# Notes on fixed-point procedure

We want to obtain estimates on $n_i - n_e$. We make the following assumptions:

- The electron density $f_e$ satisfies the boundary condition, and is bounded by a constant $c \geqslant 0$.
- The potential $\varphi$ is strongly concave, i.e. there exists $\alpha > 0$ such that $\varphi''(x) \leqslant -\alpha$ uniformly over $x \in [0,1]$.
- We have $\varphi(0) = \varphi'(0) = 0$.

The assumptions on $\varphi$ yield that

$$\varphi'(x) \leqslant -\alpha x, \quad \varphi(x) \leqslant -\alpha \frac{x^2}{2}. \tag{0.1}$$

Let us first focus on $n_e(x)$. The characteristics of the electron density are the level lines of

$$\mathcal{L}_e(x,v) := \frac{v^2}{2} - \frac{1}{\mu}\varphi(x). \tag{0.2}$$

Since $\varphi$ is strongly concave, these curves are closed. Since $f_e$ satisfies the homogeneous boundary condition, its support is embedded in $\left\{(x,v) \mid \frac{v^2}{2} - \frac{1}{\mu}\varphi(x) \leqslant \frac{0^2}{2} - \frac{1}{\mu}\varphi(1)\right\}$. In particular, we denote by $\underline{v}_e$ the extremal speed of the support,

$$\underline{v}_e := \sqrt{-\frac{2}{\mu}\varphi(1)}. \tag{0.3}$$

We can roughly majorize

$$n_e(x) = \int_{v\in\mathbb{R}} f_e(x,v)dv \leqslant \int_{v=-\underline{v}_e}^{\underline{v}_e} c dv = 2c\underline{v}_e \leqslant 2c\sqrt{-\frac{2}{\mu}\varphi(1)}.$$

The estimates on $n_i$ are slightly more technical. Let the ion Lyapunov function be defined as

$$\mathcal{L}_i(x,v) := \frac{v^2}{2} + \varphi(x). \tag{0.4}$$

In the sequel, we will heavily rely on the level lines of $\mathcal{L}_i$ to partition the space. We distinguish the *critical characteristic* as the curve $\{\mathcal{L}_i = 0\}$. Let $x \in [0,1]$ and $v \in \mathbb{R}_-$. We denote by $(x_b(x,v), v_b(x,v))$ the intersection of the boundary $\{x=0\} \cap \{v=0\}$ with the characteristic issued from $(x,v)$, equal to

$$\begin{pmatrix} x_b(x,v) \\ v_b(x,v) \end{pmatrix} := \begin{cases} \begin{pmatrix} \varphi^{-1}\left(\frac{v^2}{2} + \varphi(x)\right) \\ 0 \end{pmatrix} & \text{if } \mathcal{L}_i(x,v) \leqslant 0, \\ \begin{pmatrix} 0 \\ -\sqrt{\frac{v^2}{2} + \varphi(x)} \end{pmatrix} & \text{if } \mathcal{L}_i(x,v) > 0. \end{cases}$$

In the following paragraph, we use $(x(t), v(t))_{t\leqslant 0}$ to denote the characteristic reaching $(x_b(x,v), v_b(x,v))$ at $t = 0$. We use the symmetry of $f_i$ to write

$$n_i(x) = 2\int_{v\in\mathbb{R}^-} f_i(x_b(x,v), v_b(x,v))dv = 2\int_{v\in\mathbb{R}^-}\int_{t=-\infty}^{0} f_e(x(t), v(t))dtdv.$$

The lower bound $t \to -\infty$ is artificial, since the characteristic exits the support of $f_e$ in finite time. We will split the double integral in three domains:

1. $\mathcal{D}_1$ will be $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) \leqslant 0 \text{ and } x(t) \geqslant x\right\}$. This is the region contained between the $x-$axis, the critical characteristic and the vertical line going through $x$.

2. $\mathcal{D}_2$ is $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) \leqslant 0 \text{ and } x < x(t) \geqslant 1\right\}$. It is exactly $\{\mathcal{L}_i \leqslant 0\} \setminus \mathcal{D}_1$.

3. $\mathcal{D}_3$ is defined by $\left\{(v,t) \in \mathbb{R}^2_- \mid \mathcal{L}_i(x,v) > 0 \text{ and } \underline{v}_e \geqslant v(t)\right\}$.

**Figure de la décomposition en domaines**

Since $f_e(x(t),v(t))$ vanishes outside $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, we may exactly decompose $n_i$ in

$$\frac{n_i(x)}{2} = \underbrace{\iint_{(v,t)\in\mathcal{D}_1} f_e(x(t),v(t))dtdv}_{\mathcal{I}_1} + \underbrace{\iint_{(v,t)\in\mathcal{D}_2} f_e(x(t),v(t))dtdv}_{\mathcal{I}_2} + \underbrace{\iint_{(v,t)\in\mathcal{D}_3} f_e(x(t),v(t))dtdv}_{\mathcal{I}_3}$$

Each term will be bound separately.

**Bound on $\mathcal{I}_1$** Let $v_0(x) := \sqrt{-2\varphi(x)}$ be the velocity such that $(x, -v_0(x))$ belongs to the critical characteristic. The characteristics in the domain $\mathcal{D}_1$ are joining points $(x,v)$, with $v \in [-v_0, 0]$, with points $(x_b(x,v), 0)$. Therefore, we may use the reparametrization

$$y = x(t) \quad dy = \dot{x}(t)dt = v(t)dt = -\left(v^2 + 2\left(\varphi(x) - \varphi(x(t))\right)\right)^{1/2} dt = -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2} dt$$

The integral $\mathcal{I}_1$ becomes

$$\mathcal{I}_1 = \int_{v=-v_0(x)}^{0} \int_{y=x}^{x_b(x,v)} f_e(y, -\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}) \frac{-1}{\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}} dydv.$$

By exchanging the bounds of the integrals along $y$, and using $f_e \leqslant c$, we get

$$\mathcal{I}_1 \leqslant c \int_{v=-v_0(x)}^{0} \int_{y=x_b(x,v)}^{x} \frac{1}{\left(v^2 + 2\left(\varphi(x) - \varphi(y)\right)\right)^{1/2}} dydv.$$

In order to use the explicit $v^2$, we use Fubini theorem to switch the order of integration (since everything is positive). To do this, we write

$$\begin{cases} -v_0(x) \leqslant v \leqslant 0 \\ x_b(x,v) \leqslant y \leqslant x \end{cases} \iff \begin{cases} 0 \leqslant y \leqslant x \\ -v_0(x) \leqslant v \leqslant -g_x(y) \end{cases}$$

where $y = x_b(x,v) = \varphi^{-1}\left(\frac{v^2}{2} + \varphi(x)\right)$ is equivalent to $v = -g_x(y) := -\left(2\left(\varphi(y) - \varphi(x)\right)\right)^{1/2}$. In the sequel, we drop the $x$ and simply write $g(y)$. The function $g : [0,x] \mapsto \mathbb{R}^+$ is well-defined, since $y \leqslant x \implies \varphi(y) \geqslant \varphi(x)$. By the assumption of strong concavity of $\varphi$, $g$ is positive whenever $y < x$. Finally, using that $v_0(x) = g(0)$, may now write

$$\mathcal{I}_1 \leqslant c \int_{y=0}^{x} \int_{v=-v_0(x)}^{-g(y)} \frac{1}{\left(v^2 - g^2(y)\right)^{1/2}} dvdy \underset{\text{with } w:=g(y)-v}{=} c \int_{y=0}^{x} \int_{w=0}^{g(0)-g(y)} \frac{1}{\left(w^2 + 2wg(y)\right)^{1/2}} dvdy.$$

By integration with $\frac{d}{dy} \sinh^{-1}\left(\sqrt{\frac{y}{a}}\right) = \frac{1}{y^2 - 2ay}$, we obtain

$$\mathcal{I}_1 \leqslant c \int_{y=0}^{x} \left[\sinh^{-1}\left(\sqrt{\frac{v}{g(y)}}\right)\right]_0^{g(0)-g(y)} dv = c \int_{y=0}^{x} \sinh^{-1}\left(\sqrt{\frac{g(0)-g(y)}{g(y)}}\right) dv.$$

We use the coarse estimates $\sinh^{-1}(z) \leqslant z$ and $\sqrt{\frac{a-b}{b}} \leqslant \sqrt{\frac{a}{b}}$ to reduce the expression to

$$\mathcal{I}_1 \leqslant c\sqrt{g(0)} \int_{y=0}^{x} \frac{1}{\sqrt{g(y)}} dv = c\sqrt{g(0)} \int_{y=0}^{x} \frac{1}{\left(2\left(\varphi(y)-\varphi(x)\right)\right)^{1/4}} dv.$$

Using the strong concavity of $\varphi$, and the sign $-\varphi'(y) \geqslant 0$, we get

$$\varphi(y)-\varphi(x) \geqslant \frac{\alpha}{2}|x-y|^2 - \varphi'(y)(x-y) \geqslant \frac{\alpha}{2}(x-y)^2.$$

With this, we may finally write

$$\mathcal{I}_1 \leqslant c\sqrt{g(0)} \int_{y=0}^{x} \frac{1}{\alpha^{1/4}\left(x-y\right)^{1/2}} dv = \frac{2c}{\alpha^{1/4}}\sqrt{g(0)x} = \frac{2c}{\alpha^{1/4}}\sqrt{\sqrt{-2\varphi(x)}x} \leqslant \frac{2c}{\alpha^{1/4}}\left(-2\varphi(1)\right)^{1/4}.$$

**Bound on $\mathcal{I}_2$** We use the same reparametrization as for $\mathcal{I}_2$, but with $y \in [x,1]$, to obtain

$$\mathcal{I}_2 \leqslant c \int_{v=-v_0(x)}^{0} \int_{y=x}^{1} \frac{1}{\left(v^2+2(\varphi(x)-\varphi(y))\right)^{-1/2}} dydv.$$

On $y \geqslant x$, we may directly use

$$\varphi(x)-\varphi(y) \geqslant \frac{\alpha}{2}|y-x|^2 - \varphi'(x)(y-x) \geqslant \frac{\alpha}{2}(y-x)^2$$

to get

$$\mathcal{I}_2 \leqslant c \int_{v=-v_0(x)}^{0} \int_{y=x}^{1} \frac{1}{\left(v^2+\alpha(y-x)^2\right)^{1/2)}} dydv \leqslant c \max\left(1, \frac{1}{\sqrt{\alpha}}\right) \int_{v=0}^{v_0(x)} \int_{z=0}^{1-x} \frac{1}{\left(v^2+z^2\right)^{1/2}} dydv.$$

By switching to polar coordinates over the (larger) domain $(\theta, r) \in [0, \frac{\pi}{2}] \times [0, \sqrt{v_0(x)^2+(1-x)^2}]$, we conclude to

$$\mathcal{I}_2 \leqslant c \max\left(1, \frac{1}{\sqrt{\alpha}}\right) \frac{\pi}{2} \sqrt{v_0(x)^2+(1-x)^2} \leqslant c \max\left(1, \frac{1}{\sqrt{\alpha}}\right) \frac{\pi}{2} \sqrt{1-2\varphi(1)}.$$

**Bound on $\mathcal{I}_3$** The domain $\mathcal{D}_3$ is covered by characteristics linking $x=1$ to $x=0$. We may use the same reparametrization within fixed bounds over $y$:

$$\mathcal{I}_3 \leqslant c \int_{v=-\underline{v}_e}^{-v_0(x)} \int_{y=0}^{1} \frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}} dydv = c \int_{y=0}^{1} \int_{v=-\underline{v}_e}^{-v_0(x)} \frac{1}{\left(v^2+2\left(\varphi(x)-\varphi(y)\right)\right)^{1/2}} dydv.$$

We will use the same argument as for $\mathcal{I}_2$, but with characteristics ending on $x=0$ instead of $x=x$. Our first step is then to replace $v$ by $w$ the velocity at $x=0$, defined by $\frac{v^2}{2}+\varphi(x) = \frac{w^2}{2}+0$. We have

$$w = -\left(v^2+2\varphi(x)\right)^{1/2} \in [-\underline{w}_e, 0], \quad v = -\left(v^2-2\varphi(x)\right)^{1/2}, \quad dv = \frac{-w}{\left(w^2-2\varphi(x)\right)^{1/2}} dw,$$

where $\underline{w}_e \left(\underline{v}_e{}^2+2\varphi(x)\right)^{1/2} \leqslant \underline{v}_e \leqslant \sqrt{-2\varphi(1)}$. Using the estimate $-\varphi(y) \geqslant \alpha\frac{y^2}{2}$, we conclude similarly that

$$\mathcal{I}_3 \leqslant c \int_{y=0}^{1} \int_{w=-\underline{w}_e}^{0} \frac{1}{\left(w^2-\varphi(y)\right)^{1/2}} dydv = c \int_{y=0}^{1} \int_{w=0}^{\underline{w}_e} \frac{1}{\left(w^2+\alpha y^2\right)^{1/2}} dydv \leqslant c \max\left(1, \frac{1}{\sqrt{\alpha}}\right) \frac{\pi}{2} \sqrt{1-2\varphi(1)}.$$

In conclusion, we obtained

$$n_i(x) \leqslant \frac{4}{\alpha^{1/4}}\left(-2\varphi(1)\right)^{1/4} + 4c \max\left(1, \frac{1}{\sqrt{\alpha}}\right) \frac{\pi}{2} \sqrt{1-2\varphi(1)}$$

## References