Engineering Calculus III

MATH 251 at Texas A&M

Using *Calculus: Early Trancendentials, 8^{th} Edition*

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Lecture 1

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1.1 Three-Dimensional Coordinate System

Definition 1.1 Three Dimensional Coordinate System

A point in three dimensions is represented by a triple, (x, y, z), of numbers. The set of all triples is denoted by $\mathbb{R}^3$.

Definition 1.2 Right Hand Rule

The **right hand rule** describes the relationship between the three axes. To use the rule, curl the fingers of your right hand around the z axis.

Example 1.1

Plot the point $(-3, 4, 2)$ in the coordinate system.

Definition 1.3 Coordinate Planes

In $\mathbb{R}^3$ there exists the following coordinate planes:

- xy-planedefined by $(x, y, 0)$.
- xz-planedefined by $(x, 0, z)$.
- yz-planedefined by $(0, y, z)$.

These coordinate planes split $\mathbb{R}^3$ into **octants**. Octant 1 is defined to be where the x, y, z are positive.

Definition 1.4 Projection onto Coordinate Planes

Given a point $P(a, b, c)$:

- $Q(a, b, 0)$ is the **projection** of P onto the xy-plane
- $R(0, b, c)$ is the projection of P onto the yz-plane
- $S(a, 0, c)$ is the projection of P onto the xz-plane

Example 1.2

Find the projections of $(3, -1, 4)$ onto the coordinate planes.

1.2 Surfaces

Definition 1.5 Surfaces

An equation given in x, y, z gives a **surface** in $\mathbb{R}^3$.

The coordinate planes can be represented by:

- yz-plane $\iff x = 0$, or all points $(0, y, z)$
- xz-plane $\iff y = 0$, or all points $(x, 0, z)$
- xy-plane $\iff z = 0$, or all points $(x, y, 0)$

The equations in the form of $x = c, y = c, z = c$ in $\mathbb{R}^3$ give planes.

Example 1.3

Describe and sketch the following planes

1. $x = 5$, a plane 5 units ahead and parallel to yz-plane
2. $z = 2$, a plane 2 units above and parallel to xy-plane

Example 1.4

How do the planes $x = 5, z = 2$ intersect?

$$(5, y, z) \cap (x, y, 2) = (5, y, 2) \tag{1.1}$$

$(5, y, 2)$ represents a line parallel to the y-axis.

Method 1.1 Two-Variable Equations

A surface formed by a two-variable equation can be graphed by first creating the graph in $\mathbb{R}^2$ corresponding to the two variables. This can then be extended to $\mathbb{R}^3$ w.r.t. the missing axis.

Example 1.5

Sketch and describe the following surfaces in $\mathbb{R}^3$

1. $y + x = 2$
2. $x^2 + y^2 = 9$
3. $x^2 + (z - 4)^2 = 4$

1.3 Spheres

Definition 1.6 Equation of a Sphere

The equation of a sphere with a center $C(h, k, l)$ and radius r is given by

$$(x - h)^2 + (y - r)^2 + (z - l)^2 = r^2 \tag{1.2}$$

Example 1.6

Find an equation of the sphere with center (5, 3, 2) and radius 3. Find and describe the intersection of this sphere and the coordinate planes.

Lecture 2

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2.1 Spheres (cont.)

Example 2.1

Show that

$$x^2 + y^2 + z^2 - 6x + 8y - 4z = 20 \tag{2.1}$$

represents a sphere, find the centre and radius.

Example 2.2

What is the equation of the sphere with center $(1, 2, 3)$ that touches the xy-plane

2.2 Formulae

Definition 2.1 Distance Formula

The distance between points $P(a, b, c)$ and $Q(x, y, z)$ is given by

$$|PQ| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \tag{2.2}$$

Example 2.3

Consider the points $P(5, 5, 1), Q(3, 3, 2), R(1, 4, 4)$

1. Find the lengths of the triangle
2. What type of triangle is it?

Definition 2.2 Midpoint Formula

The midpoint of $P(a, b, c)$ and $Q(x, y, z)$ is given by

$$\left(\frac{x+a}{2}, \frac{y+b}{2}, \frac{z+c}{2}\right) \tag{2.3}$$

Example 2.4

Find an equation of a sphere if one of its diameters has endpoints $(2, 1, 4)$, and $(4, 3, 10)$.

2.3 Solids

Definition 2.3 Solids

Inequalities involving x, y, z gives solids in $\mathbb{R}^3$.

Like a circle is just a line outlining the boundary and a disk is the circle and everything inside. A sphere refers to the shell, while a ball refers to both the sphere and everything inside.

Example 2.5

Sketch and describe the following regions in $\mathbb{R}^3$

1. $x^2 + y^2 \leq 25$
2. $1 < x^2 + y^2 + z^2 < 16$

2.4 Vectors

Definition 2.4 Vectors

A **vector** has both magnitude and direction.
Algebraically:
In three dimensions, a vector is given by

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \tag{2.4}$$

where a_1, a_2, a_3 are the components of $\vec{a}$, each corresponding to the x, y, z respectively.
Geometrically:
A vector has initial and terminal points. For example, a vector starting at the origin O, and terminal point $P(a_1, a_2, a_3)$ is referred to the position vector of P.
Given the points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$, the vector with initial point A and terminal point B is given by

$$a\vec{b} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \tag{2.5}$$

Definition 2.5 Magnitude

The magnitude or length of a vector

$$|\vec{a}| = \sqrt{a_1^2, a_2^2, a_3^2} \tag{2.6}$$

2.4.1 Vector Algebra

Addition

$$\vec{a} + \vec{b} = \langle a_1 + b1, a_2 + b2, a_3 + b3 \rangle \tag{2.7}$$

Scalar Multiplication

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle \tag{2.8}$$

Difference

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \langle a_1 - b1, a_2 - b2, a_3 - b3 \rangle \tag{2.9}$$

Definition 2.6 Standard Basis Vectors

$$\vec{i} = \langle 1, 0, 0 \rangle; \quad \vec{j} = \langle 0, 1, 0 \rangle; \quad \vec{k} = \langle 0, 0, 1 \rangle \tag{2.10}$$

A vector can be written as

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \tag{2.11}$$

$$\langle 23, 2, 2, 3, , 4, \rangle \tag{2.12}$$