Notes on fixed-point procedure

Contents

1 Notations and assumptions

We suppose that

1. $f_{e,b}$ is nonnegative.
2. $f_{e,b}(v) = f_{e,b}(-v)$ for all $v \in \mathbb{R}$.
3. $f_{e,b}$ is compactly supported, i.e. there exists $v^* \geqslant 0$ such that $f_{e,b}(v) = 0$ for all $|v| \geqslant v^*$.
4. There exists $\overline{c} \leqslant 0$ such that $f_{e,b}(v) \leqslant \overline{c}$ for all $v \in \mathbb{R}$.
5. There exists $\underline{c} > 0$ and $0 \leqslant \underline{v} < \overline{v}$ such that $f_{e,b}(v) \geqslant \underline{c}$ for all $v \in [-\overline{v}, -\underline{v}]$.
6. The function φ satisfies $\varphi(0) = \varphi'(0) = 0$.
7. The function φ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi((1-\tau)x + \tau y) \geqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\alpha}{2}(1-\tau)\tau\,|x-y|^2$ for all $(x, y, \tau) \in [0,1]^3$.
8. There exists $\beta > \alpha$ such that φ is $(-\beta)-$convex, i.e. $\varphi((1-\tau)x + \tau y) \leqslant (1-\tau)\varphi(x) + \tau\varphi(y) + \frac{\beta}{2}(1-\tau)\tau\,|x-y|^2$ for all $(x, y, \tau) \in [0,1]^3$. Notice that in this case, $\varphi(x) \geqslant -\beta\frac{x^2}{2}$, and in particular, $\varphi(1) \geqslant -\beta$.

For the reader: notations $\underline{v}, \overline{v}$ indicates a domain on which $f_{e,b}$ is *lower bounded*, while v_*, v^* denote a domain on which $f_{e,b}$ is *upper bounded by 0* (with a global bound by $\overline{c}$ anyway).

We use the following notations: the Liapunov functions of the electrons and the ions are

$$\mathcal{L}_e(x, v) := \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{1.1}$$

$$\mathcal{L}_i(x,v) \coloneqq \frac{v^2}{2} + \varphi(x). \tag{1.2}$$

For a given point $(x,v) \in [0,1] \times \mathbb{R}^-$, we denote by (x_b, v_b) the intersection of the boundary $\{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ with the ion characteristic issued from (x,v). The values are given by

$$\begin{pmatrix} x_b(x,v) \\ v_b(x,v) \end{pmatrix} \coloneqq \begin{pmatrix} \varphi^{-1}\left(\min\left(0, \frac{v^2}{2} + \varphi(x)\right)\right) \\ -\sqrt{\max\left(0, v^2 + 2\varphi(x)\right)} \end{pmatrix} \tag{1.3}$$

The satisfaction of the non-emitting boundary condition in $(x = 1, v \in \mathbb{R}_-)$ may interfere with the value on the line $(x = 0, v \in \mathbb{R})$. **In the following notes, we consider that the non-emitting boundary condition has the precedence**, i.e. $f_e(x,v) = 0$ if $\mathcal{L}_e(x,v) \geqslant \mathcal{L}_e(1,0)$, regardless of the prescribed value of $f_{e,b}$ along this line.

2 Estimates

2.1 Useful elementary lemmas

Lemma 2.1. *Let $\alpha > 0$, $\overline{x} \geqslant 0$ and $\overline{y} \geqslant 0$. Then*

$$\mathcal{I} \coloneqq \int_{x=0}^{\overline{x}} \int_{y=0}^{\overline{y}} \frac{1}{\left(x^2 + \alpha y^2\right)^{1/2}} dy dx \leqslant 2\frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

Demonstration By the change of variable $z = \sqrt{\alpha} y$, with $\overline{z} \coloneqq \sqrt{\alpha}\overline{y}$, we have

$$\mathcal{I} = \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{1}{\left(x^2 + z^2\right)^{1/2}} dz dx.$$

Notice that $x + z \leqslant \sqrt{2}\left(x^2 + y^2\right)^{1/2}$. Then,

$$\mathcal{I} \leqslant \frac{1}{\sqrt{\alpha}} \int_{x=0}^{\overline{x}} \int_{z=0}^{\overline{z}} \frac{\sqrt{2}}{x+z} dz dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(\frac{x + \overline{z}}{x}\right) dx = \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \ln\left(1 + \frac{\overline{z}}{x}\right) dx.$$

Using $\ln(1+a) \leqslant \sqrt{a}$, we get

$$\mathcal{I} \leqslant \sqrt{\frac{2}{\alpha}} \int_{x=0}^{\overline{x}} \sqrt{\frac{\overline{z}}{x}} dx = \sqrt{\frac{2}{\alpha}} 2\sqrt{\overline{x}\overline{z}} = 2\frac{\sqrt{2\overline{x}\overline{y}}}{\alpha^{1/4}}.$$

$\square$

Remark 2.1 (Exact value)**.** *Let $\overline{r} \coloneqq \sqrt{\overline{x}^2 + \overline{z}^2}$. Then*

$$\mathcal{I} = \overline{x} \ln\left(\frac{1 + \overline{z}/\overline{r}}{\overline{x}/\overline{r}}\right) + \overline{z} \ln\left(\frac{1 + \overline{x}/\overline{r}}{\overline{z}/\overline{r}}\right).$$

Lemma 2.2. *If $a \geqslant 0$ and $b \geqslant 0$, then $|a - b| \leqslant \sqrt{|a^2 - b^2|}$.*

Demonstration If $a \geqslant b$, then $|a-b| = \sqrt{(a-b)(a-b)} \leqslant \sqrt{(a-b)(a+b)} = \sqrt{a^2 - b^2}$, else $|a-b| = |b-a|$. $\square$

Lemma 2.3. *Let $\alpha > 0$, $L \in \mathbb{R}$ and $0 \leqslant a \leqslant 1$. Suppose that $L + \alpha a^2 \geqslant 0$, and $a > 0$ if $L = 0$. Then*

$$\mathcal{I} \coloneqq \int_{z=a}^{1} \frac{1}{\left(L + \alpha z^2\right)^{1/2}} dz \leqslant \min\left(|L|^{-1/2}, \frac{2}{\sqrt{\alpha a}}\right).$$

Demonstration If $L>0$, we have

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\int_{z=a}^{1}\frac{1}{\left(1+\left(\sqrt{\frac{\alpha}{L}}z\right)^2\right)^{1/2}}\frac{\sqrt{\alpha}dz}{\sqrt{L}}=\frac{1}{\alpha}\int_{w=a\sqrt{\frac{\alpha}{L}}}^{\sqrt{\frac{\alpha}{L}}}\frac{1}{(1+w^2)^{1/2}}dw=\frac{1}{\sqrt{\alpha}}\left(\sinh^{-1}\left(\sqrt{\frac{\alpha}{L}}\right)-\sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right)\right).$$

Using the positivity of $\sinh^{-1}\left(a\sqrt{\frac{\alpha}{L}}\right)$, and the coarse estimate $\sinh^{-1}(x)\leqslant x$, we get $\mathcal{I}\leqslant L^{-1/2}$. Moreover, using that $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$, we get

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\log\left(\frac{\sqrt{\frac{\alpha}{L}}+\sqrt{\frac{\alpha}{L}+1}}{a\sqrt{\frac{\alpha}{L}}+\sqrt{a^2\frac{\alpha}{L}+1}}\right)=\frac{1}{\sqrt{\alpha}}\log\left(\frac{1+\sqrt{1+\frac{L}{\alpha}}}{a+\sqrt{a^2+\frac{L}{\alpha}}}\right)\leqslant\frac{1}{\sqrt{\alpha}}\log\left(\frac{2+\sqrt{\frac{L}{\alpha}}}{2a}\right)\leqslant\frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}.$$

I shamelessly used $\frac{1}{2}\leqslant 1$. Then $\mathcal{I}\leqslant\min\left(L^{-1/2},\frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}\right)$ on $L>0$. But whenever $L\geqslant\alpha$, the min is attained in its first argument: indeed, $L^{-1/2}\leqslant\frac{1}{\sqrt{\alpha a}}\leqslant\frac{1+\left(\frac{L}{\alpha}\right)^{1/4}}{\sqrt{\alpha a}}$. Then we obtained $\mathcal{I}\leqslant\min(L^{-1/2},\frac{2}{\sqrt{\alpha a}})$ on $L>0$. If $L=0$, we have

$$\int_{z=a}^{1}\frac{1}{\sqrt{\alpha z}}dz=\frac{\log(1/a)}{\sqrt{\alpha}}\frac{2}{\sqrt{\alpha a}}.$$

Finally, if $L<0$, the integral becomes

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\int_{a}^{1}\frac{1}{\left(\left(\sqrt{\frac{\alpha}{|L|}}z\right)^2-1\right)^{1/2}}\frac{\sqrt{\alpha}dz}{\sqrt{|L|}}=\frac{1}{\alpha}\int_{a\sqrt{\frac{\alpha}{|L|}}}^{\sqrt{\frac{\alpha}{|L|}}}\frac{1}{(w^2-1)^{1/2}}dw=\frac{\cosh^{-1}\left(\sqrt{\frac{\alpha}{|L|}}\right)-\cosh^{-1}\left(a\sqrt{\frac{\alpha}{|L|}}\right)}{\sqrt{\alpha}}.$$

The expression is well-defined, since $L+\alpha a^2\geqslant 0$ implies $a\sqrt{\frac{\alpha}{-L}}\geqslant 1$ when $-L>0$. Since $\cosh^{-1}(x)\leqslant x$, we obtain the coarse estimate $\mathcal{I}\leqslant|L|^{-1/2}$. Using that $\cosh^{-1}(x)=\log\left(x+\sqrt{x^2-1}\right)$, we get

$$\mathcal{I}=\frac{1}{\sqrt{\alpha}}\log\left(\frac{1+\sqrt{1-\frac{|L|}{\alpha}}}{a+\sqrt{a^2-\frac{|L|}{\alpha}}}\right)\leqslant\frac{1}{\sqrt{\alpha}}\log\left(\frac{2}{a}\right)\leqslant\sqrt{\frac{2}{\alpha a}}\leqslant\frac{2}{\sqrt{\alpha a}}.$$

$\square$

Lemma 2.4 (MAGIC Change of variable)**.** *Let $(x(\tau),v(\tau))$ be an ion characteristic issued from $(x,w)\in[0,1]\times\mathbb{R}_-$, and define $\varepsilon:=\mathcal{L}_e(x(\tau),v(\tau))$ and $z:=x(\tau)$. Then*

$$d\tau\wedge dw=\frac{dz\wedge d\varepsilon}{2\left(\varepsilon+\frac{1}{\mu}\varphi(z)\right)^{1/2}\left(\varepsilon-\varphi(x)+\left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}.$$

Demonstration We have $\varepsilon=\frac{(v(\tau))^2}{2}-\frac{1}{\mu}\varphi(z)$, so that

$$d\varepsilon=v(\tau)\dot{v}(\tau)d\tau-\frac{1}{\mu}\varphi'(z)dz=-v(\tau)\varphi'(z)d\tau-\frac{1}{\mu}\varphi'(z)dz\quad\Longrightarrow\quad d\tau=-\frac{1}{v(\tau)\varphi'(z)}d\varepsilon-\frac{1}{\mu v(\tau)}dz.$$

Moreover, $\mathcal{L}_i(x(\tau),v(\tau))=\frac{(v(\tau))^2}{2}+\varphi(z)$, so $\left(1+\frac{1}{\mu}\right)\varphi(z)=\mathcal{L}_i(x(\tau),v(\tau))-\mathcal{L}_e(x(\tau),v(\tau))=\mathcal{L}_i(x,w)-\varepsilon$ and

$$w=-\sqrt{2}\left(\varepsilon-\varphi(x)+\left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}\quad\Longrightarrow\quad=\frac{d\varepsilon+\left(1+\frac{1}{\mu}\right)\varphi'(z)dz}{w}.$$

Then, using $v(\tau)=-\sqrt{2}\left(\varepsilon+\frac{1}{\mu}\varphi(z)\right)^{1/2}$, we get

$$d\tau\wedge dw=-\frac{1+\frac{1}{\mu}}{wv(\tau)}d\varepsilon\wedge dz-\frac{\frac{1}{\mu}}{v(\tau)w}dz\wedge d\varepsilon=\frac{dz\wedge d\varepsilon}{v(\tau)(-w)}=\frac{dz\wedge d\varepsilon}{2\left(\varepsilon+\frac{1}{\mu}\varphi(z)\right)^{1/2}\left(\varepsilon-\varphi(x)+\left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}}.$$

$\square$

Remark 2.2. *Notice that* $\mathcal{L}_i(x(\tau), v(\tau)) = \varepsilon + \left(1 + \frac{1}{\mu}\right)\varphi(z)$*. Moreover, we have* $-\frac{1}{\mu}\varphi(z) = \mathcal{L}_e(z, 0)$*, and* $\varphi(x) = \mathcal{L}_i(z, 0)$*. Then the change of variable rewrites*

$$d\tau \wedge dw = \frac{dz \wedge d\varepsilon}{2\left(\mathcal{L}_e(x(\tau), v(\tau)) - \mathcal{L}_e(z, 0)\right)^{1/2}\left(\mathcal{L}_i(x(\tau), v(\tau)) - \mathcal{L}_i(x, 0)\right)^{1/2}},$$

and is well-defined for (τ, w) *such that both square roots are real.*

2.2 Boundedness estimates

2.2.1 Electronic density n_e

Lemma 2.5. *The electron density* n_e *is bounded independently of* φ*. More precisely,*

$$n_e(x) \leqslant 2\bar{c}v^* \qquad \forall x \in [0, 1].$$

Demonstration Let $v \in \mathbb{R}$ such that $|v| > v_*$, with $f_{e,b}\left(\mathbb{R} \setminus [-v^*, v^*]\right) = 0$. Then

$$\mathcal{L}_e(x, v) = \frac{v^2}{2} - \frac{1}{\mu}\varphi(x) \geqslant \frac{v^2}{2} = \mathcal{L}_e(v, 0) \qquad \Longrightarrow \qquad f_e(x, v) = 0.$$

Then we may write for all $x \in [0, 1]$ that

$$n_e(x) = \int_{v=-v^*}^{v^*} f_e(x, v) dv \leqslant 2\bar{c}v^*.$$

□

2.2.2 Ion density n_i

The estimates will rely on two particular cases, the we treat independently as lemmas. For a given x, we define $g_x : [0, x] \mapsto \mathbb{R}^+$ by

$$\mathcal{L}_i(x, -g_x(y)) = \mathcal{L}_i(y, 0), \quad \text{i.e.} \quad g_x(y) = \left(2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

For the reader: draw y to the left of x. Then $-g_x(y)$ is the intersection of the ion characteristic ending on $(y, 0)$ with the vertical line going through x.

Lemma 2.6. *Let* $0 \leqslant y < x \leqslant 1$*. We have*

$$\mathcal{I} := \int_{v=-g_x(y)}^{0} \int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dz\, dv \leqslant 2\left(\frac{\beta}{\alpha}\right)^{1/4}(x - y).$$

Demonstration Let us first use Fubini's theorem to switch the order of integration. The lower bound $x_b(x, v) \geqslant z$ becomes an upper bound $v \leqslant -g_x(z)$, and we have

$$\mathcal{I} = \int_{z=y}^{x} \int_{v=-g_x(y)}^{-g_x(z)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz = \int_{z=y}^{x} \int_{v=g_x(z)}^{g_x(y)} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}} dv\, dz.$$

With the change of variable $w = v - g_x(z)$, and using $\frac{d}{dw}\left[\sinh^{-1}\left(\sqrt{\frac{w}{a}}\right)\right] = \left(w^2 + 2aw\right)^{-1/2}$, we get

$$\mathcal{I} = \int_{z=y}^{x} \int_{w=0}^{g_x(y)-g_x(z)} \frac{1}{\left(w^2 + 2wg_x(z)\right)^{1/2}} dw\, dz = \int_{z=y}^{x} \sinh^{-1}\left(\sqrt{\frac{g_x(y) - g_x(z)}{g_x(z)}}\right) dz.$$

Using the elementary estimate $\sinh^{-1}(a) \leqslant p\, a^{1/p}$ for all $p \in \mathbb{N}^*$, and the coarse estimate $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$, we get

$$\mathcal{I} \leqslant \int_{z=y}^{x} p\left(\frac{g_x(y)}{g_x(z)}\right)^{1/2p} dz = \int_{z=y}^{x} p\left(\frac{\varphi(y) - \varphi(x)}{\varphi(z) - \varphi(x)}\right)^{1/4p} dz \quad \forall p \in \mathbb{N}^*.$$

The assumption of strong concavity yields $\varphi(z)-\varphi(x)\geqslant -\varphi'(z)(x-z)+\frac{\alpha}{2}|x-z|^2\geqslant \frac{\alpha}{2}(x-z)^2$, so that

$$\mathcal{I}\leqslant\left(\frac{2}{\alpha}\right)^{1/4p}(\varphi(y)-\varphi(x))^{1/4p}\int_{z=y}^{x}p\frac{1}{(x-z)^{1/2p}}dz=\left(\frac{2}{\alpha}\right)^{1/4p}(\varphi(y)-\varphi(x))^{1/4p}\frac{p^2}{p-1/2)}(x-y)^{1-1/2p}.$$

By the strong $(-\beta)-$convexity, we may write $\varphi(y)-\varphi(x)\leqslant\beta\frac{(x-y)^2}{2}$, so that

$$\mathcal{I}\leqslant\left(\frac{\beta}{\alpha}\right)^{1/4p}\frac{p^2}{p-1/2}(x-y)\quad\forall p\in\mathbb{N}^*.$$

Using the coarse estimates $\sinh^{-1}(a)\leqslant a$ and $\sqrt{\frac{a-b}{b}}\leqslant\sqrt{\frac{a}{b}}$, we get

$$\mathcal{I}\leqslant\int_{z=y}^{x}\sqrt{\frac{g_x(y)}{g_x(z)}}dz=\int_{z=y}^{x}\left(\frac{\varphi(y)-\varphi(x)}{\varphi(z)-\varphi(x)}\right)^{1/4}dz.$$

The assumption of strong concavity yields $\varphi(z)-\varphi(x)\geqslant -\varphi'(z)(x-z)+\frac{\alpha}{2}|x-z|^2\geqslant \frac{\alpha}{2}(x-z)^2$, so that

$$\mathcal{I}\leqslant\left(\frac{2}{\alpha}\right)^{1/4}(\varphi(y)-\varphi(x))^{1/4}\int_{z=y}^{x}\frac{1}{(x-z)^{1/2}}dz=2\left(\frac{2}{\alpha}\right)^{1/4}(\varphi(y)-\varphi(x))^{1/4}\sqrt{x-y}.$$

By the strong $(-\beta)-$convexity, we may write $\varphi(y)-\varphi(x)\leqslant\beta\frac{(x-y)^2}{2}$, so that

$$\mathcal{I}\leqslant 2\left(\frac{\beta}{\alpha}\right)^{1/4}(x-y).$$

□

Lemma 2.7. *Let $0\leqslant y\leqslant x\leqslant 1$, and $-v_0<-g_x(y)$. We have*

$$\mathcal{I}:=\int_{v=-v_0}^{-g_x(y)}\int_{z=y}^{1}\frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}dz\,dv\leqslant\frac{2\sqrt{2}}{\alpha^{1/4}}\left(v_0^2+\beta\left(x-y\right)^2\right)^{1/4}.$$

Demonstration We first shift the $v-$integration from the vertical line $z=x$ to $z=y$. Let $w=w(v)$ be such that

$$\mathcal{L}_i(y,w(v))=\mathcal{L}_i(x,v),\quad \text{i.e.}\quad w(v)=-\left(v^2+2\left(\varphi(x)-\varphi(y)\right)\right),\ \text{and } dv=\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Then, defining $w_0:=\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$, and noticing that $w(-g_x(y))=0$, we get

$$\mathcal{I}=\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+2\left(\varphi(y)-\varphi(z)\right)\right)^{1/2}}dz\frac{-w}{\left(w^2+2\left(\varphi(y)-\varphi(x)\right)\right)^{1/2}}dw.$$

Since $y\leqslant x$, we have $\varphi(y)\geqslant\varphi(x)$, and $\frac{-w}{\left(w^2+2(\varphi(y)-\varphi(x))\right)^{1/2}}\leqslant\frac{-w}{|w|}=1$. By the strong concavity assumption, we have $\varphi(y)-\varphi(z)\geqslant-\varphi'(y)(z-y)+\frac{\alpha}{2}|z-y|^2\geqslant\frac{\alpha}{2}(z-y)^2$, so that

$$\mathcal{I}\leqslant\int_{w=-w_0}^{0}\int_{z=y}^{1}\frac{1}{\left(w^2+\alpha(z-y)^2\right)^{1/2}}dzdw=\int_{w=0}^{w_0}\int_{z=0}^{1-y}\frac{1}{\left(w^2+\alpha z^2\right)^{1/2}}dzdw.$$

Using lemma (2.1), and the $(-\beta)-$convexity, we conclude that

$$\mathcal{I}\leqslant 2\frac{\sqrt{2w_0(1-y)}}{\alpha^{1/4}}=\frac{2\sqrt{2(1-y)}}{\alpha^{1/4}}\left(v_0^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/4}\leqslant\frac{2\sqrt{2}}{\alpha^{1/4}}\left(v_0^2+\beta\left(x-y\right)^2\right)^{1/4}.$$

□

Proposition 2.1. *The density n_i is bounded. More precisely,*

$$n_i(x)\leqslant 2\nu\left(2\left(\frac{\beta}{\alpha}\right)^{1/4}+2\sqrt{2}\left(\frac{\beta}{\alpha}\right)^{1/4}+\frac{2\sqrt{2}}{\alpha^{1/4}}\sqrt{v^*}\right)\qquad\forall x\in[0,1].$$

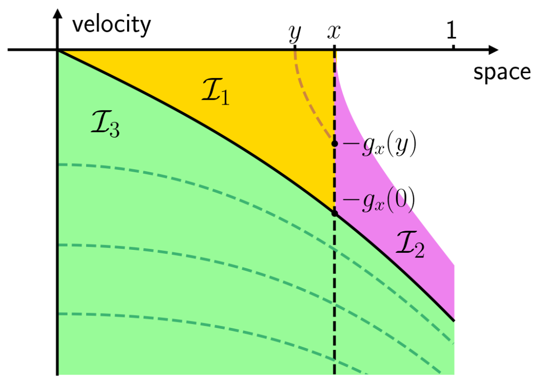

Figure 1: Decomposition of the integral defining n_i.

The phase space is divided by the critical characteristic (in solid black). Whenever $v \leqslant -g_x(0)$, the characteristics (dotted green lines) are reaching the boundary with $x_b(x,v) = 0$.

Demonstration We use the symmetry of f_i to write

$$n_i(x) = \int_{v=-\infty}^{\infty} f_i(x,v)dv = 2\int_{v=-\infty}^{0} f_i(x_b(x,v), v_b(x,v))dv = 2\nu \int_{v=-\infty}^{0}\int_{t=-\infty}^{0} f_e(x(t),v(t))dtdv,$$

where $(x(t),v(t))_{t\leqslant 0}$ is the ion characteristic reaching $(x_b(x,v), v_b(x,v))$ at $t=0$. Notice that the lower bounds are artificial, since the characteristic enters the support of f_e in finite time: we may use $v \geqslant -v^*$, and consider only times t for which $x(t) \in [0,1]$.

We first reparametrize $(x(t),v(t))$ using the space variable. Define $z = x(t) \in [x_b, 1]$, and observe that

$$dz = \dot{x}(t)dt = v(t)dt, \quad \text{with} \quad \mathcal{L}_i(z, v(t)) = \mathcal{L}_i(x,v) \quad \iff \quad v(t) = -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}.$$

Then, the density rewrites

$$n_i(x) = 2\nu \int_{v=-v^*}^{0}\int_{z=x_b(x,v)}^{1} \frac{f_e\left(z, -\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}\right)}{\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}\, dzdv.$$

Let us show that n_i is bounded. We use the coarse estimate $f_e \leqslant \bar{c}$, and decompose the integral in three:

$$n_i(x) \leqslant 2\nu\bar{c}\left[\underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x}}_{\mathcal{I}_1} + \underbrace{\int_{v=-g_x(0)}^{0}\int_{z=x}^{1}}_{\mathcal{I}_2} + \underbrace{\int_{v=-v^*}^{-g_x(0)}\int_{z=x_b(x,v)}^{1}}_{\mathcal{I}_3}\right] \frac{1}{\left(v^2 + 2\left(\varphi(x)-\varphi(z)\right)\right)^{1/2}}\, dzdv.$$

The corresponding domains are represented figure (3).

Notice that whenever $z \leqslant x$, we have $0 \geqslant 2\left(\varphi(x)-\varphi(z)\right) = -\left(2\left(\varphi(z)-\varphi(x)\right)\right)^{2/2} = -g_x^2(z)$. Then, the integral $\mathcal{I}_1$ may be bounded using lemma (2.6) with $y=0$:

$$\mathcal{I}_1 = \int_{v=-g_x(0)}^{0}\int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dzdv \leqslant 2\left(\frac{\beta}{\alpha}\right)^{1/4} x \leqslant 2\left(\frac{\beta}{\alpha}\right)^{1/4}.$$

We use lemma (2.7) to bound $\mathcal{I}_2$ and $\mathcal{I}_3$. In the first case, we take $y = x$ and $v_0 = g_x(0)$, and notice that $-g_x(x) = 0$. In the second case, we take $v_0 = v^*$ and $y = 0$, and notice that on $v \leqslant -g_x(0)$, we have $x_b(x,v) = 0$ (the velocity is low enough so that the characteristic ends on $x_b = 0$). Finally, we use that $g_x^2(0) = -2\varphi(1) \leqslant \beta$ to write

$$\mathcal{I}_2 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\left(g_x^2(0)\right)^{1/4} \leqslant 2\sqrt{2}\left(\frac{\beta}{\alpha}\right)^{1/4}, \quad \text{and} \quad \mathcal{I}_3 \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\left((v^*)^2 + 2\varphi(x)\right)^{1/4} \leqslant \frac{2\sqrt{2}}{\alpha^{1/4}}\sqrt{v^*}.$$

$\square$

Lemma 2.8. *The density n_i is uniformly bounded away from 0. More precisely,*

$$n_i(x) \geqslant \frac{\sqrt{2}\nu\underline{c}(\overline{v}^2 - \underline{v}^2)}{\overline{v}\left(\frac{\overline{v}^2}{2} + \beta\right)^{1/2}} \frac{\underline{v}}{\sqrt{\beta\left(1+\frac{1}{\mu}\right)}} \qquad \forall x \in [0,1].$$

Demonstration By assumption, there exists a constant $\underline{c} > 0$ such that $f_{e,b}(v) \geqslant \underline{c}\mathbf{1}_{\{\underline{v}\leqslant|v|\leqslant\overline{v}\}}$, which implies

$$f_e(x,v) \geqslant \underline{c}\mathbf{1}_{\{\mathcal{L}_e(0,\underline{v})\leqslant\mathcal{L}_e(x,v)\leqslant\mathcal{L}_e(0,\overline{v})\}}.$$

Notice that we may always take $\underline{v} > 0$ as long as $\overline{v} > 0$. Let us denote $\underline{\varepsilon} \coloneqq \mathcal{L}_e(0,\underline{v})$ and $\overline{\varepsilon} \coloneqq \mathcal{L}_e(0,\overline{v})$. For any $x \in [0,1]$, we have

$$n_i(x) = 2\nu \int_{w\in\mathbb{R}_-} \int_{\tau=-\infty}^{0} f_e(x(\tau), v(\tau)) d\tau dw, \quad \text{with} \quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x, w).$$

Notice that for $w \in [-g_x(-\overline{v}), -g_x(0)]$, the characteristic issued from (x,w) enters the domain $\{\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}\}$. We will restrict the domain of integration to

$$\mathcal{D} \coloneqq \left\{(w,\tau) \in \mathbb{R}_-^2 \mid \underline{\varepsilon} \leqslant \mathcal{L}_e(x(\tau), v(\tau)) \leqslant \overline{\varepsilon} \text{ and } 0 \leqslant x(\tau) \leqslant a\right\},$$

where (a, v_a) is the unique point such that $\mathcal{L}_e(a, v_a) = \mathcal{L}_e(0, \underline{v})$ and $\mathcal{L}_i(a, v_a) = 0$ (note that a depends on β). This (somehow coarse) estimate will give us trivial bounds when applying the change of variable of lemma (2.4), namely $z = x(\tau)$ and $\varepsilon = \mathcal{L}_e(x(\tau), v(\tau))$. Denoting by $[\underline{\tau}, \overline{\tau}] \subset]-\infty, 0]$ the time interval where $(x(\tau), v(\tau)) \in \mathcal{D}$, we have

$$n_i(x) \geqslant \int_{w=-g_x(-\overline{v})}^{-g_x(0)} \int_{\tau=\underline{\tau}}^{\overline{\tau}} 2\nu\underline{c}\, d\tau dw = \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{2\nu\underline{c}}{2\left(\varepsilon + \frac{1}{\mu}\varphi(z)\right)^{1/2} \left(\varepsilon - \varphi(x) + \left(1+\frac{1}{\mu}\right)\varphi(z)\right)^{1/2}} dz d\varepsilon.$$

Using remark (2.2), we are assured that the square roots are well-defined, since

$$\varepsilon \geqslant \underline{\varepsilon} = \mathcal{L}_e(0,\underline{v}) = \mathcal{L}_e(a, v_a) \geqslant \mathcal{L}_e(a, 0) \geqslant \mathcal{L}_e(z, 0) = -\frac{1}{\mu}\varphi(z) \quad \text{and} \quad \mathcal{L}_i(x(\tau), v(\tau)) = \mathcal{L}_i(x,w) \geqslant \mathcal{L}_i(x, 0).$$

Using that $\varepsilon + \frac{1}{\mu}\varphi(z) \leqslant \overline{\varepsilon} + 0$ and $\varepsilon - \varphi(x) + \left(1 + \frac{1}{\mu}\varphi(z)\right) \leqslant \overline{\varepsilon} + \beta + 0$, we obtain

$$n_i(x) \geqslant 2\nu \int_{\varepsilon=\underline{\varepsilon}}^{\overline{\varepsilon}} \int_{z=0}^{a} \frac{\underline{c}}{2\overline{\varepsilon}^{1/2}\left(\overline{\varepsilon}+\beta\right)^{1/2}} dz d\varepsilon = \frac{\nu a\underline{c}(\overline{\varepsilon} - \underline{\varepsilon})}{\overline{\varepsilon}^{1/2}\left(\overline{\varepsilon}+\beta\right)^{1/2}}.$$

Let us make explicit the dependance over β by estimating a using the $(-\beta)-$convexity assumption: we have

$$\varphi(x) \geqslant -\beta\frac{x^2}{2} \quad \Longrightarrow \quad \varphi^{-1}(y) \geqslant \sqrt{\frac{-2y}{\beta}}, \quad \text{so that} \quad a = \varphi^{-1}\left(-\frac{\underline{v}^2}{2\left(1+\frac{1}{\mu}\right)}\right) \geqslant \frac{\underline{v}}{\sqrt{\beta\left(1+\frac{1}{\mu}\right)}}.$$

and the uniform lower bound behaves like β^{-1} when $\beta \to \infty$. □

Lemma 2.9. ***We probably can do better, but maybe not win any exponent.*** *Suppose that there exists $v_* > 0$ such that $f_{e,b}(v) = 0$ for all $|v| \leqslant v_*$. Then the density f_i satisfies the uniform bound*

$$f_i(x,v) \leqslant 2^{9/4}\overline{c}\sqrt{\frac{\frac{\beta}{\alpha}(1+\frac{1}{\mu})+1}{v_*}} \quad \forall (x,v) \in [0,1]\times\mathbb{R}.$$

Demonstration Let $(x,v) \in [0,1]\times\mathbb{R}_-$, and denote by $(x(t), v(t))_{t\leqslant 0}$ the ion characteristic going through $(x,v) = (x(-T), v(-T))$, with the convention $(x_b(x,v), v_b(x,v)) = (x(0), v(0))$. Owing to the positivity of f_e,

$$f_i(x(0), v(0)) = \int_{t=-T}^{0} f_e(x(t), v(t)) dt + f_i(x,v) \geqslant f_i(x,v).$$

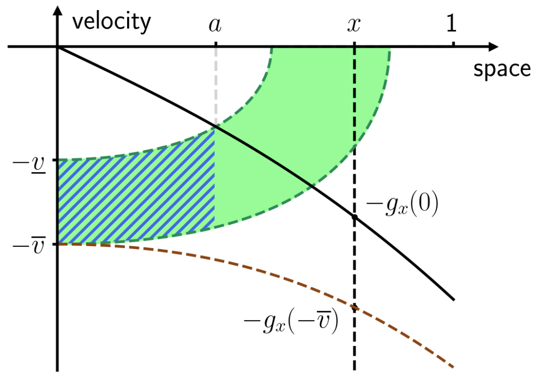

Figure 2: Notations for the lower bound on n_i.

The coloured area corresponds to the domain $\underline{\varepsilon} \leqslant \mathcal{L}_e \leqslant \overline{\varepsilon}$, on which we know that $f_e \geqslant \underline{c}$. The hatched area represents the domain $\mathcal{D}$. The solid black line is the critical ion characteristic.

On the other hand, we use $f_i(x,v) + f_i(x,-v) = 2f_i(x(0),v(0))$ to write $f_i(x,-v) \leqslant 2f_i(x(0),v(0))$. This shows that it is enough to bound f_i on the boundary $\mathcal{B} := \{x \geqslant 0, v = 0\} \cup \{x = 0, v \leqslant 0\}$ to obtain an uniform bound.

Let then $(x,v) \in \mathcal{B}$. By hypothesis, $f_e(x(t),v(t))$ vanishes whenever $\mathcal{L}_e(x(t),v(t)) \notin [\mathcal{L}_e(0,v_*), \mathcal{L}_e(0,v^*)]$, and is bounded by $\overline{c}$ otherwise. Then, we may use the reparametrization by space

$$f_i(x,v) \leqslant \int_{z=a(x,v)}^{1} \frac{\overline{c}}{\left(v^2 + 2\varphi(x) - 2\varphi(z)\right)^{1/2}} dz, \quad \text{with} \quad a(x,v) := \begin{cases} \varphi^{-1}\left(\frac{\frac{v^2}{2} + \varphi(x) - \frac{{v_*}^2}{2}}{1+1/\mu}\right) & \text{if } \mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,v_*) \\ x & \text{otherwise.} \end{cases}$$

The function a gives the smallest spatial coordinate of the characteristic $(x(t),v(t))_{t\leqslant 0}$ such that $f_e > 0$. Using that $\varphi(z) \leqslant -\alpha\frac{z^2}{2}$, we have

$$f_i(x,v) \leqslant \overline{c} \int_{z=a(x,v)}^{1} \frac{1}{\left(v^2 + 2\varphi(x) - \alpha z^2\right)^{1/2}} dz \leqslant \overline{c} \min\left(\frac{1}{|v^2 + 2\varphi(x)|^{1/2}}, \frac{2}{\sqrt{\alpha a(x,v)}}\right)$$

where we used lemma (2.3) with $L := v^2 + 2\varphi(x)$.

We rely on the concavity estimate

$$\varphi(x) \geqslant (1-x)\varphi(0) + x\varphi(1) + \frac{\alpha}{2}x(1-x) \geqslant -x\beta \quad \Longrightarrow \quad -\frac{y}{\beta} \leqslant \varphi^{-1}(y)$$

to write that for (x,v) satisfying $\mathcal{L}_e(x,v) \leqslant \mathcal{L}_e(0,v_*)$,

$$a(x,v) = \varphi^{-1}\left(\frac{\frac{v^2}{2} + \varphi(x) - \frac{{v_*}^2}{2}}{1+1/\mu}\right) \geqslant \frac{\frac{{v_*}^2}{2} - (\frac{v^2}{2} + \varphi(x))}{\beta(1+1/\mu)}.$$

Then

$$f_i(x,v) \leqslant \overline{c} \min\left(\frac{1}{\sqrt{|X|}}, \frac{A}{\sqrt{B-X}}\right) \quad \text{with} \quad X := \frac{v^2}{2} + \varphi(x), \quad A := \frac{2}{\sqrt{\frac{\alpha}{\beta(1+1/\mu)}}}, \quad \text{and} \quad B := \frac{{v_*}^2}{2}.$$

The elementary study of the function $X \to \min\left(|X|^{-1/2}, A(B-X)^{-1/2}\right)$ reveals a global maximum at $X = \frac{B}{A^2+1} < B$, and we conclude to the result.

$\square$

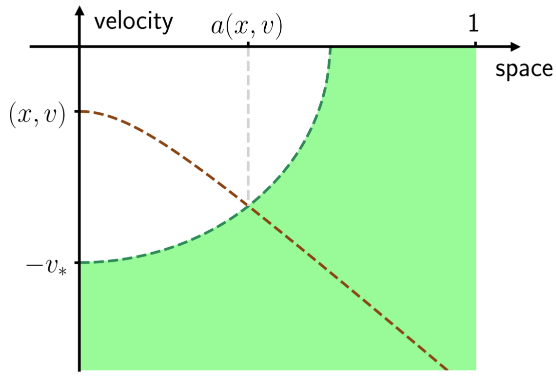

Figure 3: Notations for the boundedness of f_i.

The coloured area corresponds to $\mathcal{L}_e(x,v) \geqslant \mathcal{L}_e(0,v_*)$. The function $a(x,v)$ gives the point of the ion characteristic (in brown) where f_e vanishes.

2.3 Continuity estimates

2.3.1 Electron density n_e

Lemma 2.10. *The electronic density n_e is continuous.*

Demonstration We already know that n_e is bounded. Moreover, using the symmetry $f_e(x,v) = f_e(x,-v)$, we may write

$$n_e(x) - n_e(y) = 2\underbrace{\int_{v=0}^{v_x(y)} f_e(x,v)dv}_{=:\mathcal{I}^+} + 2\left(\underbrace{\int_{v=v_x(y)}^{v_x(1)} f_e(x,v)dv - \int_{v=0}^{v_y(1)} f_e(y,v)dv}_{=:\mathcal{I}^-}\right).$$

The term $\mathcal{I}^+$ is bounded by $\bar{c}v_x(y) = \bar{c}\left(\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$, and by continuity of φ, we have $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. On the first integral of $\mathcal{I}^-$, we apply the change of variable

$$w = \left(v^2 - \frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2} \quad \Longleftrightarrow \quad dv = \frac{w}{\left(w^2 + \frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}dw, \quad w \in [0, v_y(1)]$$

to get

$$\mathcal{I}^- = \int_{w=0}^{v_y(1)} f_e\left(x, \left(w^2 + \frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}\right)\frac{w}{\left(w^2 + \frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}dw - \int_{v=0}^{v_y(1)} f_e(y,v)dv.$$

Notice that

$$f_e\left(x, \left(w^2 + \frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}\right) = f_{e,b}\left(\frac{w^2}{2} + \frac{1}{\mu}\left(\varphi(x)-\varphi(y)\right) - \frac{1}{\mu}\varphi(x)\right) = f_e(y,w).$$

Renaming w in v, we obtain the (clearly nonpositive) expression

$$\mathcal{I}^- = \int_{v=0}^{v_y(1)} f_e(y,v)\left(\frac{v}{\left(v^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}-1\right)dv \geqslant \overline{c}\int_{v=0}^{v_y(1)}\left(\frac{v}{\left(v^2+\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}}-1\right)dv$$
$$=\overline{c}\left(\left(v_y(1)^2+\frac{2}{\mu}\left(\underbrace{\varphi(x)-\varphi(y)}_{\geqslant 0}\right)\right)^{1/2}-\left(\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}-v_y(1)\right)\geqslant -\overline{c}\left(\frac{2}{\mu}\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}$$

and this shows that $\mathcal{I}^- \underset{y\to x}{\longrightarrow} 0$. □

Lemma 2.11. *Let n_e^φ and n_e^ψ be the electron densities generated by potentials φ and ψ satisfying the assumptions. Then*

1. *If $f_{e,b}$ is Lipschitz-continuous with constant $[f_{e,b}]$, then*

$$|n_e^\varphi(x)-n_e^\psi(x)| \leqslant 2[f_{e,b}]v^*\sqrt{\frac{2}{\mu}}\,|\varphi(x)-\psi(x)|^{1/2} \qquad \forall (x,y)\in[0,1]^2.$$

2. *If there exists a constant $[f_{e,b}]$ such that $|f_{e,b}(x)-f_{e,b}(y)| \leqslant [f_{e,b}]|x^2-y^2|$ (or equivalently, $x\to f_{e,b}(\sqrt{x})$ is a lipschitz function, as for instance e^{-x^2}), then*

$$|n_e^\varphi(x)-n_e^\psi(x)| \leqslant \frac{4v^*[f_{e,b}]_2}{\mu}\,|\varphi(x)-\psi(x)| \qquad \forall (x,y)\in[0,1]^2.$$

Demonstration We have

$$\left|n_e^\varphi(x)-n_e^\psi(x)\right| \leqslant 2\int_{v=0}^{v^*}\left|f_e^\varphi(x,v)-f_e^\psi(x,v)\right|dv = 2\int_{v=0}^{v^*}\left|f_{e,b}\left(\left(v^2-\frac{2}{\mu}\varphi(x)\right)^{1/2}\right)-f_{e,b}\left(\left(v^2-\frac{2}{\mu}\psi(x)\right)^{1/2}\right)\right|dv.$$

Then, if $f_{e,b}$ is Lipschitz with constant $[f_{e,b}]$, we obtain

$$\left|n_e^\varphi(x)-n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{v^*}\left|\left(v^2-\frac{2}{\mu}\varphi(x)\right)^{1/2}-\left(v^2-\frac{2}{\mu}\psi(x)\right)^{1/2}\right|dv.$$

Using lemma (2.2) yields

$$\left|n_e^\varphi(x)-n_e^\psi(x)\right| \leqslant 2[f_{e,b}]\int_{v=0}^{v^*}\left|-\frac{2}{\mu}\varphi(x)+\frac{2}{\mu}\psi(x)\right|^{1/2}dv = 2[f_{e,b}]v^*\sqrt{\frac{2}{\mu}}\,|\varphi(x)-\psi(x)|^{1/2}.$$

If $f_{e,b}(\sqrt{\cdot})$ is Lipschitz with constant $[f_{e,b}]_2$, we may directly write

$$\left|n_e^\varphi(x)-n_e^\psi(x)\right| \leqslant 2v^*[f_{e,b}]_2\frac{2}{\mu}\,|\varphi(x)-\psi(x)|.$$

□

2.3.2 Ion density n_i

Proposition 2.2. *The density n_i is continuous.*

Demonstration Let $0 \leqslant y < x \leqslant 1$. For convenience, we represent $n_i(x)$ (resp. $n_i(y)$) as an integral with the artificial lower bound $-g_x(-v^*) \leqslant -v^*$ (resp. $-g_y(-v^*)$). Then

$$\begin{aligned} n_i(x) - n_i(y) &= 2\int_{v=-g_x(-v^*)}^{0} f_i(x_b(x,v), v_b(x,v))dv - 2\int_{v=-g_y(-v^*)}^{0} f_i(x_b(y,v), v_b(y,v))dv \\ &= 2\underbrace{\left[\int_{v=-g_x(-v^*)}^{-g_x(y)} f_i(x_b(x,v), v_b(x,v))dv - \int_{v=-g_y(-v^*)}^{0} f_i(x_b(y,v), v_b(y,v))dv\right]}_{=:\mathcal{I}^-} \qquad (2.1) \\ &\quad + 2\underbrace{\int_{v=-g_x(y)}^{0} f_i(x_b(x,v), v_b(x,v))dv}_{=:\mathcal{I}^+}. \end{aligned}$$

The term $\mathcal{I}^+$ is clearly nonnegative, and may be adressed using our lemmas. Indeed, using the integral representation of $f_i(x_b, v_b)$ and the reparametrization by a space variable z, we have

$$\begin{aligned} \mathcal{I}^+ &= \int_{v=-g_x(y)}^{0} \left[\int_{z=x_b(x,v)}^{x} + \int_{z=x}^{1}\right] \frac{f_e(z, -\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}))}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\ &\leqslant \bar{c}\int_{v=-g_x(y)}^{0}\int_{z=x_b(x,v)}^{x} \frac{1}{\left(v^2 - g_x^2(z)\right)^{1/2}}\, dzdv + \bar{c}\int_{v=-g_x(y)}^{0}\int_{z=x}^{1} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(z)\right)\right)^{1/2}}\, dzdv \\ &\leqslant \bar{c}\left(2\sqrt{\frac{2}{\alpha}}\left(\varphi(y) - \varphi(x)\right)^{1/4}\sqrt{x-y} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(2(\varphi(y) - \varphi(x))\right)^{1/2}\right), \end{aligned}$$

where we used lemma (2.6) for the first term, and lemma (2.7) for the second term (with $y = x$ and $v_0 = -g_x(y)$ under the notations of the lemma). Since φ is continuous, we deduce that $\mathcal{I}^+ \underset{y\to x}{\longrightarrow} 0$. Taking the extreme case $y = 0$ and $x = 1$, we obtain that

$$\mathcal{I}^+ \leqslant \bar{c}\left(2\sqrt{\frac{2}{\alpha}}\left(-\varphi(1)\right)^{1/4} + \frac{2\sqrt{2}}{\alpha^{1/4}}\left(-2\varphi(1)\right)^{1/2}\right) =: K.$$

Let us now focus on $\mathcal{I}^-$. On the first integral, we make the change of variable

$$w = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2} \qquad v = -\left(w^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}.$$

Since $\mathcal{L}_i(x,v) = \mathcal{L}_i(y,w)$, this yields $x_b(x,v) = x_b(y,w)$ and $v_b(x,v) = v_b(y,w)$. The bounds $v \in [-g_x(-v^*), -g_x(y)]$ are exactly transported to $w \in [-g_y(-v^*), 0]$. Renaming w in v, we get

$$\mathcal{I}^- = \int_{v=-g_y(-v^*)}^{0} f_i(x_b(y,v), v_b(y,v))\left(\frac{-v}{\left(v^2 + 2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}} - 1\right) dv.$$

Since $\varphi(y) \geqslant \varphi(x)$, the factor of f_i is nonpositive, and so is $\mathcal{I}^-$. Moreover,

$$n_i(x) - n_i(y) = 2\mathcal{I}^- + 2\mathcal{I}^+ \leqslant 2\mathcal{I}^- + 2K \quad \iff \quad \mathcal{I}^- = -K + n_i(x) - n_i(y) \geqslant -K - |n_i|_\infty$$

and $\mathcal{I}^-$ is bounded. The function $\frac{-v}{\left(v^2+2(\varphi(y)-\varphi(x))\right)^{1/2}} - 1$ converges pointwise to 0 when $x \to y$ **and f_i is almost everywhere finite**, and by Lebesgue's dominated convergence, $\mathcal{I}^- \underset{x\to y}{\longrightarrow} 0$. Then n_i is continuous. $\square$

3 Results

Let $0 < \alpha \leqslant \beta$, and define the following convex set:

$$\mathcal{K} := \left\{\varphi \in \mathcal{C}^1([0,1],\mathbb{R}) \quad | \quad \varphi(0) = \varphi'(0) = 0, \quad \varphi\ \alpha - \text{concave et } (-\beta) - \text{convex.}\right\}.$$

Both variation conditions rewrite

$$\forall (x,y,\gamma) \in [0,1]^3, \quad \begin{cases} \varphi((1-\gamma)x + \gamma y) \geqslant (1-\gamma)\varphi(x) + \gamma\varphi(y) + \frac{\alpha}{2}\gamma(1-\gamma)|x-y|^2, \\ \varphi((1-\gamma)x + \gamma y) \leqslant (1-\gamma)\varphi(x) + \gamma\varphi(y) + \frac{\beta}{2}\gamma(1-\gamma)|x-y|^2. \end{cases}$$

In the case $\varphi \in \mathcal{C}^2$, it is equivalent to $\varphi'' \in [-\beta, -\alpha]$.

Lemma 3.1. *$\mathcal{K}$ is closed for the topology induced by the sup-norm $|\cdot|_\infty := \max_{[0,1]} |\cdot|$.*

Demonstration Since $\mathcal{K} \subset \mathcal{C}^0([0,1],\mathbb{R})$, we know that any Cauchy sequence $(\varphi_n)_n \subset \mathcal{K}$ admits a limit $\varphi \in \mathcal{C}([0,1],\mathbb{R})$. The pointwise condition $\varphi_n(0) = 0$ and the pointwise grows conditions are preserved when $n \to \infty$. Since the family of continuous fonctions $(\varphi_n')_n$ is equilipschitz, we may use Arzelà-Ascoli to extract an uniformly converging subsequence $\varphi_{n_k}' \to \varphi_\infty'$. Using that for all $(x,y) \in [0,1]$,

$$\left|\varphi(y) - \varphi(x) - \int_x^y \varphi_\infty'(z)dz\right| \leqslant |\varphi(y) - \varphi_n(y)| + |\varphi(x) - \varphi_n(x)| + \int_x^y |\varphi_\infty'(z) - \varphi_n'(z)|dz \underset{n\to\infty}{\longrightarrow} 0,$$

we get that φ_∞' is the (continuous) derivative of φ. Finally, the uniform convergence $\varphi_{n_k}' \to \varphi_\infty'$ gives $\varphi'(0) = 0$. $\square$

We define an operator $F : \mathcal{K} \mapsto \mathcal{C}^2([0,1],\mathbb{R})$ by the solution of the Poisson problem

$$-\lambda^2 F'' = n_i[\varphi] - n_e[\varphi],$$

where n_i and n_e are the ion and electron densities obtained with the characteristics induced by φ.

Stability The function F is solution to a Poisson problem with bounded continuous source term, so it enjoys $\mathcal{C}^2$ regularity. The boundary condition $F_\varphi(0) = F_\varphi'(0) = 0$ are satisfied by construction. We turn to the stability of the variation estimates: owing to the boundedness of n_i and n_e stated above, we have

$$\begin{cases} -F''(x) \geqslant \frac{\kappa_1}{\sqrt{\beta}(\kappa_2+\beta)^{1/2}} - \kappa_3, \\ -F''(x) \leqslant \frac{\kappa_4}{\alpha^{1/4}}\left(\beta^{1/4} + \kappa_5\right) \end{cases}$$

where

$$\kappa_1 := \frac{\nu\sqrt{2}\underline{c}(\overline{v}^2 - \underline{v}^2)\underline{v}}{\lambda^2\overline{v}\sqrt{1+\frac{1}{\mu}}}, \quad \kappa_2 := \frac{\overline{v}^2}{2}, \quad \kappa_3 := \frac{2\overline{c}v^*}{\lambda^2}, \quad \kappa_4 := \frac{4\nu(1+\sqrt{2})}{\lambda^2}, \quad \kappa_5 := 4\sqrt{2v^*}.$$

A sufficient condition for the stability $F(\mathcal{K}) \subset \mathcal{K}$ is to obtain $\alpha \leqslant -F'' \leqslant \beta$. We may not always have a solution (α,β)... It depends on the coefficients.