Notes on fixed-point procedure

Contents

1 Notations and assumptions

We suppose that

1. $f_{e,b}$ satisfies the boundary condition, i.e. $f_{e,b}(v) = 0$ as soon as $\mathcal{L}_e(0, v) \leqslant \mathcal{L}_e(1, 0)$.
2. $f_{e,b}$ is continuous. **Lipschitz may be required?**
3. $f_{e,b}$ is bounded from above by a constant $\overline{c} \geqslant 0$.
4. There exists $\underline{c} > 0$ and $0 \leqslant \underline{v} < \overline{v}$ such that $f_{e,b}(v) \geqslant \underline{c}$ for all $v \in [-\overline{v}, -\underline{v}]$.
5. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$.
6. The function φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi((1-\tau)x + \tau y) \geqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau\,|x-y|^2$.
7. There exists $\beta > \alpha$ such that $\varphi(1) \geqslant -\beta$. **Uniformly over the fixed-point iterations**

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x, v) \coloneqq \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{1.1}$$

$$\mathcal{L}_i(x, v) \coloneqq \frac{v^2}{2} + \varphi(x). \tag{1.2}$$

We define an upper bound $\overline{v}_e$ over the velocities in the support of f_e, given as

$$\mathcal{L}_e(0, \overline{v}_e) \coloneqq \frac{1}{\mu}\beta \geqslant -\frac{1}{\mu}\varphi(1) = \mathcal{L}_e(1, 0), \quad \text{i.e.} \quad \overline{v}_e \coloneqq \sqrt{\frac{2}{\mu}\beta}. \tag{1.3}$$

For a given point $(x, v) \in [0, 1] \times \mathbb{R}^-$, we denote by (x_b, v_b) the intersection of the boundary $\{x = 0\} \cup \{v = 0\}$ with the ion characteristic issued from (x, v). The values are given by

$$\begin{pmatrix} x_b(x, v) \\ v_b(x, v) \end{pmatrix} \coloneqq \begin{pmatrix} \varphi^{-1}\left(\min\left(0, \frac{v^2}{2} + \varphi(x)\right)\right) \\ -\sqrt{\max\left(0, v^2 + 2\varphi(x)\right)} \end{pmatrix} \tag{1.4}$$

2 Estimates

2.1 Useful elementary lemmas

Lemma 2.1. *Let $\alpha > 0$, $\overline{x} \geqslant 0$ and $\overline{y} \geqslant 0$. Then*

$$\mathcal{I} \coloneqq \int_{x=0}^{\overline{x}} \int_{y=0}^{\overline{y}} \frac{1}{\left(x^2 + \alpha y^2\right)^{1/2}} dy dx \leqslant 2 \frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

Demonstration By the change of variable $z = \sqrt{\alpha} y$, with $\overline{z} \coloneqq \sqrt{\alpha}\overline{y}$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{1}{\left(x^2 + z^2\right)^{1/2}} dz dx.$$

Notice that $x + z \leqslant \sqrt{2} \left(x^2 + y^2\right)^{1/2}$. Then,

$$\mathcal{I} \leqslant \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{\sqrt{2}}{x+z} dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(\frac{x+\overline{z}}{x}\right) dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(1 + \frac{\overline{z}}{x}\right) dx.$$

Using $\ln(1+a) \leqslant \sqrt{a}$, we get

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \sqrt{\frac{\overline{z}}{x}} dx = \sqrt{\frac{2}{\alpha}} 2\sqrt{\overline{x}\overline{z}} = 2\frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

$\square$

Remark 2.1 (Exact value)**.** *Let $\overline{r} \coloneqq \sqrt{\overline{x}^2 + \overline{z}^2}$. Then*

$$\mathcal{I} = \overline{x} \ln\left(\frac{1 + \overline{z}/\overline{r}}{\overline{x}/\overline{r}}\right) + \overline{z} \ln\left(\frac{1 + \overline{x}/\overline{r}}{\overline{z}/\overline{r}}\right).$$

2.2 Integrals along ion characteristics

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given x, we define $g_x : [0, x] \mapsto \mathbb{R}^+$ by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = \left(2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Lemma 2.2. *Let $0 \leqslant y < x \leqslant 1$. We have*

$$\mathcal{I} \coloneqq \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dz\, dv \leqslant 2\sqrt{\frac{2}{\alpha}} \left(\varphi(y) - \varphi(x)\right)^{1/4} \sqrt{x-y}.$$

Demonstration Let us first use Fubini's theorem to switch the order of integration. The lower bound $x_b(x,v) \geqslant z$ becomes an upper bound $v \leqslant -g_x(z)$, and we have

$$\mathcal{I} = \int_{z=y}^{x} \int_{v=-g_x(y)}^{-g_x(z)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz = \int_{z=y}^{x} \int_{v=g_x(z)}^{g_x(y)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz.$$

With the change of variable $w = v - g_x(z)$, and using $\frac{d}{dw}\left[\sinh^{-1}\left(\sqrt{\frac{w}{a}}\right)\right] = \left(w^2 + 2aw\right)^{-1/2}$, we get

$$\mathcal{I} = \int_{z=y}^{x} \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{\left(w^2 + 2wg_x(z)\right)^{1/2}} dw\, dz = \int_{z=y}^{x} \sinh^{-1}\left(\sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}}\right) dz.$$

Using the coarse estimates $\sinh^{-1}(a) \leqslant a$ and $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$, we get

$$\mathcal{I} \leqslant \int_{z=y}^{x} \sqrt{\frac{g_x(y)}{g_x(z)}} dz = \int_{z=y}^{x} \left(\frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)}\right)^{1/4} dz.$$

The assumption of strong convexity yields $\varphi(z)-\varphi(x)\geqslant -\varphi'(z)(x-z)+\frac{\alpha}{2}|x-z|^2\geqslant\frac{\alpha}{2}(x-z)^2$, so that

$$\mathcal{I}\leqslant\sqrt{\frac{2}{\alpha}}\left(\varphi(y)-\varphi(x)\right)^{1/4}\int_{z=y}^{x}\frac{1}{(x-z)^{1/2}}dz=2\sqrt{\frac{2}{\alpha}}\left(\varphi(y)-\varphi(x)\right)^{1/4}\sqrt{x-y}.$$

□

Lemma 2.3. *Let $0\leqslant y\leqslant x\leqslant 1$, and $-v_0<-g_x(y)$. We have*

$$\mathcal{I}:=\int_{v=-v_0}^{-g_x(y)}\int_{z=y}^{1}\frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dz\,dv\leqslant\frac{2\sqrt{2(1-y)}}{\alpha^{1/4}}\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/4}.$$

Demonstration We first shift the $v-$integration from the vertical line $z=x$ to $z=y$. Let $w=w(v)$ be such that

$$\mathcal{L}_i(y,w(v))=\mathcal{L}_i(x,v),\quad\text{i.e.}\quad w(v)=-\left(v^2+2\left(\varphi(x)-\varphi(y)\right)\right),\ \text{and}\ dv=\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Then, defining $w_0:=\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$, and noticing that $w(-g_x(y))=0$, we get

$$\mathcal{I}=\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+2\left(\varphi(y)-\varphi(z)\right)\right)^{1/2}}dz\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Since $y\leqslant x$, we have $\varphi(y)\geqslant\varphi(x)$, and $\frac{-w}{\left(w^2+2(\varphi(y)-\varphi(x))\right)^{1/2}}\leqslant\frac{-w}{|w|}=1$. By the strong convexity assumption, we have $\varphi(y)-\varphi(z)\geqslant-\varphi'(y)(z-y)+\frac{\alpha}{2}|z-y|^2\geqslant\frac{\alpha}{2}(z-y)^2$, so that

$$\mathcal{I}\leqslant\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+\alpha(z-y)^2\right)^{1/2}}dzdw=\int_{w=0}^{w_0}\int_{z=0}^{1-y}\frac{1}{\left(w^2+\alpha z^2\right)^{1/2}}dzdw.$$

Using lemma (2.1), we conclude that

$$\mathcal{I}\leqslant 2\frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}}=\frac{2\sqrt{2(1-y)}}{\alpha^{1/4}}\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/4}.$$

□

Proposition 2.1. *The density n_i is bounded.* ***Be more precise***

Demonstration We use the symmetry of f_i to write

$$n_i(x)=\int_{v=-\infty}^{\infty}f_i(x,v)dv=2\int_{v=-\infty}^{0}f_i(x_b(x,v),v_b(x,v))dv=2\int_{v=-\infty}^{0}\int_{t=-\infty}^{0}f_e(x(t),v(t))dtdv,$$

where $(x(t),v(t))_{t\leqslant 0}$ is the ion characteristic reaching $(x_b(x,v),v_b(x,v))$ at $t=0$. Notice that the lower bounds are artificial, since the characteristic enters the support of f_e in finite time: we may use $v\geqslant-\overline{v}_e$, and consider only times t for which $x(t)\in[0,1]$.

We first reparametrize $(x(t),v(t))$ using the space variable. Define $z=x(t)\in[x_b,1]$, and observe that

$$dz=\dot{x}(t)dt=v(t)dt,\quad\text{with}\quad\mathcal{L}_i(z,v(t))=\mathcal{L}_i(x,v)\quad\Longleftrightarrow\quad v(t)=-\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}.$$

Then, the density rewrites

$$n_i(x)=2\int_{v=-\overline{v}_e}^{0}\int_{z=x_b(x,v)}^{1}\frac{f_e\left(z,-\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}\right)}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dzdv.$$

Let us show that n_i is bounded. We use the coarse estimate $f_e\leqslant\overline{c}$, and decompose the integral in three:

$$n_i(x)\leqslant 2\overline{c}\left[\underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x}}_{\mathcal{I}_1}+\underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x}^{1}}_{\mathcal{I}_2}+\underbrace{\int_{v=-\overline{v}_e}^{-g_x(0)}\int_{z=x_b(x,v)}^{1}}_{\mathcal{I}_3}\right]\frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dzdv.$$

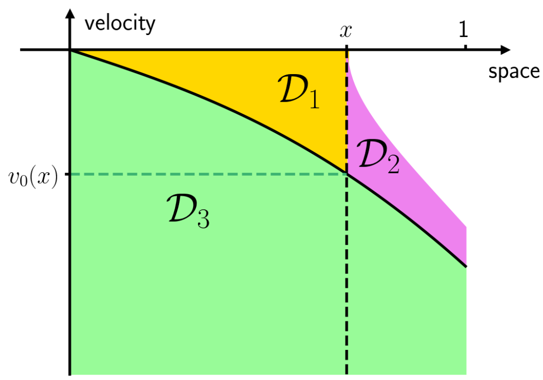

Figure 1: Decomposition of the integral defining n_i.

o boy so many things to say

The corresponding domains are represented figure (1).

Notice that whenever $z \leqslant x$, we have $0 \geqslant 2(\varphi(x) - \varphi(z)) = -(2(\varphi(z) - \varphi(x)))^{2/2} = -g_x^2(z)$. Then, the integral $\mathcal{I}_1$ may be bounded using lemma (2.2) with $y = 0$:

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{(v^2 - g_x^2(z))^{1/2}} \, dz dv \leqslant 2\sqrt{\frac{2}{\alpha}} (-\varphi(x))^{1/4} \sqrt{x} \leqslant 2\sqrt{\frac{2}{\alpha}} (-\varphi(1))^{1/4} .$$

We use lemma (2.3) to bound $\mathcal{I}_2$ and $\mathcal{I}_3$. In the first case, we take $y = x$ and $v_0 = g_x(0)$, and notice that $-g_x(x) = 0$. In the second case, we take $v_0 = \overline{v}_e$ and $y = 0$, and notice that on $v \leqslant -g_x(0)$, we have $x_b(x,v) = 0$ (the velocity is low enough so that the characteristic ends on $x_b = 0$). This yields

$$\mathcal{I}_2 \leqslant \frac{2\sqrt{2(1-x)}}{\alpha^{1/4}} \left(g_x^2(0)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}} (-\varphi(1))^{1/2}, \quad \text{and} \quad \mathcal{I}_3 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}} \left(\overline{v}_e{}^2 + 2\varphi(x)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}} \left(\frac{\beta}{\mu}\right)^{1/4} .$$

□

Proposition 2.2. *The density n_i is continuous.*

Demonstration Let $0 \leqslant y < x \leqslant 1$. For convenience, we represent $n_i(x)$ (resp. $n_i(y)$) as an integral with the artificial lower bound $-g_x(-\overline{v}_e) \leqslant -\overline{v}_e$ (resp. $-g_y(-\overline{v}_e)$). Then

$$\begin{aligned} n_i(x) - n_i(y) &= 2\int_{v=-g_x(-\overline{v}_e)}^{0} f_i(x_b(x,v), v_b(x,v)) dv - 2\int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v)) dv \\ &= \underbrace{2\left[\int_{v=-g_x(-\overline{v}_e)}^{-g_x(y)} f_i(x_b(x,v), v_b(x,v)) dv - \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v)) dv\right]}_{=: \mathcal{I}^-} + \underbrace{2\int_{v=-g_x(y)}^{0} f_i(x_b(x,v), v_b(x,v)) dv}_{=: \mathcal{I}^+} . \end{aligned}$$

□

Let us first focus on $\mathcal{I}^-$. On the first integral, we make the change of variable

$$w = -\left(v^2 + 2(\varphi(x) - \varphi(y))\right)^{1/2} \qquad v = -\left(w^2 + 2(\varphi(y) - \varphi(w))\right)^{1/2} .$$

Since $\mathcal{L}_i(x,v) = \mathcal{L}_i(y,w)$, this yields $x_b(x,v) = x_b(y,w)$ and $v_b(x,v) = v_b(y,w)$. The bounds $v \in [-g_x(-\overline{v}_e), -g_x(y)]$ are exactly transported to $w \in [-g_y(-\overline{v}_e), 0]$. Renaming w in v, we get

$$\mathcal{I}^- = \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v)) \left(\frac{-v}{(v^2 + 2(\varphi(y) - \varphi(v)))^{1/2}} - 1 \right) dv.$$

Since $\varphi(y) \geqslant \varphi(x)$, the factor of f_i is nonpositive, and so is $\mathcal{I}^-$. **go on**

The second term $\mathcal{I}^+$ is clearly nonnegative, and may be adressed using our lemmas. Indeed, using the integral representation of $f_i(x_b, v_b)$ and the reparametrization by a space variable z, we have

$$\begin{aligned}
\mathcal{I}^+ &= \int_{v=-g_x(y)}^{0} \left[\int_{z=x_b(x,v)}^{x} + \int_{z=x}^{1} \right] \frac{f_e(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2})}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} \, dz dv \\
&\leqslant \bar{c} \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} \, dz dv + \bar{c} \int_{v=-g_x(y)}^{0} \int_{z=x}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} \, dz dv \\
&\leqslant 2\sqrt{\frac{2}{\alpha}} \left(\varphi(y) - \varphi(x)\right)^{1/4} \sqrt{x - y} + \frac{2\sqrt{2}}{\alpha^{1/4}} \left(2(\varphi(y) - \varphi(x))\right)^{1/2},
\end{aligned}$$

where we used lemma (2.2) for the first term, and lemma (2.3) for the second term (with $y = x$ and $v_0 = -g_x(y)$ under the notations of the lemma).