Notes on fixed-point procedure

Let us define for all $0 < \alpha \leqslant \beta$ the set

$$\mathcal{K}^{\alpha,\beta} \coloneqq \left\{ \varphi \in \mathcal{C}^2\left([0,1], \mathbb{R}^-\right) \mid \varphi(0) = \varphi'(0) = 0, \quad -\beta \leqslant \varphi'' \leqslant -\alpha. \right\}$$

Estimates on the set $\mathcal{K}$ Since $\varphi : [0,1] \mapsto \mathbb{R}^-$ is decreasing, its inverse $\varphi^{-1} : \mathbb{R}^- \mapsto [0,1]$ is well-defined. By integration and using $\left(\varphi^{-1}\right)' = (\varphi' \circ \varphi^{-1})^{-1}$, we have

$$\begin{cases} -\beta x \leqslant \varphi'(x) \leqslant -\alpha x & \text{(0.1a)} \\ -\beta \dfrac{x^2}{2} \leqslant \varphi(x) \leqslant -\alpha \dfrac{x^2}{2} & \text{(0.1b)} \\ \sqrt{-\dfrac{2y}{\beta}} \leqslant \varphi^{-1}(y) \leqslant \sqrt{-\dfrac{2y}{\alpha}} & \text{(0.1c)} \\ \dfrac{-1}{\alpha\sqrt{-\frac{2y}{\beta}}} \leqslant (\varphi^{-1})'(y) \leqslant \dfrac{-1}{\beta\sqrt{-\frac{2y}{\alpha}}} & \text{(0.1d)} \end{cases}$$

We want to obtain estimates on $n_i - n_e$. We make the following assumptions:

- The electron density f_e satisfies the boundary condition, and is bounded by a constant $c \geqslant 0$.
- The potential φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi''(x) \leqslant -\alpha$ uniformly over $x \in [0,1]$.

Let us first focus on $n_e(x)$. The characteristics of the electron density are the level lines of

$$\mathcal{L}_e(x,v) \coloneqq \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{0.2}$$

Since φ is strongly concave, these curves are closed. Since f_e satisfies the homogeneous boundary condition, its support is embedded in $\left\{(x,v) \mid \frac{v^2}{2} - \frac{1}{\mu}\varphi(x) \leqslant \frac{0^2}{2} - \frac{1}{\mu}\varphi(1)\right\}$. In particular, we denote by $\underline{v}_e$ the extremal speed of the support,

$$\underline{v}_e \coloneqq \sqrt{-\frac{2}{\mu}\varphi(1)}. \tag{0.3}$$

We can roughly majorize

$$n_e(x) = \int_{v\in\mathbb{R}} f_e(x,v)dv \leqslant \int_{v=-\underline{v}_e}^{\underline{v}_e} c dv = 2c\underline{v}_e \leqslant 2c\sqrt{-\frac{2}{\mu}\varphi(1)}.$$

The estimates on n_i are slightly more technical. Let the ion Lyapunov function be defined as

$$\mathcal{L}_i(x,v) \coloneqq \frac{v^2}{2} + \varphi(x). \tag{0.4}$$

Let $x \in [0,1]$ and $v \in \mathbb{R}_-$. We denote by $(x_b(x,v), v_b(x,v))$ the intersection of the boundary $\{x=0\} \cap \{v=0\}$ with the ion characteristic issued from (x,v), equal to

$$\begin{pmatrix} x_b(x,v) \\ v_b(x,v) \end{pmatrix} := \begin{cases} \begin{pmatrix} \varphi^{-1}\left(\frac{v^2}{2} + \varphi(x)\right) \\ 0 \end{pmatrix} & \text{if } \mathcal{L}_i(x,v) \leqslant 0, \\ \begin{pmatrix} 0 \\ -\sqrt{\frac{v^2}{2} + \varphi(x)} \end{pmatrix} & \text{if } \mathcal{L}_i(x,v) > 0. \end{cases}$$

In the following paragraph, we use $(x(t), v(t))_{t \leqslant 0}$ to denote the characteristic reaching $(x_b(x,v), v_b(x,v))$ at $t=0$. We use the symmetry of f_i to write

$$n_i(x) = 2\int_{v \in \mathbb{R}^-} f_i(x_b(x,v), v_b(x,v))dv = 2\int_{v \in \mathbb{R}^-} \int_{t=-\infty}^{0} f_e(x(t), v(t))dtdv.$$

The lower bound $t \to -\infty$ is artificial, since the characteristic exits the support of f_e in finite time. We will split the double integral in three domains:

1. $\mathcal{D}_1$ will be $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) \leqslant 0 \text{ and } x(t) \geqslant x\right\}$. This is the region contained between the $x-$axis, the critical characteristic and the vertical line going through x.
2. $\mathcal{D}_2$ is $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) \leqslant 0 \text{ and } x < x(t) \geqslant 1\right\}$. It is exactly $\{\mathcal{L}_i \leqslant 0\} \setminus \mathcal{D}_1$.
3. $\mathcal{D}_3$ is defined by $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) > 0 \text{ and } \underline{v}_e \geqslant v(t)\right\}$.

Figure de la décomposition en domaines

References