at angle π/n from the positive real axis.

Conjugate structure. The roots satisfy $\overline{\omega_j} = \omega_{n-1-j}$. When n is odd, exactly one root is real: $\omega_{(n-1)/2} = e^{i\pi} = -1$. When n is even, all roots are non-real.

Index convention. For each $k \in \{0, 1, \ldots, \lfloor (n-2)/2 \rfloor\}$, define:

$$\theta_k := \frac{\pi(2k+1)}{n} \in (0, \pi)$$

$$c_k := \cos\theta_k, \qquad s_k := \sin\theta_k > 0$$

$$Q_k(x) := x^2 - 2c_k x + 1 = (x - c_k)^2 + s_k^2$$

The restriction $\theta_k \in (0, \pi)$ ensures $s_k > 0$ and selects exactly one representative from each conjugate pair of non-real roots.

Lemma (Real Factorisation). *The polynomial $x^n + 1$ factors over $\mathbb{R}$ as follows:*

Case I (n odd):

$$x^n + 1 = (x+1) \prod_{k=0}^{(n-3)/2} Q_k(x)$$

Case II (n even):

$$x^n + 1 = \prod_{k=0}^{n/2-1} Q_k(x)$$

Proof. Each conjugate pair $\omega_j, \overline{\omega_j}$ with argument $\theta \in (0, \pi)$ (i.e., $\omega_j = e^{i\theta}$, $\overline{\omega_j} = e^{-i\theta}$ for a unique $\theta = \theta_k$) contributes the real quadratic $(x - \omega_j)(x - \overline{\omega_j}) = x^2 - 2\,\mathrm{Re}(\omega_j)x + |\omega_j|^2 = Q_k(x)$. The quadratics are irreducible over $\mathbb{R}$ since their discriminant $4c_k^2 - 4 = -4s_k^2 < 0$. For odd n, the real root $\omega_{(n-1)/2} = -1$ contributes the linear factor $(x+1)$. Counting degrees: for odd n, $(n-1)/2$ quadratics contribute degree $n-1$, plus one linear factor, totalling n; for even n, $n/2$ quadratics contribute degree n. ■

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2 Main Theorem

Theorem. *An antiderivative of* $\dfrac{1}{x^n+1}$ *is given by:*

Case I (n odd):

$$\int \frac{dx}{x^n+1} = \frac{1}{n}\ln|x+1| + \sum_{k=0}^{(n-3)/2}\left[-\frac{c_k}{n}\ln Q_k(x) + \frac{2s_k}{n}\arctan\frac{x-c_k}{s_k}\right] + C$$

Case II (n even):

$$\int \frac{dx}{x^n+1} = \sum_{k=0}^{n/2-1}\left[-\frac{c_k}{n}\ln Q_k(x) + \frac{2s_k}{n}\arctan\frac{x-c_k}{s_k}\right] + C$$

where $c_k = \cos\theta_k$, $s_k = \sin\theta_k$, $\theta_k = \pi(2k+1)/n$, *and* $Q_k(x) = x^2 - 2c_kx + 1$.

Domain of validity. For odd n, the formula holds on $(-\infty,-1)$ and $(-1,\infty)$ separately, with potentially different constants of integration on each interval. For even n, the formula holds on all of $\mathbb{R}$.

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3 Proof

The argument proceeds in three steps: partial fraction decomposition via residues, reduction to real form by pairing conjugate roots, and integration of each component term.

3.1 Step 1. Partial Fraction Decomposition

Since $P(x) = x^n + 1$ has derivative $P'(x) = nx^{n-1}$, and $P'(\omega_j) = n\omega_j^{n-1} \neq 0$ for all j, the roots are simple. For a polynomial with simple roots $\{\alpha_j\}$, the partial fraction decomposition of $1/P(x)$ takes the form:

$$\frac{1}{P(x)} = \sum_j \frac{r_j}{x-\alpha_j}, \qquad r_j = \operatorname{Res}_{x=\alpha_j}\frac{1}{P(x)} = \frac{1}{P'(\alpha_j)}$$

To verify this identity, observe that both sides are rational functions with the same simple poles and the same principal part at each pole. Their difference is therefore a rational function with no poles, hence a polynomial. Since this difference tends to 0 as $x \to \infty$, it must be identically 0.

Applying this to $P(x) = x^n + 1$ with roots ω_j:

$$r_j = \frac{1}{P'(\omega_j)} = \frac{1}{n\omega_j^{n-1}}$$

To simplify, observe that $\omega_j^n = -1$ implies $\omega_j^{n-1} = \omega_j^n/\omega_j = -\omega_j^{-1}$. Therefore:

$$r_j = \frac{1}{n(-\omega_j^{-1})} = -\frac{\omega_j}{n}$$

This yields the complex partial fraction decomposition:

$$\frac{1}{x^n + 1} = -\frac{1}{n}\sum_{j=0}^{n-1} \frac{\omega_j}{x - \omega_j}$$

When n is odd, the index $j = (n-1)/2$ corresponds to $\omega_j = e^{i\pi} = -1$, and the associated term is:

$$-\frac{1}{n} \cdot \frac{-1}{x - (-1)} = \frac{1}{n} \cdot \frac{1}{x + 1}$$

3.2 Step 2. Pairing Conjugate Roots

For each index k in the range $\{0, 1, \ldots, \lfloor (n-2)/2 \rfloor\}$, let $\omega = \omega_k = e^{i\theta_k}$ and $\bar{\omega} = e^{-i\theta_k}$ be the corresponding conjugate pair (indeed $\bar{\omega} = \omega_{n-1-k}$). Their combined contribution to the partial fraction decomposition is:

$$-\frac{1}{n}\left(\frac{\omega}{x - \omega} + \frac{\bar{\omega}}{x - \bar{\omega}}\right) = -\frac{1}{n} \cdot \frac{\omega(x - \bar{\omega}) + \bar{\omega}(x - \omega)}{(x - \omega)(x - \bar{\omega})}$$

Denominator. Using $\omega + \bar{\omega} = 2\cos\theta_k = 2c_k$ and $\omega\bar{\omega} = |\omega|^2 = 1$:

$$(x - \omega)(x - \bar{\omega}) = x^2 - (\omega + \bar{\omega})x + \omega\bar{\omega} = x^2 - 2c_k x + 1 = Q_k(x)$$

Numerator. Expanding:

$$\omega(x - \bar{\omega}) + \bar{\omega}(x - \omega) = (\omega + \bar{\omega})x - \omega\bar{\omega} - \bar{\omega}\omega = 2c_k x - 2$$

Therefore the paired fraction becomes:

$$-\frac{1}{n} \cdot \frac{2c_k x - 2}{Q_k(x)} = \frac{A_k x + B_k}{Q_k(x)}$$

with coefficients:

$$A_k = -\frac{2c_k}{n}, \qquad B_k = \frac{2}{n}$$

3.3 Step 3. Integration of Quadratic Terms

Fix an index k, and write $c = c_k$, $s = s_k$, $Q(x) = Q_k(x) = (x-c)^2 + s^2$, $A = A_k$, $B = B_k$. Note that $Q'(x) = 2(x-c)$.

Numerator decomposition. We seek constants α and β such that:

$$Ax + B = \alpha \cdot Q'(x) + \beta = 2\alpha(x-c) + \beta$$

Matching coefficients: $2\alpha = A$ and $-2\alpha c + \beta = B$. Hence $\alpha = A/2$ and $\beta = B + Ac$.

Evaluation of β:

$$\beta = B + Ac = \frac{2}{n} + \left(-\frac{2c}{n}\right)c = \frac{2}{n}(1-c^2) = \frac{2s^2}{n}$$

Integration. The integral separates into two standard forms:

$$\int \frac{Ax+B}{Q(x)}\,dx = \frac{A}{2}\int \frac{Q'(x)}{Q(x)}\,dx + \beta \int \frac{dx}{(x-c)^2 + s^2}$$

The first integral is $\ln Q(x)$. For the second, the substitution $u = (x-c)/s$ gives $du = dx/s$, whence:

$$\int \frac{dx}{(x-c)^2+s^2} = \int \frac{s\,du}{s^2(u^2+1)} = \frac{1}{s}\arctan u = \frac{1}{s}\arctan\frac{x-c}{s}$$

Substituting $A/2 = -c/n$ and $\beta/s = 2s^2/(ns) = 2s/n$:

$$\int \frac{Ax+B}{Q(x)}\,dx = -\frac{c}{n}\ln Q(x) + \frac{2s}{n}\arctan\frac{x-c}{s} + C$$

Assembly. Summing over all conjugate pairs (indexed by k) and including the linear term $(1/n)\ln|x+1|$ when n is odd yields the stated formula. ∎

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4 Complex Logarithmic Form

Corollary. *On any simply connected domain $D \subset \mathbb{C}$ avoiding the roots $\{\omega_j\}$, with a fixed branch of* $\log$ *on D:*

$$\int \frac{dx}{x^n+1} = -\frac{1}{n}\sum_{j=0}^{n-1}\omega_j \log(x-\omega_j) + C$$

Proof. Integrate the complex partial fraction decomposition term by term. ∎

Remark on branches. On a simply connected domain avoiding the poles, fixing a branch of log renders the expression single-valued. Changing branches shifts the result by a constant, since the derivative is unchanged. Choose D to be conjugation-invariant (i.e., $z \in D \Rightarrow \bar{z} \in D$) and choose a branch of log on D satisfying $\log(\bar{z}) = \overline{\log z}$. Then for real $x \in D$, we have $\overline{x - \omega} = x - \bar{\omega}$, so $\log(x - \bar{\omega}) = \overline{\log(x - \omega)}$. Hence $\omega \log(x - \omega) + \bar{\omega} \log(x - \bar{\omega})$ is real, and pairing conjugates yields a real-valued antiderivative; branch changes affect only the additive constant on each component.

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5 Application: The Case $n = 5$

Setting $n = 5$, the index set is $k \in \{0, 1\}$, corresponding to angles $\theta_0 = \pi/5$ and $\theta_1 = 3\pi/5$.

Trigonometric values. The cosine values follow from the identity $\cos(\pi/5) = (1 + \sqrt{5})/4$ and $\cos(3\pi/5) = -\cos(2\pi/5) = (1 - \sqrt{5})/4$. The sine values follow from $\sin^2\theta + \cos^2\theta = 1$:

	$k = 0$	$k = 1$
θ_k	$\pi/5$	$3\pi/5$
c_k	$(1+\sqrt{5})/4$	$(1-\sqrt{5})/4$
s_k	$\sqrt{10-2\sqrt{5}}/4$	$\sqrt{10+2\sqrt{5}}/4$
$-c_k/5$	$-(1+\sqrt{5})/20$	$(\sqrt{5}-1)/20$
$2s_k/5$	$\sqrt{10-2\sqrt{5}}/10$	$\sqrt{10+2\sqrt{5}}/10$

Arctangent arguments. For $k = 0$:

$$\frac{x - c_0}{s_0} = \frac{x - \frac{1+\sqrt{5}}{4}}{\frac{\sqrt{10-2\sqrt{5}}}{4}} = \frac{4x - 1 - \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}$$

For $k = 1$:

$$\frac{x - c_1}{s_1} = \frac{x - \frac{1-\sqrt{5}}{4}}{\frac{\sqrt{10+2\sqrt{5}}}{4}} = \frac{4x - 1 + \sqrt{5}}{\sqrt{10 + 2\sqrt{5}}}$$

Final formula:

$$\int \frac{dx}{x^5 + 1} = \frac{1}{5}\ln|x + 1| - \frac{1+\sqrt{5}}{20}\ln\left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right) + \frac{\sqrt{5}-1}{20}\ln\left(x^2 + \frac{\sqrt{5}-1}{2}x + 1\right)$$

$$+\frac{\sqrt{10-2\sqrt{5}}}{10}\arctan\frac{4x - 1 - \sqrt{5}}{\sqrt{10-2\sqrt{5}}} + \frac{\sqrt{10+2\sqrt{5}}}{10}\arctan\frac{4x - 1 + \sqrt{5}}{\sqrt{10+2\sqrt{5}}} + C$$

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6 Generalisations

6.1 6.1. The family $x^n + a^n$

For real $a \neq 0$, the substitution $x = au$ yields:

$$\int \frac{dx}{x^n + a^n} = \frac{1}{a^{n-1}} \int \frac{du}{u^n + 1}$$

The main theorem then applies directly to the transformed integral.

6.2 6.2. The family $x^n - 1$

The roots of $x^n - 1 = 0$ are $\zeta_j = e^{2\pi i j/n}$ for $j = 0, 1, \dots, n-1$. The same method yields residues:

$$r_j = \frac{1}{n\zeta_j^{n-1}} = \frac{\zeta_j}{n}$$

since $\zeta_j^n = 1$ implies $\zeta_j^{n-1} = \zeta_j^{-1}$. The decomposition becomes:

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{\zeta_j}{x - \zeta_j}$$

Real roots. The polynomial $x^n - 1$ always has the real root $\zeta_0 = 1$, contributing a term:

$$\frac{1}{n} \cdot \frac{1}{x - 1}$$

When n is even, $\zeta_{n/2} = -1$ is also a root, contributing:

$$\frac{1}{n} \cdot \frac{-1}{x - (-1)} = -\frac{1}{n(x + 1)}$$

Quadratic factors. The non-real roots pair at angles $\phi_k = 2\pi k/n$ for appropriate k, yielding quadratics $x^2 - 2\cos\phi_k \cdot x + 1$. The integration proceeds analogously, with coefficients determined by the same residue-pairing method.

6.3 6.3. General rational integrands

The methodology extends to any integral $\int \frac{R(x)}{P(x)}\,dx$ where P has simple roots $\{\alpha_j\}$. After polynomial long division if $\deg R \geq \deg P$, the proper rational part decomposes as:

$$\frac{R(x)}{P(x)} = \sum_j \frac{R(\alpha_j)}{P'(\alpha_j)} \cdot \frac{1}{x - \alpha_j} + \text{(polynomial terms)}$$

The procedure then follows three steps:

1. **Decompose** using the residue formula above.
2. **Pair conjugates** to obtain real quadratic denominators.
3. **Integrate** each quadratic term via the identity:

$$\int \frac{Ax+B}{(x-c)^2+s^2}\,dx = \frac{A}{2}\ln\big((x-c)^2+s^2\big) + \frac{B+Ac}{s}\arctan\frac{x-c}{s} + C$$

When P has repeated roots, the partial fraction decomposition includes higher-order terms $(x-\alpha)^{-m}$, which integrate to rational functions (for $m \geq 2$) plus logarithms (for $m = 1$).

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7 Concluding Remarks

The integral $\displaystyle\int \frac{dx}{x^n+1}$ admits a closed form expressible as a finite sum of logarithmic and arctangent terms. The derivation rests on three principles:

1. **Residue-based partial fractions.** For simple roots, $r_j = 1/P'(\alpha_j)$.
2. **Conjugate pairing.** Non-real roots $\omega, \bar{\omega}$ contribute the real quadratic $(x-\omega)(x-\bar{\omega}) = x^2 - 2\,\mathrm{Re}(\omega)x + 1$.
3. **Quadratic integration.** The numerator decomposes into a multiple of the derivative (yielding ln) plus a constant (yielding arctan).

The particularly elegant coefficient formulae $A_k = -2c_k/n$ and $B_k = 2/n$ arise from the algebraic relation $\omega_j^n = -1$, which forces $r_j = -\omega_j/n$. For the family $x^n - 1$, the relation $\zeta_j^n = 1$ yields $r_j = \zeta_j/n$ instead. For general polynomials, the residues $r_j = R(\alpha_j)/P'(\alpha_j)$ do not simplify to such universal expressions, though the decomposition-and-pairing method remains fully applicable.

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8 Appendix: Algorithmic Summary

To evaluate $\int \frac{dx}{x^n + 1}$:

1. **Compute** $\theta_k = \pi(2k+1)/n$ for $k = 0, 1, \ldots, \lfloor (n-2)/2 \rfloor$.
2. **Evaluate** $c_k = \cos\theta_k$ and $s_k = \sin\theta_k$.
3. **If** n **is odd**, include the term $\frac{1}{n}\ln|x+1|$.
4. **For each** k, add:
   - $-\frac{c_k}{n}\ln(x^2 - 2c_k x + 1)$
   - $\frac{2s_k}{n}\arctan\frac{x - c_k}{s_k}$
5. **Add** the constant of integration C.