Notes on fixed-point procedure

Contents

1 Notations and assumptions

We suppose that

1. $f_{e,b}$ is nonnegative.
2. $f_{e,b}(v) = f_{e,b}(-v)$ for all $v \in \mathbb{R}$.
3. $f_{e,b}$ satisfies the boundary condition, i.e. $f_{e,b}(v) = 0$ as soon as $\mathcal{L}_e(0,v) \leqslant \mathcal{L}_e(1,0)$.
4. There exists $\overline{c} \leqslant 0$ such that $f_{e,b}(v) \leqslant \overline{c}$ for all $v \in \mathbb{R}$.
5. There exists $\underline{c} > 0$ and $0 \leqslant \underline{v} < \overline{v}$ such that $f_{e,b}(v) \geqslant \underline{c}$ for all $v \in [-\overline{v}, -\underline{v}]$.
6. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$.
7. The function φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi((1-\tau)x + \tau y) \geqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau\,|x-y|^2$.
8. There exists $\beta > \alpha$ such that $\varphi(1) \geqslant -\beta$. **Uniformly over the fixed-point iterations**

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x,v) \coloneqq \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{1.1}$$

$$\mathcal{L}_i(x,v) \coloneqq \frac{v^2}{2} + \varphi(x). \tag{1.2}$$

We define an upper bound $\overline{v}_e$ over the velocities in the support of f_e, given as

$$\mathcal{L}_e(0,\overline{v}_e) \coloneqq \frac{1}{\mu}\beta \geqslant -\frac{1}{\mu}\varphi(1) = \mathcal{L}_e(1,0), \quad \text{i.e.} \quad \overline{v}_e \coloneqq \sqrt{\frac{2}{\mu}\beta}. \tag{1.3}$$

For a given point $(x,v) \in [0,1] \times \mathbb{R}^-$, we denote by (x_b, v_b) the intersection of the boundary $\{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ with the ion characteristic issued from (x,v). The values are given by

$$\begin{pmatrix} x_b(x,v) \\ v_b(x,v) \end{pmatrix} \coloneqq \begin{pmatrix} \varphi^{-1}\left(\min\left(0, \frac{v^2}{2} + \varphi(x)\right)\right) \\ -\sqrt{\max\left(0, v^2 + 2\varphi(x)\right)} \end{pmatrix} \tag{1.4}$$

2 Estimates

2.1 Useful elementary lemmas

Lemma 2.1. *Let $\alpha > 0$, $\overline{x} \geqslant 0$ and $\overline{y} \geqslant 0$. Then*

$$\mathcal{I} := \int_{x=0}^{\overline{x}} \int_{y=0}^{\overline{y}} \frac{1}{(x^2 + \alpha y^2)^{1/2}} dy dx \leqslant 2 \frac{\sqrt{2 \overline{x} \overline{y}}}{\alpha^{1/4}}.$$

Demonstration By the change of variable $z = \sqrt{\alpha} y$, with $\overline{z} := \sqrt{\alpha} \overline{y}$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{1}{(x^2 + z^2)^{1/2}} dz dx.$$

Notice that $x + z \leqslant \sqrt{2} \left(x^2 + y^2\right)^{1/2}$. Then,

$$\mathcal{I} \leqslant \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{\sqrt{2}}{x+z} dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln \left(\frac{x + \overline{z}}{x} \right) dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln \left(1 + \frac{\overline{z}}{x}\right) dx.$$

Using $\ln(1+a) \leqslant \sqrt{a}$, we get

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \sqrt{\frac{\overline{z}}{x}} dx = \sqrt{\frac{2}{\alpha}} 2 \sqrt{\overline{x}\overline{z}} = 2 \frac{\sqrt{2 \overline{x} \overline{y}}}{\alpha^{1/4}}.$$

$\square$

Remark 2.1 (Exact value). *Let $\overline{r} := \sqrt{\overline{x}^2 + \overline{z}^2}$. Then*

$$\mathcal{I} = \overline{x} \ln \left(\frac{1 + \overline{z}/\overline{r}}{\overline{x}/\overline{r}} \right) + \overline{z} \ln \left(\frac{1 + \overline{x}/\overline{r}}{\overline{z}/\overline{r}} \right).$$

Lemma 2.2. *If $a \geqslant 0$ and $b \geqslant 0$, then $|a-b| \leqslant \sqrt{|a^2 - b^2|}$.*

Demonstration If $a \geqslant b$, then $|a-b| = \sqrt{(a-b)(a-b)} \leqslant \sqrt{(a-b)(a+b)} = \sqrt{a^2 - b^2}$, else $|a-b| = |b-a|$. $\square$

Lemma 2.3. *Let $\alpha > 0$, $L \in \mathbb{R}$ and $0 \leqslant a \leqslant 1$. Suppose that $L + \alpha a^2 \geqslant 0$, and $a > 0$ if $L = 0$. Then*

$$\mathcal{I} := \int_{z=a}^{1} \frac{1}{(L + \alpha z^2)^{1/2}} dz \leqslant \min \left(|L|^{-1/2}, \frac{2}{\sqrt{\alpha a}} \right).$$

Demonstration If $L > 0$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{z=a}^{1} \frac{1}{\left(1 + \left(\sqrt{\frac{\alpha}{L}} z\right)^2\right)^{1/2}} \frac{\sqrt{\alpha} dz}{\sqrt{L}} = \frac{1}{\alpha} \int_{w=a\sqrt{\frac{\alpha}{L}}}^{\sqrt{\frac{\alpha}{L}}} \frac{1}{(1+w^2)^{1/2}} dw = \frac{1}{\sqrt{\alpha}} \left(\sinh^{-1}\left(\sqrt{\frac{\alpha}{L}}\right) - \sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right) \right).$$

Using the positivity of $\sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right)$, and the coarse estimate $\sinh^{-1}(x) \leqslant x$, we get $\mathcal{I} \leqslant L^{-1/2}$. Moreover, using that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$, we get

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \log \left(\frac{\sqrt{\frac{\alpha}{L}} + \sqrt{\frac{\alpha}{L} + 1}}{a\sqrt{\frac{\alpha}{L}} + \sqrt{a^2 \frac{\alpha}{L} + 1}} \right) = \frac{1}{\sqrt{\alpha}} \log \left(\frac{1 + \sqrt{1 + \frac{L}{\alpha}}}{a + \sqrt{a^2 + \frac{L}{\alpha}}} \right) \leqslant \frac{1}{\sqrt{\alpha}} \log \left(\frac{2 + \sqrt{\frac{L}{\alpha}}}{2a} \right) \leqslant \frac{1 + \left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}.$$

I shamelessly used $\frac{1}{2} \leqslant 1$. Then $\mathcal{I} \leqslant \min\left(L^{-1/2}, \frac{1 + \left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}\right)$ on $L > 0$. But whenever $L \geqslant \alpha$, the min is attained in its first argument: indeed, $L^{-1/2} \leqslant \frac{1}{\sqrt{\alpha a}} \leqslant \frac{1 + \left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}$. Then we obtained $\mathcal{I} \leqslant \min(L^{-1/2}, \frac{2}{\sqrt{\alpha a}})$ on $L > 0$. If $L = 0$, we have

$$\int_{z=a}^{1} \frac{1}{\sqrt{\alpha} z} dz = \frac{\log(1/a)}{\sqrt{\alpha}} \frac{2}{\sqrt{\alpha a}}.$$

Finally, if $L<0$, the integral becomes

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\int_a^1\frac{1}{\left(\left(\sqrt{\frac{\alpha}{|L|}}z\right)^2-1\right)^{1/2}}\frac{\sqrt{\alpha}dz}{\sqrt{|L|}}=\frac{1}{\alpha}\int_{a\sqrt{\frac{\alpha}{|L|}}}^{\sqrt{\frac{\alpha}{|L|}}}\frac{1}{(w^2-1)^{1/2}}dw=\frac{\cosh^{-1}\left(\sqrt{\frac{\alpha}{|L|}}\right)-\cosh^{-1}\left(a\sqrt{\frac{\alpha}{|L|}}\right)}{\sqrt{\alpha}}.$$

The expression is well-defined, since $L+\alpha a^2\geqslant 0$ implies $a\sqrt{\frac{\alpha}{-L}}\geqslant 1$ when $-L>0$. Since $\cosh^{-1}(x)\leqslant x$, we obtain the coarse estimate $\mathcal{I}\leqslant|L|^{-1/2}$. Using that $\cosh^{-1}(x)=\log\left(x+\sqrt{x^2-1}\right)$, we get

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\log\left(\frac{1+\sqrt{1-\frac{|L|}{\alpha}}}{a+\sqrt{a^2-\frac{|L|}{\alpha}}}\right)\leqslant\frac{1}{\sqrt{\alpha}}\log\left(\frac{2}{a}\right)\leqslant\sqrt{\frac{2}{\alpha a}}\leqslant\frac{2}{\sqrt{\alpha a}}.$$

□

2.2 Properties of the electron density n_e

Lemma 2.4. *The electron density n_e is bounded and continuous.*

Demonstration For each $0\leqslant x\leqslant y\leqslant 1$, we define the velocity $v_x(y)\geqslant 0$ such that

$$\mathcal{L}_e(x,v_x(y))=\mathcal{L}_e(y,0)\qquad\Longleftrightarrow\qquad v_x(y)=\left(\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}.$$

In particular, owing to the boundary conditions, the function $v\to f_e(x,v)$ vanishes for $|v|\geqslant v_x(1)$. Then

$$n_e(x)=\int_{v=-v_x(1)}^{v_x(1)}f_e(x,v)dv\leqslant 2\overline{c}v_x(1)=2\overline{c}\left(\frac{2}{\mu}(\varphi(x)-\varphi(1))\right)^{1/2}\leqslant 2\overline{c}\sqrt{\frac{2\beta}{\mu}}.$$

Moreover, using the symmetry $f_e(x,v)=f_e(x,-v)$, we may write

$$n_e(x)-n_e(y)=2\underbrace{\int_{v=0}^{v_x(y)}f_e(x,v)dv}_{=:\mathcal{I}^+}+2\left(\underbrace{\int_{v=v_x(y)}^{v_x(1)}f_e(x,v)dv-\int_{v=0}^{v_y(1)}f_e(y,v)dv}_{=:\mathcal{I}^-}\right).$$

The term $\mathcal{I}^+$ is bounded by $\overline{c}v_x(y)=\overline{c}\left(\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$, and by continuity of φ, we have $\mathcal{I}^+\underset{y\to x}{\longrightarrow}0$. On the first integral of $\mathcal{I}^-$, we apply the change of variable

$$w=\left(v^2-\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}\qquad\Longleftrightarrow\qquad dv=\frac{w}{\left(w^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}dw,\quad w\in[0,v_y(1)]$$

to get

$$\mathcal{I}^-=\int_{w=0}^{v_y(1)}f_e\left(x,\left(w^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}\right)\frac{w}{\left(w^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}dw-\int_{v=0}^{v_y(1)}f_e(y,v)dv.$$

Notice that

$$f_e\left(x,\left(w^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}\right)=f_{e,b}\left(\frac{w^2}{2}+\frac{1}{\mu}\left(\varphi(x)-\varphi(y)\right)-\frac{1}{\mu}\varphi(x)\right)=f_e(y,w).$$

Renaming w in v, we obtain the (clearly nonpositive) expression

$$\mathcal{I}^- = \int_{v=0}^{v_y(1)} f_e(y,v) \left(\frac{v}{\left(v^2 + \frac{2}{\mu}\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}} - 1 \right) dv \geqslant \overline{c} \int_{v=0}^{v_y(1)} \left(\frac{v}{\left(v^2 + \frac{2}{\mu}\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}} - 1 \right) dv$$

$$= \overline{c} \left(\left(v_y(1)^2 + \frac{2}{\mu} \left(\underbrace{\varphi(x) - \varphi(y)}_{\geqslant 0} \right) \right)^{1/2} - \left(\frac{2}{\mu}\left(\varphi(x) - \varphi(y)\right) \right)^{1/2} - v_y(1) \right) \geqslant -\overline{c} \left(\frac{2}{\mu}\left(\varphi(x) - \varphi(y)\right) \right)^{1/2}$$

and this shows that $\mathcal{I}^- \underset{y \to x}{\longrightarrow} 0$. □

Lemma 2.5. *Let n_e^φ and n_e^ψ be the electron densities generated by potentials φ and ψ satisfying the assumptions. Then*

1. *If $f_{e,b}$ is Lipschitz-continuous with constant $[f_{e,b}]$, then*

$$|n_e^\varphi(x) - n_e^\psi(x)| \leqslant 2[f_{e,b}]\overline{v}_e \sqrt{\frac{2}{\mu}} \, |\varphi(x) - \psi(x)|^{1/2} \qquad \forall (x,y) \in [0,1]^2.$$

2. *If there exists a constant $[f_{e,b}]$ such that $|f_{e,b}(x) - f_{e,b}(y)| \leqslant [f_{e,b}]|x^2 - y^2|$ (or equivalently, $x \to f_{e,b}(\sqrt{x})$ is a lipschitz function, as for instance e^{-x^2}), then*

$$|n_e^\varphi(x) - n_e^\psi(x)| \leqslant \frac{4\overline{v}_e[f_{e,b}]_2}{\mu} \, |\varphi(x) - \psi(x)| \qquad \forall (x,y) \in [0,1]^2.$$

Demonstration We have

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leqslant 2 \int_{v=0}^{\overline{v}_e} \left| f_e^\varphi(x,v) - f_e^\psi(x,v) \right| dv = 2 \int_{v=0}^{\overline{v}_e} \left| f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2} \right) - f_{e,b}\left(\left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2} \right) \right| dv.$$

Then, if $f_{e,b}$ is Lipschitz with constant $[f_{e,b}]$, we obtain

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leqslant 2[f_{e,b}] \int_{v=0}^{\overline{v}_e} \left| \left(v^2 - \frac{2}{\mu}\varphi(x)\right)^{1/2} - \left(v^2 - \frac{2}{\mu}\psi(x)\right)^{1/2} \right| dv.$$

Using lemma (2.2) yields

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leqslant 2[f_{e,b}] \int_{v=0}^{\overline{v}_e} \left| -\frac{2}{\mu}\varphi(x) + \frac{2}{\mu}\psi(x) \right|^{1/2} dv = 2[f_{e,b}]\overline{v}_e \sqrt{\frac{2}{\mu}} \, |\varphi(x) - \psi(x)|^{1/2}.$$

If $f_{e,b}(\sqrt{\cdot})$ is Lipschitz with constant $[f_{e,b}]_2$, we may directly write

$$\left| n_e^\varphi(x) - n_e^\psi(x) \right| \leqslant 2\overline{v}_e[f_{e,b}]_2 \frac{2}{\mu} \, |\varphi(x) - \psi(x)|.$$

□

2.3 Properties of the ion density n_i

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given x, we define $g_x : [0,x] \mapsto \mathbb{R}^+$ by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = \left(2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Lemma 2.6. *Let $0 \leqslant y < x \leqslant 1$. We have*

$$\mathcal{I} := \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dz\, dv \leqslant 2\sqrt{\frac{2}{\alpha}} \left(\varphi(y) - \varphi(x)\right)^{1/4} \sqrt{x-y}.$$

Demonstration Let us first use Fubini's theorem to switch the order of integration. The lower bound $x_b(x,v) \geqslant z$ becomes an upper bound $v \leqslant -g_x(z)$, and we have

$$\mathcal{I} = \int_{z=y}^{x} \int_{v=-g_x(y)}^{-g_x(z)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz = \int_{z=y}^{x} \int_{v=g_x(z)}^{g_x(y)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz.$$

With the change of variable $w = v - g_x(z)$, and using $\frac{d}{dw}\left[\sinh^{-1}\left(\sqrt{\frac{w}{a}}\right)\right] = \left(w^2 + 2aw\right)^{-1/2}$, we get

$$\mathcal{I} = \int_{z=y}^{x} \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{\left(w^2 + 2wg_x(z)\right)^{1/2}} dw\, dz = \int_{z=y}^{x} \sinh^{-1}\left(\sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}}\right) dz.$$

Using the coarse estimates $\sinh^{-1}(a) \leqslant a$ and $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$, we get

$$\mathcal{I} \leqslant \int_{z=y}^{x} \sqrt{\frac{g_x(y)}{g_x(z)}} dz = \int_{z=y}^{x} \left(\frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)}\right)^{1/4} dz.$$

The assumption of strong concavity yields $\varphi(z) - \varphi(x) \geqslant -\varphi'(z)(x-z) + \frac{\alpha}{2}|x-z|^2 \geqslant \frac{\alpha}{2}(x-z)^2$, so that

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \left(\varphi(y) - \varphi(x)\right)^{1/4} \int_{z=y}^{x} \frac{1}{(x-z)^{1/2}} dz = 2\sqrt{\frac{2}{\alpha}} \left(\varphi(y) - \varphi(x)\right)^{1/4} \sqrt{x-y}.$$

□

Lemma 2.7. *Let $0 \leqslant y \leqslant x \leqslant 1$, and $-v_0 < -g_x(y)$. We have*

$$\mathcal{I} := \int_{v=-v_0}^{-g_x(y)} \int_{z=y}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}} dz\, dv \leqslant \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/4}.$$

Demonstration We first shift the $v-$integration from the vertical line $z = x$ to $z = y$. Let $w = w(v)$ be such that

$$\mathcal{L}_i(y, w(v)) = \mathcal{L}_i(x, v), \quad \text{i.e.} \quad w(v) = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right), \text{ and } dv = \frac{-w}{\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} dw.$$

Then, defining $w_0 := \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}$, and noticing that $w(-g_x(y)) = 0$, we get

$$\mathcal{I} = \int_{w=-w_0}^{0} \int_{z=y}^{1} \frac{1}{\left(w^2 + 2\left(\varphi(y) - \varphi(z)\right)\right)^{1/2}} dz \frac{-w}{\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} dw.$$

Since $y \leqslant x$, we have $\varphi(y) \geqslant \varphi(x)$, and $\frac{-w}{\left(w^2 + 2(\varphi(y) - \varphi(x))\right)^{1/2}} \leqslant \frac{-w}{|w|} = 1$. By the strong concavity assumption, we have $\varphi(y) - \varphi(z) \geqslant -\varphi'(y)(z-y) + \frac{\alpha}{2}|z-y|^2 \geqslant \frac{\alpha}{2}(z-y)^2$, so that

$$\mathcal{I} \leqslant \int_{w=-w_0}^{0} \int_{z=y}^{1} \frac{1}{\left(w^2 + \alpha(z-y)^2\right)^{1/2}} dz dw = \int_{w=0}^{w_0} \int_{z=0}^{1-y} \frac{1}{\left(w^2 + \alpha z^2\right)^{1/2}} dz dw.$$

Using lemma (2.1), we conclude that

$$\mathcal{I} \leqslant 2\frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}} = \frac{2\sqrt{2(1-y)}}{\alpha^{1/4}} \left(v_0^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/4}.$$

□

Proposition 2.1. *The density n_i is bounded by a constant depending on φ only through $\varphi(1)$.*

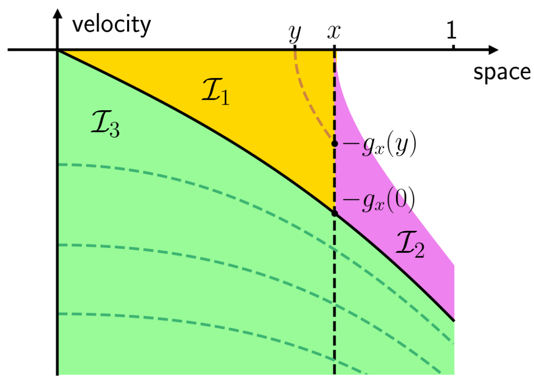

Figure 1: Decomposition of the integral defining n_i.

The phase space is divided by the critical characteristic (in solid black). Whenever $v \leqslant -g_x(0)$, the characteristics (dotted green lines) are reaching the boundary with $x_b(x,v) = 0$.

Demonstration We use the symmetry of f_i to write

$$n_i(x) = \int_{v=-\infty}^{\infty} f_i(x,v)dv = 2\int_{v=-\infty}^{0} f_i(x_b(x,v), v_b(x,v))dv = 2\int_{v=-\infty}^{0}\int_{t=-\infty}^{0} f_e(x(t),v(t))dtdv,$$

where $(x(t),v(t))_{t\leqslant 0}$ is the ion characteristic reaching $(x_b(x,v), v_b(x,v))$ at $t=0$. Notice that the lower bounds are artificial, since the characteristic enters the support of f_e in finite time: we may use $v \geqslant -\overline{v}_e$, and consider only times t for which $x(t) \in [0,1]$.

We first reparametrize $(x(t),v(t))$ using the space variable. Define $z = x(t) \in [x_b, 1]$, and observe that

$$dz = \dot{x}(t)dt = v(t)dt, \quad \text{with} \quad \mathcal{L}_i(z, v(t)) = \mathcal{L}_i(x,v) \quad \iff \quad v(t) = -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}.$$

Then, the density rewrites

$$n_i(x) = 2\int_{v=-\overline{v}_e}^{0}\int_{z=x_b(x,v)}^{1} \frac{f_e\left(z, -\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}\right)}{\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}\, dzdv.$$

Let us show that n_i is bounded. We use the coarse estimate $f_e \leqslant \overline{c}$, and decompose the integral in three:

$$n_i(x) \leqslant 2\overline{c}\left[\underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x}}_{\mathcal{I}_1} + \underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x}^{1}}_{\mathcal{I}_2} + \underbrace{\int_{v=-\overline{v}_e}^{-g_x(0)}\int_{z=x_b(x,v)}^{1}}_{\mathcal{I}_3}\right] \frac{1}{\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}\, dzdv.$$

The corresponding domains are represented figure (3).

Notice that whenever $z \leqslant x$, we have $0 \geqslant 2\left(\varphi(x) - \varphi(z)\right) = -\left(2\left(\varphi(z)-\varphi(x)\right)\right)^{2/2} = -g_x^2(z)$. Then, the integral $\mathcal{I}_1$ may be bounded using lemma (2.6) with $y=0$:

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dzdv \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(x)\right)^{1/4}\sqrt{x} \leqslant 2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4}.$$

We use lemma (2.7) to bound $\mathcal{I}_2$ and $\mathcal{I}_3$. In the first case, we take $y = x$ and $v_0 = g_x(0)$, and notice that $-g_x(x) = 0$. In the second case, we take $v_0 = \overline{v}_e$ and $y=0$, and notice that on $v \leqslant -g_x(0)$, we have $x_b(x,v) = 0$ (the velocity is low enough so that the characteristic ends on $x_b = 0$). This yields

$$\mathcal{I}_2 \leqslant \frac{2\sqrt{2(1-x)}}{\alpha^{1/4}}\left(g_x^2(0)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(-\varphi(1)\right)^{1/2}, \quad \text{and} \quad \mathcal{I}_3 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\left(\overline{v}_e{}^2 + 2\varphi(x)\right)^{1/4} \leqslant \frac{4}{\alpha^{1/4}}\left(\frac{\beta}{\mu}\right)^{1/4}.$$

$\square$

Proposition 2.2. *The density n_i is continuous.*

Demonstration Let $0 \leqslant y < x \leqslant 1$. For convenience, we represent $n_i(x)$ (resp. $n_i(y)$) as an integral with the artificial lower bound $-g_x(-\overline{v}_e) \leqslant -\overline{v}_e$ (resp. $-g_y(-\overline{v}_e)$). Then

$$\begin{aligned} n_i(x) - n_i(y) &= 2\int_{v=-g_x(-\overline{v}_e)}^{0} f_i(x_b(x,v), v_b(x,v))dv - 2\int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv \\ &= \underbrace{2\left[\int_{v=-g_x(-\overline{v}_e)}^{-g_x(y)} f_i(x_b(x,v), v_b(x,v))dv - \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))dv\right]}_{=:\mathcal{I}^-} + \underbrace{2\int_{v=-g_x(y)}^{0} f_i(x_b(x,v), v_b(x,v))dv}_{=:\mathcal{I}^+}. \end{aligned} \tag{2.1}$$

$$\tag{2.2}$$

The term $\mathcal{I}^+$ is clearly nonnegative, and may be adressed using our lemmas. Indeed, using the integral representation of $f_i(x_b, v_b)$ and the reparametrization by a space variable z, we have

$$\begin{aligned} \mathcal{I}^+ &= \int_{v=-g_x(y)}^{0}\left[\int_{z=x_b(x,v)}^{x} + \int_{z=x}^{1}\right] \frac{f_e(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}))}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\ &\leqslant \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dzdv + \overline{c}\int_{v=-g_x(y)}^{0}\int_{z=x}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\ &\leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\left(\varphi(y) - \varphi(x)\right)^{1/4}\sqrt{x-y} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(2(\varphi(y) - \varphi(x))\right)^{1/2}\right), \end{aligned}$$

where we used lemma (2.6) for the first term, and lemma (2.7) for the second term (with $y = x$ and $v_0 = -g_x(y)$ under the notations of the lemma). Since φ is continuous, we deduce that $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. Taking the extreme case $y = 0$ and $x = 1$, we obtain that

$$\mathcal{I}^+ \leqslant \overline{c}\left(2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(-2\varphi(1)\right)^{1/2}\right) =: K.$$

Let us now focus on $\mathcal{I}^-$. On the first integral, we make the change of variable

$$w = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2} \qquad v = -\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Since $\mathcal{L}_i(x,v) = \mathcal{L}_i(y,w)$, this yields $x_b(x,v) = x_b(y,w)$ and $v_b(x,v) = v_b(y,w)$. The bounds $v \in [-g_x(-\overline{v}_e), -g_x(y)]$ are exactly transported to $w \in [-g_y(-\overline{v}_e), 0]$. Renaming w in v, we get

$$\mathcal{I}^- = \int_{v=-g_y(-\overline{v}_e)}^{0} f_i(x_b(y,v), v_b(y,v))\left(\frac{-v}{\left(v^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} - 1\right)dv.$$

Since $\varphi(y) \geqslant \varphi(x)$, the factor of f_i is nonpositive, and so is $\mathcal{I}^-$. Moreover,

$$n_i(x) - n_i(y) = 2\mathcal{I}^- + 2\mathcal{I}^+ \leqslant 2\mathcal{I}^- + 2K \qquad \Longleftrightarrow \qquad \mathcal{I}^- = -K + n_i(x) - n_i(y) \geqslant -K - |n_i|_\infty$$

and $\mathcal{I}^-$ is bounded. The function $\frac{-v}{\left(v^2+2(\varphi(y)-\varphi(x))\right)^{1/2}} - 1$ converges pointwise to 0 when $x \to y$ **and f_i is almost everywhere finite**, and by Lebesgue's dominated convergence, $\mathcal{I}^- \underset{x\to y}{\longrightarrow} 0$. Then n_i is continuous. □

In the future:

- **Now that we have estimates, write it in function of α and β, and see how to get stability of the set of strongly concave functions satisfying all the hypotheses by the Poisson problem (essentially, find tweaks of λ, μ, ν, α and/or β such that the estimates are propagated).**

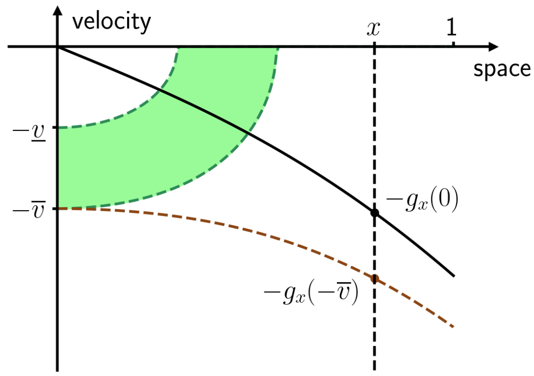

Figure 2: Notations for the lower bound on n_i.

The coloured area corresponds to the domain $\mathcal{L}_e(0,\underline{v}) \leqslant \mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,\overline{v})$, on which we know that $f_e \geqslant \underline{c}$. The solid black line is the critical ion characteristic.

- **If $f_{e,b}$ vanishes in a neighbourhood of $(0,0)$ (or decreases fast enough, see how fast), show that f_i is bounded.**
- **Under that same assumption, we should obtain stronger continuity over n_i, and n_i may be continuous with respect to φ (in the same sense as n_e in lemma (2.5)).**
- **If the estimates with respect to $|\varphi - \psi|_\infty$ succeed, define numerical scheme and see what we cas say about it.**

Lemma 2.8. *The density n_i is bounded away from 0 uniformly over $x \in [0,1]$.*

Demonstration By assumption, there exists a constant $\underline{c} > 0$ such that $f_{e,b}(v) \geqslant \underline{c}\mathbf{1}_{\{\underline{v} \leqslant |v| \leqslant \overline{v}\}}$, which implies

$$f_e(x,v) \geqslant \underline{c}\mathbf{1}_{\{\mathcal{L}_e(0,\underline{v}) \leqslant \mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,\overline{v})\}}.$$

Let $g_x : \mathbb{R}^- \mapsto \mathbb{R}^+$ be such that $\mathcal{L}_i(0,v) = \mathcal{L}_i(x, -g_x(v))$. As in (2.1), we define $n_i(x)$ by an integral over $v \in \mathbb{R}^- =]-\infty, -g_x(0)] \cup]-g_x(0), 0]$. Since we want an uniform lower bound, we will sacrify the (nonnegative) integral on $]-g_x(0), 0]$. Moreover,

$$w \leqslant -g_x(-\overline{v}) \quad \Longrightarrow \quad \mathcal{L}_i(0, v_b(x,w)) = \mathcal{L}_i(x,w) \geqslant \mathcal{L}_i(x, -g_x(-\overline{v})) = \mathcal{L}_i(0, -\overline{v}) = \mathcal{L}_e(0, \overline{v}),$$

and f_e vanishes identically along the ion characteristic crossing (x,w). Then, we may restrict the $w-$integral over the domain $[-g_x(-\overline{v}), -g_x(0)]$ for all x, and write

$$n_i(x) \geqslant \int_{w=-g_x(-\overline{v})}^{-g_x(0)} f_i(0, v_b(x,w))dw = \int_{v=-\overline{v}}^{0} f_i(0,v) \frac{-v}{\left(v^2 - 2\varphi(x)\right)^{1/2}} dv \geqslant \int_{v=-\overline{v}}^{0} f_i(0,v) \frac{-v}{\left(v^2 + 2\beta\right)^{1/2}} dv$$

where we used the change of variable $v = v_b(x,w) = -\left(w^2 + 2\varphi(x)\right)^{1/2}$, and the monotonicity $-\varphi(x) \leqslant -\varphi(1) \leqslant \beta$.

NOT FINISHED because I am not satisfied with my lower bound. $\square$

Lemma 2.9. *Suppose that there exists $v_* > 0$ such that $f_{e,b}(v) = 0$ for all $|v| \leqslant v_*$. Then the density f_i satisfies the uniform bound*

$$f_i(x,v) \leqslant 2^{9/4}\overline{c}\sqrt{\frac{\frac{\beta}{\alpha}(1 + \frac{1}{\mu}) + 1}{v_*}} \quad \forall (x,v) \in [0,1] \times \mathbb{R}.$$

Demonstration Let $(x,v)\in[0,1]\times\mathbb{R}_-$, and denote by $(x(t),v(t))_{t\leqslant 0}$ the ion characteristic going through $(x,v)=(x(-T),v(-T))$, with the convention $(x_b(x,v),v_b(x,v))=(x(0),v(0))$. Owing to the positivity of f_e,

$$f_i(x(0),v(0))=\int_{t=-T}^{0} f_e(x(t),v(t))dt+f_i(x,v)\geqslant f_i(x,v).$$

On the other hand, we use $f_i(x,v)+f_i(x,-v)=2f_i(x(0),v(0))$ to write $f_i(x,-v)\leqslant 2f_i(x(0),v(0))$. This shows that it is enough to bound f_i on the boundary $\mathcal{B}\coloneqq\{x\geqslant 0, v=0\}\cup\{x=0,v\leqslant 0\}$ to obtain an uniform bound.

Let then $(x,v)\in\mathcal{B}$. By hypothesis, $f_e(x(t),v(t))$ vanishes whenever $\mathcal{L}_e(x(t),v(t))\leqslant\mathcal{L}_e(0,v_*)$, and is bounded by $\overline{c}$ otherwise. Then, we may use the reparametrization by space

$$f_i(x,v)\leqslant\int_{z=a(x,v)}^{1}\frac{\overline{c}}{(v^2+2\varphi(x)-2\varphi(z))^{1/2}}dz,\quad\text{with}\quad a(x,v)\coloneqq\begin{cases}\varphi^{-1}\left(\frac{\frac{v^2}{2}+\varphi(x)-\frac{v_*^2}{2}}{1+1/\mu}\right) & \text{if } \mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,v_*)\\ x & \text{otherwise.}\end{cases}$$

The function a gives the smallest spatial coordinate of the characteristic $(x(t),v(t))_{t\leqslant 0}$ such that $f_e>0$. Using that $\varphi(z)\leqslant -\alpha\frac{z^2}{2}$, we have

$$f_i(x,v)\leqslant\overline{c}\int_{z=a(x,v)}^{1}\frac{1}{(v^2+2\varphi(x)-\alpha z^2)^{1/2}}dz\leqslant\overline{c}\min\left(\frac{1}{|v^2+2\varphi(x)|^{1/2}},\frac{2}{\sqrt{\alpha a(x,v)}}\right)$$

where we used lemma (2.3) with $L\coloneqq v^2+2\varphi(x)$.

We rely on the concavity estimate

$$\varphi(x)\geqslant(1-x)\varphi(0)+x\varphi(1)+\frac{\alpha}{2}x(1-x)\geqslant -x\beta\quad\Longrightarrow\quad -\frac{y}{\beta}\leqslant\varphi^{-1}(y)$$

to write that for (x,v) satisfying $\mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,v_*)$,

$$a(x,v)=\varphi^{-1}\left(\frac{\frac{v^2}{2}+\varphi(x)-\frac{v_*^2}{2}}{1+1/\mu}\right)\geqslant\frac{\frac{v_*^2}{2}-(\frac{v^2}{2}+\varphi(x))}{\beta(1+1/\mu)}.$$

Then

$$f_i(x,v)\leqslant\overline{c}\min\left(\frac{1}{\sqrt{|X|}},\frac{A}{\sqrt{B-X}}\right)\quad\text{with}\quad X\coloneqq\frac{v^2}{2}+\varphi(x),\quad A\coloneqq\frac{2}{\sqrt{\frac{\alpha}{\beta(1+1/\mu)}}},\quad\text{and}\quad B\coloneqq\frac{v_*^2}{2}.$$

The elementary study of the function $X\to\min\left(|X|^{-1/2},A(B-X)^{-1/2}\right)$ reveals a global maximum at $X=\frac{B}{A^2+1}<B$, and we conclude to the result.

$\square$

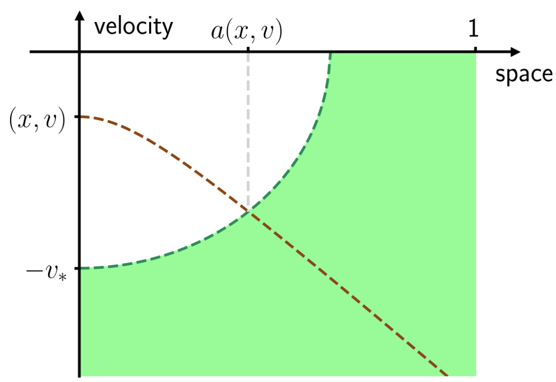

Figure 3: Notations for the boundedness of f_i.

The coloured area corresponds to $\mathcal{L}_e(x,v) \geqslant \mathcal{L}_e(0,v_*)$. The function $a(x,v)$ gives the point of the ion characteristic (in brown) where f_e vanishes.