

The Integral of $1/(x^n + 1)$

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0.1 Roots of $x^n + 1$

We seek the solutions to $x^n = -1$. Writing $-1 = e^{i\pi(2m+1)}$ for any integer m , we obtain:

$$x = e^{i\pi(2k+1)/n}, \quad k = 0, 1, \dots, n-1$$

Definition 0.1.1. Let $\omega_k = e^{i\pi(2k+1)/n}$ denote the k -th root of $x^n + 1 = 0$.

Proposition 0.1.2. *The roots satisfy:*

1. $|\omega_k| = 1$ for all k
2. ω_k and ω_{n-1-k} are complex conjugates
3. For odd n , exactly one root is real, namely $\omega_{(n-1)/2} = -1$
4. For even n , no root is real

Proof. Statement (1) follows from $|e^{i\theta}| = 1$. For (2), observe that:

$$\overline{\omega_k} = e^{-i\pi(2k+1)/n}$$

and

$$\omega_{n-1-k} = e^{i\pi(2(n-1-k)+1)/n} = e^{i\pi(2n-2k-1)/n} = e^{i\pi(2-(2k+1)/n)} = e^{-i\pi(2k+1)/n}$$

since $e^{2\pi i} = 1$. For (3), a real root requires $(2k+1)/n \in \mathbb{Z}$; with odd n , this occurs when $k = (n-1)/2$, yielding $e^{i\pi} = -1$. For even n , $(2k+1)/n$ is never an integer. ■

0.2 Factorisation over \mathbb{R}

Theorem 0.2.1 (Real Factorisation). *The polynomial $x^n + 1$ factors over \mathbb{R} as follows:*

****Case I**** (n odd):

$$x^n + 1 = (x + 1) \prod_{k=0}^{(n-3)/2} \left(x^2 - 2 \cos \frac{\pi(2k+1)}{n} x + 1 \right)$$

****Case II**** (n even):

$$x^n + 1 = \prod_{k=0}^{n/2-1} \left(x^2 - 2 \cos \frac{\pi(2k+1)}{n} x + 1 \right)$$

Proof. Each conjugate pair $\omega_k, \bar{\omega}_k$ contributes the real quadratic:

$$\begin{aligned} (x - \omega_k)(x - \bar{\omega}_k) &= x^2 - (\omega_k + \bar{\omega}_k)x + \omega_k \bar{\omega}_k \\ &= x^2 - 2 \operatorname{Re}(\omega_k) x + 1 = x^2 - 2 \cos \frac{\pi(2k+1)}{n} x + 1 \end{aligned}$$

The quadratics are irreducible over \mathbb{R} since their discriminant $4 \cos^2(\pi(2k+1)/n) - 4 < 0$ for $0 < (2k+1)/n < 1$. The stated index ranges enumerate precisely one representative from each

conjugate pair. ■

Notation. Henceforth, define:

$$\theta_k = \frac{\pi(2k+1)}{n}, \quad Q_k(x) = x^2 - 2 \cos \theta_k x + 1$$

0.3 Partial Fraction Decomposition

Lemma 0.3.1 (Residue at Simple Pole). *If $P(a) = 0$ and $P'(a) \neq 0$, then:*

$$\frac{1}{P(x)} = \frac{1}{P'(a)(x-a)} + (\text{terms regular at } a)$$

Proposition 0.3.2. *For the real root $x = -1$ (when n is odd):*

$$\text{Res}_{x=-1} \frac{1}{x^n + 1} = \frac{1}{n(-1)^{n-1}} = \frac{1}{n}$$

Proof. Since $\frac{d}{dx}(x^n + 1) = nx^{n-1}$, and for odd n we have $(-1)^{n-1} = 1$. ■

Theorem 0.3.3 (Partial Fraction Coefficients). *The decomposition takes the form:*

****Case I**** (n odd):

$$\frac{1}{x^n + 1} = \frac{1/n}{x+1} + \sum_{k=0}^{(n-3)/2} \frac{A_k x + B_k}{Q_k(x)}$$

****Case II**** (n even):

$$\frac{1}{x^n + 1} = \sum_{k=0}^{n/2-1} \frac{A_k x + B_k}{Q_k(x)}$$

where in both cases:

$$A_k = -\frac{2 \cos \theta_k}{n}, \quad B_k = \frac{2}{n}$$

Proof. The coefficient over the linear factor (when n is odd) follows from the previous proposition. For the quadratic factors, we employ the method of complex residues.

Consider the partial fraction over \mathbb{C} :

$$\frac{1}{x^n + 1} = \sum_{j=0}^{n-1} \frac{r_j}{x - \omega_j}$$

where, by the residue lemma:

$$r_j = \frac{1}{n\omega_j^{n-1}} = \frac{\omega_j}{n\omega_j^n} = \frac{\omega_j}{n(-1)} = -\frac{\omega_j}{n}$$

Combining conjugate pairs ω_k and $\bar{\omega}_k = \omega_{n-1-k}$:

$$\frac{r_k}{x - \omega_k} + \frac{\bar{r}_k}{x - \bar{\omega}_k} = \frac{r_k(x - \bar{\omega}_k) + \bar{r}_k(x - \omega_k)}{(x - \omega_k)(x - \bar{\omega}_k)}$$

The numerator is:

$$(r_k + \bar{r}_k)x - (r_k\bar{\omega}_k + \bar{r}_k\omega_k) = 2 \operatorname{Re}(r_k)x - 2 \operatorname{Re}(r_k\bar{\omega}_k)$$

Now:

$$\operatorname{Re}(r_k) = \operatorname{Re}\left(-\frac{\omega_k}{n}\right) = -\frac{\cos \theta_k}{n}$$

$$\operatorname{Re}(r_k\bar{\omega}_k) = \operatorname{Re}\left(-\frac{\omega_k\bar{\omega}_k}{n}\right) = \operatorname{Re}\left(-\frac{1}{n}\right) = -\frac{1}{n}$$

Therefore:

$$A_k = 2 \operatorname{Re}(r_k) = -\frac{2 \cos \theta_k}{n}$$

$$B_k = -2 \operatorname{Re}(r_k\bar{\omega}_k) = \frac{2}{n}$$

as claimed. ■

0.4 Integration of Component Terms

Lemma 0.4.1 (Linear Factor).

$$\int \frac{dx}{x+1} = \ln|x+1| + C$$

Lemma 0.4.2 (Quadratic Factor — Logarithmic Part). For $Q(x) = x^2 + bx + c$ with $\Delta = b^2 - 4c < 0$:

$$\int \frac{x dx}{x^2 + bx + c} = \frac{1}{2} \ln(x^2 + bx + c) - \frac{b}{2} \int \frac{dx}{x^2 + bx + c}$$

Proof. Write $x = \frac{1}{2}(2x + b) - \frac{b}{2}$ and observe that $\frac{d}{dx}(x^2 + bx + c) = 2x + b$. ■

Lemma 0.4.3 (Quadratic Factor — Arctangent Part). For $Q(x) = x^2 + bx + c$ with $\Delta = b^2 - 4c < 0$:

$$\int \frac{dx}{x^2 + bx + c} = \frac{2}{\sqrt{-\Delta}} \arctan \frac{2x + b}{\sqrt{-\Delta}} + C$$

Proof. Complete the square: $x^2 + bx + c = (x + b/2)^2 + (c - b^2/4) = (x + b/2)^2 + (-\Delta/4)$. Substituting $u = x + b/2$ and recognising the standard arctangent integral yields the result. ■

Proposition 0.4.4. For $Q_k(x) = x^2 - 2 \cos \theta_k x + 1$, we have $\Delta_k = 4 \cos^2 \theta_k - 4 = -4 \sin^2 \theta_k$,

whence:

$$\int \frac{dx}{Q_k(x)} = \frac{1}{\sin \theta_k} \arctan \frac{x - \cos \theta_k}{\sin \theta_k} + C$$

Theorem 0.4.5 (Integration of Quadratic Term).

$$\int \frac{A_k x + B_k}{Q_k(x)} dx = -\frac{\cos \theta_k}{n} \ln Q_k(x) + \frac{1}{n \sin \theta_k} \arctan \frac{x - \cos \theta_k}{\sin \theta_k} + C$$

0.5 The General Solution

Theorem 0.5.1 (Main Result). *The integral of $1/(x^n + 1)$ is given by:*

****Case I**** (n odd):

$$\int \frac{dx}{x^n + 1} = \frac{1}{n} \ln |x + 1| + \sum_{k=0}^{(n-3)/2} \left[-\frac{\cos \theta_k}{n} \ln Q_k(x) + \frac{1}{n \sin \theta_k} \arctan \frac{x - \cos \theta_k}{\sin \theta_k} \right] + C$$

****Case II**** (n even):

$$\int \frac{dx}{x^n + 1} = \sum_{k=0}^{n/2-1} \left[-\frac{\cos \theta_k}{n} \ln Q_k(x) + \frac{1}{n \sin \theta_k} \arctan \frac{x - \cos \theta_k}{\sin \theta_k} \right] + C$$

where $\theta_k = \pi(2k + 1)/n$ and $Q_k(x) = x^2 - 2 \cos \theta_k x + 1$.

0.6 Unified Compact Form

Corollary 0.6.1. *The integral admits the symmetric representation:*

$$\int \frac{dx}{x^n + 1} = \frac{1}{n} \sum_{j=0}^{n-1} \overline{\omega_j} \ln(x - \omega_j) + C$$

where the sum is interpreted over \mathbb{C} and the logarithm is complex. Taking real parts and combining conjugates recovers the main theorem.

0.7 Application to $n = 5$

Setting $n = 5$, we have $\theta_0 = \pi/5$ and $\theta_1 = 3\pi/5$. The exact values are:

$$\cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}, \quad \sin \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$\cos \frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}, \quad \sin \frac{3\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Substitution into the main theorem (Case I) yields the explicit formula.

0.8 Algorithmic Summary

To integrate $1/(x^n + 1)$:

1. **Compute** $\theta_k = \pi(2k + 1)/n$ for $k = 0, 1, \dots, \lfloor (n - 2)/2 \rfloor$
2. **Form** $Q_k(x) = x^2 - 2 \cos \theta_k x + 1$
3. **If** n **odd**, include $(1/n) \ln |x + 1|$
4. **For each** k , add:
 - $-(\cos \theta_k)/n \cdot \ln Q_k(x)$
 - $(1/(n \sin \theta_k)) \cdot \arctan((x - \cos \theta_k)/\sin \theta_k)$

This procedure generalises immediately to integrals of the form $1/(x^n + a^n)$ via the substitution $u = x/a$.