

# Rational Functions

*A Complete Treatment*

December 2025

## 1 Preliminaries

Let  $n \geq 2$  be a positive integer. The polynomial  $P(x) = x^n + 1$  has  $n$  distinct roots in  $\mathbb{C}$ , obtained by solving  $x^n = -1 = e^{i\pi(2m+1)}$  for any integer  $m$ :

$$\omega_j := e^{i\pi(2j+1)/n}, \quad j = 0, 1, \dots, n-1$$

These roots lie on the unit circle, equally spaced at angular intervals of  $2\pi/n$ , with the first root at angle  $\pi/n$  from the positive real axis.

**Conjugate structure.** The roots satisfy  $\overline{\omega_j} = \omega_{n-1-j}$ . When  $n$  is odd, exactly one root is real:  $\omega_{(n-1)/2} = e^{i\pi} = -1$ . When  $n$  is even, all roots are non-real.

**Index convention.** For each  $k \in \{0, 1, \dots, \lfloor (n-2)/2 \rfloor\}$ , define:

$$\theta_k := \frac{\pi(2k+1)}{n} \in (0, \pi)$$

$$c_k := \cos \theta_k, \quad s_k := \sin \theta_k > 0$$

$$Q_k(x) := x^2 - 2c_kx + 1 = (x - c_k)^2 + s_k^2$$

The restriction  $\theta_k \in (0, \pi)$  ensures  $s_k > 0$  and selects exactly one representative from each conjugate pair of non-real roots.

**Lemma (Real Factorisation).** *The polynomial  $x^n + 1$  factors over  $\mathbb{R}$  as follows:*

*Case I ( $n$  odd):*

$$x^n + 1 = (x + 1) \prod_{k=0}^{(n-3)/2} Q_k(x)$$

*Case II ( $n$  even):*

$$x^n + 1 = \prod_{k=0}^{n/2-1} Q_k(x)$$

*Proof.* Each conjugate pair  $\omega_j, \overline{\omega_j}$  with argument  $\theta \in (0, \pi)$  (i.e.,  $\omega_j = e^{i\theta}, \overline{\omega_j} = e^{-i\theta}$  for a unique  $\theta = \theta_k$ ) contributes the real quadratic  $(x - \omega_j)(x - \overline{\omega_j}) = x^2 - 2 \operatorname{Re}(\omega_j)x + |\omega_j|^2 = Q_k(x)$ . The quadratics are irreducible over  $\mathbb{R}$  since their discriminant  $4c_k^2 - 4 = -4s_k^2 < 0$ . For odd  $n$ , the real root  $\omega_{(n-1)/2} = -1$  contributes the linear factor  $(x + 1)$ . Counting degrees: for odd  $n$ ,  $(n-1)/2$  quadratics contribute degree  $n-1$ , plus one linear factor, totalling  $n$ ; for even  $n$ ,  $n/2$  quadratics contribute degree  $n$ . ■

## 2 Main Theorem

**Theorem.** An antiderivative of  $\frac{1}{x^n + 1}$  is given by:

Case I ( $n$  odd):

$$\int \frac{dx}{x^n + 1} = \frac{1}{n} \ln|x + 1| + \sum_{k=0}^{(n-3)/2} \left[ -\frac{c_k}{n} \ln Q_k(x) + \frac{2s_k}{n} \arctan \frac{x - c_k}{s_k} \right] + C$$

Case II ( $n$  even):

$$\int \frac{dx}{x^n + 1} = \sum_{k=0}^{n/2-1} \left[ -\frac{c_k}{n} \ln Q_k(x) + \frac{2s_k}{n} \arctan \frac{x - c_k}{s_k} \right] + C$$

where  $c_k = \cos \theta_k$ ,  $s_k = \sin \theta_k$ ,  $\theta_k = \pi(2k+1)/n$ , and  $Q_k(x) = x^2 - 2c_k x + 1$ .

**Domain of validity.** For odd  $n$ , the formula holds on  $(-\infty, -1)$  and  $(-1, \infty)$  separately, with potentially different constants of integration on each interval. For even  $n$ , the formula holds on all of  $\mathbb{R}$ .

---

## 3 Proof

The argument proceeds in three steps: partial fraction decomposition via residues, reduction to real form by pairing conjugate roots, and integration of each component term.

### 3.1 Step 1. Partial Fraction Decomposition

Since  $P(x) = x^n + 1$  has derivative  $P'(x) = nx^{n-1}$ , and  $P'(\omega_j) = n\omega_j^{n-1} \neq 0$  for all  $j$ , the roots are simple. For a polynomial with simple roots  $\{\alpha_j\}$ , the partial fraction decomposition of  $1/P(x)$  takes the form:

$$\frac{1}{P(x)} = \sum_j \frac{r_j}{x - \alpha_j}, \quad r_j = \text{Res}_{x=\alpha_j} \frac{1}{P(x)} = \frac{1}{P'(\alpha_j)}$$

To verify this identity, observe that both sides are rational functions with the same simple poles and the same principal part at each pole. Their difference is therefore a rational function with no poles, hence a polynomial. Since this difference tends to 0 as  $x \rightarrow \infty$ , it must be identically 0.

Applying this to  $P(x) = x^n + 1$  with roots  $\omega_j$ :

$$r_j = \frac{1}{P'(\omega_j)} = \frac{1}{n\omega_j^{n-1}}$$

To simplify, observe that  $\omega_j^n = -1$  implies  $\omega_j^{n-1} = \omega_j^n/\omega_j = -\omega_j^{-1}$ . Therefore:

$$r_j = \frac{1}{n(-\omega_j^{-1})} = -\frac{\omega_j}{n}$$

This yields the complex partial fraction decomposition:

$$\frac{1}{x^n + 1} = -\frac{1}{n} \sum_{j=0}^{n-1} \frac{\omega_j}{x - \omega_j}$$

When  $n$  is odd, the index  $j = (n-1)/2$  corresponds to  $\omega_j = e^{i\pi} = -1$ , and the associated term is:

$$-\frac{1}{n} \cdot \frac{-1}{x - (-1)} = \frac{1}{n} \cdot \frac{1}{x + 1}$$

### 3.2 Step 2. Pairing Conjugate Roots

For each index  $k$  in the range  $\{0, 1, \dots, \lfloor (n-2)/2 \rfloor\}$ , let  $\omega = \omega_k = e^{i\theta_k}$  and  $\bar{\omega} = e^{-i\theta_k}$  be the corresponding conjugate pair (indeed  $\bar{\omega} = \omega_{n-1-k}$ ). Their combined contribution to the partial fraction decomposition is:

$$-\frac{1}{n} \left( \frac{\omega}{x - \omega} + \frac{\bar{\omega}}{x - \bar{\omega}} \right) = -\frac{1}{n} \cdot \frac{\omega(x - \bar{\omega}) + \bar{\omega}(x - \omega)}{(x - \omega)(x - \bar{\omega})}$$

**Denominator.** Using  $\omega + \bar{\omega} = 2 \cos \theta_k = 2c_k$  and  $\omega\bar{\omega} = |\omega|^2 = 1$ :

$$(x - \omega)(x - \bar{\omega}) = x^2 - (\omega + \bar{\omega})x + \omega\bar{\omega} = x^2 - 2c_k x + 1 = Q_k(x)$$

**Numerator.** Expanding:

$$\omega(x - \bar{\omega}) + \bar{\omega}(x - \omega) = (\omega + \bar{\omega})x - \omega\bar{\omega} - \bar{\omega}\omega = 2c_k x - 2$$

Therefore the paired fraction becomes:

$$-\frac{1}{n} \cdot \frac{2c_k x - 2}{Q_k(x)} = \frac{A_k x + B_k}{Q_k(x)}$$

with coefficients:

$$A_k = -\frac{2c_k}{n}, \quad B_k = \frac{2}{n}$$

### 3.3 Step 3. Integration of Quadratic Terms

Fix an index  $k$ , and write  $c = c_k, s = s_k, Q(x) = Q_k(x) = (x - c)^2 + s^2, A = A_k, B = B_k$ . Note that  $Q'(x) = 2(x - c)$ .

**Numerator decomposition.** We seek constants  $\alpha$  and  $\beta$  such that:

$$Ax + B = \alpha \cdot Q'(x) + \beta = 2\alpha(x - c) + \beta$$

Matching coefficients:  $2\alpha = A$  and  $-2\alpha c + \beta = B$ . Hence  $\alpha = A/2$  and  $\beta = B + Ac$ .

**Evaluation of  $\beta$ :**

$$\beta = B + Ac = \frac{2}{n} + \left(-\frac{2c}{n}\right)c = \frac{2}{n}(1 - c^2) = \frac{2s^2}{n}$$

**Integration.** The integral separates into two standard forms:

$$\int \frac{Ax + B}{Q(x)} dx = \frac{A}{2} \int \frac{Q'(x)}{Q(x)} dx + \beta \int \frac{dx}{(x - c)^2 + s^2}$$

The first integral is  $\ln Q(x)$ . For the second, the substitution  $u = (x - c)/s$  gives  $du = dx/s$ , whence:

$$\int \frac{dx}{(x - c)^2 + s^2} = \int \frac{s du}{s^2(u^2 + 1)} = \frac{1}{s} \arctan u = \frac{1}{s} \arctan \frac{x - c}{s}$$

Substituting  $A/2 = -c/n$  and  $\beta/s = 2s^2/(ns) = 2s/n$ :

$$\int \frac{Ax + B}{Q(x)} dx = -\frac{c}{n} \ln Q(x) + \frac{2s}{n} \arctan \frac{x - c}{s} + C$$

**Assembly.** Summing over all conjugate pairs (indexed by  $k$ ) and including the linear term  $(1/n) \ln |x + 1|$  when  $n$  is odd yields the stated formula. ■

---

## 4 Complex Logarithmic Form

**Corollary.** On any simply connected domain  $D \subset \mathbb{C}$  avoiding the roots  $\{\omega_j\}$ , with a fixed branch of  $\log$  on  $D$ :

$$\int \frac{dx}{x^n + 1} = -\frac{1}{n} \sum_{j=0}^{n-1} \omega_j \log(x - \omega_j) + C$$

*Proof.* Integrate the complex partial fraction decomposition term by term. ■

**Remark on branches.** On a simply connected domain avoiding the poles, fixing a branch of  $\log$  renders the expression single-valued. Changing branches shifts the result by a constant, since the derivative is unchanged. Choose  $D$  to be conjugation-invariant (i.e.,  $z \in D \Rightarrow \bar{z} \in D$ ) and choose a branch of  $\log$  on  $D$  satisfying  $\log(\bar{z}) = \overline{\log z}$ . Then for real  $x \in D$ , we have  $\overline{x - \omega} = x - \bar{\omega}$ , so  $\log(x - \bar{\omega}) = \overline{\log(x - \omega)}$ . Hence  $\omega \log(x - \omega) + \bar{\omega} \log(x - \bar{\omega})$  is real, and pairing conjugates yields a real-valued antiderivative; branch changes affect only the additive constant on each component.

---

## 5 Application: The Case $n = 5$

Setting  $n = 5$ , the index set is  $k \in \{0, 1\}$ , corresponding to angles  $\theta_0 = \pi/5$  and  $\theta_1 = 3\pi/5$ .

**Trigonometric values.** The cosine values follow from the identity  $\cos(\pi/5) = (1 + \sqrt{5})/4$  and  $\cos(3\pi/5) = -\cos(2\pi/5) = (1 - \sqrt{5})/4$ . The sine values follow from  $\sin^2 \theta + \cos^2 \theta = 1$ :

	$k = 0$	$k = 1$
$\theta_k$	$\pi/5$	$3\pi/5$
$c_k$	$(1 + \sqrt{5})/4$	$(1 - \sqrt{5})/4$
$s_k$	$\sqrt{10 - 2\sqrt{5}}/4$	$\sqrt{10 + 2\sqrt{5}}/4$
$-c_k/5$	$-(1 + \sqrt{5})/20$	$(\sqrt{5} - 1)/20$
$2s_k/5$	$\sqrt{10 - 2\sqrt{5}}/10$	$\sqrt{10 + 2\sqrt{5}}/10$

**Arctangent arguments.** For  $k = 0$ :

$$\frac{x - c_0}{s_0} = \frac{x - \frac{1+\sqrt{5}}{4}}{\frac{\sqrt{10-2\sqrt{5}}}{4}} = \frac{4x - 1 - \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}}$$

For  $k = 1$ :

$$\frac{x - c_1}{s_1} = \frac{x - \frac{1-\sqrt{5}}{4}}{\frac{\sqrt{10+2\sqrt{5}}}{4}} = \frac{4x - 1 + \sqrt{5}}{\sqrt{10 + 2\sqrt{5}}}$$

**Final formula:**

$$\begin{aligned} \int \frac{dx}{x^5 + 1} &= \frac{1}{5} \ln|x + 1| - \frac{1 + \sqrt{5}}{20} \ln \left( x^2 - \frac{1 + \sqrt{5}}{2}x + 1 \right) + \frac{\sqrt{5} - 1}{20} \ln \left( x^2 + \frac{\sqrt{5} - 1}{2}x + 1 \right) \\ &\quad + \frac{\sqrt{10 - 2\sqrt{5}}}{10} \arctan \frac{4x - 1 - \sqrt{5}}{\sqrt{10 - 2\sqrt{5}}} + \frac{\sqrt{10 + 2\sqrt{5}}}{10} \arctan \frac{4x - 1 + \sqrt{5}}{\sqrt{10 + 2\sqrt{5}}} + C \end{aligned}$$

## 6 Generalisations

### 6.1 The family $x^n + a^n$

For real  $a \neq 0$ , the substitution  $x = au$  yields:

$$\int \frac{dx}{x^n + a^n} = \frac{1}{a^{n-1}} \int \frac{du}{u^n + 1}$$

The main theorem then applies directly to the transformed integral.

### 6.2 The family $x^n - 1$

The roots of  $x^n - 1 = 0$  are  $\zeta_j = e^{2\pi ij/n}$  for  $j = 0, 1, \dots, n-1$ . The same method yields residues:

$$r_j = \frac{1}{n\zeta_j^{n-1}} = \frac{\zeta_j}{n}$$

since  $\zeta_j^n = 1$  implies  $\zeta_j^{n-1} = \zeta_j^{-1}$ . The decomposition becomes:

$$\frac{1}{x^n - 1} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{\zeta_j}{x - \zeta_j}$$

**Real roots.** The polynomial  $x^n - 1$  always has the real root  $\zeta_0 = 1$ , contributing a term:

$$\frac{1}{n} \cdot \frac{1}{x - 1}$$

When  $n$  is even,  $\zeta_{n/2} = -1$  is also a root, contributing:

$$\frac{1}{n} \cdot \frac{-1}{x - (-1)} = -\frac{1}{n(x + 1)}$$

**Quadratic factors.** The non-real roots pair at angles  $\phi_k = 2\pi k/n$  for appropriate  $k$ , yielding quadratics  $x^2 - 2 \cos \phi_k \cdot x + 1$ . The integration proceeds analogously, with coefficients determined by the same residue-pairing method.

### 6.3 General rational integrands

The methodology extends to any integral  $\int \frac{R(x)}{P(x)} dx$  where  $P$  has simple roots  $\{\alpha_j\}$ . After polynomial long division if  $\deg R \geq \deg P$ , the proper rational part decomposes as:

$$\frac{R(x)}{P(x)} = \sum_j \frac{R(\alpha_j)}{P'(\alpha_j)} \cdot \frac{1}{x - \alpha_j} + (\text{polynomial terms})$$

The procedure then follows three steps:

1. **Decompose** using the residue formula above.
2. **Pair conjugates** to obtain real quadratic denominators.
3. **Integrate** each quadratic term via the identity:

$$\int \frac{Ax + B}{(x - c)^2 + s^2} dx = \frac{A}{2} \ln((x - c)^2 + s^2) + \frac{B + Ac}{s} \arctan \frac{x - c}{s} + C$$

When  $P$  has repeated roots, the partial fraction decomposition includes higher-order terms  $(x - \alpha)^{-m}$ , which integrate to rational functions (for  $m \geq 2$ ) plus logarithms (for  $m = 1$ ).

---

## 7 Concluding Remarks

The integral  $\int \frac{dx}{x^n + 1}$  admits a closed form expressible as a finite sum of logarithmic and arctangent terms. The derivation rests on three principles:

1. **Residue-based partial fractions.** For simple roots,  $r_j = 1/P'(\alpha_j)$ .
2. **Conjugate pairing.** Non-real roots  $\omega, \bar{\omega}$  contribute the real quadratic  $(x - \omega)(x - \bar{\omega}) = x^2 - 2 \operatorname{Re}(\omega)x + 1$ .
3. **Quadratic integration.** The numerator decomposes into a multiple of the derivative (yielding  $\ln$ ) plus a constant (yielding  $\arctan$ ).

The particularly elegant coefficient formulae  $A_k = -2c_k/n$  and  $B_k = 2/n$  arise from the algebraic relation  $\omega_j^n = -1$ , which forces  $r_j = -\omega_j/n$ . For the family  $x^n - 1$ , the relation  $\zeta_j^n = 1$  yields  $r_j = \zeta_j/n$  instead. For general polynomials, the residues  $r_j = R(\alpha_j)/P'(\alpha_j)$  do not simplify to such universal expressions, though the decomposition-and-pairing method remains fully applicable.

---

## 8 Appendix: Algorithmic Summary

To evaluate  $\int \frac{dx}{x^n + 1}$ :

1. **Compute**  $\theta_k = \pi(2k + 1)/n$  for  $k = 0, 1, \dots, \lfloor (n - 2)/2 \rfloor$ .
2. **Evaluate**  $c_k = \cos \theta_k$  and  $s_k = \sin \theta_k$ .
3. **If**  $n$  **is odd**, include the term  $\frac{1}{n} \ln |x + 1|$ .
4. **For each**  $k$ , add:
  - $-\frac{c_k}{n} \ln(x^2 - 2c_k x + 1)$
  - $\frac{2s_k}{n} \arctan \frac{x - c_k}{s_k}$
5. **Add** the constant of integration  $C$ .