Notes on fixed-point procedure

Contents

1 Notations and assumptions

We suppose that

1. $f_{e,b}$ is nonnegative.

2. $f_{e,b}(v) = f_{e,b}(-v)$ for all $v \in \mathbb{R}$.

3. $f_{e,b}$ satisfies the boundary condition, i.e. $f_{e,b}(v) = 0$ as soon as $\mathcal{L}_e(0,v) \leqslant \mathcal{L}_e(1,0)$.

4. There exists $\bar{c} \leqslant 0$ such that $f_{e,b}(v) \leqslant \bar{c}$ for all $v \in \mathbb{R}$.

5. There exists $\underline{c} > 0$ and $0 \leqslant \underline{v} < \bar{v}$ such that $f_{e,b}(v) \geqslant \underline{c}$ for all $v \in [-\bar{v}, -\underline{v}]$.

6. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$.

7. The function φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi((1-\tau)x + \tau y) \geqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau |x-y|^2$.

8. There exists $\beta > \alpha$ such that $\varphi(1) \geqslant -\beta$. **Uniformly over the fixed-point iterations**

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x,v) := \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{1.1}$$

$$\mathcal{L}_i(x,v) := \frac{v^2}{2} + \varphi(x). \tag{1.2}$$

We define an upper bound $\bar{v}_e$ over the velocities in the support of f_e, given as

$$\mathcal{L}_e(0, \bar{v}_e) := \frac{1}{\mu}\beta \geqslant -\frac{1}{\mu}\varphi(1) = \mathcal{L}_e(1,0), \quad \text{i.e.} \quad \bar{v}_e := \sqrt{\frac{2}{\mu}\beta}. \tag{1.3}$$

For a given point $(x, v) \in [0, 1] \times \mathbb{R}^-$, we denote by (x_b, v_b) the intersection of the boundary $\{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ with the ion characteristic issued from (x, v). The values are given by

$$\begin{pmatrix} x_b(x, v) \\ v_b(x, v) \end{pmatrix} := \begin{pmatrix} \varphi^{-1} \left(\min \left(0, \frac{v^2}{2} + \varphi(x) \right) \right) \\ -\sqrt{\max \left(0, v^2 + 2\varphi(x) \right)} \end{pmatrix} \tag{1.4}$$

2 Estimates

2.1 Useful elementary lemmas

Lemma 2.1. *Let $\alpha > 0$, $\overline{x} \geqslant 0$ and $\overline{y} \geqslant 0$. Then*

$$\mathcal{I} := \int_{x=0}^{\overline{x}} \int_{y=0}^{\overline{y}} \frac{1}{(x^2 + \alpha y^2)^{1/2}} \, dy dx \leqslant 2 \frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

Demonstration By the change of variable $z = \sqrt{\alpha} y$, with $\overline{z} := \sqrt{\alpha}\overline{y}$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{1}{(x^2 + z^2)^{1/2}} \, dz dx.$$

Notice that $x + z \leqslant \sqrt{2} \left(x^2 + y^2 \right)^{1/2}$. Then,

$$\mathcal{I} \leqslant \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{\sqrt{2}}{x + z} \, dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln \left(\frac{x + \overline{z}}{x} \right) \, dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln \left(1 + \frac{\overline{z}}{x} \right) \, dx.$$

Using $\ln (1 + a) \leqslant \sqrt{a}$, we get

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \sqrt{\frac{\overline{z}}{x}} \, dx = \sqrt{\frac{2}{\alpha}} 2\sqrt{\overline{x}\overline{z}} = 2 \frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

$\square$

Remark 2.1 (Exact value). *Let $\overline{r} := \sqrt{\overline{x}^2 + \overline{z}^2}$. Then*

$$\mathcal{I} = \overline{x} \ln \left(\frac{1 + \overline{z}/\overline{r}}{\overline{x}/\overline{r}} \right) + \overline{z} \ln \left(\frac{1 + \overline{x}/\overline{r}}{\overline{z}/\overline{r}} \right).$$

Lemma 2.2. *If $a \geqslant 0$ and $b \geqslant 0$, then $|a - b| \leqslant \sqrt{|a^2 - b^2|}$.*

Demonstration If $a \geqslant b$, then $|a - b| = \sqrt{(a - b)(a - b)} \leqslant \sqrt{(a - b)(a + b)} = \sqrt{a^2 - b^2}$, else $|a - b| = |b - a|$. $\square$

Lemma 2.3. *Let $\alpha > 0$, $L \in \mathbb{R}$ and $0 \leqslant a \leqslant 1$. Suppose that $L + \alpha a^2 \geqslant 0$, and $a > 0$ if $L = 0$. Then*

$$\mathcal{I} := \int_{z=a}^{1} \frac{1}{(L + \alpha z^2)^{1/2}} \, dz \leqslant \min \left(|L|^{-1/2}, \frac{2}{\sqrt{\alpha}a} \right).$$

Demonstration If $L > 0$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{z=a}^{1} \frac{1}{\left(1 + \left(\sqrt{\frac{\alpha}{L}} z \right)^2 \right)^{1/2}} \frac{\sqrt{\alpha} dz}{\sqrt{L}} = \frac{1}{\alpha} \int_{w=a\sqrt{\frac{\alpha}{L}}}^{\sqrt{\frac{\alpha}{L}}} \frac{1}{(1 + w^2)^{1/2}} \, dw = \frac{1}{\sqrt{\alpha}} \left(\sinh^{-1} \left(\sqrt{\frac{\alpha}{L}} \right) - \sinh^{-1} \left(a\sqrt{\frac{\alpha}{L}} \right) \right).$$

Using the positivity of $\sinh^{-1} \left(a\sqrt{\frac{\alpha}{L}} \right)$, and the coarse estimate $\sinh^{-1}(x) \leqslant x$, we get $\mathcal{I} \leqslant L^{-1/2}$. Moreover, using that $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$, we get

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \log \left(\frac{\sqrt{\frac{\alpha}{L}} + \sqrt{\frac{\alpha}{L} + 1}}{a\sqrt{\frac{\alpha}{L}} + \sqrt{a^2\frac{\alpha}{L} + 1}} \right) = \frac{1}{\sqrt{\alpha}} \log \left(\frac{1 + \sqrt{1 + \frac{L}{\alpha}}}{a + \sqrt{a^2 + \frac{L}{\alpha}}} \right) \leqslant \frac{1}{\sqrt{\alpha}} \log \left(\frac{2 + \sqrt{\frac{L}{\alpha}}}{2a} \right) \leqslant \frac{1 + \left(\frac{L}{\alpha} \right)^{1/4}}{\sqrt{\alpha}a}.$$

2

$\frac{1}{2} \leqslant 1$. Then $\mathcal{I} \leqslant \min \left(L^{-1/2}, \frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}} \right)$ on $L > 0$. But whenever $L \geqslant \alpha$, the min is attained in

its first argument: indeed, $L^{-1/2} \leqslant \frac{1}{\sqrt{\alpha a}} \leqslant \frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}$. Then we obtained $\mathcal{I} \leqslant \min(L^{-1/2}, \frac{2}{\sqrt{\alpha a}})$ on $L > 0$. If $L = 0$, we have

$$\int_{z=a}^{1} \frac{1}{\sqrt{\alpha}z} dz = \frac{\log(1/a)}{\sqrt{\alpha}} \frac{2}{\sqrt{\alpha a}}.$$

Finally, if $L < 0$, the integral becomes

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_a^1 \frac{1}{\left(\left(\sqrt{\frac{\alpha}{|L|}} z \right)^2 - 1 \right)^{1/2}} \frac{\sqrt{\alpha}dz}{\sqrt{|L|}} = \frac{1}{\alpha} \int_{a\sqrt{\frac{\alpha}{|L|}}}^{\sqrt{\frac{\alpha}{|L|}}} \frac{1}{(w^2 - 1)^{1/2}} dw = \frac{\cosh^{-1}\left(\sqrt{\frac{\alpha}{|L|}} \right) - \cosh^{-1}\left(a\sqrt{\frac{\alpha}{|L|}} \right)}{\sqrt{\alpha}}.$$

The expression is well-defined, since $L + \alpha a^2 \geqslant 0$ implies $a\sqrt{\frac{\alpha}{-L}} \geqslant 1$ when $-L > 0$. Since $\cosh^{-1}(x) \leqslant x$, we obtain the coarse estimate $\mathcal{I} \leqslant |L|^{-1/2}$. Using that $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2 - 1}\right)$, we get

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \log \left(\frac{1 + \sqrt{1 - \frac{|L|}{\alpha}}}{a + \sqrt{a^2 - \frac{|L|}{\alpha}}} \right) \leqslant \frac{1}{\sqrt{\alpha}} \log \left(\frac{2}{a} \right) \leqslant \sqrt{\frac{2}{\alpha a}} \leqslant \frac{2}{\sqrt{\alpha a}}.$$

$\square$

Lemma 2.4 (MAGIC Change of variable). *Let* $(x(\tau), v(\tau))$ *be an ion characteristic issued from* $(x, w) \in [0, 1] \times \mathbb{R}_-$, *and define* $\varepsilon := \mathcal{L}_e(x(\tau), v(\tau))$ *and* $z := x(\tau)$. *Then*

$$d\tau \wedge dw = \frac{dz \wedge d\varepsilon}{2 \left(\varepsilon + \frac{1}{\mu}\varphi(z) \right)^{1/2} \left(\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu} \right) \varphi(z) \right)^{1/2}}.$$

Demonstration We have $\varepsilon = \frac{(v(\tau))^2}{2} - \frac{1}{\mu}\varphi(z)$, so that

$$d\varepsilon = v(\tau)\dot{v}(\tau)d\tau - \frac{1}{\mu}\varphi'(z)dz = -v(\tau)\varphi'(z)d\tau - \frac{1}{\mu}\varphi'(z)dz \quad \implies \quad d\tau = -\frac{1}{v(\tau)\varphi'(z)}d\varepsilon - \frac{1}{\mu v(\tau)}dz.$$

Moreover, $\mathcal{L}_i(x(\tau), v(\tau)) = \frac{(v(\tau))^2}{2} + \varphi(z)$, so $\left(1 + \frac{1}{\mu} \right) \varphi(z) = \mathcal{L}_i(x(\tau), v(\tau)) - \mathcal{L}_e(x(\tau), v(\tau)) = \mathcal{L}_i(x, w) - \varepsilon$ and

$$w = -\sqrt{2} \left(\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu} \right) \varphi(z) \right)^{1/2} \quad \implies \quad = \frac{d\varepsilon + \left(1 + \frac{1}{\mu} \right) \varphi'(z)dz}{w}.$$

Then, using $v(\tau) = -\sqrt{2} \left(\varepsilon + \frac{1}{\mu}\varphi(z) \right)^{1/2}$, we get

$$d\tau \wedge dw = -\frac{1 + \frac{1}{\mu}}{wv(\tau)}d\varepsilon \wedge dz - \frac{\frac{1}{\mu}}{v(\tau)w}dz \wedge d\varepsilon = \frac{dz \wedge d\varepsilon}{v(\tau)(-w)} = \frac{dz \wedge d\varepsilon}{2 \left(\varepsilon + \frac{1}{\mu}\varphi(z) \right)^{1/2} \left(\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu} \right) \varphi(z) \right)^{1/2}}.$$

$\square$

Remark 2.2. *Notice that* $\mathcal{L}_i(x(\tau), v(\tau)) = \varepsilon + \left(1 + \frac{1}{\mu} \right) \varphi(z)$. *Moreover, we have* $-\frac{1}{\mu}\varphi(z) = \mathcal{L}_e(z, 0)$, *and* $\varphi(x) = \mathcal{L}_i(z, 0)$. *Then the change of variable rewrites*

$$d\tau \wedge dw = \frac{dz \wedge d\varepsilon}{2 \left(\mathcal{L}_e(x(\tau), v(\tau)) - \mathcal{L}_e(z, 0) \right)^{1/2} \left(\mathcal{L}_i(x(\tau), v(\tau)) - \mathcal{L}_i(x, 0) \right)^{1/2}},$$

and is well-defined for (τ, w) *such that both square roots are real.*

2.2 Boundedness estimates

2.2.1 Electronic density n_e

Lemma 2.5. *The electron density n_e is bounded. More precisely,*

$$n_e(x) \leqslant 2\bar{c}\sqrt{\frac{2\beta}{\mu}} \qquad \forall x \in [0,1].$$

Demonstration For each $0 \leqslant x \leqslant y \leqslant 1$, we define the velocity $v_x(y) \geqslant 0$ such that

$$\mathcal{L}_e(x, v_x(y)) = \mathcal{L}_e(y, 0) \quad \Longleftrightarrow \quad v_x(y) = \left(\frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}.$$

In particular, owing to the boundary conditions, the function $v \to f_e(x, v)$ vanishes for $|v| \geqslant v_x(1)$. Then

$$n_e(x) = \int_{v=-v_x(1)}^{v_x(1)} f_e(x, v) dv \leqslant 2\bar{c} v_x(1) = 2\bar{c}\left(\frac{2}{\mu}(\varphi(x) - \varphi(1))\right)^{1/2} \leqslant 2\bar{c}\sqrt{\frac{2\beta}{\mu}}.$$

$\square$

2.2.2 Ion density n_i

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given x, we define $g_x : [0, x] \mapsto \mathbb{R}^+$ by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = (2(\varphi(y) - \varphi(x)))^{1/2}.$$

Lemma 2.6. *Let $0 \leqslant y < x \leqslant 1$. We have*

$$\mathcal{I} := \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{(v^2 - g_x^2(z))^{1/2}} dz\, dv \leqslant 2\sqrt{\frac{2}{\alpha}}(\varphi(y) - \varphi(x))^{1/4}\sqrt{x-y}.$$

Demonstration Let us first use Fubini's theorem to switch the order of integration. The lower bound $x_b(x, v) \geqslant z$ becomes an upper bound $v \leqslant -g_x(z)$, and we have

$$\mathcal{I} = \int_{z=y}^{x} \int_{v=-g_x(y)}^{-g_x(z)} \frac{1}{(v^2 - g_x^2(z))^{1/2}} dv\, dz = \int_{z=y}^{x} \int_{v=g_x(z)}^{g_x(y)} \frac{1}{(v^2 - g_x^2(z))^{1/2}} dv\, dz.$$

With the change of variable $w = v - g_x(z)$, and using $\frac{d}{dw}\left[\sinh^{-1}\left(\sqrt{\frac{w}{a}}\right)\right] = (w^2 + 2aw)^{-1/2}$, we get

$$\mathcal{I} = \int_{z=y}^{x} \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{(w^2 + 2wg_x(z))^{1/2}} dw\, dz = \int_{z=y}^{x} \sinh^{-1}\left(\sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}}\right) dz.$$

Using the coarse estimates $\sinh^{-1}(a) \leqslant a$ and $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$, we get

$$\mathcal{I} \leqslant \int_{z=y}^{x} \sqrt{\frac{g_x(y)}{g_x(z)}} dz = \int_{z=y}^{x} \left(\frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)}\right)^{1/4} dz.$$

The assumption of strong concavity yields $\varphi(z) - \varphi(x) \geqslant -\varphi'(z)(x - z) + \frac{\alpha}{2}|x - z|^2 \geqslant \frac{\alpha}{2}(x - z)^2$, so that

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}}(\varphi(y) - \varphi(x))^{1/4} \int_{z=y}^{x} \frac{1}{(x-z)^{1/2}} dz = 2\sqrt{\frac{2}{\alpha}}(\varphi(y) - \varphi(x))^{1/4}\sqrt{x-y}.$$

$\square$

Lemma 2.7. *Let $0 \leqslant y \leqslant x \leqslant 1$, and $-v_0 < -g_x(y)$. We have*

$$\mathcal{I} := \int_{v=-v_0}^{-g_x(y)} \int_{z=y}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} dz\, dv \leqslant \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/4}.$$

Demonstration We first shift the $v-$integration from the vertical line $z = x$ to $z = y$. Let $w = w(v)$ be such that

$$\mathcal{L}_i(y, w(v)) = \mathcal{L}_i(x, v), \quad \text{i.e.} \quad w(v) = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right), \text{ and } dv = \frac{-w}{\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} dw.$$

Then, defining $w_0 := \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}$, and noticing that $w(-g_x(y)) = 0$, we get

$$\mathcal{I} = \int_{w=-w_0}^{0} \int_{z=y}^{1} \frac{1}{\left(w^2 + 2\left(\varphi(y) - \varphi(z)\right)\right)^{1/2}} dz \frac{-w}{\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} dw.$$

Since $y \leqslant x$, we have $\varphi(y) \geqslant \varphi(x)$, and $\frac{-w}{\left(w^2 + 2(\varphi(y) - \varphi(x))\right)^{1/2}} \leqslant \frac{-w}{|w|} = 1$. By the strong concavity assumption, we have $\varphi(y) - \varphi(z) \geqslant -\varphi'(y)(z - y) + \frac{\alpha}{2}|z - y|^2 \geqslant \frac{\alpha}{2}(z - y)^2$, so that

$$\mathcal{I} \leqslant \int_{w=-w_0}^{0} \int_{z=y}^{1} \frac{1}{\left(w^2 + \alpha(z - y)^2\right)^{1/2}} dzdw = \int_{w=0}^{w_0} \int_{z=0}^{1-y} \frac{1}{\left(w^2 + \alpha z^2\right)^{1/2}} dzdw.$$

Using lemma (2.1), we conclude that

$$\mathcal{I} \leqslant 2\frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}} = \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/4}.$$

$\square$

Proposition 2.1. *The density n_i is bounded by a constant depending on φ only through $\varphi(1)$. More precisely,*

$$n_i(x) \leqslant 8 \left(\frac{\beta}{\alpha}\right)^{1/4} \left(\frac{1}{\alpha^{1/4}} + \beta^{1/4} + \frac{1}{\mu^{1/4}}\right) \qquad \forall x \in [0, 1].$$

Demonstration We use the symmetry of f_i to write

$$n_i(x) = \int_{v=-\infty}^{\infty} f_i(x, v) dv = 2 \int_{v=-\infty}^{0} f_i(x_b(x, v), v_b(x, v)) dv = 2 \int_{v=-\infty}^{0} \int_{t=-\infty}^{0} f_e(x(t), v(t)) dtdv,$$

where $(x(t), v(t))_{t \leqslant 0}$ is the ion characteristic reaching $(x_b(x, v), v_b(x, v))$ at $t = 0$. Notice that the lower bounds are artificial, since the characteristic enters the support of f_e in finite time: we may use $v \geqslant -\bar{v}_e$, and consider only times t for which $x(t) \in [0, 1]$.

We first reparametrize $(x(t), v(t))$ using the space variable. Define $z = x(t) \in [x_b, 1]$, and observe that

$$dz = \dot{x}(t)dt = v(t)dt, \quad \text{with} \quad \mathcal{L}_i(z, v(t)) = \mathcal{L}_i(x, v) \iff v(t) = -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}.$$

Then, the density rewrites

$$n_i(x) = 2 \int_{v=-\bar{v}_e}^{0} \int_{z=x_b(x,v)}^{1} \frac{f_e\left(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}\right)}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} dzdv.$$

Let us show that n_i is bounded. We use the coarse estimate $f_e \leqslant \bar{c}$, and decompose the integral in three:

$$n_i(x) \leqslant 2\bar{c} \left[\underbrace{\int_{v=-g_x(0)}^{0} \int_{z=x_b(x,v)}^{x}}_{\mathcal{I}_1} + \underbrace{\int_{v=-g_x(0)}^{0} \int_{z=x}^{1}}_{\mathcal{I}_2} + \underbrace{\int_{v=-\bar{v}_e}^{-g_x(0)} \int_{z=x_b(x,v)}^{1}}_{\mathcal{I}_3}\right] \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} dzdv.$$

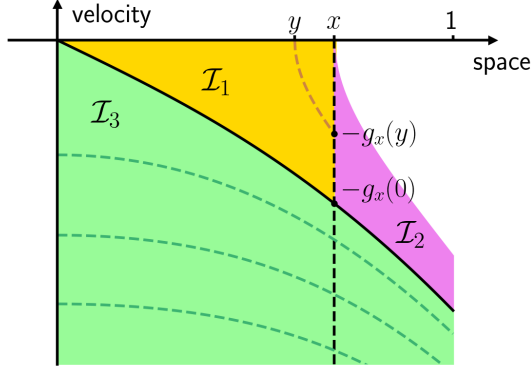The corresponding domains are represented figure (3).

Figure 1: Decomposition of the integral defining n_i.

The phase space is divided by the critical characteristic (in solid black). Whenever $v \leqslant -g_x(0)$, the characteristics (dotted green lines) are reaching the boundary with $x_b(x,v) = 0$.

Notice that whenever $z \leqslant x$, we have $0 \geqslant 2\left(\varphi(x) - \varphi(z)\right) = -\left(2\left(\varphi(z) - \varphi(x)\right)\right)^{2/2} = -g_x^2(z)$. Then, the integral $\mathcal{I}_1$ may be bounded using lemma (2.6) with $y = 0$:

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^0 \int_{z=x_b(x,v)}^x \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dz dv \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(x)\right)^{1/4}\sqrt{x} \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4}.$$

We use lemma (2.7) to bound $\mathcal{I}_2$ and $\mathcal{I}_3$. In the first case, we take $y = x$ and $v_0 = g_x(0)$, and notice that $-g_x(x) = 0$. In the second case, we take $v_0 = \overline{v}_e$ and $y = 0$, and notice that on $v \leqslant -g_x(0)$, we have $x_b(x,v) = 0$ (the velocity is low enough so that the characteristic ends on $x_b = 0$). This yields

$$\mathcal{I}_2 \leqslant \frac{2\sqrt{2(1-x)}}{\alpha^{1/4}}\left(g_x^2(0)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(-\varphi(1)\right)^{1/2}, \quad\text{and}\quad \mathcal{I}_3 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\left(\overline{v}_e^{\,2} + 2\varphi(x)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(\frac{\beta}{\mu}\right)^{1/4}.$$

$\square$

Lemma 2.8. *The density n_i is uniformly bounded away from 0. More precisely,*

$$n_i(x) \geqslant \frac{\underline{c}(\overline{v}^2 - \underline{v}^2)}{\overline{v}\left(\overline{v}^2 + 2\beta\right)^{1/2}}\frac{\underline{v}^2}{2\beta\left(1 + \frac{1}{\mu}\right)} \qquad \forall x \in [0,1].$$

Demonstration By assumption, there exists a constant $\underline{c} > 0$ such that $f_{e,b}(v) \geqslant \underline{c}\mathbf{1}_{\{\underline{v} \leqslant |v| \leqslant \overline{v}\}}$, which implies

$$f_e(x,v) \geqslant \underline{c}\mathbf{1}_{\{\mathcal{L}_e(0,\underline{v}) \leqslant \mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,\overline{v})\}}.$$

Notice that we may always take $\underline{v} > 0$ as long as $\overline{v} > 0$. Let us denote $\underline{\varepsilon} := \mathcal{L}_e(0,\underline{v})$ and $\overline{\varepsilon} := \mathcal{L}_e(0,\overline{v})$. For any $x \in [0,1]$, we have

$$n_i(x) = 2\int_{w \in \mathbb{R}_-}\int_{\tau=-\infty}^0 f_e(x(\tau), v(\tau))d\tau dw, \quad\text{with}\quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x,w).$$

Notice that for $w \in [-g_x(-\overline{v}), -g_x(0)]$, the characteristic issued from (x,w) enters the domain $\{\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}\}$. We will restrict the domain of integration to

$$\mathcal{D} := \left\{(w,\tau) \in \mathbb{R}_-^2 \mid \underline{\varepsilon} \leqslant \mathcal{L}_e(x(\tau), v(\tau)) \leqslant \overline{\varepsilon} \text{ and } 0 \leqslant x(\tau) \leqslant a\right\},$$
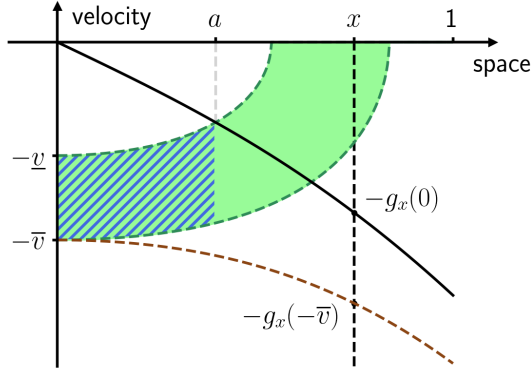
6

Figure 2: Notations for the lower bound on n_i.

The coloured area corresponds to the domain $\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}$, on which we know that $f_e \geqslant \underline{c}$. The hatched area represents the domain $\mathcal{D}$. The solid black line is the critical ion characteristic.

where (a, v_a) is the unique point such that $\mathcal{L}_e(a, v_a) = \mathcal{L}_e(0, \underline{v})$ and $\mathcal{L}_i(a, v_a) = 0$ (note that a depends on β). This (somehow coarse) estimate will give us trivial bounds when applying the change of variable of lemma (2.4), namely $z = x(\tau)$ and $\varepsilon = \mathcal{L}_e(x(\tau), v(\tau))$. Denoting by $[\underline{\tau}, \overline{\tau}] \subset] - \infty, 0]$ the time interval where $(x(\tau), v(\tau)) \in \mathcal{D}$, we have

$$n_i(x) \geqslant \int_{w=-g_x(-\overline{v})}^{-g_x(0)} \int_{\tau=\underline{\tau}}^{\overline{\tau}} \underline{c}\, d\tau dw = \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{\underline{c}}{2\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2} \left(\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}\, dzd\varepsilon.$$

Using remark (2.2), we are assured that the square roots are well-defined, since

$$\varepsilon \geqslant \underline{\varepsilon} = \mathcal{L}_e(0, \underline{v}) = \mathcal{L}_e(a, v_a) \geqslant \mathcal{L}_e(a, 0) \geqslant \mathcal{L}_e(z, 0) = -\frac{1}{\mu}\varphi(z) \quad \text{and} \quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x, w) \geqslant \mathcal{L}_i(x, 0).$$

Using that $\varepsilon + \frac{1}{\mu}\varphi(z) \leqslant \overline{\varepsilon} + 0$ and $\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu}\varphi(z)\right) \leqslant \overline{\varepsilon} + \beta + 0$, we obtain

$$n_i(x) \geqslant \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{\underline{c}}{2\overline{\varepsilon}^{1/2} (\overline{\varepsilon} + \beta)^{1/2}}\, dzd\varepsilon = \frac{a\underline{c}(\overline{\varepsilon} - \underline{\varepsilon})}{\overline{\varepsilon}^{1/2} (\overline{\varepsilon} + \beta)^{1/2}}.$$

Let us make explicit the dependance over β by estimating a using the concavity assumption: we have

$$\varphi(x) \geqslant (1-x)\varphi(0) + x\varphi(1) \quad \Longrightarrow \quad \varphi^{-1}(y) \geqslant \frac{-y}{\beta}, \quad \text{so that} \quad a = \varphi^{-1}\left(-\frac{\underline{v}^2}{2\left(1 + \frac{1}{\mu}\right)}\right) \geqslant \frac{\underline{v}^2}{2\beta\left(1 + \frac{1}{\mu}\right)}.$$

and the uniform lower bound behaves like $\beta^{-3/2}$ when $\beta \to \infty$. $\qquad\square$

Lemma 2.9. *Suppose that there exists $v_* > 0$ such that $f_{e,b}(v) = 0$ for all $|v| \leqslant v_*$. Then the density f_i satisfies the uniform bound*

$$f_i(x, v) \leqslant 2^{9/4}\overline{c}\sqrt{\frac{\frac{\beta}{\alpha}\left(1 + \frac{1}{\mu}\right) + 1}{v_*}} \quad \forall(x, v) \in [0, 1] \times \mathbb{R}.$$

Demonstration Let $(x, v) \in [0, 1] \times \mathbb{R}_-$, and denote by $(x(t), v(t))_{t \leqslant 0}$ the ion characteristic going through $(x, v) = (x(-T), v(-T))$, with the convention $(x_b(x, v), v_b(x, v)) = (x(0), v(0))$. Owing to the positivity of f_e,

$$f_i(x(0), v(0)) = \int_{t=-T}^{0} f_e(x(t), v(t))dt + f_i(x, v) \geqslant f_i(x, v).$$
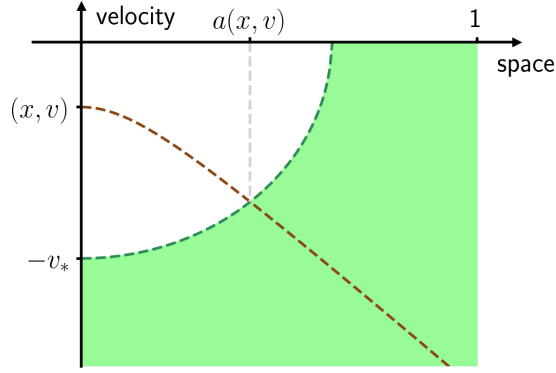
7

Figure 3: Notations for the boundedness of f_i.

The coloured area corresponds to $\mathcal{L}_e(x,v) \geqslant \mathcal{L}_e(0,v_*)$. The function $a(x,v)$ gives the point of the ion characteristic (in brown) where f_e vanishes.

On the other hand, we use $f_i(x,v) + f_i(x,-v) = 2f_i(x(0),v(0))$ to write $f_i(x,-v) \leqslant 2f_i(x(0),v(0))$. This shows that it is enough to bound f_i on the boundary $\mathcal{B} := \{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ to obtain an uniform bound.

Let then $(x,v) \in \mathcal{B}$. By hypothesis, $f_e(x(t),v(t))$ vanishes whenever $\mathcal{L}_e(x(t),v(t)) \leqslant \mathcal{L}_e(0,v_*)$, and is bounded by $\bar{c}$ otherwise. Then, we may use the reparametrization by space

$$f_i(x,v) \leqslant \int_{z=a(x,v)}^1 \frac{\bar{c}}{(v^2 + 2\varphi(x) - 2\varphi(z))^{1/2}} dz, \quad \text{with} \quad a(x,v) := \begin{cases} \varphi^{-1}\left(\frac{\frac{v^2}{2} + \varphi(x) - \frac{v_*^2}{2}}{1+1/\mu}\right) & \text{if } \mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,v_*) \\ x & \text{otherwise.} \end{cases}$$

The function a gives the smallest spatial coordinate of the characteristic $(x(t),v(t))_{t\leqslant 0}$ such that $f_e > 0$. Using that $\varphi(z) \leqslant -\alpha\frac{z^2}{2}$, we have

$$f_i(x,v) \leqslant \bar{c}\int_{z=a(x,v)}^1 \frac{1}{(v^2 + 2\varphi(x) - \alpha z^2)^{1/2}} dz \leqslant \bar{c}\min\left(\frac{1}{|v^2 + 2\varphi(x)|^{1/2}}, \frac{2}{\sqrt{\alpha a(x,v)}}\right)$$

where we used lemma (2.3) with $L := v^2 + 2\varphi(x)$.

We rely on the concavity estimate

$$\varphi(x) \geqslant (1-x)\varphi(0) + x\varphi(1) + \frac{\alpha}{2}x(1-x) \geqslant -x\beta \quad \Longrightarrow \quad -\frac{y}{\beta} \leqslant \varphi^{-1}(y)$$

to write that for (x,v) satisfying $\mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,v_*)$,

$$a(x,v) = \varphi^{-1}\left(\frac{\frac{v^2}{2} + \varphi(x) - \frac{v_*^2}{2}}{1+1/\mu}\right) \geqslant \frac{\frac{v_*^2}{2} - (\frac{v^2}{2} + \varphi(x))}{\beta(1+1/\mu)}.$$

Then

$$f_i(x,v) \leqslant \bar{c}\min\left(\frac{1}{\sqrt{|X|}}, \frac{A}{\sqrt{B-X}}\right) \quad \text{with} \quad X := \frac{v^2}{2} + \varphi(x), \quad A := \frac{2}{\sqrt{\frac{\alpha}{\beta(1+1/\mu)}}}, \quad \text{and} \quad B := \frac{v_*^2}{2}.$$

The elementary study of the function $X \to \min\left(|X|^{-1/2}, A(B-X)^{-1/2}\right)$ reveals a global maximum at $X = \frac{B}{A^2+1} < B$, and we conclude to the result.

$\square$

8

2.3 Continuity estimates

2.3.1 Electron density n_e

Lemma 2.10. *Let n_e^φ and n_e^ψ be the electron densities generated by potentials φ and ψ satisfying the assumptions. Then*

1. *If $f_{e,b}$ is Lipschitz-continuous with constant $[f_{e,b}]$, then*

$$|n_e^\varphi(x) - n_e^\psi(x)| \leqslant 2[f_{e,b}]\overline{v}_e\sqrt{\frac{2}{\mu}}\,|\varphi(x) - \psi(x)|^{1/2} \quad \forall (x,y) \in [0,1]^2.$$

2. *If there exists a constant $[f_{e,b}]$ such that $|f_{e,b}(x) - f_{e,b}(y)| \leqslant [f_{e,b}]|x^2 - y^2|$ (or equivalently, $x \to f_{e,b}(\sqrt{x})$ is a lipschitz function, as for instance e^{-x^2}), then*

$$|n_e^\varphi(x) - n_e^\psi(x)| \leqslant \frac{4\overline{v}_e[f_{e,b}]_2}{\mu}\,|\varphi(x) - \psi(x)| \quad \forall (x,y) \in [0,1]^2.$$

Lemma 2.11. *The electronic density n_e is continuous.*

Demonstration We already know that n_e is bounded. Moreover, using the symmetry $f_e(x,v) = f_e(x,-v)$, we may write

$$n_e(x) - n_e(y) = 2\underbrace{\int_{v=0}^{v_x(y)} f_e(x,v)dv}_{=:\,\mathcal{I}^+} + 2\left(\underbrace{\int_{v=v_x(y)}^{v_x(1)} f_e(x,v)dv - \int_{v=0}^{v_y(1)} f_e(y,v)dv}_{=:\,\mathcal{I}^-}\right).$$

The term $\mathcal{I}^+$ is bounded by $\overline{c}v_x(y) = \overline{c}\left(\frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}$, and by continuity of φ, we have $\mathcal{I}^+ \xrightarrow[y \to x]{} 0$. On the first integral of $\mathcal{I}^-$, we apply the change of variable

$$w = \left(v^2 - \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2} \quad \Longleftrightarrow \quad dv = \frac{w}{\left(w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}}\,dw, \quad w \in [0, v_y(1)]$$

to get

$$\mathcal{I}^- = \int_{w=0}^{v_y(1)} f_e\left(x, \left(w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}\right)\frac{w}{\left(w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}}dw - \int_{v=0}^{v_y(1)} f_e(y,v)dv.$$

Notice that

$$f_e\left(x, \left(w^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}\right) = f_{e,b}\left(\frac{w^2}{2} + \frac{1}{\mu}(\varphi(x) - \varphi(y)) - \frac{1}{\mu}\varphi(x)\right) = f_e(y,w).$$

Renaming w in v, we obtain the (clearly nonpositive) expression

$$\mathcal{I}^- = \int_{v=0}^{v_y(1)} f_e(y,v)\left(\frac{v}{\left(v^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}} - 1\right)dv \geqslant \overline{c}\int_{v=0}^{v_y(1)}\left(\frac{v}{\left(v^2 + \frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}} - 1\right)dv$$

$$= \overline{c}\left(\left((v_y(1))^2 + \frac{2}{\mu}\left(\underbrace{\varphi(x) - \varphi(y)}_{\geqslant 0}\right)\right)^{1/2} - \left(\frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2} - v_y(1)\right) \geqslant -\overline{c}\left(\frac{2}{\mu}(\varphi(x) - \varphi(y))\right)^{1/2}$$

and this shows that $\mathcal{I}^- \xrightarrow[y \to x]{} 0$. $\qquad\square$

Demonstration We have

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2\int_{v=0}^{\overline{v}_e}\left|f_e^\varphi(x,v) - f_e^\psi(x,v)\right|dv = 2\int_{v=0}^{\overline{v}_e}\left|f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2}\right) - f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2}\right)\right|dv.$$

Then, if $f_{e,b}$ is Lipschitz with constant $[f_{e,b}]$, we obtain

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{\overline{v}_e}\left|\left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2} - \left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2}\right|dv.$$

Using lemma (2.2) yields

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{\overline{v}_e}\left|-\frac{2}{\mu}\varphi(x) + \frac{2}{\mu}\psi(x)\right|^{1/2}dv = 2[f_{e,b}]\overline{v}_e\sqrt{\frac{2}{\mu}}\,|\varphi(x) - \psi(x)|^{1/2}.$$

If $f_{e,b}(\sqrt{\cdot})$ is Lipschitz with constant $[f_{e,b}]_2$, we may directly write

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2\overline{v}_e[f_{e,b}]_2\frac{2}{\mu}\,|\varphi(x) - \psi(x)|.$$

$\square$

2.3.2 Ion density n_i

Proposition 2.2. *The density n_i is continuous.*

Demonstration Let $0 \leqslant y < x \leqslant 1$. For convenience, we represent $n_i(x)$ (resp. $n_i(y)$) as an integral with the artificial lower bound $-g_x(-\overline{v}_e) \leqslant -\overline{v}_e$ (resp. $-g_y(-\overline{v}_e)$). Then

$$
\begin{aligned}
n_i(x) - n_i(y) &= 2\int_{v=-g_x(-\overline{v}_e)}^{0} f_i(x_b(x,v), v_b(x,v))dv - 2\int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv \\
&= 2\underbrace{\left[\int_{v=-g_x(-\overline{v}_e)}^{-g_x(y)} f_i(x_b(x,v), v_b(x,v))dv - \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv\right]}_{=:\,\mathcal{I}^-} \\
&\quad + 2\underbrace{\int_{v=-g_x(y)}^{0} f_i(x_b(x,v), v_b(x,v))dv}_{=:\,\mathcal{I}^+}.
\end{aligned}
\tag{2.1}
$$

The term $\mathcal{I}^+$ is clearly nonnegative, and may be adressed using our lemmas. Indeed, using the integral representation of $f_i(x_b, v_b)$ and the reparametrization by a space variable z, we have

$$
\begin{aligned}
\mathcal{I}^+ &= \int_{v=-g_x(y)}^{0}\left[\int_{z=x_b(x,v)}^{x} + \int_{z=x}^{1}\right]\frac{f_e(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}))}{(v^2 + 2\left(\varphi(x) - \varphi(z)\right))^{1/2}}\,dzdv \\
&\leqslant \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x_b(x,v)}^{x}\frac{1}{(v^2 - g_x^2(z))^{1/2}}\,dzdv + \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x}^{1}\frac{1}{(v^2 + 2\left(\varphi(x) - \varphi(z)\right))^{1/2}}\,dzdv \\
&\leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\,(\varphi(y) - \varphi(x))^{1/4}\sqrt{x - y} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(2(\varphi(y) - \varphi(x))\right)^{1/2}\right),
\end{aligned}
$$

where we used lemma (2.6) for the first term, and lemma (2.7) for the second term (with $y = x$ and $v_0 = -g_x(y)$ under the notations of the lemma). Since φ is continuous, we deduce that $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. Taking the extreme case $y = 0$ and $x = 1$, we obtain that

$$\mathcal{I}^+ \leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\,(-\varphi(1))^{1/4} + \frac{2\sqrt{2}}{\alpha^{1/4}}\,(-2\varphi(1))^{1/2}\right) =: K.$$

Let us now focus on $\mathcal{I}^-$. On the first integral, we make the change of variable

$$w = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2} \quad v = -\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Since $\mathcal{L}_i(x,v) = \mathcal{L}_i(y,w)$, this yields $x_b(x,v) = x_b(y,w)$ and $v_b(x,v) = v_b(y,w)$. The bounds $v \in [-g_x(-\overline{v}_e), -g_x(y)]$ are exactly transported to $w \in [-g_y(-\overline{v}_e), 0]$. Renaming w in v, we get

$$\mathcal{I}^- = \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v)) \left(\frac{-v}{(v^2 + 2\left(\varphi(y) - \varphi(x)\right))^{1/2}} - 1 \right) dv.$$

Since $\varphi(y) \geqslant \varphi(x)$, the factor of f_i is nonpositive, and so is $\mathcal{I}^-$. Moreover,

$$n_i(x) - n_i(y) = 2\mathcal{I}^- + 2\mathcal{I}^+ \leqslant 2\mathcal{I}^- + 2K \quad \Longleftrightarrow \quad \mathcal{I}^- = -K + n_i(x) - n_i(y) \geqslant -K - |n_i|_\infty$$

and $\mathcal{I}^-$ is bounded. The function $\frac{-v}{\left(v^2 + 2(\varphi(y) - \varphi(x))\right)^{1/2}} - 1$ converges pointwise to 0 when $x \to y$ **and f_i is almost everywhere finite**, and by Lebesgue's dominated convergence, $\mathcal{I}^- \xrightarrow[x \to y]{} 0$. Then n_i is continuous. $\qquad\square$

3 Results

Let $0 < \alpha \leqslant \beta$, and define the following convex set:

$$\mathcal{K} := \left\{ \varphi \in \mathcal{C}^1([0,1], \mathbb{R}) \quad | \quad \varphi(0) = \varphi'(0) = 0, \quad \varphi \; \alpha - \text{concave}, \quad \varphi(1) \geqslant -\beta \right\}.$$

Lemma 3.1. *$\mathcal{K}$ is closed for the topology induced by the sup-norm $|\cdot|_\infty := \max_{[0,1]} |\cdot|$.*

Demonstration Since $\mathcal{K} \subset \mathcal{C}^0([0,1], \mathbb{R})$, we know that any Cauchy sequence $(\varphi^n)_n \subset \mathcal{K}$ admits a limit $\varphi \in \mathcal{C}([0,1], \mathbb{R})$. The pointwise conditions

$$\varphi^n(0) = 0, \quad \varphi^n(1) \geqslant -\beta, \quad \varphi^n((1-\gamma)x + \gamma y) \geqslant (1-\gamma)\varphi^n(x) + \gamma\varphi^n(y) + \frac{\alpha}{2}\gamma(1-\gamma)|x-y|^2 \quad \forall (x,y,\gamma) \in [0,1]^3$$

are preserved when $n \to \infty$ under this topology. Let us show that $\varphi'(0) = 0$: first, notice that $\varphi^n(x) \leqslant 0$ implies $\varphi(x) \leqslant 0$. Since φ is $\alpha-$concave, we have $\forall x \in [0,1]$

$$\varphi(x) \geqslant (1-x)\varphi(0) + x\varphi(1) + \frac{\alpha}{2}x(1-x) \geqslant -\beta x + \frac{\alpha}{2}x(1-x),$$

and

$$\frac{-\beta\varepsilon + \frac{\alpha}{2}\varepsilon(1-\varepsilon)}{\varepsilon} \leqslant \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} \leqslant 0.$$

$\qquad\square$

In the future:

- **Now that we have estimates, write it in function of α and β, and see how to get stability of the set of strongly concave functions satisfying all the hypotheses by the Poisson problem (essentially, find tweaks of λ, μ, ν, α and/or β such that the estimates are propagated).**

- **If $f_{e,b}$ vanishes in a neighbourhood of $(0,0)$ (or decreases fast enough, see how fast), show that f_i is bounded.**

- **Under that same assumption, we should obtain stronger continuity over n_i, and n_i may be continuous with respect to φ (in the same sense as n_e in lemma (2.10)).**

- **If the estimates with respect to $|\varphi - \psi|_\infty$ succeed, define numerical scheme and see what we can say about it.**