Swirling measures

The quotient structure of the tangent cone to the Wasserstein space

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This presentation contains errors. I let it online for the record, but in the general case, $\mathrm{Tan}_\mu \subsetneq \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu / \sim_\mu$.

July 5, 2024
Journée de la fédération Normandie Mathématiques, Rouen

Helmholtz decomposition

Theorem – HH decomposition [Lad87] Let $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. There exists two uniquely defined vector fields $g, h \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$f = g + h, \qquad g \in \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2}, \qquad h \in \overline{\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{div}\varphi = 0\}}^{L^2}.$$

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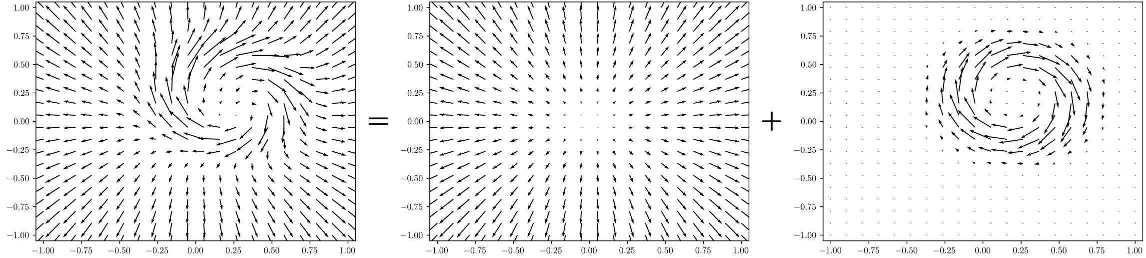

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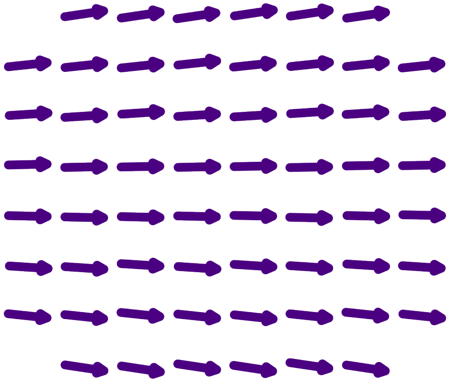

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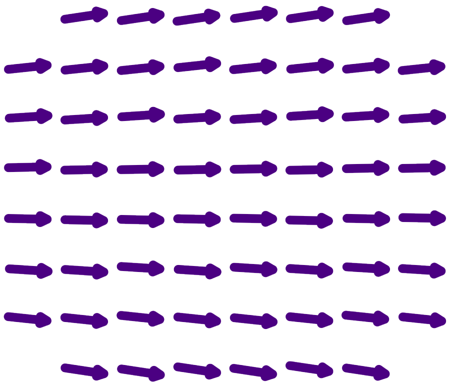

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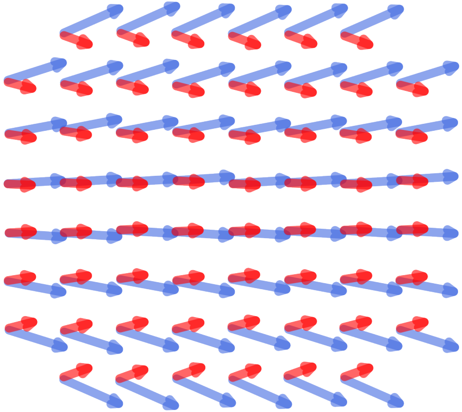

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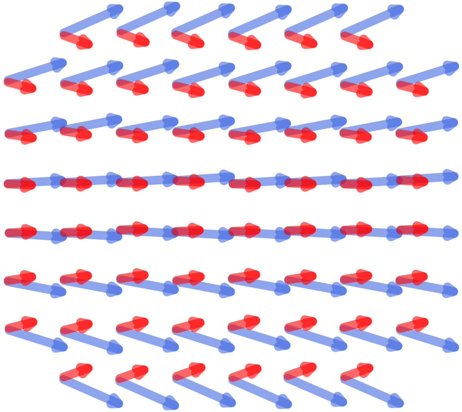

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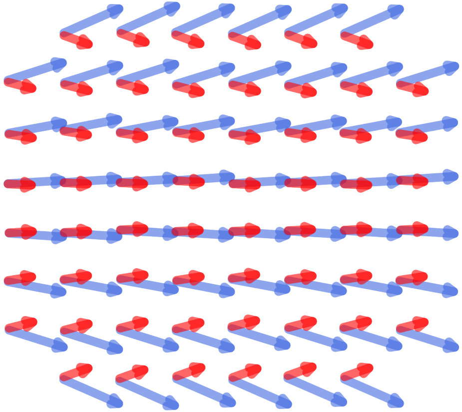
- Derive a formulation of the tangent cone to the Wasserstein space.



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$$d_{\mathcal{W}}^2(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$

$\left(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}}\right)$ is a complete geodesic metric space.

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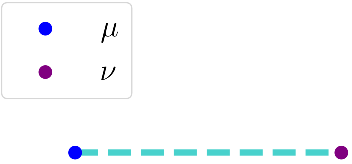

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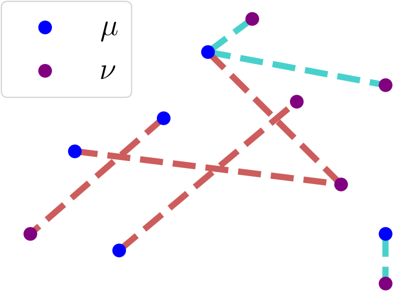

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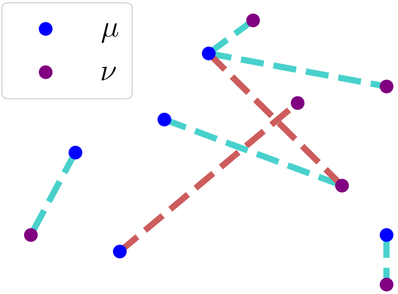

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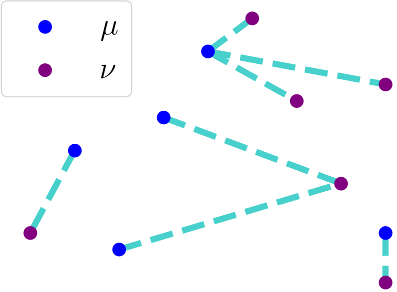

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Measure fields

Let $\mathsf{T}\mathbb{R}^d := \left\{(x,v) \mid x \in \mathbb{R}^d,\ v \in \mathsf{T}_x\mathbb{R}^d\right\}$. Denote $f\#\alpha$ the measure $\alpha(f^{-1}(\cdot))$.

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Any vector field $f \in L^2_\mu$ identifies with $\xi = f\#\mu$, for which $\xi_x = \delta_{(x,f(x))}$.

Def – Distance between measure fields [Gig08]

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In particular, $W_\mu(f\#\mu, g\#\mu) = \|f-g\|_{L^2_\mu}$.

Tangent cone: the geometric approach

Canonical construction of the tangent cone [AGS05, Gig08]

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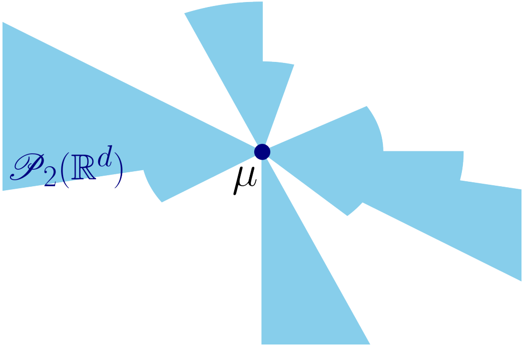

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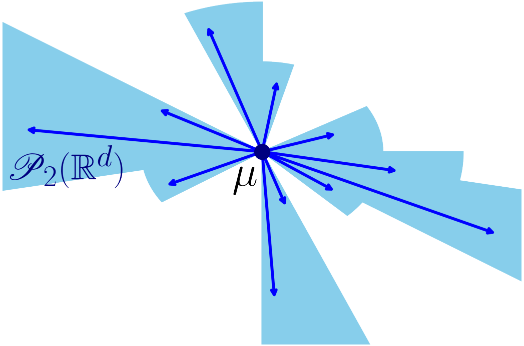

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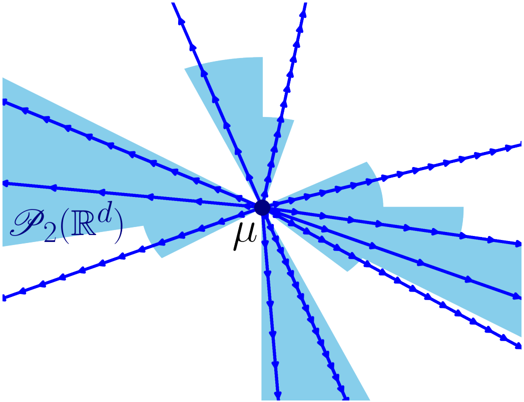

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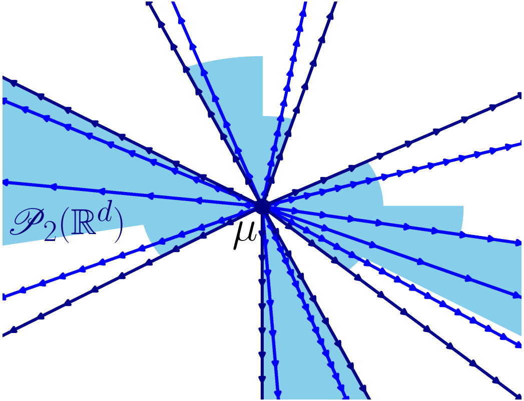
  is a geodesic;
- the positive cone $\alpha \cdot \eta$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.



Link with gradient fields

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Theorem – Tangent space to a regular measure [Bre91] Assume that μ is absolutely continuous with respect to the Lebesgue measure. Then

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) = \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu} \# \mu =: \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

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In the general case, one has the following.

Theorem – Vertical superposition of the tangent cone For any $\eta \in \mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$, there exists $\varpi \in \mathscr{P}_2(\mathrm{Tan}_\mu)$ such that for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d;\mathbb{R})$,

$$\int_{(x,v)\in\mathrm{T}\mathbb{R}^d} \varphi(x,v)d\eta(x,v) = \int_{b\in\mathrm{Tan}_\mu}\int_{x\in\mathbb{R}^d} \varphi(x,b(x))d\mu(x)d\varpi(b).$$

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We may now define the set $\mathbf{Sol}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Def – Solenoidal measure fields An element $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ is said solenoidal if

$$\langle \eta, \zeta \rangle_\mu = 0 \qquad \forall \eta \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

Examples

- If $\mu = \delta_x$, then $\textbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone.

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Hence $\zeta \in \mathbf{Sol}_\mu$ iff its barycenter is a solenoidal vector field. **Very large!**

HH decomposition for measure fields

Recall that in the classical case, $f = g + h$, or in weak form,

$$\int_x \varphi(x, f(x))d\mu = \int_x \varphi(x, g(x) + h(x))d\mu \qquad \forall \varphi \in \mathcal{C}_b(\mathsf{T}\mathbb{R}^d; \mathbb{R}).$$

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Theorem – HH decomposition For any $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$, there exists an unique pair $\eta \in \textbf{Tan}_\mu$ and $\zeta \in \textbf{Sol}_\mu$ such that for some measurable family $(\alpha_x)_x$ with $\alpha_x \in \Gamma(\eta_x, \zeta_x)$ for a.e. $x \in \operatorname{supp}\mu$,

$$\int_{(x,v)} \varphi(x, v)d\xi = \int_{\substack{x \in \mathbb{R}^d, \\ (v,w) \in (\mathsf{T}_x\mathbb{R}^d)^2}} \varphi(x, v + w)d[\alpha_x \otimes \mu] \qquad \forall \varphi \in \mathcal{C}_b(\mathsf{T}\mathbb{R}^d; \mathbb{R}).$$

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Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

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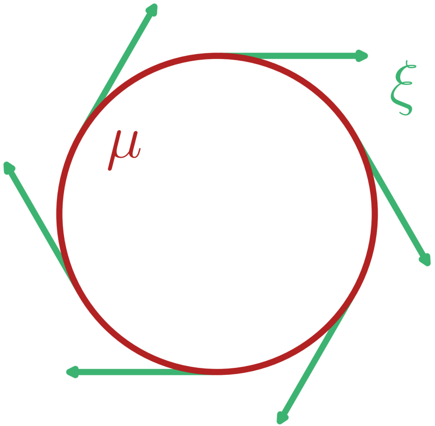

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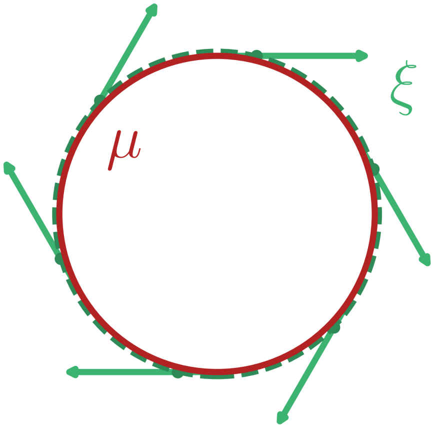

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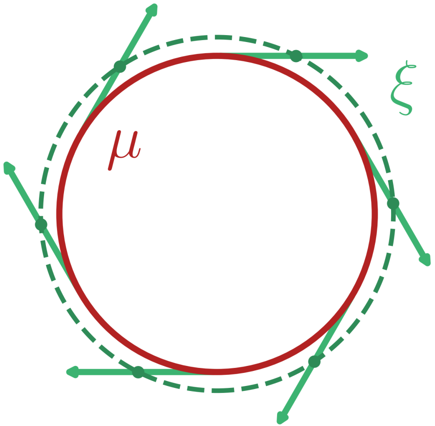

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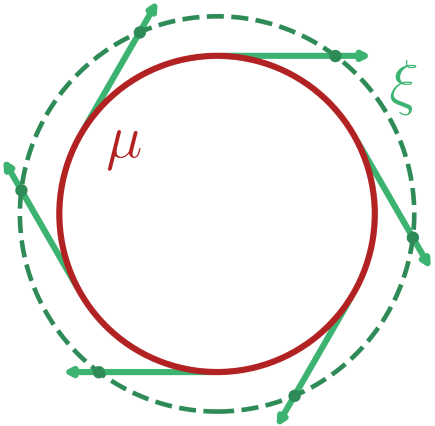

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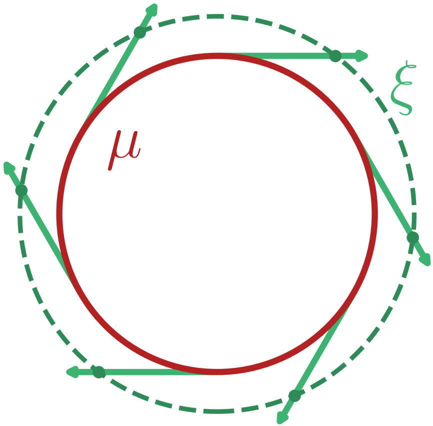

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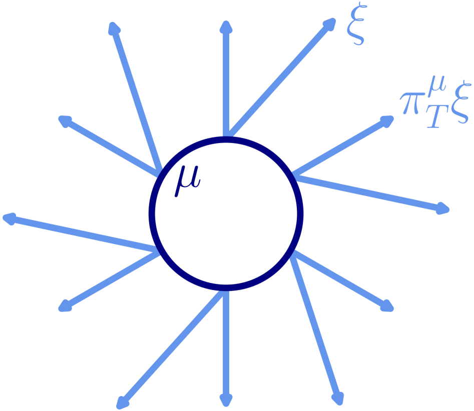
In general, if $\pi^\mu \xi$ denotes the tangent component,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}((\pi_x + h\pi_v)\#\pi^\mu\xi, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



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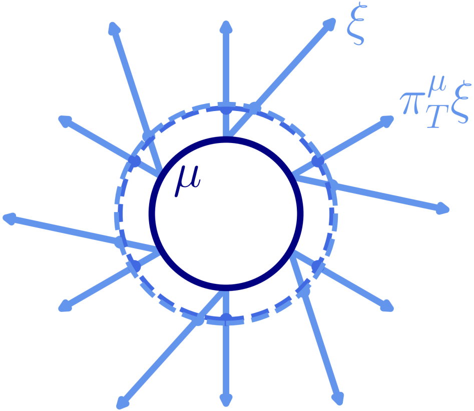
$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

In this case, the tangent component of ξ is 0_μ.
In general, if $\pi^\mu \xi$ denotes the tangent component,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}((\pi_x + h\pi_v)\#\pi^\mu\xi, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



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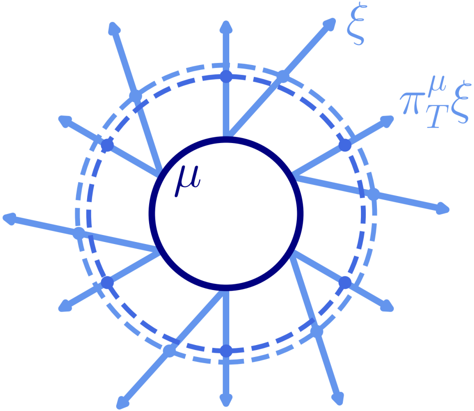

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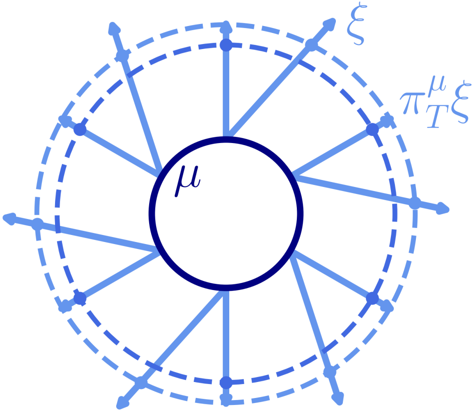

Tangent cone: quotient approach

Construction of the tangent cone

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\xi$$

  is a geodesic;
- the positive cone $\alpha \cdot \xi$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

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That's it!

`False in general. One can craft a counterexample on a Cantor measure. Sorry if you lost time`

Thank you!

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