Differential Equations

MATH 308 at Texas A&M

Using *Elementary Differential Equations, 11th Edition*

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Contents

Lecture 1

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1.1 First Order Differential Equations

Definition 1.1 First Order Differential Equation

The generic form of a **first order differential equation** is

$$y' = f(x, y) \tag{1.1}$$

Sometimes, t is substituted for x, especially if the function relates to time.

Definition 1.2 General Solution

A solution to a differential equation is considered to be **general** if there is an arbitrary constant present in the final answer, i.e. a problem without initial values.

Example 1.1

$$\begin{align} y' &= 1 \tag{1.2} \\ y &= \int y' \, dx \tag{1.3} \\ &= 1 \, dx \tag{1.4} \\ &= x + C \tag{1.5} \end{align}$$

Definition 1.3 Open Differential Equations

Equations without solutions are considered to be **open**. Many differential equations are without solutions.

Example 1.2 Open Differential Equation

$$y' = x'y - x^3 \tag{1.6}$$

This differential equation does not have a solution; thusly open.

Example 1.3

$$y' = y \tag{1.7}$$

$$\int y' \, dy = \int y \, dy \not\Leftrightarrow y' = y \tag{1.8}$$

Notice above that the integration of both sides is not the same as the differential equation.

$$y' = e^x \implies y \int y' \, dx = \int e^x \, dx \tag{1.9}$$

Using the above, the general solution can be found

$$y = Ce^x \tag{1.10}$$

Remark 1.1 Regarding Example 1.3

If both sides of a differential equation are dependent on the same variable — i.e. the same variable appearing on both sides of the equation, then taking the intergal of both sides is not a valid method to solve the equation.

Definition 1.4 Initial Value Problems

An **initial value problem** (IVP), or **initial condition problem**, is a problem where an initial condition of the equation is defined which leads to a **unique solution** to the equation.

Example 1.4 Initial Value Problem

$$y' = x,\ y(0) = 1 \tag{1.11}$$

Notice that this is an **initial value problem**, because $y(0) = 1$. Also notice that y is an anti-derivative w.r.t. x; because each side of the equation is independent of one another (unlike *Example 1.3*).

$$\int y'\,dx = \int x\,dx \tag{1.12}$$

$$\implies \quad y = \frac{1}{2}x^2 + C \tag{1.13}$$

$$y(0) = 1 \tag{1.14}$$

$$\implies \quad = \frac{1}{2}(0^2) + C \tag{1.15}$$

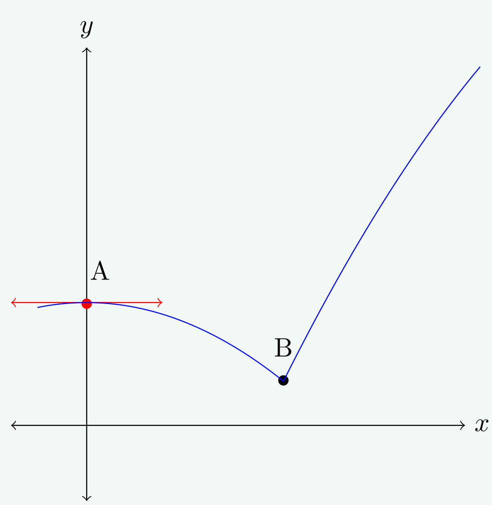
$$\implies \quad C = 1 \tag{1.16}$$

$$\implies \quad y = \frac{1}{2}x^2 + 1 \tag{1.17}$$

1.2 Differentiable Functions

Definition 1.5 Differentiability

Given $f : \mathbb{R} \to \mathbb{R}$ and point a, $\exists f'(a) \iff \exists T_1(a)$, where T_1 is a tangent line (Taylor polynomial of degree one).



In the example, point A has a singular tangent line and is therefore differentiable. Point B has infinitely many tangent lines, and is therefore both undefined and not differentiable.

1.3 Kinematics

Example 1.5 Kinematics with Differential Equations

Given an object with a velocity v_0, and acceleration a, find the position s at any time t.

$$\begin{aligned}
\frac{d}{dt}v(t) &= a && (1.18)\\
\implies \quad v(t) &= \int a\,dt && (1.19)\\
&= at + C && (1.20)\\
\because \quad v(0) &= v_0 && (1.21)\\
v_0 &= a(0) + C && (1.22)\\
C &= v_0 && (1.23)\\
\frac{d}{dt}s(t) &= \int v\,dt && (1.24)\\
\implies \quad s(t) &= \int v\,dt && (1.25)\\
&= \int (at + v_0)\,dt && (1.26)\\
&= \frac{1}{2}at^2 + v_0 t && (1.27)
\end{aligned}$$

Lecture 2

16 January 2020

2.1 Linear Differential Equations

Definition 2.1 First Order Linear Differential Equations

$$\underbrace{y' + p(t)y = g(t)}_{\text{Usual form}} \iff y' = g(t) - p(t)y \tag{2.1}$$

A **first order linear differential equation** (LDE) is linear due to y being dependent on only one variable, t.

Notice that t is typically used in place of x as most differential equations are used in models dependent on time; as such, most differential equations are in the form $y' = f(t, y)$ as opposed to $y' = f(x, y)$.

Example 2.1

$$\text{Solve} \quad (4 + t^2)y' + 2ty = 4t \tag{2.2}$$

$$\begin{aligned}
\text{Notice:} \qquad (4y + t^2y)' &= \frac{d}{dt}(4y + t^2y) = 4t && (2.3)\\
&= 4y' + (t^2y)^2 && (2.4)\\
&= 4y' + (2ty + t^2y') && (2.5)\\
&= (4 + t^2)y' + 2ty && (2.6)
\end{aligned}$$

Example 2.1

The original problem can now be reduced to:

$$\frac{d}{dt}(4y + t^2 y) = 4t \tag{2.7}$$

$$\text{let} \quad z(t) = 4y + t^2 y \tag{2.8}$$

$$= 2t^2 + C \tag{2.9}$$

$$\implies \quad 4y + t^2 y = 2t^2 + C \tag{2.10}$$

$$\therefore \quad y = \frac{1}{4 + t^2}(2t^2 + C) \tag{2.11}$$

Remark 2.1 Constants

Notice in the above example that the constant, C, is being multiplied by $\frac{1}{4+t^2}$. When expanding the answer, it now becomes $y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$. Notice how the constant is dependent on the variable t, and is therefore not the same as just C.

Definition 2.2 Integrating Factors with LDEs

An **integrating factor**, $\mu(t)$ is a function $\mu(t) : \mathbb{R} \to \mathbb{R}$, that satisfies $\frac{d}{dt}\mu(t) = \mu(t)y' + \mu(t)p(t)y$.

Remark 2.2

There are infinitely many integrating factors due to the arbitrary constant C from indefinite integration, see **Method 2.1** and **Example 2.2** on the following page.

Method 2.1 Solution of the General LDE Case

Solve $y' + p(t)y = g(t)$.

1. Multiply the LDE by $\mu(t)$ results in:
$$\mu(t)(y' + p(t)y) = \mu(t)g(t) \tag{2.12}$$

2. Letting $z(t) = \mu(t)y$, and $z' = \mu(t)g(t)$ yields:
$$z(t) = \int \mu(t)g(t)\,dt \tag{2.13}$$
$$\implies y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt \tag{2.14}$$
$$\implies \mu(t) = \exp\left(\int p(t)\,dt\right) \tag{2.15}$$

3. Therefore the solution of the general case is
$$y(t) = \left(\exp\left(\int p(t)\,dt\right)\right)^{-1} \cdot \int \exp\left(\int p(t)\,dt\right) g(t)\,dt \tag{2.16}$$

Example 2.2 Solving an IVP involving LDEs

Working with example 2.1.4 from the textbook:
$$ty' + 42y = 4t^2,\ y(1) = 2 \tag{2.17}$$

1. Compute the integrating factor ($\mu(t)$)
$$\mu(t) = \exp\left(\int p(t)\,dt\right) \tag{2.18}$$
$$= \exp\left(\int 2t^{-1}\,dt\right) \tag{2.19}$$
$$= \exp(2\ln(t) + C) \Leftrightarrow e^{2\ln(t)+C} \tag{2.20}$$

2. Find the general case
When solving, 0 can be subsituted in for C to simplify calculations; for $C \neq 0$ it is trivially shown that the constant will cancel out in

Example 2.2 Solving an IVP involving LDEs

computing the solution.

$$\begin{aligned} y_c(t) &= \frac{1}{\mu(t)} \int \mu(t) g(t)\, dt && (2.21) \\ &= \frac{1}{t^2} \left(\int t^2 \cdot 4t\, dt \right) && (2.22) \\ &= \frac{1}{t^2} (t^4 + C) && (2.23) \end{aligned}$$

Note: $y_c(t)$ is used to denote the general case.

3. Find formula w.r.t. intial value

$$\begin{aligned} y(1) &= 2 && (2.24) \\ \implies y(1) &= (1)^2 + \frac{C}{(1)^2} && (2.25) \\ \implies \quad C &= 1 && (2.26) \\ \therefore \quad y(t) &= t^2 + t^{-2} && (2.27) \end{aligned}$$

Lecture 3

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3.1 Linear Differential Equations (cont).

Example 3.1

Given $y' - 2y = t^2 e^{2t}$ find:

1. The general solution

$$p(t) = -2,\ g(t) = t^2 e^{2t} \tag{3.1}$$
$$\mu(t) = \exp\left(\int -2\,dt\right) \tag{3.2}$$
$$= e^{-2t+C} \tag{3.3}$$
$$y_c(t) = e^{2t} \int t^2\,dt \tag{3.4}$$
$$= e^{2t}\left(\frac{1}{3}t^3 + C\right) \tag{3.5}$$

2. What is $\lim_{t\to\infty} y_c(t)$?
   There are infinitely many $y_c(t)$; the answer may vary with the value of C. In this case, the value of C does not matter.

$$\lim_{t\to\infty} y_c(t) = +\infty$$

3.2 Separable Differential Equations

Definition 3.1 Separable Differential Equations

A **separable differential equation** (SDE) can be defined by

$$\frac{dy}{dx} = y' = f(x,y) = -\frac{M(x,y)}{N(x,y)} \tag{3.6}$$

where

$$M(x,y) = -f(x,y) \tag{3.7}$$
$$N(x,y) = 1 \tag{3.8}$$

it is **separable** because it can be written in the **differential form**

$$M(x)\,dx + N(y)\,dy = 0 \tag{3.9}$$

Theorem 3.1

If $\frac{dy}{dx} = \frac{M(x)}{N(y)}$, then $\int N(y)\,dy = \int M(x)\,dx$

Proof 3.1

Choose $\widetilde{N}$ such that $\frac{d\widetilde{N}(y)}{dx} = M(x)$:

$$\frac{d\widetilde{N}(y)}{dy} = \frac{d\widetilde{N}(y)}{dx}\frac{dx}{dy} = \frac{d\widetilde{N}(y)}{dy}\frac{dy}{dx} = \frac{d\widetilde{N}(x)}{dx} \tag{3.10}$$

$$\frac{d\widetilde{N}(y)}{dy} = \frac{dy}{dx} \tag{3.11}$$

$$\implies \frac{d\widetilde{N}(y)}{dx} = M(x) \tag{3.12}$$

Example 3.2

Find a particular solution that passes through the point $(0, 1)$.

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y} \tag{3.13}$$

$$\implies \int (4 + y)\, dy = \int (4x - x^3)\, dx \tag{3.14}$$

$$4y + \frac{1}{2}y^2 + C_1 = 2x^2 - \frac{1}{4}x^4 + C_2 \tag{3.15}$$

$$4y + \frac{1}{2}y^2 = 2x^2 - \frac{1}{4}x^4 + (C_2 - C_1) \tag{3.16}$$

$$\implies 2y + 16y + x^4 - 8x^2 + C = 0 \tag{3.17}$$

$$(0, 1) \implies 2(1) + 16(1) + 0^4 - 8(0)^2 + C = 0 \tag{3.18}$$

$$C = -18 \tag{3.19}$$

$$\therefore 2y + 16y + x^4 - 8x^2 = 18 \tag{3.20}$$

Homework 3.1

$$y' = \frac{dy}{dx} = \frac{x^2}{y} \tag{3.21}$$

$$y\, dy = x^2\, dx \tag{3.22}$$

$$\int y\, dy = \int x^2\, dx \tag{3.23}$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C \tag{3.24}$$

$$y(x) = \pm\sqrt{\frac{2}{3}x^3 + C} \tag{3.25}$$

Lecture 4

23 January 2020

4.1 Separable Equations (cont.)

Example 4.1

From the textbook, 2.2, ex. 2.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad y(0) = -1 \tag{4.1}$$

Given the above, determine the interval in which the solution exists.

$$\int 2(y-1)\, dy = \int (3x^2 + 4x + 2)\, dx \tag{4.2}$$

$$\implies y^2 - 2y + C_1 = x^3 + 2x^2 + 2x + C_2 \tag{4.3}$$

The solution above is the **general implicit solution**. The constants, C_1 and C_2 can be combined into one constant, C, because they are independent.
Next, use the initial value to solve for C

$$y(0) = -1 \tag{4.4}$$

$$\implies (-1)^2 - 2(-1) = 0^3 + 2(0)^2 + 2(0) + C \tag{4.5}$$

$$\implies C = 3 \tag{4.6}$$

Example 4.1 (cont.)

Then complete the square on the left hand side to get the **explicit solution**.

$$\begin{aligned}
& (y^2 - 2y + 1) - 1 = x^3 + 2x^2 + 2x + 3 && (4.7)\\
\Longrightarrow \quad & (y-1)^2 = x^3 + 2x^2 + 2x + 4 && (4.8)\\
\Longrightarrow \quad & y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4} && (4.9)\\
\Longrightarrow \quad & y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} && (4.10)\\
\Longrightarrow \quad & y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} && (4.11)\\
\because \quad & y(0) = -1 && (4.12)
\end{aligned}$$

Note: It is also possible to use the quadratic formula in order to convert this instance of an implicit into an explicit solution.
Observation: Because the unique solution involves a square root, a function defined for $x \in [0, \infty)$, it is possible to reduce the original question to findinding when the radicand is non-negative.

$$\begin{aligned}
& x^3 + 2x^2 + 2x + 4 \ = 0 && (4.13)\\
& (x^2 + 2)(x + 2) = 0 && (4.14)\\
\Longrightarrow \quad & x \geq -2 && (4.15)
\end{aligned}$$

The factor $x^2 + 2$ will always be positive, so now the question is further reduced to when $x + 2$ will be non-negative, which is $x \in [-2, \infty)$.
Therefore, the interval of which the solution exists is $(-2, \infty)$

Remark 4.1 Solutions to Differential Equations

In **Example 4.1**, notice the final answer was an open interval, $(-2, \infty)$, rather than a half closed interval, $[-2, \infty)$, even if the solution would be defined if $x = -2$. The reason for this is that **solutions to differential equations must also be differentiable**.
At point $x = -2$, the unique solution is defined, however, it is not differentiable as $\lim_{x \to -2^-}$ does not exist, because the function is not defined for $x < -2$.

4.2 Mathematical Modelling

Example 4.2 Modelling

Consider a pond fille with 10 million gallons of fresh water. A flow of 5 million gallons per year with water that is contaminated wiht a chemical enters the pond. There is also an outflow of this mixture on the order of 5 million gallons per year.

Let $\gamma(t)$ be the concentration of the fluid entering the pod at time t, and let $Q(t)$ be the quantity of chemicals in the pod at time t.

It is determined that

$$\gamma(t) = 2 + \sin(2t) \text{ g} \cdot \text{gal}^{-1}$$

Find $Q(t)$ using the given information.

We can infer that $Q(0) = 0$ because the water starts off fresh at $t = 0$. We know that $\frac{dQ}{dt}$ is equal to the rate at which chemicals are entering minus the rate at which they leave, leading us to

$$\frac{dQ}{dt} = I(t)\gamma(t) - \frac{O(t)}{V(t)}\left[Q(t)\right]$$

Where $I(t)$ describes the rate at which the contaminated water enters, $O(t)$ describes the rate at which the water mixture leaves the pond, and $V(t)$ describes the total volume of the pond at any given time.
In this case,

$$I(t) = 5 \times 10^6 \text{ gal year}^{-1} \tag{4.16}$$
$$O(t) = 5 \times 10^6 \text{ gal year}^{-1} \tag{4.17}$$
$$V(t) = 10^7 \text{ gal} \tag{4.18}$$
$$\tag{4.19}$$

Plugging in the values yields the following,

$$\frac{dQ}{dt} = 5 \times 10^6\gamma(t) - \frac{1}{2}Q(t) \tag{4.20}$$
$$\tag{4.21}$$

Example 4.2 Modelling

Solving the linear differential equation,

$$\frac{dQ}{dt} + \frac{1}{2}Q(t) = 5 \times 10^6 \gamma(t) \tag{4.22}$$

$$\implies \quad Q_c(t) = 5 \times 10^6 e^{-\frac{1}{2}t} \int e^{\frac{1}{2}t}(2 + \sin(2t))\, dt \tag{4.23}$$

$$\implies \quad Q_c(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17}cos(2t) + Ce^{-\frac{1}{2}t} \tag{4.24}$$

$$Q_c(0) = 2 \times 10^7 - \frac{4 \times 10^7}{17} + C = 0 \tag{4.25}$$

$$\implies \quad C = \frac{-3 \cdot 10^8}{17} \tag{4.26}$$

$$Q(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17}cos(2t) - \frac{3 \cdot 10^8}{17}e^{-\frac{1}{2}t} \tag{4.27}$$

Remark 4.2 Behavior of Example 4.2

When graphing this equation, it can be seen that in the long term the equation becomes periodic despite beginning with an irregular pattern. This is due to the fact that the term $-\frac{3 \cdot 10^8}{17}e^{-\frac{1}{2}t}$ is able to affect the behavior in the short term, however, it is decaying exponentially and tends towards 0. The sin and cos functions are periodic which cause the sinusoidial shape of the graph as $t \to \infty$.

Lecture 5

28 January 2020

5.1 Mathematical Modelling (cont.)

Example 5.1

Example 2.3.1 from the textbook.

1. Find the amount of salt in the tank at a time t ($Q(t)$).
   Inference: $Q(0) = Q_0$

$$\frac{dQ}{dt} = \frac{1}{4}r - \frac{rQ}{100} \tag{5.1}$$

$$\implies Q' + \frac{r}{100}Q = \frac{1}{4}r \tag{5.2}$$

$$\implies \quad Q_c = \exp\left(-\frac{r}{100}t\right)\int\left(\exp\left(\frac{r}{100}t\right)\frac{1}{4}r\right)dt \tag{5.3}$$

$$= \frac{r}{4}\exp\left(\frac{-r}{100}t\right)\left(\frac{100}{r}\exp\left(\frac{r}{100}t\right)+C\right) \tag{5.4}$$

$$= 25 + \frac{r}{4}\exp\left(-\frac{r}{100}t\right)C \tag{5.5}$$

$$= 25 + C\exp\left(-\frac{r}{100}t\right) \tag{5.6}$$

$$Q(0) = Q_0 \tag{5.7}$$

$$\implies \quad C = (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.8}$$

$$\implies \quad Q(t) = 25 + (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.9}$$

Example 5.1

2. Find the limiting amount, Q_l, after a long time.

$$\lim_{t\to\infty}(Q_c(t)) = Q_c = 25 \tag{5.10}$$

Remark 5.1 Regarding Example 5.1

Notice that no matter the amount of salt that the system starts with, it will always tend towards 25 lbs of salt in the tank.

5.2 Exact Differential Equations

Definition 5.1 Exact Differential Equations

A differential equation is exact iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \Leftrightarrow N(x,y)y' + M(x,y) = 0 \tag{5.11}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{5.12}$$

Given $\psi(x,y)$, parameterize by using $\delta(t) = \psi(f_1(t), f_2(t))$.

$$\begin{aligned} \frac{d\psi(x,y)}{dt} &= \frac{d\delta}{dt} && (5.13)\\ &= \frac{\partial\psi(x,y)}{\partial x}\frac{df_1}{dt} + \frac{\partial\psi(x,y)}{\partial y}\frac{df_2}{dt} && (5.14) \end{aligned}$$

Example 5.2

$$\begin{aligned} \psi(x,y) &= x^2y + xy && (5.15)\\ f_1(t) = t, &\quad f_2(t) = t^2 && (5.16)\\ \delta(t) &= \psi(f_1, f_2) && (5.17)\\ &= t^2t^2 + tt^2 && (5.18)\\ \delta'(t) &= 4t^3 + 3t^2 && (5.19)\\ \frac{\partial\psi(x,y)}{\partial x}\cdot 1 &+ \frac{\partial\psi(x,y)}{\partial y}\cdot 2t && (5.20)\\ &= (2f_1f_2 + f_2)\cdot 1 + (f_1^2 + f_1)\cdot 2ty && (5.21)\\ &= 4t^3 + 3t^2 && (5.22) \end{aligned}$$

Example 5.2

Notice how **Equation 5.19** and **Equation 5.22** are the same, but derived via different methods.

Example 5.3

1. $y' = \frac{\frac{1}{y}}{x}$ is an exact differential equation.
   Let $M(x,y) = \frac{1}{x}$, and $N(x,y) = y$.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial \frac{1}{x}}{\partial y} = 0$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial y}{\partial x} = 0$$

   Because both partial derivatives are equal, they are exact.

2. $y' = x$ is exact.
3. $y' = \frac{xy}{x+y} \iff (x+y)\,dy + xy\,dx = 0$ is exact.
4. $y' = \frac{xy+x}{\frac{1}{2}x^2+y}$ is exact.

Theorem 5.1 Exactness

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if, and only if, $\exists\,\psi(x,y)$ s.t.

$$\frac{\partial \psi(x,y)}{\partial x} = M(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = N(x,y)$$

Remark 5.2 Relationship

Exact differential equations are a superset of the separable differential equations, i.e. all separable differential equations are exact differential equations.

Lecture 6

30 January 2020

6.1 Exact Differential Equations (cont.)

Example 6.1

Solve

$$(y\cos x + 2xe^y) + (\sin x + x^2 + x^2e^y - 1)y' = 0 \tag{6.1}$$

Checking if **Equation 6.1** is exact,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(y\cos x + 2xe^y)}{\partial y} = \cos x + 2xe^y \tag{6.2}$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(\sin x + x^2e^y - 1)}{\partial y} = \cos x + 2xe^y \tag{6.3}$$

From the above, this is an exact differential equation.

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.4}$$

$$= \int (y\cos x + 2xe^y)\,dx + h(y) \tag{6.5}$$

$$= h(y) + y\sin x + x^2e^y + C \tag{6.6}$$

$$= h(y) + y\sin x + x^2e^y \tag{6.7}$$

Notice that the constant can be neglected as it can be contained in $h(y)$.

Example 6.1

Now solving for $h(y)$,

$$\psi_y(x,y) = N(x,y) \tag{6.8}$$
$$\implies \frac{dh}{dy} + \frac{\partial(y\sin x + e^y x^2)}{\partial y} = \sin x + x^2 e^y - 1 \tag{6.9}$$
$$\frac{dh}{dy} + \sin x + x^2 e^y = \sin x + x^2 e^y - 1 \tag{6.10}$$
$$\frac{dh}{dy} = -1 \tag{6.11}$$
$$h = -y + C \tag{6.12}$$

Then,

$$\psi(x,y) = y\sin x + x^2 e^y - y + C \tag{6.13}$$

Finally, $y(x)$ is given by the implicit expression

$$y\sin x + x^2 e^y - y = C \tag{6.14}$$

Example 6.2

Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{6.15}$$

Checking if the equation is exact,

$$\frac{\partial(3xy + y^2)}{\partial y} = 3x + 2y \tag{6.16}$$
$$\frac{\partial(x^2 + xy)}{\partial x} = 2x + y \tag{6.17}$$

Notice that they are not equal; however,

$$\mu(x)(3xy + y^2) + \mu(x)(x^2 + xy)y' = 0 \tag{6.18}$$

is an exact differential equation if

$$-\frac{N_x(x,y) + M_y)x,y}{N(x,y)} \tag{6.19}$$

is a function dependent only on x. However, $\forall M(x,y), N(x,y) \nexists \mu(x)$. $\mu(x)$

Example 6.2

can be found by solving the differential equation,

$$\frac{d\mu}{dx} = \frac{-N_x(x,y) + M_y(x,y)}{N(x,y)}\mu \tag{6.20}$$

$$\mu(x) = \exp\left(\int \frac{N_x - M_y}{N}\, dx\right) \tag{6.21}$$

In this problem,

$$\frac{M_y - N_x}{N} = \frac{(3x+2y) - (2x+y)}{x^2+xy} = \frac{1}{x} \tag{6.22}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = x + C \tag{6.23}$$

Multiplying **Equation 6.15** by $\mu(x)$ yields,

$$(3x^2y + y^2x) + (x^3 + x^2y)y' = 0 \tag{6.24}$$

Checking if the equation is exact yields the following,

$$\frac{\partial 3x^2y + xy^2}{\partial y} = 3x^2 + 2xy \tag{6.25}$$

$$\frac{\partial x^3 + x^2y}{\partial x} = 3x^2 + 2xy \tag{6.26}$$

and is therefore exact.

$$\psi(x,y) = \int (3x^2y + xy^2)\, dx + h(y) \tag{6.27}$$

$$= x^3y + \frac{1}{2}x^2y^2 + h(y) \tag{6.28}$$

$$\frac{\partial \psi(x,y)}{\partial y} = x^3 + x^2y + \frac{dh}{dy} \tag{6.29}$$

$$= N(x,y) \tag{6.30}$$

$$\frac{dh}{dy} = x^3 + x^2y = x^3 + x^2y \tag{6.31}$$

$$h = 0 \tag{6.32}$$

Finally, $y(x)$ can be expressed as,

$$x^3y + \frac{1}{2}x^2y^2 = C \tag{6.33}$$

Method 6.1 Solving Exact Differential Equations

1. Step 1: Determine if the equation is exact

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{6.34}$$

2. Step 2: Find $\psi(x,y)$ such that $\psi_x(x,y) = M(x,y)$, and $\psi_y(x,y) = N(x,y)$. Generally,

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.35}$$

this works because

$$\frac{\partial \psi(x,y)}{\partial x} = \frac{\partial \int M(x,y)\,dx}{\partial x} + \frac{\partial h(y)}{\partial x} \tag{6.36}$$

$$= M(x,y) + 0 \tag{6.37}$$

Then find $h(y)$ such that $\psi_y(x,y) = N(x,y)$.

Remark 6.1

Note in step 2 of **Method 6.1**

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.38}$$

can also be defined as

$$\psi(x,y) = \int N(x,y)\,dy + h(x) \tag{6.39}$$

$$\frac{\partial \psi(x,y)}{\partial y} = \psi_y(x,y) = \frac{\partial \int N(x,y)\,dy}{\partial y} + \frac{\partial h(x)}{\partial y} \tag{6.40}$$

$$= N(x,y) + 0 \tag{6.41}$$

Remark 6.2

$y(x)$ is a solution for $M(x,y)\,dx + N(x,y)\,dy = 0$ iff $\psi(x, y(x)) = c$. Consider the following,

$$\frac{d\psi(f_1, f_2)}{dt} = \frac{\partial \psi(x,y)}{\partial x}\frac{df_1}{dt} + \frac{\partial \psi(x,y)}{\partial y}\frac{df_2}{dt}$$

Remark 6.2

we can replace t with x, let $f_1 \equiv x$ and $f_2 \equiv y(x)$, then

$$\frac{d\psi(f_1(x), f_2(x))}{dx} = \frac{\partial\psi(x,y)}{\partial x}\frac{df_1}{dx} + \frac{\partial\psi(x,y)}{\partial y}\frac{df_2}{dx} \tag{6.42}$$

$$= \frac{\partial\psi(x,y)}{\partial x} + \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx} \tag{6.43}$$

finally,

$$N(x,y)\frac{dy}{dx} = \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx} = \frac{d\psi(x,y(x))}{dx} - \frac{\partial\psi(x,y)}{\partial x}$$

Remark 6.3

Notice in **Equation 6.31** has 3 variables: h, x, y; however, the terms with x cancel, leaving just h and y. This occurs due to the equation being exact.

Lecture 7

4 February 2020

Recall in the last lecture:

$$M_y(x,y) = N_x(x,y) \tag{7.1}$$
$$\implies \exists\, \psi(x, y(x)) := \psi_x = M(x,y); \psi_y = N(x,y) \tag{7.2}$$
$$\psi(x) = \psi(x,y) \equiv C \tag{7.3}$$

For example,

$$\psi(x,y) = x + y \tag{7.4}$$
$$x + y(x) = C \tag{7.5}$$
$$y = C - x \tag{7.6}$$

And if $\frac{M_y(x,y)-N_x(x,y)}{N(x,y)}$ depends only on x, then $\exists\, \mu(x) : \frac{d\mu}{dx} = \frac{M_y(x,y)-N_x(x,y)}{N(x,y)}\mu$. Thus, the differential equation $\mu M + \mu N y' = 0$ is an exact differential equation.

7.1 Uniqueness and Exactness

Theorem 7.1 Uniqueness of Linear Differential Equations

Consider the linear first order differential equation,

$$y' + p(t)y = g(t); \quad y(t_0) = y_0 \tag{7.7}$$

such that in some open interval, $I = (\alpha; \beta)$, $p(t)$ and $g(t)$ are continuous and $t_0 \in I$.
Then,

$$\exists!\, y(t) : y(t_0) = y_0 \wedge y' + p(t)y = g(t) \tag{7.8}$$

Theorem 7.2 Uniqueness of Non-linear Differential Equations

Consider the following,

$$y' = f(t, y) \wedge y(t_0) = y_0 \tag{7.9}$$

such that $f(t, y)$ and $\frac{\partial f(t,y)}{\partial y}$ are continuous over the domains $t \in (\alpha; \beta)$, and $y \in (\gamma; \delta)$.
Then, $h > 0, I = (t_0 - h, t_0 + h) : \exists t_0 \in I, y(t_0) = y_0$.

Example 7.1

$$ty' + 2y = 4t^2; \quad y(1) = 2 \tag{7.10}$$

Use **Theorem 7.1** to find an interval $\exists! y(t)$.

$$y' + \frac{2}{t} y = 4t \tag{7.11}$$

$$p(t) = \frac{2}{t}, \quad g(t) = 4t \tag{7.12}$$

In the interval $I := (\alpha, \beta)$, $\exists t \in I : p(t), g(t) \implies \exists! y(t)$

1. $\forall t \in (-\infty, 0) \cup (0, \infty), p(t)$
2. $\forall t \in (-\infty, \infty), g(t)$
3. $1 \in (\alpha, \beta)$
4. Therefore, $\alpha = 0, \beta = \infty \implies I = (0, \infty) = \mathbb{R}^+$

Lecture 8

6 February 2020

8.1 First Order Differential Equation Review

Topics covered in First Order Differential Equations.

- $y' = f(x, y) \quad y(x_0) = y_0$
- First Order LDE, $y' + p(t)y = g(t)$
- Separable, $y' = \frac{M(x,y)}{N(x,y)}$
- Exact, $M(x, y) + N(x, y)y' = 0$; $M_y(x, y) = N_x(x, y)$
- Uniqueness and Existance Theorems
- Modelling

8.2 Second Order Differential Equations

Definition 8.1 Second Order Differential Equations

The general form of a **second order differential equation** (SODE) is

$$y'' = f(x, y, y'); \quad y(x_0) = y_0; \quad y'(x_0) = y_1 \tag{8.1}$$

Example 8.1

The following are SODEs,

$$y'' = 1 \tag{8.2}$$
$$y'' = 1 + y' \tag{8.3}$$
$$y'' = \frac{x}{t} \tag{8.4}$$

An example of a SODE IVP,

$$y'' = x + y + y'; \quad y(0) = 1; \quad y'(0) = -3 \tag{8.5}$$

Definition 8.2 Second Order Linear Differential Equations

A **second order linear differential equation** (SOLDE) has the general form

$$y'' + p(t)y' + q(t)y = g(t) \tag{8.6}$$

where $p(t)$, $q(t)$, $g(t)$ are continuous over some interval I.

Theorem 8.1 SOLDE Uniqueness Theorem

If $p(t), q(t), g(t)$ are continuous in some interval $I : (\alpha, \beta)$
Then, for any $t_0 \in I$, the IVP defined by

$$y'' + p(t)y' + q(t)y = g(t), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{8.7}$$

has a unique solution.

Definition 8.3 Cases of SOLDEs

Homogeneous SOLDEs (HSOLDE) are of the following form

$$y'' + p(t)y' + q(t)y = 0 \tag{8.8}$$

If a Homogeneous SODE is defined where $p(t)$, and $q(t)$ are constants, it is considered as a **homogeneous SOLDE with Constant Coefficients** (CHSOLDE).

8.3 Homogeneous Second Order Linear Differential Equations with Constant Coefficients

Example 8.2 CHSOLDE

Find the general solution of

$$L[y] = y'' + 5y' + 6y = 0 \tag{8.9}$$

Consider the following quadratic (characteristic function, or characteristic polynomial).

$$f(r) = r^2 + 5r + 6 = 0 \tag{8.10}$$

There are 2 different roots to the characteristic function,

$$r_1 = -3; \quad r_2 = -2 \tag{8.11}$$

Now consider the equations,

$$y_1(t) = e^{r_1 t} = e^{-2t} \tag{8.12}$$

$$y_2(t) = e^{r_1 t} = e^{-3t} \tag{8.13}$$

Then, $y_1(t)$ and $y_2(t)$ are solutions of **Equation 8.9**.
Proof:

$$y_1'(t) = -3e^{-3t}; \quad y_1'' = 9e^{-3t} \tag{8.14}$$

$$L[y_1] = 9e^{-3t} + 5(-3)e^{-3t} + 6e^{-3t} = 0 \tag{8.15}$$

$$0 = (9 - 15 + 6)e^{-3t} \tag{8.16}$$

Therefore, the general solution to **Equation 8.9**

$$y_c = C_1 e^{-3t} + C_2 e^{-2t} \tag{8.17}$$

where C_1 and C_2 are constants.

Example 8.3

$$L[y] = y'' + ay' + by = 0 \tag{8.18}$$

$$f(r) = r^2 + ar + b = 0 \tag{8.19}$$

Suppose that r_0 is a root of $f(r) = 0$
Consider

$$y_0(t) = e^{r_0 t} \tag{8.20}$$

$$y_0'(t) = r_0 e^{r_0 t} \tag{8.21}$$

$$y_0''(t) = r_0^2 e^{r_0 t} \tag{8.22}$$

$$\implies L(y_0) = r_0^2 e^{r_0 t} + ar_0 e^{r_0 t} + be^{r_0 t} \tag{8.23}$$

$$= e^{r_0 t}(r_0^2 + ar_0 + b) \tag{8.24}$$

Things to consider, what if $r_0 \in \mathbb{C}$ or $r_0 = r_1$?

Example 8.4 CHSOLDE IVP

Find the solution of the CHSOLDE IVP,

$$L[y] = y'' + 5y' + 6y = 0; \quad y(0) = 2; \quad y'(0) = 3 \tag{8.25}$$

1. Find the general solution

$$y_c(t) = C_1 y_1 + C_2 y_2 \implies y_c(t) = C_1 e^{-3t} + C_2 e^{-2t} \tag{8.26}$$

2. Find the particular values of C_1 and C_2 such that $C_1 y_1(0) + C_2 y_2(0) = 2$ and $(C_1 y_1(0) + C_2 y_2(0))' = 3$.

$$\begin{cases} C_1 e^{-3(0)} + C_2 e^{-2(0)} = 2 \\ -3C_1 e^{-3(0)} + -2C_2 e^{-2(0)} = 3 \end{cases} \tag{8.27}$$

Solving the linear combination yields $C_1 = 7$, $C_2 = 9$. Then, the solution to this IVP is

$$y(t) = -7e^{-3t} + 9e^{-2t} \tag{8.28}$$

Lecture 9

11 February 2020

9.1 Second Order Linear Differential Equations

Theorem 9.1 Principle of Superposition

Suppose that y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.1}$$

Then, $C_1y_1 + C_2y_2$ is another solution for $L[y] = 0$ where C_1 and C_2 are constants. ($C_1y_1 + C_2y_2$ is the liear combination of y_1 and y_2)

Proof 9.1 Principle of Superposition

Show that

$$L[C_1y_1 + C_2y_2] = 0 \tag{9.2}$$

$$\begin{aligned}
&L[C_1y_1 + C_2y_2] && (9.3)\\
&= (C_1y_1 + C_2y_2)'' + p(t)(C_1y_1 + C_2y_2)' + q(t)(C_1y_1 + C_2y_2) && (9.4)\\
&= C_1y_1'' + C_2y_2'' + p(t)C_1y_1' + p(t)C_2y_2' + q(t)C_1y_1 + q(t)C_2y_2 && (9.5)\\
&= (C_1y_1'' + p(t)C_1y_1' + q(t)C_1y_1) + (C_2y_2'' + p(t)C_2y_2' + q(t)C_2y_2) && (9.6)\\
&= C_1L[y_1] + C_2L[y_2] = 0 && (9.7)
\end{aligned}$$

From the above, $C_1L[y_1] = 0$ and $C_2L[y_2] = 0$, therefore

$$L[C_1y_1 + C_2y_2] = 0 \tag{9.8}$$

Theorem 9.2 Existance and Uniqueness Theorem

Given

$$L[y] = y'' + p(t)y' + q(t)y = g(t); \quad y(t_0) = z_0; \quad y'(t_0) = z_1 \tag{9.9}$$

suppose $t_0 \in I$.
Then, this IVP has exactly 1 solution. Moreover, this solution will be defined throughout the interval.

Example 9.1 Application of Existance and Uniqueness Theorem

Find the longest interval in which the solution of the IVP is certain to exist.

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0; \quad y(1) = 2; \quad y(1) = 1 \tag{9.10}$$

The equation is equivalent to

$$L[y] = y'' + \frac{t}{t^2 - 3t}y' - \frac{t+3}{t^2 - 3t}y = 0 \tag{9.11}$$

1. $\forall t \in (-\infty, \infty), \lim_{a \to t}(g(a))$
2. $\forall\ t \in (-\infty, 0) \cup (0, 3) \cup (3, \infty), \lim_{a \to t}(q(a))$
3. $\forall\ t \in (-\infty, 3) \cup (3, \infty), \lim_{a \to t}(p(a))$

From the above,

$$I = (0, 3) \tag{9.12}$$

Example 9.2

Find the unique solution of the IVP given by

$$L[y] = y'' + p(t)y' + q(t)y = 0; \quad y(t_0) = 0; \quad y'(t_0) = 0 \tag{9.13}$$

where $p(t)$ and $q(t)$ are continuous for $t \in (-\infty, \infty)$.
The solution is

$$y(t) = 0 \tag{9.14}$$

and because of the uniqueness theorem, this is the only answer.

9.2 Linear Algebra with 2 Unknowns Detour

Definition 9.1

General form of a linear system with 2 Unknowns

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{9.15}$$

where x and y are the two unknowns. In matrix form, the linear combination above can be rewritten as

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{9.16}$$

Definition 9.2 Matricies

A $n \times m$ **matrix** is a $n \times m$ table filled with numbers or functions. They are written with parenthesis or brackets around the numbers, such as

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \tag{9.17}$$

When $n = m$, the matrix is considered to be a **square matrix**.

Definition 9.3 Determinant

An import concept involved with square matricies is the determinant, in the case of

$$\det(A) = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = ad - bc \tag{9.18}$$

Theorem 9.3

The solution to

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{9.19}$$

is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.20}$$

where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. No other solution exists.

9.3 Wronskian

Definition 9.4 Wronskian

For two differentiable functions $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$, the **Wronskian** of y_1 and y_2 is defined by

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{9.21}$$

9.4 Miscellaneous Definitions

Additional notes that were either not covered or were missed from previous lectures.

Definition 9.5 Differential Operator

Let p and q are continuous over the open interval I, where $t \in (\alpha, \beta)$, where $\alpha = -\infty$ or $\beta = \infty$ are included. Then for any function ϕ that is twice differentiable on I. The **differential operator** is defined by

$$L[\phi] = \phi'' + p\phi' + q\phi \tag{9.22}$$

Note that result of the operator is a function itself, so the value of $L[\phi]$ at point t is

$$L[\phi] = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \tag{9.23}$$

Lecture 10

13 February 2020

10.1 Applications of the Wronskian

Corollary 10.1

Recall in the last lecture, **Theorem 9.3**. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{10.1}$$

If $|A| = 0$, then for **some** values of c_1 and c_2 the linear system

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{10.2}$$

does not have a solution (inconsistent).

Theorem 10.1

Assume that y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.3}$$

where p and q are continuous, and t_0 is a fixed point.
Then, $\forall z_0 \wedge \forall z_1$, it is possible to find c_1 and c_2 such that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{10.4}$$

satisfies the IVP

$$L[y], \quad y(t_0) = z_0, \quad y'(t_0) = z_1 \tag{10.5}$$

if and only if

$$W[y_1, y_2](t_0) \neq 0 \tag{10.6}$$

Proof 10.1

Suppose that $\forall z_0, z_1 \implies \exists\, C_1, C_2$ such that

$$\begin{cases} C_1 y_1(t_0) + C_2 y_{2(t_0)} = z_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) = z_1 \end{cases} \tag{10.7}$$

then, $\exists!\, C_1, C_2$ iff $W[y_1, y_2](t_0) \neq 0$

Theorem 10.2

Suppose that y_1 and y_2 are solutions to

$$L[y] = 0 \tag{10.8}$$

Then, the family of solutions

$$y = C_1 y_1 + C_2 y_2 \tag{10.9}$$

includes all solutions of $L[y] = 0$ iff $\exists t_0 \implies W[y_1, y_2](t_0) \neq 0$

Example 10.1 Application of 10.2

The solutions to

$$y'' - 5y' + 6y = 0 \tag{10.10}$$

are

$$y_1 = e^{2t}, \quad y_2 = e^{3t} \tag{10.11}$$

$$y = C_1 e^{2t} + C_2 e^{3t} \tag{10.12}$$

Calculating the Wronskian

$$W[y_1, y_2] = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} \tag{10.13}$$

$$= e^{5t} \tag{10.14}$$

$$e^{5t} \neq 0 \tag{10.15}$$

Therefore, there does not exist other solutions to this CHSOLDE.

Theorem 10.3 Abel's Theorem

If y_1, y_2 are solutions to a SOLDE,

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.16}$$

where p, t are continuous over an open interval, I, then the Wronskian at point t is given by Abel's Formula,

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\, dt\right) \tag{10.17}$$

where c is some arbitrary constant dependent on y_1, y_2, but not on t.

$$\forall t \in I, W[y_1, y_2](t) \equiv 0 \iff c = 0 \tag{10.18}$$

$$\forall t \in I, W[y_1, y_2](t) \not\equiv 0 \iff c \neq 0 \tag{10.19}$$

Example 10.2

Suppose that

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t} \tag{10.20}$$

are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.21}$$

Show that if $r_1 \neq r_2$, then $C_1 y_1 + C_2 y_2$ includes all solutions of $L[y] = 0$.

$$W[e^{r_1 t}, e^{r_2 t}] = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \tag{10.22}$$

$$= (r_2 - r_1)e^{(r_1+r_2)t} \tag{10.23}$$

$$\neq 0 \tag{10.24}$$

Definition 10.1 Fundamental Set of Solutions

If y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.25}$$

such that $C_1 y_1 + C_2 y_2$ includes all possible solutions of $L[y] = 0$, then y_1 and y_2 form a **fundamental set of solutions** (FSS).
Alternatively, if and only if

$$W[y_1, y_2] \neq 0 \tag{10.26}$$

then there exists fundamental set containing y_1 and y_2.

Example 10.3

Show that $y_1(t) = t^{\frac{1}{2}}$, $y_2(t) = t^{-1}$ form a FSS of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0 \tag{10.27}$$

1. Ensure they are solutions of $L[y] = 0$
   i.e. $L[t^{\frac{1}{2}}] = 0$, $L[t^{-1}] = 0$

$$L[t^{\frac{1}{2}}] = 2t^2(-\frac{1}{4})t^{-\frac{3}{4}} + 3t(\frac{1}{2})t^{-\frac{1}{2}} + t^{\frac{1}{2}} \tag{10.28}$$

$$= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \tag{10.29}$$

$$\equiv 0 \tag{10.30}$$

Example 10.3

$$L[t^{-1}] = (2t^2)(2t^{-3}) + 3t(-1)t^{-2} - t^{-1} \tag{10.31}$$
$$= 4t^{-1} - 3t^{-1} - t^{-1} \tag{10.32}$$
$$\equiv 0 \tag{10.33}$$

2. Ensure that the Wronskian is not constantly equal to 0

$$W[t^{\frac{1}{2}}, t^{-1}] = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-1} \end{vmatrix} \tag{10.34}$$
$$= -t^{-\frac{1}{2}} - -\frac{1}{2}t^{-\frac{3}{2}} \tag{10.35}$$
$$\not\equiv 0 \tag{10.36}$$

Then,

$$L[y] = y'' + ay' + by = 0; \quad f(r) = r^2 + ar + b = 0 \tag{10.37}$$

has only one solution of degree 2.

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}, \quad r_1 = r_2 \iff \sqrt{a^2 - 4b} \equiv 0 \tag{10.38}$$

Example 10.4

1. If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt} \tag{10.39}$$

When $r = r_1$, $L[e^{rt}] = 0$, therefore e^{rt} is a solution to

$$L[y] = ay'' + by' + cy = 0 \tag{10.40}$$

2. Then,

$$\frac{\partial}{\partial r} L[e^{rt}] = L[\frac{\partial}{\partial r} e^{rt}] = L[te^{rt}] \tag{10.41}$$
$$= ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1) \tag{10.42}$$

Because $r = r_1 \implies L[te^{rt}] = 0$, te^{rt} is another solution to $L[y] = 0$. Show that e^{rt}, te^{rt} form a FSS.

a. $L[e^{rt}] = 0$

Example 10.4

b. $L[te^{rt}] = 0$

$$
\begin{aligned}
(te^{rt})' &= e^{rt} + rte^{rt} && (10.43)\\
(te^{rt})'' &= 2re^{rt} + r^2te^{rt} && (10.44)\\
L[te^{rt}] &= (2re^{rt} + r^2te^{rt}) + a(e^{rt} + rte^{rt}) + b && (10.45)\\
&= e^{rt}(2r + a) + te^{rt}(r^2 + ar + b) && (10.46)
\end{aligned}
$$

Lecture 11

18 February 2020

11.1 Test Corrections

7. Prove that $y_1 = t^{-1}$, $y_2 = t^{1/2}$ form a fundamental set for $L[y] = 0$. Wrong:

$$L[c_1 t^{-1} + c_2 t^{1/2}] = 0 \implies c_1 L[t^{-1}] + c_2 L[t^{1/2}] = 0 \tag{11.1}$$

Correct:

$$L[t^{-1}] = 0; \quad L[t^{1/2}] = 0 \tag{11.2}$$

Then, prove that

$$W[t^{-1}, t^{1/2}] \neq 0 \tag{11.3}$$

Lecture 12

20 February 2020

Lecture 13

25 February 2020

Lecture 14

27 February 2020

Lecture 15

3 March 2020

Lecture 16

5 March 2020

Lecture 17

17 March 2020

Lecture 18

22 March 2020

Lecture 19

24 March 2020

Lecture 20

29 March 2020

Lecture 21

31 March 2020