Hamilton-Jacobi via neural networks

Master's thesis defense

Averil PROST

directed by

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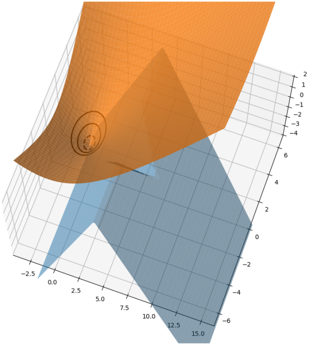
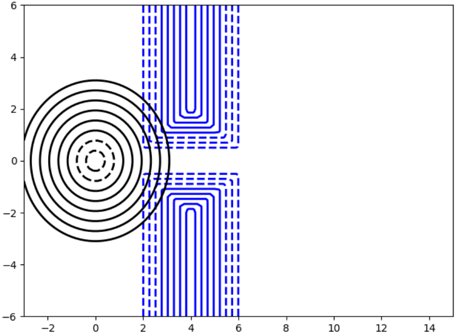
INSTITUT NATIONAL DES SCIENCES APPLIQUÉES ROUEN NORMANDIE

Table of Contents

Deterministic control problems

Level set formulation

- target region : $\varphi \leqslant 0$

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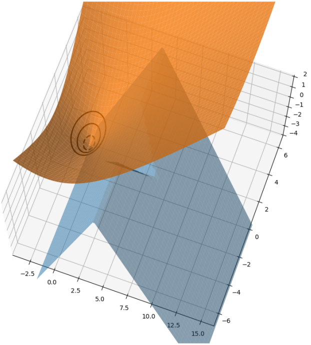

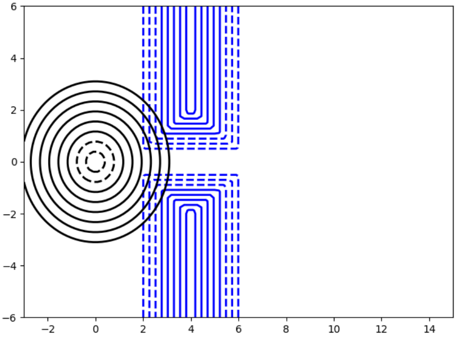

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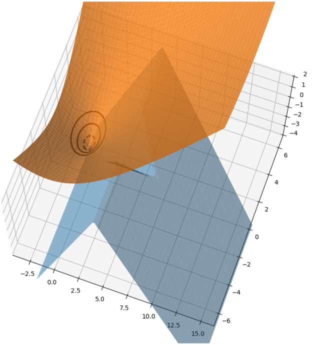

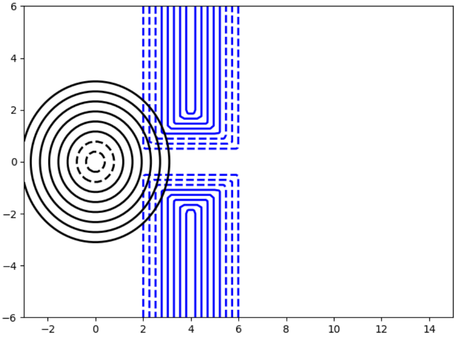

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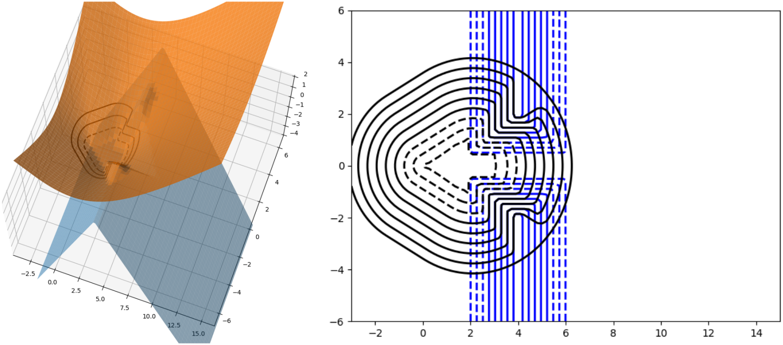

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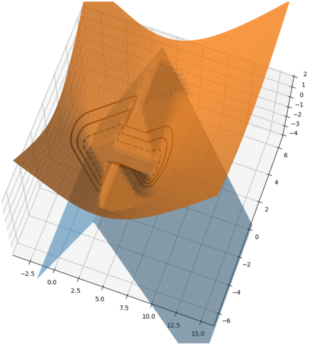

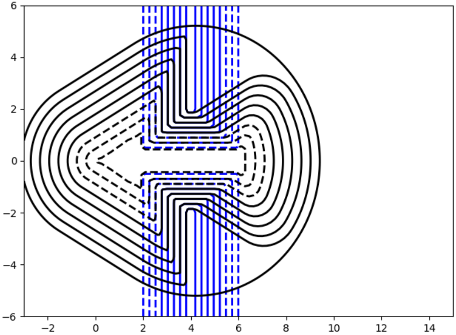

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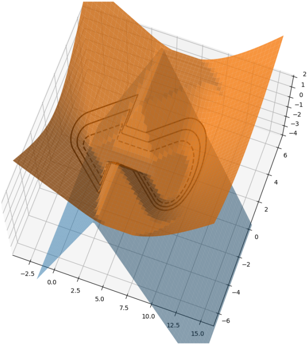

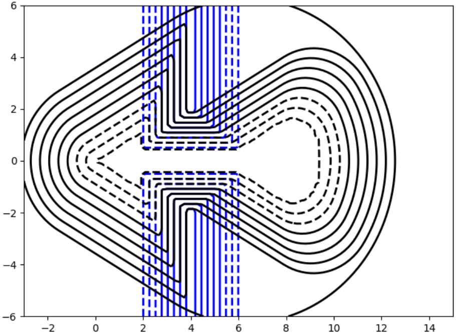

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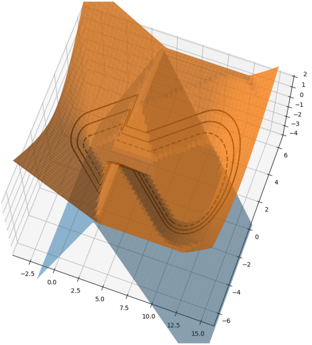

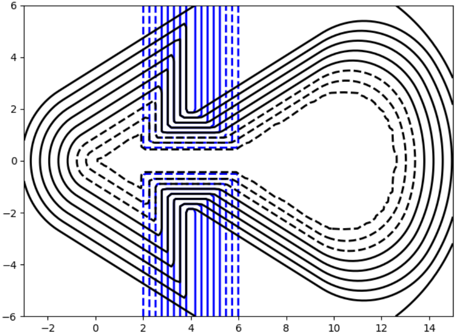

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  Curse of dimensionality.

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- Reinforcement/learning methods.

Case of study

Obstacle problem ([ABZ13]) Find $u = u(t,x)$ s.t.

$$\begin{cases} \min\left(\partial_t u + \max_{a \in A} \nabla u \cdot f(x,a), u - g(x)\right) = 0, \\ u(T,x) = \varphi(x) \vee g(x). \end{cases}$$

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- $f(x,a) = a$, $g \equiv -\infty$: $u(t,x) = \min_{b\in\mathscr{B}(0,t)} \varphi(x - tb)$.

Table of Contents

DPP

We consider the integral version of our model :

Dynamical programming principle $\forall h \in [0, T-t]$,

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- May not be defined everywhere (no uniqueness).
- If $\alpha(\cdot,\cdot) \in \mathcal{A}_{[t,t+h]} := L^\infty([0,T], A)$, equivalent formulation.

Functional formulation

Let μ a positive measure such that $\mu(\Omega) < \infty$, with Ω the space domain. New equivalent formulation :

$$\begin{cases} J(t,x,\alpha) := u(t+h, y_x^a(h)) \bigvee \max_{\theta \in [0,h]} g(y_x^a(\theta)) \\ \alpha^* \in \operatorname{argmin}_{\alpha \in \mathcal{A}_{[t,t+h]}} \int_{x \in \Omega} J(t,x,\alpha)\mu(dx), \\ u(t,x) := J(t,x,\alpha^*). \end{cases}$$

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Two-step discretization :

- Fonction space : finite-dimensional space $\hat{\mathcal{A}} \subset \mathcal{A}$ for the controls.
- Time : let $h = \Delta t$, computation of $u^n(\cdot) \simeq u(n\Delta t, \cdot)$ with u^{n+1} by induction.

Lagrangian scheme

Let $\hat{A}$ a finite-dimensional space (FE, spectral basis, neural networks...), $N \coloneqq T/\Delta t$, and $M \in \mathbb{N}^*$. Let $\hat{y}_x^\alpha(\Delta t) \simeq y_x^\alpha(\Delta t)$.

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Choose $\hat{u}^N \coloneqq \varphi \vee g$.
for $n \in [\![N-1, 0]\!]$ **do**
    Draw $(X_k)_k$ M iid samples of $X \sim \mu$.
    Pick $\hat{\alpha}^n \in \operatorname{argmin}_{\hat{\alpha} \in \hat{\mathcal{A}}} \sum_{i=1}^M \hat{u}^{n+1}(\hat{y}_{X_i}^{\hat{\alpha}}(\Delta t)) \vee g(X_i)$.
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- Quadratic complexity.

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- Linear complexity (with double computational/memory cost).
- Projection step induces diffusion.
- Hybrid algorithm if the evaluation step is fully Lagrangian.

Table of Contents

Definition

$$\mathscr{R}(x) = \sigma_1 \circ L_1 \circ \cdots \circ \sigma_p \circ L_p, \quad L_i \text{ affine, } \sigma_i \text{ nonlinear.}$$

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- Activation $\sigma(x) = \max(0, x)$ (ReLu) often used for function approximation.
- The regularity of $\mathbb{R}^m \mapsto \mathbb{R}^p$ stems from that of σ (continuity/Lipschitz-continuity with GroupSort).

Density

Let $\Omega \subset \mathbb{R}$ be compact.

Squasher $\lim\limits_{x\to-\infty} \sigma(x) = 0$, σ nondecreasing and $\lim\limits_{x\to\infty} \sigma(x) = 1$.

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- Gap between the estimates and the numerical efficiency of networks.

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- The functions f, g and φ are Lipschitz-continuous.
- The training densities μ_n are bounded by $0 < \nu \leqslant \mu_n \leqslant C$.

Moreover, we assume that the support of the densities "follows" the dynamics, i.e. $\forall \omega \subset \Omega$ open, $y_\omega^{a^*}(T - t_n) \in \{\varphi \leqslant 0\}$ implies $\mu_n(\omega) > 0$.

Result

Let $N \in \mathbb{N}$ be fixed. Let Θ the dimension of the control space.

Proposition Let $(X_n)_n$ a random variable sequence following the laws μ_n.

$$\lim_{\Theta \to \infty} \inf_{\hat{a} \in \hat{\mathcal{A}}_\Theta} \mathbb{E}[|\hat{V}_n(X_n) - V_n(X_n)|] = 0$$

Idea of the proof

Lack of smoothness on the control : let a_n^ε be the regularization of a_n^* by convolution. We decompose $\hat{V}_n - V_n$ in

$$0 \leqslant \mathbb{E}(\hat{V}_n - V_n) \leqslant C_1 \inf_{\hat{a} \in \hat{\mathcal{A}}} \mathbb{E}\,|a_n^\varepsilon - a_n^*| \quad \text{regularization error}$$

$$+ \dots$$

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$$
\begin{aligned}
0 \leqslant \mathbb{E}(\hat{V}_n - V_n) \leqslant\ & C_1 \inf_{\hat{a} \in \hat{\mathcal{A}}} \mathbb{E}\,|a_n^\varepsilon - a_n^*| \quad \text{regularization error} \\
& + C_2 \inf_{\hat{a} \in \hat{\mathcal{A}}} \mathbb{E}\,|\hat{a} - a_n^\varepsilon| \quad \text{approximation of } \mathcal{A} \text{ by } \hat{\mathcal{A}} \\
& + \dots
\end{aligned}
$$

Idea of the proof

Lack of smoothness on the control : let a_n^ε be the regularization of a_n^* by convolution. We decompose $\hat{V}_n - V_n$ in

$$\begin{aligned} 0 \leqslant \mathbb{E}(\hat{V}_n - V_n) \leqslant C_1 \inf_{\hat{a}\in\hat{\mathcal{A}}} \mathbb{E}\,|a_n^\varepsilon - a_n^*| & \quad \text{regularization error} \\ + C_2 \inf_{\hat{a}\in\hat{\mathcal{A}}} \mathbb{E}\,|\hat{a} - a_n^\varepsilon| & \quad \text{approximation of } \mathcal{A} \text{ by } \hat{\mathcal{A}} \\ + C_3\,\mathbb{E}\left(\hat{V}_{n+1} - V_{n+1}\right) & \quad \text{induction term} \end{aligned}$$

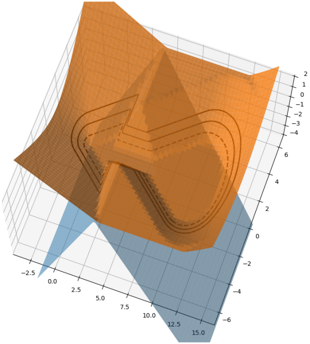
Letting $\Theta \to \infty$, we may control the growth of C_3 when $\varepsilon \searrow 0$. Hence convergence for fixed N.

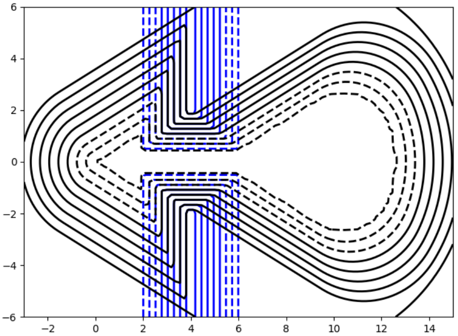
Table of Contents

Advection-eikonal equation with obstacle term

Let $A := \mathscr{B}_{\mathbb{R}^n}(0,1)$, $b \in \mathbb{R}^n$, $c \geqslant 0$. Find $u = u(t,x)$ s. t.

$$\min(-\partial_t u + \max_{a \in A} \nabla u \cdot [b + ca], u - g) = 0$$





One simulation example

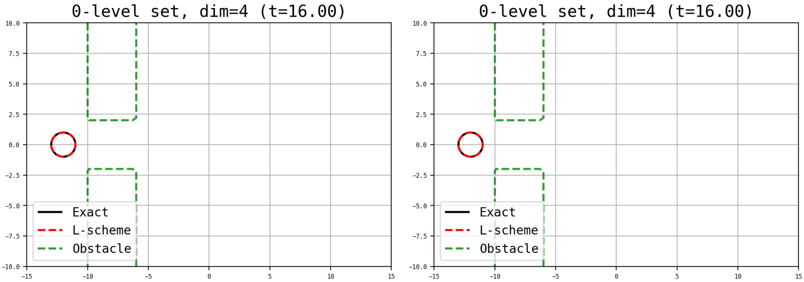
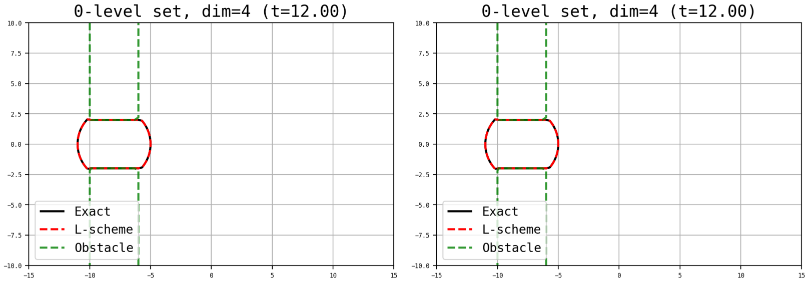


Figure – Left : $N_{\text{it}} = 8$, right : $N_{\text{it}} = 16$, Lagrangian scheme.

Parameters : $b = e_1$, $c = 1/2$, 3 layers of 60 neurons, ReLu activation, 10^5 iterations of S.G. with 4000 points.

One simulation example



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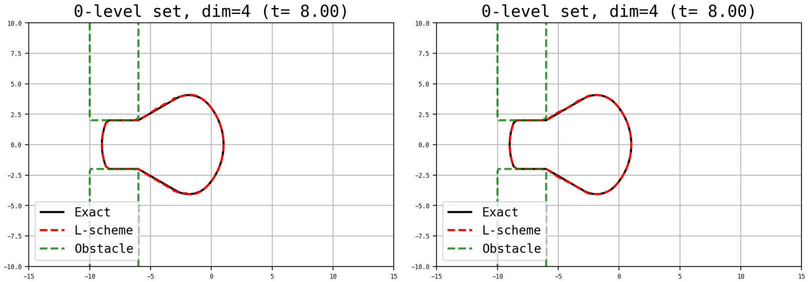

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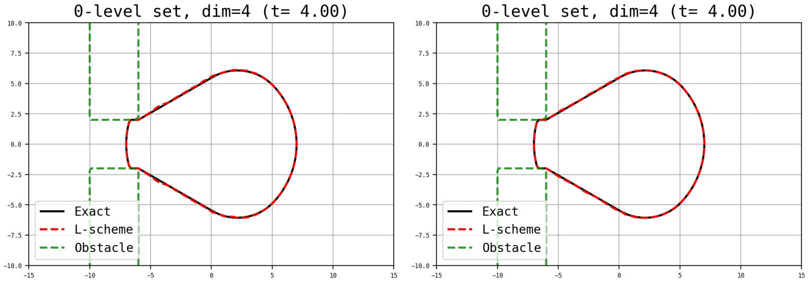

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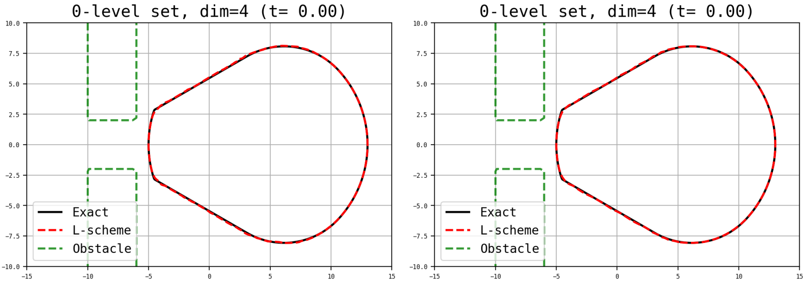

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Error

d	Global errors		Local errors		Time
	L_∞	L_1 rel.	L_∞	L_1 rel.	
2	2.66e-01	5.99e-03	1.19e-01	4.61e-02	3h02
4	3.90e-01	6.77e-03	1.16e-01	2.69e-02	8h13
6	9.69e-01	1.09e-02	1.78e-01	2.88e-02	35h20

Table – Errors when the dimension d increases.

Parameters : $b = e_1$, $c = 1/2$, *3 layers of 60 neurons, ReLu activation,* 10^5 *iterations of S.G. with 4000 points.*

Conclusion & future work

Done :

- Lagrangian scheme for obstacle problems in high dimension.

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To do :

- Influence of the training measure μ.
- Scheme of order > 1 for discontinuous right hand-side ODE !

Thank you

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