Differential Equations

MATH 308 at Texas A&M

Using *Elementary Differential Equations, 11th Edition*

Brandon Nguyen

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Contents

Lecture 1

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1.1 First Order Differential Equations

Definition 1.1 First Order Differential Equation

The generic form of a **first order differential equation** is

$$y' = f(x, y) \tag{1.1}$$

Sometimes, t is substituted for x, especially if the function relates to time.

Definition 1.2 General Solution

A solution to a differential equation is considered to be **general** if there is an arbitrary constant present in the final answer, i.e. a problem without initial values.

Example 1.1

$$\begin{aligned} y' &= 1 && (1.2) \\ y &= \int y' \, dx && (1.3) \\ &= 1 \, dx && (1.4) \\ &= x + C && (1.5) \end{aligned}$$

Definition 1.3 Open Differential Equations

Equations without solutions are considered to be **open**. Many differential equations are without solutions.

Example 1.2 Open Differential Equation

$$y' = x'y - x^3 \tag{1.6}$$

This differential equation does not have a solution; thusly open.

Example 1.3

$$y' = y \tag{1.7}$$

$$\int y'\,dy = \int y\,dy \nLeftrightarrow y' = y \tag{1.8}$$

Notice above that the integration of both sides is not the same as the differential equation.

$$y' = e^x \implies y \int y'\,dx = \int e^x\,dx \tag{1.9}$$

Using the above, the general solution can be found

$$y = Ce^x \tag{1.10}$$

Remark 1.1 Regarding Example 1.3

If both sides of a differential equation are dependent on the same variable — i.e. the same variable appearing on both sides of the equation, then taking the intergal of both sides is not a valid method to solve the equation.

Definition 1.4 Initial Value Problems

An **initial value problem** (IVP), or **initial condition problem**, is a problem where an initial condition of the equation is defined which leads to a **unique solution** to the equation.

Example 1.4 Initial Value Problem

$$y' = x,\ y(0) = 1 \tag{1.11}$$

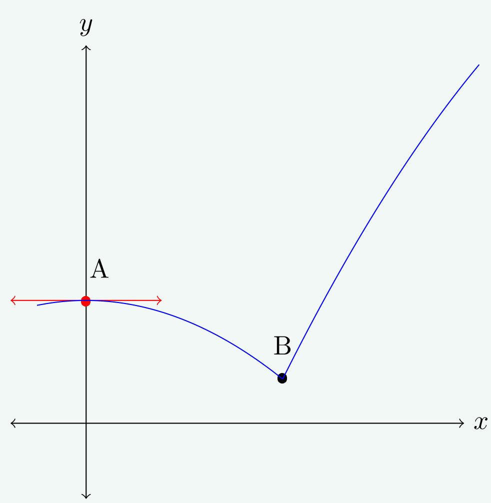
Notice that this is an **initial value problem**, because $y(0) = 1$. Also notice that y is an anti-derivative w.r.t. x; because each side of the equation is independent of one another (unlike *Example 1.3*).

$$\begin{aligned}
& \int y'\,dx = \int x\,dx && (1.12)\\
\implies \quad & y = \frac{1}{2}x^2 + C && (1.13)\\
& y(0) = 1 && (1.14)\\
\implies \quad & = \frac{1}{2}(0^2) + C && (1.15)\\
\implies \quad & C = 1 && (1.16)\\
\implies \quad & y = \frac{1}{2}x^2 + 1 && (1.17)
\end{aligned}$$

1.2 Differentiable Functions

Definition 1.5 Differentiability

Given $f : \mathbb{R} \to \mathbb{R}$ and point a, $\exists f'(a) \iff \exists T_1(a)$, where T_1 is a tangent line (Taylor polynomial of degree one).



In the example, point A has a singular tangent line and is therefore differentiable. Point B has infinitely many tangent lines, and is therefore both undefined and not differentiable.

1.3 Kinematics

Example 1.5 Kinematics with Differential Equations

Given an object with a velocity v_0, and acceleration a, find the position s at any time t.

$$\begin{aligned}
\frac{d}{dt}v(t) &= a && (1.18)\\
\implies \quad v(t) &= \int a\,dt && (1.19)\\
&= at + C && (1.20)\\
\because \quad v(0) &= v_0 && (1.21)\\
v_0 &= a(0) + C && (1.22)\\
C &= v_0 && (1.23)\\
\frac{d}{dt}s(t) &= \int v\,dt && (1.24)\\
\implies \quad s(t) &= \int v\,dt && (1.25)\\
&= \int (at + v_0)\,dt && (1.26)\\
&= \frac{1}{2}at^2 + v_0 t && (1.27)
\end{aligned}$$

Lecture 2

16 January 2020

2.1 Linear Differential Equations

Definition 2.1 First Order Linear Differential Equations

$$\underbrace{y' + p(t)y = g(t)}_{\text{Usual form}} \iff y' = g(t) - p(t)y \tag{2.1}$$

A **first order linear differential equation** (LDE) is linear due to y being dependent on only one variable, t.

Notice that t is typically used in place of x as most differential equations are used in models dependent on time; as such, most differential equations are in the form $y' = f(t, y)$ as opposed to $y' = f(x, y)$.

Example 2.1

$$\text{Solve} \quad (4 + t^2)y' + 2ty = 4t \tag{2.2}$$

$$\begin{aligned}
\text{Notice:} \qquad (4y + t^2y)' &= \frac{d}{dt}(4y + t^2y) = 4t && (2.3)\\
&= 4y' + (t^2y)^2 && (2.4)\\
&= 4y' + (2ty + t^2y') && (2.5)\\
&= (4 + t^2)y' + 2ty && (2.6)
\end{aligned}$$

Example 2.1

The original problem can now be reduced to:

$$\frac{d}{dt}(4y + t^2y) = 4t \tag{2.7}$$

$$\text{let} \quad z(t) = 4y + t^2y \tag{2.8}$$

$$= 2t^2 + C \tag{2.9}$$

$$\implies \quad 4y + t^2y = 2t^2 + C \tag{2.10}$$

$$\therefore \quad y = \frac{1}{4+t^2}(2t^2 + C) \tag{2.11}$$

Remark 2.1 Constants

Notice in the above example that the constant, C, is being multiplied by $\frac{1}{4+t^2}$. When expanding the answer, it now becomes $y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$. Notice how the constant is dependent on the variable t, and is therefore not the same as just C.

Definition 2.2 Integrating Factors with LDEs

An **integrating factor**, $\mu(t)$ is a function $\mu(t) : \mathbb{R} \to \mathbb{R}$, that satisfies $\frac{d}{dt}\mu(t) = \mu(t)y' + \mu(t)p(t)y$.

Remark 2.2

There are infinitely many integrating factors due to the arbitrary constant C from indefinite integration, see **Method 2.1** and **Example 2.2** on the following page.

Method 2.1 Solution of the General LDE Case

Solve $y' + p(t)y = g(t)$.

1. Multiply the LDE by $\mu(t)$ results in:
$$\mu(t)(y' + p(t)y) = \mu(t)g(t) \tag{2.12}$$

2. Letting $z(t) = \mu(t)y$, and $z' = \mu(t)g(t)$ yields:
$$z(t) = \int \mu(t)g(t)\,dt \tag{2.13}$$
$$\implies y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt \tag{2.14}$$
$$\implies \mu(t) = \exp\left(\int p(t)\,dt\right) \tag{2.15}$$

3. Therefore the solution of the general case is
$$y(t) = \left(\exp\left(\int p(t)\,dt\right)\right)^{-1} \cdot \int \exp\left(\int p(t)\,dt\right) g(t)\,dt \tag{2.16}$$

Example 2.2 Solving an IVP involving LDEs

Working with example 2.1.4 from the textbook:
$$ty' + 42y = 4t^2,\ y(1) = 2 \tag{2.17}$$

1. Compute the integrating factor ($\mu(t)$)
$$\mu(t) = \exp\left(\int p(t)\,dt\right) \tag{2.18}$$
$$= \exp\left(\int 2t^{-1}\,dt\right) \tag{2.19}$$
$$= \exp(2\ln(t) + C) \Leftrightarrow e^{2\ln(t)+C} \tag{2.20}$$

2. Find the general case
When solving, 0 can be subsituted in for C to simplify calculations; for $C \neq 0$ it is trivially shown that the constant will cancel out in

Example 2.2 Solving an IVP involving LDEs

computing the solution.

$$y_c(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)\, dt \tag{2.21}$$
$$= \frac{1}{t^2} \left(\int t^2 \cdot 4t\, dt \right) \tag{2.22}$$
$$= \frac{1}{t^2} (t^4 + C) \tag{2.23}$$

Note: $y_c(t)$ is used to denote the general case.

3. Find formula w.r.t. intial value

$$y(1) = 2 \tag{2.24}$$
$$\implies y(1) = (1)^2 + \frac{C}{(1)^2} \tag{2.25}$$
$$\implies \quad C = 1 \tag{2.26}$$
$$\therefore \quad y(t) = t^2 + t^{-2} \tag{2.27}$$

Lecture 3

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3.1 Linear Differential Equations (cont).

Example 3.1

Given $y' - 2y = t^2e^{2t}$ find:

1. The general solution

$$p(t) = -2,\ g(t) = t^2e^{2t} \tag{3.1}$$

$$\mu(t) = \exp\left(\int -2\,dt\right) \tag{3.2}$$

$$= e^{-2t+C} \tag{3.3}$$

$$y_c(t) = e^{2t}\int t^2\,dt \tag{3.4}$$

$$= e^{2t}\left(\frac{1}{3}t^3 + C\right) \tag{3.5}$$

2. What is $\lim_{t\to\infty} y_c(t)$?
   There are infinitely many $y_c(t)$; the answer may vary with the value of C. In this case, the value of C does not matter.

$$\lim_{t\to\infty} y_c(t) = +\infty$$

3.2 Separable Differential Equations

Definition 3.1 Separable Differential Equations

A **separable differential equation** (SDE) can be defined by

$$\frac{dy}{dx} = y' = f(x, y) = -\frac{M(x, y)}{N(x, y)} \tag{3.6}$$

where

$$M(x, y) = -f(x, y) \tag{3.7}$$

$$N(x, y) = 1 \tag{3.8}$$

it is **separable** because it can be written in the **differential form**

$$M(x)\,dx + N(y)\,dy = 0 \tag{3.9}$$

Theorem 3.1

If $\frac{dy}{dx} = \frac{M(x)}{N(y)}$, then $\int N(y)\,dy = \int M(x)\,dx$

Proof 3.1

Choose $\widetilde{N}$ such that $\frac{d\widetilde{N}(y)}{dx} = M(x)$:

$$\frac{d\widetilde{N}(y)}{dy} = \frac{d\widetilde{N}(y)}{dx}\frac{dx}{dy} = \frac{d\widetilde{N}(y)}{dy}\frac{dy}{dx} = \frac{d\widetilde{N}(x)}{dx} \tag{3.10}$$

$$\frac{d\widetilde{N}(y)}{dy} = \frac{dy}{dx} \tag{3.11}$$

$$\implies \frac{d\widetilde{N}(y)}{dx} = M(x) \tag{3.12}$$

Example 3.2

Find a particular solution that passes through the point $(0, 1)$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{4x - x^3}{4 + y} && (3.13)\\
\implies \int (4 + y)\, dy &= \int (4x - x^3)\, dx && (3.14)\\
4y + \frac{1}{2}y^2 + C_1 &= 2x^2 - \frac{1}{4}x^4 + C_2 && (3.15)\\
4y + \frac{1}{2}y^2 &= 2x^2 - \frac{1}{4}x^4 + (C_2 - C_1) && (3.16)\\
\implies 2y + 16y + x^4 - 8x^2 + C &= 0 && (3.17)\\
(0, 1) \implies 2(1) + 16(1) + 0^4 - 8(0)^2 + C &= 0 && (3.18)\\
C &= -18 && (3.19)\\
\therefore 2y + 16y + x^4 - 8x^2 &= 18 && (3.20)
\end{aligned}
$$

Homework 3.1

$$
\begin{aligned}
y' = \frac{dy}{dx} &= \frac{x^2}{y} && (3.21)\\
y\, dy &= x^2\, dx && (3.22)\\
\int y\, dy &= \int x^2\, dx && (3.23)\\
\frac{1}{2}y^2 &= \frac{1}{3}x^3 + C && (3.24)\\
y(x) &= \pm\sqrt{\frac{2}{3}x^3 + C} && (3.25)
\end{aligned}
$$

Lecture 4

23 January 2020

4.1 Separable Equations (cont.)

Example 4.1

From the textbook, 2.2, ex. 2.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad y(0) = -1 \tag{4.1}$$

Given the above, determine the interval in which the solution exists.

$$\int 2(y-1)\,dy = \int (3x^2 + 4x + 2)\,dx \tag{4.2}$$

$$\implies y^2 - 2y + C_1 = x^3 + 2x^2 + 2x + C_2 \tag{4.3}$$

The solution above is the **general implicit solution**. The constants, C_1 and C_2 can be combined into one constant, C, because they are independent.
Next, use the initial value to solve for C

$$y(0) = -1 \tag{4.4}$$

$$\implies (-1)^2 - 2(-1) = 0^3 + 2(0)^2 + 2(0) + C \tag{4.5}$$

$$\implies \quad C = 3 \tag{4.6}$$

Example 4.1 (cont.)

Then complete the square on the left hand side to get the **explicit solution**.

$$
\begin{aligned}
& (y^2 - 2y + 1) - 1 = x^3 + 2x^2 + 2x + 3 && (4.7)\\
\implies & (y-1)^2 = x^3 + 2x^2 + 2x + 4 && (4.8)\\
\implies & y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4} && (4.9)\\
\implies & y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} && (4.10)\\
\implies & y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} && (4.11)\\
\because & y(0) = -1 && (4.12)
\end{aligned}
$$

Note: It is also possible to use the quadratic formula in order to convert this instance of an implicit into an explicit solution.
Observation: Because the unique solution involves a square root, a function defined for $x \in [0, \infty)$, it is possible to reduce the original question to findinding when the radicand is non-negative.

$$
\begin{aligned}
& x^3 + 2x^2 + 2x + 4 = 0 && (4.13)\\
& (x^2 + 2)(x + 2) = 0 && (4.14)\\
\implies & x \geq -2 && (4.15)
\end{aligned}
$$

The factor $x^2 + 2$ will always be positive, so now the question is further reduced to when $x + 2$ will be non-negative, which is $x \in [-2, \infty)$.
Therefore, the interval of which the solution exists is $(-2, \infty)$

Remark 4.1 Solutions to Differential Equations

In **Example 4.1**, notice the final answer was an open interval, $(-2, \infty)$, rather than a half closed interval, $[-2, \infty)$, even if the solution would be defined if $x = -2$. The reason for this is that **solutions to differential equations must also be differentiable**.
At point $x = -2$, the unique solution is defined, however, it is not differentiable as $\lim_{x \to -2^-}$ does not exist, because the function is not defined for $x < -2$.

4.2 Mathematical Modelling

Example 4.2 Modelling

Consider a pond fille with 10 million gallons of fresh water. A flow of 5 million gallons per year with water that is contaminated wiht a chemical enters the pond. There is also an outflow of this mixture on the order of 5 million gallons per year.

Let $\gamma(t)$ be the concentration of the fluid entering the pod at time t, and let $Q(t)$ be the quantity of chemicals in the pod at time t.

It is determined that

$$\gamma(t) = 2 + \sin(2t) \ \mathrm{g} \cdot \mathrm{gal}^{-1}$$

Find $Q(t)$ using the given information.

We can infer that $Q(0) = 0$ because the water starts off fresh at $t = 0$. We know that $\frac{dQ}{dt}$ is equal to the rate at which chemicals are entering minus the rate at which they leave, leading us to

$$\frac{dQ}{dt} = I(t)\gamma(t) - \frac{O(t)}{V(t)}\left[Q(t)\right]$$

Where $I(t)$ describes the rate at which the contaminated water enters, $O(t)$ describes the rate at which the water mixture leaves the pond, and $V(t)$ describes the total volume of the pond at any given time.
In this case,

$$I(t) = 5 \times 10^6 \ \mathrm{gal\,year}^{-1} \tag{4.16}$$

$$O(t) = 5 \times 10^6 \ \mathrm{gal\,year}^{-1} \tag{4.17}$$

$$V(t) = 10^7 \ \mathrm{gal} \tag{4.18}$$

(4.19)

Plugging in the values yields the following,

$$\frac{dQ}{dt} = 5 \times 10^6 \gamma(t) - \frac{1}{2}Q(t) \tag{4.20}$$

(4.21)

Example 4.2 Modelling

Solving the linear differential equation,

$$\frac{dQ}{dt} + \frac{1}{2}Q(t) = 5 \times 10^6 \gamma(t) \tag{4.22}$$

$$\implies \quad Q_c(t) = 5 \times 10^6 e^{-\frac{1}{2}t} \int e^{\frac{1}{2}t}(2 + \sin(2t))\, dt \tag{4.23}$$

$$\implies \quad Q_c(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17} cos(2t) + Ce^{-\frac{1}{2}t} \tag{4.24}$$

$$Q_c(0) = 2 \times 10^7 - \frac{4 \times 10^7}{17} + C = 0 \tag{4.25}$$

$$\implies \quad C = \frac{-3 \cdot 10^8}{17} \tag{4.26}$$

$$Q(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17} cos(2t) - \frac{3 \cdot 10^8}{17} e^{-\frac{1}{2}t} \tag{4.27}$$

Remark 4.2 Behavior of Example 4.2

When graphing this equation, it can be seen that in the long term the equation becomes periodic despite beginning with an irregular pattern. This is due to the fact that the term $-\frac{3 \cdot 10^8}{17} e^{-\frac{1}{2}t}$ is able to affect the behavior in the short term, however, it is decaying exponentially and tends towards 0. The sin and cos functions are periodic which cause the sinusoidial shape of the graph as $t \to \infty$.

Lecture 5

28 January 2020

5.1 Mathematical Modelling (cont.)

Example 5.1

Example 2.3.1 from the textbook.

1. Find the amount of salt in the tank at a time t ($Q(t)$).
   Inference: $Q(0) = Q_0$

$$\frac{dQ}{dt} = \frac{1}{4}r - \frac{rQ}{100} \tag{5.1}$$

$$\implies Q' + \frac{r}{100}Q = \frac{1}{4}r \tag{5.2}$$

$$\implies \quad Q_c = \exp\left(-\frac{r}{100}t\right)\int\left(\exp\left(\frac{r}{100}t\right)\frac{1}{4}r\right)dt \tag{5.3}$$

$$= \frac{r}{4}\exp\left(\frac{-r}{100}t\right)\left(\frac{100}{r}\exp\left(\frac{r}{100}t\right) + C\right) \tag{5.4}$$

$$= 25 + \frac{r}{4}\exp\left(-\frac{r}{100}t\right)C \tag{5.5}$$

$$= 25 + C\exp\left(-\frac{r}{100}t\right) \tag{5.6}$$

$$Q(0) = Q_0 \tag{5.7}$$

$$\implies \quad C = (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.8}$$

$$\implies \quad Q(t) = 25 + (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.9}$$

Example 5.1

2. Find the limiting amount, Q_l, after a long time.

$$\lim_{t\to\infty}(Q_c(t)) = Q_c = 25 \tag{5.10}$$

Remark 5.1 Regarding Example 5.1

Notice that no matter the amount of salt that the system starts with, it will always tend towards 25 lbs of salt in the tank.

5.2 Exact Differential Equations

Definition 5.1 Exact Differential Equations

A differential equation is exact iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \Leftrightarrow N(x,y)y' + M(x,y) = 0 \tag{5.11}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{5.12}$$

Given $\psi(x,y)$, parameterize by using $\delta(t) = \psi(f_1(t), f_2(t))$.

$$\begin{aligned} \frac{d\psi(x,y)}{dt} &= \frac{d\delta}{dt} && (5.13)\\ &= \frac{\partial\psi(x,y)}{\partial x}\frac{df_1}{dt} + \frac{\partial\psi(x,y)}{\partial y}\frac{df_2}{dt} && (5.14) \end{aligned}$$

Example 5.2

$$\begin{aligned} \psi(x,y) &= x^2y + xy && (5.15)\\ f_1(t) = t, \quad f_2(t) &= t^2 && (5.16)\\ \delta(t) &= \psi(f_1, f_2) && (5.17)\\ &= t^2t^2 + tt^2 && (5.18)\\ \delta'(t) &= 4t^3 + 3t^2 && (5.19)\\ \frac{\partial\psi(x,y)}{\partial x}\cdot 1 + \frac{\partial\psi(x,y)}{\partial y}\cdot 2t & && (5.20)\\ &= (2f_1f_2 + f_2)\cdot 1 + (f_1^2 + f_1)\cdot 2ty && (5.21)\\ &= 4t^3 + 3t^2 && (5.22) \end{aligned}$$

Example 5.2

Notice how **Equation 5.19** and **Equation 5.22** are the same, but derived via different methods.

Example 5.3

1. $y' = \frac{\frac{1}{y}}{x}$ is an exact differential equation.
   Let $M(x,y) = \frac{1}{x}$, and $N(x,y) = y$.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial \frac{1}{x}}{\partial y} = 0$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial y}{\partial x} = 0$$

   Because both partial derivatives are equal, they are exact.

2. $y' = x$ is exact.

3. $y' = \frac{xy}{x+y} \iff (x+y)\,dy + xy\,dx = 0$ is exact.

4. $y' = \frac{xy+x}{\frac{1}{2}x^2+y}$ is exact.

Theorem 5.1 Exactness

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if, and only if, $\exists\,\psi(x,y)$ s.t.

$$\frac{\partial \psi(x,y)}{\partial x} = M(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = N(x,y)$$

Remark 5.2 Relationship

Exact differential equations are a superset of the separable differential equations, i.e. all separable differential equations are exact differential equations.

Lecture 6

30 January 2020

6.1 Exact Differential Equations (cont.)

Example 6.1

Solve

$$(y\cos x + 2xe^y) + (\sin x + x^2 + x^2e^y - 1)y' = 0 \tag{6.1}$$

Checking if **Equation 6.1** is exact,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial (y\cos x + 2xe^y)}{\partial y} = \cos x + 2xe^y \tag{6.2}$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial (\sin x + x^2e^y - 1)}{\partial y} = \cos x + 2xe^y \tag{6.3}$$

From the above, this is an exact differential equation.

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.4}$$

$$= \int (y\cos x + 2xe^y)\,dx + h(y) \tag{6.5}$$

$$= h(y) + y\sin x + x^2e^y + C \tag{6.6}$$

$$= h(y) + y\sin x + x^2e^y \tag{6.7}$$

Notice that the constant can be neglected as it can be contained in $h(y)$.

Example 6.1

Now solving for $h(y)$,

$$\psi_y(x,y) = N(x,y) \tag{6.8}$$
$$\implies \frac{dh}{dy} + \frac{\partial(y\sin x + e^y x^2)}{\partial y} = \sin x + x^2 e^y - 1 \tag{6.9}$$
$$\frac{dh}{dy} + \sin x + x^2 e^y = \sin x + x^2 e^y - 1 \tag{6.10}$$
$$\frac{dh}{dy} = -1 \tag{6.11}$$
$$h = -y + C \tag{6.12}$$

Then,

$$\psi(x,y) = y\sin x + x^2 e^y - y + C \tag{6.13}$$

Finally, $y(x)$ is given by the implicit expression

$$y\sin x + x^2 e^y - y = C \tag{6.14}$$

Example 6.2

Solve

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{6.15}$$

Checking if the equation is exact,

$$\frac{\partial(3xy + y^2)}{\partial y} = 3x + 2y \tag{6.16}$$
$$\frac{\partial(x^2 + xy)}{\partial x} = 2x + y \tag{6.17}$$

Notice that they are not equal; however,

$$\mu(x)(3xy + y^2) + \mu(x)(x^2 + xy)y' = 0 \tag{6.18}$$

is an exact differential equation if

$$-\frac{N_x(x,y) + M_y)x,y}{N(x,y)} \tag{6.19}$$

is a function dependent only on x. However, $\forall M(x,y), N(x,y) \nexists \mu(x)$. $\mu(x)$

Example 6.2

can be found by solving the differential equation,

$$\frac{d\mu}{dx} = \frac{-N_x(x,y) + M_y(x,y)}{N(x,y)}\mu \tag{6.20}$$

$$\mu(x) = \exp\left(\int \frac{N_x - M_y}{N}\,dx\right) \tag{6.21}$$

In this problem,

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{1}{x} \tag{6.22}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = x + C \tag{6.23}$$

Multiplying **Equation 6.15** by $\mu(x)$ yields,

$$(3x^2y + y^2x) + (x^3 + x^2y)y' = 0 \tag{6.24}$$

Checking if the equation is exact yields the following,

$$\frac{\partial 3x^2y + xy^2}{\partial y} = 3x^2 + 2xy \tag{6.25}$$

$$\frac{\partial x^3 + x^2y}{\partial x} = 3x^2 + 2xy \tag{6.26}$$

and is therefore exact.

$$\psi(x,y) = \int (3x^2y + xy^2)\,dx + h(y) \tag{6.27}$$

$$= x^3y + \frac{1}{2}x^2y^2 + h(y) \tag{6.28}$$

$$\frac{\partial\psi(x,y)}{\partial y} = x^3 + x^2y + \frac{dh}{dy} \tag{6.29}$$

$$= N(x,y) \tag{6.30}$$

$$\frac{dh}{dy} = x^3 + x^2y = x^3 + x^2y \tag{6.31}$$

$$h = 0 \tag{6.32}$$

Finally, $y(x)$ can be expressed as,

$$x^3y + \frac{1}{2}x^2y^2 = C \tag{6.33}$$

Method 6.1 Solving Exact Differential Equations

1. Step 1: Determine if the equation is exact

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{6.34}$$

2. Step 2: Find $\psi(x,y)$ such that $\psi_x(x,y) = M(x,y)$, and $\psi_y(x,y) = N(x,y)$. Generally,

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.35}$$

this works because

$$\frac{\partial \psi(x,y)}{\partial x} = \frac{\partial \int M(x,y)\,dx}{\partial x} + \frac{\partial h(y)}{\partial x} \tag{6.36}$$

$$= M(x,y) + 0 \tag{6.37}$$

Then find $h(y)$ such that $\psi_y(x,y) = N(x,y)$.

Remark 6.1

Note in step 2 of **Method 6.1**

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.38}$$

can also be defined as

$$\psi(x,y) = \int N(x,y)\,dy + h(x) \tag{6.39}$$

$$\frac{\partial \psi(x,y)}{\partial y} = \psi_y(x,y) = \frac{\partial \int N(x,y)\,dy}{\partial y} + \frac{\partial h(x)}{\partial y} \tag{6.40}$$

$$= N(x,y) + 0 \tag{6.41}$$

Remark 6.2

$y(x)$ is a solution for $M(x,y)\,dx + N(x,y)\,dy = 0$ iff $\psi(x,y(x)) = c$. Consider the following,

$$\frac{d\psi(f_1,f_2)}{dt} = \frac{\partial \psi(x,y)}{\partial x}\frac{df_1}{dt} + \frac{\partial \psi(x,y)}{\partial y}\frac{df_2}{dt}$$

Remark 6.2

we can replace t with x, let $f_1 \equiv x$ and $f_2 \equiv y(x)$, then

$$\begin{aligned} \frac{d\psi(f_1(x), f_2(x))}{dx} &= \frac{\partial\psi(x,y)}{\partial x}\frac{df_1}{dx} + \frac{\partial\psi(x,y)}{\partial y}\frac{df_2}{dx} && (6.42) \\ &= \frac{\partial\psi(x,y)}{\partial x} + \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx} && (6.43) \end{aligned}$$

finally,

$$N(x,y)\frac{dy}{dx} = \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx} = \frac{d\psi(x,y(x))}{dx} - \frac{\partial\psi(x,y)}{\partial x}$$

Remark 6.3

Notice in **Equation 6.31** has 3 variables: h, x, y; however, the terms with x cancel, leaving just h and y. This occurs due to the equation being exact.

Lecture 7

4 February 2020

Recall in the last lecture:

$$M_y(x,y) = N_x(x,y) \tag{7.1}$$
$$\implies \exists\, \psi(x,y(x)) : = \psi_x = M(x,y); \psi_y = N(x,y) \tag{7.2}$$
$$\psi(x) = \psi(x,y) \equiv C \tag{7.3}$$

For example,

$$\psi(x,y) = x + y \tag{7.4}$$
$$x + y(x) = C \tag{7.5}$$
$$y = C - x \tag{7.6}$$

And if $\frac{M_y(x,y)-N_x(x,y)}{N(x,y)}$ depends only on x, then $\exists\, \mu(x) : \frac{d\mu}{dx} = \frac{M_y(x,y)-N_x(x,y)}{N(x,y)}\mu$. Thus, the differential equation $\mu M + \mu N y' = 0$ is an exact differential equation.

7.1 Uniqueness and Exactness

Theorem 7.1 Uniqueness of Linear Differential Equations

Consider the linear first order differential equation,

$$y' + p(t)y = g(t); \quad y(t_0) = y_0 \tag{7.7}$$

such that in some open interval, $I = (\alpha; \beta)$, $p(t)$ and $g(t)$ are continuous and $t_0 \in I$.
Then,

$$\exists!\, y(t) : y(t_0) = y_0 \wedge y' + p(t)y = g(t) \tag{7.8}$$

Theorem 7.2 Uniqueness of Non-linear Differential Equations

Consider the following,

$$y' = f(t, y) \wedge y(t_0) = y_0 \tag{7.9}$$

such that $f(t, y)$ and $\frac{\partial f(t,y)}{\partial y}$ are continuous over the domains $t \in (\alpha; \beta)$, and $y \in (\gamma; \delta)$.
Then, $h > 0, I = (t_0 - h, t_0 + h) : \exists t_0 \in I, y(t_0) = y_0$.

Example 7.1

$$ty' + 2y = 4t^2; \quad y(1) = 2 \tag{7.10}$$

Use **Theorem 7.1** to find an interval $\exists! y(t)$.

$$y' + \frac{2}{t} y = 4t \tag{7.11}$$

$$p(t) = \frac{2}{t}, \quad g(t) = 4t \tag{7.12}$$

In the interval $I := (\alpha, \beta)$, $\exists t \in I : p(t), g(t) \implies \exists! y(t)$

1. $\forall\ t \in (-\infty, 0) \cup (0, \infty), p(t)$
2. $\forall\ t \in (-\infty, \infty), g(t)$
3. $1 \in (\alpha, \beta)$
4. Therefore, $\alpha = 0, \beta = \infty \implies I = (0, \infty) = \mathbb{R}^+$

Lecture 8

6 February 2020

8.1 First Order Differential Equation Review

Topics covered in First Order Differential Equations.

- $y' = f(x, y) \quad y(x_0) = y_0$
- First Order LDE, $y' + p(t)y = g(t)$
- Separable, $y' = \frac{M(x,y)}{N(x,y)}$
- Exact, $M(x, y) + N(x, y)y' = 0$; $M_y(x, y) = N_x(x, y)$
- Uniqueness and Existance Theorems
- Modelling

8.2 Second Order Differential Equations

Definition 8.1 Second Order Differential Equations

The general form of a **second order differential equation** (SODE) is

$$y'' = f(x, y, y'); \quad y(x_0) = y_0; \quad y'(x_0) = y_1 \tag{8.1}$$

Example 8.1

The following are SODEs,

$$y'' = 1 \tag{8.2}$$
$$y'' = 1 + y' \tag{8.3}$$
$$y'' = \frac{x}{t} \tag{8.4}$$

An example of a SODE IVP,

$$y'' = x + y + y'; \quad y(0) = 1; \quad y'(0) = -3 \tag{8.5}$$

Definition 8.2 Second Order Linear Differential Equations

A **second order linear differential equation** (SOLDE) has the general form

$$y'' + p(t)y' + q(t)y = g(t) \tag{8.6}$$

where $p(t)$, $q(t)$, $g(t)$ are continuous over some interval I.

Theorem 8.1 SOLDE Uniqueness Theorem

If $p(t), q(t), g(t)$ are continuous in some interval $I : (\alpha, \beta)$
Then, for any $t_0 \in I$, the IVP defined by

$$y'' + p(t)y' + q(t)y = g(t), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{8.7}$$

has a unique solution.

Definition 8.3 Cases of SOLDEs

Homogeneous SOLDEs (HSOLDE) are of the following form

$$y'' + p(t)y' + q(t)y = 0 \tag{8.8}$$

If a Homogeneous SODE is defined where $p(t)$, and $q(t)$ are constants, it is considered as a **homogeneous SOLDE with Constant Coefficients** (CHSOLDE).

8.3 Homogeneous Second Order Linear Differential Equations with Constant Coefficients

Example 8.2 CHSOLDE

Find the general solution of

$$L[y] = y'' + 5y' + 6y = 0 \tag{8.9}$$

Consider the following quadratic (characteristic function, or characteristic polynomial).

$$f(r) = r^2 + 5r + 6 = 0 \tag{8.10}$$

There are 2 different roots to the characteristic function,

$$r_1 = -3; \quad r_2 = -2 \tag{8.11}$$

Now consider the equations,

$$y_1(t) = e^{r_1 t} = e^{-2t} \tag{8.12}$$

$$y_2(t) = e^{r_1 t} = e^{-3t} \tag{8.13}$$

Then, $y_1(t)$ and $y_2(t)$ are solutions of **Equation 8.9**.
Proof:

$$y_1'(t) = -3e^{-3t}; \quad y_1'' = 9e^{-3t} \tag{8.14}$$

$$L[y_1] = 9e^{-3t} + 5(-3)e^{-3t} + 6e^{-3t} = 0 \tag{8.15}$$

$$0 = (9 - 15 + 6)e^{-3t} \tag{8.16}$$

Therefore, the general solution to **Equation 8.9**

$$y_c = C_1 e^{-3t} + C_2 e^{-2t} \tag{8.17}$$

where C_1 and C_2 are constants.

Example 8.3

$$L[y] = y'' + ay' + by = 0 \tag{8.18}$$
$$f(r) = r^2 + ar + b = 0 \tag{8.19}$$

Suppose that r_0 is a root of $f(r) = 0$
Consider

$$\begin{aligned} y_0(t) &= e^{r_0 t} && (8.20)\\ y_0'(t) &= r_0 e^{r_0 t} && (8.21)\\ y_0''(t) &= r_0^2 e^{r_0 t} && (8.22)\\ \implies L(y_0) &= r_0^2 e^{r_0 t} + a r_0 e^{r_0 t} + b e^{r_0 t} && (8.23)\\ &= e^{r_0 t}(r_0^2 + a r_0 + b) && (8.24) \end{aligned}$$

Things to consider, what if $r_0 \in \mathbb{C}$ or $r_0 = r_1$?

Example 8.4 CHSOLDE IVP

Find the solution of the CHSOLDE IVP,

$$L[y] = y'' + 5y' + 6y = 0; \quad y(0) = 2; \quad y'(0) = 3 \tag{8.25}$$

1. Find the general solution

$$y_c(t) = C_1 y_1 + C_2 y_2 \implies y_c(t) = C_1 e^{-3t} + C_2 e^{-2t} \tag{8.26}$$

2. Find the particular values of C_1 and C_2 such that $C_1 y_1(0) + C_2 y_2(0) = 2$ and $(C_1 y_1(0) + C_2 y_2(0))' = 3$.

$$\begin{cases} C_1 e^{-3(0)} + C_2 e^{-2(0)} = 2 \\ -3C_1 e^{-3(0)} + -2C_2 e^{-2(0)} = 3 \end{cases} \tag{8.27}$$

Solving the linear combination yields $C_1 = 7$, $C_2 = 9$. Then, the solution to this IVP is

$$y(t) = -7e^{-3t} + 9e^{-2t} \tag{8.28}$$

Lecture 9

11 February 2020

9.1 Second Order Linear Differential Equations

Theorem 9.1 Principle of Superposition

Suppose that y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.1}$$

Then, $C_1y_1 + C_2y_2$ is another solution for $L[y] = 0$ where C_1 and C_2 are constants. ($C_1y_1 + C_2y_2$ is the liear combination of y_1 and y_2)

Proof 9.1 Principle of Superposition

Show that

$$L[C_1y_1 + C_2y_2] = 0 \tag{9.2}$$

$$\begin{aligned}
&L[C_1y_1 + C_2y_2] && (9.3)\\
&= (C_1y_1 + C_2y_2)'' + p(t)(C_1y_1 + C_2y_2)' + q(t)(C_1y_1 + C_2y_2) && (9.4)\\
&= C_1y_1'' + C_2y_2'' + p(t)C_1y_1' + p(t)C_2y_2' + q(t)C_1y_1 + q(t)C_2y_2 && (9.5)\\
&= (C_1y_1'' + p(t)C_1y_1' + q(t)C_1y_1) + (C_2y_2'' + p(t)C_2y_2' + q(t)C_2y_2) && (9.6)\\
&= C_1L[y_1] + C_2L[y_2] = 0 && (9.7)
\end{aligned}$$

From the above, $C_1L[y_1] = 0$ and $C_2L[y_2] = 0$, therefore

$$L[C_1y_1 + C_2y_2] = 0 \tag{9.8}$$

Theorem 9.2 Existance and Uniqueness Theorem

Given

$$L[y] = y'' + p(t)y' + q(t)y = g(t); \quad y(t_0) = z_0; \quad y'(t_0) = z_1 \tag{9.9}$$

suppose $t_0 \in I$.
Then, this IVP has exactly 1 solution. Moreover, this solution will be defined throughout the interval.

Example 9.1 Application of Existance and Uniqueness Theorem

Find the longest interval in which the solution of the IVP is certain to exist.

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0; \quad y(1) = 2; \quad y(1) = 1 \tag{9.10}$$

The equation is equivalent to

$$L[y] = y'' + \frac{t}{t^2 - 3t}y' - \frac{t+3}{t^2 - 3t}y = 0 \tag{9.11}$$

1. $\forall\ t \in (-\infty, \infty), \lim_{a \to t}(g(a))$
2. $\forall\ t \in (-\infty, 0) \cup (0, 3) \cup (3, \infty), \lim_{a \to t}(q(a))$
3. $\forall\ t \in (-\infty, 3) \cup (3, \infty), \lim_{a \to t}(p(a))$

From the above,

$$I = (0, 3) \tag{9.12}$$

Example 9.2

Find the unique solution of the IVP given by

$$L[y] = y'' + p(t)y' + q(t)y = 0; \quad y(t_0) = 0; \quad y'(t_0) = 0 \tag{9.13}$$

where $p(t)$ and $q(t)$ are continuous for $t \in (-\infty, \infty)$.
The solution is

$$y(t) = 0 \tag{9.14}$$

and because of the uniqueness theorem, this is the only answer.

9.2 Linear Algebra with 2 Unknowns Detour

Definition 9.1

General form of a linear system with 2 Unknowns

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \tag{9.15}$$

where x and y are the two unknowns. The linear combination above can be rewritten as

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9.16}$$

Definition 9.2 Matricies

A $n \times m$ **matrix** is a $n \times m$ table filled with numbers or functions. They are written with parenthesis or brackets around the numbers, such as

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \tag{9.17}$$

When $n = m$, the matrix is considered to be a **square matrix**.

Definition 9.3 Determinant

An import concept involved with square matricies is the determinant, in the case of

$$\det(A) = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = ad - bc \tag{9.18}$$

Theorem 9.3

The solution to

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{9.19}$$

is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.20}$$

where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. No other solution exists.

9.3 Wronskian

Definition 9.4 Wronskian

For two differentiable functions $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$, the **Wronskian** of y_1 and y_2 is defined by

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{9.21}$$

9.4 Miscellaneous Definitions

Additional notes that were either not covered or were missed from previous lectures.

Definition 9.5 Differential Operator

Let p and q are continuous over the open interval I, where $t \in (\alpha, \beta)$, where $\alpha = -\infty$ or $\beta = \infty$ are included. Then for any function ϕ that is twice differentiable on I. The **differential operator** is defined by

$$L[\phi] = \phi'' + p\phi' + q\phi \tag{9.22}$$

Note that result of the operator is a function itself, so the value of $L[\phi]$ at point t is

$$L[\phi] = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \tag{9.23}$$

Lecture 10

13 February 2020

10.1 Applications of the Wronskian

Corollary 10.1

Recall in the last lecture, **Theorem 9.3**. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{10.1}$$

If $|A| = 0$, then for **some** values of c_1 and c_2 the linear system

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{10.2}$$

does not have a solution (inconsistent).

Theorem 10.1

Assume that y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.3}$$

where p and q are continuous, and t_0 is a fixed point.
Then, $\forall z_0 \wedge \forall z_1$, it is possible to find c_1 and c_2 such that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{10.4}$$

satisfies the IVP

$$L[y], \quad y(t_0) = z_0, \quad y'(t_0) = z_1 \tag{10.5}$$

if and only if

$$W[y_1, y_2](t_0) \neq 0 \tag{10.6}$$

Proof 10.1

Suppose that $\forall z_0, z_1 \implies \exists\, C_1, C_2$ such that

$$\begin{cases} C_1 y_1(t_0) + C_2 y_{2(t_0)} = z_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) = z_1 \end{cases} \tag{10.7}$$

then, $\exists!\, C_1, C_2$ iff $W[y_1, y_2](t_0) \neq 0$

Theorem 10.2

Suppose that y_1 and y_2 are solutions to

$$L[y] = 0 \tag{10.8}$$

Then, the family of solutions

$$y = C_1 y_1 + C_2 y_2 \tag{10.9}$$

includes all solutions of $L[y] = 0$ iff $\exists t_0 \implies W[y_1, y_2](t_0) \neq 0$

Example 10.1 Application of 10.2

The solutions to

$$y'' - 5y' + 6y = 0 \tag{10.10}$$

are

$$y_1 = e^{2t}, \quad y_2 = e^{3t} \tag{10.11}$$

$$y = C_1 e^{2t} + C_2 e^{3t} \tag{10.12}$$

Calculating the Wronskian

$$\begin{aligned} W[y_1, y_2] &= \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} \qquad (10.13) \\ &= e^{5t} \qquad (10.14) \end{aligned}$$

$$e^{5t} \neq 0 \tag{10.15}$$

Therefore, there does not exist other solutions to this CHSOLDE.

Theorem 10.3 Abel's Theorem

If y_1, y_2 are solutions to a SOLDE,

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.16}$$

where p, t are continuous over an open interval, I, then the Wronskian at point t is given by Abel's Formula,

$$W[y_1, y_2](t) = c \exp\left(- \int p(t)\, dt \right) \tag{10.17}$$

where c is some arbitrary constant dependent on y_1, y_2, but not on t.

$$\forall t \in I, W[y_1, y_2](t) \equiv 0 \iff c = 0 \tag{10.18}$$

$$\forall t \in I, W[y_1, y_2](t) \not\equiv 0 \iff c \neq 0 \tag{10.19}$$

Example 10.2

Suppose that
$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t} \tag{10.20}$$
are solutions of
$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.21}$$
Show that if $r_1 \neq r_2$, then $C_1 y_1 + C_2 y_2$ includes all solutions of $L[y] = 0$.

$$W[e^{r_1 t}, e^{r_2 t}] = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \tag{10.22}$$
$$= (r_2 - r_1)e^{(r_1 + r_2)t} \tag{10.23}$$
$$\neq 0 \tag{10.24}$$

Definition 10.1 Fundamental Set of Solutions

If y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.25}$$

such that $C_1 y_1 + C_2 y_2$ includes all possible solutions of $L[y] = 0$, then y_1 and y_2 form a **fundamental set of solutions** (FSS).
Alternatively, if and only if

$$W[y_1, y_2] \neq 0 \tag{10.26}$$

then there exists fundamental set containing y_1 and y_2.

Example 10.3

Show that $y_1(t) = t^{\frac{1}{2}}$, $y_2(t) = t^{-1}$ form a FSS of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0 \tag{10.27}$$

1. Ensure they are solutions of $L[y] = 0$
   i.e. $L[t^{\frac{1}{2}}] = 0$, $L[t^{-1}] = 0$

$$L[t^{\frac{1}{2}}] = 2t^2(-\frac{1}{4})t^{-\frac{3}{4}} + 3t(\frac{1}{2})t^{-\frac{1}{2}} + t^{\frac{1}{2}} \tag{10.28}$$
$$= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \tag{10.29}$$
$$\equiv 0 \tag{10.30}$$

Example 10.3

$$L[t^{-1}] = (2t^2)(2t^{-3}) + 3t(-1)t^{-2} - t^{-1} \tag{10.31}$$
$$= 4t^{-1} - 3t^{-1} - t^{-1} \tag{10.32}$$
$$\equiv 0 \tag{10.33}$$

2. Ensure that the Wronskian is not constantly equal to 0

$$W[t^{\frac{1}{2}}, t^{-1}] = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-1} \end{vmatrix} \tag{10.34}$$
$$= -t^{-\frac{1}{2}} - -\frac{1}{2}t^{-\frac{3}{2}} \tag{10.35}$$
$$\not\equiv 0 \tag{10.36}$$

Then,

$$L[y] = y'' + ay' + by = 0; \quad f(r) = r^2 + ar + b = 0 \tag{10.37}$$

has only one solution of degree 2.

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}, \quad r_1 = r_2 \iff \sqrt{a^2 - 4b} \equiv 0 \tag{10.38}$$

Example 10.4

1. If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt} \tag{10.39}$$

When $r = r_1$, $L[e^{rt}] = 0$, therefore e^{rt} is a solution to

$$L[y] = ay'' + by' + cy = 0 \tag{10.40}$$

2. Then,

$$\frac{\partial}{\partial r} L[e^{rt}] = L[\frac{\partial}{\partial r} e^{rt}] = L[te^{rt}] \tag{10.41}$$
$$= ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1) \tag{10.42}$$

Because $r = r_1 \implies L[te^{rt}] = 0$, te^{rt} is another solution to $L[y] = 0$. Show that e^{rt}, te^{rt} form a FSS.

a. $L[e^{rt}] = 0$

Example 10.4

b. $L[te^{rt}] = 0$

$$\begin{aligned} (te^{rt})' &= e^{rt} + rte^{rt} && (10.43) \\ (te^{rt})'' &= 2re^{rt} + r^2te^{rt} && (10.44) \\ L[te^{rt}] &= (2re^{rt} + r^2te^{rt}) + a(e^{rt} + rte^{rt}) + b && (10.45) \\ &= e^{rt}(2r + a) + te^{rt}(r^2 + ar + b) && (10.46) \end{aligned}$$

Lecture 11

18 February 2020

11.1 Test Corrections

No points were missed, however, to clear a misunderstanding:

7. Show that
$$y_1 = t^{-1}; \quad y_2 = t^{1/2} \tag{11.1}$$
form a fundamental set of solutions for
$$L[y] = 2ty'' + 3y' - \frac{1}{t}y = 0; t > 0 \tag{11.2}$$
Wrong:
$$L[c_1 t^{-1} + c_2 t^{1/2}] = 0 \implies c_1 L[t^{-1}] + c_2 L[t^{1/2}] = 0 \tag{11.3}$$
Correct:
$$L[t^{-1}] = 0; \quad L[t^{1/2}] = 0 \tag{11.4}$$
Then, prove that
$$W[t^{-1}, t^{1/2}] \neq 0 \tag{11.5}$$

The wrong assumption proved the principle of superposition (the linear combination of y_1, y_2 are also solutions to $L[y]$), however, it does not prove that y_1, y_2 are indeed solutions to $L[y]$. To correctly do the problem, one must compute the linear operator on y_1 and y_2, and ensure that they equal 0.

Lecture 12

20 February 2020

12.1 Imaginary Roots

Now consider a characteristic function,

$$f(r) = r^2 + ar + b = 0 \tag{12.1}$$

where $r \in \mathbb{C}$. The roots of $f(r) = 0$ can then be expressed by

$$r_{1,2} = \lambda \pm i\tau \tag{12.2}$$

Then, what is the general solution of

$$L[y] = y'' + ay' + by = 0 \tag{12.3}$$

where the characteristic function yields non-real roots?

Theorem 12.1 Imaginary Roots

In the case that the solutions to $f(r) = 0$ are

$$r_{1,2} = \lambda \pm i\tau \tag{12.4}$$

the general solution of the corresponding $L[y] = 0$ is given by

$$y_1 = e^{\lambda t}\cos(\tau t); \quad y_2 = e^{\lambda t}\sin(\tau t) \tag{12.5}$$

The solutions y_1 and y_2 are a fundamental set of solutions.

Homework 12.1

Show that $y_1 = e^{\lambda t}\cos(\tau t)$; $y_2 = e^{\lambda t}\sin(\tau t)$ are solutions to $L[y] = 0$, with a characteristic function that has imaginary roots.

Proof 12.1 Fundamental Set

In **Theorem 12.1**, $y_1 = e^{\lambda t}\cos(\tau t); \quad y_2 = e^{\lambda t}\sin(\tau t)$

$$
\begin{aligned}
&W[y_1, y_2] && (12.6)\\
&= \begin{vmatrix} e^{\lambda t}\cos(\tau t) & e^{\lambda t}\sin(\tau t) \\ \lambda e^{\lambda t}\cos(\tau t) - \tau e^{\lambda t}\sin(\tau t) & \lambda e^{\lambda t}\sin(\tau t) + \tau e^{\lambda t}\cos(\tau t) \end{vmatrix} && (12.7)\\
&= e^{2\lambda t}\begin{vmatrix} \cos(\tau t) & \sin(\tau t) \\ \lambda\cos(\tau t) - \tau\sin(\tau t) & \lambda\sin(\tau t) + \tau\cos(\tau t) \end{vmatrix} && (12.8)\\
&= e^{2\lambda t}[\lambda\cos(\tau t)\sin(\tau t) + \tau\cos^2(\tau t) - \lambda\sin(\tau t)\cos(\tau t) + \tau\sin^2(\tau t)] && (12.9)\\
&= \tau e^{2\lambda t} \not\equiv 0 && (12.10)
\end{aligned}
$$

Therefore, $y_{1,2}$ form a fundamental set of solutions.

Example 12.1

Solve

$$L[y] = y'' - 2y' + 6y = 0 \tag{12.11}$$

The characteristic function is given by

$$f(r) = r^2 - 2r + 6 = 0 \tag{12.12}$$

The roots are

$$r_{1,2} = \frac{2 \pm \sqrt{-20}}{2} = 1 \pm i\sqrt{5} \tag{12.13}$$

Therefore,

$$y_1 = e^t \cos\left(\sqrt{5}t\right); \quad y_2 = e^t \sin(\sqrt{5}t) \tag{12.14}$$

Then, the general solution can be given as

$$y_c = c_1 e^t \cos(\sqrt{5}t) + c_2 e^t \sin(\sqrt{5}t) \tag{12.15}$$

Example 12.2

Solve

$$L[y] = 9y'' + 6y' + y = 0 \tag{12.16}$$

The characteristic function is given by

$$f(r) = r^2 + 6y' + y = 0 \tag{12.17}$$

The roots are

$$(3r + 1)^2 \tag{12.18}$$

$$r_1 = r_2 = -\frac{1}{3} \tag{12.19}$$

This implies

$$y_1 = \exp(-\frac{1}{3}t); \quad y_2 = t\exp(-\frac{1}{3}t) \tag{12.20}$$

Then, the general solution can be given as

$$y_c = c_1 \exp(-\frac{1}{3}t) + c_2 t \exp(-\frac{1}{3}t) \tag{12.21}$$

Lecture 13

25 February 2020

Note: Lectures from 25 February to 3 March, 2020 are not presented by Dr. Darbinyan

13.1 Nonhomogeneous SOLDEs

Definition 13.1 Nonhomogeneous SOLDE

The general form of a Nonhomogeneous SOLDE (NSOLDE) is given by

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.1}$$

Theorem 13.1

Suppose that y_1, y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.2}$$

then,

$$y_1 - y_2 = 0 \tag{13.3}$$

Proof 13.1

$$L[y_1] = g \tag{13.4}$$

$$L[y_2] = g \tag{13.5}$$

Then,

$$L[y_1] - L[y_2] = L[y_1, y_2] = g - g = 0 \tag{13.6}$$

Theorem 13.2 General Solutions to NSOLDEs

Given the following NSOLDE

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.7}$$

and its corresponding HSOLDE,

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{13.8}$$

The general solution involves a particular solution of **Equation 13.7** consists of a particular solution and a solution of the corresponding HSODLE,

$$y = \phi(t) = C_1 y_1 + C_2 y_2 + Y \tag{13.9}$$

where y_1, y_2 are solutions to **Equation 13.8**, and Y is a particular solution of **Equation 13.7**

Proof 13.2

From the **Theorem 13.1**, we get

$$\phi - Y = c_1 y_1 + c_2 y_2 \tag{13.10}$$

this is the same as **Equation 13.9**.

From **Theorem 13.2**, solving NSOLDEs involves

1. Finding the general solution of the corresponding HSOLDE. (complementary solution; y_c)
2. Find any particular solution, Y, to the NSOLDE.
3. Then, the solution to the NSOLDE is the sum of the particular and general solutions.

13.2 Method of Undetermined Coefficients

Example 13.1

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} \tag{13.11}$$

The corresponding HSOLDE is given by

$$y'' - 3y' - 4y = 0 \tag{13.12}$$

It is possible to "guess" the particular solution to a NSOLDE based on the form of $g(t)$. In this case, a reasonable guess would be

$$Y = Ae^{2t} \tag{13.13}$$

Substituting Y into the original equation yields

$$L[Ae^{2t}] = (Ae^{2t})'' - 3(Ae^{2t})' - 4(Ae^{2t}) = 3e^{2t} \tag{13.14}$$

Solving for the undetermined coefficient,

$$(Ae^{2t})' = (2Ae^{2t}), (Ae^{2t})'' = (4Ae^{2t}) \tag{13.15}$$

$$4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) = 3e^{2t} \tag{13.16}$$

$$4A - 6A - 4A = 3 \tag{13.17}$$

$$-6A = 3 \tag{13.18}$$

$$A = -\frac{1}{2} \tag{13.19}$$

Therefore,

$$Y = -\frac{1}{2}e^{2t} \tag{13.20}$$

is a particular solution to this NSOLDE.

Example 13.2

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin(t) \tag{13.21}$$

Guess,

$$\begin{align} Y &= A\sin(t) \tag{13.22}\\ \implies 2\sin(t) &= -A\sin(t) - 3A\cos(t) - 6A\sin(t) \tag{13.23}\end{align}$$

As can be seen above, the guess $A\sin(t)$ is incorrect due to the appearance of the $\cos(t)$ term, creating an open subspace. Creating a closed subspace yields

$$Y = A\cos(t) + B\sin(t) \tag{13.24}$$

Then,

$$\begin{align} Y' &= -A\sin(t) + B\cos(t) \tag{13.25}\\ Y'' &= -A\cos(t) + B\sin(t) \tag{13.26}\\ L[Y] &= (-A\cos(t) + B\sin(t)) - 3(-A\sin(t) + B\cos(t)) \tag{13.27}\\ &\quad - 4(A\cos(t) + B\sin(t)) = 2\sin(t) \tag{13.28}\\ 2\sin(t) &= \cos(t)(-A - 3B - 4A) + \sin(t)(-B + 3A - 4B) \tag{13.29}\end{align}$$

$$\begin{cases} -A - 3B - 4A &= 0 \\ -B + 3A - 4B &= 2 \end{cases} \tag{13.30}$$

By solving the linear combination,

$$A = \frac{3}{17}, \quad B = -\frac{5}{17} \tag{13.31}$$

Finally, a particular solution given by this method is

$$y = \frac{3}{17}\cos(t) = \frac{5}{7}\sin(t) \tag{13.32}$$

Example 13.3

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos(2t) \tag{13.33}$$

A guess at a particular solution would be

$$Y = Ae^t \cos(2t) + Be^t \sin(2t) \tag{13.34}$$

Then, substitution

$$Y' = Ae^t \cos(2t) - 2Ae^t \sin(2t) + Be^t \sin(2t) + 2Be^t \cos(2t) \tag{13.35}$$

$$Y'' = (-3A + 4B)e^t \cos(2t) - (4A + 3B) \sin(2t) \tag{13.36}$$

$$\begin{cases} 10A + 2B & = 8 \\ 2A - 10B & = 0 \end{cases} \tag{13.37}$$

Solving the linear combination yields

$$A = \frac{10}{13}; \quad B = \frac{2}{13} \tag{13.38}$$

Finally, a particular solution given by this method is

$$y = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t) \tag{13.39}$$

Example 13.4

Find a particular solution of

$$L[y] = y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) \tag{13.40}$$

It is possible to separate this NSOLDE into two,

$$L[y] = 3e^{2t}; L[y] = 2\sin(t) \tag{13.41}$$

And a particular solution to **Equation 13.40** is the linear combination of the solutions to the two NSOLDEs in **13.41**. Having solved the two NSOLDEs previously,

$$Y_1 = -\frac{1}{2}e^{2t} \tag{13.42}$$

$$Y_2 = -\frac{5}{17}\cos(t) + \frac{3}{17}\sin(t) \tag{13.43}$$

Then, a particular solution is given by

$$y = -\frac{1}{2}e^{2t} - \frac{5}{17}\cos(t) + \frac{3}{17}\sin(t) \tag{13.44}$$

Example 13.5

Find a particular solution of

$$L[y] = y'' - 3y' - 4y = 2e^{2t} \tag{13.45}$$

if

$$Y = Ae^{-t}; \quad Y' = -Ae^{-t}; \quad Y'' = Ae^{-t} \tag{13.46}$$

then,

$$L[Y] = Ae^{-t} - 3(-Ae^{-t}) - 4(Ae^{-t}) = 2e^{-t} \tag{13.47}$$

When solving for A, we get $0 = 2$, which is false, therefore Ae^{-t} is not a form of a particular solution. Then,

$$Y = Ate^{-t}; \quad L[Y] = 2e^{-t} \tag{13.48}$$

$$L[Y] = (Ate^{-t} - 2Ae^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} \tag{13.49}$$

$$-5Ae^{t} = 2e^{-t} \tag{13.50}$$

$$A = -\frac{2}{5} \tag{13.51}$$

$$y = -\frac{2}{5}te^{-t} \tag{13.52}$$

In the previous example, the corresponding homogeneous equation is

$$L[y] = y'' - 3y' - 4y = 0 \tag{13.53}$$

And the solutions to this equation are

$$y_1 = e^{-t}; \quad y_2 = e^{4t} \tag{13.54}$$

As it can be seen, the guess of $Y = Ae^{-t} = Ay_1$,

$$L[Ae^{-t}] = L[Ay_1] = AL[y_1] = 0 \tag{13.55}$$

Therefore,

$$L[Y] \neq 2e^{-t} \tag{13.56}$$

Lecture 14

27 February 2020

14.1 Method of Undetermined Coefficients (cont)

From the last lecture, we have considered

$$L[y] = y'' - 3y' + 4y = g(t) \tag{14.1}$$

with the following cases for $g(t)$

1. $g =$ polynomial
2. $g = e^{\alpha t}$
3. $g = \sin(\alpha t)$
4. $g = e^{\alpha t} \sin(\beta t)$
5. $g = \alpha e^{-t}$

Example 14.1

Given

$$L[y] = y'' - 2y' + y = e^t \tag{14.2}$$

Possible guesses following the previously considered equation and cases,

$$Y = \begin{cases} Ae^t \\ Ate^t \end{cases} \tag{14.3}$$

However, neither one of these cases work, because the corresponding homogeneous equation has solutions $y_1 = e^t, y_2 = te^t$. Therefore, another guess would be At^2e^t.

$$L[At^2e^t] \implies A = \frac{1}{2} \tag{14.4}$$

A general rule of thumb is to increase the order of t until $L[Y] \neq 0$.

14.2 Variation of Parameters

What if a NSOLDE is given by

$$L[Y] = \frac{P(t)}{e^{\alpha t}} \tag{14.5}$$

or some other complicated function?

Example 14.2

Find the general solution of

$$L[y] = y'' + 4y = 8\tan(t); \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{14.6}$$

Solution:

$$L[Y] = 0 \implies f(r) = r^2 + 4 = 0 \implies r_{1,2} = \pm 2i \tag{14.7}$$

Then, a general solution to the CHSOLDE

$$y = c_1\cos(2t) + c_2\sin(2t) \tag{14.8}$$

If the constants $c_{1,2}$ are replaced by some functions $u_{1,2}$ then,

$$y = u_1(t)\cos(2t) + u_2(t)\sin(2t) \tag{14.9}$$

is a solution of the original NSOLDE.

$$y' = -2u_1\sin(2t) + 2u_2\cos(2t) + u_1'\cos(2t) + u_2'\sin(2t) \tag{14.10}$$

Then, choosing a restriction and applying it yields

$$u_1'\cos(2t) + u_2'\sin(2t) = 0 \tag{14.11}$$

$$y' = -2u_1\sin(2t) + 2u_2\cos(2t) \tag{14.12}$$

$$y'' = -4u_1\cos(2t) - 4u_2\sin(2t) - 2u_1'\sin(2t) + 2u_2'\cos(2t) \tag{14.13}$$

Substituting the values into the NSOLDE,

$$\begin{aligned} y'' + 4y = &- 4u_1\cos(2t) - 4u_2\sin(2t) - 2u_1'\sin(2t) + 2u_2'\cos(2t) \\ &+ 4u_1\cos(2t) + 4u_2\sin(2t) = 8\tan(t) \end{aligned} \tag{14.14}$$

Solving for u_2'

$$u_2' = -u_1'\cot(2t) \tag{14.15}$$

Solving for u_1'

$$u_1' = -\frac{8\tan(t)\sin(2t)}{2} = -8\sin^2 t \tag{14.16}$$

Example 14.2

Plugging in $u_{1,2}$

$$u_2' = 4\frac{\sin(t)(2\cos^2(t)-1)}{\cos(t)} = 4\sin(t)\left(2\cos(t) - \frac{1}{\cos(t)}\right) \tag{14.17}$$

Then,

$$u_1 = 4\sin(t)\cos(t) - 4t + c_1 \tag{14.18}$$

$$u_2 = 4\ln(\cos(t)) - 4\cos^2 t + c_2 \tag{14.19}$$

Finally, a solution can be written as

$$y = -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos(2t))\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \tag{14.20}$$

Theorem 14.1

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{14.21}$$

If p, q, g are continuous on the open interval I; and the solutions to the corresponding HSOLDE satisfy $W[y_1, y_2] \not\equiv 0$. Then, a particular solution is given by

$$Y = y_2\int_{t_0}^{t}\frac{y_1(s)g(s)}{W[y_1,y_2](s)}\,ds - y_1\int_{t_0}^{t}\frac{y_2(s)g(s)}{W[y_1,y_2](s)}\,ds \tag{14.22}$$

$\forall t_0 \in I$ the general solution is

$$y = c_1y_1 + c_2y_2 + Y \tag{14.23}$$

Proof 14.1

Given

$$L[y] = y'' + py' + qy = g \tag{14.24}$$

Where p, q, g are continuous functions, in the case that $g = 0$, gives us a HSOLDE which has a general solution

$$y_c = c_1y_1 + c_2y_2 \tag{14.25}$$

(Keep in mind that only CHSOLDEs, or cases where p, q are constants have been discussed.) Substituting $c_{1,2}$ for $u_{1,2}$

$$y = u_1y_1 + u_2y_2 \tag{14.26}$$

Proof 14.1

To ensure that the solution is one of the NSOLDE and not the HSOLDE, we take the derivative

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \tag{14.27}$$

Setting a restriction

$$u_1'y_1 + u_2'y_2 = 0 \tag{14.28}$$

Then, y' can be simplified and be differentiated once more,

$$y' = u_1y_1' + u_2y_2' \tag{14.29}$$

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \tag{14.30}$$

Then substituting for y, y', y'' yields

$$\begin{aligned} g = {} & u_1\left(y_1'' + py_1' + qy_1\right) \\ & + u_2\left(y_2'' + py_2' + qy_2\right) \\ & + u_1'y_1' + u_2'y_2' \end{aligned} \tag{14.31}$$

In the above equation, the lines with coefficients $u_{1,2}$ are equal to zero, as $y_{1,2}$ are solutions to the HSOLDE, giving

$$u_1'y_1' + u_2'y_2' = g \tag{14.32}$$

Then, solving the system gives

$$u_1' = -\frac{y_2 g}{W[y_1, y_2]}; \quad u_2' = \frac{y_1 g}{W[y_1, y_2]} \tag{14.33}$$

Note that because $y_{1,2}$ form a FSS, $W[y_1, y_2] \not\equiv 0$. Now to solve for $u_{1,2}$

$$u_1 = -\int \frac{y_2 g}{W[y_1, y_2]}\, dt + c_1; \quad u_2 = \int \frac{y_1 g}{W[y_1, y_2]}\, dt + c_2 \tag{14.34}$$

Lecture 15

3 March 2020

15.1 Indefinite Integrals

Definition 15.1 Improper Integrals

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt \tag{15.1}$$

If the limit exists then it is convergent, otherwise it is considered to be divergent

Example 15.1

Does the following integral converge?

$$\int_1^\infty t^{-1}\,dt \tag{15.2}$$

Solution:

$$\lim_{A\to\infty}\int_1^A (t^{-1})\,dt \tag{15.3}$$

$$\lim_{A\to\infty}\left(\ln(t)|_1^A\right) = \infty \tag{15.4}$$

Therefore, it is divergent.

Example 15.2

For what values does the following integral converge?

$$\int_0^\infty e^{ct}\,dt \tag{15.5}$$

$$= \lim_{A\to\infty}\int_0^A e^{ct}\,dt \tag{15.6}$$

$$= \lim_{A\to\infty} \frac{1}{c}e^{ct}\bigg|_0^A \tag{15.7}$$

$$= \lim_{A\to\infty}\left[\frac{1}{c}e^{ct} - \frac{1}{c}\right] \tag{15.8}$$

$$= \frac{1}{c}, c < 0 \tag{15.9}$$

Example 15.3

For what values of p does the integral converge?

$$\int_1^\infty t^{-p}\,dt \tag{15.10}$$
$$= \lim_{A\to\infty} \int_1^A t^{-p}\,dt \tag{15.11}$$
$$= \lim_{A\to\infty} \frac{1}{1-p} t^{1-p}\bigg|_1^A \tag{15.12}$$
$$= \lim_{A\to\infty} \frac{1}{1-p}\left(A^{1-p}-1\right), p \neq -1 \tag{15.13}$$
$$= \frac{1}{p-1}, p > 1 \tag{15.14}$$

15.2 Laplace Transformation

In algebra, we were introduced to the concept of factorization,

$$x^2 + 4x + 3 = 0 \tag{15.15}$$
$$(x+3)(x+1) = 0 \tag{15.16}$$
$$\implies x \in \{-2, -1\} \tag{15.17}$$

Factorization is useful because it was a tool to solve for the roots of a polynomial. The Laplace tranformation can also be thought of as a tool that would help in solving ODEs.

Definition 15.2

The **Laplace Transformation** is an integral transformation, given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{15.18}$$

Theorem 15.1

Suppose that

1. f is piecewise continuous on the intervals $t \in [0, A], A \in \mathbb{R}^+$
2. $\exists (k, a, M), (K, M) > 0, |f(t)| \leq ke^{at}, t \geq M$

Then, $\forall s > a, F(s)$

Example 15.4

Find

$$\mathcal{L}\{1\} = \frac{1}{s^2} \tag{15.19}$$

5 - 19 March 2020

Lectures on dates 5, 17, 19 March were cancelled.

Lecture 16

24 March 2020

16.1 Laplacians

$$\mathcal{L} \circ f(t) \longrightarrow F(s), f : \mathbb{R} \to \mathbb{R} \tag{16.1}$$

Some uses for Laplacians include stochastic processes and probability.

Note: As the inverse Laplace transform ($\mathcal{L}^{-1}$) will not be introduced in this lecture, all steps involving $\mathcal{L}\{y\} \to y$ will be skipped in the examples in this lecture.

As defined previously,

$$\begin{array}{ccc} \mathcal{L}\{f(t)\} & = & \displaystyle\int_0^\infty e^{-st} f(t)\, dt \\ \| & & \| \\ F(s) & = & \displaystyle\lim_{x\to\infty} \int_0^x e^{-st} f(t)\, dt \end{array}$$

Example 16.1 Basic Examples

$$f \equiv 0 \tag{16.2}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty (e^{-st})(0)\, dt \tag{16.3}$$

$$= 0 \tag{16.4}$$

Example 16.1 Basic Examples

$$
\begin{aligned}
f &\equiv c && (16.5)\\
\mathcal{L}\{f(t)\} &= \int_0^\infty ce^{-st}\,dt && (16.6)\\
&= \lim_{x\to\infty} \int_0^x ce^{-st}\,dt && (16.7)\\
&= \lim_{x\to\infty} -\frac{c}{s}e^{-st}|_0^x && (16.8)\\
&= \lim_{x\to\infty} \left(\frac{1}{s}e^{-st}\right) && (16.9)\\
&= cs^{-1} && (16.10)
\end{aligned}
$$

Theorem 16.1

The Laplacian of a differential:

$$
\begin{aligned}
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) && (16.11)\\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) && (16.12)
\end{aligned}
$$

Example 16.2

Find the solution of the following using Laplacians.

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0 \qquad (16.13)$$

Solution:

$$
\begin{aligned}
\mathcal{L}\{y'' - y' - 2y\} &= \mathcal{L}\{0\} && (16.14)\\
\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} &= 0 && (16.15)\\
(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - (s\mathcal{L}\{y\} - y(0)) - 2\mathcal{L}\{y\} &= 0 && (16.16)
\end{aligned}
$$

$$
\begin{aligned}
(s^2 - s - 2)\mathcal{L}\{y\} - s + 1 &= 0 && (16.17)\\
\mathcal{L}\{y\} &= \frac{s-1}{s^2 - s - 2} && (16.18)\\
&= \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} && (16.19)\\
\implies y &= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} && (16.20)
\end{aligned}
$$

Example 16.3

Solve using Laplacians.

$$y'' + y = \sin(2t), y(0) = 2, y'(0) = 1 \tag{16.21}$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\} \tag{16.22}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{2}{s^2+4} \tag{16.23}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1}\left(\frac{1}{s^2+4} + 2s + 1\right) \tag{16.24}$$

$$= \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \tag{16.25}$$

The full solution would involve the inverse Laplacian of the last equation (**16.25**).

Lecture 17

26 March 2020

17.1 Inverse Laplace Transform

Definition 17.1

If

$$F(s) = \mathcal{L}\{f(t)\} \tag{17.1}$$

then, the **inverse Laplace transform** is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{17.2}$$

Addendum to the previous lecture, when solving for:

$$y'' + y = \sin(2t), y(0) = 2, y'(0) = 1 \tag{17.3}$$

We came to the point,

$$\mathcal{L}\{y\} = \frac{2s+1}{s^2+1} \cdot \frac{2}{(s^2+1)(s^2+4)} \tag{17.4}$$

So then,

$$y = \mathcal{L}^{-1}\{\frac{2s+1}{s^2+1} \cdot \frac{2}{(s^2+1)(s^2+4)}\} \tag{17.5}$$

17.1.1 Step Functions

For every function $f : \mathbb{R} \to \mathbb{R}$, there exists a graph. Most graphs we are accustomed to seeing are continuous. However, there exists step functions where there are a finite number of steps.

Definition 17.2

The unit step function can be defined as follows:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \tag{17.6}$$

Example 17.1

Consider

$$f(t) = \begin{cases} 2 & t \in [0, 4) \\ 5 & t \in [4, 7) \\ -1 & t \in [7, 9) \\ 1 & t \geq 9 \end{cases} \tag{17.7}$$

It can be written as the linear combination of multiple unit step functions:

$$f(t) = 2u_0(t) + 3u_4(t) - 6u_7(t) + 2u_9(t) \tag{17.8}$$

Theorem 17.1

$$\mathcal{L}\{u_c(t) \cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}, t > c \tag{17.9}$$

Corollary 17.1

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c) \tag{17.10}$$

Example 17.2

Compute the following

$$\mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s}\} \tag{17.11}$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \tag{17.12}$$

$$\mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s}\} = u_c(t) \cdot f(t-2) \tag{17.13}$$

Theorem 17.2

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c) \tag{17.14}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{ct}\mathcal{L}^{-1}\{F(s+c)\} \tag{17.15}$$

Also, in the last lecture

$$\mathcal{L}\{y\} = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \tag{17.16}$$

Notice that in both fractions, it can be expressed as a linear combination of the following function

$$F(s) = \frac{1}{s} \tag{17.17}$$

rewriting the expression yields:

$$\mathcal{L}\{y\} = \frac{1}{3}F(s-2) + \frac{2}{3}F(s+1) \tag{17.18}$$

Now solving for y

$$y = \mathcal{L}^{-1}\left\{\frac{1}{3}F(s-2) + \frac{2}{3}F(s+1)\right\} \tag{17.19}$$

The inverse Laplacian can also be written as a linear combination

$$y = \mathcal{L}^{-1}\left\{\frac{1}{3}F(s-2)\right\} + \mathcal{L}^{-1}\left\{\frac{2}{3}F(s+1)\right\} \tag{17.20}$$

Remember to **Theorem 17.2**, we can adjust $F(s+c)$ back to $F(s)$:

$$y = \frac{1}{3}e^{2t}\mathcal{L}^{-1}\{F(s+2)\} + \frac{2}{3}e^{t}\mathcal{L}^{-1}\{F(s+1)\} \tag{17.21}$$

Now, notice that $\mathcal{L}^{-1}\{F(s)\} = 1$, and rearranging **Theorem 17.2**, we find that $\mathcal{L}^{-1}\{F(s+c)\} = e^{-ct}\mathcal{L}^{-1}\{F(s)\}$. If we apply this to our solution, we get the final solution:

$$y = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-t} \tag{17.22}$$

Lecture 18

31 March 2020

18.1 Examples Involving Inverse Laplacians

For examples 1-4, find the inverse Laplacian of the given.

Example 18.1

$$F(s) = \frac{3}{s^2+4} \tag{18.1}$$

$$\mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = \sin(at) \tag{18.2}$$

$$F(s) = \frac{3}{2}\cdot\frac{2}{s^2+2^2} \tag{18.3}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{3}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+2^2}\} \tag{18.4}$$

$$f(t) = \frac{3}{2}\sin(2t) \tag{18.5}$$

Example 18.2

$$F(s) = \frac{2}{s^2-3s-4} \tag{18.6}$$

First, complete the square

$$F(s) = \frac{2}{\left(s^2-\frac{3}{2}\right)-4-\frac{9}{4}} \tag{18.7}$$

$$= \frac{2}{\left(s+\frac{3}{2}\right)^2-\frac{25}{4}} \tag{18.8}$$

Example 18.2

$$\mathcal{L}^{-1}\{F(s)\} = e^{-\frac{3}{2}t}\mathcal{L}^{-1}\{\underbrace{\frac{2}{s^2 - \frac{25}{4}}}_{E(s)}\} \tag{18.9}$$

We obtained the above from **Theorem 17.1**:

$$\mathcal{L}^{-1}\{E(s+c)\} = e^{-ct}\mathcal{L}^{-1}\{E(s)\} \tag{18.10}$$

$$f(t) = e^{-\frac{3}{2}t} \cdot 2 \cdot \frac{2}{5} \cdot \mathcal{L}^{-1}\left\{\frac{\frac{5}{2}}{s^2 - \left(\frac{5}{2}\right)^2}\right\} \tag{18.11}$$

Then, by using the table, we see that

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2 - a^2}\right\} = \sinh(at) \tag{18.12}$$

Therefore, the inverse Laplacian

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{4}{5}e^{-\frac{3}{2}t}\sinh(\frac{5}{2}t) \tag{18.13}$$

Example 18.3

$$F(s) = \frac{8s^2 - 4s + 12}{s(s^2+4)} \tag{18.14}$$

In order to solve this problem, the method of partial fractions is required

$$\begin{aligned} F(s) &= \frac{8s^2 - 4s + 12}{s(s^2+4)} && (18.15)\\ &= \frac{A}{s} + \frac{Bs + C}{s^2+4} && (18.16)\\ &= \frac{(A+B)s^2 + Cs + 4A}{s(s^2+4)} && (18.17)\\ \implies & A = 3, B = 5, C = -4 && (18.18)\\ F(s) &= \frac{3}{s} + \frac{5s - 4}{s^2+4} && (18.19) \end{aligned}$$

Example 18.3

Then, the inverse Laplacian

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{5s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{-4}{s^2+4}\right\} \tag{18.20}$$

$$= 3 + \frac{5}{2}\cos(2t) - 2\sin(2t) \tag{18.21}$$

Example 18.4

$$F(s) = \frac{1-2s}{s^2+4s+1} \tag{18.22}$$

Employing both completing the square, and partial fractions

$$F(s) = \frac{1-2s}{s^2+4s+1} \tag{18.23}$$

$$= \frac{1-2s}{s^2+1} \tag{18.24}$$

$$= \frac{1}{(s+2)^2+1} - \frac{2s}{(s+2)^2+1} \tag{18.25}$$

Then, the inverse Laplacian

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2s}{(s+2)^2+1}\right\} \tag{18.26}$$

$$= e^{-2t}\sin t \qquad -2e^{-2t}\left(\cos t - 2\sin t\right) \tag{18.27}$$

18.2 Using Laplacians to solve ODEs

For the following examples, solve the SOLDE using Laplacians

Example 18.5

$$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1 \tag{18.28}$$

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\} \tag{18.29}$$

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{y''\} - 2\mathcal{L}^{-1}\{y'\} + 2\mathcal{L}^{-1}\{y\} \tag{18.30}$$

$$= (s^2\mathcal{L}^{-1}\{y\} - sy(0) - y'(0)) \tag{18.31}$$

$$- 2s\mathcal{L}^{-1}\{y\} - 2y(0)) + 2\mathcal{L}^{-1}\{y\}$$

$$= (s^2 - 2s + 2)\mathcal{L}^{-1}\{y\} - 1 \tag{18.32}$$

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \tag{18.33}$$

Then,

$$(s^2 - 2s + 2)\mathcal{L}\{y\} - 1 = \frac{1}{s+1} \tag{18.34}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2} + \frac{1}{(s+1)(s^2 - 2s + 2)} \tag{18.35}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2 - 2s + 2)}\right\} \tag{18.36}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1}\right\} \tag{18.37}$$

$$= e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} \tag{18.38}$$

$$= e^t \sin(t) \tag{18.39}$$

Example 18.5

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2-2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2-2s+2}\right\} \tag{18.40}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}s+\frac{3}{5}}{s^2-2s+2}\right\} \tag{18.41}$$

$$= \frac{1}{5}e^{-t} + \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}(s-1)+\frac{1}{5}+\frac{3}{5}}{(s-1)^2+1}\right\} \tag{18.42}$$

$$= \frac{1}{5}e^{-t} + e^t\mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}s-\frac{1}{5}+\frac{3}{5}}{s^2+1}\right\} \tag{18.43}$$

$$= \frac{1}{5}e^{-t} + e^t\left(-\frac{1}{5}\cos(t)+\frac{2}{5}\sin(t)\right) \tag{18.44}$$

Therefore, the solution is

$$y = e^t\sin(t) + \frac{1}{5}e^{-t} + e^t\left(-\frac{1}{5}\cos(t)+\frac{2}{5}\sin(t)\right) \tag{18.45}$$

Example 18.6

$$y'' + 4y = \begin{cases} t & t \in [0,1) \\ 2-t & t \in [1,2) \\ 0 & t \in [2,\infty) \end{cases} \tag{18.46}$$

Let $g(t)$ represent the piecewise function on the right hand side.

$$g(t) = \begin{cases} f_1(t) & t \in [0,1) \\ f_2(t)-t & t \in [1,2) \\ f_3(t) & t \in [2,\infty) \end{cases} \tag{18.47}$$

$$= u_0(t)f_1(t) + u_1(t)\left(f_2(t)-f_1(t)\right) + u_2(t)\left(f_3(t)-f_2(t)\right) \tag{18.48}$$

$$= u_0(t)\cdot t + u_1(t)(2-2t) + u_2(t)(t-2) \tag{18.49}$$

Example 18.6

Then,

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_0(t)\cdot t\} + \mathcal{L}\{u_1(2-2t) + \mathcal{L}\{u_2(t)(t-2)\}\} \tag{18.50}$$

$$\mathcal{L}\{u_0(t)\cdot t\} = \frac{1}{s^2} \tag{18.51}$$

$$\mathcal{L}\{u_1(2-2t)\} = e^{-s}\mathcal{L}\{2-2t\} \tag{18.52}$$

$$= -\frac{2e^{-s}}{s^2} \tag{18.53}$$

$$\mathcal{L}\{u_2(t)(t-2)\} = \frac{e^{-2s}}{s^2} \tag{18.54}$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \tag{18.55}$$

$$\tag{18.56}$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} \tag{18.57}$$

$$= s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} \tag{18.58}$$

$$= (s^2+4)\mathcal{L}\{y\} \tag{18.59}$$

Remember,

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\} \tag{18.60}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s^2+4)}\right\} \tag{18.61}$$

$$= \frac{1}{4}\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) \tag{18.62}$$

$$- \frac{1}{2}\left(\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) \tag{18.63}$$

$$+ \frac{1}{4}\left(\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4}\right\}\right)$$

Lecture 19

2 April 2020

19.1 Laplacian Properties

Given functions f, g, then the Laplacian of $f + g$ can be written as the linear combination of the Laplacian of f and g, i.e.:

$$\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} \tag{19.1}$$

However, does the following hold?

$$\mathcal{L}\{f \cdot g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\} \tag{19.2}$$

Example 19.1 Counter Example

Consider the case of

$$f = g = 1 \tag{19.3}$$

Then,

$$\mathcal{L}\{f \cdot g\} = \mathcal{L}\{1\} = \frac{1}{s} \tag{19.4}$$

$$\mathcal{L}\{f\}\mathcal{L}\{g\} = \frac{1}{s^2} \tag{19.5}$$

$$\frac{1}{s^2} \neq \frac{1}{s} \tag{19.6}$$

19.2 Convolutions

However, there does exist a function h such that

$$\mathcal{L}\{h\} = \mathcal{L}\{f\}\mathcal{L}\{g\} \tag{19.7}$$

Theorem 19.1

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau \tag{19.8}$$

Then,

$$\mathcal{L}\{h\} = \mathcal{L}\{f\}\mathcal{L}\{g\} \tag{19.9}$$

The function h is defined to be the **convolution** of f and g, denoted by $f * g$.

19.2.1 Convolution Properties

$$f * g = g * f \qquad \text{commutivity} \tag{19.10}$$
$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{distributivity} \tag{19.11}$$
$$(f * g) * h = f * (g * h) \qquad \text{associativity} \tag{19.12}$$
$$0 * f = f * 0 = 0 \tag{19.13}$$

Example 19.2

Compute

$$\mathcal{L}^{-1}\{H(s)\}, \quad H(s) = \frac{a}{s^2(s^2+a^2)} \tag{19.14}$$

Solution

$$H(s) = \underbrace{\frac{1}{s^2}}_{F(s)} \cdot \underbrace{\frac{a}{s^2+a^2}}_{G(s)} \tag{19.15}$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\} \tag{19.16}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} \tag{19.17}$$
$$= t * \sin(at) \tag{19.18}$$
$$= \int_0^t (t-\tau)\sin(a\tau)d\tau \tag{19.19}$$
$$= \frac{at - \sin(at)}{a^2} \tag{19.20}$$

Example 19.3

Solve

$$y'' + 4y = g(t), \quad y(0) = 3, \quad y'(0) = -1 \tag{19.21}$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g\} \tag{19.22}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = \mathcal{L}\{g\} \tag{19.23}$$

$$\mathcal{L}\{y\} = 3\frac{s}{s^2+4} - \frac{1}{2}\cdot\frac{2}{s^2+4} + \frac{1}{2}\cdot\frac{2}{s^2+4}\cdot G(s) \tag{19.24}$$

$$y = \mathcal{L}^{-1}\left\{3\frac{s}{s^2+4} - \frac{1}{2}\cdot\frac{2}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}G(s)\right\} \tag{19.25}$$

$$= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}(f * g) \tag{19.26}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin(2t) \tag{19.27}$$

Then the final solution is,

$$y = 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t-\tau))g(\tau)\,d\tau \tag{19.28}$$

Example 19.4

Find $\mathcal{L}\{f\}$ given

$$f = \int_0^t \sin(t-\tau)\cos(\tau)\,d\tau \tag{19.29}$$

Solution,

$$f = \sin(t) * \cos(t) \tag{19.30}$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\sin(t)\}\mathcal{L}\{\cos(t)\} \tag{19.31}$$

$$= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} \tag{19.32}$$

$$= \frac{s}{(s^2+1)^2} \tag{19.33}$$

Lecture 20

7 April 2020

20.1 Intro to Systems of Differential Equations

Definition 20.1

A basic system of differential equations is

$$\begin{cases} x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = F_1(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ x_n' = F_n(t, x_1, x_2, \ldots, x_n) \end{cases} \tag{20.1}$$

Example 20.1

$$\begin{cases} x_1' = 1 x_2' = x_2 \end{cases} \tag{20.2}$$

This system of equations can be solved by solving each differential separately,

$$x_1' = 1 \implies x_1 = t + c_1 \tag{20.3}$$

$$x_2' = x_2 \implies x_2 = c_2 e^t \tag{20.4}$$

Example 20.2

Not all systems can be solved in this way, take the following

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 + 3x_2 \end{cases} \tag{20.5}$$

20.2 Linear Algebra (again)

20.2.1 Matrices

The content covered is similar to that we covered previously.

Operations

If A and B are $m \, \mathrm{x} \, n$ matrices, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} \tag{20.6}$$

Definition 20.2 Addition

The addition of A and B can be written as

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix} \tag{20.7}$$

Definition 20.3 Scalar Multiplication

The multiplication of A by a scalar c (a number), results in

$$cA = \begin{pmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{pmatrix} \tag{20.8}$$

Definition 20.4 Subtraction

Subtraction can be thought of as first multiplying the matrix by a scalar of -1, then adding, i.e.

$$A - A = A + (-1)A \tag{20.9}$$

Definition 20.5 Zero Matrix

In the previous example,

$$A + (-1)A = 0 \tag{20.10}$$

However, the 0 is not the number 0, it is the zero matrix, i.e.

$$A + (-1)A = 0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{20.11}$$

Where the size of the zero matrix is $m \,\mathrm{x}\, n$, or the same size as matrix A. The zero is used as a shorthand, and in some literature is denoted by a bolded zero, i.e. **0**.

Definition 20.6 1x1 Matrices

In general, most $1 \,\mathrm{x}\, 1$ matrices are written as just the element, rather than being surrounded by the matrix symbol, i.e. given the following matrix

$$A = \begin{pmatrix} 23 \end{pmatrix} \tag{20.12}$$

it is acceptable to write

$$A = 23 \tag{20.13}$$

Definition 20.7 Vectors and Row Matrices

A **vector** is defined as a matrix of size $m \,\mathrm{x}\, 1$, i.e.

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \tag{20.14}$$

Whilst a **row matrix** is defined as a matrix of size $1 \,\mathrm{x}\, n$, i.e.

$$r = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix} \tag{20.15}$$

Definition 20.8 Matrix Multiplication

1. Matrix multiplication can only occur between two matrices of size $m \,\mathrm{x}\, n$ and $n \,\mathrm{x}\, k$. Or, the number of columns of the first matrix must equal the number of rows of the second.

2. It is non-commutive, i.e.

$$A\,B \neq B\,A \tag{20.16}$$

3. The resulting matrix between the multiplication of $m \,\mathrm{x}\, n$ and $n \,\mathrm{x}\, k$ matrices is of size $m \,\mathrm{x}\, k$.

Therefore, the multiplication of matrices is written as

$$A\,B = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{pmatrix} \tag{20.17}$$

Let c_{ij} be defined as the dot product of the i^{th} row of matrix A, and the j^{th} column of matrix B.

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} \tag{20.18}$$

$$= \sum_{k=0}^{n} a_{ik} b_{kj} \tag{20.19}$$

Matrix Multiplication Examples

Example 20.3

This is a basic example of matrix multiplication.

$$A = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \end{pmatrix} \tag{20.20}$$

$$A\,B = \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & -6 \\ 2 & -4 \end{pmatrix} \tag{20.21}$$

Example 20.4

In the event that the resulting matrix is a 1 x 1, i.e.

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} \tag{20.22}$$

Then, $B\,A$ is a 1 x 1 matrix, therefore it can be computed as the dot product between B and A, i.e.

$$B \cdot A = \sum a_n b_n \tag{20.23}$$

Example 20.5 Non-commutativity

In the previous example,

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} \tag{20.24}$$

We found $B\,A$ equal to the dot product between the two matrices, however, it is not equal to $A\,B$:

$$A\,B = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mk} \end{pmatrix} \tag{20.25}$$

Determinants

Definition 20.9

Determinants are a number associated with square matrices, i.e. matrices of size $n \, \text{x} \, n$. Given a matrix A, the determinant can be denoted as

$$\det(A) = |A| \tag{20.26}$$

or by replacing the parentheses/brackets around a matrix with a straight line, i.e.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix} \tag{20.27}$$

While the true significance and the generalized method to compute determinants for any square matrix is outside of the scope of this class, determinants still prove useful in solving systems of differential equations.

Example 20.6 Determinant of a 2 x 2 Matrix

Given a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{20.28}$$

The determinant is computed as,

$$|A| = a11a_{22} - a_{21}a_{12} \tag{20.29}$$

Example 20.7 Determinant of a 3 x 3 Matrix

Given a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{20.30}$$

The determinant can be computed via the Laplace Formula, i.e.

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{20.31}$$

Also, you may encounter another method using the diagonals formed by the matrix, in that case see Sarus' rule or Sarus' Scheme.

Lecture 21

9 April 2020

21.1 Linear Algebra

21.1.1 Inverse Matrices

Definition 21.1 Identity Matrix

An identity matrix of size n is defined as an $n \text{x} n$ matrix whose main diagonal is filled with 1s and all other entries are 0,

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{21.1}$$

It is called either the identity matrix or the unit matrix, however, the latter is much less common. It is called as such because it satisfies the following identity

$$I\,A = A\,I = A \tag{21.2}$$

Definition 21.2 Inverse Matrix

For a square matrix A of size $n \text{x} n$, then the matrix B also of size $n \text{x} n$ is the inverse matrix if it satisfies the following identity:

$$A\,B = I_n \tag{21.3}$$

The inverse matrix is not defined for all matrices, take the zero matrix for example.

Theorem 21.1

A square matrix, A, has an inverse iff $\det(A) \neq 0$.

Example 21.1

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad \det(A) = 0 \implies \nexists A^{-1} \tag{21.4}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \det(A) = 1 \implies \exists A^{-1} \tag{21.5}$$

Example 21.2 Finding the Inverse Matrix

The inverse matrix of A can be found via the augmented matrix of A and I,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \tag{21.6}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \tag{21.7}$$

Then, to compute the inverse is to make the left hand size of the augmented matrix look like the identity matrix on the right hand size.

1. $R_3 - R_1$
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \tag{21.8}$$

2. $R_1 - 2R_3$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & 0 & -2 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \tag{21.9}$$

3. $R_3 + R_2$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 3 & 0 & -2 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right) \tag{21.10}$$

4. $R_1 + 5R_3$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & 3 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right) \tag{21.11}$$

Example 21.2 Finding the Inverse Matrix

5. $R_2 - 2R_2$ and $R_3 - 2R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & 3 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right) \tag{21.12}$$

Then confirm that the final result is the inverse matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -2 & 5 & 3 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21.13}$$

21.1.2 Eigenvectors and Eigenvalues

First some review over matrices, a matrix $(m \, \text{x} \, n)$ multiplied by a vector $(n \, \text{x} \, 1)$, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{21.14}$$

Then,

$$A \, \bar{x} = \begin{pmatrix} x_1 a_{11} & x_2 a_{12} & \dots & x_n a_{1n} \\ x_1 a_{21} & x_2 a_{22} & \dots & x_n a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 a_{m1} & x_2 a_{m2} & \dots & x_n a_{mn} \end{pmatrix} \tag{21.15}$$

A system of equations can be written as follows

$$\begin{cases} 2x & +3y & +z & = 1 \\ & 4y & -z & = 0 \\ 4x & & -5z & = 12 \end{cases} \iff \begin{pmatrix} 2 & 3 & 1 \\ 0 & 4 & -1 \\ 4 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 12 \end{pmatrix} \tag{21.16}$$

Definition 21.3 Eigenvectors and Eigenvalues

Let A be a square $(n \, \text{x} \, n)$ matrix, then $\bar{x}$ as a vector with n elements is an **eigenvector** if $\exists \lambda, \lambda \neq 0$, such that

$$(A - \lambda I)\bar{x} = 0 \tag{21.17}$$

The value of λ is defined to be the **eigenvalue**. Generally, there are n eigenvalues, and infinitely many eigenvectors for each eigenvalue.

Remark 21.1 Note about Eigenvectors

Although the zero matrix satisfies

$$(A - \lambda I)\bar{x} = 0 \tag{21.18}$$

it is not considered to be an eigenvector, therefore

$$\bar{x} \neq 0 \tag{21.19}$$

Theorem 21.2 Eigenvalue

λ is an eigenvalue of a matrix A, if and only if

$$\det(A - \lambda I) = 0 \tag{21.20}$$

Example 21.3

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} \tag{21.21}$$

Solution: Find λ such that, $|A - \lambda I| = 0$.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0 \tag{21.22}$$

$$= (3-\lambda)(-2-\lambda) - (-1)(4) \tag{21.23}$$

$$0 = \lambda^2 - \lambda - 2 \tag{21.24}$$

Solving the quadratic yields $\lambda_1 = -1, \lambda_2 = 2$.

1. Finding the eigenvectors for λ_1.

$$(A - \lambda I)\bar{x} = 0 \iff \begin{cases} 4x_1 - x_2 = 0 \\ 4x_1 - x_2 = 0 \end{cases} \implies \bar{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{21.25}$$

Therefore, the eigenvectors for $\lambda = -1$ are

$$c\bar{x} = c\begin{pmatrix} 1 \\ 4 \end{pmatrix}, c \neq 0 \tag{21.26}$$