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COLLISIONAL SHEATH SOLUTIONS OF A BI-SPECIES VLASOV-POISSON-BOLTZMANN BOUNDARY VALUE PROBLEM

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Abstract. The mathematical description of the interaction between a collisional plasma and an absorbing wall is a challenging issue. In this paper, we propose to model this interaction by considering a stationary bi-species Vlasov-Poisson-Boltzmann boundary value problem with boundary conditions that are consistent with the physics. In particular, we show that the wall potential can be uniquely determined from the ambipolarity of the particles flows as the unique solution of a nonlinear equation. We also prove that it is an increasing function of the electrons re-emission coefficient at the wall. Based on the Schauder fixed point theorem, our analysis establishes the existence of a solution provided, on the one hand that the incoming ions density satisfies a moment condition that generalizes the Historical Bohm criterion, and on the other hand that the collision frequency does not exceed a critical value whose definition is subordinated to the strict validity of our generalized Bohm criterion.

1. **Introduction.** The mathematical description of the interaction between a plasma and an absorbing wall is a challenging issue with many practical applications. Applications range from the design of laboratory plasma devices to the fundamental understanding of solar wind. One of the main physical feature of plasmas in contact with an isolated absorbing wall, is the development near the wall of a thin net positively charged layer. This layer of several Debye length in thickness is called a sheath [2, 1, 8, 10]. It results from the relative difference of mobility between electrons and ions. Indeed, electrons being much lighter than ions, they are prone to exit the plasma to the wall at a higher rate. As a consequence, a negative charge accumulates at the wall and leaves the surrounding plasma with a net positive charge. As this phenonemon alone would result in a growing positive charge in the core plasma, a mechanism of regulation settles. Namely, the accumulated negative charge at the wall causes the electric potential to drop and repel a significant fraction of the electrons. The magnitude of the drop is then such that the flow is ambipolar, in the sense that positive and negative charges exit the core plasma at the same rate. As a result, the plasma reaches an equilibrium.

Plasma sheaths have been the focus of many studies in the plasma physics literature [6, 11, 9, 5]. On the mathematical side sheaths are now better understood

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: existence and stability theories can be found in [12, 13, 14, 3]. A common observation that is supported both at the numerical level and the theoretical level is that the formation of a sheath requires the ions to be fast enough. This requirement is often expressed mathematically by an inequality satisfied by some moment in velocity of the ions distribution function. This inequality is called the Bohm criterion. When particles suffer collisions or friction effects, the role played by the Bohm criterion is unclear [15]. On the mathematical side, existence theories for kinetic models describing plasmas is by now well-established, see e.g [16, 7, 4, 12, 17, 18, 19, 20, 21, 22, 23, 24] for a non exhaustive list. We mention the work of Mischler [23] and the one of Jiang and Zhiang [24] which deal with the existence of renormalized solutions for the non stationary Vlasov-Poisson-Boltzmann system and the Boltzmann equation in bounded domain. The one dimensional stationnary Vlasov-Poisson-Boltzmann in bounded domain has been studied by Bostan and al in [18]. In the context of plasma sheaths, the aforementioned studies cannot be directly applied because the boundary condition at the wall on the electric potential is not prescribed a priori. The electric potential at the wall adjusts itself according to the accumulated flow of charges at the wall leading to a boundary condition that depends on the solution itself.

In this work, we consider a simplified model of bounded plasma interacting with a metallic wall where ions suffer collisions with a cold neutral gas. We consider a stationary one dimensional two species Vlasov-Poisson-Boltzmann system with a linear collision term for the ions which has the form of a relaxation operator towards a mono-kinetic distribution. This collision operator is also refered to as the Charge eXchange collision operator [15, 27, 28]. Boundary conditions are imposed to reflect the interaction with the wall : for particles densities, incoming flow of particles is considered at one side (the core plasma) while at the other side (the wall) partial absorption is considered. On the electrostatic potential, Dirichlet boundary conditions are considered and the wall potential is computed so as to ensure the ambipolarity of the flows.

To perform the mathematical analysis, our methodology is constructive and follows a standard approach: using the characteristics curves, we integrate the Vlasov-Boltzmann equations with respect to the electrostatic potential, and then consider a nonlinear Poisson equation that we formulate as a fixed point problem. An originality of our construction is the derivation of a weakly singular Volterra integral equation of the second kind satisfied by the ions macroscopic density [25, 26]. It results in a nonlinear Poisson problem that is an integro-differential equation.

The first of our two main results concerns the existence and the uniqueness of a wall potential under the condition that in the core plasma, the ions mean velocity should not exceed the electrons mean velocity. It is further shown that the wall potential is an increasing function of the electrons re-emission coefficient at the wall. This is performed by solving two algebraic equations : one is the mathematical expression of the neutrality in the core plasma (11) and the other one expresses the ambipolarity of the flow (12).

The second result is the existence of a solution for the Vlasov-Poisson-Boltzmann in the class of sufficiently decreasing potentials when a semi-Maxwellian electrons incoming particles density is considered. This result holds under two conditions : the first one is that the incoming particles density satisfies a kinetic version of the Bohm criterion. The second one is that the collision frequency does not exceed

a critical collision frequency. These conditions are sufficient to ensure a sufficient decrease of the electrostatic potential so that it yields a non negative charge density.

The plan of this paper is as follows. In section 2, we introduce the mathematical model based on the Vlasov-Poisson-Boltzmann equations and define mathematically the notion of sheath solutions that is considered in this work. We then precise, in section 3, the mathematical framework needed for the analysis and introduce our main result in Theorem 3.1. Section 4 is devoted to the analysis of the linear Vlasov-Boltzmann equations (1)-(2) when the potential ϕ is assumed to be fixed and decreasing. In section 5, it is shown that the wall potential is the unique solution of a nonlinear equation and that it is an increasing function of the fraction of re-emitted electrons from the wall. In section 6, we consider the nonlinear Poisson problem that results from the study of the linear Vlasov-Boltzmann equations and we prove that it has a decreasing and concave solution. Finally, the proof of the main result is given in Section 7. Some technical results needed for the analysis are in the Appendix A.

2. **The Vlasov-Boltzmann-Poisson system.** We consider a bounded plasma in contact with a planar absorbing wall. We use a kinetic approach to describe this plasma where particles positions and velocities are denoted (x, v) and belong to $[0, 1] \times \mathbb{R}$. The unknowns are particles densities in the phase space $f_i : (x, v) \in [0, 1] \times \mathbb{R} \mapsto f_i(x, v) \in \mathbb{R}^+$, $f_e : (x, v) \in [0, 1] \times \mathbb{R} \mapsto f_e(x, v) \in \mathbb{R}^+$, the electrostatic potential $\phi : x \in [0, 1] \mapsto \phi(x) \in \mathbb{R}$, the reference density $n_0 > 0$ and the wall potential $\phi_{\text{wall}} \in \mathbb{R}$. The unknowns f_i, f_e and ϕ are assumed to obey the Vlasov-Poisson-Boltzmann equations (1)-(3):

$$\begin{cases} v\partial_x f_i - \partial_x \phi \partial_v f_i = -\nu Q(f_i), \quad (x, v) \in (0, 1) \times \mathbb{R}, & (1) \\ v\partial_x f_e + \frac{1}{\mu}\partial_x \phi \partial_v f_e = 0, \qquad (x, v) \in (0, 1) \times \mathbb{R}, & (2) \end{cases}$$

$$- \varepsilon^2 \partial_{xx}\phi = n_i(x) - n_e(x), \quad x \in (0, 1), \tag{3}$$

where $\nu > 0$ is a normalized collision frequency between ions and neutrals, $\mu > 0$ denotes the mass ratio between electrons and ions, $\varepsilon > 0$ is a normalized Debye length, $Q(f_i)$ is a collision operator which here has the form of a linear relaxation operator towards a mono-kinetic distribution (see [22, 29, 18, 15] for more details),

$$\forall (x, v) \in (0, 1) \times \mathbb{R}, \quad Q(f_i)(x, v) := f_i(x, v) - \left(\int_{\mathbb{R}} f_i(x, v)dv\right)\delta_{v=0}, \tag{4}$$

where $\delta_{v=0}$ is the Dirac measure supported at the point $v = 0$. This collision operator preserves the mass :

$$\forall x \in (0, 1), \quad \int_{\mathbb{R}} Q(f_i)(x, v)dv = 0. \tag{5}$$

The macroscopic densities are defined by :

$$\forall x \in [0, 1] \quad n_i(x) := \int_{\mathbb{R}} f_i(x, v)dv, \quad n_e(x) := \int_{\mathbb{R}} f_e(x, v)dv. \tag{6}$$

Since the densities f_i and f_e solve the Vlasov-Boltzmann equations, and the collision operator $Q(f_i)$ preserves the mass, an integration in velocity yields that the current densities defined by

$$\forall x \in [0, 1] \quad J_i(x) := \int_{\mathbb{R}} f_i(x, v)vdv, \quad J_e(x) := \int_{\mathbb{R}} f_e(x, v)vdv, \tag{7}$$

are constant, namely for all $x \in [0,1]$, $J_i(x) \equiv J_i$ and $J_e(x) \equiv J_e$.

2.0.1. *Boundary conditions in the core plasma and at the wall.* The system (1)-(3) is supplemented with the following boundary conditions:

$$\begin{cases} f_i(0, v > 0) = f_i^{\text{inc}}(v), \qquad f_i(1, v < 0) = 0, & (8) \\ f_e(0, v > 0) = n_0 f_e^{\text{inc}}(v), \quad f_e(1, v < 0) = \alpha f_e(1, -v), & (9) \end{cases}$$

$$\phi(0) = 0, \quad \phi(1) = \phi_{\text{wall}}, \tag{10}$$

where $f_i^{\text{inc}} : (0, +\infty) \to \mathbb{R}^+$, $f_e^{\text{inc}} : (0, +\infty) \to \mathbb{R}^+$ stand for incoming particles densities that model the flows of particles that come from the plasma ($x = 0$). At the wall ($x = 1$), ions particles are absorbed while for the electrons a fraction $\alpha \in [0,1]$ of the particles is re-emitted from the wall specularly.

For an arbitrarily given pair $(n_0, \phi_{\text{wall}})$, the system (8)-(10) is a well defined set of boundary conditions. However, in this work the pair $(n_0, \phi_{\text{wall}})$ plays the role of an unknown. Therefore, it will be determined in such a way that the solutions f_i, f_e, ϕ to (1)-(10) satisfy the additional equations:

$$\begin{cases} n_i(0) = n_e(0), & (11) \\ J_i = J_e. & (12) \end{cases}$$

The first equation models the neutrality in the core plasma while the second one translates the ambipolarity of the particles flows [10]. This latter equation is necessary if one wants to model stationary solutions that are compatible with the Maxwell-Ampère equations.

2.0.2. *Sheath solutions for the Vlasov-Boltzmann-Poisson system.* We define what we call a sheath solution for the Vlasov-Poisson-Boltzman boundary value problem (1)-(12) accordingly to the plasma sheath physics [8, 2, 10].

Definition 2.1 (Sheath solutions)**.** We say that $(f_i, f_e, \phi, n_0, \phi_{\text{wall}}) \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R})^2 \times C^2[0,1] \times (0, +\infty) \times (-\infty, 0)$ is a sheath solution of the Vlasov-Poisson-Boltzmann system (1)-(12) iff

- f_i is a mild solution of the Vlasov-Boltzmann equation (1) in the sense of definition A.1.
- f_e is a mild solution of the Vlasov equation (2) in the sense of defintion A.2.
- $n_i, n_e \in C^0[0,1]$ and $n_i - n_e$ is non negative everywhere in [0,1].
- ϕ is a decreasing solution of the Poisson equation (3) with the boundary condition (10).
- The algebraic equations (11)-(12) hold.

3. **Main result.** For incoming particles densities, we introduce the set

$$\mathcal{P} := \{f \in (L^1 \cap L^\infty)(\mathbb{R}^+),\ v^2 f(v) \in L^1(\mathbb{R}^+),\ f \geq 0,\ f \neq 0\} \tag{13}$$

which is made of non zero, bounded, integrable and of finite kinetic energy particles densities. For $0 \leq \alpha \leq 1$ and $f_e^{\text{inc}} \in \mathcal{P}$, we will need the subset of $\mathcal{P}$ for incoming ions densities

$$\mathcal{I}(\alpha, f_e^{\text{inc}}) := \{f \in \mathcal{P}\ :\ \frac{\int_0^{+\infty} f(v) v\, dv}{\int_0^{+\infty} f(v)\, dv} < s_1(\alpha, f_e^{\text{inc}})\} \tag{14}$$

where

$$s_1(\alpha, f_e^{\text{inc}}) := \frac{(1-\alpha)}{(1+\alpha)} \frac{\int_0^{+\infty} f_e^{\text{inc}}(v) v\, dv}{\int_0^{+\infty} f_e^{\text{inc}}(v)\, dv}. \tag{15}$$

Given $f_e^{\text{inc}} \in \mathcal{P}$, $0 \leq \alpha < 1$ and $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ the algebraic equations (11) and (12) will be shown in sections 4 and 5 to be equivalent to the following system of unknown $(n_0, \phi_{\text{wall}}) \in (0, +\infty) \times (-\infty, 0)$,

$$\begin{cases} \int_0^{+\infty} f_i^{\text{inc}}(v)dv = n_0 \left(2 \int_0^{+\infty} f_e^{\text{inc}}(v)dv - (1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} f_e^{\text{inc}}(v)dv \right), \\ \int_0^{+\infty} f_i^{\text{inc}}(v)vdv = n_0(1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} f_e^{\text{inc}}(v)vdv. \end{cases} \tag{16}$$

As shown in Theorem 5.1, this system has a unique solution and the wall potential ϕ_{wall} is an increasing function of the parameter α.

To study the nonlinear Vlasov-Poisson-Boltzmann system (1)-(12), we will focus on an incoming electron density which is a normalized semi-Maxwellian,

$$f_e^{\text{inc}}(v) = \sqrt{\frac{2\mu}{\pi}} e^{-\frac{\mu v^2}{2}} \text{ for } v > 0. \tag{17}$$

It belongs to $\mathcal{P}$, satisfies $\int_0^{+\infty} f_e^{\text{inc}}(v)dv = 1$ and implies that $s_1(\alpha, f_e^{\text{inc}}) = \frac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\mu\pi}}$. We define the critical re-emission coefficient

$$\alpha_c := \frac{1 - \sqrt{\frac{\pi\mu}{2}}}{1 + \sqrt{\frac{\pi\mu}{2}}} \tag{18}$$

which is for all the physical mass ratio μ, such that $0 < \alpha_c < 1$ and $s_1(\alpha_c, f_e^{\text{inc}}) = 1$. For $0 \leq \alpha < 1$ and $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ the pair $(n_0, \phi_{\text{wall}})$ being uniquely defined by the system of equations (16), we define the function

$$\forall u \in [\phi_{\text{wall}}, 0], \quad m_\alpha(u) = 2 - (1-\alpha)\text{erfc}(\sqrt{u - \phi_{\text{wall}}}). \tag{19}$$

The main result is then the following.

Theorem 3.1. *Consider f_e^{inc} given by (17). Let $0 \leq \alpha \leq \alpha_c$, $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ so that the pair $(n_0, \phi_{\text{wall}})$ is uniquely defined by (16). Assume moreover the kinetic Bohm criterion,*

$$\frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v^2}dv}{\int_0^{+\infty} f_i^{\text{inc}}(v)dv} < \frac{m_\alpha(0) + m_\alpha'(0)}{m_\alpha(0)}, \tag{20}$$

and denote the Bohm number

$$B_\alpha(f_i^{\text{inc}}) := \frac{m_\alpha(0) + m_\alpha'(0)}{m_\alpha(0)} - \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v^2}dv}{\int_0^{+\infty} f_i^{\text{inc}}(v)dv} > 0. \tag{21}$$

Define the critical collision frequency

$$\nu^c := -\phi_{\text{wall}} B_\alpha(f_i^{\text{inc}}) \left(\frac{\int_0^{+\infty} f_i^{\text{inc}}(v)dv}{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v}dv} \right) > 0. \tag{22}$$

Then for all $0 < \nu < \nu^c$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \geq \varepsilon^*$ the Vlasov-Poisson-Boltzmann system (1)-(12) has a sheath solution $(f_i, f_e, \phi, n_0, \phi_{\text{wall}}) \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R})^2 \times C^2[0,1] \times (0,+\infty) \times (-\infty, 0)$ in the sense of definition 2.1. Moreover, the electrostatic potential ϕ is solution of the nonlinear Poisson problem $(\text{NLP})_{\nu,\varepsilon}$ (59) and satisfies the estimate:

$$\forall x \in [0,1], \quad \partial_x \phi(x) \leq -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \left(\frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v}dv}{\int_0^{+\infty} f_i^{\text{inc}}(v)dv} \right) < 0. \tag{23}$$

The proof of Theorem 3.1 is given in section 7. The strategy of the proof is somehow standard, we nevertheless sketch here after the main steps of the proof and provide additional comments about the technical hypothesis of the Theorem 3.1.

- The linear Vlasov-Boltzmman equations. We assume ϕ to be given and decreasing. Using the characteristics curves, we integrate the Vlasov-Boltzmann and Vlasov equations (1)-(2). For the ions, it yields a weakly singular Volterra integral equation (41) satisfied by the macroscopic ion density. It is proven to be well-posed in $C^0[0,1]$. For the electrons, the particles density being constant along the characteristics curves, we obtain an explicit expression of the particles density as function of the potential ϕ (54). We then denote $n_i[\phi]$, $n_e[\phi]$, J_i and J_e, the resulting macroscopic densities and currents.
- The neutrality constrain. Considering the densities currents $n_i[\phi]$, $n_e[\phi]$, J_i and J_e, we write down the algebraic equations (11)-(12). It yields the system (16) of unknown $(n_0, \phi_{\text{wall}})$. This system is proven in Theorem 5.1 to admit a unique solution in $(0, +\infty) \times (-\infty, 0)$ if and only if the boundary conditions f_i^{inc} and f_e^{inc} are related by an inequality on their first moment in velocity that is expressed by the fact that $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ for $0 \le \alpha < 1$.
- The fixed point procedure. We consider a mapping $T : \phi \mapsto T(\phi)$ where $T(\phi)$ is solution to the Poisson equation $-\varepsilon^2 \partial_{xx} T(\phi) = n_i[\phi] - n_e[\phi]$ with the boundary condition $T(\phi)(0) = 0$ and $T(\phi)(1) = \phi_{\text{wall}} < 0$ where n_0 and ϕ_{wall} are now determined. We then prove the existence of a fixed point for the mapping T by applying the Schauder fixed point theorem A.6. The main difficulty lies in the fact that one has to find a convex set $\mathcal{C}$ that is stable by the mapping T, namely $T(\mathcal{C}) \subset \mathcal{C}$, and such that for any function ϕ belonging to $\mathcal{C}$, the charge density $n_i[\phi] - n_e[\phi]$ is non negative. This is where the two technical assumptions, the kinetic Bohm criterion (20) and the fact that the collision frequency $0 < \nu < \nu^c$ cannot be too large, show up.
- Self-Consistency. We show that the above construction is not based on the empty set. Namely, we remind in Proposition 5 a result from [12] that states that the set of admissible incoming boundary conditions $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ that additionally verify the Bohm criterion (20) is not empty provided $0 \le \alpha \le \alpha_c$.

Let us finish this section with some comments on the range of applicability of this result.

- The critical re-emission parameter α_c is very close to 1 since μ is small. For example, in a Deuterium plasma $\mu = 1/3672$.
- The kinetic Bohm criterion (20) is exactly the same as in the work of Badsi, Campos-Pinto and Després in [12]. Since the upperbound $\frac{m_\alpha(0)+m'_\alpha(0)}{m_\alpha(0)} > 1$, it is weaker than the Historical Bohm criterion of plasma physics [8, 2, 1, 10] which writes $\frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v^2} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} < 1$. It implies that there is essentially no ions with null velocity at $x = 0$.
- The critical collision frequency (22) depends both on the wall potential $\phi_{\text{wall}} < 0$ and on the Bohm number $B_\alpha(f_i^{\text{inc}}) > 0$. It is up to our knowledge the first time that such a precise restriction ($\nu < \nu_c$) on the collision frequency is highlighted. The strict inequality for the Bohm criterion (20) is crucial in the definition of the critical collision frequency. The equality case in the inequality (20) yields $\nu^c = 0$ and our analysis fails in this case.

- We point out that the more restrictive condition is on the normalized Debye length $\varepsilon > 0$. Our theorem provides the existence of $\varepsilon^* > 0$ that depends obviously on $0 < \nu < \nu^c$ such that our system (1)-(12) has a solution. This restriction comes from the the fact that we look for a potential ϕ solution to the Poisson equation that is sufficiently decreasing to ensure the non negativity of the charge density (see Proposition 3). So far, we have not been able to overcome this restriction. We conjecture that some scaling assumption between $\varepsilon > 0$ and $\nu > 0$ is needed for this model to provide sheaths solutions, for arbitrarily small ε. We refer to [15] for a discussion on this subject. We stress that in the limit case $\nu = 0$, it was proven in [12] that no restriction is needed on $\varepsilon > 0$ provided the Bohm criterion (20) holds and it is possible to obtain quasi-neutrality estimates in the regime $\varepsilon \to 0$.

4. **The linear Vlasov-Boltzmann equations.** When the potential ϕ is known, the Vlasov-Boltzman equations (1)-(2) are linear transport equations with a linear collision term. Their solutions correspond to the transport of incoming particles densities by the characteristics with additional effects due to the collision term. We will need to study the characteristics. In this respect, for an arbitrarily given $\phi_{\text{wall}} < 0$ we define the set

$$W^{\phi_{\text{wall}}} := \{\psi \in W^{2,\infty}(0,1) \ : \ \partial_x \psi < 0 \text{ in } [0,1], \ \psi(0) = 0, \psi(1) = \phi_{\text{wall}}\}. \quad (24)$$

and for $\psi \in W^{\phi_{\text{wall}}}$, we denote

$$M_\psi := \sup_{x \in [0,1]} \partial_x \psi(x) < 0. \quad (25)$$

In this section, we show that given $\phi_{\text{wall}} < 0$ and $\phi \in W^{\phi_{\text{wall}}}$ there exists a unique mild solution (f_i, f_e) to (1)-(2) with the boundary conditions (8)-(9). Namely, for the ions, the integration of the equation (1) yields a weakly singular Volterra integral equation of the second kind (41) satisfied by the macroscopic density. We prove that this equation has a unique non negative solution in $C^0[0,1]$ and give some pointwise estimates on the solution. For the electrons, the integration of the equation (2) yields an explicit representation of the particles density as a function of the potential ϕ.

4.1. **Construction of a solution for the ions.** Let $\phi_{\text{wall}} < 0$ and $\phi \in W^{\phi_{\text{wall}}}$. The ionic characteristics curves are the solutions to

$$\begin{cases} X'(t,x,v) = V(t,x,v), \\ V'(t,x,v) = -\partial_x \phi(X(t,x,v)), \\ X(0,x,v) = x, \ V(0,x,v) = v. \end{cases} \quad (26)$$

where $x \in (0,1)$, $v \in \mathbb{R}$. The function $\partial_x \phi$ being Lipschitz-continuous, the Cauchy-Lipschitz theorem ensures that for each $(x,v) \in (0,1) \times \mathbb{R}$ there is a unique solution $(X,V) \in C^1[t_{\text{inc}}(x,v), t_{\text{out}}(x,v)]^2$ that satisfies (26) where

$$\begin{aligned} t_{\text{inc}}(x,v) &= \inf\{t \le 0 \ : \ X(t',x,v) \in (0,1) \ \forall t' \in (t,0)\}, \\ t_{\text{out}}(x,v) &= \sup\{t \ge 0 \ : \ X(t',x,v) \in (0,1) \ \forall t' \in (0,t)\}, \end{aligned} \quad (27)$$

denote the incoming and the exit time of the characteristic from the domain $(0,1)$. Since $\partial_x \phi < 0$ in $[0,1]$ these times are finite. The characteristics satisfy for all

$x \in (0,1)$, $v \in \mathbb{R}$ and $t \in [t_{\text{inc}}(x,v), t_{\text{out}(x,v)}]$,

$$\frac{V^2(t,x,v)}{2} + \phi(X(t,x,v)) = \frac{v^2}{2} + \phi(x). \tag{28}$$

Characteristics curves therefore belong to the level set of the microscopic energy $I(x,v) = \frac{v^2}{2} + \phi(x)$. Since $\partial_x \phi < 0$ in $[0,1]$ with $\phi(0) = 0$, one has $\phi(x) < 0$ for all $x \in (0,1)$. One therefore naturally splits the phase space has follows :

$$(0,1) \times \mathbb{R} = D_{i,0} \cup D_{i,1} \tag{29}$$

with

$$D_{i,0} := \{(x,v) \in (0,1) \times \mathbb{R} \ : \ v > \sqrt{-2\phi(x)}\}, \tag{30}$$

$$D_{i,1} := \{(x,v) \in (0,1) \times \mathbb{R} \ : \ v \le \sqrt{-2\phi(x)}\}; \tag{31}$$

This decomposition splits the phase space into two domains as sketched in figure 1:

- For $(x,v) \in D_{i,0}$, there is a characteristic that passes through (x,v) and originates at $x = 0$ with a positive velocity $v^0 = \sqrt{v^2 + 2\phi(x)}$.
- For $(x,v) \in D_{i,1} \setminus \{(x, \sqrt{-2\phi(x)}), \ x \in (0,1)\}$, there is a characteristic that passes through (x,v) and originate at $x = 1$ with a negative velocity $v^1 = -\sqrt{v^2 + 2(\phi(x) - \phi_{\text{wall}})}$.

Remark 1. The curve

$$S_i := \{(x, \sqrt{-2\phi(x)}), \ x \in (0,1)\} \tag{32}$$

is a singular set. Points on this curve originates at $x = 0$ with a null velocity.

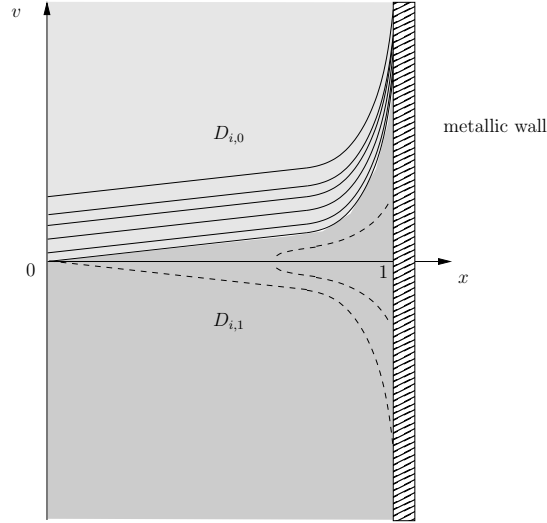FIGURE 1. Schematic characteristic ions trajectories associated with a decreasing potential ϕ. The solid lines corresponds to characteristic curves originating from $x = 0$ with positive velocities, and they span $D_{i,0}$ the lighter gray region. The dashed lines correspond to characteristic curves originating from the wall with negative velocities, and they span the darker gray region $D_{i,1}$.

Assuming f_i is a smooth enough solution to the Vlasov-Boltzmann equation (1), using the characteristics curves (26) one has for all $(x, v) \in (0, 1) \times \mathbb{R}$ and $t \in [t_{\mathrm{inc}}(x, v), 0]$:

$$\frac{d}{dt} f_i(X(t, x, v), V(t, x, v)) = -\nu Q(f_i)(X(t, x, v), V(t, x, v)) \tag{33}$$

where the linear collision term writes

$$Q(f_i)(X(t, x, v), V(t, x, v)) = f_i(X(t, x, v), V(t, x, v)) - n_i(X(t, x, v))\delta_{V(t,x,v)=0}.$$

Using the Duhamel's principle, we are going to integrate the equation (33) for $t \in [t^{\mathrm{inc}}(x, v), 0]$. One thus needs to give a precise meaning of the measure $\delta_{V(t,x,v)=0}$. To do so, we note that $\phi \in W^{2,\infty}(0, 1)$ with $\partial_x \phi < 0$ in $[0, 1]$. Therefore for any $(x, v) \in (0, 1) \times \mathbb{R}$, the function $t \mapsto V(t, x, v)$ is C^1 and increasing. One can thus define without ambiguity the measure $\delta_{V(t,x,v)=0}$ by composition as follows.

Definition 4.1 (Cancelation time and Dirac measure). Let $(x, v) \in (0, 1) \times \mathbb{R}$. Define the set

$$\mathcal{S}(x, v) := \{s \in [t_{\mathrm{inc}}(x, v), 0] : V(s, x, v) = 0\} \tag{34}$$

which is either empty or made of one element called the cancelation time and denoted $s_0(x, v)$. The measure $\delta_{V(t,x,v)=0}$ is defined by

$$\delta_{V(t,x,v)=0} := \begin{cases} \frac{1}{V'(s_0(x,v),x,v)} \delta_{t=s_0(x,v)} & \text{if } \mathcal{S}(x, v) \neq \emptyset, \\ 0 & \text{if } \mathcal{S}(x, v) = \emptyset, \end{cases} \tag{35}$$

where $\delta_{t=s_0(x,v)}$ denotes the usual Dirac measure in time supported at $\{s_0(x, v)\}$.

We will now compute the set $\mathcal{S}(x, v)$ (34). It is easily seen that it is empty for $(x, v) \in D_{i,0}$. For $(x, v) \in D_{i,1} \setminus S_i$ there is two cases :

- It has one element if the initial velocity is non negative.
- It is empty if if the initial velocity is negative.

Indeed, one has the following.

Lemma 4.2. *Let $\phi_{\mathrm{wall}} < 0$, $\phi \in W^{\phi_{\mathrm{wall}}}$ and let $(x, v) \in D_{i,1} \setminus S_i$. There exists a unique cancelation time $s_0(x, v) \in (t_{\mathrm{inc}}(x, v), 0)$ i.e, such that $V(s_0(x, v), x, v) = 0$ if and only if $v > 0$.*

Proof. Let $(x, v) \in D_{i,1} \setminus S_i$. For all $s \in [t_{\mathrm{inc}}(x, v), 0)$ one has

$$V(s, x, v) = v + \int_s^0 \partial_x \phi(X(t, x, v)) dt.$$

Since $\partial_x \phi < 0$ in $[0, 1]$ one has obviously $V(s, x, v) < v$ and it yields the necessary condition. We now prove the sufficient condition. So assume $v > 0$, and let us prove that $V(t_{\mathrm{inc}}(x, v), x, v) < 0$. Firstly remark that the function ϕ being decreasing there exists a unique $0 < x^* < x$ such that $\phi(x^*) = \phi(x) + \frac{v^2}{2}$. Therefore using the invariant one deduces

$$\frac{V^2(t_{\mathrm{inc}}(x, v), x, v)}{2} + \phi(X(t_{\mathrm{inc}}(x, v), x, v)) = \phi(x^*).$$

So it yields $\phi(x^*) \geq \phi(X(t_{\mathrm{inc}}(x, v), x, v))$ and by monotonicity of ϕ, we have $X(t_{\mathrm{inc}}(x, v), x, v) \geq x^*$. Since $x^* > 0$ it yields necessarily by definition of the incoming time that $X(t_{\mathrm{inc}}(x, v), x, v) = 1$ and thus that $V(t_{\mathrm{inc}}(x, v), x, v) < 0$. By continuity of the function $s \in (t_{\mathrm{inc}}(x, v), 0) \mapsto V(s, x, v)$ there exists $s_0(x, v) \in$

$(t_{\mathrm{inc}}(x,v),0)$ such that $V(s_0(x,v),x,v) = 0$. Its uniqueness follows from the fact that the function $s \in (t_{\mathrm{inc}}(x,v),0) \mapsto V(s,x,v)$ is increasing. $\square$

Using the previous Lemma (4.2), one therefore has

$$
\mathcal{S}(x,v) = \begin{cases} \emptyset & \text{if } (x,v) \in D_{i,0}, \\ \{s_0(x,v)\} & \text{if } (x,v) \in D_{i,1} \setminus S_i \text{ and } v > 0, \\ \{0\} & \text{if } (x,v) \in D_{i,1} \setminus S_i \text{ and } v = 0, \\ \emptyset & \text{if } (x,v) \in D_{i,1} \setminus S_i \text{ and } v < 0. \end{cases}
$$

where $s_0(x,v)$ is the cancelation time of Lemma 4.2. Using the Definition 4.1 and the Duhamel's principle to integrate the differential equation (33) on $[t_{\mathrm{inc}}(x,v),0]$, one obtains

$$
f_i(x,v) = \begin{cases} f_i^{\mathrm{inc}}(\sqrt{v^2 + 2\phi(x)})e^{\nu t_{\mathrm{inc}}(x,v)} & \text{if } (x,v) \in D_{i,0}, \\ \dfrac{-\nu e^{\nu s_0(x,v)} n_i(X(s_0(x,v),x,v))}{\partial_x \phi(X(s^0(x,v),x,v))} & \text{if } (x,v) \in D_{i,1} \setminus S_i \text{ and } v > 0, \\ 0 & \text{if } (x,v) \in D_{i,1} \setminus S_i \text{ and } v < 0. \end{cases}
\tag{36}
$$

Remark 2. Note that f_i given by (36) is defined almost everywhere except on the two sets S_i and $(D_{i,1} \setminus S_i) \cap \{v = 0\}$ which are of measure zero.

In the subsequent analysis, it will be useful to derive explicit formulas for both the incoming time $t_{\mathrm{inc}}(x,v)$ and the cancelation time $s_0(x,v)$. Using an implicit function theorem, one can prove the following.

Proposition 1. *Let $\phi_{\mathrm{wall}} < 0$ and $\phi \in W^{\phi_{\mathrm{wall}}}$. Then one has:*

$$
\forall (x,v) \in D_{i,0}, \quad t_{\mathrm{inc}}(x,v) = -\int_0^x \frac{du}{\sqrt{v^2 + 2(\phi(x) - \phi(u))}}.
\tag{37}
$$

$$
\forall (x,v) \in (D_{i,1} \setminus S_i) \cap \{v > 0\},
$$

$$
s_0(x,v) = -\int_{x_0(x,v)}^x \frac{du}{\sqrt{2(\phi(x_0(x,v)) - \phi(u))}},
\tag{38}
$$

with

$$
x_0(x,v) = \phi^{-1}(\phi(x) + \frac{v^2}{2})
\tag{39}
$$

where $\phi^{-1} : [\phi_{\mathrm{wall}}, 0] \to [0,1]$ denotes the inverse of ϕ. Moreover, $x_0(x,v)$ verifies

$$
X(s_0(x,v),x,v) = x_0(x,v).
\tag{40}
$$

In view of the formula (36), one sees that to end up with the construction of the solution f_i one has to provide a closed equation for n_i. In this regard, an integration in velocity of (36) yields the following.

Proposition 2. *Let $\phi_{\mathrm{wall}} < 0$, $\phi \in W^{\phi_{\mathrm{wall}}}$, $f_i^{\mathrm{inc}} \in \mathcal{P}$, $\nu > 0$ and f_i be defined by (36). Then the macroscopic ionic density satisfies the weakly singular Volterra integral equation of the second kind :*

$$
\forall x \in (0,1), \quad n_i(x) = \int_0^{+\infty} \frac{f_i^{\mathrm{inc}}(w)w}{\sqrt{w^2 - 2\phi(x)}} e^{\nu t_{\mathrm{inc}}(x, \sqrt{w^2 - 2\phi(x)})} dw
$$

$$
+ \nu \int_0^x \frac{e^{\nu s_0\left(x, \sqrt{2(\phi(\hat{x}) - \phi(x))}\right)} n_i(\hat{x})}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x}.
\tag{41}
$$

Proof. Integrating in velocity f_i given by (36), we decompose the integral for $x \in (0, 1)$ as follows:

$$n_i(x) = \int_{\mathbb{R}} f_i(x, v) dv = \int_{\sqrt{-2\phi(x)}}^{+\infty} f_i^{\mathrm{inc}}(\sqrt{v^2 + 2\phi(x)}) e^{\nu t_{\mathrm{inc}}(x,v)} dv$$

$$- \int_0^{\sqrt{-2\phi(x)}} \frac{\nu e^{\nu s_0(x,v)} n_i(X(s_0(x,v),x,v))}{\partial_x \phi(X(s_0(x,v),x,v))} dv.$$

We then use we the change of variable $w = \sqrt{v^2 + 2\phi(x)}$ for the first integral. For the second integral, we notice that thanks to Proposition (1) $X(s_0(x,v),x,v) = x_0(x,v) = \phi^{-1}(\phi(x) + \frac{v^2}{2})$ and then use the change of variable $\hat{x} = x_0(x,v)$. $\square$

The integral equation (41) shows up a singularity at the upper bound $\hat{x} = x$ of the second integral. Since $\phi \in W^{\phi_{\mathrm{wall}}}$, one has for $0 < \hat{x} \le x < 1$, $\phi(\hat{x}) - \phi(x) = -\partial_x \phi(\hat{x})(x - \hat{x}) + \mathrm{o}(x - \hat{x})$ with $\partial_x \phi(\hat{x}) < 0$. The singularity is therefore integrable and the formula (41) makes sense everywhere in $(0, 1)$. To solve the integral equation (41), we define the following linear operator:

$$\forall \rho \in C^0[0,1], \forall x \in [0,1], \quad \mathcal{V}_\phi^\nu(\rho)(x) := \int_0^x \frac{K_\phi^\nu(x,\hat{x})}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} \rho(\hat{x}) d\hat{x}, \qquad (42)$$

where K_ϕ^ν is defined on the triangle

$$T := \{(x, \hat{x}) \in [0,1] \times [0,1] : \hat{x} \le x\} \qquad (43)$$

by

$$\forall (x, \hat{x}) \in T, \quad K_\phi^\nu(x,\hat{x}) = e^{\nu s_0\left(x, \sqrt{2(\phi(\hat{x}) - \phi(x))}\right)}. \qquad (44)$$

It satisfies $K_\phi^\nu \in C^0(T)$ and since $s_0\left(x, \sqrt{2(\phi(\hat{x}) - \phi(x))}\right) \le 0$, it is such that $\sup_{(x,\hat{x}) \in T} |K_\phi^\nu(x,\hat{x})| \le 1$. The singular part $(x, \hat{x}) \in (T \setminus \{(x,x) : x \in [0,1]\}) \mapsto \frac{1}{\sqrt{2(\phi(\hat{x}) - \phi(x))}}$ belongs to $L^1(T)$.

The integral equation (41) equivalently reformulates:

$$n_i - \nu \mathcal{V}_\phi^\nu(n_i) = b_\phi^\nu, \qquad (45)$$

where

$$\forall x \in [0,1] \quad b_\phi^\nu(x) = \int_0^{+\infty} \frac{f_i^{\mathrm{inc}}(w) w}{\sqrt{w^2 - 2\phi(x)}} e^{\nu t_{\mathrm{inc}}(x, \sqrt{w^2 - 2\phi(x)})} dw. \qquad (46)$$

The two main results of this section are the following.

Theorem 4.3. *Let $\phi_{\mathrm{wall}} < 0$, $\phi \in W^{\phi_{\mathrm{wall}}}$, $f_i^{\mathrm{inc}} \in \mathcal{P}$ and $\nu > 0$. Then the integral equation (45) has a unique solution $n_i \in C^0[0,1]$. Moreover, it is such that :*

a) n_i is non negative and satisfies for all $x \in [0,1]$ $n_i(x) \ge b_\phi^\nu(x)$.

b) There exists $\tau > 0$ such that for all $x \in [0,1]$,

$$n_i(x) \le 2 \int_0^{+\infty} f_i^{\mathrm{inc}}(w) dw e^{-\tau \phi(x)}. \qquad (47)$$

Theorem 4.4. *Let $\phi_{\mathrm{wall}} < 0$, $\phi \in W^{\phi_{\mathrm{wall}}}$, $f_i^{\mathrm{inc}} \in \mathcal{P}$ and $\nu > 0$. Then f_i defined by (36) where n_i is the solution to (41) is the unique mild solution to the linear Vlasov-Boltzmann equation (1). Moreover, the current density is given by*

$$\forall x \in (0, 1), \quad J_i(x) = \int_0^{+\infty} f_i^{\mathrm{inc}}(v) v dv. \qquad (48)$$

The proof of the theorem 4.4 relies on our construction by the characteristics that we use to define f_i (36). The key point is to prove that the weakly singular Volterra integral equation (45) is well-posed in $C^0[0,1]$. We thus only give the proof of the Theorem 4.3.

Proof theorem (4.3). The proof is in three steps.

Existence and uniqueness of a solution. Thanks to Lemma A.4, there exists $\tau > 0$ such that for all $x \in [0,1]$,

$$e^{\tau\phi(x)}\int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x})-\phi(x))}}d\hat{x} \leq \frac{1}{2\nu}.$$

Consider the Banach space $X = C^0[0,1]$ endowed with the norm defined for all $\rho \in C^0[0,1]$ by $\|\rho\|_{X,\tau} := \sup_{x\in[0,1]} e^{\tau\phi(x)}|\rho(x)|$. Thanks to Lemma A.3, the linear operator $\mathcal{V}_\phi^\nu$ is well defined on X. Let us show that the operator $\nu\mathcal{V}_\phi^\nu$ is contractant for the norm $\|\cdot\|_{X,\tau}$. One has for all $\rho \in X$ and $x \in [0,1]$,

$$e^{\tau\phi(x)}\mathcal{V}_\phi^\nu(\rho)(x) = e^{\tau\phi(x)}\int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x})-\phi(x))}}K_\phi^\nu(x,\hat{x})e^{\tau\phi(\hat{x})}\rho(\hat{x})d\hat{x}$$

Since $\sup_{(x,\hat{x})\in T} |K_\phi^\nu(x,\hat{x})| \leq 1$, we obtain thanks to Lemma A.4,

$$e^{\tau\phi(x)}\left|\mathcal{V}_\phi^\nu(\rho)(x)\right| \leq e^{\tau\phi(x)}\int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x})-\phi(x))}}d\hat{x}\|\rho\|_{X,\tau} \leq \frac{1}{2\nu}\|\rho\|_{X,\tau}.$$

We therefore deduce $\|\nu\mathcal{V}_\phi^\nu(\rho)(x)\|_{X,\tau} < \frac{1}{2}\|\rho\|_{X,\tau}$. The operator $\mathbf{I}_d - \nu\mathcal{V}_\phi^\nu$ is thus invertible with inverse $(\mathbf{I}_d - \nu\mathcal{V}_\phi^\nu)^{-1} = \sum_{k=0}^{+\infty}\nu^k\left(\mathcal{V}_\phi^\nu\right)^k$ which is a convergent serie in $\mathcal{L}(X)$ for the subordinate norm $\|\cdot\|_{X,\tau}$. The solution is given by $n_i(x) = \sum_{k=0}^{+\infty}\nu^k\left(\mathcal{V}_\phi^\nu\right)^k(b_\phi^\nu)(x)$.

Proof of a). One has $n_i = b_\phi^\nu + \sum_{k=1}^{+\infty}\nu^k\left(\mathcal{V}_\phi^\nu\right)^k(b_\phi^\nu)$. Since $\mathcal{V}_\phi^\nu$ is a non negative operator and b_ϕ^ν is non negative we deduce $n_i \geq b_\phi^\nu$.

Proof of b). One has

$$\|n_i\|_{X,\tau} \leq \|b_\phi^\nu\|_{X,\tau}\sum_{k=0}^{+\infty}\|\nu\mathcal{V}_\phi^\nu\|_{\mathcal{L}(X)}^k \leq \|b_\phi^\nu\|_{X,\tau}\sum_{k=0}^{+\infty}\frac{1}{2^k}.$$

Therefore $\|n_i\|_{X,\tau} \leq 2\|b_\phi^\nu\|_{X,\tau}$. One has moreover,

$$b_\phi^\nu(x) = \int_0^{+\infty} \frac{f_i^{\mathrm{inc}}(w)w}{\sqrt{w^2-2\phi(x)}}e^{\nu t_{\mathrm{inc}}(x,\sqrt{w^2-2\phi(x)})}dw.$$

Since for all $x \in [0,1]$, $t_{\mathrm{inc}}(x,\sqrt{w^2-2\phi(x)}) \leq 0$ and ϕ is non positive, one obtains

$$b_\phi^\nu(x) \leq \int_0^{+\infty} f_i^{\mathrm{inc}}(w)dw.$$

It yields that $\|b_\phi^\nu\|_{X,\tau} \leq \int_0^{+\infty} f_i^{\mathrm{inc}}(w)dw$. We eventually obtain for all $x \in [0,1]$,

$$n_i(x) = e^{-\tau\phi(x)}e^{\tau\phi(x)}n_i(x) \leq e^{-\tau\phi(x)}\|n_i\|_{X,\tau} \leq 2e^{-\tau\phi(x)}\|b_\phi^\nu\|_{X,\tau},$$

and thus $n_i(x) \leq 2e^{-\tau\phi(x)} \int_0^{+\infty} f_i^{\text{inc}}(w)dw$. $\qquad\square$

4.2. **Construction of a solution for the electrons.** Let $\phi_{\text{wall}} < 0$ and $\phi \in W^{\phi_{\text{wall}}}$. The electronic characteristics curves are the solutions to

$$\begin{cases} X'(t,x,v) = V(t,x,v), \\ V'(t,x,v) = \frac{1}{\mu}\partial_x\phi(X(t,x,v)), \\ X(0,x,v) = x, \ V(0,x,v) = v. \end{cases} \tag{49}$$

where $x \in (0,1)$, $v \in \mathbb{R}$. The function $\partial_x\phi$ being Lipschitz-continuous, the Cauchy-Lipschitz theorem ensures that for each $(x,v) \in (0,1) \times \mathbb{R}$ there is a unique solution $(X,V) \in C^1[t_{\text{inc}}(x,v), t_{\text{out}}(x,v)]^2$ that satisfies (49) where

$$\begin{aligned} t_{\text{inc}}(x,v) &= \inf\{t \leq 0 : X(t',x,v) \in (0,1) \ \forall t' \in (t,0)\}, \\ t_{\text{out}}(x,v) &= \sup\{t \geq 0 : X(t',x,v) \in (0,1) \ \forall t' \in (0,t)\}, \end{aligned} \tag{50}$$

denote the incoming and the exit time of the characteristic from the domain $(0,1)$. Since $\partial_x\phi < 0$ in $[0,1]$ these times are finite. The characteristics satisfy for all $x \in (0,1)$, $v \in \mathbb{R}$ and $t \in [t_{\text{in}}(x,v), t_{\text{out}(x,v)}]$,

$$\frac{V^2(t,x,v)}{2} - \frac{1}{\mu}\phi(X(t,x,v)) = \frac{v^2}{2} - \frac{1}{\mu}\phi(x).$$

Characteristics curves therefore belong to the level set of the microscopic energy $E(x,v) = \frac{v^2}{2} - \frac{1}{\mu}\phi(x)$. Since $\partial_x\phi < 0$ in $[0,1]$ with $\phi(0) = 0$, one has $\phi_{\text{wall}} < \phi(x) < 0$ for all $0 < x < 1$. One therefore naturally splits the phase space has follows :

$$(0,1) \times \mathbb{R} = D_{e,0} \cup D_{e,1}$$

with

$$D_{e,0} := \{(x,v) \in (0,1) \times \mathbb{R} : v > -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_{\text{wall}})}\}, \tag{51}$$

$$D_{e,1} := \{(x,v) \in (0,1) \times \mathbb{R} : v \leq -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_{\text{wall}})}\}. \tag{52}$$

This decomposition splits the phase space into two domains as sketched in figure 2:

- For $(x,v) \in D_{e,0}$ there is a characteristic that passes through (x,v) and originates at $x = 0$ with a positive velocity $v^0 = \sqrt{v^2 - \frac{2}{\mu}\phi(x)}$.

- For $(x,v) \in D_{e,1} \setminus \{(x, -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_{\text{wall}})}), x \in (0,1)\}$, there is a characteristic that passes through (x,v) and originate at $x = 1$ with a negative velocity $v^1 = -\sqrt{v^2 - \frac{2}{\mu}(\phi(x) - \phi_{\text{wall}})}$.

Remark 3. The curve

$$S_e := \{(x, -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_{\text{wall}})}), x \in (0,1)\} \tag{53}$$

is a singular set. Points on this curve originates at $x = 1$ with a null velocity.

Using the fact f_e is constant along the characteristics curves and following [12], we define:

$$f_e(x,v) = n^0 \begin{cases} f_e^{\text{inc}}(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}) & \text{if } (x,v) \in D_{e,0}, \\ \alpha f_e^{\text{inc}}(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}) & \text{if } (x,v) \in D_{e,1} \setminus S_e. \end{cases} \tag{54}$$
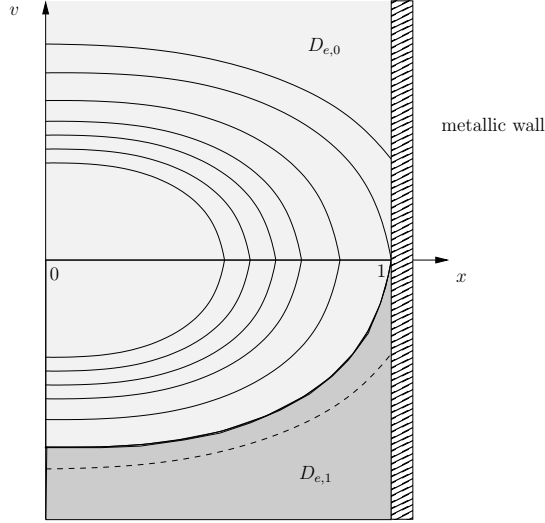
One has then the following, we refer to [12] for a proof.

FIGURE 2. Schematic characteristic electrons trajectories associated with a decreasing potential ϕ. The solid lines corresponds to characteristic curves originating from $x = 0$ with positive velocities, and they span $D_{e,0}$ the lighter gray region. The dashed lines correspond to characteristic curves originating from the wall with negative velocities, and they span the darker gray region $D_{e,1}$.

Theorem 4.5. *Let $\phi_{\text{wall}} < 0$, $\phi \in W^{\phi_{\text{wall}}}$, $f_e^{\text{inc}} \in \mathcal{P}$ and f_e be defined by* (54). *Then f_e defined by* (54) *is the unique mild solution to* (2). *Moreover one has for all $x \in (0,1)$:*

$$n_e(x) = 2n_0 \int_{\sqrt{-\frac{2}{\mu}\phi(x)}}^{+\infty} \frac{f_e^{\text{inc}}(w)w}{\sqrt{w^2 + \frac{2}{\mu}\phi(x)}} dw$$

$$- n_0(1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} \frac{f_e^{\text{inc}}(w)w}{\sqrt{w^2 + \frac{2}{\mu}\phi(x)}} dw. \tag{55}$$

$$J_e(x) = n_0(1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} f_e^{\text{inc}}(w)w\,dw. \tag{56}$$

5. **Determination of the pair** $(n_0, \phi_{\text{wall}})$. Considering the form of n_i, J_i, n_e, J_e given by (41),(48),(55) and (56), the equations (11)-(12) yields the system

$$\begin{cases} \int_0^{+\infty} f_i^{\text{inc}}(v)dv = n_0 \left(2 \int_0^{+\infty} f_e^{\text{inc}}(v)dv - (1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} f_e^{\text{inc}}(v)dv \right), \\ \int_0^{+\infty} f_i^{\text{inc}}(v)v\,dv = n_0(1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_{\text{wall}}}}^{+\infty} f_e^{\text{inc}}(v)v\,dv. \end{cases}$$

The unknowns are $n_0 > 0$ and $\phi_{\text{wall}} \leq 0$. We notice that the case $\alpha = 1$ yields a degenerate system. Physically, it means that all electrons are re-emitted from the wall. We thus consider $0 \leq \alpha < 1$. Eliminating n_0 from the first equation and substituting it in the second yields a nonlinear equation $\mathcal{W}_\alpha(\phi_{\text{wall}}) = 0$ where $\mathcal{W}_\alpha$

is the function defined for $t \leq 0$ by:

$$
\begin{aligned}
\mathcal{W}_\alpha(t) = & \int_0^{+\infty} f_i^{\mathrm{inc}}(v)vdv \left(2 \int_0^{+\infty} f_e^{\mathrm{inc}}(v)dv - (1-\alpha) \int_{\sqrt{-\frac{2}{\mu}t}}^{+\infty} f_e^{\mathrm{inc}}(v)dv \right) \\
& - (1-\alpha) \int_0^{+\infty} f_i^{\mathrm{inc}}(v)dv \int_{\sqrt{-\frac{2}{\mu}t}}^{+\infty} f_e^{\mathrm{inc}}(v)vdv.
\end{aligned}
\tag{57}
$$

One has then following.

Theorem 5.1. *Let $f_i^{\mathrm{inc}}, f_e^{\mathrm{inc}} \in \mathcal{P}$ and $0 \leq \alpha < 1$. Then :*

a) *The function $\mathcal{W}_\alpha$ defined by (57) has a unique non positive root if and only if*

$$
\frac{\int_0^{+\infty} f_i^{\mathrm{inc}}(v)vdv}{\int_0^{+\infty} f_i^{\mathrm{inc}}(v)dv} \leq \frac{(1-\alpha)}{(1+\alpha)} \frac{\int_0^{+\infty} f_e^{\mathrm{inc}}(v)vdv}{\int_0^{+\infty} f_e^{\mathrm{inc}}(v)dv}.
\tag{58}
$$

b) *The function $\alpha \in [0,1) \mapsto \phi_{\mathrm{wall}}(\alpha) \in (-\infty, 0)$ where $\phi_{\mathrm{wall}}(\alpha)$ is the unique root of $\mathcal{W}_\alpha$ (57) is increasing.*

Proof. a) The function $\mathcal{W}_\alpha$ is continuous on $(-\infty, 0]$ and decreasing. It has a limit as $t \to -\infty$ which is such that $\lim_{t \to -\infty} \mathcal{W}_\alpha(t) > 0$. Therefore $\mathcal{W}_\alpha$ has a unique non positive root if and only if $\mathcal{W}_\alpha(0) \leq 0$ which yields the inequality (58).

b) We define the function $(\alpha, t) \in [0,1) \times \mathbb{R}^- \mapsto R(\alpha, t) = W_\alpha(t)$. For all $t \geq 0$, $R(\cdot, t)$ is increasing with respect to α, and for all $\alpha \in [0,1)$, $R(\alpha, \cdot)$ is decreasing with respect to t. Let $0 \leq \alpha_1 < \alpha_2 < 1$ and consider $\phi_{\mathrm{wall}}(\alpha_1)$ and $\phi_{\mathrm{wall}}(\alpha_2)$ such that $R(\alpha_1, \phi_{\mathrm{wall}}(\alpha_1)) = R(\alpha_2, \phi_{\mathrm{wall}}(\alpha_2)) = 0$. Since for all $t \leq 0$, $R(\alpha_1, t) < R(\alpha_2, t)$, one has necessarily $\phi_{\mathrm{wall}}(\alpha_1) \neq \phi_{\mathrm{wall}}(\alpha_2)$. Let us now show that $\phi_{\mathrm{wall}}(\alpha_1) < \phi_{\mathrm{wall}}(\alpha_2)$. We argue by contradiction and assume that $\phi_{\mathrm{wall}}(\alpha_1) > \phi_{\mathrm{wall}}(\alpha_2)$. Since $R(\alpha_1, \cdot)$ is decreasing, one has on the one hand

$$
0 = R(\alpha_1, \phi_{\mathrm{wall}}(\alpha_1)) < R(\alpha_1, \phi_{\mathrm{wall}}(\alpha_2)).
$$

And on the other hand since $R(\cdot, \phi_{\mathrm{wall}}(\alpha_2))$ is increasing that

$$
R(\alpha_1, \phi_{\mathrm{wall}}(\alpha_2)) < R(\alpha_2, \phi_{\mathrm{wall}}(\alpha_2)) = 0.
$$

Therefore $R(\alpha_1, \phi_{\mathrm{wall}}(\alpha_2))$ is both positive and negative which yields a contradiction. $\square$

Remark 4. The case of equality in the inequality (58) yields $\phi_{\mathrm{wall}}(\alpha) = 0$.

6. The nonlinear Poisson problem with a semi-Maxwellian incoming electron density.

In this section, we study the existence of solutions for the nonlinear Poisson equation (3) where the density n_i and n_e result from the construction of Section 4. Moreover, we consider the specific case of an incoming electrons particles density f_e^{inc} which is a semi Maxwellian (17). The problem is as follows:

Let f_e^{inc} be given by (17).
Let $0 \leq \alpha < 1$ and $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ so that the pair
$(n_0, \phi_{\mathrm{wall}}) \in (0, +\infty) \times (-\infty, 0)$ is uniquely defined by (16).
Let $\nu > 0$ and $\varepsilon > 0$. We consider the nonlinear Poisson problem:

$$(\mathrm{NLP})_{\nu,\varepsilon} : \begin{cases} \text{Find } \phi \in W^{\phi_{\mathrm{wall}}} \text{ a concave function such that ,} \\ -\varepsilon^2 \partial_{xx} \phi(x) = n_i[\phi](x) - n_e[\phi](x), \ x \in (0,1), \end{cases} \tag{59}$$

where $n_i[\phi] \in C^0[0,1]$ is the solution of the integral equation (41),
and $n_e[\phi]$ is now given by the so called truncated Boltzmann density [10]:

$$\forall x \in [0,1], \quad n_e[\phi](x) = n_0 e^{\phi(x)} m_\alpha(\phi(x)), \tag{60}$$

where the function m_α is defined by (19).

To prove the existence of a concave and decreasing potential ϕ to (59), we will apply the Schauder fixed point theorem A.6. In this respect, we need a priori estimates on the charge density $n_i[\phi] - n_e[\phi]$ that are uniform in ϕ. The keystone is the kinetic Bohm inequality (20) combined with the estimate (23). It enables us to bound by below the incoming time (37) one the one hand and to prove that for not too large $\nu > 0$, the solutions to (59) with not too small $\varepsilon > 0$ are concave and decreasing on the other hand.

6.1. **A priori estimates.** First of all, since the pair $(n_0, \phi_{\mathrm{wall}}) \in (0, +\infty) \times (-\infty, 0)$ is uniquely determined from (16). It is easy to see that $n_e[\phi]$ given by (60) is bounded uniformly with respect to $\phi \in W^{\phi_{\mathrm{wall}}}$. More precisely, one has the bounds

$$n_0 e^{\phi_{\mathrm{wall}}} m_\alpha(\phi_{\mathrm{wall}}) \leq n_e[\phi](x) \leq n_0 m_\alpha(0), \ \forall \phi \in W^{\phi_{\mathrm{wall}}}, \forall x \in [0,1]. \tag{61}$$

We now provide several a priori estimates. One has the following point-wise lower bound on the incoming time (37).

Lemma 6.1. *Let $0 \leq \alpha < 1$ and $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ so that the pair $(n_0, \phi_{\mathrm{wall}}) \in (0, +\infty) \times (-\infty, 0)$ is uniquely defined by (16). Let $\nu > 0$, $\varepsilon > 0$ and $\phi \in W^{\phi_{\mathrm{wall}}}$ a solution to (59). Therefore the incoming time (37) satisfies the point-wise lower bound:*

$$\forall (x,w) \in (0,1) \times (0, +\infty), \ t_{\mathrm{inc}}(x, \sqrt{w^2 - 2\phi(x)}) \geq \frac{1}{M_\phi} \left(\sqrt{w^2 - 2\phi(x)} - w \right). \tag{62}$$

where M_ϕ is defined as in (25).

Proof. For all $(x,w) \in (0,1) \times (0, +\infty)$ one has

$$t_{\mathrm{inc}} \left(x, \sqrt{w^2 - 2\phi(x)} \right) = -\int_0^x \frac{1}{\sqrt{w^2 - 2\phi(u)}} du.$$

Using the change of variable $e = \phi(u)$, we obtain

$$t_{\mathrm{inc}} \left(x, \sqrt{w^2 - 2\phi(x)} \right) = \int_{\phi(x)}^0 \frac{1}{\partial_x \phi(u)} \frac{1}{\sqrt{w^2 - 2e}} de.$$

Since $\frac{1}{\partial_x \phi(u)} \geq \frac{1}{M_\phi}$, it yields

$$t_{\mathrm{inc}} \left(x, \sqrt{w^2 - 2\phi(x)} \right) \geq \frac{1}{M_\phi} \int_{\phi(x)}^0 \frac{1}{\sqrt{w^2 - 2e}} de = \frac{1}{M_\phi} \left(\sqrt{w^2 - 2\phi(x)} - w \right).$$

$\square$

We are now going to give an estimate on the charge density. Namely, we shall prove that provided the electrostatic potential ϕ is sufficiently decreasing, the charge density $n_i[\phi] - n_e[\phi]$ is non negative. To do so, we define for $w > 0$ the following function

$$a_w : u \in [\phi_{\text{wall}}, 0] \mapsto a_w(u) = \frac{e^{-u + \frac{\nu}{M_\phi}(\sqrt{w^2 - 2u} - w)}}{\sqrt{w^2 - 2u}} \tag{63}$$

for which we will need the following technical lemma.

Lemma 6.2. *Let $0 \leq \alpha < 1$ and $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ so the pair $(n_0, \phi_{\text{wall}}) \in (0, +\infty) \times (-\infty, 0)$ is uniquely defined by (16). For all $w > 0$, the function a_w defined by (63) is convex. Consequently, it satisfies for all $u \in [\phi_{\text{wall}}, 0]$,*

$$a_w(u) \geq \frac{1}{w} + u\left(-\frac{1}{w} - \frac{\nu}{M_\phi}\frac{1}{w^2} + \frac{1}{w^3}\right). \tag{64}$$

Proof. The pointwise bound is a consequence of the convexity of the function a_w. For all $u \in [\phi_{\text{wall}}, 0]$, one has $a_w(u) \geq a_w(0) + u a'_w(0)$ which is the inequality (64). $\square$

We are now able to prove the charge density is non negative.

Proposition 3. *Let $0 \leq \alpha < 1$ and $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ so that the pair $(n_0, \phi_{\text{wall}})$ is uniquely defined by (16). Assume moreover that f_i^{inc} satisfies the kinetic Bohm criterion (20). Let $\nu > 0$, $\varepsilon > 0$ and $\phi \in W^{\phi_{\text{wall}}}$ a solution to (59) such that*

$$M_\phi \leq -\frac{\nu}{B_\alpha(f_i^{\text{inc}})}\left(\frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}\right) \tag{65}$$

where M_ϕ is defined as in (25) and $B_\alpha(f_i^{\text{inc}})$ is given by (21).
Then, the charge density $n_i[\phi] - n_e[\phi]$ is non negative everywhere in $[0, 1]$.

Proof. By definition $n_i[\phi]$ is the unique solution to (41). Since $\phi \in W^{\phi_{\text{wall}}}$, in virtue of Theorem 4.3, it satisfies the pointwise bound $n_i[\phi](x) \geq b_\phi^\nu(x)$ for all $x \in [0, 1]$. It is then sufficient to prove that $b_\phi^\nu(x) - n_e[\phi](x) \geq 0$ for all $x \in [0, 1]$. One has for $x \in [0, 1]$,

$$b_\phi^\nu(x) - n_e[\phi](x) = e^{\phi(x)}\left(\int_0^{+\infty} f_i^{\text{inc}}(w)w\frac{e^{-\phi(x)}e^{\nu t_{\text{inc}}(x, \sqrt{w^2 - 2\phi(x)})}}{\sqrt{w^2 - 2\phi(x)}} dw - n_0 m_\alpha(\phi(x))\right)$$

Using the Lemma 6.1, we have $e^{\nu t_{\text{inc}}(x, \sqrt{w^2 - 2\phi(x)})} \geq e^{\frac{\nu}{M_\phi}\left(\sqrt{w^2 - 2\phi(x)} - w\right)}$ and therefore

$$b_\phi^\nu(x) - n_e[\phi](x) \geq e^{\phi(x)}\left(\int_0^{+\infty} f_i^{\text{inc}}(w)w a_w(\phi(x)) dw - n_0 m_\alpha(\phi(x))\right).$$

Using the convexity inequality (64) and the fact that the function m_α is concave, we glean for all $x \in [0, 1]$:

$$b_\phi^\nu(x) - n_e[\phi](x) \geq e^{\phi(x)}\left(\int_0^{+\infty} f_i^{\text{inc}}(w)w\left[\frac{1}{w} + \phi(x)\left(-\frac{1}{w} - \frac{\nu}{M_\phi}\frac{1}{w^2} + \frac{1}{w^3}\right)\right] dw - n_0\left[m_\alpha(0) + \phi(x)m'_\alpha(0)\right]\right).$$

Since by definition $(n_0, \phi_{\text{wall}})$ solves the system (16), one has $\int_0^{+\infty} f_i^{\text{inc}}(w) dw = n_0 m_\alpha(0)$. Rearranging the terms of the right hand side, we obtain

$$b_\phi^\nu(x) - n_e[\phi](x) \geq e^{\phi(x)}\phi(x)\left(\int_0^{+\infty} f_i^{\text{inc}}(w)\left[-\frac{\nu}{M_\phi}\frac{1}{w} + \frac{1}{w^2}\right] dw - n_0(m'_\alpha(0) + m_\alpha(0))\right).$$

Since ϕ is non positive, the right hand side of the inequality is non negative everywhere in $[0, 1]$ provided the inequalities (20) and (65) hold. $\qquad\square$

To apply the Schauder fixed point theorem we shall need also need a uniform upper bound with respect to ϕ for the density $n_i[\phi]$. In this regard, one would like to make the constant $\tau > 0$ in the estimate (47) of Theorem 4.3 independent of ϕ. One has the following.

Proposition 4. *Let $0 \le \alpha < 1$ and $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ so that the pair $(n_0, \phi_{\mathrm{wall}})$ is uniquely defined by (16). Let $\nu > 0$, $\varepsilon > 0$ and $\phi \in W^{\phi_{\mathrm{wall}}}$ a solution to (59) that verifies (65). Then the constant $\tau > 0$ in the estimate (47) can be made independent of ϕ.*

Proof. Regarding the proof of Theorem 4.3 which uses the Lemma (A.4). The condition to obtain the estimate (47) is that

$$\tau \ge \frac{1}{q}\left(\frac{2\nu C_0(\phi_{\mathrm{wall}})}{-M_\phi}\right)^q,$$

where $q > 1$ is arbitrarily chosen and the constant $C_0(\phi_{\mathrm{wall}})$ is given by (69) and depends only on ϕ_{wall}. Since ϕ verifies (65), we see that it is sufficient to choose τ such that

$$\tau \ge \frac{1}{q}\left(2\nu C_0(\phi_{\mathrm{wall}})\frac{B_\alpha(f_i^{\mathrm{inc}})}{\nu}\frac{\int_0^{+\infty} f_i^{\mathrm{inc}}(v)dv}{\int_0^{+\infty}\frac{f_i^{\mathrm{inc}}(v)}{v}dv}\right)^q.$$

$\qquad\square$

For the self consistency of the analysis, we finish this section by characterizing the set of incoming ions densities $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ that satisfy the kinetic Bohm criterion (20). The following result is a compilation of Theorem 3.7 and Corollary 1 of [12]. They are reminded here after for the completeness of this work.

Proposition 5. *Let $0 \le \alpha < 1$ and $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ so that the pair $(n_0, \phi_{\mathrm{wall}})$ is uniquely defined by (16). Define*

$$s_2(\alpha) := \frac{m_\alpha(0) + m_\alpha'(0)}{m_\alpha(0)}.$$

Then there exists $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ that satisfies the kinetic Bohm criterion (20) if and only if $s_1(\alpha)^2 s_2(\alpha) \ge 1$. Consequently, for $0 \le \alpha \le \alpha_c$ where α_c is given by (18) there exists $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ that satisfies the kinetic Bohm criterion (20).

Proof. See Theorem 3.7 and Corollary 1 of [12]. $\qquad\square$

6.2. **Existence of a solution.** Using the a priori estimates of Section 6.1, we are ready to prove the existence of a solution to (59). Let $0 \le \alpha \le \alpha_c$ and consider $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ that satisfies the kinetic Bohm inequality (20). Then the pair $(n_0, \phi_{\mathrm{wall}})$ is uniquely defined by (16) and $B_\alpha(f_i^{\mathrm{inc}})$ given by (21) is positive. For $0 < \nu < \nu^c$ where ν^c is given by (22), we define the set

$$\mathcal{C}_\nu = \left\{\psi \in C^1[0,1] : M_\psi \le -\frac{\nu\int_0^{+\infty}\frac{f_i^{\mathrm{inc}}(v)}{v}dv}{B_\alpha(f_i^{\mathrm{inc}})\int_0^{+\infty} f_i^{\mathrm{inc}}(v)dv}, \ \psi(0) = 0, \ \psi(1) = \phi_{\mathrm{wall}}\right\}, \tag{66}$$

where we remind that $M_\psi := \sup\limits_{x \in [0,1]} \partial_x \psi(x)$. It is a closed convex subset of $C^1[0,1]$ where $C^1[0,1]$ is a Banach space equipped with the usual norm:

$$\forall \psi \in C^1[0,1], \quad \|\psi\|_{C^1[0,1]} := \sup_{x \in [0,1]} |\psi(x)| + \sup_{x \in [0,1]} |\partial_x \psi(x)|.$$

For $0 < \nu < \nu^c$ and $\varepsilon > 0$, we define the map

$$\begin{aligned} T_{\nu,\varepsilon} : \mathcal{C}_\nu &\to C^1[0,1] \\ \phi &\mapsto T_{\nu,\varepsilon}(\phi) \end{aligned} \tag{67}$$

where $T_{\nu,\varepsilon}(\phi)$ is the unique solution to the Poisson problem

$$\begin{cases} -\varepsilon^2 \partial_{xx} T_{\nu,\varepsilon}(\phi)(x) = n_i[\phi](x) - n_e[\phi](x), \ x \in (0,1), \\ T_{\nu,\varepsilon}(\phi)(0) = 0, \quad T_{\nu,\varepsilon}(\phi)(1) = \phi_{\text{wall}} \end{cases} \tag{68}$$

where $n_i[\phi] \in C^0[0,1]$ is the unique solution to the integral equation (41) and $n_e[\phi] \in C^0[0,1]$ is defined by (60). Since $n_i[\phi] - n_e[\phi]$ is a continuous function on $[0,1]$, one has $T_{\nu,\varepsilon}(\phi) \in C^2[0,1] \subset C^1[0,1]$ and $T_{\nu,\varepsilon}$ is well-defined. We shall now apply the Schauder fixed point theorem A.6. To do so, we begin with the following remark.

Remark 5. The Lemma 6.1 and the Proposition 3 of Section 6.1 assumes the regularity $W^{\phi_{\text{wall}}} \subset W^{2,\infty}(0,1)$. We nevertheless mention that the regularity needed in the proof of these estimates is only $C^1[0,1]$. We will use these two results with the $C^1[0,1]$ regularity assumption.

One then has the following.

Proposition 6. Let $0 < \nu < \nu^c$. Then, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \geq \varepsilon^*$ the convex set $\mathcal{C}_\nu$ is stable by $T_{\nu,\varepsilon}$. Namely, $T_{\nu,\varepsilon}(\mathcal{C}_\nu) \subset \mathcal{C}_\nu$.

Proof. Let $0 < \nu < \nu^c$ and $\phi \in \mathcal{C}_\nu$. Since $M_\phi \leq -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}$, reproducing exactly the same proof as in the Proposition 3 with $\phi \in C^1[0,1]$ yields that $n_i[\phi] - n_e[\phi]$ is non negative everywhere in $[0,1]$. By definition, for all $\varepsilon > 0$, $T_{\nu,\varepsilon}(\phi)$ is the solution to (68). It therefore satisfies the boundary conditions $T_{\nu,\varepsilon}(\phi)(0) = 0$, $T_{\nu,\varepsilon}(\phi)(1) = \phi_{\text{wall}} < 0$ and it is concave. We then have for all $x \in [0,1]$, $\partial_x T_{\nu,\varepsilon}(\phi)(x) \leq \partial_x T_{\nu,\varepsilon}(\phi)(0)$. It is therefore sufficient to bound from above $\partial_x T_{\nu,\varepsilon}(\phi)(0)$ to prove that $T_{\nu,\varepsilon}(\phi) \in \mathcal{C}_\nu$. Integrating the Poisson equation yields for all $x \in [0,1]$,

$$-\varepsilon^2 T_{\nu,\varepsilon}(\phi)(x) + \varepsilon^2 \partial_x T_{\nu,\varepsilon}(\phi)(0)x = \int_0^x \int_0^u n_i[\phi](y) - n_e[\phi](y) dy du.$$

Especially taking $x = 1$ and carrying an integration by parts yields

$$\partial_x T_{\nu,\varepsilon}(\phi)(0) = \phi_{\text{wall}} + \frac{1}{\varepsilon^2} \int_0^1 (1-u)(n_i[\phi] - n_e[\phi])(u) du.$$

Since $\phi \in \mathcal{C}_\nu$, ϕ verifies (65) and by Proposition 4 there exists $\tau > 0$ that is independent of ϕ such that such that for all $x \in [0,1]$,

$$n_i[\phi](x) \leq 2 \int_{\mathbb{R}^+} f_i^{\text{inc}}(w) dw e^{-\tau \phi(x)} \leq 2 \int_{\mathbb{R}^+} f_i^{\text{inc}}(w) dw e^{-\tau \phi_{\text{wall}}}.$$

Using the bounds (61) for $n_e[\phi]$, we eventually deduce the upper bound for the charge density. Namely for all $x \in [0,1]$,

$$n_i[\phi](x) - n_e[\phi](x) \le \left(2 \int_{\mathbb{R}^+} f_i^{\text{inc}}(w) dw e^{-\tau \phi_{\text{wall}}} - n_0 e^{\phi_{\text{wall}}} m_\alpha(\phi_{\text{wall}}) \right).$$

The constant $C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) := \left(2 \int_{\mathbb{R}^+} f_i^{\text{inc}}(w) dw e^{-\tau \phi_{\text{wall}}} - n_0 e^{\phi_{\text{wall}}} m_\alpha(\phi_{\text{wall}}) \right)$ is independant of ϕ and positive as proven in Proposition A.5. One then deduces,

$$\partial_x T_{\nu,\varepsilon}(\phi)(0) \le \phi_{\text{wall}} + \frac{1}{2\varepsilon^2} C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu).$$

Consider the function g defined for all $\varepsilon > 0$ by

$$g(\varepsilon) = \phi_{\text{wall}} + \frac{1}{2\varepsilon^2} C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) + \frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}.$$

Since $\nu < \nu^c$ where ν^c is given by (22) one has $-\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} - \phi_{\text{wall}} > 0$. Therefore one has $g(\varepsilon) \le 0$ for all $\varepsilon \ge \varepsilon^*$ with

$$\varepsilon^* = \sqrt{ 2 \frac{ -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} - \phi_{\text{wall}} }{ C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) } } > 0.$$

We eventually glean that

$$\partial_x T_{\nu,\varepsilon}(\phi)(0) \le \phi_{\text{wall}} + \frac{1}{2\varepsilon^2} C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) \le -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}$$

for $\varepsilon \ge \varepsilon^*$. It proves that for all $0 < \nu < \nu^c$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \ge \varepsilon^*$, and for all $\phi \in \mathcal{C}_\nu$, $T_{\nu,\varepsilon}(\phi) \in \mathcal{C}_\nu$. $\qquad \square$

Proposition 7. *Let $0 < \nu < \nu^c$. Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \ge \varepsilon^*$, $T_{\nu,\varepsilon}(\mathcal{C}_\nu)$ is a relatively compact part of $C^1[0,1]$.*

Proof. Let $0 < \nu < \nu^c$. Using Proposition 6, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \ge \varepsilon^*$, $T_{\nu,\varepsilon}(\mathcal{C}_\nu) \subset \mathcal{C}_\nu$. Using moreover the Proposition 4 and the estimate (61), we have for all $\phi \in \mathcal{C}_\nu$ and $x \in [0,1]$;

$$0 \le n_i[\phi](x) - n_e[\phi](x) \le C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu)$$

where $C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) := \left(2 \int_{\mathbb{R}^+} f_i^{\text{inc}}(w) dw e^{-\tau \phi_{\text{wall}}} - n_0 e^{\phi_{\text{wall}}} m_\alpha(\phi_{\text{wall}}) \right)$ where τ does not depend on ϕ. It yields that $-\varepsilon^2 \partial_{xx} T_{\nu,\varepsilon}(\phi)$ is uniformly bounded in ϕ. Functions of the set $T_{\nu,\varepsilon}(\mathcal{C}_\nu)$ are thus uniformly Lipschitz continuous and their first derivative as well. Therefore the set $T_{\nu,\varepsilon}(\mathcal{C}_\nu)$ is equi-continuous in $C^1[0,1]$. It is moreover bounded in $C^1[0,1]$. The conclusion follows from the Arzela-Ascoli theorem. $\qquad \square$

One has eventually.

Theorem 6.3. *Let $0 < \nu < \nu^c$. Then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \ge \varepsilon^*$ there exists $\phi \in \mathcal{C}_\nu \cap C^2[0,1]$ solution to the nonlinear Poisson problem (59).*

Proof. It is a consequence of Proposition 6 and 7. Let $0 < \nu < \nu^c$, then there exists $\varepsilon^* > 0$ such that for all $\varepsilon \ge \varepsilon^*$, the operator $T_{\nu,\varepsilon}$ defined by (67) is such that $T_{\nu,\varepsilon}(\mathcal{C}_\nu) \subset \mathcal{C}_\nu$, it is continuous and $T_{\nu,\varepsilon}(\mathcal{C}_\nu)$ is relatively compact. Thus the Schauder fixed point theorem A.6 applies and $T_{\nu,\varepsilon}$ has a fixed point. $\qquad \square$

7. **Proof of the main result.** Consider f_e^{inc} given by (17). Let $0 \leq \alpha \leq \alpha_c$ and consider $f_i^{\text{inc}} \in \mathcal{I}(\alpha, f_e^{\text{inc}})$ that satisfies the kinetic Bohm inequality (20). The proof of the main result is as follows :

- The pair $(n_0, \phi_{\text{wall}})$ is uniquely defined by (16). The Bohm number $B_\alpha(f_i^{\text{inc}})$ is therefore positive and the critical collision frequency (22) is well-defined.
- In virtue of Theorem 6.3, for $0 < \nu < \nu^c$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \geq \varepsilon^*$, the nonlinear Poisson problem (59) has a decreasing solution
  $\phi \in C^2[0,1]$ such that for all $x \in [0,1]$, $\partial_x \phi(x) \leq -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}$. It
  thus verifies $\phi \in W^{\phi_{\text{wall}}}$.
- So consider $\phi \in W^{\phi_{\text{wall}}}$ a solution to the problem $(\text{NLP})_{\nu,\varepsilon}$ (59).
  - One has that f_i defined by (36) is bounded since f_i^{inc} is, $n_i[\phi]$ also is and $\partial_x \phi$ does not vanish. One remarks that $n_i[\phi]$ is the unique solution to the integral equation (41) and is continuous on $[0,1]$. Since it is moreover non negative, we deduce by the Fubini-Tonneli theorem that that $f_i \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R})$. One verifies that it is a mild solution of the Vlasov-Boltzmann equation (1) and that by construction it satisfies the boundary conditions (8).
  - One has that f_e defined by (54) is bounded and integrable since f_e^{inc} and $n_e[\phi]$ is given by (60) which is also bounded. One verifies that it is a mild solution of the Vlasov equation (2) and that it satisfies the boundary conditions (9).
  - We remark that the ions density is such that $n_i = n_i[\phi]$ since both are solutions of the integral equation (41) which has a unique solution. For the electrons density, we remark that their expressions are such that $n_e = n_e[\phi]$. Thus (f_i, f_e, ϕ) solves the Vlasov-Poisson-Botzmann equations (1)-(10).
  - In virtue of Proposition 3, $n_i - n_e$ is non negative everywhere in $[0,1]$ and ϕ is concave.
- By construction of the pair $(n_0, \phi_{\text{wall}})$, it is such that $n_i(0) = n_e(0)$ and $J_i = J_e$.

The proof is achieved.

A. **Appendix.**

A.1. **Definition of mild solutions.**

Definition A.1 (Mild solutions for the ions)**.** Let $\phi \in W^{\phi_{\text{wall}}}$ and $f_i^{\text{inc}} \in \mathcal{P}$. We say that a non negative function $f_i \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R})$ is a mild solution to the linear Vlasov-Boltzmann equation (1) with the boundary condition (8) iff :

- $x \in [0,1] \mapsto n_i(x) = \int_{\mathbb{R}} f_i(x,v) dv$ is a continuous function.
- It satistfies for almost every $(x,v) \in (0,1) \times \mathbb{R}$ and $t \in [t_{\text{inc}}(x,v), 0]$:

$$f_i(X(t,x,v), V(t,x,v)) = f_i(x,v) - \nu \int_0^t f_i(X(\tau,x,v), V(\tau,x,v)) d\tau$$

$$+ \nu \int_0^t n_i(X(\tau,x,v)) \delta_{V(\tau,x,v)=0} d\tau.$$

  where the measure $\delta_{V(\tau,x,v)=0}$ is defined in (4.1).
- f_i has a trace on $\{0\} \times (0, +\infty)$ and $\{1\} \times (-\infty, 0)$ which belongs to $\mathcal{P}$ and the boundary conditions (8) holds almost everywhere.

Definition A.2 (Mild solutions for the electrons)**.** Let $\phi \in W^{\phi_{\text{wall}}}$ and $f_e^{\text{inc}} \in \mathcal{P}$. We say that a non negative function $f_e \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R})$ is a mild solution to the linear Vlasov equation (2) with the boundary condition (9) iff :

- It satistfies for almost every $(x,v) \in (0,1) \times \mathbb{R}$ and $t \in [t_{\text{inc}}(x,v), 0]$:
$$f_e(X(t,x,v), V(t,x,v)) = f_e(x,v).$$

- f_e has a trace on $\{0\} \times (0,+\infty)$ and $\{1\} \times (-\infty, 0)$ which belongs to $\mathcal{P}$ and the boundary conditions (9) holds almost everywhere.

A.2. **Technical results for the linear Vlasov-Boltzmann equation.**

Lemma A.3. *Let $\phi_{\text{wall}} < 0$, $\phi \in W^{\phi_{\text{wall}}}$ and $\nu > 0$. Then $\mathcal{V}_\phi^\nu$ is a well-defined operator from $C^0[0,1]$ to $C^0[0,1]$.*

Proof. We first remark $M_\phi = \sup\limits_{x \in [0,1]} \partial_x \phi(x) < 0$ and thus for all $x \in [0,1]$, $-\frac{1}{\partial_x \phi(x)} \le \frac{1}{|M_\phi|}$. Let $\rho \in C^0[0,1]$ we are going to prove that $\mathcal{V}_\phi(\rho)$ is a continuous function on $[0,1]$. We distinguish two cases:

Continuity at $x = 0$. Since $\sup\limits_{(x,\hat{x}) \in T} |K_\phi^\nu(x,\hat{x})| \le 1$, one has for $x > 0$

$$|\mathcal{V}_\phi(\rho)(x)| \le -\int_0^x \frac{-\partial_x \phi(\hat{x})}{\partial_x \phi(\hat{x})\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x} \|\rho\|_{C^0[0,1]} \le \frac{1}{|M_\phi|} \sqrt{-2\phi(x)} \|\rho\|_{C^0[0,1]}.$$

Since ϕ is continuous on $[0,1]$ with $\phi(0) = 0$, we deduce that $\lim\limits_{x \to 0^+} \mathcal{V}_\phi(\rho)(x) = 0$ and thus $\mathcal{V}_\phi(\rho)$ is continuous at $x = 0$ with $\mathcal{V}_\phi(\rho)(0) = 0$.

Continuity in $(0,1]$. For the sake of conciseness, we set for all $0 < \hat{x} < x \le 1$, $S(x,\hat{x}) := \frac{-\partial_x \phi(\hat{x})}{-\partial_x \phi(\hat{x})\sqrt{2(\phi(\hat{x}) - \phi(x))}}$. Let $0 < x < y \le 1$, then one has the decomposition in two terms:

$$\mathcal{V}_\phi(\rho)(x) - \mathcal{V}_\phi(\rho)(y) = -I_1(x,y) + I_2(x,y)$$

where

$$I_1(x,y) = \int_x^y S(y,\hat{x}) K_\phi^\nu(y,\hat{x}) \rho(\hat{x}) d\hat{x},$$

$$I_2(x,y) = \int_0^x \left(S(x,\hat{x}) K_\phi^\nu(x,\hat{x}) - S(y,\hat{x}) K_\phi^\nu(y,\hat{x}) \right) \rho(\hat{x}) d\hat{x},$$

We may estimate each terms. Since $\sup\limits_{(x,\hat{x}) \in T} |K_\phi^\nu(x,\hat{x})| \le 1$, for the first term, one has

$$|I_1(x,y)| \le \frac{1}{|M_\phi|} \sup\limits_{x \in [0,1]} |\rho(x)| \int_x^y \frac{-\partial_x \phi(\hat{x})}{\sqrt{2(\phi(\hat{x}) - \phi(y))}} d\hat{x}$$

$$\le \frac{1}{|M_\phi|} \sup\limits_{x \in [0,1]} |\rho(x)| \sqrt{2(\phi(x) - \phi(y))}.$$

For the second term, we decompose it has follows:

$$I_2(x,y) = \int_0^x \left(S(x,\hat{x}) - S(y,\hat{x}) \right) K_\phi^\nu(x,\hat{x}) \rho(\hat{x}) d\hat{x}$$

$$+ \int_0^x S(y,\hat{x}) \left(K_\phi^\nu(x,\hat{x}) - K_\phi^\nu(y,\hat{x}) \right) \rho(\hat{x}) d\hat{x}$$

For the first integral one has

$$\left| \int_0^x \left(S(x, \hat{x}) - S(y, \hat{x}) \right) K_\phi^\nu(x, \hat{x}) \rho(\hat{x}) d\hat{x} \right| \le \sup_{x \in [0,1]} |\rho(x)| \int_0^x |S(x, \hat{x}) - S(y, \hat{x})| d\hat{x}.$$

Now remark that because ϕ is decreasing, for $\hat{x} < x$, $S(\cdot, \hat{x})$ is a decreasing function, therefore $x < y \Rightarrow S(x, \hat{x}) - S(y, \hat{x}) > 0$. One has therefore

$$\int_0^x |S(x, \hat{x}) - S(y, \hat{x})| d\hat{x} = \int_0^x S(x, \hat{x}) - S(y, \hat{x}) d\hat{x} \le$$

$$\frac{1}{|M_\phi|} \left(\int_0^x \frac{-\partial_x \phi(\hat{x})}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} + \frac{\partial_x \phi(\hat{x})}{\sqrt{2(\phi(\hat{x}) - \phi(y))}} d\hat{x} \right)$$

$$\le \frac{1}{|M_\phi|} \left(\sqrt{2(\phi(x) - \phi(y))} + \sqrt{-2\phi(x)} - \sqrt{-2\phi(y)} \right).$$

We thus glean

$$\left| \int_0^x \left(S(x, \hat{x}) - S(y, \hat{x}) \right) K_\phi^\nu(x, \hat{x}) \rho(\hat{x}) d\hat{x} \right| \le$$

$$\frac{1}{|M_\phi|} \sup_{x \in [0,1]} |\rho(x)| \left(\sqrt{2(\phi(x) - \phi(y))} + \sqrt{-2\phi(x)} - \sqrt{-2\phi(y)} \right).$$

For the second integral, one remark that the function K_ϕ^ν belongs to $C^0(T)$ and where T is compact, it is therefore uniformly continuous. Consequently for all $\epsilon > 0$, there exists $\delta_T > 0$ such that

$$(x, y) \in T \quad |x - y| < \delta_T \text{ and } \hat{x} \le x \Rightarrow$$

$$|K_\phi^\nu(x, \hat{x}) - K_\phi^\nu(y, \hat{x})| < \frac{\epsilon}{\displaystyle\sup_{x \in [0,1]} \sqrt{-2\phi(x)} \sup_{x \in [0,1]} |\rho(x)|}.$$

This yields the estimate

$$\left| \int_0^x S(y, \hat{x}) \left(K_\phi^\nu(x, \hat{x}) - K_\phi^\nu(y, \hat{x}) \right) \rho(\hat{x}) d\hat{x} \right| \le \frac{\epsilon}{\displaystyle\sup_{x \in [0,1]} \sqrt{-2\phi(x)}} \int_0^x S(y, \hat{x}) d\hat{x}$$

$$\le \frac{1}{|M_\phi|} \frac{\epsilon}{\displaystyle\sup_{x \in [0,1]} \sqrt{-2\phi(x)}} \left(-\sqrt{2(\phi(x) - \phi(y))} + \sqrt{-2\phi(y)} \right).$$

Since ϕ is uniformly continuous on $[0, 1]$ for any $\epsilon > 0$, there are :

$$\delta_1 > 0 \text{ such that } (|x - y| < \delta_1 \text{ and } x < y) \Rightarrow |I_1(x, y)| < \epsilon,$$

$$\delta_{2,a} > 0 \text{ such that } (|x - y| < \delta_{2,a} \text{ and } x < y) \Rightarrow$$

$$\left| \int_0^x \left(S(x, \hat{x}) - S(y, \hat{x}) \right) K_\phi^\nu(x, \hat{x}) \rho(\hat{x}) d\hat{x} \right| < \epsilon,$$

$$\delta_{2,b} > 0 \text{ such that } (|x - y| < \delta_{2,b} \text{ and } x < y) \Rightarrow$$

$$\left| \left(-\sqrt{2(\phi(x) - \phi(y))} + \sqrt{-2\phi(y)} \right) \right| < \sqrt{-2\phi(x)}(1 + \epsilon).$$

Taking $\delta = \min(\delta_T, \delta_1, \delta_{2,a}, \delta_{2,b})$ yields $|x - y| < \delta$ and $x < y \Rightarrow |\mathcal{V}_\phi(\rho)(x) - \mathcal{V}_\phi(\rho)(y)| < \frac{1}{|M_\phi|} \left(2\epsilon + \epsilon(1 + \epsilon) \right)$. It ends the proof. $\qquad \square$

Lemma A.4. *Let $\phi_{\mathrm{wall}} < 0$, $\phi \in W^{\phi_{\mathrm{wall}}}$ and $\nu > 0$. Then, there exists $\tau > 0$ such that for all $x \in [0,1]$, $e^{\tau\phi(x)} \int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x} < \frac{1}{2\nu}$.*

Proof. If $x = 0$ there is nothing to prove. So let $x \in (0,1]$. Using the change of variable $u = \phi(\hat{x})$ one has

$$\int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x} = -\int_{\phi(x)}^0 \frac{1}{\partial_x\phi(\hat{x})} \frac{e^{-\tau u}}{\sqrt{2(u - \phi(x))}} du \leq$$

$$-\frac{1}{M_\phi} \int_{\phi(x)}^0 \frac{e^{-\tau u}}{\sqrt{2(u - \phi(x))}} du$$

where $M_\phi = \sup_{x \in [0,1]} \partial_x\phi(x) < 0$. Let $1 < p < 2$ and denote $q > 1$ the Hölder exponent of p which is such that $\frac{1}{q} + \frac{1}{p} = 1$. Using the Hölder inequality, one has

$$\int_{\phi(x)}^0 \frac{e^{-\tau u}}{\sqrt{2(u - \phi(x))}} du \leq \left(\int_{\phi(x)}^0 e^{-\tau q u} du\right)^{\frac{1}{q}} \left(\int_{\phi(x)}^0 [2(u - \phi(x))]^{-\frac{p}{2}} du\right)^{\frac{1}{p}}.$$

We thus obtain

$$\int_{\phi(x)}^0 \frac{e^{-\tau u}}{\sqrt{2(u - \phi(x))}} \leq \frac{1}{(\tau q)^{\frac{1}{q}}} \left(e^{-\tau q \phi(x)} - 1\right)^{\frac{1}{q}} \left(\frac{(-2\phi(x))^{1-\frac{p}{2}}}{2(1 - \frac{p}{2})}\right)^{\frac{1}{p}}.$$

It yields after multiplication by $e^{\tau\phi(x)}$:

$$e^{\tau\phi(x)} \int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x} \leq -\frac{1}{M_\phi} \frac{1}{(\tau q)^{\frac{1}{q}}} \left(1 - e^{\tau q \phi(x)}\right)^{\frac{1}{q}} \left(\frac{(-2\phi(x))^{1-\frac{p}{2}}}{2(1 - \frac{p}{2})}\right)^{\frac{1}{p}}.$$

Since $\phi_{\mathrm{wall}} \leq \phi(x) \leq 0$ one has,

$$e^{\tau\phi(x)} \int_0^x \frac{e^{-\tau\phi(\hat{x})}}{\sqrt{2(\phi(\hat{x}) - \phi(x))}} d\hat{x} \leq -\frac{1}{M_\phi} \frac{1}{(\tau q)^{\frac{1}{q}}} \left(\frac{(-2\phi_{\mathrm{wall}})^{1-\frac{p}{2}}}{2(1 - \frac{p}{2})}\right)^{\frac{1}{p}}.$$

Set

$$C_0(\phi_{\mathrm{wall}}) := \left(\frac{(-2\phi_{\mathrm{wall}})^{1-\frac{p}{2}}}{2(1 - \frac{p}{2})}\right)^{\frac{1}{p}}. \tag{69}$$

Then the right hand side of the latter inequality is lower than $\frac{1}{2\nu}$ if and only if $\tau \geq \frac{1}{q} \left(\frac{2\nu C_0(\phi_{\mathrm{wall}})}{-M_\phi}\right)^q$. It ends the proof. $\qquad\square$

Lemma A.5 (Positivity of the constant). *Let $0 \leq \alpha < 1$ and $f_i^{\mathrm{inc}} \in \mathcal{I}(\alpha, f_e^{\mathrm{inc}})$ so that the pair $(n_0, \phi_{\mathrm{wall}})$ is uniquely defined by (16). Then the constant*

$$C(\alpha, f_i^{\mathrm{inc}}, \phi_{\mathrm{wall}}, \nu) := \left(2\int_{\mathbb{R}^+} f_i^{\mathrm{inc}}(w) dw e^{-\tau\phi_{\mathrm{wall}}} - n_0 e^{\phi_{\mathrm{wall}}} m_\alpha(\phi_{\mathrm{wall}})\right)$$

is positive.

Proof. By construction of the pair $(n_0, \phi_{\mathrm{wall}})$, $\int_{\mathbb{R}^+} f_i^{\mathrm{inc}}(w) dw = n_0 m_\alpha(0)$. Thus we have

$$C(\alpha, f_i^{\mathrm{inc}}, \phi_{\mathrm{wall}}, \nu) = n_0 \left(2m_\alpha(0) e^{-\tau\phi_{\mathrm{wall}}} - e^{\phi_{\mathrm{wall}}} m_\alpha(\phi_{\mathrm{wall}})\right)$$

$$\geq n_0 \left(2m_\alpha(0) - e^{\phi_{\mathrm{wall}}} m_\alpha(\phi_{\mathrm{wall}})\right).$$

The function m_α is increasing therefore $m_\alpha(\phi_{\text{wall}}) \leq m_\alpha(0)$. We thus infer since $\phi_{\text{wall}} < 0$

$$C(\alpha, f_i^{\text{inc}}, \phi_{\text{wall}}, \nu) \geq n_0 m_\alpha(0) \left(2 - e^{\phi_{\text{wall}}}\right) > 0.$$

$\square$

A.3. The Schauder fixed point theorem.

Theorem A.6 (Schauder fixed point theorem)**.** *Let E be a Banach space and C be a closed convex subset of E. Let moreover $T : C \to C$ be a continuous map such that $T(C)$ is relatively compact. Then T has a fixed point, i.e, there exists $\phi \in C$ such that $T(\phi) = \phi$.*

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