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CORE MATHEMATICS
FOR
SENIOR HIGH SCHOOLS

NEW INTERNATIONAL EDITION
(Sixth Edition)

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PREFACE

This book is intended to encourage an understanding and appreciation of core mathematics at the Senior High School level in West Africa. Providing appropriate solutions to examination problems is of particular importance in the study of mathematics. As a mathematics lecturer, the author has discovered the weaknesses and shortcomings of students in the handling of examination questions. Subsequently, to guide students in answering typical questions in core mathematics as set out in recent examinations, the writer has paid particular attention to those areas of the syllabus, which many students find difficult.

A prominent feature of this book is the inclusion of many examples. Each example is carefully selected to illustrate the application of a particular mathematical technique and or interpretation of results. Another feature is that each chapter has an extensive collection of exercises. It is important that students have several exercises to practice.

This book is therefore designed to help students to:

1. acquire the basic skills and understanding which is vital to examination success.
2. appreciate the use of mathematics as a tool for analysis and effective thinking.
3. discover order, patterns and relations.
4. communicate their thoughts through symbolic expressions and graphs.
5. develop mathematical abilities useful in commerce, industry and public service.

I have gone to great lengths to make this text both pedagogically sound and error-free. If you have any suggestions, or find potential errors, please contact the writer at **akrongh@yahoo.com**.

C. A. Hesse
January, 2011

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First and foremost, I express my sincere thanks to the Almighty God both for the gift of writing he has endowed me with and for the endurance to complete this particular book.

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CHAPTER ONE

Sets

1.1 Introduction

If you want to prepare a cake, you need flour, eggs, margarine, baking powder and sugar. When you buy these ingredients in a shop, you probably do not buy them one at a time. It is easier and cheaper to buy them in a set. We often use the word '***set***' to describe a collection of objects, quantities or numbers. We have a set of living room furniture: what does it contain? We speak of cutlery set, a math set, the set of students under 14 in your class and so on. Can you think of any more?

> A *set* is a collection or list of objects, quantities or numbers with specified properties.

A set is usually denoted by capital letters such as A, B, P, Q, X and Y. The objects that make up a set are called ***members*** or ***elements*** of the set. The elements of a set may be named in a list or may be given by a description enclosed in braces { }. For instance, the set of numbers between 1 and 6 may be given as $\{2, 3, 4, 5\}$ or as {the numbers between 1 and 6}.

For example, in the set $Q = \{2, 4, 6, 8, 10\}$, 4 is a member or element of the set Q. In set operations, the symbol $\in$ is used to denote the phrase '***is a member of***' or '***is an element of***' or '***belongs to***'. So the statement '4 is a member of Q' can be written as $4 \in Q$. Can you name the other elements of the set Q? Similarly the statement '***5 is not a member of Q***' may be abbreviated to $5 \notin Q$, $\notin$ standing for '***is not an element of***' or '***does not belong to***.'

Example 1.1

If $P = \{2, 4, 6, 8, 10\}$ and $Q = \{3, 5, 7, 9\}$, complete the following statements by inserting $\in$, $\notin$, P, Q or elements of the sets P and Q

(a) $4 \ldots P$	(b) $6 \in \ldots$	(c) $2 \notin \ldots$	(d) $8 \ldots Q$	(e) $\ldots \notin P$
(f) $10 \ldots Q$	(g) $5 \in \ldots$	(h) $7 \notin \ldots$	(i) $7 \ldots P$	(j) $\ldots \notin Q$.

The following are some few definitions that will enable us to define the elements of sets in problems.

1. An ***odd number*** is a number which when divided by two (2) leaves a remainder of one (1).
   Example: $\ldots -5, -3, -1, 1, 3, 5 \ldots$
2. An ***even number*** is a number which leaves no remainder when it is divided by two (2).
   Example: $\ldots -6, -4, -2, 0, 2, 4, 6 \ldots$

3. An integer x is said to be ***a factor*** of another integer y if x can divide y without leaving any remainder. Example: The set factors of 48 = {1, 2, 4, 6, 8, 12, 24, 48}
4. A ***prime number*** is any positive number that is exactly divisible only by itself and one. Example: 2, 3, 5, 7, 11, 13, 17, etc.
5. The ***prime factors*** of a number n refer to the factors of n that are prime numbers. Example: The set of prime factors of 36 = {2, 3}
6. ***Multiple of a number*** x refers to any number formed when x is multiplied by any integer. Example: Multiples of 3 are 3, 6, 9, 12, 16, etc.

1.1.1 Set-builder notation

We can describe a set by using some of the above definitions or properties. For example the set P of even numbers between 1 and 20 can be written in the following two ways

(1) P = {even numbers between 1 and 20}

(2) $P = \{2, 4, 6, \ldots, 18\}$

The set P can also be described using a notation or symbol such as 'x' to represent any member of the set of even numbers between 1 and 20. Thus we can write P as follows:

$P = \{x$: x is an even number, and $1 < x < 20\}$

In the above expression for P, the colon ':' means 'such that', and is followed by the property that x is an even number between 1 and 20. The set Q of integers greater than 100 can be written as

$Q = \{x : x$ is an integer, and $x > 100\}$

The set R of regions in Ghana can also be written as

$R = \{x : x$ is a region in Ghana$\}$

Other letters such as y and z can also be used as notations for representing sets.

Example 1.2

Use set builder notation to describe the following:

(a) The set of odd numbers greater than 30,

(b) The set of prime numbers greater than 2 but less than 24,

(c) The set of triangles,

(d) The set of positive integers less than 100,

(e) The set of rivers in Ghana.

Solution

(a) $\{x : x$ is an odd number, and $x > 30\}$

(b) $\{x : x$ is a prime number, and $2 < x < 24\}$

(c) $\{x : x$ is a triangle$\}$

(d) $\{x : x$ is a positive integer, $x < 100\}$

(e) $\{x : x$ is a river in Ghana$\}$

Example 1.3

List the elements of the following sets

(a) $A = \{x: x \text{ is a factor of } 12\}$, (b) $B = \{x: x \text{ is a multiple of 4 less than } 20\}$

Solution

(a) $A = \{x: x \text{ is a factor of } 12\} = \{1, 2, 3, 4, 6, 12\}$
(b) $B = \{x: x \text{ is a multiple of 4 less than } 20\} = \{4, 8, 12, 16\}$

Example 1.4

$P = \{x: 2(x - 1) \leq 8\}$ and $Q = \{x: 2x - 2 \leq 3x + 6\}$ are subsets of $U = \{\text{integers}\}$. List the elements of P and Q.

Solution

$$2(x - 1) \leq 8 \Rightarrow 2x - 2 \leq 8 \Rightarrow 2x \leq 10 \Rightarrow x \leq 5$$

$P = \{x: 2(x - 1) \leq 8\} = \{x: x \leq 5\} = \{\ldots 1, 2, 3, 4, 5\}$

$$2x - 2 \leq 3x + 6 \Rightarrow 2x - 3x \leq 6 + 2 \Rightarrow -x \leq 8 \Rightarrow x \geq -8.$$

$Q = \{x: 2x - 2 \leq 3x + 6\} = \{x: x \geq -8\} = \{-8, -7, -6, -5\ldots\}$

1.1.2 Subsets

Consider the following example.

Example 1.5

(a) If all Ghanaians are Africans, then {Ghanaians} is a subset of {Africans}.
(b) All prime numbers are whole numbers; therefore {prime numbers} is a subset of {whole numbers}.

> A set P is said to be the subset of the set Q if all the elements of P belong to the set Q.

The symbol $\subset$ is used to denote the phrase '***subset of***'. P is a subset of Q is therefore written as $P \subset Q$. For example, If $P = \{2, 5, 8\}$ and $Q = \{1, 2, 3, 5, 7, 8\}$, then $P \subset Q$. If the set A is not a subset of the set B, we write $A \not\subset$ B.
It is important not to confuse the symbols $\subset$ and $\in$. The symbol $\subset$ connects two sets while $\in$ connect a member and its set.

The set of all objects under discussion is called the universe or ***universal set***. We use the letter $\mathcal{U}$ or the symbol ξ to denote the universal set. In Example 1.5 (a) $\mathcal{U}$ = {Africans}.

A set, which contains no elements, is called an ***empty (or null) set***. It is usually denoted by { } or $\varnothing$.

The ***complement of a set A*** is defined as the set of all elements of the universal set $\mathcal{U}$, which are not elements of A. The complement of A is written as A'. For example, if $A = \{1, 3, 5\}$ and $B = \{2, 4, 5\}$ are subsets of the universal set $\mathcal{U} = \{1, 2, 3, 4, 5, 7\}$ then the complements of A and B are $A' = \{2, 4, 7\}$ and $B' = \{1, 3, 7\}$ respectively. ***The complement of the universal set is the empty set***.

Example 1.6

Suggest a universal set for each of the following subsets.

(a) {Francophone countries in Africa}, (b) {isosceles triangles},
(c) {students in your class}, (d) {horses}.

1.1.3 Venn diagrams

So far, we have discovered that a set can be described by using words/set builder notation or listing the members of the set. We also discussed the connection between two sets. The ideas that we have met so far can be represented very simply by means of a diagram.

Consider: $\mathcal{U}$ = {all men}, Q = {people who wear uniform}, P = {policemen}.

If all policemen wear uniform, then {policemen} $\subset$ {people who wear uniform} or we write $P \subset Q$. We know that P and Q are both subsets of $\mathcal{U}$, that is $P, Q \subset \mathcal{U}$. This information can be represented diagrammatically.

Fig. 1.1 shows the relationship between the sets P, Q and $\mathcal{U}$. Since $Q \subset \mathcal{U}$, the circle representing Q is drawn inside the rectangle which represents $\mathcal{U}$. Furthermore, since $P \subset Q$, the circle representing P is inside that of Q. A diagram like this is called a ***Venn diagram***, after the English mathematician John Venn (1834 – 1923).

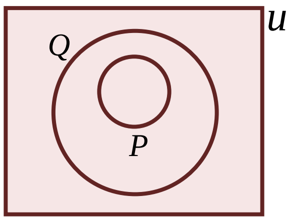


Fig. 1.1

Example 1.7

Draw a Venn diagram of $\mathcal{U}$ = {plane geometrical figures}, P = {polygons}, T = {triangles}.

Solution

We know that $T \subset P \subset U$. Therefore *Fig*. 1.2 shows the required Venn diagram. Notice that since $T \subset P$, the circle representing T lies entirely inside that of P.

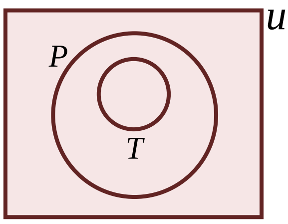


Fig. 1.2

1.1.4 Power set

Given a set S, the power set of S is the set that contains all subsets of S. The power set of S is usually denoted 2^S or $P(S)$. If S is a finite set with $n(S) = k$ elements, then the power set of S contains $|P(S)| = 2^k$ elements. Power sets are larger than the sets associated with them.

Example 1.8

If S is the set $\{x, y, z\}$, then the complete list of subsets of S is as follows:

(i) { } (also denoted $\varnothing$, the empty set), (ii) $\{x\}$, (iii) $\{y\}$, (iv) $\{z\}$, (v) $\{x, y\}$,
(vi) $\{x, z\}$, (vii) $\{y, z\}$, (viii) $\{x, y, z\}$.

Hence the power set of S is $P(S) = \{\{\ \}, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$.
The number of elements in $P(S)$, $n(P(S))$, is $2^3 = 8$.

Definitions

1. When the elements of a set are arranged in increasing order of magnitude, the first element (the least member) is called the **lower limit** whilst the last element (the greatest member) is the **upper limit**.
   Example: If $A = \{2, 4, 6, 8\}$, then the Lower Limit = 2 and the Upper Limit = 8
2. A set is said to be **finite** if it has both lower and upper limits. In other words, a set is finite if the first and the last members can be found. A finite set is also called a **bounded set**. For example, the set $A = \{2, 4, 6, 8\}$ is a finite set.
3. A set without a lower or upper limit or both is called an **infinite set**. An infinite set is also called an **unbounded set**.
   For example, the sets $N = \{1, 2, 3, 4,\ldots\}$, $Z = \{\ldots,-3, -2, -1, 0, 1, 2, 3,\ldots\}$ and $P = \{\ldots,7, 9, 13, 15\}$ are infinite sets.

Exercise 1.1

1. List the elements of the following sets
   (i) $A = \{x: x \text{ is a factor of } 44\}$ (ii) $B = \{x: x \text{ is a multiple of 3 less than } 20\}$
2. $P = \{x: 2x + 3 \leq 13\}$ and $Q = \{x: 5x + 4 \leq 18 - 2x\}$ are subsets of $U = \{\text{integers}\}$. List the elements of P and Q.
3. Write out the following statements in full.
   (a) $36 \in \{\text{multiple of 4}\}$, (b) Togo $\notin$ {state where English is the official language},
   (c) A snake $\notin$ {bird}, (d) Canada $\notin$ {African countries},
   (e) $6 \in \{\text{factor of 48}\}$, (f) A quadrilateral $\in$ {polygons},
   (g) Lizard $\in$ {reptiles}, (h) $7 \in \{\text{prime numbers}\}$.
4. Rewrite the following using set notation.
   (a) My cat is not a bird, (b) Ghana is a country in West Africa,
   (c) 4 is an even number, (d) Mensah is a student at Methodist High School,
   (e) My dog is an animal, (f) June is a month in the year.
5. Rewrite the following in 'set language'
   (a) All Akans are Ghanaians, (b) All rectangles are parallelograms,
   (c) All goats eat grass, (d) All students are hardworking,
   (e) All my friends are intelligent, (f) Not all prefects play football,
   (g) Not all prime numbers are odd,
   (h) Not all Senior High School pupils are well-behaved,
   (i) All Senior High School pupils wear uniform,
   (j) Not all bullies are strong people.

6. Suggest a universal set for each of the following subsets
   (a) {squares}, (b) {the football team of your school},
   (c) {three sided figures}, (d) {eagles},
   (e) {odd numbers}, (f) {equilateral triangle}.

7. Let $\mathcal{U}$ = {1, 2, 3, …, 50}, A = {1, 2, 3, 4, 6, 8, 12, 24}, B = {factors of 12}, C = {factors of 24}, D = {factors of 72}. Rewrite the following in symbols.
   (a) 1, 2, 3, 4, 6, 8, 12 and 24 are all factors of 72,
   (b) All factors of 24 are also factors of 72,
   (c) Some factors of the factors of 72 are {1, 2, 3, 4, 6, 12, 24},
   (d) All factors of 12 are included in the list 1, 2, 3, 4, 6, 8, 12, 24.

8. Use a Venn diagram to illustrate the following statements:
   (a) All good Mathematics students are in the science class,
   (b) All bullies are strong people,
   (c) All university graduates are wise,
   (d) All pastors are compassionate,
   (e) All men use guns,
   (f) All students suffering from malaria go to the clinic,
   (g) All the good students of Mathematics are in the football team,
   (h) All students are hardworking.

9. Consider the following statements
   p: All scientists are introverts, *q*: All introverts are anti-social.
   Draw a Venn diagram to illustrate the above statements.

10. Look at these statements:
    s: All final year students are in the SRC, *t*: All SRC students are good students
    Represent the statements on a Venn diagram.

11. Consider the following statements:
    a: All my friends like Coca-cola, *b*: All who like Coca-cola are very studious.
    Draw a Venn diagram to illustrate the above statements.

1.2 Operations on sets

1.2.1 Intersection of Sets

The senior housemaster of Pentecost High School invited the school athletic team for a dinner at his residence. The team is made up of 8 sprinters and 5 hurdlers. He realised that there were 10 athletes in his residence. He checked and found that all the athletes were present. But $8 + 5 > 10$. Can you explain it?

The solution is much easier using a Venn diagram. We shall use S and H to denote the sets of students who are sprinters and hurdler respectively.

The information is illustrated in *Fig*. 1.3. If every member of the athletic team were present at the dinner, then it follows that 3 members of the team who are sprinters must also be hurdlers. The set of elements which are common to both S and N is called the ***intersection*** of S and N.

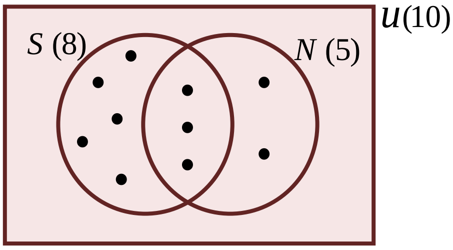
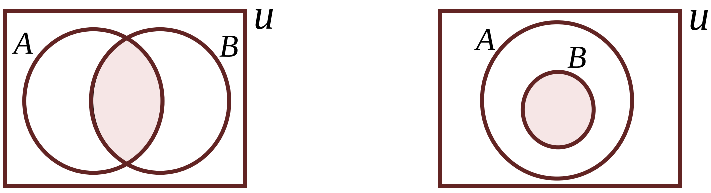


Fig. 1.3

> The intersection of two sets A and B is defined as the set of all elements that belong to both A and B.

The operation $\cap$ is used to define the intersection between two sets. Intersection of A and B is written as $A \cap B$. For example, if $A = \{1, 2, 3, 4, 5, 8\}$ and $B = \{2, 4, 6, 8, 10\}$, then $A \cap B = \{2, 4, 8\}$. The shaded regions Fig. 1.4, show the intersection between the sets A and B.



(a) (b)

Fig. 1.4: $A \cap B$ is shaded vertically

As illustrated in Fig. 1.4(b), if $B \subset A$, then $A \cap B = B$.

Example 1.9

Let $A = \{a, b, c, d, e\}$ and $B = \{b, c, f, g\}$

(a) Draw a Venn diagram of the two sets A and B. Show all the members of each set.

(b) Using this diagram, find the intersection of A and B.

Solution

(a) Fig. 1.5 shows the required Venn diagram.

(b) From the diagram $A \cap B = \{b, c\}$

Can you see another way of finding this intersection, without drawing the diagram?

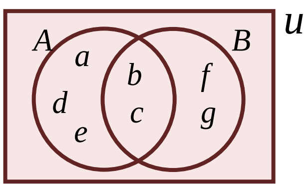


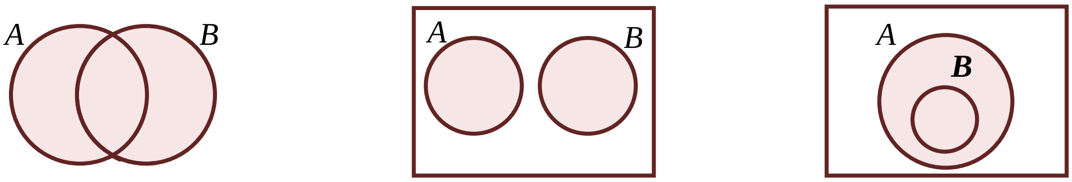
Fig. 1.5

1.2.2 Union of sets

The union of two sets A and B is defined as the set of all elements that belong to either A or B or both. The operation $\cup$ is used to define the union between two sets. The union of A

and B is written as $A \cup B$. For example, if A = {1, 2, 3, 4} and B = {2, 4, 6, 8}, then $A \cup B$ = {1, 2, 3, 4, 6, 8}

The union of A and B is shaded in Fig. 1.6.



(a) (b) (c)

Fig. 1.6: $A \cup B$ is shaded

It can be seen from *Fig*. 1.6(c) that if $B \subset A$, then $A \cup B = A$.
The Complement A' union of A shaded in Fig. 1.7.

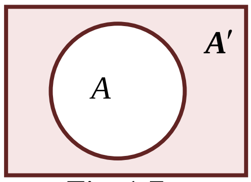


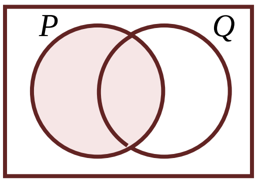
Fig. 1.7

Example 1.10

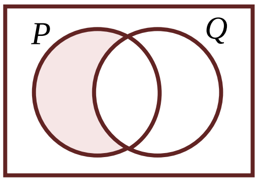
If P and Q are the subsets of a universal set $\mathcal{U}$, shade the sets:
(a) $P \cap (P \cup Q)$, (b) $P \cap Q'$, (c) $P' \cup Q$, (d) $(P' \cup Q)'$, (e) $P' \cap (P \cup Q)$.

Solution

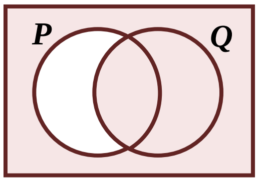
(a) $P \cap (P \cup Q)$ is shaded



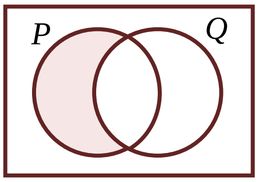
(b) $P \cap Q'$ is shaded



(b) $P' \cup Q$ is shaded



(d) $(P' \cup Q)'$ is shaded



(d) $P' \cap (P \cup Q)$ is shaded

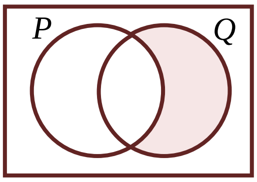


Fig. 1.8

Example 1.11

M and N are two intersecting sets. If $n(M) = 20$, $n(N) = 30$ and $n(M \cup N) = 40$, find $n(M \cap N)$. **Nov. 2002**

Solution

Let $n(M \cap N) = x$. From Fig. 1.9,

$(20-x)+x+(30-x)=40$

$50-x=40$

$x=50-40=10.$

$\therefore \quad n(M \cap N)=10.$

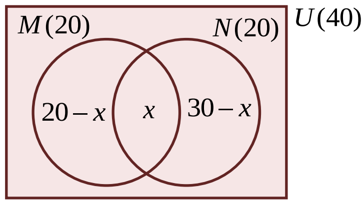


Fig. 1.9

Alternative approach

$n(M \cup N)=n(M)+n(N)-n(M \cap N)$

$40=20+30-n(M \cap N) \qquad \Rightarrow \qquad n(M \cap N)=20+30-40=10.$

Example 1.12

The set $A=\{...,-6,-4,-2,0,2,4,6,...\}$, $B=\{0 \le x \le 9\}$ and $C=\{x:-4<x \le 0\}$ are subsets of Z, the set of integers.

(a) (i) Describe the members of the set A', where A' is a complement of A.

(ii) Find $A' \cap B$.

(b) Represent the set B and C on a Venn diagram. **June 1997**

Solution

(a) $Z=\{...,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ ...\}$,

(b) $B=\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$,

$C=\{-3,\ -2,\ -1,\ 0\}$,

(i) $A'=\{...,\ -7,\ -5,\ -3,\ -1,\ 1,\ 3,\ 5,\ 7,...\}$.

A' is the set of odd numbers.

(ii) $A' \cap B=\{1,\ 3,\ 5,\ 7,\ 9\}$.

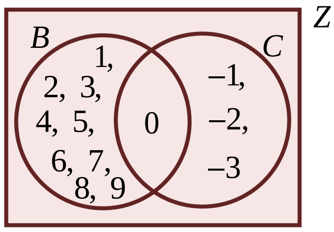


Fig. 1.10

Example 1.13

Fig. 1.11 shows the results of an interview of Methodist High School C = {students who like Chemistry} and P = {Students who like Physics}.

(a) How many students were interviewed?

(b) How many students like Chemistry?

(c) How many students like only one subject?

(d) How many students like Physics only?

(e) How many students like Physics and Chemistry?

(f) How many students like Physics or Chemistry?

(g) How many students like none of the two subjects?

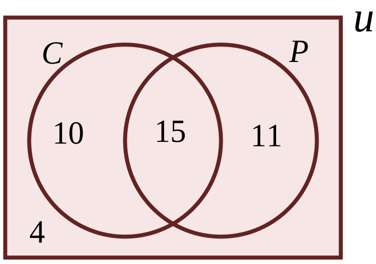


Fig. 1.11

Solution

(a) The number of students interviewed = 10 + 15 + 11 + 4 = 40
(b) The number of students who like Chemistry = 10 + 15 = 25
(c) The number of students who like only one subject = 10 + 11 = 21
(d) The number of students who like Physics only = 11
(e) In set operations the word 'and' implies intersection ($\cap$)
$\therefore$ The number of students who like Physics and Chemistry = $n(P \cap C)$ = 15
(f) In set operations the word 'or' implies union ($\cup$)
$\therefore$ The number of students who like Physics or Chemistry = $n(P \cup C)$
= 10 + 15 + 11 = 36
(g) The number of students who like none of the two subjects = 4.

Example 1.14

(a) The sets $P = \{2, 5\}$ and $Q = \{5, 7\}$ are subsets of the universal set $U = \{2, 3, 5, 7\}$. Find: (i) $(P \cap Q)'$, (ii) $P' \cup Q'$. State the relationship between (i) and (ii).
(b) In a class of 50 students, 30 offer Economics, 17 offer Government and 7 offer neither Economics nor Government. How many students offer both subject. **June 1994**

Solution

(a) $U = \{2, 3, 5, 7\}$, $P = \{2, 5\}$, $P' = \{3, 7\}$, $Q = \{5, 7\}$, $Q' = \{2, 3\}$.

(i) $P \cap Q = \{2, 5\} \cap \{5, 7) = \{5\} \Rightarrow (P \cap Q)' = \{2, 3, 7\}$.

(ii) $P' \cup Q' = \{3, 7\} \cup \{2, 3\} = \{2, 3, 7\}$.

It can be seen that $(P \cap Q)' = P' \cup Q'$.

(b) Let $\mathcal{U}$ = {students in the class},
E = {students who offer Economics},
G = {students who offer Government}.

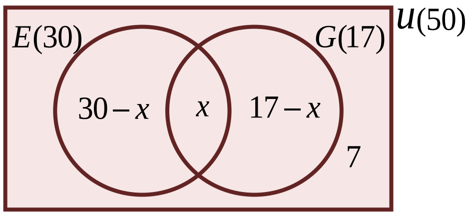


Fig. 1.12

Then $n(\mathcal{U}) = 50$, $n(E) = 30$ and $n(G) = 17$. Let x denote the number of students who offer both Economics and Government, that is $n(E \cap G) = x$. Since 7 students offer neither Economics nor Government, It follows that $n(E \cup G)' = 7$. The Venn diagram is as shown in Fig. 2.1. Notice that $(30 - x)$ students Economics only and $(17 - x)$ Government only.

$$n(\mathcal{U}) = (30 - x) + x + (17 - x) + 7 = 30 + 17 + 7 - x = 54 - x.$$

But $n(\mathcal{U}) = 50$. Hence,

$$54 - x = 50, \quad \text{which gives} \quad x = 54 - 50 = 4.$$

Thus, 4 students offer both Economics and Government.

Example 1.15

The set P = {multiples of 3}, Q = {factors of 72} and R = {even numbers} are the subset of $U = \{18 \le x \le 36\}$

(a) List the elements of P, Q and R.

(b) Find: (i) $P \cap Q$, (ii) $Q \cap R$, (iii) $P \cap R$.

(c) What is the relationship between $P \cap Q$ and $Q \cap R$. **June 2000**

Solution

$U = \{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$.

(a) $P = \{18, 21, 24, 27, 30, 33, 36\}$, $Q = \{18, 24, 36\}$,
$R = \{18, 20, 22, 24, 26, 28, 30, 32, 34, 36\}$.

(b) (i) $P \cap Q = \{18, 24, 36\}$, (ii) $Q \cap R = \{18, 24, 36\}$, (iii) $P \cap R = \{18, 24, 30, 36\}$.

(c) $P \cap Q = Q \cap R$.

Example 1.16

(a) If $P = \{1, 2, 3, 4\}$, write down all the subsets of P which have exactly two elements.

(b) A = {Prime numbers less than 15}, B = {Even numbers less than 15} and C = {x: $3 \le x < 12$, x is an integer} are subsets of U = {positive integers less than 15}. List the elements of (i) $A \cap C$, (ii) $B \cap C$, (iii) $(A \cup B)' \cap C$. **June 2001**

Solution

(a) The subsets of P with exactly two elements are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, and {3, 4}

(b) $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.
$A = \{2, 3, 5, 7, 11, 13\}$, $B = \{2, 4, 6, 8, 10, 12, 14\}$, $C = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$.
(i) $A \cap C = \{3, 5, 7, 11\}$, (ii) $B \cap C = \{4, 6, 8, 10\}$
(iii) $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
$(A \cup B)' = \{1, 9\} \Rightarrow (A \cup B)' \cap C = \{9\}$.

Example 1.17

A survey of the reading habits of 130 students showed that 30 read both Comics and Novels, 10 read neither Comics nor Novels and twice as many students read Comics as read Novels.

(a) How many students read Novels? (b) How many read Comics?

(c) How many read only Comics?

Solution

$\mathcal{U}$ = {students}, $n(\mathcal{U}) = 130$

C = {those who like Comics}, $n(C) = ?$

N = {those who like Novels}, $n(N) = ?$

$n(C \cup N)' = 10$

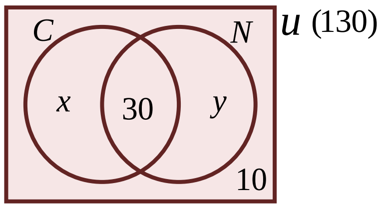


Fig. 1.13

Let x = the number of students who read Comics only
y = the number of students who read Novels only
$n(C) = 30 + x$ and $n(N) = 30 + y$
Twice as many students read Comics as read Novels $\Rightarrow n(C) = 2n(N)$
$30 + x = 2(30 + y) \Rightarrow 30 + x = 60 + 2y$
$x - 2y = 30$ ……………………………………………………………………………..(1)

Total number of students $n(\mathcal{U}) = 130$. Thus,
$30 + 10 + x + y = 130 \Rightarrow x + y = 130 - 40$
$x + y = 90$ ……………………………………………………………………………..(2)
$(2) - (1) \Rightarrow 3y = 60 \Rightarrow y = 20$
From (2) $x + 20 = 90 \Rightarrow x = 70$

(a) The number of students who read Novels $= y + 30 = 20 + 30 = 50$
(b) The number of students who read Comics $= x + 30 = 70 + 30 = 100$
(c) The number of students who read Comics only $= x = 70$

1.2.3 Set identities

1. *Commutative properties*

The union ($\cup$) *and the intersection* ($\cap$) *are both commutative*. It follows that, for any two sets A and B,

(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$.

2. *Associative properties*

The union ($\cup$) *and the intersection* ($\cap$) *are also associative*. For any three sets A, B and C,

(a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Properties*

The intersection ($\cap$) *is distributive over the union* ($\cup$). For three sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Also the union ($\cup$) *is distributive over the intersection* ($\cap$). That is

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. *If $\varnothing$ is an empty set then for every set $P \subset \mathcal{U}$, then*

(i) $P \cup P = P$, (ii) $P \cap P = P$, (iii) $P \cup \varnothing = P$

(iv) $P \cap \mathcal{U} = P$, (v) $P \cup \mathcal{U} = \mathcal{U}$, (vi) $P \cap \varnothing = \varnothing$

(vii) $P \cup P' = \mathcal{U}$ (viii) $P \cap P' = \varnothing$, (ix) $(A')' = A$

(x) $U' = \varnothing$ and $\varnothing' = \mathcal{U}$.

5. *Since $P \cap Q \subset P$ and $P \subset P \cup Q$, it follows that*:

(i) $P \cup (P \cap Q) = P$ (ii) $P \cap (P \cup Q) = P$

6. *De Morgan's Law*

(i) $(P \cup Q)' = P' \cap Q'$ (ii) $(P \cap Q)' = P' \cup Q'$

Exercise 1.2

1. P = {multiples of 3} and Q = {factors of 12} are subsets of the universal set $U = \{x: 1 \leq x \leq 12\}$.
   (a) Draw a Venn diagram to illustrate the above information.
   (b) List the elements of (i) $P \cap Q$ (ii) $P \cup Q$ (iii) $P \cup Q'$
   (iv) $P' \cap Q$ (v) $(P \cap Q)'$ (vi) $(P \cup Q)'$

2. In a group of 50 traders, 30 sell gari, and 40 sell rice. Each trader sells at least one of the two items. How many traders sell both gari and rice?

3. The sets P = {x: x is a prime factor of 42} and Q = {x: x is a factor of 24} are subsets of the U = {x: x is an integer}. List the elements of (a) $P \cap Q$, (b) $P \cup Q$.

4. The sets A = {x: x is an odd number}, B = {x: x is a factor of 60} and C = {x: x is a prime number} are the subsets of $\mathcal{U}$ = {x: x is a natural number and $x < 9$}. Find
   (a) $A \cap B$, (b) $B' \cap C$, (c) $A \cap B \cap C$, (d) $B \cup C$.

5. $A = \{10, 11, 12, 13, 14\}$ and $B = \{10, 12, 14, 16, 18\}$ are subsets of the universal set $\mathcal{U} = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$
   List the elements of: (a) $A' \cap B$, (b) $(A' \cap B)'$.

6. P, Q, and R are subsets of the universal set $\mathcal{U}$ = {$1 \leq x \leq 12$: x is an integer} and P = {x: x is a factor of 12}, Q = {x: x is a multiple of 3} and $R = \{x: x \geq 4\}$. Find:
   (a) $P \cup Q$ (b) $P' \cap R'$ (c) $Q \cap R$ (d) $(P \cap Q')'$ (e) $P' \cup Q$

7. Given that the universal set $\mathcal{U}$ = {x: $5x - 3 \leq 47$, where x is a natural number} and the subsets A, B, and C are defined as A = {prime numbers less than 10}, B = {odd numbers less than 10} and $C = \{x: 5 < x < 10\}$, find
   (a) $A \cap B$, (b) $A \cup C$, (c) $A' \cup B'$, (d) $(A \cap B)'$.

8. P, Q, and R are subsets of $\mathcal{U}$, where $\mathcal{U}$ = {x: $4x - 42 \leq 58 - 6x$, *x is a natural number*}, P = {prime factors of 36}, Q = {x: x is a factor of 15} and R = {multiple of 3 }.
   (a) find: (i) $Q \cap R$, (ii) $P \cup Q$, (iii) $P \cup R$,
   (iv) $Q \cup R$, (v) $P \cap Q$, (vi) $P \cap R$.
   (b) Show that (i) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$,
   (ii) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$,

9. The universal set $\mathcal{U}$ = {x: x is an integer and $0 < x \leq 20$} and P, Q and R are subsets of $\mathcal{U}$ such that P = {factors of 48} , Q = {multiples of 3} and R = {x: x is divisible by 4}. Find:
   (a) (i) $Q \cup R$, (ii) $Q \cap P$, (iii) $R \cap P$, (iv) $(Q \cap P)'$.
   (b) Show that (i) $(Q \cup R) \cap P = (Q \cap P) \cup (R \cap P)$,
   (ii) $(Q \cap P)' = Q' \cup P'$.

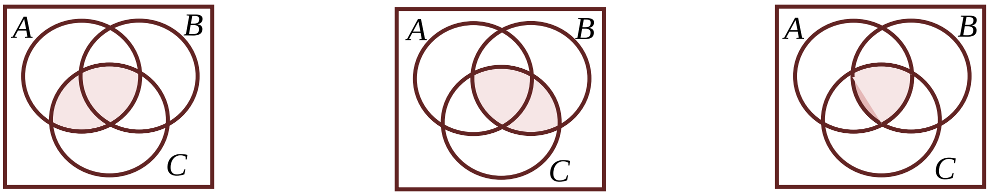
10. The universal set $\mathcal{U}$ = {x: x is a non-negative integer and $(7x - 5 \leq 15 + 5x)$} and P, Q and R are subsets of $\mathcal{U}$ such that P = {x: x is prime}, Q = {x: x is odd} and R = {x: x is a factor of 42}. Find: (a) $Q \cup R$, (b) $Q \cap P$, (c) $R \cap P'$, (d) $Q' \cup P$.

11. In a class of 42 students, 26 offer Mathematics and 28 offer Chemistry. If each student offers at least one of the two subjects, find the number of students who offer both subjects.

12. In a class of 42 students each student studies either Economics or Accounting or both. If 12 students study both subjects and the number of students who study Accounting only is twice that of those who study Economics only, find how many students study (i) Economics, (ii) Accounting.

1.3 Three set problems

Fig. 1.14 shows Venn diagrams of three intersecting sets A, B and C.



(a) $A \cap C$ is shaded (b) $B \cap C$ is shaded (c) $A \cap B \cap C$ is shaded

Fig. 1.14

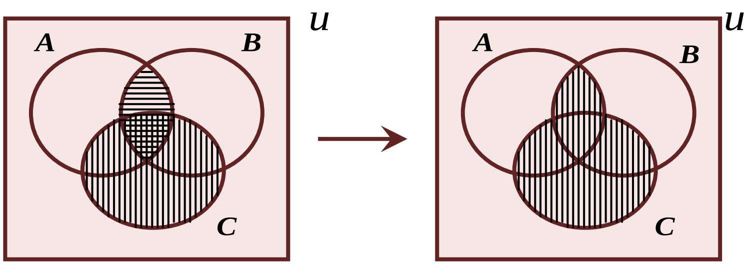
Example 1.18

If A, B and C are subsets of the universal set $\mathcal{U}$, shade the sets:

(a) $(A \cap B) \cup C$, (b) $(A \cup C) \cap (B \cup C)$, (c) $(A \cap B) \cup C'$, (d) $A' \cap (B \cup C)$.

Solution

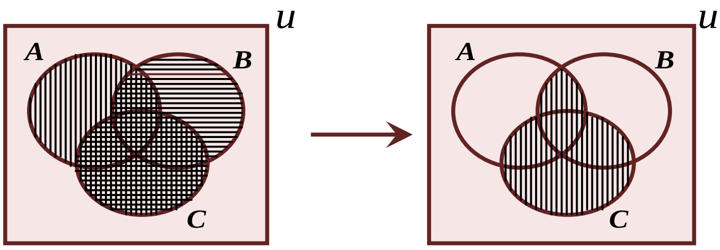
(a)



$A \cap B$ is shaded horizontally
C is shaded vertically.

$(A \cap B) \cup C$ is shaded vertically

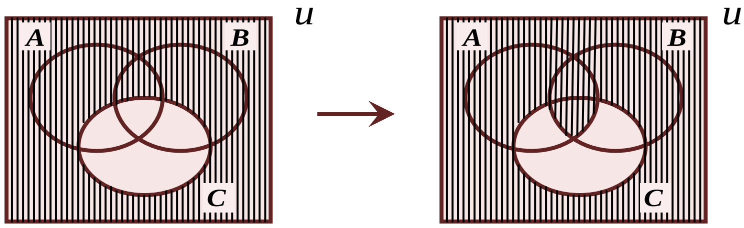
(b)



$A \cup C$ is shaded vertically
$B \cup C$ is shaded horizontally

$(A \cup C) \cap (B \cup C)$ is shaded vertically

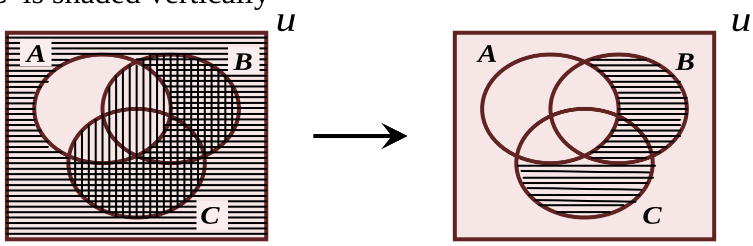
(c)



$A \cap B$ is shaded horizontally
C' is shaded vertically

$(A \cap B) \cup C'$ is shaded vertically

(d)



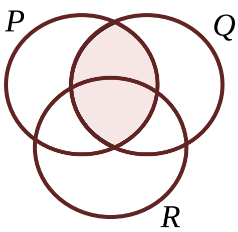
A' is shaded horizontally
$B \cup C$ is shaded vertically

$A' \cap (B \cup C)$ is shaded horizontally

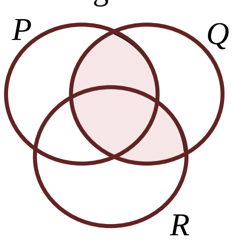
Example 1.19

Describe the shaded regions in the Venn diagrams below using P, Q and R.

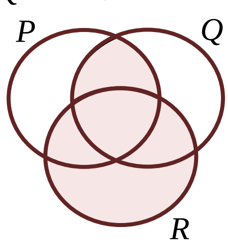
(a)



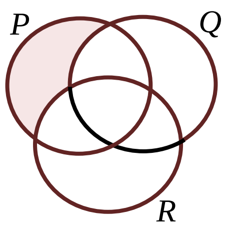
(b)



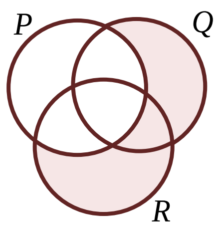
(c)



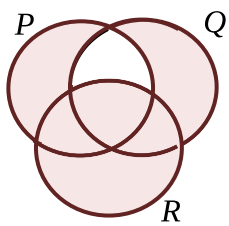
(d)



(e)



(f)



Solution

(a) $P \cap Q$

(b) $(P \cap Q) \cup (R \cap Q)$ or $Q \cap (P \cup R)$

(c) $R \cup (P \cap Q)$ or $(R \cup P) \cap (R \cup Q)$

(d) $P \cap Q' \cap R'$ or $P \cap (Q \cup R)'$

(e) $(R \cup Q) \cap P'$

(f) $R \cup Q' \cup P'$ or $R \cup (Q \cap P)'$.

Example 1.20

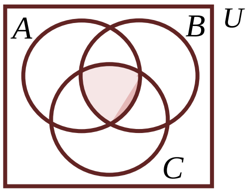
In a certain class, each student offers at least one of the following subjects: Accounting, Business management (BM) and Commerce. Represent by shading, on a Venn diagram, the region that represent the number of students who offer the following:

(1) All three subject,
(2) Accounting and BM only,
(3) Accounting and BM,
(4) Commerce only,
(5) Accounting or BM only,
(6) Accounting or BM.

Solution

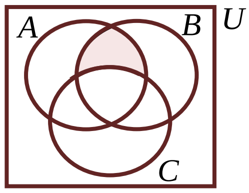
Let A, B and C denote Accounting, Business management (BM) and Commerce respectively. The required shaded regions are as shown in the Venn diagrams below.

(1)



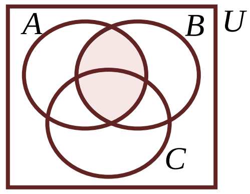
All three subjects
$A \cap B \cap C$

(2)



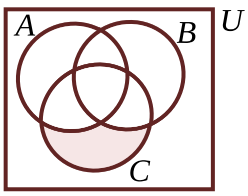
Accounting and BM only
$A \cap B \cap C'$

(3)



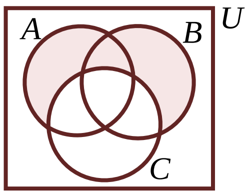
Accounting and BM
$A \cap B$

(4)



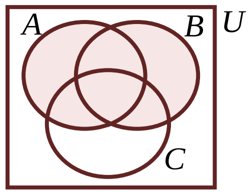
Commerce only
$C \cap A' \cap B'$ or
$C \cap (A \cup B)'$

(5)



Accounting or BM only
$A \cup B \cap C'$

(6)



Accounting or BM
$A \cup B$

Example 1.21

Fig. 1.15, on the next page, shows the result of interviewing some students in a certain school to ask which channels they watch on television. G = {students who watch GTV), T = {students who watch TV3} and M = {students who watch Metro TV}

(a) How many students were interviewed?
(b) How many students watch TV3?
(c) How many students watch Metro TV only?
(d) How many students watch Metro TV and GTV?
(e) How many students watch Metro TV and GTV only?
(f) How many students watch only one channel?
(g) How many students watch only two channels?
(h) How many students watch all three channels?
(i) How many students watch at least two channels?
(j) How many students watch Metro TV or GTV but not TV3?
(k) How many students watch Metro TV or GTV?

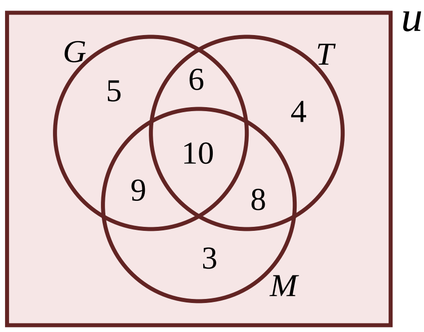


Fig. 1.15

Solution

(a) The number of students interviewed = 5 + 4 + 3 + 6 + 8 + 9 + 10 = 45
(b) The number of students who watch TV3 = 6 + 10 + 8 + 4 = 28
(c) The number of students who watch Metro TV only = 3
(d) The number of students who watch Metro TV and GTV = 9 + 10 = 19
(e) The number of students who watch Metro TV and GTV only = 9
(f) The number of students who watch only one channel = 5 + 4 + 3 = 12
(g) The number of students who watch only two channels = 6 + 8 + 9 = 23
(h) The number of students who watch all three channels = 10
(i) The number of students who watch at least two channels = 6 + 9 + 8 + 10 =33
(j) The number of students who watch Metro TV or GTV but not TV.3 = 3 + 9 + 5
= 17
(k) The number of students who watch Metro TV or GTV = 5 + 6 + 10 + 9 + 8 + 3
= 41

Example 1.22

Some students were interviewed to find out which of the following three sports they liked: football, boxing and volleyball. 70% of the students liked football, 60% boxing and 45% volleyball, 45% liked football and boxing, 15% boxing and volleyball, 25% football and volleyball and 5% liked all three sports.

(a) Draw a Venn diagram to illustrate this information.
(b) Use your diagram to find the percentage of students who liked
  (i) football or boxing but not volleyball, (ii) exactly two sports,
  (iii) none of the three sports. **Nov. 2001**

Solution

(a) Let U = {students interviewed},
F = {students who liked football},
B = {students who liked boxing} and
V = {students who liked volleyball}.

Then $n(U)$ = 100%, $n(F)$ = 70%, $n(B)$ = 60% and $n(V)$ = 45%. $F \cap B \cap V = 5\%$.

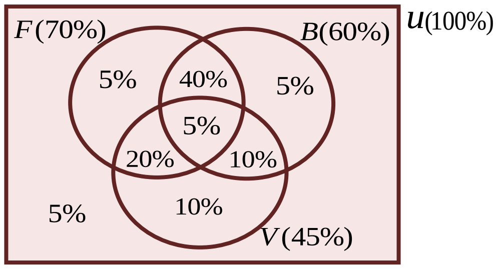


Fig. 1.16

Percentage of students who liked football only $= 70\% - (40\% + 5\% + 20\%) = 5\%$.
Percentage of students who liked boxing only $= 60 - (40\% + 5\% + 10\%) = 5\%$.
Percentage of students who liked volleyball only
$= 45 - (20\% + 5\% + 10\%) = 10\%$.

(b) (i) The percentage of students who liked football or boxing but not volleyball
= 5% + 40% + 5% = 50%

(ii) The percentage of students who liked exactly two sports
= 40% + 10% + 20% = 70%

(iii)The percentage of students who liked none of the three sports
= 100% – (5% + 40% + 5% + 20% + 5% + 10% + 10%) = 5%.

Example 1.23

In a Senior Secondary School there are 174 students in form two. Of these, 86 play table tennis, 84 play football and 94 play volleyball; 30 play table tennis and volleyball, 34 play volleyball and football and 42 play table tennis and football. Each student plays at least one of the three games and x students play all three games.

(a) Illustrate this information on a Venn diagram.
(b) Write down an equation in x and hence solve for x.
(c) If a student is chosen at random from form two, what is the probability that he plays two games? **June 1993**

Solution

(a) Let U = {students in form 2},
T = {students who Play Table tennis},
F = {students who play Football} and
V = {students who play volleyball}.

Then $n(U)$ = 174, $n(T)$ = 86, $n(F)$ = 84, $n(V)$ = 94, $n(T \cap V) = 30$, $n(V \cap F) = 34$ and $n(T \cap F) = 42$. If x is the number of students who play all three games, then $T \cap F \cap V = x$.

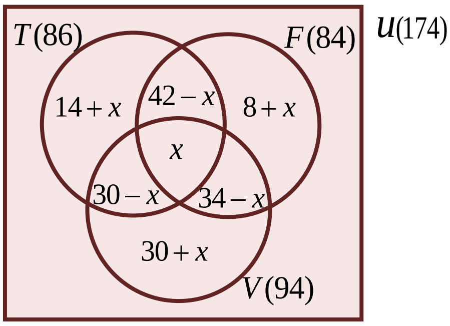


Fig. 1.17

Number of students who play Table tennis only $= 86-(42-x+x+30-x)$
$= 86-(72-x) = 14+x.$

Number of students who play Football only $= 84-(42-x+x+34-x)$
$= 84-(76-x) = 8+x.$

Number of students who play Volleyball only $= 94-(30-x+x+34-x)$
$= 94-(64-x) = 30+x.$

The Venn diagram is as shown in Fig. 1.1.

(b) From Fig. 1.1, the required equation in x can be written as

$$n(T)+(8+x)+(34-x)+(30+x) = n(u)$$

$86+(8+x)+(34-x)+(30+x) = 174,$ which simplifies to

$158+x = 174$ which given $x=174-158=16.$

(c) The number of students who play exactly two games $= (42-16)+(34-16)+(30-16)$
$= 26 + 18 + 14 = 58.$

The probability a student selected at random plays two games $= \frac{58}{174} = \frac{1}{3}.$

Example 1.24

In a class of 32 students, 18 offer Chemistry, 16 offer Physics and 22 offer Mathematics. 6 offer all three subjects, 3 offer Chemistry and Physics only and 5 study Physics only. Each student offers at least one subject. Find the number of students who offer: (a) Chemistry only, (b) only one subject, (c) only two subject. **June 1995**

Solution

(a) Let U = {students in the class},
C = {students who offer Chemistry},
P = {students who offer Physics} and
M ={students who offer Mathematics}.

Then $n(U)$ = 32, $n(C)$ = 18, $n(P)$ = 16 and $n(M)$ = 22. $C\cap P\cap M = 6$. Let x denote the number of students who offer Chemistry and Mathematics only. Fig. 1.18 is the Venn diagram illustrating the given information.

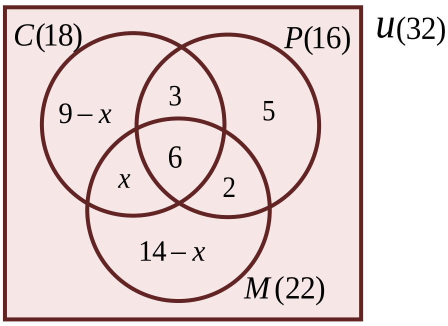


Fig. 1.18

Number of students who offer Physics only = 16 – (5 + 3 + 6) = 2.

Number of students who offer Chemistry only $=18-(3+6+x) = 9-x.$

Number of students who offer Mathematics only $= 22-(6+2+x)=14-x.$

From Fig. 1.1, the required equation in x can be written as

$$n(C)+5+2+(14-x) = n(u)$$

$$18+5+2+14-x = 32 \Rightarrow 39-x = 32 \Rightarrow x = 39-32 = 7.$$

(a) Number of students who offer Chemistry only $= 9-x = 9-7 = 2$.
(b) Number of students who offer only one subject $= (9-x)+5+(14-x)$
$= 2+5+7 = 14$.
(c) Number of students who offer only two subjects $= 3+2+x = 3+2+7 = 12$.

Example 1.25

100 members of a community were asked to state the activities they undertake during the day.

38 go to School. 18 go to School and also Trade.
54 go for Fishing. 22 go to Fishing and also Trade.
50 engage in Trading.

Each of these members undertakes at least one of the activities. The number of people who go to school only is the same as the number who engages in Trading only. Use the information to find the number of people who
(a) undertake all the three activities, (b) go to school only. **June 2003**

Solution

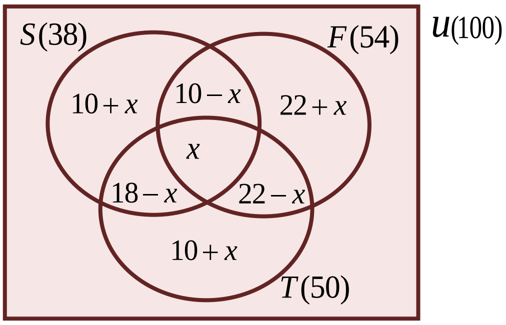


Fig. 1.19

(a) Let U = {members of the community},
S = {members who go to School},
F = {members who go to Fishing} and
T ={members who engage in Trading}.

Then $n(U) = 100$, $n(S) = 38$, $n(F) = 54$ and $n(T) = 50$. Let $n(C \cap P \cap M) = x$. Fig. 1.19 is the Venn diagram illustrating the given information.

Number of member who Trade only $= 50 - (18 - x + x + 22 - x) = 10 + x$.

Since the number of people who go to school only is the same as the number who engages in Trading only, it follows that

Number of members who go to school only $= 10 + x$.
Number of members who go to school and also Fish
$= 38 - (18 - x + x + 10 + x) = 10 - x$.
Number of member who go to Fishing only $= 54 - (10 - x + x + 22 - x)$
$= 22 + x$.

From Fig. 13.1,

$$n(T) + (10 + x) + (10 - x) + (22 + x) = n(u)$$

$$50 + (10 + x) + (10 - x) + (22 + x) = 100$$

$$92 + x = 100 \quad \Rightarrow \quad x = 100 - 92 = 8.$$

Thus, the number of members who undertake all the three activities is 8.

(b) The number of members who go to school only $= 10 + x = 10 + 8 = 18$.

Example 1.26

In a class of 60 students, some study at least one of the following subjects: Mathematics, Economics and Accounting. 8 students study none of them. The following table gives further details of the subjects studied.

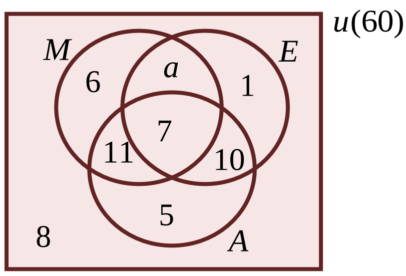
Mathematics only	6	All three subjects	7
Economics only	1	Mathematics & Accounting	18
Accounting only	5	Economics & Accounting	17

(a) Illustrate the above data on a Venn diagram.

(b) Find the number of students who study:

(i) Mathematics or Accounting or both but not Economics, (ii) Economics.

Solution



$\mathcal{U}$ = {students in the class} $\Rightarrow n(\mathcal{U}) = 60$

M = {students who study Mathematics}

E = {students who study Economics}

A = {students who study Accounts}

$n(A) = 7 + 10 + 5 + 11 = 33$

$n(\mathcal{U}) = n(A) + 6 + 1 + 8 + a$

$60 = 33 + 15 + a$

$a = 60 - 48 = 12$

(b) (i) The number of students who study Mathematics or Accounting or both but not Economics = 6 + 11 + 5 = 22

(ii) The number of students who study Economics $= 10 + 7 + 1 + a$

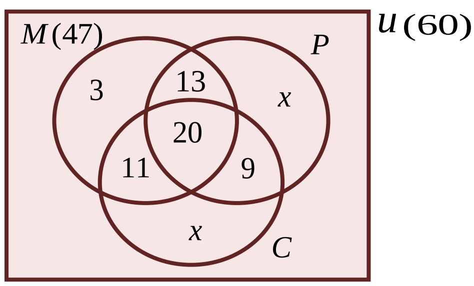
$= 18 + 12 = 30$.

Example 1.27

In a class of 60 students, 47 study Mathematics, 33 study Mathematics and Physics, 31 study Mathematics and Chemistry, 29 study Physics and Chemistry and 20 study all the three subjects. If the number of students who study only Physics is equal to that of those who study only Chemistry, Illustrate the given information on a Venn diagram and find the number of students who study

(i) Only Physics, (ii) Chemistry, (iii) Only one subject.

Solution



$\mathcal{U}$ = {students in the class} $\Rightarrow n(\mathcal{U}) = 60$

M = {those who study Mathematics} $\Rightarrow n(M) = 47$

P = {those who study Physics}

C = {those who study Chemistry}

Let the number of students who study Physics only $= x$

$\Rightarrow$ the number of students who study
Chemistry only = x
Number of students who study Mathematics only = $47 - (13 + 20 + 11) = 3$

Note: $n(M) = 13 + 20 + 11 + 3 = 47$

Total number of students in the class $n(\mathcal{U}) = 60$

$$n(M) + 9 + x + x = 60$$
$$\Rightarrow \quad 47 + 9 + 2x = 60$$
$$2x = 60 - 56 \quad \Rightarrow \quad 2x = 4 \quad \Rightarrow \quad x = 2$$

(i) The number of students who study only Physics $= x = 2$
(ii) The number of students who study Chemistry $= 11 + 20 + 9 + x = 42$
(iii)The number of students who study only one subject $= 3 + x + x = 7$

Exercise 1.3

1. Mathematics, English and Life Skills books were distributed to 50 students in a class. 22 had Mathematics books, 21 English books and 25 Life Skills books, 7 had Mathematics and English books, 6 Mathematics and Life Skill books and 9 English and Life Skill books. Find the number of students who had: (a) all three books, (b) exactly two of the books, (c) only Life Skills books. **June 1996.**

2. There are 30 students in a class. 20 of them play football, 16 play hockey and 16 play volley, 9 play all three games, 15 play football and volley, 11 play football and hockey, while 10 play hockey and volley.
   (a) Illustrate the information on a Venn diagram.
   (b) Using your Venn diagram, find the number of students who play at least two games.
   (c) What is the probability that a student chosen at random from the class does not play any of the three games? **Nov. 2003.**

3. The set A, B, and C are defined as $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20, 25\}$.
   (a) Draw a Venn diagram to illustrate the above information.
   (b) Find: (i) $B \cap C$, (ii) $(A \cup B)' \cap C$, (iii) the number of elements in $A \cup B$. **Nov. 2004.**

4. The set $P = \{n: 10 < n < 20\}$, where n is an integer. The set Q is a subset of P such that $Q = \{n: 55 - 2n \geq 25\}$. Find Q. **Nov. 2005.**

5. A survey of 150 traders in a market shows that 90 of them sell cassava, 70 sell maize and 80 sell yam. Also, 26 sell cassava and maize, 30 sell cassava and yam and 40 sell yam and maize. Each of the traders sells at least one of these crops.
   (a) Represent the information on a Venn diagram.
   (b) Find the number of traders who sell all the three food crops.

(c) How many of the traders sell one food crop only? **June 2006**.

6. There are 100 boys in a sports club. 65 of them play soccer, 50 play hockey and 40 play basketball. 25 of them play soccer and hockey, 20 play hockey and basketball and 5 play all three games. Each boy plays at least one of the three games.
   (a) Draw a Venn diagram to illustrate this information.
   (b) Find the number of boys who play:
   (i) soccer only, (ii) basketball only, (iii) exactly two games. **Nov. 2006**.

7. Given that A, B, C are subsets of the universal set U of real numbers such that $A = \{1, 2, \ldots, 16\}$, $B = \{x: 0 < x < 16$, where x is an odd $\}$, $C = \{p: p < 16$, where p is prime$\}$
   (a) List all the elements of B, (b) Find $B \cap C$, (c) Find $(A \cap B)'$.

8. In a class of 52 students, 34 offer Mathematics, 31 offer Chemistry and 36 offer Physics. 5 offer all the three subjects, 15 Physics and Chemistry only, 2 offer Physics only. Each student offers at least one of the three subjects. Illustrate the information on a Venn diagram. Find the number of students who offer:
   (a) Chemistry only, (b) only one subject, (c) only two subjects.

9. There are 22 players in a football team. 9 play defence, 10 play midfield and 11 play attack. 5 play defence only, 4 play midfield only and 6 play attack only.
   (a) Represent this information on a Venn diagram
   (b) How many play all the three positions?

10. In an athletic team, there are 20 sprinters, 12 hurdlers and 10 pole-vaulters. 12 are sprinters only, 4 are hurdlers only, 5 are pole-vaulters only and 2 are sprinters and pole-vaulters only. Each athlete does at least one of the three. Find the number of athletes in the team.

11. In a class of 80 students, 40 study Physics, 48 study Mathematics and 44 study Chemistry. 20 study Physics and Mathematics, 24 study Physics and Chemistry and 32 study only two of the three subjects. If every student studies at least one of the three subjects,
    find: (a) the number of students who study all the three subjects,
    (b) the number of students who study only Mathematics and Chemistry.

12. 56 teachers in Methodist High School were asked their preferences for three FM stations in Accra, Joy, Peace and Unique. 20 liked Unique, 8 liked Joy and Unique, and 2 liked Peace and Unique only. 6 liked Peace only, 24 liked Joy only and each teacher liked at least one of the three stations. If the number of teachers who liked Unique only was double that of those who preferred all the three stations, illustrate this information on a Venn diagram. Find the number of teachers who liked:
    (a) Joy, (b) Joy and Peace.

13. In a class of 70 students, 45 offer Mathematics, 37 offer Chemistry and 43 offer Physics. 5 offer all the three subjects, 20 offer Physics and Chemistry only, 3 offer Physics only. Each student offers at least one of the three subjects. Illustrate the information in a Venn diagram. Find the number of students who offer:
    (i) Chemistry only, (ii) only one subject, (iii) only two subjects.

14. There are 65 pupils in a class. 29 of them do Arts, 37 do Business and 38 do Science. All the students do at least one of the three programs. 10 do all the three programs while 18 do Arts and Business. 7 do Business and Science but not Arts and 14 do Arts and Science. Represent this information on a Venn diagram
    (i) How many pupils do only two subjects
    (ii) If a pupil is selected at random, what is the probability that he studies either Arts or Science?

15. There are 40 players in Presec football team. 22 play defence, 5 play midfield and defence, 8 play defence and attack, 5 play midfield and attack and 3 play all the three positions. If the number of students who play only midfield is equal to that of those who play only attack, represent this information on a Venn diagram. How many play: (a) only midfield, (b) attack, (c) only one position.

16. In a class of 54 students, 22 offer Mathematics, 27 offer Chemistry and 26 offer Physics. 4 offer all the three subjects, 5 offer Physics and Chemistry only, 15 offer Physics only. Each student offers at least one of the three subjects. Illustrate the information on a Venn diagram. Find the number of students who offer:
    (i) Chemistry only, (ii) only one subject, (iii) only two subjects.

17. In an athletic team, there are 16 sprinters, 16 hurdlers and 15 pole-vaulters. 6 are sprinters only, 4 are hurdlers only, 1 is a pole-vaulter only and 5 are sprinters and pole-vaulters only. Each athlete does at least one of the three. Find:
    (a) the number of athletes in the team,
    (b) the probability of selecting from the team an athlete who does only one event.

18. A class of 49 boys were each required to have certain textbooks in English, French and Mathematics. 28 boys had the English book, 24 had the French book and 26 the Mathematics book. 10 boys had both English and French books, 11 had French and Mathematics books 14 had the Mathematics and English books. Illustrate the information in a Venn diagram.
    How many boys in the class possessed:
    (i) all the three books, (ii) one book only, (iii) English and French only.

19. In a class of 36 students, 25 study Chemistry, 22 study Mathematics and 25 study Physics. 17 study Physics and Mathematics, 18 study Physics and Chemistry and 15 study only one of the three subjects. If every student studies at least one of the three subjects, find:
    (a) the number of students who study all the three subjects,

(b) the number of students who study only Mathematics and Chemistry,
(c) the probability that a student selected at random studies only two of the three subjects.

20. In a class, 39 study Physics, 35 study Chemistry and 33 study Biology. 13 study Chemistry and Biology, 12 study Chemistry only, 9 study Biology only and 34 study only one of the three subjects. If 12 students study none of the three subjects,
find: (a) (i) total number of students in the class;
(ii) the number of students who study all the three subjects,
(b) If a student is selected at random, what is the probability that he studies either all the three subjects nor none of the three?

21. There are 40 pupils in a class. 30 of them study Biology, 22 study Physics and 21 study Chemistry. 15 study Physics and Biology, 10 study Physics and Chemistry, and 13 study Biology and Chemistry. Each student in the class studies at least one of the three subjects.
(a) Represent this information on a Venn diagram.
(b) How many pupils study all three subjects?
(c) If a pupil is selected at random, what is the probability that he studies either Physics or Chemistry?

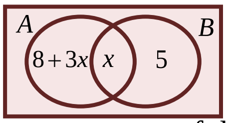
Revision Exercises 1

1. The sets $P = \{x: x$ is a prime factor of $30\}$ and $Q = \{x: x$ is a factor of $36\}$ are subsets of $\mathcal{U} = \{x: x$ is an integer$\}$. List the elements of (a) $P \cap Q$, (b) $P \cup Q$.
2. The sets $A = \{x: x$ is an even number$\}$, $B = \{x: x$ is a factor of $42\}$ and $C = \{x: x$ is a prime number$\}$ are the subsets of $\mathcal{U} = \{x: x$ is a natural number and $x < 10\}$. Find:
(a) $A \cap B$, (b) $B' \cap C$, (b) $A \cap B \cap C$, (c) $B \cup C$.
3. The universal set $\mathcal{U} = \{5, 7, 11, 15\}$, $P = \{5, 11\}$ and $Q = \{11, 15\}$. Find:
(a) $P \cap Q$, (b) $P' \cup Q'$. State the relation between (a) and (b).
4. $A = \{2, 3, 4, 5, 7\}$ and $B = \{2, 3, 4, 7, 10, 19\}$ are subsets of the universal set $\mathcal{U} = \{2, 3, 4, 5, 7, 10, 13, 19, 37\}$. List the elements of: (a) $A' \cap B$ (b) $(A' \cap B)'$.
5. P, Q, and R are subsets of the universal set $\mathcal{U} = \{1 \le x \le 10$: x is an integer$\}$ and $P = \{x: x$ is a factor of $20\}$, $Q = \{x: x$ is a multiple of $5\}$ and $R = \{x: x \ge 5\}$. Find:
(a) $P \cup Q$, (b) $P' \cap R'$, (c) $Q \cap R$, (d) $(P \cap Q')'$, (e) $P' \cup Q$.
6. Given that the universal set $\mathcal{U} = \{x: x \le 15$, where x is a natural number$\}$ and the subsets A, B, and C are defined as $A = \{$prime numbers less than 15$\}$, $B = \{$odd numbers less than 15$\}$ and $C = \{x: 4 < x < 15\}$, find
(a) $A \cap B$, (b) $A \cup C$, (c) $A' \cup B'$, (d) $(A \cap B)'$.

7. P, Q, and R are subsets of $\mathcal{U}$, where $\mathcal{U} = \{x: x \leq 10, x$ is a natural number$\}$, $P = \{$prime factors of 42$\}$, $Q = \{x: x$ is a factor of 9$\}$ and $R = \{$multiples of 3 less than 9$\}$.
(a) Find: (i) $Q \cap R$, (ii) $P \cup Q$, (iii) $P \cup R$, (iv) $Q \cup R$, (v) $P \cap Q$, (vi) $P \cap R$.
(b) Show that (i) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
(ii) $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$
(iii) $(P \cup Q)' = P' \cap Q'$.

8. In a class of 40 students, 23 offer Biology and 27 offer Chemistry. Each student offers at least one of the two subjects. How many students offer both subjects?

9. In the diagram $n(A) = 2n(B)$. Find the value of x.



10. Find the number of possible subsets of $A = \{2, 3, 4, 5, 6\}$.

11. A recent survey of 50 students revealed that the number studying one or more of the three subjects Mathematics, English and Integrated Science is as follows:

Subject	***Number of Students***
Mathematics	25
English	21
Integrated Science	24
English & Mathematics	7
Mathematics & Integrated Science	8
Only Two Subjects	20

Find: (a) the number of students who study all three subjects,
(b) the number of students who study two or three subjects,

12. In a school, 27 students were asked their preferences for three brands of soft drinks: Fanta, Coca-Cola, and Sprite. 15 liked Sprite,16 liked Fanta and 5 liked all the three. 12 preferred Coca-Cola and Fanta, 6 preferred Coca-Cola and Sprite and 6 preferred Sprite only. Illustrate the information on a Venn diagram.
Find how many students liked:
(a) Coca-Cola, (b) Fanta or Sprite but not Coca-Cola,
(c) Fanta and Sprite but not Coca-Cola, (d) only one brand, (e) only two brands.

13. In a group of 59 traders, 26 sell gari, 8 sell only rice, and 15 sell only maize. 10 sell both gari and rice, 16 sell rice and maize, and 42 sell maize. Each trader sells at least one of the three items. Find the number of traders who sell:
(i) gari or maize, (ii) gari and maize, (iii) only two items.

14. There are 28 pupils in a class. 3 do all the three programs while 5 do Arts and Business, and 7 do Arts and Science. 5 do Business only and all the students do at least one of the three programs. If the number of pupils who do Business is twice that of those who do Arts and the number of pupils who do Business is equal to that of those who do Science, represent this information on a Venn diagram.

Find the number of pupils who do (a) only two programs, (b) Science,
(c) Arts only, (d) Business and Science only.

15. There are 28 players in the national football team. 14 play midfield and defence, 15 play defence and attack and 3 play midfield only. The number of players who play attack only is twice that of those who play defence only, and the number who play defence is equal to that of those who play attack. If 18 play midfield, represent this information on a Venn diagram. How many players play:
(a) defence, (b) attack and midfield, (c) only one position.

16. In a class of 10 students, 4 offer Mathematics, and 1 offers Chemistry and Mathematics. 1 offers Physics and Chemistry, and 3 offer Physics and Mathematics. Each student offers at least one of the three subjects. If the number of students who offer Mathematics is equal to that of those who offer Physics only and $n(M) + n(C) = n(P)$, Illustrate the information in a Venn diagram. Find the number of students who offer:
(a) Chemistry only, (b) only one subject, (c) only two subjects,
(d) Physics, (e) Chemistry, (f) Chemistry and Physics only.

17. 30 teachers were asked their preferences for three newspapers, Graphic, Times and Chronicle. 20 liked Graphic, 6 liked Graphic and Chronicle, and 4 liked Times and Chronicle. 5 liked Times only, and 4 liked Graphic and Times Only. All teachers liked at least one of the three papers. If the number of teachers who liked all the three newspapers was 3 times that of those who preferred Times and Chronicle only, and the number of teachers who liked Times exceed those who preferred Chronicle by 2, illustrate this information on a Venn diagram.
Find the number of teachers who liked:
(a) Times, (b) Chronicle, (c) Chronicle only,
(d) Graphic and Chronicle only.

18. In a class, each student was required to have certain textbooks in French, History and Geography. 24 boys had the History book, 27 had the French book and 30 the Geography book. 11 boys had both History and French books, 8 had History and Geography books only and 12 had the French and Geography books. If 5 had all the three, Illustrate the information in a Venn diagram.
(a) Find the total number of students in the class.
(b) How many boys in the class possessed:
(i) only one of the three books, (ii) exactly two books.

19. In an examination each of the 35 students sat for Biology, Chemistry and Physics. 21 passed Biology, 8 passed Chemistry and Physics only, and 5 passed Biology and Physics only. 7 passed Biology and Chemistry only and 20 passed Chemistry. The number of students who passed Chemistry is equal to the number that passed Physics. If all the

students passed at least one of the three subjects, illustrate this information on a Venn diagram.

Find the number of students who passed in

(a) Physics, (b) Physics and Chemistry, (c) Physics or Chemistry.

12. In a class, 11 study Physics, 15 study Chemistry and 16 study Biology. 7 study Chemistry and Biology, 8 study Chemistry only, 3 study Biology only and 16 study only one of the three subjects. If 6 students study none of the three subjects,

find: (a) total number of students in the class,

(b) the number of students who study all the three subjects.

21. A recent survey of 160 students revealed that the number studying one or more of the three subjects Mathematics, English and Integrated Science is as follows:

Subject	*Number of Students*
Integrated Science	100
English	70
Mathematics	70
Integrated Science only	40
English & Mathematics only	10
Mathematics & Integrated Science	40
Only One Subjects	90

Find: (a) the number of students who study all three subjects,

(b) the number of students who study only two subjects.

22. 400 students in a Senior High School were asked to indicate which of the hobbies, reading, dancing and singing they liked. The results revealed that:

Hobbies	*Number of SSS1 Students*
Reading	200
Dancing	160
Singing	175
Reading only	75
Reading & Dancing	75
Dancing & Singing	65
Reading & Singing	80

Find how many students liked

(a) none of the three hobbies, (b) all the three hobbies,

(c) only one hobby, (d) Reading or Singing but not Dancing.

CHAPTER TWO

Real Number System

In Junior High School, you studied about natural numbers, whole numbers and integers. ***Natural numbers*** were defined as the basic set of numbers. The set of natural numbers is $N = \{1, 2, 3, 4, 5, \ldots\}$. This set is also called the set of ***counting numbers***. If we add zero to this set, we obtain the set $W = \{0, 1, 2, 3, 4, 5, \ldots\}$. This set is called the set of ***whole numbers***. It follows that $N \subset W$, and so $N \cap W = N$ and $N \cup W = W$. *Any number (negative or positive) without a fraction is called an* ***integer***. The set $\{\ldots -4, \; -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$ of integers is denoted by $\boldsymbol{Z}$. Notice that $\boldsymbol{W} \subset \boldsymbol{Z}$.

2.1 Rational and irrational numbers

2.1.1 Rational numbers

At the Junior High School level, you learnt that a ***rational number*** is any number that can be expressed as a quotient (fraction) of two integers. If r is a rational number, then there are two integers a and b such that $r = \frac{a}{b}$, $b \neq 0$. Rational numbers are recognized to be ***fractions***. The numbers $\frac{2}{3}$, $\frac{5}{7}$, and $-\frac{8}{5}$ are examples of rational numbers. Integers can be recognized as rational numbers since every integer can be expressed as a quotient of two numbers $\left(\text{For example: } 2 = \frac{6}{3}, \; 3 = \frac{12}{4}, \; 4 = \frac{24}{6}, \text{ etc}\right)$. However not every rational number can be said to be an integer. It therefore follows that the set $\boldsymbol{Z}$ of integers is a subset of the set $\boldsymbol{Q}$ of rational numbers (i.e. $\boldsymbol{Z} \subset \boldsymbol{Q}$).

Types of fraction

Proper Fraction: In a proper fraction the numerator is less than the denominator, as in $\frac{2}{3}$, $\frac{1}{2}$, $\frac{4}{9}$ etc.

Improper Fraction: In an improper fraction the numerator is greater than the denominator, as in $\frac{5}{3}$, $\frac{11}{8}$, $\frac{12}{5}$ etc.

Mixed Fraction: A mixed fraction contains both a whole number and a fractional part, as in $2\frac{2}{3}$, $5\frac{1}{2}$, $7\frac{5}{8}$ etc.

Terminal decimal fractions

Terminal decimals such as 0.6, 0.25, 0.125 and 1.35 are recognized as rational numbers. If the denominator of a rational number (written in the simplest fractional form) has no prime factors other than 2 and 5, then the rational number can be recognized as a terminal fraction.

Example: $\frac{7}{8}, \frac{3}{5}, \frac{9}{40}$ etc. are terminal decimal fractions. Terminal decimals can be converted into fractions

(a) $0.6 = \frac{6}{10} = \frac{3}{5}$ (b) $0.25 = \frac{25}{100} = \frac{1}{4}$

(c) $0.125 = \frac{125}{1000} = \frac{1}{8}$ (d) $1.35 = \frac{135}{100} = \frac{27}{20}$

Recurrent decimal fractions

Recurrent decimals are repeated decimals. It can be seen that every recurrent decimal represents a rational number. Every rational number can be represented by a terminal decimal or a recurrent decimal. If the denominator of a rational number (written in simplest form) has prime factors other than 2 and 5, then the rational number is recognized as a recurrent decimal. ***Example***: $\frac{5}{8} = 0.555\,555\ldots$ or $0.\dot{5}$, $\frac{2}{11} = 0.18\,18\,18\ldots$ or $0.\dot{1}\dot{8}$, $\frac{8}{15} = 0.53333\ldots$ or $0.5\dot{3}$ etc. Every recurrent decimal can be expressed as the quotient of two integers.

Example 2.1

Express $0.\dot{2}\dot{3}$ in the form $\frac{a}{b}$, where a and b are integers with no common factors.

Solution

$$
\begin{aligned}
\text{Let} \quad x &= 0.\dot{2}\dot{3} \\
x &= 0.23\,23\,23\ldots\ldots \quad (1) \\
100x &= 23.23\,23\,23\ldots\ldots \quad (2) \\
(2)-(1) \quad 99x &= 23 \Rightarrow x = \frac{23}{99}
\end{aligned}
$$

Example 2.2

Express $0.1\dot{8}$ in the form $\frac{a}{b}$, where a and b are integers with no common factors.

Solution

$$
\begin{aligned}
\text{Let} \quad x &= 0.1\dot{8} \\
x &= 0.188\,888\,8\ldots \\
10x &= 1.88\,88\,88\ldots\ldots\ldots\ldots \quad (1) \\
100x &= 18.88\,88\,88\ldots\ldots\ldots\ldots \quad (2) \\
(2)-(1) \quad 90x &= 17 \Rightarrow x = \frac{17}{90}.
\end{aligned}
$$

Percentages

Percentages such as 60%, 45 % and 25% are recognized as rational numbers.

Example $60\% = \frac{60}{100} = \frac{3}{5}$, $45\% = \frac{45}{100} = \frac{9}{20}$, $25\% = \frac{25}{100} = \frac{1}{4}$

Example 2.3

Express the following percentages as fractions

(a) 35%, (b) 5%, (c) 22.5%, (d) 75. %,

Solution

(a) $35\% = 0.35 = \frac{35}{100} = \frac{7}{20}$, (b) $5\% = 0.05 = \frac{5}{100} = \frac{1}{20}$,

(c) $22.5\% = 0.225 = \frac{225}{1000} = \frac{9}{40}$, (d) $75.2\% = 0.752 = \frac{752}{1000} = \frac{94}{125}$.

Comparing and ordering rational numbers

We can re-write given rational numbers in forms that can be compared. That is, as common fraction with the same Least Common Denominator (LCD), as decimals or as percentages.

Example 2.4

Arrange in ascending order 60%, $\frac{7}{10}$, 0.65 and $0.\dot{6}$.

Solution

We can express all the four numbers as decimals and then compare them

i.e. $60\ \% = 0.6$, $\frac{7}{10} = 0.7$, 0.65, $0.\dot{6} = 0.666\ldots$

We can also express all the numbers as percentages and then compare them

i.e. 60%, $\frac{7}{10} = 0.7 = 70\%$, $0.65 = 65\%$, $0.\dot{6} = 0.666\ldots = 66.6\ldots\%$

It follows that arranging in ascending order we have $60\ \%, 0.65, 0.\dot{6}$ and $\frac{7}{10}$

2.1.2 Irrational numbers

Numbers such as $\pi, \sqrt{2}, \sqrt{3}, \sqrt{5}$ etc cannot be expressed as ratios of two integers. Such numbers are called irrational numbers. Irrational numbers have no termination and are non-recurring.

Example: $\sqrt{2} = 1.414\ 213\ 562\ \ldots$ $\sqrt{3} = 1.732\ 050\ 808\ \ldots$

$\sqrt{5} = 2.236\ 067\ 977\ \ldots$ π (pi) = $3.141\ 592\ 654\ \ldots$

The set of irrational numbers is denoted by Q'.

Exercise 2.1

1. Which of the following is/are rational?

   $P = \{3, \sqrt{5}, 7\}$, $Q = \left\{19, \frac{3}{7}, \sqrt{25}\right\}$, $R = \{-5 < x \leq 9\}$

2. Sort out the following numbers into rational and irrational.

   (a) 5.234 234 …, (b) 2.321 435 …, (c) 7.463 127 …,

(d) 0.456 123 456 123 …, (e) 8.030 121 212 1…, (f) $\sqrt{6}$,
(g) $\sqrt{0.81}$, (h) $\sqrt{50}$, (i) $\sqrt{\frac{9}{4}}$ (j) 5π.

3. Arrange in ascending order 75 %, $\frac{18}{25}$, $0.\dot{6}$, 0.667.
4. Express $0.\dot{5}\dot{5}$ in the form $\frac{a}{b}$, where a and b are integers with no common factors.
5. Express $0.4\dot{6}$ in the form $\frac{a}{b}$, where a and b are integers with no common factors.
6. Express the recurring decimal $0.\dot{2}\dot{1}$ in the form $\frac{p}{q}$, where p and q are integers.
7. Express the recurring decimal $0.\dot{1}\dot{6}$ in the form $\frac{p}{q}$, where p and q are integers.
8. Arrange in ascending order $0.5\dot{3}$, 50%, $\frac{7}{15}$, 0.534.
9. Express the following recurrent decimals as the quotient of two integers
(a) $0.1\dot{2}$, (b) $0.\dot{1}\dot{2}$, (c) $0.\dot{9}\dot{9}$, (d) $0.\dot{2}\dot{5}$, (e) $0.1\dot{2}\dot{3}$, (f) $0.\dot{1}\dot{2}\dot{3}$, (g) $0.\dot{6}\dot{7}$, (h) $0.6\dot{7}$.
10. Express the following decimals as fractions
(a) 0.8 (b) 0.05 (c) 0.55 (d) 0.0625

2.2 Real Numbers

The set R of real numbers consists of rational numbers and irrational numbers. The set R of real numbers is defined as the union of the set Q of rational numbers and the set Q' of irrational numbers. (i.e. $R = Q \cup Q'$). It therefore follows that the sets of natural numbers, whole numbers, integers, rational and irrational numbers are subsets of the set of real numbers, i.e. $N \subset W \subset Z \subset Q \subset R$. The set of real numbers therefore consists of terminating decimals, recurring decimals and decimals which neither repeat nor terminate. Natural numbers, whole numbers, rational and irrational numbers are all recognized as real numbers. The relationship between the set of natural numbers (***N***), Integers (***Z***), rational numbers (***Q***) and real numbers (***R***) is given in Fig. 2.1.

(1) The region with the cross lines represents the set of irrational numbers.
(2) The region shaded with slanted lines represents rational numbers which are not integers.
(3) The region shaded with only vertical lines represents integers which are not natural numbers.
(4) The unshaded region represents the set of natural numbers.

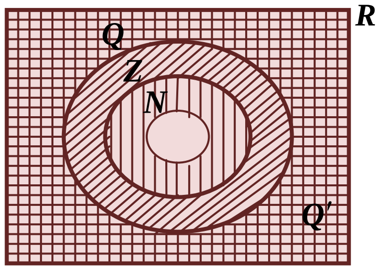


Fig. 2.1

The relation between all these sets of numbers is summarized in Fig.2.2.

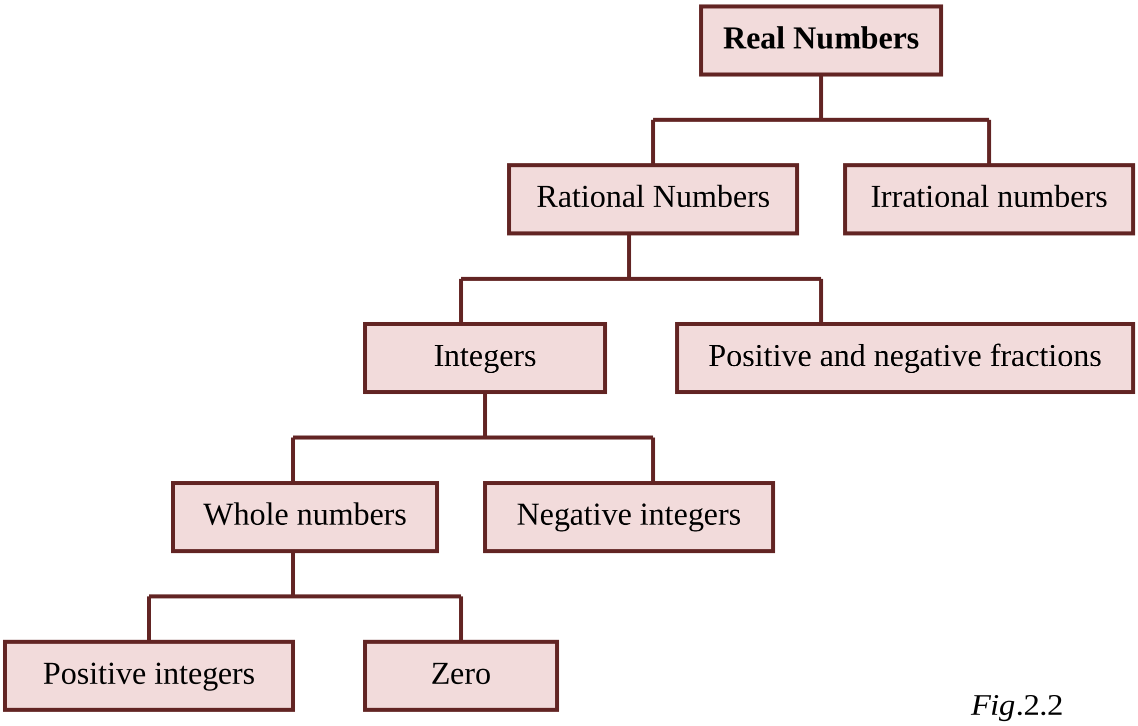


Fig.2.2

2.2.1 Real numbers on the number Line

The real number system is continuous. It is therefore impossible to list all the members of a set if it comprises real numbers within an interval. This is due to the fact that in between any two real numbers, there are innumerably many real numbers that can never be listed entirely. For instance, there are innumerably many real numbers between 1 and 2. Some of them are: 1.01, 1.001, 1.0901, 1.203, 1.099, 1.099, 1.0998, 1.9909 etc. We can write over a trillion of numbers between 1 and 2. On the real number line there are no breaks. Every point on the number line represents a real number. Thus any real number can be represented by a point on the real number line. Numbers to the right of zero are positive numbers and those to the left of zero are negative numbers. Fig. 2.3 shows part of the real number line with some rational numbers located on it.

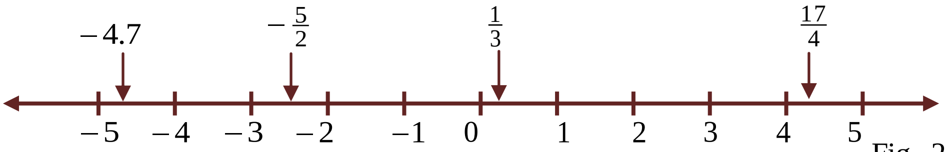


Fig. 2.3

We can show subsets of the set of real numbers by using intervals on the number line. For instance, the set between 1 and 2 is written as $\{x: x \in R, -1 < x < 2\}$. This set is shown on

Fig. 2.4 below. Note that an open circle 'o' is used at the points where $x = -1$ and $x = 2$. This is to show that the end points are not included in the set.

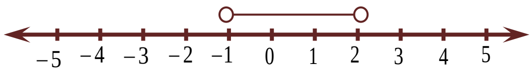


Fig. 2.4

When an end point is included in the set we use a shaded circle '•' to indicate that the end point is included. The set of all real numbers greater or equal to –2.5 is written as $\{x: x \in R, x \geq -2.5\}$. The graph for this inequality is illustrated in *Fig*. 2.5 below. The arrow head indicates that the line goes on and on.

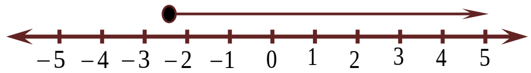


Fig. 2.5

The set $\{x: x \in R, -3 \leq x < 2\}$ can be represented on the number as shown in Fig. 2.6.

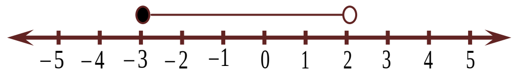


Fig. 2.6

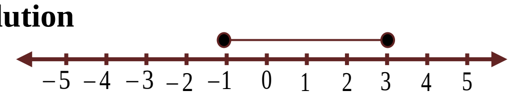
Example 2.5

Represent the following sets on the number line:

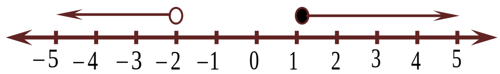
(a) $A = \{x: x \in R, -1 \leq x \leq 3\}$, (b) $B = \{x: x \in R, x < -2 \text{ or } x \geq 1\}$

Solution

(a)



(b)

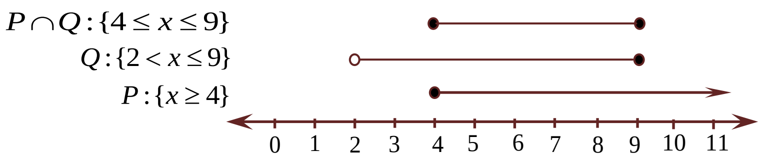


Example 2.6

Let U = {real numbers}, $P = \{x: x \geq 4\}$ and $Q = \{x: 2 < x \leq 9\}$. By using the real number line, find (a) $P \cap Q$, (b) $P \cup Q$, (c) $P' \cup Q$, (d) $P \cap Q'$.

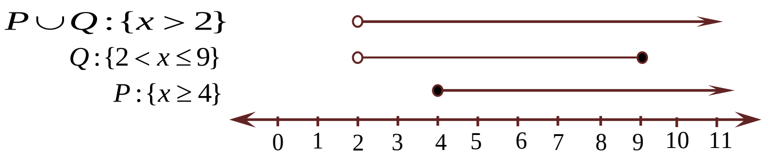
Solution

(a)



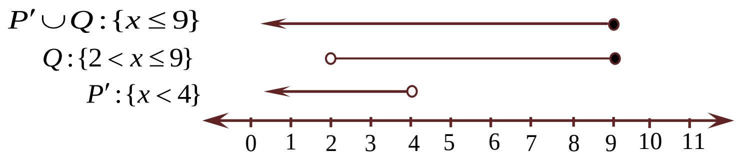
$P \cap Q = \{x: 4 \leq x \leq 9\}$.

(b)



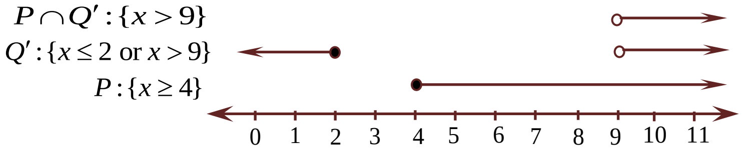
$P \cup Q = \{x: x > 2\}$.

(c)



$P' \cup Q = \{x:\ x \leq 9\}.$

(d)



$P \cap Q' = \{x:\ x > 9\}$

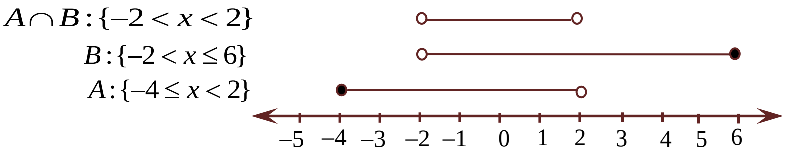
Example 2.7

Let U = {real numbers}, $A = \{x: -4 \leq x < 2\}$ and $B = \{x: -2 < x \leq 6\}$. By using the real number line, find: (a) $A \cap B$, (b) $A \cup B$, (c) $A' \cup B$, (d) $A \cap B'$, (e) $(A \cup B)'$.

Solution

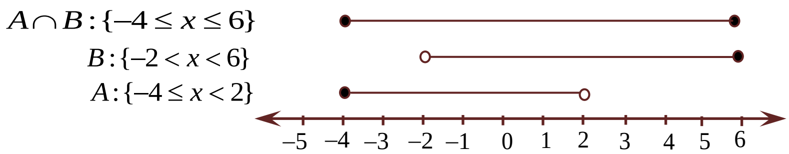
Since the universal set is made up of real numbers, the set operation is performed using the number line.

(a)



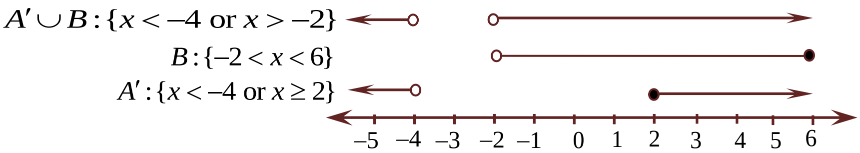
$A \cap B = \{x: -2 < x < 2\}.$

(b)



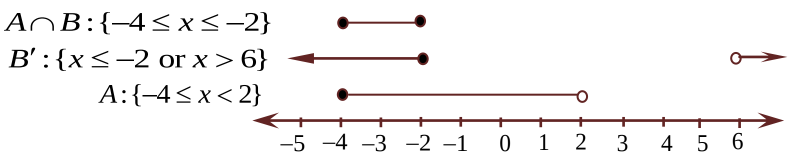
$A \cup B = \{x: -4 \leq x \leq 6\}.$

(c)



$A' \cup B = \{x: x < -4 \text{ or } x > -2\}.$

(d)



$A \cap B' = \{x: -4 \le x \le -2\}$

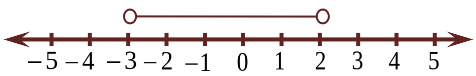
(e) From (ii) $A \cup B = \{x: -4 \le x \le 6\}$

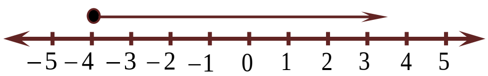
$(A \cup B)'$ is the complement of $A \cup B$

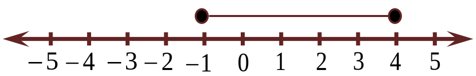
Hence $(A \cup B)' = \{x: x < -4 \text{ or } x > 6\}$.

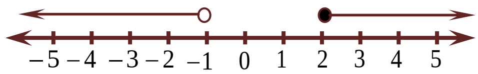
Exercise 2.2

1. Write down the range or interval for the set of real numbers represented on the following number lines.

(a)


(b)


(c)


(d)


2. Let U = {real numbers}, $A = \{x: -3 \le x < 3\}$ and $B = \{x: -2 < x \le 4\}$. By using the real number line, find
   (a) $A \cap B$, (b) $A \cup B'$, (c) $A' \cup B$, (d) $A' \cap B'$, (e) $(A \cap B)'$.

3. Let U = {real numbers}, $A = \{x: -2 \le x < 5\}$ and $B = \{x: 2 < x \le 8\}$. By using the real number line, find (a) $A \cap B$, (b) $A \cup B$, (c) $A' \cup B'$,
   (d) $A' \cap B'$, (e) $(A \cup B)'$, (f) $A \cup B'$, (g) $A' \cap B$.

2.3 Approximations

An approximation implies nearly correct or accurate. In this section we shall study how to reduce a real number first to a given number of decimal places and then to a given number of significant figures.

2.3.1 Decimal Places

Decimal places refer to the digits on the right of the decimal point. The number of decimal places is the number of digits after the decimal point. To correct a decimal to a stated number of decimal places:

1. Locate the digit of the specified number of decimal place
2. Discard any digit beyond the specified number decimal places
3. Add 1 to the last digit retained if the first digit discarded is equal to or greater than 5.

Consider the number 3.237;

The digit in the third decimal place is 7 which is greater than 5. Therefore to correct 3.237 to two decimal places we discard the third digit, 7, and add 1 to the second digit to give 3.24.

The digit in the 1st decimal place is 2. The figure after the 1st decimal place (i.e. the 2nd decimal place) is 3 which is less than 5. Therefore to correct 3.237 to one decimal place, we discard the 2nd and the 3rd digits (i.e. 3 and 7) and leave the 1st digit (i.e. 2) as it is to give 3.2.

Example 2.8

Correct 9.4268 to (a) one decimal place (b) two decimal places (c) three decimal places.

Solution

(a) The digit in the 2nd decimal place is 2 which is less than 5. We therefore leave the digit in the 1st decimal place and discard the digits in the 2nd, 3rd and 4th decimal places (i.e. 2, 6 and 8 respectively).

9.4268 = 9.4 (correct to one decimal place)

(b) The digit in the 3rd decimal place is 6 which is greater than 5. We therefore discard the digits in the 3rd and 4th decimal places (i.e. 6 and 8 respectively) and add 1 to the digit in the 2nd decimal place (i.e. 2 + 1 = 3)

9.4268 = 9.43 (correct to two decimal places)

(c) The digit in the 4th decimal place is 8 which is greater than 5. We therefore discard the digits in the 4th decimal place (i.e. 8) and add 1 to the digit in the 3rd decimal place (i.e. 6 + 1 = 7)

9.4268 = 9.427 (correct to three decimal places)

Example 2.9

Correct 3.795 to two decimal places.

Solution

The digit in the 3rd decimal place is equal to 5. We discard the digits in the 3rd decimal place (i.e. 5) and add 1 to the digit in the 2nd decimal place (i.e. 9 + 1 =10). Since the result is 10, we carry 1 and add it to the digit in the first decimal place (i.e. 7 + 1 = 8), so that we have 0 in the 2nd decimal place.

3.795 = 3.80 (correct to two decimal places)

2.3.2 Significant figures

Significant figures refer to the number of digits in a number. This excludes zeros holding places at the end of a number or at the beginning of a number to the right of the decimal point.

Consider the numbers 2048 and 2548. Both numbers have four significant figures. The zero in 2048 is significant since it tells us that there are no hundreds though there are 4 tens and 8 units.

Let us also consider the two numbers 45 500 and 40 500. Both have zeros holding places at the end. These zeros at the end are not significant figures, even though the zero between the four and five in 40500 is important. We therefore conclude that both numbers have three significant figures.

In the numbers 0.00205 and 0.00245, the zeros at the beginning, to the right of the decimal points are not significant figures, though the zero sandwiched between 2 and 5 in the number 0.00205 still is. So again we have three significant figures.

To correct a number to a stated number of significant figures:

1. locate the last significant figure you want,
2. replace all figures beyond the specified number of significant figures by 0,
3. if the next significant figure after the specified number of significant is greater than or equal to 5, then add 1 to the last significant figure you want.

Example 2.10

Correct to two significant figures 0.0006875.

Solution

The 2^{nd} significant figure in the number 0.0006875 is 8. Discard the digits beyond the 2^{nd} significant figure (i.e. 7 and 5). Since the figure after the 2^{nd} significant digit is greater than 5 (i.e. $7 > 5$), we add 1 to 8 to give 9 in the 2^{nd} significant figure.

0.0006875 = 0.00069 (to two significant figures)

Example 2.11

Correct to three significant figures 63429.71.

Solution

The 3^{rd} significant figure in the number 63429.71 is 4. Discard the digits beyond the decimal point and replace all other figures beyond the 3^{rd} significant figure by 0 (i.e. 2 and 9). Since the figure after the 3^{rd} significant digit is less than 5 (i.e. $2 < 5$), we leave the 4 as it is.

63429.71 = 63400 (to three significant figures)

2.3.3 Rounding a number to the nearest whole number

To round a figure to the nearest whole number, discard all digits which come after the decimal point. If the digit just after the decimal point is greater than or equal to 5, then add 1 to the last digit before the decimal point.

Example 2.12

Correct the following numbers to the nearest whole number.

(a) 23.4, (b) 324.97, (c) 569.851.

Solution

(a) 23.4

The digit just after the decimal point is 4, which is less than 5. We therefore discard the 4 and leave all the digits before the decimal points.

23.4 = 23 (to the nearest whole number)

(b) 324.97

The digit just after the decimal point is 9, which is greater than 5. We therefore add 1 to the digit just before the decimal point (i.e. 4 + 1 = 5) and discard all digits after the decimal point.

324.97 = 325 (to the nearest whole number)

(c) 569.851

The digit just after the decimal point is 8, which is greater than 5. We discard the all the digits after the decimal point and add 1 to the digit just before the decimal point (i.e. 9 + 1 = 10). Since the result is 10, we carry 1 and add it to the 2nd digit before the decimal point (i.e. 6 + 1 = 7), so that 0 replaces 9.

569.851 = 570 (to the nearest whole number)

2.3.4 Rounding Numbers to Nearest 10, 100, 1000, etc.

Numbers can be rounded to the nearest 10, 100 or 1000 etc. For example, the numbers 10, 11, 12, 13 and 14 are nearer to 10 than 20, while the numbers 16, 17, 18 and 19 are nearer to 20 than 10. So if we want to write numbers as multiples of 10 then we say to the nearest 10.
The sign ≈ means approximately equal to.

$12 \approx 10$, $13 \approx 10$, $14 \approx 10$, $15 \approx 20$, $16 \approx 20$, $17 \approx 20$

$21 \approx 20$, $24 \approx 20$, $27 \approx 30$, $35 \approx 40$, $59 \approx 60$, $94 \approx 90$

Exercise 2.3

1. Correct the following to three decimal places:
   (a) 3.8976, (b) 5.982341, (c) 12.76993, (d) 7.99986.
2. Correct the following to two decimal places:
   (a) 6.1829, (b) 5.8962, (c) 89.995, (d) 9.1234.
3. Correct the following to three significant figures:
   (a) 3.8976, (b) 5.982341, (c) 12.76993, (d) 7.19986.
4. Correct the following to two significant figures:
   (a) 6.1829, (b) 5.8962, (c) 87.995, (d) 9.1234.
5. Correct the following to three significant figures:
   (a) 0.003567, (b) 45634, (c) 78689, (d) 923736.
6. Correct the following numbers to the nearest whole numbers
   (a) 45.94, (b) 567.87, (c) 9.24, (d) 699.573.
7. Correct the following numbers to the nearest whole numbers
   (a) 790.63, (b) 399.61, (c) 224.231, (d) 9899.5.
8. Round the following numbers to the nearest 10
   (a) 32, (b) 76, (c) 123, (d) 656, (e) 1234, (f) 17866.
9. Round the following numbers to the nearest 100:

(a) 32, (b) 76, (c) 123, (d) 656, (e) 1234, (f) 17866.

10. Round the following numbers to the nearest 1000:
(a) 124, (b) 1420, (c) 1650, (d) 134765, (e) 179981, (f) 892678.

11. Round the following to the nearest 10,000
(a) 4,500, (b) 6,320, (c) 14,102, (d) 17,893, (e) 5,767,988, (f) 45,898,654.

2.4 Numbers in standard form

When we have a very large or small number, it is often convenient to write in the form $m \times 10^n$, where $1 \leq m < 10$ and n is an integer. A number expressed in this form is said to be in ***standard form***.

For example consider the number 945000. This can be expressed as $945000 = \frac{945000}{100000} \times 100000 = 9.45 \times 10^5$. We say that $945000 = 9.45 \times 10^5$.

Example 2.13
Express the following numbers in standard form.
(a) 7345, (b) 0.4567, (c) 8470000, (d) 0.000000841.

Solution
(a) $7345 = 7.345 \times 10^3$, (b) $0.4567 = 4.567 \times 10^{-1}$,
(c) $8470000 = 8.47 \times 10^6$, (d) $0.000000841 = 8.41 \times 10^{-7}$.

Example 2.14
Evaluate the following leaving your answers in standard form.
(a) $8100000 \div 0.00027$, (b) 256000×0.000004,
(c) $\frac{625000 \times 0.00064}{0.00025 \times 160000}$, (d) $\frac{0.0048 \times 0.81}{0.0027 \times 0.004}$.

Solution

(a) $\frac{8100000}{0.00027} = \frac{81 \times 10^5}{27 \times 10^{-5}} = 3 \times 10^{6+4} = 3.0 \times 10^{10}$

(b) $256000 \times 0.000004 = (256 \times 10^3) \times (4 \times 10^{-6})$
$= 1024 \times 10^{3-6} = 1024 \times 10^{-3}\, 1.024 = 1.024.$

(c) $\frac{625000 \times 0.00064}{0.00025 \times 160000} = \frac{(625 \times 10^3) \times (64 \times 10^{-5})}{(25 \times 10^{-5}) \times (16 \times 10^4)} = \frac{625}{25} \times \frac{64}{16} \times \frac{10^{-2}}{10^{-1}}$
$= 25 \times 4 \times 10^{-2+1} = 100 \times 10^{-1} = 1.0 \times 10.$

(d) $\dfrac{0.0048 \times 0.81}{0.0027 \times 0.004} = \dfrac{(48\times10^{-4})\times(81\times10^{-2})}{(27\times10^{-4})\times(4\times10^{-3})} = \dfrac{48}{4}\times\dfrac{81}{27}\times\dfrac{10^{-6}}{10^{-7}}$

$= 12 \times 3 \times 10^{-6+7} = 36 \times 10 = 3.6 \times 10^{2}$.

Exercise 2.4

1. Express the following numbers in standard form.

(a) 894000 (b) 0.00984, (c) 38600000, (d) 0.0000000924.

2. Evaluate the following leaving your answers in standard form.

(a) $0.00024 \div 16$, (b) 0.00236×0.0045,

(c) $\dfrac{0.0008\times0.0036}{0.04\times0.6}$, (d) $\dfrac{0.54\times0.0672}{0.0009\times0.0000096}$.

2.5 Operations on rational numbers

Equal fractions give the same amount. So $\frac{4}{5}$ is equal to $\frac{12}{15}$. We write $\frac{12}{15} = \frac{4}{5}$. Equal fractions may be found by multiplying or dividing the numerator and denominator by the same amount: $\frac{2}{3} = \frac{2\times4}{3\times4} = \frac{8}{12}$, $\frac{12}{15} = \frac{12\div3}{15\div3} = \frac{4}{5}$.

2.5.1 Addition and Subtraction of Rational Numbers

You can add or subtract fractions easily if the bottom number (that is, the denominator) is the same. To add two fractions with the same denominator, you simply add the numerators and keep the denominator the same.

Example 2.15

(a) $\frac{5}{7} + \frac{4}{7} = \frac{5+4}{7} = \frac{9}{7} = 1\frac{2}{7}$, (b) $\frac{5}{8} - \frac{3}{8} = \frac{5-3}{8} = \frac{2}{8} = \frac{1}{4}$.

If the denominators are ***not*** the same, you may apply the technique mentioned above to make them the same before doing the addition or the subtraction. To achieve this, we can use the following steps:

1. Find the LCM of the denominators of the fractions.
2. Express each of the fractions to an equivalent fraction with the LCM as the denominator.
3. Add or subtract as illustrated in Example 3.14.

Example 2.16

Simplify the following:

(a) $\frac{1}{4} + \frac{2}{5}$, (b) $\frac{3}{8} + \frac{5}{6}$, (c) $\frac{5}{6} + \frac{4}{9}$, (d) $\frac{6}{15} + \frac{2}{9}$.

Solution

(a) The LCM of the denominators, 4 and 5, is 20.

$$\frac{1}{4} + \frac{2}{5} = \frac{1 \times 5}{4 \times 5} + \frac{2 \times 4}{5 \times 4} = \frac{5}{20} + \frac{8}{20} = \frac{13}{20}.$$

(b) The LCM of the denominators, 8 and 6, is 24.

$$\frac{3}{8} + \frac{5}{6} = \frac{3 \times 3}{8 \times 3} + \frac{5 \times 4}{6 \times 4} = \frac{9}{24} + \frac{20}{24} = \frac{9 + 20}{24} = \frac{29}{24} = 1\frac{5}{24}.$$

(c) The LCM of the denominators, 6 and 9, is 18.

$$\frac{5}{6} + \frac{4}{9} = \frac{5 \times 3}{6 \times 3} + \frac{4 \times 2}{9 \times 2} = \frac{15}{18} + \frac{8}{18} = \frac{15 + 8}{18} = \frac{23}{18} = 1\frac{5}{18}.$$

(d) The LCM of the denominators, 15 and 9, is 45.

$$\frac{6}{15} + \frac{2}{9} = \frac{6 \times 3}{15 \times 3} + \frac{2 \times 5}{9 \times 5} = \frac{18}{45} + \frac{10}{45} = \frac{18 + 10}{45} = \frac{28}{45}.$$

Example 2.17

Simplify the following:

(a) $\frac{3}{4} - \frac{1}{6}$, (b) $\frac{5}{6} - \frac{7}{12}$, (c) $\frac{5}{8} - \frac{5}{12}$, (d) $\frac{7}{9} - \frac{11}{15}$.

Solution

(a) The LCM of the denominators, 4 and 6, is 12.

$$\frac{3}{4} - \frac{1}{6} = \frac{3 \times 3}{4 \times 3} - \frac{1 \times 2}{6 \times 2} = \frac{9}{12} - \frac{2}{12} = \frac{9 - 2}{12} = \frac{7}{12}.$$

(b) The LCM of the denominators, 6 and 12, is 12.

$$\frac{5}{6} - \frac{7}{12} = \frac{5 \times 2}{6 \times 2} - \frac{7 \times 1}{12 \times 1} = \frac{10}{12} - \frac{7}{12} = \frac{10 - 7}{12} = \frac{3}{12} = \frac{1}{4}.$$

(c) The LCM of the denominators, 8 and 12, is 24.

$$\frac{5}{8} - \frac{5}{12} = \frac{5 \times 3}{8 \times 3} - \frac{5 \times 2}{12 \times 2} = \frac{15}{24} - \frac{10}{24} = \frac{15 - 10}{24} = \frac{5}{24}.$$

(d) The LCM of the denominators, 9 and 15, is 45.

$$\frac{7}{9} - \frac{11}{15} = \frac{7 \times 5}{9 \times 5} - \frac{11 \times 3}{15 \times 3} = \frac{35}{45} - \frac{33}{45} = \frac{35 - 33}{45} = \frac{2}{45}.$$

Example 2.18

Simplify the following: (a) $6\frac{2}{5} + 3\frac{3}{10}$, (b) $5\frac{2}{3} - 3\frac{1}{2}$.

Solution

Note carefully how these sums are done;

(a) ***First method***

We first express each mixed fraction as improper fraction and then simplify.

$$6\frac{2}{5} + 3\frac{3}{10} = \frac{32}{5} + \frac{33}{10} = \frac{32 \times 2}{5 \times 2} + \frac{33 \times 1}{10 \times 1} = \frac{64}{10} + \frac{33}{10} = \frac{97}{10} = 9\frac{7}{10}.$$

Second method

$$6\frac{2}{5} + 3\frac{3}{10} = (6+3) + \left(\frac{2}{5} + \frac{3}{10}\right) = 9 + \left(\frac{2 \times 2}{5 \times 2} + \frac{3}{10}\right) = 9 + \left(\frac{4}{10} + \frac{3}{10}\right) = 9 + \frac{7}{10} = 9\frac{7}{10}.$$

(b) ***First method***

We first express each mixed fraction as improper fraction and then simplify.

$$5\frac{2}{3} - 3\frac{1}{2} = \frac{17}{3} - \frac{7}{2} = \frac{17 \times 2}{3 \times 2} - \frac{7 \times 3}{2 \times 3} = \frac{34}{6} - \frac{21}{6} = \frac{13}{6} = 2\frac{1}{6}.$$

Second method

$$5\frac{2}{3} - 3\frac{1}{2} = (5-3) + \frac{2}{3} - \frac{1}{2} = 2 + \frac{2 \times 2}{3 \times 2} - \frac{1 \times 3}{2 \times 3} = 2 + \frac{4}{6} - \frac{3}{6} = 2 + \frac{1}{6} = 2\frac{1}{6}.$$

Example 2.19

Find (a) $3\frac{3}{5} + 5\frac{3}{4} - 7\frac{1}{2}$ (b) $10\frac{1}{6} + 8\frac{2}{3} - 15\frac{2}{3}$ (c) $16\frac{1}{2} - 9\frac{1}{5} - 6\frac{1}{3}$.

Solution

(a) $3\frac{3}{5} + 5\frac{3}{4} - 7\frac{1}{2} = \frac{18}{5} + \frac{23}{4} - \frac{15}{2} = \frac{72+115-150}{20} = \frac{37}{20} = 1\frac{17}{20}$.

(b) $10\frac{1}{6} + 8\frac{2}{3} - 15\frac{2}{3} = \frac{61}{6} + \frac{26}{3} - \frac{47}{3} = \frac{61+52-94}{6} = \frac{19}{6} = 3\frac{1}{6}$.

(c) $16\frac{1}{2} - 9\frac{1}{5} - 6\frac{1}{3} = \frac{33}{2} - \frac{46}{5} - \frac{19}{3} = \frac{495-276-190}{30} = \frac{29}{30}$.

2.6.2 Multiplication and division of rational numbers

Fractions are multiplied or divided as follows:

$$\frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7 \times 3} = \frac{10}{21}, \qquad \frac{5}{7} \div \frac{2}{3} = \frac{5}{7} \times \frac{3}{2} = \frac{15}{14} = 1\frac{1}{14}.$$

Example 2.20

Perform the following operations:

(a) $\frac{2}{3} \times \frac{4}{5} \div \frac{8}{15}$, (b) $\frac{7}{9} \div \frac{21}{5} \times \frac{3}{10}$, (c) $\frac{26}{35} \times \frac{6}{13} \div \frac{2}{15}$.

Solution

(a) $\frac{2}{3} \times \frac{4}{5} \div \frac{8}{15} = \frac{2}{3} \times \frac{4}{5} \times \frac{15}{8} = \frac{8}{15} \times \frac{15}{8} = 1.$

(b) $\frac{7}{9} \div \frac{21}{5} \times \frac{3}{10} = \frac{7}{9} \times \frac{5}{21} \times \frac{3}{10} = \frac{1}{9} \times \frac{1}{1} \times \frac{1}{2} = \frac{1}{18}.$

(c) $\frac{26}{35} \times \frac{6}{13} \div \frac{2}{15} = \frac{26}{35} \times \frac{6}{13} \times \frac{15}{2} = \frac{1}{7} \times \frac{6}{1} \times \frac{3}{1} = \frac{18}{7} = 2\frac{4}{7}.$

Example 2.21

Without using a calculator, evaluate

(a) $\dfrac{2\frac{7}{8}\times 1\frac{1}{5}}{8-2\frac{1}{4}}$, (b) $\dfrac{6\frac{3}{8}\times\frac{4}{5}-7\frac{7}{15}\div\frac{224}{135}}{\frac{2}{5}+3\frac{1}{30}\times\frac{6}{7}}$.

Solution

$$\text{(a)}\ \frac{2\frac{7}{8}\times 1\frac{1}{5}}{8-2\frac{1}{4}} = \frac{\frac{23}{8}\times\frac{6}{5}}{8-\frac{9}{4}} = \frac{\frac{23}{4}\times\frac{3}{5}}{\frac{32-9}{4}} = \frac{\frac{23}{4}\times\frac{3}{5}}{\frac{23}{4}} = \frac{3}{5}.$$

$$\text{(d)}\ \frac{6\frac{3}{8}\times\frac{4}{5}-7\frac{7}{15}\div\frac{224}{135}}{\frac{2}{5}+3\frac{1}{30}\times\frac{6}{7}} = \frac{\frac{51}{8}\times\frac{4}{5}-\frac{112}{15}\times\frac{135}{224}}{\frac{2}{5}+\frac{91}{30}\times\frac{6}{7}} = \frac{\frac{51}{2}\times\frac{1}{5}-\frac{1}{1}\times\frac{9}{2}}{\frac{2}{5}+\frac{13}{5}\times\frac{1}{1}} = \frac{\frac{51}{10}-\frac{9}{2}}{\frac{2}{5}+\frac{13}{5}}$$

$$= \frac{\frac{51-45}{10}}{\frac{2+13}{5}} = \frac{\cancel{6}/\cancel{10}}{\cancel{15}/\cancel{5}} = \frac{3/5}{3} = \frac{1}{5}.$$

2.6.3 Evaluation without calculators

Example 2.22

Without using tables or calculators, evaluate

(a) $\dfrac{20.3}{3.5\times 0.58}$, (b) $\dfrac{0.4\times 0.25}{0.5\times 0.2}$, (c) $\dfrac{1.8\times 0.\dot{3}}{1.6\times 0.75}$.

Solution

$$\text{(a)}\ \frac{20.3}{3.5\times 0.58} = \frac{20.3\times 1000}{3.5\times 0.58\times 1000} = \frac{203\times 100}{35\times 58} = \frac{29\times 100}{5\times 58} = \frac{1\times 20}{1\times 2} = 10.$$

$$\text{(b)}\ \frac{0.4\times 0.25}{0.5\times 0.2} = \frac{\frac{4}{10}\times\frac{25}{100}}{\frac{5}{10}\times\frac{2}{10}} = \frac{\frac{2}{5}\times\frac{1}{4}}{\frac{1}{2}\times\frac{1}{5}} = \frac{\frac{1}{5}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{5}} = 1.$$

(c) $0.\dot{3} = 0.333... = \dfrac{1}{3}$

$$\frac{1.8\times 0.\dot{3}}{1.6\times 0.75} = \frac{\frac{18}{10}\times\frac{1}{3}}{\frac{16}{10}\times\frac{75}{100}} = \frac{\frac{9}{5}\times\frac{1}{3}}{\frac{8}{5}\times\frac{3}{4}} = \frac{\frac{3}{5}}{\frac{2}{5}\times\frac{3}{1}} = \frac{\frac{3}{5}}{\frac{6}{5}} = \frac{3}{5}\times\frac{5}{6} = \frac{1}{2}.$$

Exercise 2.5

1. Simplify the following:

(a) $\frac{1}{8}+\frac{1}{6}$, (b) $\frac{4}{9}+\frac{1}{15}$, (c) $\frac{3}{5}+\frac{2}{7}$, (d) $\frac{5}{12}+\frac{7}{15}$,

(e) $\frac{2}{3} - \frac{5}{15}$, (f) $\frac{5}{16} - \frac{5}{24}$, (g) $\frac{4}{5} - \frac{4}{15}$, (h) $\frac{5}{12} - \frac{7}{36}$,

(i) $\frac{7}{18} + \frac{5}{24}$, (j) $\frac{2}{14} + \frac{1}{28}$, (k) $\frac{3}{15} + \frac{1}{25}$, (l) $\frac{6}{18} + \frac{1}{21}$,

(m) $\frac{6}{7} - \frac{7}{14}$, (n) $\frac{5}{9} - \frac{7}{21}$, (o) $\frac{8}{15} - \frac{5}{18}$, (p) $\frac{7}{11} - \frac{5}{22}$.

2. Simplify the following:

(a) $3\frac{1}{2} + 4\frac{1}{3}$, (b) $1\frac{1}{9} + 2\frac{1}{3}$, (c) $4\frac{3}{4} + 2\frac{1}{3}$, (d) $5\frac{1}{5} + 2\frac{2}{10}$,

(e) $3\frac{2}{3} - 2\frac{1}{6}$, (f) $5\frac{4}{5} - 3\frac{3}{4}$, (g) $5\frac{5}{6} - 4\frac{3}{8}$, (h) $8\frac{1}{12} - 6\frac{5}{24}$.

3. Find (a) $4\frac{1}{5}+2\frac{3}{4}-5\frac{1}{2}$, (b) $1\frac{5}{6}+5\frac{1}{3}-8\frac{2}{3}$, (c) $6\frac{1}{2}-2\frac{1}{5}-1\frac{1}{3}$.

4. Find (a) $2\frac{2}{3}+4\frac{4}{9}-6\frac{1}{2}$, (b) $7\frac{8}{9}-9\frac{1}{3}+2\frac{5}{18}$, (c) $19\frac{1}{2}-34\frac{3}{4}+23\frac{1}{2}$.

5. Find (a) $\dfrac{\frac{1}{7}\times\frac{1}{2}-\frac{3}{7}\times\frac{5}{2}}{\frac{1}{14}-\frac{1}{7}}$. (b) $\dfrac{\frac{1}{3}\times\frac{2}{5}-\frac{7}{3}\times\frac{1}{5}}{\frac{1}{9}\times\frac{1}{5}-\frac{2}{3}\times\frac{2}{5}}$. (c) $\dfrac{\frac{1}{4}\times\frac{1}{2}+\frac{1}{8}\times\frac{3}{2}}{\frac{1}{8}\times\frac{3}{4}-\frac{3}{2}\times\frac{7}{8}}$.

6. Find (a) $\dfrac{\frac{1}{2}-\frac{1}{4}+\frac{1}{8}}{\frac{3}{16}+\frac{5}{16}}$, (b) $\dfrac{\frac{4}{5}+\frac{1}{15}-\frac{2}{3}}{\frac{1}{15}-\frac{1}{30}}$, (c) $\dfrac{\frac{5}{7}+\frac{2}{7}-\frac{3}{14}}{\frac{2}{21}-\frac{1}{7}}$.

7. Find (a) $\dfrac{\frac{1}{2}\div\frac{1}{4}-\frac{3}{4}\times 2\frac{13}{27}}{\frac{1}{3}\times\frac{1}{2}+\frac{1}{6}\times\frac{2}{3}}$, (b) $\dfrac{1\frac{1}{2}\times 1\frac{1}{4}+2\frac{1}{2}\times\frac{1}{4}}{2\frac{1}{8}-1\frac{1}{4}}$, (c) $\dfrac{7\frac{5}{12}\times\frac{2}{3}-\frac{1}{9}\div\frac{1}{3}}{2\frac{1}{3}\times 3\frac{1}{3}-5\frac{1}{6}\times\frac{1}{6}}$.

8. Find (a) $\dfrac{1\frac{1}{2}\div 1\frac{1}{4}-2\frac{1}{2}\times\frac{1}{4}}{\frac{1}{2}\times\frac{1}{4}-\frac{1}{8}\times\frac{1}{2}}$, (b) $\dfrac{2\frac{1}{9}\div\frac{2}{3}-6\frac{1}{4}\times\frac{2}{5}}{\frac{1}{5}\times\frac{1}{3}+\frac{2}{5}\times\frac{2}{3}}$, (c) $\dfrac{\frac{1}{6}\times\frac{1}{12}+\frac{1}{3}\times\frac{2}{3}}{\frac{7}{12}\times\frac{1}{3}-\frac{5}{6}\times\frac{1}{3}}$.

9. Without using calculators, evaluate

(a) $\dfrac{0.6\times 0.\dot{3}}{0.8\times 0.25}$, (b) $\dfrac{0.12\times 3.\dot{3}}{0.32\times 2.5}$, (c) $\dfrac{1.2\times 0.\dot{6}}{0.75\times 3.2}$, (d) $\dfrac{0.12\times 0.5}{3.2\times 0.075}$.

2.6 Difference of Two Squares

A perfect square minus another perfect square is called *difference of two squares*. For example, 49 – 25 is the difference between two perfect squares and this can be written as $7^2 - 5^2$. If x^2 and y^2 are two perfect squares, then their difference can be expressed as $x^2 - y^2 = (x + y)(x - y)$ or $y^2 - x^2 = (y + x)(y - x)$.

Example 2.23

(a) $7^2 - 5^2 = (7 + 5)(7 - 5) = 12 \times 2 = 24$.

(b) $21^2 - 11^2 = (21 + 11)(21 - 11) = 32 \times 10 = 320$.

(c) $6.23^2 - 3.77^2 = (6.23 + 3.77)(6.23 - 3.77) = 10 \times 2.46 = 24.6$.

(d) $\frac{1}{9} - \frac{1}{16} = \left(\frac{1}{3}\right)^2 - \left(\frac{1}{4}\right)^2 = \left(\frac{1}{3} + \frac{1}{4}\right)\left(\frac{1}{3} - \frac{1}{4}\right) = \left(\frac{4+3}{12}\right) \times \left(\frac{4-3}{12}\right) = \frac{7}{12} \times \frac{1}{12} = \frac{7}{144}.$

(e) $46.27^2 - 53.73^2 = (46.27 + 53.73)(46.27 - 53.73) = 100 \times (-7.46) = -746.$

(f) $53.8^2 - 46.2^2 = (53.8 - 46.2)(53.8 + 46.2) = 7.6 \times 100 = 760.$

Example 2.24

Without using a calculator or a table, determine the values of x in the following:

(a) $4x + 12^2 = 16^2$ (b) $3x - 19^2 = -16^2$

(c) $25x^2 - 9 = 5x + 3$ (d) $5.76x + 2.12^2 = 7.88^2$.

Solution

(a)
$$4x + 12^2 = 16^2$$
$$\Rightarrow 4x = 16^2 - 12^2$$
$$4x = (16 - 12)(16 + 12)$$
$$4x = 4 \times 28$$
$$\Rightarrow x = 28.$$

(b)
$$3x - 19^2 = -16^2$$
$$\Rightarrow 3x = 19^2 - 16^2$$
$$3x = (19 - 16)(19 + 16)$$
$$3x = 3 \times 35$$
$$\Rightarrow x = 35.$$

(c)
$$25x^2 - 9 = 5x + 3$$
$$\Rightarrow (5x)^2 - 3^2 = 5x + 3$$
$$(5x + 3)(5x - 3) = 5x + 3$$
$$5x - 3 = 1$$
$$\Rightarrow 5x = 1 + 3$$
$$5x = 4$$
$$x = \frac{4}{5}.$$

(d)
$$5.76x + 2.12^2 = 7.88^2$$
$$\Rightarrow 5.76x = 7.88^2 - 2.12^2$$
$$5.76x = (7.88 - 2.12)(7.88 + 2.12)$$
$$5.76x = 5.76 \times 10$$
$$\Rightarrow x = 10$$

Exercise 2.6

Evaluate the following without using a calculator:

(a) $11^2 - 10^2$ (b) $56.28^2 - 43.72^2$ (c) $5.29^2 - 4.71^2$

(d) $0.64^2 - 0.36^2$ (e) $1.13^2 - 0.87^2$ (f) $3.94^2 - 6.06^2$

Solved Examination Problems

1. Without using tables or calculators, evaluate

(a) $\dfrac{20.3}{3.5 \times 0.58}$, (b) $53.8^2 - 46.2^2$, (c) $\dfrac{2\frac{7}{8} \times 1\frac{1}{5}}{5 - 2\frac{1}{4}}$. (June 1993)

Solution

(a) $\dfrac{20.3}{3.5 \times 0.58} = \dfrac{203 \times 10^{-1}}{35\times10^{-1} \times 58\times10^{-2}} = \dfrac{7 \times 29 \times 10^{-1}}{7 \times 5 \times 10^{-1} \times 2 \times 29 \times 10^{-2}} = \dfrac{1}{5 \times 2 \times 10^{-2}}$

$$= \frac{1}{10 \times 10^{-2}} = \frac{1}{10^{-1}} = 10^1 = 10.$$

(b) $53.8^2 - 46.2^2 = (53.8 - 46.2)(53.8 + 46.2) = 7.6 \times 100 = 760.$

(c) $\dfrac{2\frac{7}{8} \times 1\frac{1}{5}}{8 - 2\frac{1}{4}} = \dfrac{\frac{23}{8} \times \frac{6}{5}}{\frac{8}{1} - \frac{9}{4}} = \dfrac{\frac{23}{4} \times \frac{3}{5}}{\frac{32-9}{4}} = \dfrac{\frac{23}{4} \times \frac{3}{5}}{\frac{23}{4}} = \frac{23}{4} \times \frac{3}{5} \times \frac{4}{23} = \frac{3}{5}.$

2. Without using calculator,

(a) evaluate $\dfrac{2x - y}{z} + \dfrac{z + 2y}{x}$, when $x = 2$, $y = -3$ and $z = 4$;

(b) find Q if $3Q + 13^2 = 16^2$. (**June 1994**).

(c) evaluate $\dfrac{0.0048 \times 0.81}{0.0027 \times 0.004}$, leaving your answer in standard form. (**June 1995**).

Solution

(a) When $x = 2$, $y = -3$ and $z = 4$, we have

$$\frac{2x - y}{z} + \frac{z + 2y}{x} = \frac{2(2) - (-3)}{4} + \frac{4 + 2(-3)}{2}$$

$$= \frac{4 + 3}{4} + \frac{4 - 6}{2} = \frac{7}{4} - \frac{2}{2} = \frac{7 - 4}{4} = \frac{3}{4}.$$

(b) $3Q + 13^2 = 16^2 \Rightarrow 3Q = 16^2 - 13^2$

Applying difference of two squares, we have

$3Q = (16-13)(16+13) \Rightarrow 3Q = 3\times29 \Rightarrow Q = \dfrac{3 \times 29}{3} = 29.$

(d) $\dfrac{0.0048 \times 0.81}{0.0027 \times 0.004} = \dfrac{48 \times 10^{-4} \times 81 \times 10^{-2}}{27 \times 10^{-4} \times 4 \times 10^{-3}} = 12\times3\times10^{-2+3} = 36\times10^1 = 3.6\times10^2.$

3. Without using calculators or tables,

(a) simplify $\left(1\frac{2}{7} - \frac{1}{3}\right) \times 1\frac{3}{4}$; (b) evaluate $\dfrac{7.25 \times (0.16)^2}{0.004}$. (**June 1997**).

Solution

(a) $\left(1\frac{2}{7} - \frac{1}{3}\right) \times 1\frac{3}{4} = \left(\frac{9}{7} - \frac{1}{3}\right) \times \frac{7}{4} = \left(\frac{27-7}{21}\right) \times \frac{7}{4} = \frac{20}{21} \times \frac{7}{4} = \frac{5}{3} = 1\frac{2}{3}.$

(b) $$\frac{7.25 \times (0.16)^2}{0.004} = \frac{725 \times 10^{-2} \times \left(16 \times 10^{-2}\right)^2}{4 \times 10^{-3}} = \frac{725 \times 10^{-2} \times \left(4^2 \times 10^{-2}\right)^2}{4 \times 10^{-3}}$$

$$= \frac{725 \times 10^{-2} \times 4^4 \times 10^{-4}}{4^1 \times 10^{-3}} = 145 \times 5 \times 4^{4-1} \times 10^{-2-4+3}$$

$$= 29 \times 5 \times 5 \times 4 \times 4^2 \times 10^{-3} = 29 \times 25 \times 4\times 4^2 \times 10^{-3}$$

$$= 29 \times 100 \times 4^2 \times 10^{-3} = 29 \times 10^2 \times 4^2 \times 10^{-3}$$

$$= 29 \times 16 \times 10^{2-3} = 464 \times 10^{-1} = 46.4.$$

4. (a) Without using tables and calculators express $\frac{(0.00042 \times 10^{-8})(15{,}000)}{(5000 \times 10^{7})(0.0021 \times 10^{14})}$ in the form $a \times 10^{n}$, where $1 < a < 10$ and n is an integer. (**June 1999**).

(b) Find R, if $5R^2 + (22.55)^2 = (27.45)^2$. (**June 2000**)

Solution

(a) $\frac{(0.00042 \times 10^{-8})(15{,}000)}{(5000 \times 10^{7})(0.0021 \times 10^{14})} = \frac{(42 \times 10^{-5} \times 10^{-8})(15 \times 10^{3})}{(5 \times 10^{3} \times 10^{7})(21 \times 10^{-4} \times 10^{14})}$

$= \frac{42 \times 15 \times 10^{-5-8+3}}{5 \times 21 \times 10^{3+7-4+14}} = \frac{2 \times 3 \times 10^{-10}}{1 \times 10^{20}}$

$= 6 \times 10^{-10-20} = 6.0 \times 10^{-30}$.

(b) $5R^2 + (22.55)^2 = (27.45)^2 \Rightarrow 5R^2 = (27.45)^2 - (22.55)^2$

This is difference of two squares. Thus,

$5R^2 = (27.45 - 22.55)(27.45 + 22.55) \Rightarrow 5R^2 = 4.9 \times 50$

$\therefore \quad R^2 = \frac{4.9 \times 50}{5} = 4.9 \times 10 = 49 \Rightarrow R = \sqrt{49} = 7.$

5. Without using mathematical tables or calculators,

(a) find the value of y if $13y = 187^2 - 174^2$,

(b) evaluate $\sqrt{\frac{0.0048 \times 0.81 \times 10^{-7}}{0.027 \times 0.04 \times 10^{6}}}$, leaving your answer in standard form. (**Nov. 2001**)

Solution

(a) $13y = 187^2 - 174^2 \Rightarrow 13y = (187 - 174)(187 + 174) \Rightarrow 13y = 13 \times 361$

$\therefore \quad y = \frac{13 \times 361}{13} = 361.$

(b) $\sqrt{\frac{0.0048 \times 0.81 \times 10^{-7}}{0.027 \times 0.04 \times 10^{6}}} = \sqrt{\frac{48 \times 10^{-4} \times 81 \times 10^{-2} \times 10^{-7}}{27 \times 10^{-3} \times 4 \times 10^{-2} \times 10^{6}}} = \sqrt{\frac{48 \times 81 \times 10^{-4-2-7}}{27 \times 4 \times 10^{-3-2+6}}}$

$= \sqrt{\frac{12 \times 3 \times 10^{-13}}{1 \times 1 \times 10^{1}}} = \sqrt{12 \times 3 \times 10^{-13-1}}$

$= \sqrt{36 \times 10^{-14}} = \sqrt{36} \times \sqrt{10^{-14}} = 6 \times 10^{-7}.$

6. (a) Without using calculators, simplify $\frac{\frac{3}{4}\left(3\frac{3}{8} + 1\frac{5}{6}\right)}{2\frac{1}{8} - 1\frac{1}{2}}$. (**June 2002**).

(b) Without using mathematical tables or calculators, evaluate $\frac{0.0125 \times 0.00576}{0.0015 \times 0.32}$ leaving your answer in standard form. (**June 2003**).

Solution

(a) $$\frac{\frac{3}{4}\left(3\frac{3}{8}+1\frac{5}{6}\right)}{2\frac{1}{8}-1\frac{1}{2}} = \frac{\frac{3}{4}\left(\frac{27}{8}+\frac{11}{6}\right)}{\frac{17}{8}-\frac{3}{2}} = \frac{\frac{3}{4}\left(\frac{81+44}{24}\right)}{\frac{17-12}{8}} = \frac{\frac{3}{4}\times\frac{125}{24}}{\frac{5}{8}}$$

$$= \frac{3}{4}\times\frac{125}{24}\times\frac{8}{5} = \frac{3}{4}\times\frac{25}{3} = \frac{25}{4} = 6\frac{1}{4}.$$

(b) $$\frac{0.0125\times 0.00576}{0.0015\times 0.32} = \frac{125\times10^{-4}\times 576\times10^{-5}}{15\times10^{-4}\times 32\times10^{-2}} = \frac{5\times25\times10^{-4}\times 18\times32\times10^{-5}}{3\times5\times10^{-4}\times 32\times10^{-2}}$$

$$= \frac{25\times 18\times 10^{-5}}{3\times10^{-2}} = 25\times6\times10^{-5+2} = 150\times10^{-3} = 1.5\times10^{-1}.$$

7. (a) Without using calculators, evaluate $\left(\frac{2}{3}\text{ of }2\frac{1}{4}\right)\div\left(3\frac{1}{2}-2\frac{1}{4}\right)$. **(Nov. 2007)**.

(b) Simplify $6\frac{2}{3}\div\left(3\frac{4}{15}-1\frac{3}{5}\right)$. **(Nov. 2006)**.

Solution

(a) $$\left(\frac{2}{3}\text{ of }2\frac{1}{4}\right)\div\left(3\frac{1}{2}-2\frac{1}{4}\right) = \left(\frac{2}{3}\times\frac{9}{4}\right)\div\left(\frac{7}{2}-\frac{9}{4}\right) = \left(\frac{3}{2}\right)\div\left(\frac{14-9}{4}\right)$$

$$= \frac{3}{2}\div\frac{5}{4} = \frac{3}{2}\times\frac{4}{5} = \frac{3}{1}\times\frac{2}{5} = \frac{6}{5} = 1\frac{1}{5}.$$

(b) $$6\frac{2}{3}\div\left(3\frac{4}{15}-1\frac{3}{5}\right) = \frac{20}{3}\div\left(\frac{49}{15}-\frac{8}{5}\right) = \frac{20}{3}\div\left(\frac{49-24}{15}\right)$$

$$= \frac{20}{3}\div\frac{25}{15} = \frac{20}{3}\times\frac{15}{25} = \frac{4}{1}\times\frac{5}{5} = 4.$$

Revision Exercises 2

1. Represent the following sets on the number line:
   (i) $A = \{x: x\in R, -4\le x\le 1\}$, (ii) $B = \{x: x\in R, x<-3 \text{ or } x\ge 2\}$.
2. Correct the following to two decimal places:
   (a) 4.5683, (b) 9.523, (c) 36.996, (d) 15.008.
3. Correct the following to three significant figures:
   (a) 5.3698, (b) 56.987, (c) 369.8, (d) 6997.
4. A boy measured the length of his mathematics text book to be 24.1 cm. If the original radius is 24 cm, find the percentage error he made in measuring the radius.
5. The area of a circle was measured to be $\frac{24}{25}$ of its original area. Find the percentage error made in measuring the area.

6. Evaluate: (a) $\dfrac{\frac{1}{3}-2\frac{1}{9}+\frac{5}{3}}{\frac{2}{3}-\frac{1}{6}}$, (b) $\dfrac{\frac{3}{2}+\frac{3}{4}-\frac{5}{4}}{\frac{7}{8}-\frac{1}{4}}$, (c) $\dfrac{\left(\frac{1}{2}\times\frac{2}{3}\right)-\left(\frac{5}{12}\div\frac{3}{4}\right)}{\frac{1}{6}+\frac{7}{2}}$.

7. Without using calculators, evaluate

(a) $\dfrac{0.64\times0.81}{0.16\times0.09}$, (b) $\dfrac{0.8\times1.\dot{3}}{0.\dot{3}\times0.4}$, (c) $\dfrac{0.24\times0.1\dot{6}}{0.02\times0.5}$, (d) $\dfrac{0.32\times0.36}{0.09\times0.04}$.

8. Evaluate the following leaving your answers in standard form.

(a) $6400000 \div 0.004$, (b) 3200×0.000000045,

(c) $\dfrac{12000\times0.00081}{0.0024\times90000}$, (d) $\dfrac{0.0025\times0.36}{0.0012\times0.005}$.

9. Which of the following are rational and which are irrational?

(a) 0.35, (b) $\sqrt{8}$, (c) 0.234 324 432 1…, (d) $\sqrt{64}$,

(e) $\frac{5}{8}$, (f) 0.541 541 …, (g) 0. 225 252 525 …, (h) $-0.235\ 217...$,

10. Without using calculator, evaluate

(a) $\dfrac{8\frac{1}{3}\times\frac{3}{50}}{2\frac{7}{12}\times\frac{3}{7}-\frac{3}{4}\times\frac{1}{7}}$, (b) $\dfrac{5\frac{3}{4}\div15\frac{1}{3}-4\frac{1}{2}\times\frac{2}{15}}{\frac{1}{2}-\frac{7}{20}}$.

CHAPTER THREE

Algebraic Expressions

3.1 Introduction

Expressions such as $2x + 3y$, $\frac{1}{2}a^2 + \frac{3}{4}bc$, $2x^2 - 3x + 2$ which contain letters put together using +, –, × and ÷ are known as ***Algebraic Expressions***. Letters or symbols are used to represent quantities, and signs are used to represent relations between them. Symbols or letters which have fixed values are called ***constants***. Symbols or letters which represent changing or changeable quantities are called ***variables***. For example, in $\frac{1}{3}\pi r^2 h$, r and h are variables whilst $\frac{1}{3}$ and π are constants. The ***terms*** in an expression are separated by plus (+) or minus (–). A number that stands on its own is called the ***constant term***. For example, in the expression $3xy^2 + 2xz^3 - 4y + 5$, the first term is $3xy^2$, the second term is $2xz^3$, the third and fourth terms are $4y$ and 5 respectively. The number 5 is the constant term in the expression. The number placed in front of a letter or group of letters is called the ***coefficient***. For example, in $2x - 3x^2 + 4y^3$, the coefficient of x is 2, the coefficient of x^2 is –3 and that of y^3 is 4. Terms in an algebraic expression with the same letter(s) and power(s) are called ***like-terms***. For example $4x^2y$ and $7x^2y$ are like terms but $5x^2y$ and $9xy^2$ are not like terms since the powers of x and y in the two expressions are not the same.

Note: 1. × and ÷ signs are not usually written in algebraic expressions. $a \times b$ is usually written as ab (so as not to confuse the multiplication sign with the letter x) and $a \div b$ is also written as $\frac{a}{b}$.

2. Numbers are written first, followed by letters in alphabetical order. For example: $6x^2yz$, $9abc^3$.
3. Plus (+) or minus (–) signs in an algebraic expression go with the term which follows it.

3.1.1 Building algebraic expressions

When building an algebraic expression using mathematical symbols, always remember to state what the letters represent, giving units where necessary.

There are some problems which are stated in words. Such problems can be written as algebraic equations in mathematics. Letters or symbols are used to denote unknown quantities. For each unknown quantity, a different letter or symbol should be used to denote it. For example: If x denotes an unknown number, then

(i) Six times a number $\Rightarrow$ $6x$

(ii) Five less than a number $\Rightarrow$ $x - 5$

(iii) Seven more than a number $\Rightarrow$ $x + 7$

(iv) Two less than a number is three times the number $\Rightarrow$ $x - 2 = 3x$

(v) A certain number was divided by 2, and then 5 was added to the result. The final answer is 4 $\Rightarrow$ $\frac{x}{2} + 5 = 4$

(vi) Eight more than twice a number is four less than five times the number. Find the number $\Rightarrow$ $2x + 8 = 5x - 4$

(vii) When 3 is added to a number, and the result is multiplied by 5, you obtain two times the number $\Rightarrow$ $5(x + 3) = 2x$.

Example 3.1

Mr. Amah and his family were going on a journey by bus. He bought two tickets for himself and his wife, and six tickets at half price for his children. Express in symbols the total amount he paid in terms of the price of tickets for adults.

Solution

Suppose the cost of a ticket is x and c is the total cost of tickets bought, in cedis.
The cost of ticket for himself and the wife = $2x$

The cost of tickets for the 4 children = $\frac{1}{2}(6x) = 3x$

Therefore $c = 2x + 3x \quad \Rightarrow \quad c = 5x$

Example 3.2

Akrong needs a certain amount of money for school fees for next year. His father told him: 'I will give you GH¢ 80 to start with, and then for each GH¢ 8 that you earn by yourself, I will give you GH¢ 24 more.' Express the amount that Akrong needs in terms of the number of times he earned GH¢ 8.

Solution

Let n be the number of times Akrong earned GH¢ 8 and c the amount of money he needs for the school fees.
Initial amount given by the father = GH¢ 80
The total amount he earned = $8n$
Additional money given by the father = $24n$
$\therefore \quad c = 80 + 8n + 24n$ and hence $c = 80 + 32n$.

Example 3.3

A supermarket pays its sales personnel on a weekly basis. At the end of each week, each sales person receives a basic weekly wage of GH¢ 500, less GH¢ 150 for every day he was absent, and plus a bonus of GH¢120 for every day he worked. How much did a man earn in a week in which

(a) he worked for all the seven days,

(b) he worked for only 5 days,

(c) he worked for only 3 days?

Solution

Let x be the number of days he worked,
y be the number of days he was absent,
and c be the total amount of money he earned in a week.
Therefore $c = 500 + 120x - 150y$.

(a) $x = 7$ and $y = 0$

$\therefore$ $c = 500 + 120 \times 7 - 150 \times 0 = 500 + 840 = 1340$.

He earned GH¢ 1,340.

(b) $x = 5$ and $y = 2$

$c = 500 + 120 \times 5 - 150 \times 2 = 500 + 600 - 300 = 800$

He earned GH¢ 800

(c) $x = 3$ and $y = 4$

$c = 500 + 120 \times 3 - 150 \times 4 = 500 + 360 - 600 = 260$

He earned GH¢ 260.

Example 3.4

Kofi, Mensah and Asamoah are three brothers. Kofi is x years older than Mensah and Mensah is y years older than Asamoah.

(a) How much older than Asamoah is Kofi?

(b) If Asamoah is 5 years old, how old is: (i) Mensah, (ii) Kofi?

Solution

(a) Let k, m and a denote ages of Kofi, Mensah and Asamoah respectively. If Kofi is x years older than Mensah, then

Kofi's age = Mensah's age + x years.

Thus,

$k = m + x$ ....(1)

If Mensah is y years older than Asamoah, then

Mensah's age = Asamoah's age + y years.

Thus,

$m = a + y$ ....(2)

So, from (1) and (2),

$k = (a + y) + x$

$= a + (y + x)$

$\therefore$ Kofi is $(y + x)$ years older than Asamoah.

(b) (i) Mensah's age = Asamoah's age + y year = $(5 + y)$ years.

(ii) Kofi's age = Asamoah's age + $(y + x)$ years = $(5 + y + x)$ years

Exercise 3.1

1. Write an algebraic expression for each of the following, given that the number is x.
   (a) I thought of a number. I multiplied by 3, and then subtracted 5. The result was y.
   (b) A certain number was divided by 3, and then 3 was added to the result. The final answer is w.
   (c) Eight more than twice a number is four less than five times y.
   (d) When 3 is added to a number, and the result is multiplied by 5, you obtain k.
   (e) If a number subtracted from 100 gives a result equal to one-third of u.
   (f) 3 added to five times a number, and the result is multiplied by 4. The final result is v.
   (g) If you subtracted a number from 6, you get the same result as you get by adding the number to t.
   (h) Four times two less than a number is the same as trice one more than p.

2. Mavis, Doris and Agnes are three friends. Mavis is p years older than Doris and Doris is q years older than Agnes.
   (a) How much older than Agnes is Mavis?
   (b) If Agnes is 14 years old, how old is: (i) Doris, (ii) Mavis?

3. When 3 is added to a number, and the result is multiplied by 5, find an expression for the number you obtain.

4. A student paid a certain amount of money for 30 pencils. Some cost GH¢ 1.00 each, and the others cost GH¢ 2.00 each. Find an expression for the amount the student paid.

5. Find the area, if 200 m of fencing is to be used to make a rectangular enclosure.

6. A gardener wishes to make a rectangular hen-run of area 128 m^2 against a wall which is to serve as one of the boundaries. Find the length of wire netting required for the other three sides.

3.2 Operations on algebraic expressions

3.2.1 Addition and subtraction of algebraic expressions

We now consider how to add and subtract algebraic expressions. To do this we first gather similar terms together (grouping like terms) and then combine them. The four rules of addition, subtraction, multiplication and division can be used to simplify algebraic expressions. Only like terms can be added or subtracted to give a single term. For example: $3x^2 + 4x^2 = 7x^2$ and $9x^2y – 5x^2y = 4x^2y$.

Example $5a + 4b – 3a + 3b$ can be written as $5a – 3a + 4b + 3b = 2a + 7b$.

Only like terms can be added or subtracted

Example 3.5

Add $3xy + 4x - 5y$ to $12 - 6x + 8y - 9xy$

Solution

$$\begin{aligned}(3xy + 4x - 5y) + (12 - 6x + 8y - 9xy) &= 3xy + 4x - 5y + 12 - 6x + 8y - 9xy \\ &= 12 + 4x - 6x - 5y + 8y + 3xy - 9xy \\ &= 12 - 2x + 3y - 6xy.\end{aligned}$$

Example 3.6

Subtract $6y^2 - 9xy + 3x^2 + 3$ from $4x^2 - 5xy + 8y^2$.

Solution

$$\begin{aligned}(4x^2 - 5xy + 8y^2) - (6y^2 - 9xy + 3x^2 + 3) &= 4x^2 - 5xy + 8y^2 - 6y^2 + 9xy - 3x^2 - 3 \\ &= 4x^2 - 3x^2 - 5xy + 9xy + 8y^2 - 6y^2 - 3 \\ &= x^2 + 4xy + 2y^2 - 3.\end{aligned}$$

Example 3.7

Simplify the following expressions:

(a) $7x + 3y - 4 - 5x + 2y - 3$, (b) $5x + 7y - 2x - 9y$,

(c) $2x^2 + 7y - 4x + 5y^2 - x^2 - 2y + 6x - 9y^2$, (d) $3x^2y - 5xy^2 + 5x^2y + 4xy^2$,

(e) $6x^2y^3 - 5x^5y^2 + 4x^2y^3 - 9x^5y^2 - 2x^2y^3$.

Solution

(a) $7x + 3y - 4 - 5x + 2y - 3 = 7x - 5x + 3y + 2y - 4 - 3$ **(collecting like terms)**

$= 2x + 5y - 7$

(b) $5x + 7y - 2x - 9y = 5x - 2x + 7y - 9y$ **(collecting like terms)**

$= 3x - 2y$

(c) $2x^2 + 7y - 4x + 5y^2 - x^2 - 2y + 6x - 9y^2$

$= 2x^2 - x^2 + 5y^2 - 9y^2 - 4x + 6x + 7y - 2y$

$= x^2 - 4y^2 + 2x + 5y$

(d) $3x^2y - 5xy^2 + 5x^2y + 4xy^2 = 3x^2y + 5x^2y - 5xy^2 + 4xy^2$ **(collecting like terms)**

$= 8x^2y - xy^2$

(e) $6x^2y^3 - 5x^5y^2 + 4x^2y^3 - 9x^5y^2 - 2x^2y^3$ **(collecting like terms)**

$= 6x^2y^3 + 4x^2y^3 - 2x^2y^3 - 5x^5y^2 - 9x^5y^2$

$= (6 + 4 - 2)x^2y^3 - (5 + 9)x^5y^2$

$= 8x^2y^3 - 14x^5y^2$

Addition and subtraction of fractional terms

To add or subtract like terms of algebraic expressions with fractional coefficients, we first determine the LCM of the denominators before performing the operation. Consider the following examples.

Example 3.8

Simplify the following expressions:

(a) $\frac{1}{7}x+\frac{4}{7}x$, (b) $\frac{3}{8}a+\frac{5}{16}a$, (c) $\frac{6}{5}y^2-\frac{11}{15}y^2$, (d) $\frac{8}{12}p^2-\frac{7}{16}p^2$.

Solution

(a) $\frac{1}{7}x+\frac{4}{7}x = \left(\frac{1}{7}+\frac{4}{7}\right)x = \frac{1+4}{7}x = \frac{5}{7}x.$

(b) $\frac{3}{8}a+\frac{5}{16}a = \left(\frac{3}{8}+\frac{5}{16}\right)a = \left(\frac{3\times2+5\times1}{16}\right)a = \left(\frac{6+5}{16}\right)a = \frac{11}{16}a.$

(c) $\frac{6}{5}y^2-\frac{11}{15}y^2 = \left(\frac{6}{5}-\frac{11}{15}\right)y^2 = \left(\frac{6\times3-11\times1}{15}\right)y^2 = \left(\frac{18-11}{15}\right)y^2 = \frac{7}{15}y^2.$

(d) $\frac{8}{12}p^2-\frac{7}{16}p^2 = \left(\frac{8}{12}-\frac{7}{16}\right)p^2 = \left(\frac{8\times4-7\times3}{48}\right)p^2 = \left(\frac{32-21}{48}\right)p^2 = \frac{11}{48}p^2.$

Example 3.9

Simplify the following expressions:

(a) $\frac{1}{3}a+\frac{2}{5}b+\frac{2}{3}a-\frac{3}{5}b$, (b) $\frac{2}{5}a^2-\frac{5}{6}b^2+\frac{3}{10}a^2-\frac{1}{3}b^2$.

Solution

(a) $\frac{1}{3}a+\frac{3}{5}b+\frac{2}{3}a-\frac{2}{5}b = \frac{1}{3}a+\frac{2}{3}a+\frac{3}{5}b-\frac{2}{5}b$

$= \left(\frac{1}{3}+\frac{2}{3}\right)a+\left(\frac{3}{5}-\frac{2}{5}\right)b$

$= \left(\frac{1+2}{3}\right)a+\left(\frac{3-2}{5}\right)b = \frac{3}{3}a-\frac{1}{5}b = a-\frac{b}{5}.$

(b) $\frac{2}{5}a^2-\frac{5}{6}b^2+\frac{3}{10}a^2-\frac{1}{3}b^2 = \frac{2}{5}a^2+\frac{3}{10}a^2-\frac{5}{6}b^2-\frac{1}{3}b^2$

$= \left(\frac{2}{5}+\frac{3}{10}\right)a^2-\left(\frac{5}{6}+\frac{1}{3}\right)b^2$

$= \left(\frac{4+3}{10}\right)a^2-\left(\frac{5+2}{6}\right)b^2 = \frac{7}{10}a^2-\frac{7}{6}b^2.$

Example 3.10

Express as single fraction:

(a) $\frac{x-1}{2}+\frac{x+1}{3}$, (b) $\frac{5x-2}{3}-\frac{x+3}{6}$, (c) $\frac{a+2b}{5}+\frac{a-2b}{3}$.

Solution

(a) $\frac{x-1}{2} + \frac{x+1}{3} = \frac{3(x-1)+2(x+1)}{6}$ (the LCM of the denominators is 6)

$= \frac{3x-3+2x+2}{6} = \frac{3x+2x-3+2}{6} = \frac{5x-1}{6}.$

(b) $\frac{5x-2}{3} - \frac{x+3}{6} = \frac{2(5x-2)-(x+3)}{6}$ (the LCM of the denominators is 6)

$= \frac{10x-4-x-3}{6} = \frac{10x-x-4-3}{6} = \frac{9x-7}{6}.$

(c) $\frac{a+2b}{5} + \frac{a-2b}{3} = \frac{3(a+2b)+5(a-2b)}{15}$ (the LCM of the denominators is 15)

$= \frac{3a+6b+5a-10b}{15} = \frac{3a+5a+6b-10b}{15} = \frac{8a-4b}{15}.$

Example 3.11

Simplify the following expressions:

(a) $\frac{1}{3}a+\frac{2}{5}b+\frac{2}{3}a-\frac{3}{5}b,$ (b) $\frac{4}{9}x^3y^4-\frac{7}{15}x^2y^5-\frac{1}{3}x^3y^4+\frac{2}{5}x^2y^5.$

Solution

(a) $\frac{1}{3}a+\frac{2}{5}b+\frac{2}{3}a-\frac{3}{5}b = \frac{1}{3}a+\frac{2}{3}a+\frac{2}{5}b-\frac{3}{5}b = \frac{a+2a}{3} + \frac{2b-3b}{5}$

$= \frac{3a}{3} - \frac{b}{5} = a - \frac{b}{5} = \frac{5a-b}{5}.$

(c) $\frac{4}{9}x^3y^4-\frac{7}{15}x^2y^5-\frac{1}{3}x^3y^4+\frac{2}{5}x^2y^5 = \frac{4}{9}x^3y^4-\frac{1}{3}x^3y^4-\frac{7}{15}x^2y^5+\frac{2}{5}x^2y^5$

$= \frac{4x^3y^4-3x^3y^4}{9} + \frac{-7x^2y^5+6x^2y^5}{15}$

$= \frac{x^3y^4}{9} + \frac{-x^2y^5}{15} = \frac{5x^3y^4-3x^2y^5}{45}.$

Exercise 3.2a

1. Find the sum of
   (a) $(7x+3y-4)$ and $(-5x+2y-3)$,
   (b) $(5x+7y)$ and $(-2x-9y)$,
   (c) $(2x^2+7y-4x+5y^2)$ and $(-x^2-2y+6x-9y^2)$,
   (d) $3x^2y-5xy^2+5x^2y+4xy^2$,
   (e) $(6x^2y^3-5x^5y^2)$, $(-9x^5y^2-2x^2y^3)$ and $4x^2y^3$.

2 (a) From $3x-4xy+y$, take away $x-2xy-5y$,
  (b) From $6x^2-4x^2y+3xy^2-2y^2$, take away $5y^2-xy^2-2x^2y+2x^2$,

(c) From $2a+4ab^2-3b+5$, take away $2b+8-2ab^2-a$,
(d) From $7-5u^2+4uv-v^2$ take away $7uv-3u^2-4v^2+5$,
(e) From $p^2-8p^2q^2+2q$, take away $3p^2-6p^2q^2-6$.

3. Simplify the following expressions:
(a) $5x^2+8y+7y^2-2x^2-4y-8y^2$,
(b) $13pq^2-3pq-5pq^2-4pq$.
(c) $7x^2+9y-2-4y^2-5x^2-7y-5-2y^2$,
(d) $6u^2v-5-9u^2v+7u^2v-8$.
(e) $9xy^2+7y-5xy^2-10y-2xy^2+8y$,
(f) $8a^2b^4-5ab^3-3a^2b^4-2a^2b^4$.
(g) $5xy^2-4x^2y-3+5y^2-x^2y-3xy^2-2$,
(h) $7st^3-8s^2t-5-6st^3-s^2t-3$.

4. Simplify the following expressions:
(a) $\frac{3}{5}x+\frac{4}{7}x$, (b) $\frac{4}{9}a+\frac{1}{3}a$, (c) $\frac{7}{8}y^2-\frac{3}{16}y^2$, (d) $\frac{6}{5}p^2-\frac{7}{25}p^2$,
(e) $\frac{3}{14}v+\frac{2}{21}v$, (f) $\frac{5}{16}q+\frac{3}{24}q$, (g) $\frac{5}{18}s^3-\frac{7}{27}s^3$, (h) $\frac{7}{32}u^4-\frac{5}{48}u^4$.

5. Simplify the following expressions:
(a) $\frac{1}{6}x+\frac{2}{7}y+\frac{1}{4}x-\frac{3}{14}y$,
(b) $\frac{3}{13}v^2-\frac{7}{12}u^2+\frac{5}{26}v^2-\frac{1}{24}u^2$,
(c) $\frac{5}{12}a^3+\frac{2}{9}b^2+\frac{1}{8}a^3-\frac{1}{6}b^2$,
(d) $\frac{2}{7}p^4+\frac{1}{9}q^5-\frac{5}{21}p^4-\frac{2}{15}q^5$,
(e) $\frac{4}{15}s-\frac{5}{24}t^6+\frac{7}{30}s+\frac{13}{32}t^6$,
(f) $\frac{5}{6}x-\frac{1}{18}xy-\frac{7}{24}x-\frac{2}{27}xy$,
(g) $\frac{3}{16}m-\frac{3}{10}n-\frac{5}{32}m+\frac{9}{15}n$,
(h) $\frac{2}{9}kl^2-\frac{1}{8}gh-\frac{1}{6}kl^2+\frac{7}{12}gh$.

6. Express as single fraction:
(a) $\frac{2x-1}{3}-\frac{x-2}{3}$, (b) $\frac{x-3}{2}+\frac{x-2}{3}$, (c) $\frac{2y-x}{4}-\frac{3y+x}{5}$,
(d) $\frac{2a-3}{2}+\frac{3a+2}{4}$, (e) $\frac{3x-1}{2}-\frac{7x-2}{5}$, (f) $\frac{3-4x}{3}+\frac{6x-4}{5}$,

7. Simplify the following expressions:
(a) $4x + 6y - 8 - 9x + 5y - 2$,
(b) $6x+7y-5x-4y$,
(c) $4x^2-8y+4x+3y^2-5x^2+3y+2x-6y^2$,
(d) $\frac{3}{7}a+\frac{13}{5}b+\frac{4}{7}a-\frac{3}{5}b$,
(e) $\frac{3}{8}a^2-\frac{5}{6}b^2+\frac{21}{8}a^2-\frac{7}{6}b^2$,
(f) $4x^2y-6xy^2+9x^2y+8xy^2$.

8. (a) Take the sum of $6x - 5y - 2z$ and $2x + 4y + z$ from the sum of $3x - 2y + 5z$ and $7x + 5y - 3z$.
(b) Take the sum of $5p + 4q - 3r$ and $3p - 2q + 5r$ from the sum of $9p - 7q - 12r$ and $3p + 8q + 15r$.
(c) Take the sum of $7a^2-2ab+3b^2$ and $5ab-5a^2-7b^2$ from the sum of $3a^2+4ab-6b^2$ and $a^2-ab+9b^2$.

(d) What must be added to $2x^4+4x^3-2x^2+3$ to give $7x^4+9x^3+4x^2-5$.

9. In Fig. 3.1, the dimensions are given in terms of p and q. Find, in terms of p and q the expression for:
(a) the lengths of AB, GH and EF,
(b) the lengths of DE and FG,
(c) the perimeter of the figure.

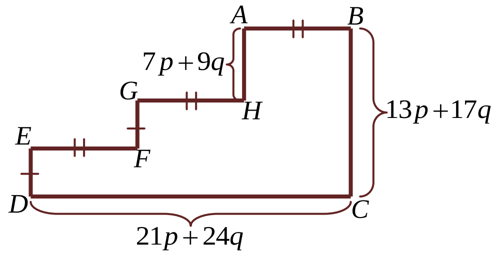


Fig. 3.1

3.2.2 Multiplication and division

In performing multiplication and division of algebraic expressions, it is important to first group the numbers and the same letters together and apply the basic operating rules of indices. The basic rules for multiplication and division of algebraic expressions are

1. $a^m \times a^n = a^{m+n}$,   2. $\dfrac{a^m}{a^n} = a^{m-n}$.

Example 3.12

(a) $5a^2 \times 4a$,   (b) $-3x \times 2y \times 4z$,   (c) $3xy^2 \times 4x^3y^4$,
(d) $32a^4b^2 \div 4a^2b$,   (e) $35a^6b^5 \div 5a^4b^2$,   (f) $36m^8n^6 \div 4m^6n^4$.

Solution

(a) $5a^2 \times 4a = 5 \times 4a^2a = 20a^{2+1} = 20a^3$

(b) $-3x \times 2y \times 4z = -3 \times 2 \times 4xyz = -24xyz$

(c) $3xy^2 \times 4x^3y^4 = 3 \times 4x^{1+3}y^{2+4} = 12x^4y^6$

(d) $32a^4b^2 \div 4a^2b = \dfrac{32a^4b^2}{4a^2b} = 8a^{4-2}b^{2-1} = 8a^2b$

(e) $35a^6b^5 \div 5a^4b^2 = \dfrac{35a^6b^5}{5a^4b^2} = 7a^{6-4}b^{5-2} = 7a^2b^3$

(f) $36m^8n^6 \div 4m^6n^4 = \dfrac{36m^8n^6}{4m^6n^4} = 9m^{8-6}n^{6-4} = 9m^2n^2$

Exercise 3.2b

Simplify the following.

(a) $2a \times 5a^2$,   (b) $3c^2 \times 2c^3 \times c^5$,   (c) $\dfrac{8d^3 \times 2d^5}{4d^2 \times d^4}$,   (d) $\dfrac{25e^6 \times e^3}{5e^2 \times e^8}$,

(e) $2a^4b \times ab^5$,   (f) $3a^2b^7 \times 2a^6b^2$,   (g) $\dfrac{6a^7b^4}{3a^3b^2}$,   (h) $\dfrac{3^5x^9y^{15}}{3^3x^5y^{12}}$,

(i) $\dfrac{2x^3y \times x^6y^9}{x^4y^2 \times x^2y^7}$, (j) $\dfrac{12p^4q^2 \times p^5q^5}{pq^4 \times 4p^6q^4}$, (k) $\dfrac{16m^4n^5 \times m^6n^7}{m^2n^4 \times 8m^3n^5}$, (l) $\dfrac{12u^4v^2 \times v^5}{uv^4 \times 4u^6}$.

3.3 Expansion of algebraic expressions

Expansion in algebra simply means removal of brackets. To remove the bracket we use the distribution law. Multiply the term outside the bracket by each of the terms inside the bracket. For example $a \times (b + c) = (a \times b) + (a \times c)$, which is simply written as $a(b + c) = ab + ac$. Likewise $x(y-z) = xy - xz$.

Note that: $-3x \times y = -3xy$ and $-3x \times -y = 3xy$

Example 3.13

Expand the following expressions:

(a) $3(a-b)$, (b) $a(3c+d)$, (c) $2x(x-3y)$, (d) $4a(2b+3c)$.

Solution

(a) $3(a-b) = 3a-3b$, (b) $a(3c+d) = 3ac+ad$,

(c) $2x(x-3y) = 2x^2-6xy$, (d) $4a(2b+3c) = 8ab+12ac$.

Example 3.14

Expand and simplify the following expressions:

(a) $2(a-b)+3(a-b)$, (b) $4(x-2y)+5(x+y)$,

(c) $a(6p-3q)-4a(p-q)$ (d) $4a(a+b)+a(2a+b)$.

Solution

(a) $2(a-b)+3(a-b) = 2a-2b+3a-3b = 2a+3a-2b-3b = 5a-(2b+3b) = 5a-5b$.

(b) $4(x-2y)+5(x+y) = 4x-8y+5x+5y = 4x+5x-(8y-5y) = 9x-3y$.

(c) $a(6p-3q)-4a(p-q) = 6ap-3ap-4ap+4aq = 6ap-4ap+4aq-3aq = 2ap-aq$.

(d) $4a(a+b)+a(2a+b) = 4a^2+4ab+2a^2+ab = 4a^2+2a^2+4ab+ab = 6a^2+5ab$.

Example 3.15

Expand and simplify the following expressions.

(a) $2y(5y-x)-3x(5x-2y)$, (b) $6p^2(2q+3p)-5q^2(-4p-5q)$

Solution

(a) $2y(5y-x)-3x(5x-2y) = 10y^2-2xy-15x^2+6xy = 10y^2+4xy-15x^2$.

(b) $6p^2(2q+3p)-5q^2(-4p-5q) = 12p^2q+18p^3+20q^2p+25q^3$

3.3.1 Multiplying two binomials

A binomial is an expression containing two terms. Example $(x+2y)$, $(3b-2a)$ and (p^2+2qr) are binomials. The expansion of the product of two binomials involves the application of the distributive property of multiplication over addition or subtraction. For example $(x + y)(a + b) = x(a + b) + y(a + b) = ax + bx + ay + by$.

Example 3.16

Expand and simplify the following expressions:

(a) $(a+2)(a+3)$, (b) $(p-3)(p+5)$, (c) $(x-y)(2x-y)$, (d) $(3x+y)(x-2y)$.

Solution

(a) $(a+2)(a+3) = a(a+3)+2(a+3) = a^2+3a+2a+6 = a^2+5a+6.$

(b) $(p-3)(p+5) = p(p+5)-3(p+5) = p^2+5p-3p+15 = p^2+2p+15.$

(c) $(x-y)(2x-y) = x(2x-y)-y(2x-y) = 2x^2-xy-2xy+y^2 = 2x^2-3xy+y^2.$

(d) $(3x+y)(x-2y) = 3x(x-2y)+y(x-2y) = 3x^2-6xy+xy-2y^2 = 3x^2-5xy-2y^2.$

Example 3.17

Expand and simplify the following expressions

(a) $(x+2)(x+4) + (x+3)(x+1)$, (b) $(y-2)(y+5) - (y-3)$.

(c) $(p + q)(4p+2q) + (3p+2q)(5p–3q)$ (d) $(4p+6q)(7p–4q) – (2p–q)(3p–q)$.

Solution

(a)
$$\begin{aligned}(x+2)(x+4) + (x+3)(x+1) &= \left[x(x+4)+2(x+4)\right] + \left[x(x+1)+3(x+1)\right]\\ &= \left(x^2+4x+2x+16\right) + \left(x^2+x+3x+3\right)\\ &= \left(x^2+6x+16\right) + \left(x^2+4x+3\right)\\ &= x^2+x^2+6x+4x+16+3 = 2x^2+10x+19.\end{aligned}$$

(b)
$$\begin{aligned}(y-2)(y+5) - (y-3) &= y(y+5)-2(y+5)-(y-3)\\ &= y^2+5y-2y-10-y+3 = y^2+3y-10-y+3\\ &= y^2+3y-y-10+3 = y^2+2y-7.\end{aligned}$$

(c) $(p + q)(4p+2q) + (3p+2q)(5p–3q)$
$$\begin{aligned}&= 4p^2 + 2pq + 4pq + 2q^2 + 15p^2 - 9pq + 10pq - 6q^2\\ &= 4p^2 + 6pq + 2q^2 + 15p^2 + pq - 6q^2\\ &= 19p^2 + 7pq - 4q^2\end{aligned}$$

(d) $(4p+6q)(7p–4q) – (2p–q)(3p–q)$
$$\begin{aligned}&= 28p^2 - 16pq + 42pq - 24q^2 - (6p^2 - 2pq - 3pq + q^2)\\ &= 28p^2 + 26pq - 24q^2 - (6p^2 - 5pq + q^2)\end{aligned}$$

$= \; 28p^2 + 26pq - 24q^2 - 6p^2 + 5pq - q^2$
$= \; 22p^2 + 31pq - 25q^2.$

Example 3.18
Expand and simplify the following expressions:
(a) $(2a - b)(3a + 2b)(a + 2b)$ (b) $(5a - 6b)(4a - 3b)(3a + 2b)$.

Solution
(a) $(3a + 2b)(a + 2b) = 3a(a + 2b) + 2b(a + 2b)$
$= 3a^2 + 6ab + 2ab + 4b^2$
$= 3a^2 + 8ab + 4b^2$

$(2a - b)(3a + 2b)(a + 2b) = (2a - b)(3a^2 + 8ab + 4b^2)$
$= 2a(3a^2 + 8ab + 4b^2) - b(3a^2 + 8ab + 4b^2)$
$= 6a^3 + 16a^2b + 8ab^2 - 3a^2b - 8ab^2 - 4b^3$
$= 6a^3 + 13a^2b - 4b^3$

(b) $(4a - 3b)(3a + 2b) = 4a(3a + 2b) - 3b(3a + 2b)$
$= 12a^2 + 8ab - 9ab - 6b^2$
$= 12a^2 - ab - 6b^2$

$\therefore (5a - 6b)(4a - 3b)(3a + 2b) = (5a - 6b)(12a^2 - ab - 6b^2)$
$= 5a(12a^2 - ab - 6b^2) - 6b(12a^2 - ab - 6b^2)$
$= 60a^3 - 5a^2b - 30ab^2 - 72a^2b + 6ab^2 + 36b^3$
$= 60a^3 - 77a^2b - 24ab^2 + 36b^3$

Note the following identities:
Perfect squares

$$(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b) = a^2+ab+ab+b^2 = a^2+2ab+b^2.$$
$$(a-b)^2 = (a-b)(a-b) = a(a-b)-b(a-b) = a^2-ab-ab+b^2 = a^2-2ab+b^2.$$

Difference of two squares

$$(a+b)(a-b) = a(a-b)+b(a-b) = a^2-ab+ab-b^2 = a^2-b^2.$$

Example 3.19
Expand the following using the identities above:
(a) $(2x - 3y)^2$, (b) $(4x + 5y)^2$, (c) $(5 - x)(5 + x)$,
(d) $(5a^2 - 2b)^2$, (e) $(6a + 7b^3)^2$, (f) $(9a - 5b^2)(9a + 5b^2)$.

Solution

(a) $(2x - 3y)^2 = (2x)^2 - 2(2x)(3y) + (3y)^2 = 4x^2 - 12xy + 9y^2$

(b) $(4x + 5y)^2 = (4x)^2 + 2(4x)(5y) + (5y)^2 = 16x^2 + 40xy + 25y^2$

(c) $(5 - x)(5 + x) = 5^2 - x^2 = 25 - x^2$

(d) $(5a^2 - 2b)^2 = (5a^2)^2 - 2(5a^2)(2b) + (2b)^2 = 25a^4 - 20a^2b + 4b^2$

(e) $(6a + 7b^3)^2 = (6a)^2 + 2(6a)(7b^3) + (7b^3)^2 = 36a^2 + 84ab^3 + 49b^6$

(f) $(9a - 5b^2)(9a + 5b^2) = (9a)^2 - (5b^2)^2 = 81a^2 - 25b^4$

Exercise 3.3

1. Expand the following expressions:
   (a) $3y(2+y)$, (b) $5x(x-3)$, (c) $2p(2p-q)$, (d) $4t(3t+2s)$, (e) $t(2t-r)$.

2. Expand and simplify the following expressions:
   (a) $3(6b+5) + 5(4b+1)$, (b) $x(x+1) + 2(x-1)$, (c) $q(2q-3) + 2(4q-1)$,
   (d) $6r(r+2) - r(r+5)$, (e) $2s(2+s)-3s(1-s)$, (f) $6t(t+1) - 2t(2t-1)$.

3. Expand the following expressions:
   (a) $4l(2l+3m-n)$, (b) $3p(4p-5q+2r)$, (c) $5a(2b-a+c)$,

4. Expand and simplify the following expressions:
   (a) $(2c-b)(c+3b)$, (b) $(5+v)(4+v)$, (c) $(2+a)^2$, (d) $(3x-1)^2$,
   (e) $(x-3)(x-3)$, (f) $(2g-f)(2g+f)$, (g) $(2x+y)(3x-4y)$, (h) $(2c+3)(3c+4)$.

5. Expand and simplify the following expressions
   (a) $(x-1)(x-2) + (x+1)(x+2)$, (b) $(y+2)(y+3) + (y+1)(y+4)$,
   (c) $(t-6)(t-2) + (t+3)(t+2)$, (d) $(2a+1)(a-2) + (a+1)(3a+1)$,
   (e) $(3x + y)(4x - 2y) - (x - y)(2x - y)$, (f) $(2p-5q)(4p+2q) - (p+2q)(5p-4q)$.

6. Expand and simplify the following expressions.
   (a) $4(2a + 5) - 3(2a - 7)$, (b) $3(2x^2 + 4) - 5(3x^2 - 2)$,
   (c) $2y(5y - x) - 3x(5x - 2y)$, (d) $(3x - 2y)(5x - 4y)$,
   (e) $(2x + 5y)(8x - 3y)$, (f) $(5x^2 - 6y)(2y^2 + 3x)$.
   (g) $(p - 4q)(3p + 5q) - (2p + 3q)(4p - 7q)$

3.4 Factorization

When an expression is written as the product of two or more factors, we say that is ***factorized***. Factorizing is the opposite of expanding.

To factorize completely an expression such as $4a + 6bx$:

1. Find the highest common factor (HCF) of all terms. The HCF of the terms $4ax$ and $6bx$ is $2x$.

2. Divide each term separately by this factor, that is $4ax/2x = 2a$ and $6bx/2x = 3b$. Thus, when each of the terms of $4ax + 6bx$ is divided by $2x$ we obtain $2a + 3b$.
3. Enclose the quotient within brackets, that is $(2a + 3b)$.
4. Place the common factor outside the bracket. This gives $2x(2a + 3b)$.

Example 3.20

Factorize completely the following expressions

(a) $ax + bx$, (b) $15p - 27pq$, (c) $18pr^2 + 6pqr$, (d) $7xy^2 + 13x^2y$.

Solution

(a) $ac + bc = c \times a + c \times b$
$= c(a + b)$.

(b) $15p - 27pq = 3p \times 5 - 3p \times 9q$
$= 3p(5 - 9q)$.

(c) $18pr^2 + 6pqr = 6pr \times 3r + 6pr \times q$
$= 6pr(3r + q)$.

(d) $7xy^2 + 13x^2y = xy \times 7y + xy \times 13x$
$= xy(7y + 13x)$.

Example 3.21

(a) $6p - 18 = 6(p - 3)$,
(b) $16x + 24x^2 = 8x(2 + 3x)$,
(c) $12ab + 4a^2 = 4a(3b + a)$,
(d) $20p^2 - 35pq = 5p(4p - 7q)$,
(e) $12a^2b - 16ab^2 = 4ab(3a - 4b)$,
(f) $x^2y^2 - 3xyz = xy(xy - 3z)$.

Factorizing by taking out the negative common factor

If a negative sign is factorize from the terms of an algebraic expression, the signs of each term inside the bracket must be changed. For example,

(a) $-a - b = -(a + b)$, (b) $-a + b = -(a - b)$, (c) $a - b = -(b - a)$.

Consider the following example.

Example 3.22

Factorise by taking out the negative common factor

(a) $-3x - 15$, (b) $-4m^2 + 12m$, (c) $-4x - 12$,
(d) $-15x^2 + 12xy$, (e) $-20y^2 - 35y$, (f) $18ac - 15ad$.

Solution

(a) $-3x - 15 = -(3x + 15) = -3(x + 5)$,
(b) $-4m^2 + 12m = -(4m^2 - 12m) = -4m(m - 3)$,
(c) $-4x - 12 = -(4x + 12) = -4(x + 3)$,
(d) $-15x^2 + 12xy = -(15x^2 - 12xy) = -3x(5x - 4y)$,
(e) $-20y^2 - 35y = -(20y^2 + 35y) = -5(4y + 7)$,
(f) $18ac - 15ad = -(15ad - 18ac) = -3a(5d - 6c)$.

Example 3.23

Factorize the following expressions

(i) $\frac{1}{2}a^2b - \frac{3}{4}ab$, (ii) $\frac{5}{4}a^2p + \frac{7}{24}ap^2$.

Solution

(i) $\frac{1}{2}a^2b - \frac{3}{4}ab \;=\; \frac{2}{4}a^2b - \frac{3}{4}ab \;=\; \frac{1}{4}ab(2a-3)$.

(ii) $\frac{5}{4}a^2p + \frac{7}{24}ap^2 \;=\; \frac{30}{24}a^2p + \frac{7}{24}ap^2 \;=\; \frac{1}{24}ap(30a+7p)$.

3.4.1 Factorization of expressions with compound factors

The expression $a(x + y) + b(x + y)$ is grouped into two terms which has a common compound factor $(x + y)$. Divide each term separately by the common factor $(x + y)$ to obtain the quotient $(a + b)$. Multiply the compound factor $(x + y)$ by the quotient $(a + b)$ to obtain the factorized expression $(x + y)(a + b)$.

Expressions such as $2y^2 + 2xy - xy - x^2$ can be grouped into compound terms as in $(2y^2 + 2xy) - (xy + x^2)$ and then factorize the common terms in each group separately to obtain $2y(y + x) - x(y + x)$. Finally, factorize the common compound factor $(y + x)$ to obtain the factorized expression $(y + x)(2y - x)$

Example 3.24

Factorize the following expressions:

(a) $3(x + 1) + 4(x + 1)$, (b) $a(2x + y) - b(2x + y)$, (c) $3r(q + 2p) + 5(2q + 4p)$.

Solution

(a) $3(x + 1) + 4(x + 1) \;=\; (x + 1)(3 + 4) \;=\; 7(x + 1)$.

(b) $a(2x + y) - b(2x + y) \;=\; (2x + y)(a - b)$.

(c) $3(q + 2p) + 5(2q + 4p) \;=\; 3r(q + 2p) + 10(q + 2p) \;=\; (q + 2p)(3r + 10)$.

Example 3.25

Factorize the following expressions:

(a) $ab+ac+bd+cd$, (b) $6pq+2p+3q+1$, (c) $12xy-8x-3y+2$,

(d) $6ab+2a+9b+3$, (e) $3p^2+p-3px-x$, (f) $2ab+2ac+3b+3c$.

Solution

(a) $$\begin{aligned} ab+ac+bd+cd &= (ab+ac)+(bd+cd) \\ &= a(b+c)+d(b+c) \\ &= (b+c)(a+d) \end{aligned}$$

(b) $$\begin{aligned} 6pq+2p+3q+1 &= (6pq+2p)+(3q+1) \\ &= 2p(3q+1)+1(3q+1) \\ &= (3q+1)(2p+1). \end{aligned}$$

(c) $$\begin{aligned} 12xy-8x-3y+2 &= (12xy-8x)-(3y-2) \\ &= 4x(3y-2)-1(3y-2) \\ &= (3y-2)(4x-1) \end{aligned}$$

(d) $$\begin{aligned} 6ab+2a+9b+3 &= (6ab+2a)+(9b+3) \\ &= 2a(3b+1)+3(3b+1) \\ &= (3b+1)(2a+3) \end{aligned}$$

(e) $3p^2 + p - 3px - x = (3p^2 - p) - (3px + x)$
$= p(3p - 1) - x(3p + 1)$
$= (3p + 1)(p - x)$

(f) $2ab + 2ac + 3b + 3c = (2ab + 2ac) + (3b + 3c)$
$= 2a(b + c) + 3(b + c)$
$= (b + c)(2a + 3)$.

Example 3.26

Factorize the following expressions

(a) $\frac{1}{4}pqx + pqy - bcx - 4ybc$, (b) $\frac{1}{6}ax^2y - \frac{1}{3}bx^2y - 2a^2 + 4ab$,

(c) $3a^2bc - \frac{3}{2}ab^2c^2 + 6axy - 3bcxy$.

Solution

(a) $\frac{1}{4}pqx + pqy - bcx - 4bcy = \frac{1}{4}pqx + \frac{4}{4}pqy - bcx - 4bcy$
$= \frac{1}{4}pq(x + 4y) - bc(x + 4y) = (x + 4y)\left(\frac{1}{4}pq - bc\right)$

(b) $\frac{1}{6}ax^2y - \frac{1}{3}bx^2y - 2a^2 + 4ab = \frac{1}{6}ax^2y - \frac{2}{6}bx^2y - 2a^2 + 4ab$
$= \frac{1}{6}x^2y(a - 2b) - 2a(a - 2b) = (a - 2b)\left(\frac{1}{6}x^2y - 2a\right)$

(c) $3a^2bc - \frac{3}{2}ab^2c^2 + 6axy - 3bcxy = \frac{6}{2}a^2bc - \frac{3}{2}ab^2c^2 + 6axy - 3bcxy$
$= \frac{3}{2}abc(2a - bc) + 3xy(2a - bc)$
$= (2a - bc)\left(\frac{3}{2}abc + 3xy\right)$
$= 3(2a - bc)\left(\frac{1}{2}abc + xy\right)$

Example 3.27

Simplify and factorize

(a) $4(4a + 5b) + 5(a + 3b)$, (b) $5(x + 3y) + 3(3x + y)$, (c) $4p + 3(2p + 5q) + 5q$.

Solution

(a) $4(4a + 5b) + 5(a + 3b) = 16a + 20b + 5a + 15b = 16a + 5a + 20b + 15b$
$= 21a + 35b = 7(3a + 5b)$.

(b) $5(x + 3y) + 3(3x + y) = 5x + 15y + 9x + 3y = 5x + 9x + 15y + 3y$
$= 14x + 18y = 2(7x + 9y)$.

(c) $4p + 3(2p + 5q) + 5q = 4p + 6p + 15q + 5q = 10p + 20q = 10(p + 2q)$.

Exercise 3.4a

1. Complete the following:
   (a) $12a + 8 = 4(...)$, (b) $15b - 10c = 5(...)$, (c) $py^2 + qy^2 = y^2(...)$,

(d) $4xy - 12x = 4x(...)$, (e) $9b^3 - 21ab = 3b(...)$, (f) $6ax^2 + 9a^2x = 3ax(...)$,

2. Factorize completely each of the following:
   (a) $6mp - 14mq$, (b) $7t + 42$, (c) $12x - 32x^2$, (d) $8a + 20a^2b$,
   (e) $7ax - 7ay^2$, (f) $u^3v^2 - u^2v^3$, (g) $p^2q^3 - 9pq$, (h) $\frac{1}{3}st + \frac{1}{3}s^2t^2$.

3. Factorize by taking out the negative common factor:
   (a) $-4x - 22x^2$, (b) $-15p - 25p^2$, (c) $-8a + 20a^2$,
   (d) $10xy - 5x^2$, (e) $-24x^2y + 8xy^2$, (f) $-3a^2 + 9ab - 12a$.

4. Factorize:
   (a) $2a(x - y) + b(x - y)$, (b) $3p(p - 1) - (p - 1)$, (c) $2(4a + 3) + b(4a + 3)$,
   (d) $5(a - 3) + b(2a - 6)$, (e) $7(x - y) + 2ax - 2ay$, (f) $2(x + 3) - xy - 3y$.

5. Factorize :
   (a) $2ac + 2ad + bc + bd$, (b) $8ac + 2ad + 12bc + 3bd$, (c) $2x^2 - xz - 2xy + yz$,
   (d) $2pq - 8p + 3q - 12$, (e) $10vu - 15v + 8u - 12$, (f) $15as - 18bs + 20at - 24bt$.

6. Factorize the following expressions
   (a) $12ac + 3ad + 4bc + bd$ (b) $35a^3 - 5ab + 28a^2b - 4b^2$
   (c) $24ac^2 - 3ab^2 - 40bc^2 + 5b^3$ (d) $72x^2 - 9xz - 8xy + yz$
   (e) $30x^2 + 40xy + 27xy^2 + 36y^3$ (f) $30ux - 35vx - 24uy + 28vy$
   (g) $15x^3 - 18x^2y - 20xy^3 + 24y^4$ (h) $55p^3 - 44p^2q + 60pq - 48q^2$.

3.4.2 Factorization of difference of two squares

A perfect square minus another perfect square is called *difference of two squares*. For example $a^2 - b^2$ is a difference between two squares. If x^2 and y^2 are two perfect squares, then their difference can be expressed as

$$x^2 - y^2 = (x + y)(x - y) \quad \text{or} \quad y^2 - x^2 = (y + x)(y - x).$$

Example: $4 - a^2 = 2^2 - a^2 = (2 + a)(2 - a)$.

$36a^2 - 25b^2 = (6a)^2 - (5b)^2 = (6a + 5b)(6a - 5b)$.

Example 3.28

Factorize the following

(a) $4a^2 - 1$, (b) $9x^2 - 25y^2$, (c) $x^2 - y^2 + x - y$,
(d) $4a^2 - b^2 - (2a - b)^2$, (e) $16a^2 - 81b^2 - 8a + 18b$, (f) $(h + 2k)^2 + 4k^2 - h^2$.

Solution

(a) $4a^2 - 1 = (2a)^2 - 1^2 = (2a - 1)(2a + 1)$

(b) $9x^2 - 25y^2 = (3x)^2 - (5y)^2 = (3x - 5y)(3x + 5y)$

(c) $x^2 - y^2 + x - y = (x - y)(x + y) + (x - y) = (x - y)(x + y + 1)$

(d) $4a^2 - b^2 - (2a-b)^2 = (2a)^2 - b^2 - (2a-b)^2 = (2a-b)(2a+b) - (2a-b)^2$
$= (2a-b)[2a+b-(2a-b)] = (2a-b)(2a+b-2a+b)$
$= (2a-b)(2b) = 2b(2a-b).$

(e) $16a^2 - 81b^2 - 8a + 18b = (4a)^2 - (9b)^2 - 2(4a-9b)$
$= (4a-9b)(4a+9b) - 2(4a-9b) = (4a-9b)(4a+9b-2)$

(f) $(h+2k)^2 + 4k^2 - h^2 = (h+2k)^2 + (2k)^2 - h^2 = (2k+h)^2 + (2k+h)(2k-h)$
$= (2k+h)(2k+h+2k-h) = (2k+h)(4k) = 4k(2k+h)$

Exercise 3.4b

Factorize completely the following expressions:

(a) $a^2 - 9b^2 = (a-3b)(a+3b)$,
(b) $16 - 25c^2 = (4-5c)(4+5c)$,
(c) $p^2 - q^2 + 2(p+q)^2$,
(d) $9t^2 - 1 + 27t - 9$,
(e) $r^3 - 4p^2r - 2p^3 + p^2r$,
(f) $v^2 - u^2 + u - v$,
(g) $16x^3y - 25xy^3 + 5y - 4x$,
(h) $6q^2 - pq - p^2$,
(i) $p^3q^5 - 4p^5q^3 + 2pq^2 - 4p^2q$,
(j) $5x^2y + 6xy^2 - 25x^2 + 36y^2$.

3.4.3 Quadratic Factorization

Quadratics are expressions that can be written in the form $ax^2 + bx + c$, where x is a variable and a, b, and c are constants. For example, in the expression $3x^2 + 5x + 6$, $a = 3$, $b = 5$ and $c = 6$. Likewise in $x^2 - 4x - 7$, $a = 1$, $b = -4$ and $c = -7$

When factorizing quadratic expressions;

1. Multiply the coefficient '***a***' of x^2 by the constant term '***c***' in the expression. That is the product ***ac***.
2. Find the pair of factors of the product ac which when added gives us the coefficient '***b***' of x.
3. Replace the coefficient '***b***' of x by the factors obtained in (ii) and use the method of grouping to complete the factorization.

Example: Factorize $6x^2 + 11x + 4$.

$a = 6$, $b = 11$ and $c = 4 \Rightarrow ac = 6 \times 4 = 24$.

Possible pairs of factors of 24 are (6, 4), (–6, –4), (12, 2), (–12, –2), (3, 8), (–3, –8), (1, 24), (–1, –24)

The pair of factors of 24 which when added gives 11 is (3, 8)

$\therefore\ 6x^2 + 11x + 4 = 6x^2 + 8x + 3x + 4$
$= 2x(3x+4) + 1(3x+4) = (3x+4)(2x+1).$

Example 3.29

Factorize the following

(a) $x^2 + x - 6$, (b) $2x^2 - 9x + 4$, (c) $6x^2 - x - 2$,
(d) $10x^2 - 23x + 12$, (e) $42x^2 - 9x - 6$, (f) $16x^2 + 38x + 21$.

Solution

(a) $x^2 + x - 6 \ = \ x^2 + 3x - 2x - 6 \ = \ x(x + 3) - 2(x + 3) \ = (x + 3)(x - 2)$.

(b) $2x^2 - 9x + 4 \ = \ 2x^2 - x - 8x + 4 \ = \ x(2x - 1) - 4(2x - 1) \ = \ (2x - 1)(x - 4)$.

(c) $6x^2 - x - 2 \ = \ 6x^2 + 3x - 4x - 2 \ = \ 3x(2x + 1) - 2(2x + 1) \ = \ (2x + 1)(3x - 2)$.

(d) $10x^2 - 23x + 12 \ = \ 10x^2 - 15x - 8x + 12 \ = \ 5x(2x - 3) - 4(2x - 3) \ = \ (2x - 3)(5x - 4)$.

(e) $42x^2 - 9x - 6 \ = \ 42x^2 + 12x - 21x - 6 \ = \ 6x(7x + 2) - 3(7x + 2) \ = \ (7x + 2)(6x - 3)$.

(f) $16x^2 + 38x + 21 \ = \ 16x^2 + 24x + 14x + 21 \ = \ 8x(2x + 3) + 7(2x + 3) \ = \ (2x + 3)(8x + 7)$.

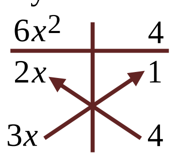
Alternative Way of Factorizing Quadratic Expressions

1. Write the factors of ax^2 and c in a square array, factors of ax^2 in the first column and c in the second.
2. Arrange the factors in such a way that when cross multiplied and added you obtained bx.

Example: Factorize $6x^2 + 11x + 4$.

Possible pairs of factors of $6x^2$ are $(x, 6x)$, $(-x, -6x)$, $(2x, 3x)$ and $(-2x, -3x)$
Possible pairs of factors of 4 are $(2, 2)$, $(-2, -2)$, $(4, 1)$ and $(-4, -1)$

By trial and error, we have:



Cross multiplying and adding, we have

$8x + 3x = 11x$.

$6x^2 + 11x + 4 \ = \ (2x + 1)(3x + 4)$

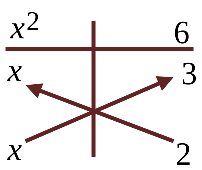
Example 3.30

Factorize the following expressions

(a) $x^2 + 5x + 6$, (b) $6x^2 - 31x + 35$, (c) $24x^2 + 5x - 14$,
(d) $36x^2 - 39x + 10$, (e) $72x^2 - 25x - 77$, (f) $42x^2 + 47x + 10$.

Solution

(a) $x^2 + 5x + 6$



$= \ 3x + 2x \ = \ 5x$.

$x^2 + 5x + 6 = (x + 3)(x + 2)$

(b) $6x^2 - 31x + 35$

$$\begin{array}{l|r} 6x^2 & 35 \\ \hline 3x & -5 \\ 2x & -7 \end{array} \quad = -21x - 10x = -31x$$

$6x^2 - 31x + 35 = (3x - 5)(2x - 7)$

(c) $24x^2 + 5x - 14$

$$\begin{array}{l|r} 24x^2 & -14 \\ \hline 8x & 7 \\ 3x & -2 \end{array} \quad = -16x + 21x = 5x$$

$24x^2 + 5x - 14 = (8x + 7)(3x - 2)$

(d) $36x^2 - 39x + 10$

$$\begin{array}{l|r} 36x^2 & 10 \\ \hline 12x & -5 \\ 3x & -2 \end{array} \quad = -24x - 15x = -39x$$

$36x^2 - 39x + 10 = (12x - 5)(3x - 2)$

(e) $72x^2 - 25x - 77$

$$\begin{array}{l|r} 72x^2 & -77 \\ \hline 9x & -11 \\ 8x & 7 \end{array} \quad = 63x - 88x = -25x$$

$72x^2 - 25x - 77 = (9x - 11)(8x + 7)$

(f) $42x^2 + 47x + 10$

$$\begin{array}{l|r} 42x^2 & 10 \\ \hline 6x & 5 \\ 7x & 2 \end{array} \quad = 12x + 35x = 47x$$

$42x^2 + 47x + 10 = (6x + 5)(7x + 2)$

Factorization of Quadratic Expressions with Two Variables

Quadratic expressions with two variables can be factorized in the same way as that of one variable. Consider the following example.

Example 3.31

Factorize the following expressions

(a) $x^2 - 3xy + 2y^2$, (b) $15a^2 + 13ab + 2b^2$, (c) $4p^2 + 4pq - 3q^2$.

Solution

(a) $x^2 - 3xy + 2y^2 = x^2 - 2xy - xy + 2y^2 = x(x - 2y) - y(x - 2y) = (x - y)(x - 2y)$.

(b) $15a^2+13ab+2b^2 = 15a^2+10ab+3ab+2b^2$
$= 5a(3a+2b)+b(3a+2b) = (5a+b)(3a+2b).$

(c) $4p^2+4pq-3q^2 = 4p^2+6pq-2pq-3q^2$
$= 2p(2p+3q)-q(2p+3q) = (2p-q)(2p+3q).$

Exercise 3.4c

Factorize the following expressions

(a) $3x^2 +10xy + 8y^2$, (b) $10x^2 -23xy + 12y^2$, (c) $12x^2 - 24xy - 15y^2$,
(d) $10p^2 - 7pq - 12q^2$, (e) $21p^2 - 38pq + 16q^2$, (f) $15p^2 - 31pq + 14q^2$,
(g) $10a^2 + 27ab + 18b^2$, (h) $18a^2 -30ab + 8b^2$, (i) $9x^2 + 30x + 25$,
(j) $8x^2 + 2x - 15$, (k) $10x^2 - 23x +12$, (l) $28y^2 - 47y + 15$,
(m) $24y^2 + 37y + 14$, (n) $21p^2 + 45p - 54$, (o) $72p^2 - 13p - 20$,

3.5 Rational algebraic expressions

An algebraic expression is said to be rational if it can be expressed as a quotient of two algebraic expressions with denominator not equal to zero.

Examples are (a) $\frac{3x}{2y}$, (b) $\frac{2x+3y}{5x-3y}$ (c) $\frac{2x^2+3x-1}{3x^2-4x+5}$ (c) $\frac{2abc}{a-b}$.

3.5.1 Simplification of rational expressions

Example 3.32

Simplify the following expressions:

(a) $\dfrac{15a^2bc}{12ab^2c}$, (b) $\dfrac{36-x^2}{6x-x^2}$, (c) $\dfrac{x^2-5x+6}{x^2-9}$, (d) $\dfrac{4x^2+12x+9}{4x^2-9}$,

(e) $\dfrac{x^2-49}{(x-7)^2}$, (f) $\dfrac{y^2-16}{xy+2y-4x-8}$, (g) $\dfrac{x^2y^2-9x^2}{x^2y-3x^2}$, (h) $\dfrac{4x-12}{2x^2+14x-60}$.

Solution

(a) $$\frac{15a^2bc}{12ab^2c} = \frac{3abc(5a)}{3abc(4b)} = \frac{5a}{4b}.$$

(b) $$\frac{36-x^2}{6x-x^2} = \frac{6^2-x^2}{6x-x^2} = \frac{(6-x)(6+x)}{x(6-x)} = \frac{6+x}{x}.$$

(c) $$\frac{x^2-5x+6}{x^2-9} = \frac{(x-2)(x-3)}{x^2-3^2} = \frac{(x-2)(x-3)}{(x-3)(x+3)} = \frac{x-2}{x+3}.$$

(d) $\dfrac{4x^2+12x+9}{4x^2-9} = \dfrac{(2x+3)(2x+3)}{(2x)^2-3^2} = \dfrac{(2x+3)(2x+3)}{(2x-3)(2x+3)} = \dfrac{2x+3}{2x-3}$.

(e) $\dfrac{x^2-49}{(x-7)^2} = \dfrac{x^2-7^2}{(x-7)^2} = \dfrac{(x-7)(x+7)}{(x-7)(x-7)} = \dfrac{x+7}{x-7}$.

(f) $\dfrac{y^2-16}{xy+2y-4x-8} = \dfrac{y^2-4^2}{xy+2y-4x-8} = \dfrac{(y-4)(y+4)}{y(x+2)-4(x+2)} = \dfrac{(y-4)(y+4)}{(x+2)(y-4)} = \dfrac{y+4}{x+2}$

(g) $\dfrac{x^2y^2-9x^2}{yx^2-3x^2} = \dfrac{x^2(y^2-9)}{x^2(y-3)} = \dfrac{x^2(y-3)(y+3)}{x^2(y-3)} = y+3.$

(h) $\dfrac{4x-12}{2x^2+14x-60} = \dfrac{4(x-3)}{2(x^2+7x-30)} = \dfrac{4(x-3)}{2(x-3)(x+10)} = \dfrac{2}{x+10}$.

Exercise 3.5a

Simplify the following expressions:

(a) $\dfrac{36u^3vw^4}{12u^2v^2w^2}$, (b) $\dfrac{81-36x^2}{9x-6x^2}$, (c) $\dfrac{2x^2-7x-15}{x^2-25}$, (d) $\dfrac{9x^2-12x+4}{9x^2-4}$,

(e) $\dfrac{x^2-36}{(x-6)^2}$, (f) $\dfrac{4x^2+4xy+y^2}{2x^2-xy-y^2}$, (g) $\dfrac{x^2y^2-49x^2}{x^2y-7x^2}$, (h) $\dfrac{3x-4}{6x^2-11x+4}$.

3.5.2 Addition and subtraction of rational algebraic expressions

Addition and subtraction of rational algebraic expressions is done in the same way as adding or subtracting ordinary fractions. This implies that we have to be able to find the ***lowest common multiple*** (LCM) of the denominators of the rational expressions, and use it as the common denominator.

Example 3.33

Simplify the following

1. $\dfrac{2}{5x^2}+\dfrac{1}{2x}$, 2. $\dfrac{1}{a^2}+\dfrac{3}{a}-\dfrac{1}{3a}$, 3. $\dfrac{2}{3b^2}-\dfrac{5}{3b^2}+\dfrac{3}{4b}$.

Solution

1. $\dfrac{2}{5x^2}+\dfrac{1}{2x} = \dfrac{4+5x}{10x^2}$.

2. $\dfrac{1}{a^2}+\dfrac{3}{a}-\dfrac{1}{3a} = \dfrac{3+9a-a}{3a^2} = \dfrac{8a+3}{3a^2}.$

3. $\dfrac{2}{3b^2}-\dfrac{5}{3b^2}+\dfrac{3}{4b} = \dfrac{8-20+9b}{12b^2} = \dfrac{9b-12}{12b^2} = \dfrac{3(3b-4)}{12b^2} = \dfrac{3b-4}{4b^2}.$

Example 3.34

Simplify the following expressions:

1. $\dfrac{2}{x+2}+\dfrac{3}{x-3},$ 2. $\dfrac{5}{2x+1}-\dfrac{6}{3x-1},$ 3. $\dfrac{5}{x-2}-\dfrac{2}{x+2},$

4. $\dfrac{2x}{x+4}+\dfrac{8x-32}{x^2-16},$ 5. $\dfrac{5ab-5b^2}{a^2-b^2}+\dfrac{5a}{a+b},$ 6. $\dfrac{10x^2+xy-24y^2}{4x^2-9y^2}-\dfrac{x+2y}{2x+3y}.$

Solution

1. $\dfrac{2}{x+2}+\dfrac{3}{x-3} = \dfrac{2(x-3)+3(x+2)}{(x+2)(x-3)} = \dfrac{2x-6+3x+6}{(x+2)(x-3)} = \dfrac{5x}{(x+2)(x-3)}.$

2. $\dfrac{5}{2x+1}-\dfrac{6}{3x-1} = \dfrac{5(3x-1)-6(2x+1)}{(2x+1)(3x-1)} = \dfrac{15x-5-12x-6}{(2x+1)(3x-1)} = \dfrac{3x-11}{(2x+1)(3x-1)}.$

3. $\dfrac{5}{x-2}-\dfrac{2}{x+2} = \dfrac{5(x+2)-2(x-2)}{(x-2)(x+2)} = \dfrac{5x+10-2x+4}{(x-2)(x+2)} = \dfrac{3x+14}{x^2-4}.$

4. $\dfrac{2x}{x+4}+\dfrac{8x-32}{x^2-16} = \dfrac{2x}{x+4}+\dfrac{8x-32}{(x-4)(x+4)} = \dfrac{2x(x-4)+8x-32}{(x-4)(x+4)}$

$$= \frac{2x^2-8x+8x-32}{(x-4)(x+4)} = \frac{2x^2-32}{x^2-16} = \frac{2(x^2-16)}{x^2-16} = 2.$$

5. $\dfrac{5ab-5b^2}{a^2-b^2}+\dfrac{5a}{a+b} = \dfrac{5ab-5b^2}{(a-b)(a+b)}+\dfrac{5a}{a+b} = \dfrac{5ab-5b^2+5a(a-b)}{(a-b)(a+b)}$

$$= \frac{5ab-5b^2+5a^2-5ab}{(a-b)(a+b)} = \frac{5a^2-5b^2}{a^2-b^2} = \frac{5(a^2-b^2)}{a^2-b^2} = 5.$$

6. $\dfrac{10x^2+xy-24y^2}{4x^2-9y^2}-\dfrac{x+2y}{2x+3y} = \dfrac{10x^2+xy-24y^2}{(2x)^2-(3y)^2}-\dfrac{x+2y}{2x+3y}$

$$= \frac{10x^2+xy-24y^2}{(2x+3y)(2x-3y)}-\frac{x+2y}{2x+3y}$$

$$= \frac{10x^2+xy-24y^2-(x+2y)(2x-3y)}{(2x+3y)(2x-3y)}$$

$$= \frac{10x^2 + xy - 24y^2 - (2x^2 - 3xy + 4xy - 6y^2)}{(2x+3y)(2x-3x)}$$

$$= \frac{10x^2 + xy - 24y^2 - 2x^2 - xy + 6y^2}{(2x+3y)(2x-3y)}$$

$$= \frac{8x^2 - 18y^2}{4x^2 - 9y^2} = \frac{2(4x^2 - 9y^2)}{4x^2 - 9y^2} = 2.$$

Example 3.35

Simplify the following expressions:

1. $\dfrac{6x^2}{x^2 - y^2} - \dfrac{3x}{x+y} + \dfrac{3y}{y-x},$
2. $\dfrac{2y(10+y)}{y^2 - 25} - \dfrac{3y}{y-5} - \dfrac{4y}{y+5},$
3. $\dfrac{3x^2 - 20x + 6}{x^2 - 3x} - \dfrac{x-2}{x} + \dfrac{3x}{x-3},$
4. $\dfrac{a^2 - 7a + 7}{a^2 - a - 6} - \dfrac{a-3}{a+2} + \dfrac{2a-5}{a-3}.$

Solution

(1) $$\frac{6x^2}{x^2 - y^2} - \frac{3x}{x+y} + \frac{3y}{y-x} = \frac{6x^2}{(x+y)(x-y)} - \frac{3x}{x+y} - \frac{3y}{x-y}$$

$$= \frac{6x^2 - 3x(x-y) - 3y(x+y)}{(x+y)(x-y)}$$

$$= \frac{6x^2 - 3x^2 + 3xy - 3xy - 3y^2}{(x+y)(x-y)}$$

$$= \frac{3x^2 - 3y^2}{(x+y)(x-y)} = \frac{3(x^2 - y^2)}{x^2 - y^2} = 3$$

(2) $$\frac{2y(10+y)}{y^2 - 25} - \frac{3y}{y-5} - \frac{4y}{y+5} = \frac{2y(10+y)}{(y-5)(y+5)} - \frac{3y}{y-5} - \frac{4y}{y+5}$$

$$= \frac{2y(10+y) - 3y(y+5) - 4y(y-5)}{(y-5)(y+5)}$$

$$= \frac{20y + 2y^2 - 3y^2 - 15y - 4y^2 + 20y}{(y-5)(y+5)}$$

$$= \frac{25y - 5y^2}{(y-5)(y+5)} = \frac{5y(5-y)}{(y-5)(y+5)}$$

$$= \frac{-5y(y-5)}{(y-5)(y+5)} = -\frac{5y}{y+5}$$

3. $\frac{3x^2-20x+6}{x^2-3x} - \frac{x-2}{x} + \frac{3x}{x-3} = \frac{3x^2-20x+6}{x(x-3)} - \frac{x-2}{x} + \frac{3x}{x-3}$

$$= \frac{3x^2-20x+6-(x-2)(x-3)+3x^2}{x(x-3)}$$

$$= \frac{3x^2-20x+6-(x^2-3x-2x+6)+3x^2}{x(x-3)}$$

$$= \frac{3x^2-20x+6-x^2+5x-6+3x^2}{x(x-3)}$$

$$= \frac{5x^2-15x}{x(x-3)} = \frac{5x(x-3)}{x(x-3)} = 5$$

4. $\frac{a^2-7a+7}{a^2-a-6} - \frac{a-3}{a+2} + \frac{2a-5}{a-3} = \frac{a^2-7a+7}{(a-3)(a+2)} - \frac{a-3}{a+2} + \frac{2a-5}{a-3}$

$$= \frac{a^2-7a+7-(a-3)(a-3)+(2a-5)(a+2)}{(a+2)(a-3)}$$

$$= \frac{a^2-7a+7-(a^2-6a+9)+2a^2+4a-5a-10}{(a+2)(a-3)}$$

$$= \frac{a^2-7a+7-a^2+6a-9+2a^2-a-10}{(a+2)(a-3)}$$

$$= \frac{2a^2-2a-12}{(a+2)(a-3)} = \frac{2(a^2-a-6)}{a^2-a-6} = 2$$

Exercise 3.5b

Express as single fractions:

(1) $\frac{3}{5x^2} - \frac{3}{2x}$, (2) $\frac{1}{y^2} - \frac{3}{2y} - \frac{1}{3y}$, (3) $\frac{3}{4t^2} - \frac{5}{6t^2} + \frac{2}{3t}$,

(4) $\frac{1}{5(x-2)} - \frac{1}{5(x+3)}$, (5) $\frac{15}{2(5x-1)} - \frac{15}{2(3x-1)}$, (6) $\frac{4}{x^2-4} + \frac{2}{x+2}$,

(7) $\frac{2}{x+2} + \frac{2x^2-2x-4}{x^2-4}$, (8) $\frac{5a+5b}{a^2-b^2} + \frac{5}{a+b}$, (9) $\frac{2x^2}{x^2-x-6} - \frac{2x}{x-3}$.

Exercise 3.5c

Express the following as single fractions:

(a) $\frac{4}{5(x+2)}+\frac{6}{5(x-3)}$, (b) $\frac{8}{(x-4)^2}+\frac{2}{x-4}$,

(c) $\frac{9x}{x^2-9}+\frac{3x}{x+3}$, (d) $\frac{2}{2x+3}-\frac{3x}{x+3}$,

(e) $\frac{6x^2}{4x^2-25y^2}-\frac{3x}{2x-5y}$, (f) $\frac{x}{x-y}-\frac{2y^2}{x^2-y^2}$,

(g) $\frac{2x}{x+y}+\frac{2y}{x-y}+\frac{4y^2}{y^2-x^2}$, (h) $\frac{x^2}{x^2-x-20}+\frac{5x}{x-5}+\frac{4x}{x+4}$,

(i) $\frac{1}{2-x}-\frac{4}{(x-2)^2}-\frac{4}{(x-2)^3}$, (j) $\frac{12}{(x+4)^2}-\frac{1}{x+4}+\frac{1}{x-2}$,

(k) $\frac{1}{4(x-1)}-\frac{x+3}{4(x^2+2x+5)}$, (l) $\frac{1}{2(x-2)}+\frac{1}{6(x+2)}-\frac{1}{3(x-1)}$,

(m) $\frac{3}{2(x+1)^2}-\frac{9}{8(x+1)}+\frac{9}{8(x-3)}$.

3.5.3 Multiplication and division of rational algebraic expressions

Multiplication and Division of rational algebraic expressions is similar to that for ordinary fractions.

Some important steps to follow

(i) Factorize completely all terms

(ii) Reduce the resulting terms to the lowest terms

Example 3.36

Simplify the following expressions:

1. $\frac{(x+3x)^2}{x^2-y^2}\times\frac{x-y}{x+3x}$, 2. $\frac{3x^2-3y^2}{12x}\div\frac{x+y}{4x}$, 3. $\frac{36x^2-9y^2}{4xy}\times\frac{20xy}{6x-3y}$.

Solution

1. $$\frac{(x+3x)^2}{x^2-y^2}\times\frac{x-y}{x+3x} = \frac{(x+3x)(x+3x)}{(x-y)(x+y)}\times\frac{x-y}{x+3x} = \frac{x+3x}{x+y}.$$

2. $$\frac{3x^2-3y^2}{12x} \div \frac{x+y}{4x} = \frac{3(x^2-y^2)}{12x} \times \frac{4x}{x+y} = \frac{3(x-y)(x+y)}{12x} \times \frac{4x}{x+y} = x-y.$$

3. $$\frac{36x^2-9y^2}{4xy} \times \frac{20xy}{6x-3y} = \frac{(6x)^2-(3y)^2}{4xy} \times \frac{20xy}{6x-3y}$$
$$= \frac{(6x-3y)(6x+3y)}{4xy} \times \frac{20xy}{6x-3y} = 5(6x+3y).$$

Example 3.37

Simplify the following expressions:

(a) $\dfrac{3x^2+7xy+2y^2}{x^2+2xy+y^2} \times \dfrac{x^2-y^2}{3x+y}$ (b) $\dfrac{10a^2+ab-3b^2}{a+2b} \div \dfrac{6a-3b}{a^2+4ab+4b^2}$

Solution

(a) $$\frac{3x^2+7xy+2y^2}{x^2+2xy+y^2} \times \frac{x^2-y^2}{3x+y} = \frac{3x^2+6xy+xy+2y^2}{x^2+xy+xy+y^2} \times \frac{(x-y)(x+y)}{3x+y}$$
$$= \frac{3x(x+2y)+y(x+2y)}{x(x+y)+y(x+y)} \times \frac{(x-y)(x+y)}{3x+y}$$
$$= \frac{(x+2y)(3x+y)}{(x+y)(x+y)} \times \frac{(x-y)(x+y)}{3x+y}$$
$$= \frac{(x+2y)(x-y)}{x+y}$$

(b) $$\frac{10a^2+ab-3b^2}{a+2b} \div \frac{6a-3b}{a^2+4ab+4b^2} = \frac{10a^2-5ab+6ab-3b^2}{a+2b} \times \frac{a^2+2ab+2ab+4b^2}{6a-3b}$$
$$= \frac{5a(2a-b)+3b(2a-b)}{a+2b} \times \frac{a(a+2b)+2b(a+2b)}{3(2a-b)}$$
$$= \frac{(2a-b)(5a+3b)}{a+2b} \times \frac{(a+2b)(a+2b)}{3(2a-b)}$$
$$= \tfrac{1}{3}(5a+3b)(a+2b).$$

Exercise 3.5d

Simplify the following expressions:

(a) $\dfrac{2x^2+5x+3}{x^2-9} \times \dfrac{x-3}{2x+3}$, (b) $\dfrac{6x^2+5xy+y^2}{5x+y} \times \dfrac{25x^2+10xy+y^2}{3x+y}$,

(c) $\dfrac{9x^2-4y^2}{3xy} \div \dfrac{3x+2y}{x^2y^2}$,

(d) $\dfrac{4ac}{a(b+c)} \div \dfrac{8ac}{ab+ac+cb+c^2}$,

(e) $\dfrac{x^2+xy-2y^2}{x+y} \times \dfrac{4x+4y}{x^2+4xy+4y^2}$,

(f) $\dfrac{15a^2-8ab+b^2}{(a+b)^2} \times \dfrac{a^2-b^2}{5a-b}$,

(g) $\dfrac{2a^2+ab-3b^2}{a^2+ab} \times \dfrac{a^3-ab^2}{2a+3b}$,

(h) $\dfrac{4x+5y}{3x^2+xy-2y^2} \times \dfrac{3x-2y}{4x^2+xy-5y^2}$,

(i) $\dfrac{3x+5y}{x^2-y^2} \times \dfrac{x+y}{12x^2+17xy-5y^2}$,

(j) $\dfrac{14x^2-9xy+y^2}{x^2-y^2} \div \dfrac{2x^2+xy-y^2}{x^2+2xy+y^2}$.

Revision Exercises 3

1. Simplify and factorize the following expressions
   (a) $3x^2 + 12xy + 10y^2 + 5x^2 + 10xy + 5y^2$,
   (b) $13x^2 - 23xy + 13y^2 + 7x^2 - 20xy + 8y^2$,
   (c) $13a^2 - 13ab - 19b^2 + 4ab + 5a^2 - 16b^2$,
   (d) $15a^2 + 12ab + 12a^2 - 6b^2 - 6ab - 10b^2$,
   (e) $8x^2 - 12x + 2 + 7x^2 - 10x + 6$,
   (f) $23x^2 + 5x - 6 + 33x^2 - 9 + 6x$,
   (g) $15x^2 - 52x + 7 + 9x^2 + 21x + 3$,
   (h) $16q^2 - 6pq - 50p^2 + 8q^2 + 4pq + 15p^2$,
   (i) $50p^2 - 30pq - 10p^2 - 32pq + 15q^2$.

2. Factorize the following expressions
   (a) $8x^2-14xy+3y^2$, (b) $2x^2y^2+xy^2-4x^2y-2xy$, (c) $6x^2y^2-5x^2y+x^2$,
   (d) $15x^2+14x-8$, (e) $x^2y^2-x^2y-xy^2+xy$, (f) $6x^4y^2-5x^2y^2+y^2$,
   (g) $20x^2+13xy-15y^2$, (h) $x^2-y^2+(x-y)^2$, (i) $10x^2y^2+13x^2y-3x^2$,
   (j) $2x^4-5x^2+3$, (k) $4x^2-y^2+4x+2y$, (l) $6x^4+x^2-1$,
   (m) $6x^2-23xy+15y^2$, (n) $xy-3x^2y-3xy^2+6x^2y^2$, (o) $9x^4-20x^2+4$,
   (p) x^6-3x^3+2, (q) $20x^2-17xy^2+3y^4$, (r) $3x^8-x^4-2$,
   (s) $14x^2+27xy-20y^2$, (t) $2x^2y^2-x^3y-x^4$, (u) $16x^4-28x^2-30$,
   (v) $35x^2+2xy-48y^2$, (w) $8x^2y^2-33x^2y+4x^2$, (x) $14x^4-5x^2-24$,
   (y) $8x^6-9x^3+1$, (z) $15x^2y^2-14xy^2-8y^2$.

3. Express the following as single fractions:

(a) $\dfrac{5x-3}{x^2-1}-\dfrac{4}{x+1}$, (b) $\dfrac{6-2x}{(x-3)^2}+\dfrac{3}{x-3}$,

(c) $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x-1}$, (d) $\dfrac{7-x}{6x^2+x-1}+\dfrac{3}{2x+1}$,

(e) $\dfrac{2}{2x+1}+\dfrac{2}{2x-1}$, (f) $\dfrac{4x+2y}{y^2-4x^2}+\dfrac{3}{2x-y}$,

(g) $\dfrac{4x}{y^2-x^2}+\dfrac{2}{x+y}-\dfrac{3}{y-x}$, (h) $\dfrac{8-6x}{x^2-3x+2}+\dfrac{4}{x-2}+\dfrac{3}{x-1}$,

(i) $\dfrac{1}{x-1}+\dfrac{2}{(x-1)^2}-\dfrac{x^2}{(x-1)^3}$, (j) $\dfrac{2x^2+3x}{(x+3)^3}-\dfrac{1}{x+3}+\dfrac{3}{(x+3)^2}$.

4. Simplify the following expressions:

(a) $\dfrac{6x^2+7x-3}{x^2-25}\times\dfrac{x+5}{2x+3}$, (b) $\dfrac{10x^2-17xy+3y^2}{4x^2-9y^2}\times\dfrac{2x+3y}{2x-3y}$,

(c) $\dfrac{8x^2-26xy+15y^2}{(x-y)^2}\div\dfrac{4x^2-4xy-15y^2}{2x^2+xy-3y^2}$, (d) $\dfrac{6x^2-19xy+8y^2}{x-3y}\div\dfrac{2x^2-7xy+3y^2}{x^2-6xy+9y^2}$,

(e) $\dfrac{8x^2+xy-9y^2}{4x^2-25y^2}\times\dfrac{2x+5y}{8x+9y}$.

5 (a) A rectangular box has a square base of sides x, a depth h and no lid. Determine the external surface area A of the box.

(b) Simplify the following expressions:
(i) $2x^2y^3-3x^5y^2+x^2y^3-5x^5y^2-4x^2y^3$, (ii) $\frac{5}{9}x^3y^4-\frac{5}{14}x^2y^5-\frac{2}{3}x^3y^4+\frac{3}{7}x^2y^5$.

(c) Expand and simplify the following expressions:
(i) $(2p+3q)(4p+5q)+(p+3q)(6p-7q)$,
(ii) $(8p+9q)(2p-q)-(7p-8q)(3p-q)$.

(d) Factorize the following expressions
(i) $65p^2v-39pqu^2v-25p^3u+15p^2qu^3$, (ii) $18u^3-45uv+8u^2v^2-20v^3$.

(e) Factorize completely the following expressions:
(i) $a^3-9ab^2-3a^2b+9ab^2$, (ii) $a^2p^2q+apqst-apst-s^2t^2$.

(f) Factorize the following expressions
(i) $40a^2+38ab-15b^2$, (ii) $4a^2+4ab+b^2$, (iii) $6q^2-29q+28$,
(iv) $55q^2-7q-24$, (v) $24q^2+78q-21$.

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