

# Notes on fixed-point procedure

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## 1 Notations and assumptions

We suppose that

1.  $f_{e,b}$  is nonnegative.
2.  $f_{e,b}$  satisfies the boundary condition, i.e.  $f_{e,b}(v) = 0$  as soon as  $\mathcal{L}_e(0, v) \leq \mathcal{L}_e(1, 0)$ .
3.  $f_{e,b}$  is bounded from above by a constant  $\bar{c} \geq 0$ .
4. There exists  $\underline{c} > 0$  and  $0 \leq \underline{v} < \bar{v}$  such that  $f_{e,b}(v) \geq \underline{c}$  for all  $v \in [-\bar{v}, -\underline{v}]$ .
5. The function  $\varphi$  satisfies  $\varphi(0) = \varphi'(0) = 0$ .
6. The function  $\varphi$  is strongly concave, i.e. there exists  $\alpha > 0$  such that  $\varphi((1-\tau)x + \tau y) \geq (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau|x-y|^2$ .
7. There exists  $\beta > \alpha$  such that  $\varphi(1) \geq -\beta$ . **Uniformly over the fixed-point iterations**

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x, v) := \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \quad (1.1)$$

$$\mathcal{L}_i(x, v) := \frac{v^2}{2} + \varphi(x). \quad (1.2)$$

We define an upper bound  $\bar{v}_e$  over the velocities in the support of  $f_e$ , given as

$$\mathcal{L}_e(0, \bar{v}_e) := \frac{1}{\mu}\beta \geq -\frac{1}{\mu}\varphi(1) = \mathcal{L}_e(1, 0), \quad \text{i.e.} \quad \bar{v}_e := \sqrt{\frac{2}{\mu}\beta}. \quad (1.3)$$

For a given point  $(x, v) \in [0, 1] \times \mathbb{R}^-$ , we denote by  $(x_b, v_b)$  the intersection of the boundary  $\{x = 0\} \cup \{v = 0\}$  with the ion characteristic issued from  $(x, v)$ . The values are given by

$$\begin{pmatrix} x_b(x, v) \\ v_b(x, v) \end{pmatrix} := \begin{pmatrix} \varphi^{-1}\left(\min\left(0, \frac{v^2}{2} + \varphi(x)\right)\right) \\ -\sqrt{\max\left(0, v^2 + 2\varphi(x)\right)} \end{pmatrix} \quad (1.4)$$

## 2 Estimates

### 2.1 Useful elementary lemmas

**Lemma 2.1.** *Let  $\alpha > 0$ ,  $\bar{x} \geq 0$  and  $\bar{y} \geq 0$ . Then*

$$\mathcal{I} := \int_{x=0}^{\bar{x}} \int_{y=0}^{\bar{y}} \frac{1}{(x^2 + \alpha y^2)^{1/2}} dy dx \leq 2 \frac{\sqrt{2\bar{x}\bar{y}}}{\alpha^{1/4}}.$$

**Demonstration** By the change of variable  $z = \sqrt{\alpha}y$ , with  $\bar{z} := \sqrt{\alpha}\bar{y}$ , we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\bar{x}} \int_{z=0}^{\bar{z}} \frac{1}{(x^2 + z^2)^{1/2}} dz dx.$$

Notice that  $x + z \leq \sqrt{2}(x^2 + y^2)^{1/2}$ . Then,

$$\mathcal{I} \leq \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\bar{x}} \int_{z=0}^{\bar{z}} \frac{\sqrt{2}}{x+z} dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\bar{x}} \ln \left( \frac{x+\bar{z}}{x} \right) dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\bar{x}} \ln \left( 1 + \frac{\bar{z}}{x} \right) dx.$$

Using  $\ln(1+a) \leq \sqrt{a}$ , we get

$$\mathcal{I} \leq \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\bar{x}} \sqrt{\frac{\bar{z}}{x}} dx = \sqrt{\frac{2}{\alpha}} 2\sqrt{\bar{x}\bar{z}} = 2 \frac{\sqrt{2\bar{x}\bar{y}}}{\alpha^{1/4}}.$$

□

**Remark 2.1** (Exact value). *Let  $\bar{r} := \sqrt{\bar{x}^2 + \bar{z}^2}$ . Then*

$$\mathcal{I} = \bar{x} \ln \left( \frac{1 + \bar{z}/\bar{r}}{\bar{x}/\bar{r}} \right) + \bar{z} \ln \left( \frac{1 + \bar{x}/\bar{r}}{\bar{z}/\bar{r}} \right).$$

**Lemma 2.2.** *If  $a \geq 0$  and  $b \geq 0$ , then  $|a - b| \leq \sqrt{|a^2 - b^2|}$ .*

**Demonstration** If  $a \geq b$ , then  $|a - b| = \sqrt{(a-b)(a+b)} \leq \sqrt{(a-b)(a+b)} = \sqrt{a^2 - b^2}$ , else  $|a - b| = |b - a|$ . □

### 2.2 Properties of the electron density $n_e$

**Lemma 2.3.** *The electron density  $n_e$  is bounded and continuous.*

**Demonstration** For each  $0 \leq x \leq y \leq 1$ , we define the velocity  $v_x(y) \geq 0$  such that

$$\mathcal{L}_e(x, v_x(y)) = \mathcal{L}_e(y, 0) \iff v_x(y) = \left( \frac{2}{\mu} (\varphi(x) - \varphi(y)) \right)^{1/2}.$$

In particular, owing to the boundary conditions, the function  $v \rightarrow f_e(x, v)$  vanishes for  $|v| \geq v_x(1)$ . Then

$$n_e(x) = \int_{v=-v_x(1)}^{v_x(1)} f_e(x, v) dv \leq 2\bar{c}v_x(1) = 2\bar{c} \left( \frac{2}{\mu} (\varphi(x) - \varphi(1)) \right)^{1/2} \leq 2\bar{c} \sqrt{\frac{2\beta}{\mu}}.$$

Moreover, using the symmetry  $f_e(x, v) = f_e(x, -v)$ , we may write

$$n_e(x) - n_e(y) = 2 \underbrace{\int_{v=0}^{v_x(y)} f_e(x, v) dv}_{=: \mathcal{I}^+} + 2 \left( \underbrace{\int_{v=v_x(y)}^{v_x(1)} f_e(x, v) dv}_{=: \mathcal{I}^-} - \underbrace{\int_{v=0}^{v_y(1)} f_e(y, v) dv}_{=: \mathcal{I}^-} \right).$$

The term  $\mathcal{I}^+$  is bounded by  $\bar{c}v_x(y) = \bar{c}\left(\frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}$ , and by continuity of  $\varphi$ , we have  $\mathcal{I}^+ \xrightarrow{y \rightarrow x} 0$ . On the first integral of  $\mathcal{I}^-$ , we apply the change of variable

$$w = \left(v^2 - \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2} \iff dv = \frac{w}{\left(w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}} dw, \quad w \in [0, v_y(1)]$$

to get

$$\mathcal{I}^- = \int_{w=0}^{v_y(1)} f_e \left( x, \left( w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2} \right) \frac{w}{\left( w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2}} dw - \int_{v=0}^{v_y(1)} f_e(y, v) dv.$$

Notice that

$$f_e \left( x, \left( w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2} \right) = f_{e,b} \left( \frac{w^2}{2} + \frac{1}{\mu}(\varphi(x) - \varphi(y)) - \frac{1}{\mu}\varphi(x) \right) = f_e(y, w).$$

Renaming  $w$  in  $v$ , we obtain the (clearly nonpositive) expression

$$\begin{aligned} \mathcal{I}^- &= \int_{v=0}^{v_y(1)} f_e(y, v) \left( \frac{v}{\left( v^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2}} - 1 \right) dv \geq \bar{c} \int_{v=0}^{v_y(1)} \left( \frac{v}{\left( v^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2}} - 1 \right) dv \\ &= \bar{c} \left( \left( v_y(1)^2 + \frac{2}{\mu} \underbrace{(\varphi(x) - \varphi(y))}_{\geq 0} \right)^{1/2} - \left( \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2} - v_y(1) \right) \geq -\bar{c} \left( \frac{2}{\mu}(\varphi(x) - \varphi(y)) \right)^{1/2} \end{aligned}$$

and this shows that  $\mathcal{I}^- \xrightarrow{y \rightarrow x} 0$ . □

**Lemma 2.4.** Let  $n_e^\varphi$  and  $n_e^\psi$  be the electron densities generated by potentials  $\varphi$  and  $\psi$  satisfying the assumptions. Then

1. If  $f_{e,b}$  is Lipschitz-continuous with constant  $[f_{e,b}]$ , then

$$|n_e^\varphi(x) - n_e^\psi(x)| \leq 2[f_{e,b}]\bar{v}_e \sqrt{\frac{2}{\mu}} |\varphi(x) - \psi(x)|^{1/2} \quad \forall (x, y) \in [0, 1]^2.$$

2. If there exists a constant  $[f_{e,b}]$  such that  $|f_{e,b}(x) - f_{e,b}(y)| \leq [f_{e,b}]|x^2 - y^2|$  (or equivalently,  $x \rightarrow f_{e,b}(\sqrt{x})$  is a Lipschitz function, as for instance  $e^{-x^2}$ ), then

$$|n_e^\varphi(x) - n_e^\psi(x)| \leq \frac{4\bar{v}_e[f_{e,b}]2}{\mu} |\varphi(x) - \psi(x)| \quad \forall (x, y) \in [0, 1]^2.$$

**Demonstration** We have

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leq 2 \int_{v=0}^{\bar{v}_e} \left| f_e^\varphi(x, v) - f_e^\psi(x, v) \right| dv = 2 \int_{v=0}^{\bar{v}_e} \left| f_{e,b} \left( \left( v^2 - \frac{2}{\mu}\varphi(x) \right)^{1/2} \right) - f_{e,b} \left( \left( v^2 - \frac{2}{\mu}\psi(x) \right)^{1/2} \right) \right| dv.$$

Then, if  $f_{e,b}$  is Lipschitz with constant  $[f_{e,b}]$ , we obtain

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leq 2[f_{e,b}] \int_{v=0}^{\bar{v}_e} \left| \left( v^2 - \frac{2}{\mu}\varphi(x) \right)^{1/2} - \left( v^2 - \frac{2}{\mu}\psi(x) \right)^{1/2} \right| dv.$$

Using lemma (2.2) yields

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leq 2[f_{e,b}] \int_{v=0}^{\bar{v}_e} \left| -\frac{2}{\mu}\varphi(x) + \frac{2}{\mu}\psi(x) \right|^{1/2} dv = 2[f_{e,b}]\bar{v}_e \sqrt{\frac{2}{\mu}} |\varphi(x) - \psi(x)|^{1/2}.$$

If  $f_{e,b}(\sqrt{\cdot})$  is Lipschitz with constant  $[f_{e,b}]_2$ , we may directly write

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leq 2\bar{v}_e[f_{e,b}]_2 \frac{2}{\mu} |\varphi(x) - \psi(x)|.$$

□

### 2.3 Properties of the ion density $n_i$

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given  $x$ , we define  $g_x : [0, x] \mapsto \mathbb{R}^+$  by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = (2(\varphi(y) - \varphi(x)))^{1/2}.$$

**Lemma 2.5.** *Let  $0 \leq y < x \leq 1$ . We have*

$$\mathcal{I} := \int_{v=-g_x(y)}^0 \int_{z=x_b(x,v)}^x \frac{1}{(v^2 - g_x^2(z))^{1/2}} dz dv \leq 2\sqrt{\frac{2}{\alpha}} (\varphi(y) - \varphi(x))^{1/4} \sqrt{x-y}.$$

**Demonstration** Let us first use Fubini's theorem to switch the order of integration. The lower bound  $x_b(x, v) \geq z$  becomes an upper bound  $v \leq -g_x(z)$ , and we have

$$\mathcal{I} = \int_{z=y}^x \int_{v=-g_x(z)}^{-g_x(y)} \frac{1}{(v^2 - g_x^2(z))^{1/2}} dv dz = \int_{z=y}^x \int_{v=g_x(z)}^{g_x(y)} \frac{1}{(v^2 - g_x^2(z))^{1/2}} dv dz.$$

With the change of variable  $w = v - g_x(z)$ , and using  $\frac{d}{dw} [\sinh^{-1}(\sqrt{\frac{w}{a}})] = (w^2 + 2aw)^{-1/2}$ , we get

$$\mathcal{I} = \int_{z=y}^x \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{(w^2 + 2wg_x(z))^{1/2}} dw dz = \int_{z=y}^x \sinh^{-1} \left( \sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}} \right) dz.$$

Using the coarse estimates  $\sinh^{-1}(a) \leq a$  and  $\sqrt{\frac{a-b}{b}} \leq \sqrt{\frac{a}{b}}$ , we get

$$\mathcal{I} \leq \int_{z=y}^x \sqrt{\frac{g_x(y)}{g_x(z)}} dz = \int_{z=y}^x \left( \frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)} \right)^{1/4} dz.$$

The assumption of strong convexity yields  $\varphi(z) - \varphi(x) \geq -\varphi'(z)(x-z) + \frac{\alpha}{2}|x-z|^2 \geq \frac{\alpha}{2}(x-z)^2$ , so that

$$\mathcal{I} \leq \sqrt{\frac{2}{\alpha}} (\varphi(y) - \varphi(x))^{1/4} \int_{z=y}^x \frac{1}{(x-z)^{1/2}} dz = 2\sqrt{\frac{2}{\alpha}} (\varphi(y) - \varphi(x))^{1/4} \sqrt{x-y}.$$

□

**Lemma 2.6.** *Let  $0 \leq y \leq x \leq 1$ , and  $-v_0 < -g_x(y)$ . We have*

$$\mathcal{I} := \int_{v=-v_0}^{-g_x(y)} \int_{z=y}^1 \frac{1}{(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}} dz dv \leq \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} (v_0^2 + 2(\varphi(x) - \varphi(y)))^{1/4}.$$

**Demonstration** We first shift the  $v$ -integration from the vertical line  $z = x$  to  $z = y$ . Let  $w = w(v)$  be such that

$$\mathcal{L}_i(y, w(v)) = \mathcal{L}_i(x, v), \quad \text{i.e.} \quad w(v) = -(v^2 + 2(\varphi(x) - \varphi(y))), \quad \text{and} \quad dv = \frac{-w}{(w^2 + 2(\varphi(y) - \varphi(x)))^{1/2}} dw.$$

Then, defining  $w_0 := (v_0^2 + 2(\varphi(x) - \varphi(y)))^{1/2}$ , and noticing that  $w(-g_x(y)) = 0$ , we get

$$\mathcal{I} = \int_{w=-w_0}^0 \int_{z=y}^1 \frac{1}{(w^2 + 2(\varphi(y) - \varphi(z)))^{1/2}} dz \frac{-w}{(w^2 + 2(\varphi(y) - \varphi(x)))^{1/2}} dw.$$

Since  $y \leq x$ , we have  $\varphi(y) \geq \varphi(x)$ , and  $\frac{-w}{(w^2 + 2(\varphi(y) - \varphi(x)))^{1/2}} \leq \frac{-w}{|w|} = 1$ . By the strong convexity assumption, we have  $\varphi(y) - \varphi(z) \geq -\varphi'(y)(z-y) + \frac{\alpha}{2}|z-y|^2 \geq \frac{\alpha}{2}(z-y)^2$ , so that

$$\mathcal{I} \leq \int_{w=-w_0}^0 \int_{z=y}^1 \frac{1}{(w^2 + \alpha(z-y)^2)^{1/2}} dz dw = \int_{w=0}^{w_0} \int_{z=0}^{1-y} \frac{1}{(w^2 + \alpha z^2)^{1/2}} dz dw.$$

Using lemma (2.1), we conclude that

$$\mathcal{I} \leq 2 \frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}} = \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} (v_0^2 + 2(\varphi(x) - \varphi(y)))^{1/4}.$$

□

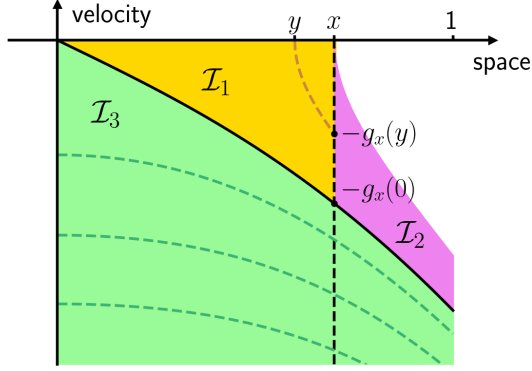


Figure 1: Decomposition of the integral defining  $n_i$ .

The phase space is divided by the critical characteristic (in solid black). Whenever  $v \leq -g_x(0)$ , the characteristics (dotted green lines) are reaching the boundary with  $x_b(x, v) = 0$ .

**Proposition 2.1.** *The density  $n_i$  is bounded by a constant depending on  $\varphi$  only through  $\varphi(1)$ .*

**Demonstration** We use the symmetry of  $f_i$  to write

$$n_i(x) = \int_{v=-\infty}^{\infty} f_i(x, v) dv = 2 \int_{v=-\infty}^0 f_i(x_b(x, v), v_b(x, v)) dv = 2 \int_{v=-\infty}^0 \int_{t=-\infty}^0 f_e(x(t), v(t)) dt dv,$$

where  $(x(t), v(t))_{t \leq 0}$  is the ion characteristic reaching  $(x_b(x, v), v_b(x, v))$  at  $t = 0$ . Notice that the lower bounds are artificial, since the characteristic enters the support of  $f_e$  in finite time: we may use  $v \geq -\bar{v}_e$ , and consider only times  $t$  for which  $x(t) \in [0, 1]$ .

We first reparametrize  $(x(t), v(t))$  using the space variable. Define  $z = x(t) \in [x_b, 1]$ , and observe that

$$dz = \dot{x}(t) dt = v(t) dt, \quad \text{with} \quad \mathcal{L}_i(z, v(t)) = \mathcal{L}_i(x, v) \quad \Longleftrightarrow \quad v(t) = - (v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}.$$

Then, the density rewrites

$$n_i(x) = 2 \int_{v=-\bar{v}_e}^0 \int_{z=x_b(x, v)}^1 \frac{f_e \left( z, - (v^2 + 2(\varphi(x) - \varphi(z)))^{1/2} \right)}{(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}} dz dv.$$

Let us show that  $n_i$  is bounded. We use the coarse estimate  $f_e \leq \bar{c}$ , and decompose the integral in three:

$$n_i(x) \leq 2\bar{c} \left[ \underbrace{\int_{v=-g_x(0)}^0 \int_{z=x_b(x, v)}^x}_{\mathcal{I}_1} + \underbrace{\int_{v=-g_x(0)}^0 \int_{z=x}^1}_{\mathcal{I}_2} + \underbrace{\int_{v=-\bar{v}_e}^{-g_x(0)} \int_{z=x_b(x, v)}^1}_{\mathcal{I}_3} \right] \frac{1}{(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}} dz dv.$$

The corresponding domains are represented figure (1).

Notice that whenever  $z \leq x$ , we have  $0 \geq 2(\varphi(x) - \varphi(z)) = -(2(\varphi(z) - \varphi(x)))^{2/2} = -g_x^2(z)$ . Then, the integral  $\mathcal{I}_1$  may be bounded using lemma (2.5) with  $y = 0$ :

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^0 \int_{z=x_b(x, v)}^x \frac{1}{(v^2 - g_x^2(z))^{1/2}} dz dv \leq 2\sqrt{\frac{2}{\alpha}} (-\varphi(x))^{1/4} \sqrt{x} \leq 2\sqrt{\frac{2}{\alpha}} (-\varphi(1))^{1/4}.$$

We use lemma (2.6) to bound  $\mathcal{I}_2$  and  $\mathcal{I}_3$ . In the first case, we take  $y = x$  and  $v_0 = g_x(0)$ , and notice that  $-g_x(x) = 0$ . In the second case, we take  $v_0 = \bar{v}_e$  and  $y = 0$ , and notice that on  $v \leq -g_x(0)$ , we have  $x_b(x, v) = 0$  (the velocity is low enough so that the characteristic ends on  $x_b = 0$ ). This yields

$$\mathcal{I}_2 \leq \frac{2\sqrt{2(1-x)}}{\alpha^{1/4}} (g_x^2(0))^{1/4} \leq \frac{4}{\alpha^{1/4}} (-\varphi(1))^{1/2}, \quad \text{and} \quad \mathcal{I}_3 \leq \frac{2\sqrt{2}}{\alpha^{1/4}} (\bar{v}_e^2 + 2\varphi(x))^{1/4} \leq \frac{4}{\alpha^{1/4}} \left( \frac{\beta}{\mu} \right)^{1/4}.$$

□

**Proposition 2.2.** *The density  $n_i$  is continuous.*

**Demonstration** Let  $0 \leq y < x \leq 1$ . For convenience, we represent  $n_i(x)$  (resp.  $n_i(y)$ ) as an integral with the artificial lower bound  $-g_x(-\bar{v}_e) \leq -\bar{v}_e$  (resp.  $-g_y(-\bar{v}_e)$ ). Then

$$\begin{aligned} n_i(x) - n_i(y) &= 2 \int_{v=-g_x(-\bar{v}_e)}^0 f_i(x_b(x, v), v_b(x, v)) dv - 2 \int_{v=-g_y(-\bar{v}_e)}^0 f_i(x_b(y, v), v_b(y, v)) dv \\ &= 2 \underbrace{\left[ \int_{v=-g_x(-\bar{v}_e)}^{-g_x(y)} f_i(x_b(x, v), v_b(x, v)) dv - \int_{v=-g_y(-\bar{v}_e)}^0 f_i(x_b(y, v), v_b(y, v)) dv \right]}_{=: \mathcal{I}^-} + 2 \underbrace{\int_{v=-g_x(y)}^0 f_i(x_b(x, v), v_b(x, v)) dv}_{=: \mathcal{I}^+}. \end{aligned}$$

The term  $\mathcal{I}^+$  is clearly nonnegative, and may be addressed using our lemmas. Indeed, using the integral representation of  $f_i(x_b, v_b)$  and the reparametrization by a space variable  $z$ , we have

$$\begin{aligned} \mathcal{I}^+ &= \int_{v=-g_x(y)}^0 \left[ \int_{z=x_b(x, v)}^x + \int_{z=x}^1 \right] \frac{f_e(z, -(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2})}{(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}} dz dv \\ &\leq \bar{c} \int_{v=-g_x(y)}^0 \int_{z=x_b(x, v)}^x \frac{1}{(v^2 - g_x^2(z))^{1/2}} dz dv + \bar{c} \int_{v=-g_x(y)}^0 \int_{z=x}^1 \frac{1}{(v^2 + 2(\varphi(x) - \varphi(z)))^{1/2}} dz dv \\ &\leq \bar{c} \left( 2\sqrt{\frac{2}{\alpha}} (\varphi(y) - \varphi(x))^{1/4} \sqrt{x-y} + \frac{2\sqrt{2}}{\alpha^{1/4}} (2(\varphi(y) - \varphi(x)))^{1/2} \right), \end{aligned}$$

where we used lemma (2.5) for the first term, and lemma (2.6) for the second term (with  $y = x$  and  $v_0 = -g_x(y)$  under the notations of the lemma). Since  $\varphi$  is continuous, we deduce that  $\mathcal{I}^+ \xrightarrow{y \rightarrow x} 0$ . Taking the extreme case  $y = 0$  and  $x = 1$ , we obtain that

$$\mathcal{I}^+ \leq \bar{c} \left( 2\sqrt{\frac{2}{\alpha}} (-\varphi(1))^{1/4} + \frac{2\sqrt{2}}{\alpha^{1/4}} (-2\varphi(1))^{1/2} \right) =: K.$$

Let us now focus on  $\mathcal{I}^-$ . On the first integral, we make the change of variable

$$w = -(v^2 + 2(\varphi(x) - \varphi(y)))^{1/2} \quad v = -(w^2 + 2(\varphi(y) - \varphi(x)))^{1/2}.$$

Since  $\mathcal{L}_i(x, v) = \mathcal{L}_i(y, w)$ , this yields  $x_b(x, v) = x_b(y, w)$  and  $v_b(x, v) = v_b(y, w)$ . The bounds  $v \in [-g_x(-\bar{v}_e), -g_x(y)]$  are exactly transported to  $w \in [-g_y(-\bar{v}_e), 0]$ . Renaming  $w$  in  $v$ , we get

$$\mathcal{I}^- = \int_{v=-g_y(-\bar{v}_e)}^0 f_i(x_b(y, v), v_b(y, v)) \left( \frac{-v}{(v^2 + 2(\varphi(y) - \varphi(x)))^{1/2}} - 1 \right) dv.$$

Since  $\varphi(y) \geq \varphi(x)$ , the factor of  $f_i$  is nonpositive, and so is  $\mathcal{I}^-$ . Moreover,

$$n_i(x) - n_i(y) = 2\mathcal{I}^- + 2\mathcal{I}^+ \leq 2\mathcal{I}^- + 2K \iff \mathcal{I}^- = -K + n_i(x) - n_i(y) \geq -K - |n_i|_\infty$$

and  $\mathcal{I}^-$  is bounded. The function  $\frac{-v}{(v^2 + 2(\varphi(y) - \varphi(x)))^{1/2}} - 1$  converges pointwise to 0 when  $x \rightarrow y$  **and  $f_i$  is almost everywhere finite**, and by Lebesgue's dominated convergence,  $\mathcal{I}^- \xrightarrow{x \rightarrow y} 0$ . Then  $n_i$  is continuous. □

**In the future:**

- Now that we have estimates, write it in function of  $\alpha$  and  $\beta$ , and see how to get stability of the set of strongly concave functions satisfying all the hypotheses by the Poisson problem (essentially, find tweaks of  $\lambda$ ,  $\mu$ ,  $\nu$ ,  $\alpha$  and/or  $\beta$  such that the estimates are propagated).
- If  $f_{e,b}$  vanishes in a neighbourhood of  $(0,0)$  (or decreases fast enough, see how fast), show that  $f_i$  is bounded.
- Under that same assumption, we should obtain stronger continuity over  $n_i$ , and  $n_i$  may be continuous with respect to  $\varphi$  (in the same sense as  $n_e$  in lemma (2.4)).
- If the estimates with respect to  $|\varphi - \psi|_\infty$  succeed, define numerical scheme and see what we can say about it.