Notes on fixed-point procedure

Contents

1 Notations and assumptions

We suppose that

1. $f_{e,b}$ is nonnegative.
2. $f_{e,b}(v) = f_{e,b}(-v)$ for all $v \in \mathbb{R}$.
3. $f_{e,b}$ satisfies the boundary condition, i.e. $f_{e,b}(v) = 0$ as soon as $\mathcal{L}_e(0,v) \leqslant \mathcal{L}_e(1,0)$.
4. There exists $\overline{c} \leqslant 0$ such that $f_{e,b}(v) \leqslant \overline{c}$ for all $v \in \mathbb{R}$.
5. There exists $\underline{c} > 0$ and $0 \leqslant \underline{v} < \overline{v}$ such that $f_{e,b}(v) \geqslant \underline{c}$ for all $v \in [-\overline{v}, -\underline{v}]$.
6. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$.
7. The function φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi((1-\tau)x + \tau y) \geqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau\,|x-y|^2$ for all $(x,y,\tau) \in [0,1]^3$.
8. There exists $\beta > \alpha$ such that φ is $(-\beta)-$convex, i.e. $\varphi((1-\tau)x + \tau y) \leqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\beta}{2}(1-\tau)\tau\,|x-y|^2$ for all $(x,y,\tau) \in [0,1]^3$. Notice that in this case, $\varphi(x) \geqslant -\beta\frac{x^2}{2}$, and in particular, $\varphi(1) \geqslant -\beta$.

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x,v) \coloneqq \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{1.1}$$

$$\mathcal{L}_i(x,v) \coloneqq \frac{v^2}{2} + \varphi(x). \tag{1.2}$$

We define an upper bound $\overline{v}_e$ over the velocities in the support of f_e, given as

$$\mathcal{L}_e(0,\overline{v}_e) \coloneqq \frac{1}{\mu}\beta \geqslant -\frac{1}{\mu}\varphi(1) = \mathcal{L}_e(1,0), \quad \text{i.e.} \quad \overline{v}_e \coloneqq \sqrt{\frac{2}{\mu}\beta}. \tag{1.3}$$

For a given point $(x,v) \in [0,1] \times \mathbb{R}^-$, we denote by (x_b, v_b) the intersection of the boundary $\{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ with the ion characteristic issued from (x,v). The values are given by

$$\begin{pmatrix} x_b(x,v) \\ v_b(x,v) \end{pmatrix} \coloneqq \begin{pmatrix} \varphi^{-1}\left(\min\left(0, \frac{v^2}{2} + \varphi(x)\right)\right) \\ -\sqrt{\max\left(0, v^2 + 2\varphi(x)\right)} \end{pmatrix} \tag{1.4}$$

2 Estimates

2.1 Useful elementary lemmas

Lemma 2.1. *Let $\alpha > 0$, $\overline{x} \geqslant 0$ and $\overline{y} \geqslant 0$. Then*

$$\mathcal{I} \coloneqq \int_{x=0}^{\overline{x}} \int_{y=0}^{\overline{y}} \frac{1}{(x^2 + \alpha y^2)^{1/2}} dy dx \leqslant 2\frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

Demonstration By the change of variable $z = \sqrt{\alpha}y$, with $\overline{z} \coloneqq \sqrt{\alpha}\overline{y}$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{1}{(x^2 + z^2)^{1/2}} dz dx.$$

Notice that $x + z \leqslant \sqrt{2}\left(x^2 + y^2\right)^{1/2}$. Then,

$$\mathcal{I} \leqslant \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{\sqrt{2}}{x + z} dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(\frac{x + \overline{z}}{x}\right) dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(1 + \frac{\overline{z}}{x}\right) dx.$$

Using $\ln(1 + a) \leqslant \sqrt{a}$, we get

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \sqrt{\frac{\overline{z}}{x}} dx = \sqrt{\frac{2}{\alpha}} 2\sqrt{\overline{x}\overline{z}} = 2\frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

□

Remark 2.1 (Exact value)**.** *Let $\overline{r} \coloneqq \sqrt{\overline{x}^2 + \overline{z}^2}$. Then*

$$\mathcal{I} = \overline{x}\ln\left(\frac{1 + \overline{z}/\overline{r}}{\overline{x}/\overline{r}}\right) + \overline{z}\ln\left(\frac{1 + \overline{x}/\overline{r}}{\overline{z}/\overline{r}}\right).$$

Lemma 2.2. *If $a \geqslant 0$ and $b \geqslant 0$, then $|a - b| \leqslant \sqrt{|a^2 - b^2|}$.*

Demonstration If $a \geqslant b$, then $|a - b| = \sqrt{(a - b)(a - b)} \leqslant \sqrt{(a - b)(a + b)} = \sqrt{a^2 - b^2}$, else $|a - b| = |b - a|$. □

Lemma 2.3. *Let $\alpha > 0$, $L \in \mathbb{R}$ and $0 \leqslant a \leqslant 1$. Suppose that $L + \alpha a^2 \geqslant 0$, and $a > 0$ if $L = 0$. Then*

$$\mathcal{I} \coloneqq \int_{z=a}^{1} \frac{1}{(L + \alpha z^2)^{1/2}} dz \leqslant \min\left(|L|^{-1/2}, \frac{2}{\sqrt{\alpha a}}\right).$$

Demonstration If $L > 0$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{z=a}^{1} \frac{1}{\left(1 + \left(\sqrt{\frac{\alpha}{L}}z\right)^2\right)^{1/2}} \frac{\sqrt{\alpha}dz}{\sqrt{L}} = \frac{1}{\alpha} \int_{w=a\sqrt{\frac{\alpha}{L}}}^{\sqrt{\frac{\alpha}{L}}} \frac{1}{(1 + w^2)^{1/2}} dw = \frac{1}{\sqrt{\alpha}}\left(\sinh^{-1}\left(\sqrt{\frac{\alpha}{L}}\right) - \sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right)\right).$$

Using the positivity of $\sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right)$, and the coarse estimate $\sinh^{-1}(x) \leqslant x$, we get $\mathcal{I} \leqslant L^{-1/2}$. Moreover, using that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$, we get

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \log\left(\frac{\sqrt{\frac{\alpha}{L}} + \sqrt{\frac{\alpha}{L}+1}}{a\sqrt{\frac{\alpha}{L}} + \sqrt{a^2\frac{\alpha}{L}+1}}\right) = \frac{1}{\sqrt{\alpha}} \log\left(\frac{1+\sqrt{1+\frac{L}{\alpha}}}{a+\sqrt{a^2+\frac{L}{\alpha}}}\right) \leqslant \frac{1}{\sqrt{\alpha}} \log\left(\frac{2+\sqrt{\frac{L}{\alpha}}}{2a}\right) \leqslant \frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}.$$

I shamelessly used $\frac{1}{2} \leqslant 1$. Then $\mathcal{I} \leqslant \min\left(L^{-1/2}, \frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}\right)$ on $L > 0$. But whenever $L \geqslant \alpha$, the min is attained in its first argument: indeed, $L^{-1/2} \leqslant \frac{1}{\sqrt{\alpha a}} \leqslant \frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}$. Then we obtained $\mathcal{I} \leqslant \min(L^{-1/2}, \frac{2}{\sqrt{\alpha a}})$ on $L > 0$. If $L = 0$, we have

$$\int_{z=a}^{1} \frac{1}{\sqrt{\alpha z}} dz = \frac{\log(1/a)}{\sqrt{\alpha}} \frac{2}{\sqrt{\alpha a}}.$$

Finally, if $L < 0$, the integral becomes

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_a^1 \frac{1}{\left(\left(\sqrt{\frac{\alpha}{|L|}}z\right)^2 - 1\right)^{1/2}} \frac{\sqrt{\alpha}dz}{\sqrt{|L|}} = \frac{1}{\alpha} \int_{a\sqrt{\frac{\alpha}{|L|}}}^{\sqrt{\frac{\alpha}{|L|}}} \frac{1}{(w^2-1)^{1/2}} dw = \frac{\cosh^{-1}\left(\sqrt{\frac{\alpha}{|L|}}\right) - \cosh^{-1}\left(a\sqrt{\frac{\alpha}{|L|}}\right)}{\sqrt{\alpha}}.$$

The expression is well-defined, since $L + \alpha a^2 \geqslant 0$ implies $a\sqrt{\frac{\alpha}{-L}} \geqslant 1$ when $-L > 0$. Since $\cosh^{-1}(x) \leqslant x$, we obtain the coarse estimate $\mathcal{I} \leqslant |L|^{-1/2}$. Using that $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$, we get

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \log\left(\frac{1+\sqrt{1-\frac{|L|}{\alpha}}}{a+\sqrt{a^2-\frac{|L|}{\alpha}}}\right) \leqslant \frac{1}{\sqrt{\alpha}} \log\left(\frac{2}{a}\right) \leqslant \sqrt{\frac{2}{\alpha a}} \leqslant \frac{2}{\sqrt{\alpha a}}.$$

□

Lemma 2.4 (MAGIC Change of variable)**.** *Let $(x(\tau), v(\tau))$ be an ion characteristic issued from $(x, w) \in [0,1] \times \mathbb{R}_-$, and define $\varepsilon := \mathcal{L}_e(x(\tau), v(\tau))$ and $z := x(\tau)$. Then*

$$d\tau \wedge dw = \frac{dz \wedge d\varepsilon}{2\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2}\left(\varepsilon - \varphi(x) + \left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}.$$

Demonstration We have $\varepsilon = \frac{(v(\tau))^2}{2} - \frac{1}{\mu}\varphi(z)$, so that

$$d\varepsilon = v(\tau)\dot{v}(\tau)d\tau - \frac{1}{\mu}\varphi'(z)dz = -v(\tau)\varphi'(z)d\tau - \frac{1}{\mu}\varphi'(z)dz \quad \Longrightarrow \quad d\tau = -\frac{1}{v(\tau)\varphi'(z)}d\varepsilon - \frac{1}{\mu v(\tau)}dz.$$

Moreover, $\mathcal{L}_i(x(\tau), v(\tau)) = \frac{(v(\tau))^2}{2} + \varphi(z)$, so $\left(1+\frac{1}{\mu}\right)\varphi(z) = \mathcal{L}_i(x(\tau), v(\tau)) - \mathcal{L}_e(x(\tau), v(\tau)) = \mathcal{L}_i(x, w) - \varepsilon$ and

$$w = -\sqrt{2}\left(\varepsilon - \varphi(x) + \left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2} \quad \Longrightarrow \quad = \frac{d\varepsilon + \left(1+\frac{1}{\mu}\right)\varphi'(z)dz}{w}.$$

Then, using $v(\tau) = -\sqrt{2}\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2}$, we get

$$d\tau \wedge dw = -\frac{1+\frac{1}{\mu}}{wv(\tau)}d\varepsilon \wedge dz - \frac{\frac{1}{\mu}}{v(\tau)w}dz \wedge d\varepsilon = \frac{dz \wedge d\varepsilon}{v(\tau)(-w)} = \frac{dz \wedge d\varepsilon}{2\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2}\left(\varepsilon - \varphi(x) + \left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}.$$

□

Remark 2.2. *Notice that* $\mathcal{L}_i(x(\tau), v(\tau)) = \varepsilon + \left(1 + \frac{1}{\mu}\right)\varphi(z)$. *Moreover, we have* $-\frac{1}{\mu}\varphi(z) = \mathcal{L}_e(z, 0)$, *and* $\varphi(x) = \mathcal{L}_i(z, 0)$. *Then the change of variable rewrites*

$$d\tau \wedge dw = \frac{dz \wedge d\varepsilon}{2\left(\mathcal{L}_e(x(\tau), v(\tau)) - \mathcal{L}_e(z, 0)\right)^{1/2}\left(\mathcal{L}_i(x(\tau), v(\tau)) - \mathcal{L}_i(x, 0)\right)^{1/2}},$$

and is well-defined for (τ, w) *such that both square roots are real.*

2.2 Boundedness estimates

2.2.1 Electronic density n_e

Lemma 2.5. *The electron density* n_e *is bounded. More precisely,*

$$n_e(x) \leqslant 2\overline{c}\sqrt{\frac{2\beta}{\mu}} \qquad \forall x \in [0, 1].$$

Demonstration For each $0 \leqslant x \leqslant y \leqslant 1$, we define the velocity $v_x(y) \geqslant 0$ such that

$$\mathcal{L}_e(x, v_x(y)) = \mathcal{L}_e(y, 0) \qquad \Longleftrightarrow \qquad v_x(y) = \left(\frac{2}{\mu}\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}.$$

In particular, owing to the boundary conditions, the function $v \to f_e(x, v)$ vanishes for $|v| \geqslant v_x(1)$. Then

$$n_e(x) = \int_{v=-v_x(1)}^{v_x(1)} f_e(x, v) dv \leqslant 2\overline{c} v_x(1) = 2\overline{c}\left(\frac{2}{\mu}(\varphi(x) - \varphi(1))\right)^{1/2} \leqslant 2\overline{c}\sqrt{\frac{2\beta}{\mu}}.$$

□

2.2.2 Ion density n_i

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given x, we define $g_x : [0, x] \mapsto \mathbb{R}^+$ by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = \left(2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Lemma 2.6. *Let* $0 \leqslant y < x \leqslant 1$. *We have*

$$\mathcal{I} := \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dz\, dv \leqslant 2\sqrt{\frac{2}{\alpha}}\left(\varphi(y) - \varphi(x)\right)^{1/4}\sqrt{x - y}.$$

Demonstration Let us first use Fubini's theorem to switch the order of integration. The lower bound $x_b(x, v) \geqslant z$ becomes an upper bound $v \leqslant -g_x(z)$, and we have

$$\mathcal{I} = \int_{z=y}^{x} \int_{v=-g_x(y)}^{-g_x(z)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz = \int_{z=y}^{x} \int_{v=g_x(z)}^{g_x(y)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz.$$

With the change of variable $w = v - g_x(z)$, and using $\frac{d}{dw}\left[\sinh^{-1}\left(\sqrt{\frac{w}{a}}\right)\right] = \left(w^2 + 2aw\right)^{-1/2}$, we get

$$\mathcal{I} = \int_{z=y}^{x} \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{\left(w^2 + 2wg_x(z)\right)^{1/2}} dw\, dz = \int_{z=y}^{x} \sinh^{-1}\left(\sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}}\right) dz.$$

Using the coarse estimates $\sinh^{-1}(a) \leqslant a$ and $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$, we get

$$\mathcal{I} \leqslant \int_{z=y}^{x} \sqrt{\frac{g_x(y)}{g_x(z)}} dz = \int_{z=y}^{x} \left(\frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)}\right)^{1/4} dz.$$

The assumption of strong concavity yields $\varphi(z)-\varphi(x)\geqslant -\varphi'(z)(x-z)+\frac{\alpha}{2}|x-z|^2\geqslant \frac{\alpha}{2}(x-z)^2$, so that

$$\mathcal{I}\leqslant\sqrt{\frac{2}{\alpha}}\left(\varphi(y)-\varphi(x)\right)^{1/4}\int_{z=y}^{x}\frac{1}{(x-z)^{1/2}}dz=2\sqrt{\frac{2}{\alpha}}\left(\varphi(y)-\varphi(x)\right)^{1/4}\sqrt{x-y}.$$

□

Lemma 2.7. *Let $0\leqslant y\leqslant x\leqslant 1$, and $-v_0<-g_x(y)$. We have*

$$\mathcal{I}\coloneqq\int_{v=-v_0}^{-g_x(y)}\int_{z=y}^{1}\frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dz\,dv\leqslant\frac{2\sqrt{2(1-y)}}{\alpha^{1/4}}\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/4}.$$

Demonstration We first shift the $v-$integration from the vertical line $z=x$ to $z=y$. Let $w=w(v)$ be such that

$$\mathcal{L}_i(y,w(v))=\mathcal{L}_i(x,v),\quad\text{i.e.}\quad w(v)=-\left(v^2+2\left(\varphi(x)-\varphi(y)\right)\right),\ \text{and}\ dv=\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Then, defining $w_0\coloneqq\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$, and noticing that $w(-g_x(y))=0$, we get

$$\mathcal{I}=\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+2\left(\varphi(y)-\varphi(z)\right)\right)^{1/2}}dz\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Since $y\leqslant x$, we have $\varphi(y)\geqslant\varphi(x)$, and $\frac{-w}{\left(w^2+2(\varphi(y)-\varphi(x))\right)^{1/2}}\leqslant\frac{-w}{|w|}=1$. By the strong concavity assumption, we have $\varphi(y)-\varphi(z)\geqslant-\varphi'(y)(z-y)+\frac{\alpha}{2}|z-y|^2\geqslant\frac{\alpha}{2}(z-y)^2$, so that

$$\mathcal{I}\leqslant\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+\alpha(z-y)^2\right)^{1/2}}dzdw=\int_{w=0}^{w_0}\int_{z=0}^{1-y}\frac{1}{\left(w^2+\alpha z^2\right)^{1/2}}dzdw.$$

Using lemma (2.1), we conclude that

$$\mathcal{I}\leqslant 2\frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}}=\frac{2\sqrt{2(1-y)}}{\alpha^{1/4}}\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/4}.$$

□

Proposition 2.1. *The density n_i is bounded by a constant depending on φ only through $\varphi(1)$. More precisely,*

$$n_i(x)\leqslant 8\nu\left(\frac{\beta}{\alpha}\right)^{1/4}\left(\frac{1}{\alpha^{1/4}}+\beta^{1/4}+\frac{1}{\mu^{1/4}}\right)\qquad\forall x\in[0,1].$$

Demonstration We use the symmetry of f_i to write

$$n_i(x)=\int_{v=-\infty}^{\infty}f_i(x,v)dv=2\int_{v=-\infty}^{0}f_i(x_b(x,v),v_b(x,v))dv=2\nu\int_{v=-\infty}^{0}\int_{t=-\infty}^{0}f_e(x(t),v(t))dtdv,$$

where $(x(t),v(t))_{t\leqslant 0}$ is the ion characteristic reaching $(x_b(x,v),v_b(x,v))$ at $t=0$. Notice that the lower bounds are artificial, since the characteristic enters the support of f_e in finite time: we may use $v\geqslant-\overline{v}_e$, and consider only times t for which $x(t)\in[0,1]$.

We first reparametrize $(x(t),v(t))$ using the space variable. Define $z=x(t)\in[x_b,1]$, and observe that

$$dz=\dot{x}(t)dt=v(t)dt,\quad\text{with}\quad\mathcal{L}_i(z,v(t))=\mathcal{L}_i(x,v)\quad\iff\quad v(t)=-\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}.$$

Then, the density rewrites

$$n_i(x)=2\nu\int_{v=-\overline{v}_e}^{0}\int_{z=x_b(x,v)}^{1}\frac{f_e\left(z,-\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}\right)}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dzdv.$$

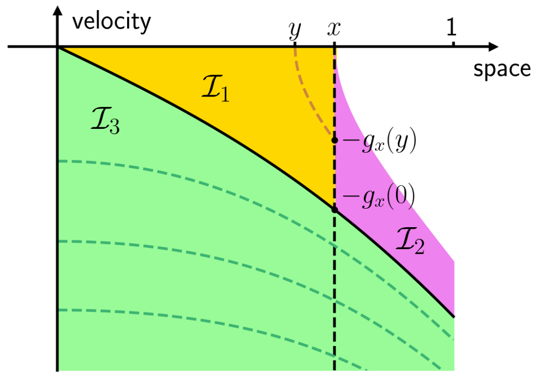

Figure 1: Decomposition of the integral defining n_i.

The phase space is divided by the critical characteristic (in solid black). Whenever $v \leqslant -g_x(0)$, the characteristics (dotted green lines) are reaching the boundary with $x_b(x, v) = 0$.

Let us show that n_i is bounded. We use the coarse estimate $f_e \leqslant \bar{c}$, and decompose the integral in three:

$$n_i(x) \leqslant 2\nu\bar{c}\left[\underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x}}_{\mathcal{I}_1} + \underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x}^{1}}_{\mathcal{I}_2} + \underbrace{\int_{v=-\overline{v}_e}^{-g_x(0)}\int_{z=x_b(x,v)}^{1}}_{\mathcal{I}_3}\right]\frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\,dzdv.$$

The corresponding domains are represented figure (3).

Notice that whenever $z \leqslant x$, we have $0 \geqslant 2\left(\varphi(x) - \varphi(z)\right) = -\left(2\left(\varphi(z) - \varphi(x)\right)\right)^{2/2} = -g_x^2(z)$. Then, the integral $\mathcal{I}_1$ may be bounded using lemma (2.6) with $y = 0$:

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x}\frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\,dzdv \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(x)\right)^{1/4}\sqrt{x} \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4}.$$

We use lemma (2.7) to bound $\mathcal{I}_2$ and $\mathcal{I}_3$. In the first case, we take $y = x$ and $v_0 = g_x(0)$, and notice that $-g_x(x) = 0$. In the second case, we take $v_0 = \overline{v}_e$ and $y = 0$, and notice that on $v \leqslant -g_x(0)$, we have $x_b(x, v) = 0$ (the velocity is low enough so that the characteristic ends on $x_b = 0$). This yields

$$\mathcal{I}_2 \leqslant \frac{2\sqrt{2(1-x)}}{\alpha^{1/4}}\left(g_x^2(0)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(-\varphi(1)\right)^{1/2}, \quad \text{and} \quad \mathcal{I}_3 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\left(\overline{v}_e{}^2 + 2\varphi(x)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(\frac{\beta}{\mu}\right)^{1/4}.$$

□

Lemma 2.8. *The density n_i is uniformly bounded away from 0. More precisely,*

$$n_i(x) \geqslant \frac{\sqrt{2}\nu\underline{c}(\overline{v}^2 - \underline{v}^2)}{\overline{v}\left(\frac{\overline{v}^2}{2} + \beta\right)^{1/2}}\frac{\underline{v}}{\sqrt{\beta\left(1 + \frac{1}{\mu}\right)}} \qquad \forall x \in [0, 1].$$

Demonstration By assumption, there exists a constant $\underline{c} > 0$ such that $f_{e,b}(v) \geqslant \underline{c}\mathbf{1}_{\{\underline{v}\leqslant|v|\leqslant\overline{v}\}}$, which implies

$$f_e(x, v) \geqslant \underline{c}\mathbf{1}_{\{\mathcal{L}_e(0,\underline{v})\leqslant\mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,\overline{v})\}}.$$

Notice that we may always take $\underline{v} > 0$ as long as $\overline{v} > 0$. Let us denote $\underline{\varepsilon} := \mathcal{L}_e(0, \underline{v})$ and $\overline{\varepsilon} := \mathcal{L}_e(0, \overline{v})$. For any $x \in [0, 1]$, we have

$$n_i(x) = 2\nu\int_{w\in\mathbb{R}_-}\int_{\tau=-\infty}^{0} f_e(x(\tau), v(\tau))d\tau dw, \quad \text{with} \quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x, w).$$

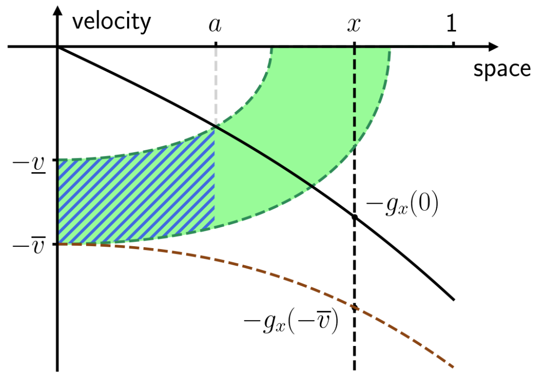

Figure 2: Notations for the lower bound on n_i.

The coloured area corresponds to the domain $\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}$, on which we know that $f_e \geqslant \underline{c}$. The hatched area represents the domain $\mathcal{D}$. The solid black line is the critical ion characteristic.

Notice that for $w \in [-g_x(-\overline{v}), -g_x(0)]$, the characteristic issued from (x, w) enters the domain $\{\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}\}$. We will restrict the domain of integration to

$$\mathcal{D} := \left\{(w, \tau) \in \mathbb{R}_-^2 \mid \underline{\varepsilon} \leqslant \mathcal{L}_e(x(\tau), v(\tau)) \leqslant \overline{\varepsilon} \text{ and } 0 \leqslant x(\tau) \leqslant a\right\},$$

where (a, v_a) is the unique point such that $\mathcal{L}_e(a, v_a) = \mathcal{L}_e(0, \underline{v})$ and $\mathcal{L}_i(a, v_a) = 0$ (note that a depends on β). This (somehow coarse) estimate will give us trivial bounds when applying the change of variable of lemma (2.4), namely $z = x(\tau)$ and $\varepsilon = \mathcal{L}_e(x(\tau), v(\tau))$. Denoting by $[\underline{\tau}, \overline{\tau}] \subset]-\infty, 0]$ the time interval where $(x(\tau), v(\tau)) \in \mathcal{D}$, we have

$$n_i(x) \geqslant \int_{w=-g_x(-\overline{v})}^{-g_x(0)} \int_{\tau=\underline{\tau}}^{\overline{\tau}} 2\nu\underline{c}\, d\tau dw = \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{2\nu\underline{c}}{2\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2}\left(\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}\, dz d\varepsilon.$$

Using remark (2.2), we are assured that the square roots are well-defined, since

$$\varepsilon \geqslant \underline{\varepsilon} = \mathcal{L}_e(0, \underline{v}) = \mathcal{L}_e(a, v_a) \geqslant \mathcal{L}_e(a, 0) \geqslant \mathcal{L}_e(z, 0) = -\frac{1}{\mu}\varphi(z) \quad \text{and} \quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x, w) \geqslant \mathcal{L}_i(x, 0).$$

Using that $\varepsilon + \frac{1}{\mu}\varphi(z) \leqslant \overline{\varepsilon} + 0$ and $\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu}\varphi(z)\right) \leqslant \overline{\varepsilon} + \beta + 0$, we obtain

$$n_i(x) \geqslant 2\nu \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{\underline{c}}{2\overline{\varepsilon}^{1/2}\left(\overline{\varepsilon} + \beta\right)^{1/2}}\, dz d\varepsilon = \frac{\nu a \underline{c}(\overline{\varepsilon} - \underline{\varepsilon})}{\overline{\varepsilon}^{1/2}\left(\overline{\varepsilon} + \beta\right)^{1/2}}.$$

Let us make explicit the dependance over β by estimating a using the $(-\beta)-$convexity assumption: we have

$$\varphi(x) \geqslant -\beta\frac{x^2}{2} \quad \Longrightarrow \quad \varphi^{-1}(y) \geqslant \sqrt{\frac{-2y}{\beta}}, \quad \text{so that} \quad a = \varphi^{-1}\left(-\frac{\underline{v}^2}{2\left(1 + \frac{1}{\mu}\right)}\right) \geqslant \frac{\underline{v}}{\sqrt{\beta\left(1 + \frac{1}{\mu}\right)}}.$$

and the uniform lower bound behaves like β^{-1} when $\beta \to \infty$. $\square$

Lemma 2.9. *Suppose that there exists $v_* > 0$ such that $f_{e,b}(v) = 0$ for all $|v| \leqslant v_*$. Then the density f_i satisfies the uniform bound*

$$f_i(x, v) \leqslant 2^{9/4}\overline{c}\sqrt{\frac{\frac{\beta}{\alpha}(1 + \frac{1}{\mu}) + 1}{v_*}} \quad \forall (x, v) \in [0, 1] \times \mathbb{R}.$$

Demonstration Let $(x,v) \in [0,1]\times\mathbb{R}_-$, and denote by $(x(t),v(t))_{t\leqslant 0}$ the ion characteristic going through $(x,v) = (x(-T),v(-T))$, with the convention $(x_b(x,v),v_b(x,v)) = (x(0),v(0))$. Owing to the positivity of f_e,

$$f_i(x(0),v(0)) = \int_{t=-T}^{0} f_e(x(t),v(t))dt + f_i(x,v) \geqslant f_i(x,v).$$

On the other hand, we use $f_i(x,v)+f_i(x,-v) = 2f_i(x(0),v(0))$ to write $f_i(x,-v) \leqslant 2f_i(x(0),v(0))$. This shows that it is enough to bound f_i on the boundary $\mathcal{B} \coloneqq \{x\geqslant 0, v=0\}\cup\{x=0, v\leqslant 0\}$ to obtain an uniform bound.

Let then $(x,v)\in\mathcal{B}$. By hypothesis, $f_e(x(t),v(t))$ vanishes whenever $\mathcal{L}_e(x(t),v(t)) \leqslant \mathcal{L}_e(0,v_*)$, and is bounded by $\bar{c}$ otherwise. Then, we may use the reparametrization by space

$$f_i(x,v) \leqslant \int_{z=a(x,v)}^{1} \frac{\bar{c}}{(v^2+2\varphi(x)-2\varphi(z))^{1/2}}dz, \quad \text{with} \quad a(x,v) \coloneqq \begin{cases} \varphi^{-1}\left(\frac{\frac{v^2}{2}+\varphi(x)-\frac{v_*^2}{2}}{1+1/\mu}\right) & \text{if } \mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,v_*) \\ x & \text{otherwise.}\end{cases}$$

The function a gives the smallest spatial coordinate of the characteristic $(x(t),v(t))_{t\leqslant 0}$ such that $f_e>0$. Using that $\varphi(z)\leqslant -\alpha\frac{z^2}{2}$, we have

$$f_i(x,v) \leqslant \bar{c}\int_{z=a(x,v)}^{1} \frac{1}{(v^2+2\varphi(x)-\alpha z^2)^{1/2}}dz \leqslant \bar{c}\min\left(\frac{1}{|v^2+2\varphi(x)|^{1/2}}, \frac{2}{\sqrt{\alpha a(x,v)}}\right)$$

where we used lemma (2.3) with $L \coloneqq v^2+2\varphi(x)$.

We rely on the concavity estimate

$$\varphi(x) \geqslant (1-x)\varphi(0)+x\varphi(1)+\frac{\alpha}{2}x(1-x) \geqslant -x\beta \quad\implies\quad -\frac{y}{\beta}\leqslant\varphi^{-1}(y)$$

to write that for (x,v) satisfying $\mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,v_*)$,

$$a(x,v) = \varphi^{-1}\left(\frac{\frac{v^2}{2}+\varphi(x)-\frac{v_*^2}{2}}{1+1/\mu}\right) \geqslant \frac{\frac{v_*^2}{2}-(\frac{v^2}{2}+\varphi(x))}{\beta(1+1/\mu)}.$$

Then

$$f_i(x,v)\leqslant\bar{c}\min\left(\frac{1}{\sqrt{|X|}},\frac{A}{\sqrt{B-X}}\right) \quad\text{with}\quad X\coloneqq\frac{v^2}{2}+\varphi(x),\quad A\coloneqq\frac{2}{\sqrt{\frac{\alpha}{\beta(1+1/\mu)}}},\quad\text{and}\quad B\coloneqq\frac{v_*^2}{2}.$$

The elementary study of the function $X\to\min\left(|X|^{-1/2}, A(B-X)^{-1/2}\right)$ reveals a global maximum at $X=\frac{B}{A^2+1}<B$, and we conclude to the result.

□

2.3 Continuity estimates

2.3.1 Electron density n_e

Lemma 2.10. *Let n_e^φ and n_e^ψ be the electron densities generated by potentials φ and ψ satisfying the assumptions. Then*

1. *If $f_{e,b}$ is Lipschitz-continuous with constant $[f_{e,b}]$, then*

$$|n_e^\varphi(x)-n_e^\psi(x)| \leqslant 2[f_{e,b}]\overline{v}_e\sqrt{\frac{2}{\mu}}\,|\varphi(x)-\psi(x)|^{1/2} \quad \forall (x,y)\in[0,1]^2.$$

2. *If there exists a constant $[f_{e,b}]$ such that $|f_{e,b}(x)-f_{e,b}(y)|\leqslant[f_{e,b}]|x^2-y^2|$ (or equivalently, $x\to f_{e,b}(\sqrt{x})$ is a lipschitz function, as for instance e^{-x^2}), then*

$$|n_e^\varphi(x)-n_e^\psi(x)| \leqslant \frac{4\overline{v}_e[f_{e,b}]_2}{\mu}\,|\varphi(x)-\psi(x)| \quad \forall (x,y)\in[0,1]^2.$$

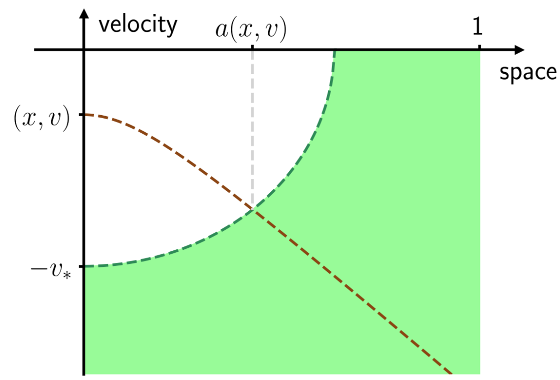

Figure 3: Notations for the boundedness of f_i.

The coloured area corresponds to $\mathcal{L}_e(x,v) \geqslant \mathcal{L}_e(0,v_*)$. The function $a(x,v)$ gives the point of the ion characteristic (in brown) where f_e vanishes.

Lemma 2.11. *The electronic density n_e is continuous.*

Demonstration We already know that n_e is bounded. Moreover, using the symmetry $f_e(x,v) = f_e(x,-v)$, we may write

$$n_e(x) - n_e(y) = 2\underbrace{\int_{v=0}^{v_x(y)} f_e(x,v)dv}_{=:\mathcal{I}^+} + 2\left(\underbrace{\int_{v=v_x(y)}^{v_x(1)} f_e(x,v)dv - \int_{v=0}^{v_y(1)} f_e(y,v)dv}_{=:\mathcal{I}^-}\right).$$

The term $\mathcal{I}^+$ is bounded by $\bar{c}v_x(y) = \bar{c}\left(\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}$, and by continuity of φ, we have $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. On the first integral of $\mathcal{I}^-$, we apply the change of variable

$$w = \left(v^2 - \frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2} \quad\Longleftrightarrow\quad dv = \frac{w}{\left(w^2 + \frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}}dw, \quad w \in [0, v_y(1)]$$

to get

$$\mathcal{I}^- = \int_{w=0}^{v_y(1)} f_e\left(x, \left(w^2 + \frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}\right)\frac{w}{\left(w^2+\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}}dw - \int_{v=0}^{v_y(1)} f_e(y,v)dv.$$

Notice that

$$f_e\left(x, \left(w^2 + \frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}\right) = f_{e,b}\left(\frac{w^2}{2} + \frac{1}{\mu}(\varphi(x)-\varphi(y)) - \frac{1}{\mu}\varphi(x)\right) = f_e(y,w).$$

Renaming w in v, we obtain the (clearly nonpositive) expression

$$\begin{aligned}\mathcal{I}^- &= \int_{v=0}^{v_y(1)} f_e(y,v)\left(\frac{v}{\left(v^2+\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}} - 1\right)dv \geqslant \bar{c}\int_{v=0}^{v_y(1)}\left(\frac{v}{\left(v^2+\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}} - 1\right)dv \\ &= \bar{c}\left(\left(v_y(1)^2 + \frac{2}{\mu}\Big(\underbrace{\varphi(x)-\varphi(y)}_{\geqslant 0}\Big)\right)^{1/2} - \left(\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2} - v_y(1)\right) \geqslant -\bar{c}\left(\frac{2}{\mu}(\varphi(x)-\varphi(y))\right)^{1/2}\end{aligned}$$

and this shows that $\mathcal{I}^- \underset{y\to x}{\longrightarrow} 0$. $\square$

Demonstration We have

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2\int_{v=0}^{\overline{v}_e} \left|f_e^\varphi(x,v) - f_e^\psi(x,v)\right| dv = 2\int_{v=0}^{\overline{v}_e} \left|f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2}\right) - f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2}\right)\right| dv.$$

Then, if $f_{e,b}$ is Lipschitz with constant $[f_{e,b}]$, we obtain

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{\overline{v}_e} \left|\left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2} - \left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2}\right| dv.$$

Using lemma (2.2) yields

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{\overline{v}_e} \left|-\frac{2}{\mu}\varphi(x) + \frac{2}{\mu}\psi(x)\right|^{1/2} dv = 2[f_{e,b}]\overline{v}_e\sqrt{\frac{2}{\mu}}\left|\varphi(x) - \psi(x)\right|^{1/2}.$$

If $f_{e,b}(\sqrt{\cdot})$ is Lipschitz with constant $[f_{e,b}]_2$, we may directly write

$$\left|n_e^\varphi(x) - n_e^\psi(x)\right| \leqslant 2\overline{v}_e[f_{e,b}]_2\frac{2}{\mu}\left|\varphi(x) - \psi(x)\right|.$$

$\square$

2.3.2 Ion density n_i

Proposition 2.2. *The density n_i is continuous.*

Demonstration Let $0 \leqslant y < x \leqslant 1$. For convenience, we represent $n_i(x)$ (resp. $n_i(y)$) as an integral with the artificial lower bound $-g_x(-\overline{v}_e) \leqslant -\overline{v}_e$ (resp. $-g_y(-\overline{v}_e)$). Then

$$\begin{aligned}
n_i(x) - n_i(y) &= 2\int_{v=-g_x(-\overline{v}_e)}^{0} f_i(x_b(x,v), v_b(x,v))dv - 2\int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv \\
&= 2\underbrace{\left[\int_{v=-g_x(-\overline{v}_e)}^{-g_x(y)} f_i(x_b(x,v), v_b(x,v))dv - \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv\right]}_{=:\mathcal{I}^-} \qquad (2.1)\\
&\quad + 2\underbrace{\int_{v=-g_x(y)}^{0} f_i(x_b(x,v), v_b(x,v))dv}_{=:\mathcal{I}^+}.
\end{aligned}$$

The term $\mathcal{I}^+$ is clearly nonnegative, and may be adressed using our lemmas. Indeed, using the integral representation of $f_i(x_b, v_b)$ and the reparametrization by a space variable z, we have

$$\begin{aligned}
\mathcal{I}^+ &= \int_{v=-g_x(y)}^{0}\left[\int_{z=x_b(x,v)}^{x} + \int_{z=x}^{1}\right] \frac{f_e(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}))}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\
&\leqslant \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dzdv + \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\
&\leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\left(\varphi(y) - \varphi(x)\right)^{1/4}\sqrt{x-y} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(2(\varphi(y) - \varphi(x))\right)^{1/2}\right),
\end{aligned}$$

where we used lemma (2.6) for the first term, and lemma (2.7) for the second term (with $y = x$ and $v_0 = -g_x(y)$ under the notations of the lemma). Since φ is continuous, we deduce that $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. Taking the extreme case $y = 0$ and $x = 1$, we obtain that

$$\mathcal{I}^+ \leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(-2\varphi(1)\right)^{1/2}\right) =: K.$$

Let us now focus on $\mathcal{I}^-$. On the first integral, we make the change of variable

$$w = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2} \qquad v = -\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Since $\mathcal{L}_i(x,v) = \mathcal{L}_i(y,w)$, this yields $x_b(x,v) = x_b(y,w)$ and $v_b(x,v) = v_b(y,w)$. The bounds $v \in [-g_x(-\overline{v}_e), -g_x(y)]$ are exactly transported to $w \in [-g_y(-\overline{v}_e), 0]$. Renaming w in v, we get

$$\mathcal{I}^- = \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v)) \left(\frac{-v}{\left(v^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} - 1 \right) dv.$$

Since $\varphi(y) \geqslant \varphi(x)$, the factor of f_i is nonpositive, and so is $\mathcal{I}^-$. Moreover,

$$n_i(x) - n_i(y) = 2\mathcal{I}^- + 2\mathcal{I}^+ \leqslant 2\mathcal{I}^- + 2K \qquad \Longleftrightarrow \qquad \mathcal{I}^- = -K + n_i(x) - n_i(y) \geqslant -K - |n_i|_\infty$$

and $\mathcal{I}^-$ is bounded. The function $\frac{-v}{\left(v^2+2(\varphi(y)-\varphi(x))\right)^{1/2}} - 1$ converges pointwise to 0 when $x \to y$ **and f_i is almost everywhere finite**, and by Lebesgue's dominated convergence, $\mathcal{I}^- \underset{x \to y}{\longrightarrow} 0$. Then n_i is continuous. $\square$

3 Results

Let $0 < \alpha \leqslant \beta$, and define the following convex set:

$$\mathcal{K} := \left\{ \varphi \in \mathcal{C}^1([0,1], \mathbb{R}) \quad | \quad \varphi(0) = \varphi'(0) = 0, \quad \varphi \ \alpha - \text{concave et } (-\beta) - \text{convex.} \right\}.$$

Both variation conditions rewrite

$$\forall (x,y,\gamma) \in [0,1]^3, \quad \begin{cases} \varphi((1-\gamma)x + \gamma y) \geqslant (1-\gamma)\varphi(x) + \gamma\varphi(y) + \frac{\alpha}{2}\gamma(1-\gamma)|x-y|^2, \\ \varphi((1-\gamma)x + \gamma y) \leqslant (1-\gamma)\varphi(x) + \gamma\varphi(y) + \frac{\beta}{2}\gamma(1-\gamma)|x-y|^2. \end{cases}$$

In the case $\varphi \in \mathcal{C}^2$, it is equivalent to $\varphi'' \in [-\beta, -\alpha]$.

Lemma 3.1. *$\mathcal{K}$ is closed for the topology induced by the sup-norm $|\cdot|_\infty := \max_{[0,1]} |\cdot|$.*

Demonstration Since $\mathcal{K} \subset \mathcal{C}^0([0,1], \mathbb{R})$, we know that any Cauchy sequence $(\varphi_n)_n \subset \mathcal{K}$ admits a limit $\varphi \in \mathcal{C}([0,1], \mathbb{R})$. The pointwise condition $\varphi_n(0) = 0$ and the pointwise grows conditions are preserved when $n \to \infty$. Since the family of continuous fonctions $(\varphi'_n)_n$ is equilipschitz, we may use Arzelà-Ascoli to extract an uniformly converging subsequence $\varphi'_{n_k} \to \varphi'_\infty$. Using that for all $(x,y) \in [0,1]$,

$$\left| \varphi(y) - \varphi(x) - \int_x^y \varphi'_\infty(z) dz \right| \leqslant |\varphi(y) - \varphi_n(y)| + |\varphi(x) - \varphi_n(x)| + \int_x^y |\varphi'_\infty(z) - \varphi'_n(z)| dz \underset{n \to \infty}{\longrightarrow} 0,$$

we get that φ'_∞ is the (continuous) derivative of φ. Finally, the uniform convergence $\varphi'_{n_k} \to \varphi'_\infty$ gives $\varphi'(0) = 0$. $\square$

We define an operator $F : \mathcal{K} \mapsto \mathcal{C}^2([0,1], \mathbb{R})$ by the solution of the Poisson problem

$$-\lambda^2 F'' = n_i[\varphi] - n_e[\varphi],$$

where n_i and n_e are the ion and electron densities obtained with the characteristics induced by φ.

Lemma 3.2 (Coarse stability)**.** *Assume all the hypotheses of the above lemmas are satisfied.* ***Be more precise.*** *For all $(\lambda, \mu) \in (\mathbb{R}_*^+)^2$, there exists parameters $(\alpha, \beta, \nu) \in (\mathbb{R}_*^+)^3$ such that $F(\mathcal{K}) \subset \mathcal{K}$.*

Demonstration The function F is solution to a Poisson problem with bounded continuous source term, so it enjoys $\mathcal{C}^2$ regularity. The boundary condition $F_\varphi(0) = F'_\varphi(0) = 0$ are satisfied by construction. We turn to the stability of the variation estimates: owing to the boundedness of n_i and n_e stated above, we have

$$\begin{cases} -\lambda^2 F''(x) \leqslant 8\nu \left(\frac{\beta}{\alpha}\right)^{1/4} \left(\frac{1}{\alpha^{1/4}} + \beta^{1/4} + \frac{1}{\mu^{1/4}} \right), \\ -\lambda^2 F''(x) \geqslant \nu \frac{\kappa_1}{\sqrt{\beta}(\kappa_2 + \beta)^{1/2}\sqrt{1 + \frac{1}{\mu}}} - 2\overline{c}\sqrt{\frac{2\beta}{\mu}} \end{cases}$$

where
$$\kappa_1 := \frac{\underline{c}(\overline{v}^2 - \underline{v}^2)\underline{v}}{\sqrt{2}\overline{v}}, \quad \kappa_2 := \frac{\overline{v}^2}{2}$$
A sufficient condition for the stability is to obtain $\alpha \leqslant -F'' \leqslant \beta$. We will restrict our search to $0 < \alpha \leqslant \beta$ such that $\alpha\beta = 1$. Then, sufficient conditions write

$$\begin{cases} 1 \leqslant \beta \\ 8\frac{\nu}{\lambda^2}\beta^{1/2}\left(\beta^{-1/4} + \beta^{1/4} + \frac{1}{\mu^{1/4}}\right) \leqslant \beta \\ \nu\frac{\kappa_1}{\sqrt{\beta}(\kappa_2+\beta)^{1/2}\sqrt{1+\frac{1}{\mu}}} - 2\overline{c}\sqrt{\frac{2\beta}{\mu}} \geqslant \beta^{-1} \end{cases} \iff \begin{cases} 1 \leqslant \beta \\ 1 + \beta^{1/2} + \frac{\beta^{1/4}}{\mu^{1/4}} - \frac{\lambda^2}{8\nu}\beta \leqslant 0 \\ \nu\frac{\kappa_1\sqrt{\beta}}{(\kappa_2+\beta)^{1/2}\sqrt{1+\frac{1}{\mu}}} \geqslant 1 + 2\overline{c}\sqrt{\frac{2}{\mu}}\beta^{3/2} \end{cases}$$

Using the coarse estimate $\beta^{1/4} \leqslant \beta^{1/2}$ on $1 \leqslant \beta$ yields
$$1 + \beta^{1/2} + \frac{\beta^{1/4}}{\mu^{1/4}} - \frac{\lambda^2}{8\nu}\beta \leqslant 1 + \beta^{1/2}\left(1 + \frac{1}{\mu^{1/4}}\right) - \frac{\lambda^2}{8\nu}\beta := 1 + X\left(1 + \frac{1}{\mu^{1/4}}\right) - \frac{\lambda^2}{8\nu}X^2,$$
and this concave polynomial is negative for β large enough. **Do the computation if useful**

Moreover, since $1 \geqslant \beta$ and the function $\beta \to \frac{\sqrt{\beta}}{\sqrt{\kappa_2+\beta}}$ is increasing, the third condition is implied by
$$\nu\frac{\kappa_1}{(\kappa_2+1)^{1/2}\sqrt{1+\frac{1}{\mu}}} \geqslant 1 + 2\overline{c}\sqrt{\frac{2}{\mu}}\beta^{3/2}.$$

□

In the future:

- **Now that we have estimates, write it in function of α and β, and see how to get stability of the set of strongly concave functions satisfying all the hypotheses by the Poisson problem (essentially, find tweaks of λ, μ, ν, α and/or β such that the estimates are propagated).**
- **If $f_{e,b}$ vanishes in a neighbourhood of $(0,0)$ (or decreases fast enough, see how fast), show that f_i is bounded.**
- **Under that same assumption, we should obtain stronger continuity over n_i, and n_i may be continuous with respect to φ (in the same sense as n_e in lemma (2.10)).**
- **If the estimates with respect to $|\varphi - \psi|_\infty$ succeed, define numerical scheme and see what we can say about it.**