The Integral of $1/(x^n + 1)$

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0.1 Roots of xⁿ + 1

We seek the solutions to $x^n = -1$. Writing $-1 = e^{i\pi(2m+1)}$ for any integer m, we obtain:

$$x = e^{i\pi(2k+1)/n}, \quad k = 0, 1, \ldots, n-1$$

Definition 0.1.1. Let $\omega_k = e^{i\pi(2k+1)/n}$ denote the k-th root of $x^n + 1 = 0$.

Proposition 0.1.2. *The roots satisfy:*

1. $|\omega_k| = 1$ for all k 2. ω_k and ω_{n-1-k} are complex conjugates 3. For odd n, exactly one root is real, namely $\omega_{(n-1)/2} = -1$ 4. For even n, no root is real

Proof. Statement (1) follows from $|e^{i\theta}| = 1$. For (2), observe that:

$$\overline{\omega_k} = e^{-i\pi(2k+1)/n}$$

and

$$\omega_{n-1-k} = e^{i\pi(2(n-1-k)+1)/n} = e^{i\pi(2n-2k-1)/n} = e^{i\pi(2-(2k+1)/n)} = e^{-i\pi(2k+1)/n}$$

since $e^{2\pi i} = 1$. For (3), a real root requires $(2k+1)/n \in \mathbb{Z}$; with odd n, this occurs when $k = (n-1)/2$, yielding $e^{i\pi} = -1$. For even n, $(2k+1)/n$ is never an integer. ∎

0.2 Factorisation over ℝ

Theorem 0.2.1 (Real Factorisation)**.** *The polynomial $x^n + 1$ factors over $\mathbb{R}$ as follows:*

***Case I** (n odd):*

$$x^n + 1 = (x+1) \prod_{k=0}^{(n-3)/2} \left(x^2 - 2\cos\frac{\pi(2k+1)}{n}\, x + 1 \right)$$

***Case II** (n even):*

$$x^n + 1 = \prod_{k=0}^{n/2-1} \left(x^2 - 2\cos\frac{\pi(2k+1)}{n}\, x + 1 \right)$$

Proof. Each conjugate pair $\omega_k, \bar{\omega}_k$ contributes the real quadratic:

$$(x - \omega_k)(x - \overline{\omega_k}) = x^2 - (\omega_k + \overline{\omega_k})x + \omega_k\overline{\omega_k}$$

$$= x^2 - 2\,\mathrm{Re}(\omega_k)\, x + 1 = x^2 - 2\cos\frac{\pi(2k+1)}{n}\, x + 1$$

The quadratics are irreducible over $\mathbb{R}$ since their discriminant $4\cos^2(\pi(2k+1)/n) - 4 < 0$ for $0 < (2k+1)/n < 1$. The stated index ranges enumerate precisely one representative from each conjugate pair. ■

Notation. Henceforth, define:

$$\theta_k = \frac{\pi(2k+1)}{n}, \qquad Q_k(x) = x^2 - 2\cos\theta_k\, x + 1$$

0.3 Partial Fraction Decomposition

Lemma 0.3.1 (Residue at Simple Pole)**.** *If $P(a) = 0$ and $P'(a) \neq 0$, then:*

$$\frac{1}{P(x)} = \frac{1}{P'(a)(x-a)} + \textit{(terms regular at } a\textit{)}$$

Proposition 0.3.2. *For the real root $x = -1$ (when n is odd):*

$$Res_{x=-1} \frac{1}{x^n+1} = \frac{1}{n(-1)^{n-1}} = \frac{1}{n}$$

Proof. Since $\frac{d}{dx}(x^n+1) = nx^{n-1}$, and for odd n we have $(-1)^{n-1} = 1$. ■

Theorem 0.3.3 (Partial Fraction Coefficients)**.** *The decomposition takes the form:*

***Case I** (n odd):*

$$\frac{1}{x^n+1} = \frac{1/n}{x+1} + \sum_{k=0}^{(n-3)/2} \frac{A_k x + B_k}{Q_k(x)}$$

***Case II** (n even):*

$$\frac{1}{x^n+1} = \sum_{k=0}^{n/2-1} \frac{A_k x + B_k}{Q_k(x)}$$

where in both cases:

$$A_k = -\frac{2\cos\theta_k}{n}, \qquad B_k = \frac{2}{n}$$

Proof. The coefficient over the linear factor (when n is odd) follows from the previous proposition. For the quadratic factors, we employ the method of complex residues.

Consider the partial fraction over $\mathbb{C}$:

$$\frac{1}{x^n+1} = \sum_{j=0}^{n-1} \frac{r_j}{x-\omega_j}$$

where, by the residue lemma:

$$r_j = \frac{1}{n\omega_j^{n-1}} = \frac{\omega_j}{n\omega_j^n} = \frac{\omega_j}{n(-1)} = -\frac{\omega_j}{n}$$

Combining conjugate pairs ω_k and $\bar{\omega}_k = \omega_{n-1-k}$:

$$\frac{r_k}{x-\omega_k} + \frac{\overline{r_k}}{x-\overline{\omega_k}} = \frac{r_k(x-\overline{\omega_k}) + \overline{r_k}(x-\omega_k)}{(x-\omega_k)(x-\overline{\omega_k})}$$

The numerator is:

$$(r_k + \overline{r_k})x - (r_k\overline{\omega_k} + \overline{r_k}\omega_k) = 2\,\mathrm{Re}(r_k)\,x - 2\,\mathrm{Re}(r_k\overline{\omega_k})$$

Now:

$$\mathrm{Re}(r_k) = \mathrm{Re}\left(-\frac{\omega_k}{n}\right) = -\frac{\cos\theta_k}{n}$$

$$\mathrm{Re}(r_k\overline{\omega_k}) = \mathrm{Re}\left(-\frac{\omega_k\overline{\omega_k}}{n}\right) = \mathrm{Re}\left(-\frac{1}{n}\right) = -\frac{1}{n}$$

Therefore:

$$A_k = 2\,\mathrm{Re}(r_k) = -\frac{2\cos\theta_k}{n}$$
$$B_k = -2\,\mathrm{Re}(r_k\overline{\omega_k}) = \frac{2}{n}$$

as claimed. ■

0.4 Integration of Component Terms

Lemma 0.4.1 (Linear Factor)**.**

$$\int \frac{dx}{x+1} = \ln|x+1| + C$$

Lemma 0.4.2 (Quadratic Factor — Logarithmic Part)**.** *For $Q(x) = x^2+bx+c$ with $\Delta = b^2-4c < 0$:*

$$\int \frac{x\,dx}{x^2+bx+c} = \frac{1}{2}\ln(x^2+bx+c) - \frac{b}{2}\int \frac{dx}{x^2+bx+c}$$

Proof. Write $x = \frac{1}{2}(2x+b) - \frac{b}{2}$ and observe that $\frac{d}{dx}(x^2+bx+c) = 2x+b$. ■

Lemma 0.4.3 (Quadratic Factor — Arctangent Part)**.** *For $Q(x) = x^2+bx+c$ with $\Delta = b^2-4c < 0$:*

$$\int \frac{dx}{x^2+bx+c} = \frac{2}{\sqrt{-\Delta}}\arctan\frac{2x+b}{\sqrt{-\Delta}} + C$$

Proof. Complete the square: $x^2+bx+c = (x+b/2)^2 + (c-b^2/4) = (x+b/2)^2 + (-\Delta/4)$. Substituting $u = x+b/2$ and recognising the standard arctangent integral yields the result. ■

Proposition 0.4.4. *For $Q_k(x) = x^2 - 2\cos\theta_k\,x + 1$, we have $\Delta_k = 4\cos^2\theta_k - 4 = -4\sin^2\theta_k$,*

whence:
$$\int \frac{dx}{Q_k(x)} = \frac{1}{\sin\theta_k} \arctan \frac{x - \cos\theta_k}{\sin\theta_k} + C$$

Theorem 0.4.5 (Integration of Quadratic Term)**.**

$$\int \frac{A_k x + B_k}{Q_k(x)}\,dx = -\frac{\cos\theta_k}{n} \ln Q_k(x) + \frac{1}{n\sin\theta_k} \arctan \frac{x - \cos\theta_k}{\sin\theta_k} + C$$

0.5 The General Solution

Theorem 0.5.1 (Main Result)**.** *The integral of* $1/(x^n + 1)$ *is given by:*

***Case I** (n odd):*

$$\int \frac{dx}{x^n + 1} = \frac{1}{n} \ln|x + 1| + \sum_{k=0}^{(n-3)/2} \left[-\frac{\cos\theta_k}{n} \ln Q_k(x) + \frac{1}{n\sin\theta_k} \arctan \frac{x - \cos\theta_k}{\sin\theta_k} \right] + C$$

***Case II** (n even):*

$$\int \frac{dx}{x^n + 1} = \sum_{k=0}^{n/2-1} \left[-\frac{\cos\theta_k}{n} \ln Q_k(x) + \frac{1}{n\sin\theta_k} \arctan \frac{x - \cos\theta_k}{\sin\theta_k} \right] + C$$

where $\theta_k = \pi(2k + 1)/n$ *and* $Q_k(x) = x^2 - 2\cos\theta_k\, x + 1$.

0.6 Unified Compact Form

Corollary 0.6.1. *The integral admits the symmetric representation:*

$$\int \frac{dx}{x^n + 1} = \frac{1}{n} \sum_{j=0}^{n-1} \overline{\omega_j} \ln(x - \omega_j) + C$$

where the sum is interpreted over $\mathbb{C}$ *and the logarithm is complex. Taking real parts and combining conjugates recovers the main theorem.*

0.7 Application to n = 5

Setting $n = 5$, we have $\theta_0 = \pi/5$ and $\theta_1 = 3\pi/5$. The exact values are:

$$\cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}, \quad \sin\frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$

$$\cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}, \quad \sin\frac{3\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Substitution into the main theorem (Case I) yields the explicit formula.

0.8 Algorithmic Summary

To integrate $1/(x^n + 1)$:

1. **Compute** $\theta_k = \pi(2k+1)/n$ for $k = 0, 1, \ldots, \lfloor (n-2)/2 \rfloor$
2. **Form** $Q_k(x) = x^2 - 2\cos\theta_k\, x + 1$
3. **If** n **odd**, include $(1/n)\ln|x+1|$
4. **For each** k, add:
   - $-(\cos\theta_k)/n \cdot \ln Q_k(x)$
   - $(1/(n\sin\theta_k)) \cdot \arctan((x - \cos\theta_k)/\sin\theta_k)$

This procedure generalises immediately to integrals of the form $1/(x^n + a^n)$ via the substitution $u = x/a$.