Lecture 1

14 April 2020

No lecture was conducted. These are some homework practice questions that can be found in Elementary Differential Equations, 11^{th} Edition.

1.1 Method of Undetermined Coefficients

For the following problems:

1. Find the general solution using the method of undetermined coefficients.
2. Let $L[\phi]$ be the linear differential operator $(L[\phi] = \phi'' + p(t)\phi' + q(t)\phi)$
3. Some computations such as derivatives may have steps skipped for the sake of brevity.

Homework 141.1

Given:

$$y'' - 2y' - 3y = 3e^{2t} \tag{1.1}$$

Solution:

$$y'' - 2y' - 3y = 0 \tag{1.2}$$
$$\implies y_1 = e^{3t}, \quad y_2 = e^{-t} \tag{1.3}$$

$$L[Ae^{2t}] = [4A - 4A - 3A]\, e^{2t} = 3e^{2t} \tag{1.4}$$
$$\implies A = -1 \tag{1.5}$$

Therefore,

$$y = c_1 e^{3t} + c_2 e^{-t} - e^{2t} \tag{1.6}$$

Homework 141.3

Given:

$$y'' + y' - 6y = 12e^{3t} + 12e^{-2t} \tag{1.7}$$

Solution:

$$y'' + y' - 6y = 0 \tag{1.8}$$
$$\implies y_1 = e^{2t}, \quad y_2 = e^{-3t} \tag{1.9}$$

$$L[Ae^{3t}] = 9Ae^{3t} + 3Ae^{3t} - 6Ae^{3t} = 12e^{3t} \tag{1.10}$$
$$6Ae^{3t} = 12e^{3t} \tag{1.11}$$
$$\implies A = 2 \tag{1.12}$$

$$LBe^{-2t} = 4Be^{-2t} - 2Be^{-2t} - 6Be^{-2t} = 12e^{-2t} \tag{1.13}$$
$$-4Be^{-2t} = 12e^{-2t} \tag{1.14}$$
$$\implies B = -3 \tag{1.15}$$

Therefore,

$$y = c_1e^{2t} + c_2e^{-3t} + 2e^{3t} - 3e^{-2t} \tag{1.16}$$

Homework 141.5

Given:

$$y'' + 2y' = 3 + 4\sin(2t) \tag{1.17}$$

Solution:

$$y'' + 2y' = 0 \tag{1.18}$$
$$\implies y_1 = 1, \quad y_2 = e^{2t} \tag{1.19}$$

$$L[A\sin(2t) + B\cos(2t)] = (4A - 4B)\cos(2t) + (-4A - 4B)\sin(2t) \tag{1.20}$$

$$\begin{cases} 4A - 4B = 0 \\ -4A - 4B = 4 \end{cases} \quad \implies A = B = \frac{1}{2} \tag{1.21}$$

Remark: I could have added the following C term in the previous, however, due to lack of space I decided to do it separately.

$$L[Ct] = 0 + 2C = 3 \implies C = \frac{3}{2} \tag{1.22}$$

Therefore,

$$y = c_1 + c_2e^{2t} + \frac{3}{2}t - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t) \tag{1.23}$$

Homework 141.7

Given:

$$y'' + y = 3\sin(2t) + t\cos(2t) \tag{1.24}$$

Solution:

$$y'' + y = 0 \tag{1.25}$$
$$\implies y_1 = \cos t, \quad y_2 = \sin t \tag{1.26}$$

Remark: For the following guess I accidentally wrote an incorrect guess, however, it ended up working. For reference, a so called proper guess as defined in Table 3.5.1 in the text would be $Y = At\cos(2t) + Bt\sin(2t)$.

$$L[At\cos(2t) + B\sin(2t)] = -4A\sin(2t) - 4At\cos(2t) - 4B\sin(2t) \tag{1.27}$$
$$+ At\cos(2t) + B\sin(2t) \tag{1.28}$$
$$\implies t\cos(2t) + 3\sin(2t) = (-4A - 3B)\sin(2t) - 3At\cos(2t) \tag{1.29}$$

$$\begin{cases} -4A - 3B &= 3 \\ -3A &= 1 \end{cases} \implies A = -\frac{1}{3}, \quad B = -\frac{5}{9} \tag{1.30}$$

Therefore,

$$y = c_2\cos t + c_2\sin t - \frac{1}{3}t\cos(2t) - \frac{5}{9}\sin(2t) \tag{1.31}$$

Homework 141.9

Given:

$$u'' + \omega_0^2 u = \cos(\omega_0 t) \tag{1.32}$$

Solution:

$$u'' + \omega_0^2 = 0 \tag{1.33}$$
$$\implies u_1 = \cos(\omega_0 t), \quad u_2 = \sin(\omega_0 t) \tag{1.34}$$

$$L[At\cos(\omega_0 t) + Bt\sin(\omega_0 t)] \tag{1.35}$$
$$= -A\sin(\omega_0 t) + 2B\omega_0\cos(\omega_0 t) = \cos(\omega_0 t) \tag{1.36}$$

$$\begin{cases} -2A\omega_0 = 0 \\ 2B\omega_0 = 1 \end{cases} \implies B = \frac{1}{2\omega_0} \tag{1.37}$$

Therefore,

$$y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{1}{2\omega_0}t\sin(\omega_0 t) \tag{1.38}$$

Homework 141.11

Given:

$$y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1 \tag{1.39}$$

Solution:

$$y'' + y' - 2y = 0 \tag{1.40}$$

$$\implies y_1 = e^t, \quad y_2 = e^{-2t} \tag{1.41}$$

$$L[At + B] = 0 + A - 2At - 2B = 2t \tag{1.42}$$

$$\begin{cases} A - 2B & = 0 \\ -2A & = 2 \end{cases} \implies A = -1, \quad A = -\frac{1}{2} \tag{1.43}$$

$$y(t) = c_1 e^t + c_2 e^{-2t} - t - \frac{1}{2} \tag{1.44}$$

$$y(0) = c_2 + c_2 - \frac{1}{2} = 0 \tag{1.45}$$

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} - 1 \tag{1.46}$$

$$y'(0) = c_1 - 2c_2 - 1 = 1 \tag{1.47}$$

$$\begin{cases} c_1 + c_2 & = \frac{1}{2} \\ c_1 - 2c_2 & = 2 \end{cases} \implies c_1 = 1, \quad c_2 = -\frac{1}{2} \tag{1.48}$$

Therefore,

$$y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2} \tag{1.49}$$

Homework 141.13

Given:

$$y'' - 2y' + y = te^t + 4, \quad y(0) = 1, y'(0) = 1 \tag{1.50}$$

Solution:

$$y'' - 2y' + y = 0 \tag{1.51}$$

$$\implies y_1 = e^t, \quad y_2 = te^t \tag{1.52}$$

$$\begin{aligned} L[At^3e^t + Bt^2e^t + C] &= At^3e^t + 3At^2e^t + 3At^2e^t + 6Ate^t && (1.53)\\ &+ Bt^2e^t + 2Bte^t + 2Bte^t + 2Be^t && (1.54)\\ &- 2At^3e^t - 6At^2e^t - 2Bt^2e^t - 4Bte^t && (1.55)\\ &+ At^3e^t + C && (1.56)\\ &= 6Ate^t - 2Be^t + C && (1.57) \end{aligned}$$

$$\begin{cases} 6A &= 1 \\ 2B &= 0 \\ C &= 4 \end{cases} \tag{1.58}$$

$$\begin{aligned} y(t) &= c_1e^t + c_2te^t + \frac{1}{6}t^3e^t + 4 && (1.59)\\ y(0) &= c_1 + 4 = 1 && (1.60)\\ y'(t) &= c_1e^t + c_2te^t + c_2e^t + \frac{1}{2}t^2e^t && (1.61)\\ y'(0) &= c_1 + c_2 = 1 && (1.62) \end{aligned}$$

$$\begin{cases} c_1 & = 3 \\ c_1 + c_2 = 1 \end{cases} \implies c_1 = -3, \quad c_2 = 4 \tag{1.63}$$

Therefore,

$$y = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4 \tag{1.64}$$

1.2 Variation of Parameters

For this section, solve using variation of parameters. For problems of the form

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1.65}$$

$$Y = -y_1 \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\, ds + y_2 \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\, ds \tag{1.66}$$

Where t_0 is defined as any point conveniently inside the open interval I where the solution exists.

1. y_1 and y_2 are general solutions found by solving the corresponding CH-SOLDE.
2. For the following examples, it is sufficient to only evaluate the upper bound t.
3. After finding a particular solution, confirm using method of undetermined coefficients.
4. If a particular solution has a term with a constant coefficient that is of the same form of either of the general solutions y_1 or y_2, it is sufficient to leave them out, as the coefficients c_1 and c_2 will take care of it. For example, Y contains e^t, but $y_1 = e^t$, it is fine to leave out the e^t term from Y.

Homework 146.1

Given:

$$y'' - 5y' + 6y = 2e^t \tag{1.67}$$

Solution:

$$y_1 = e^{2t}, \quad y_2 = e^{3t} \tag{1.68}$$

$$W[e^{2t}, e^{3t}] = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \tag{1.69}$$

$$\begin{aligned} Y &= -e^{2t} \int^{t} \frac{2e^{3s}e^{s}}{e^{5s}}\, ds + e^{3t} \int^{t} \frac{2e^{2s}e^{s}}{e^{5s}}\, ds && (1.70) \\ &= -e^{2t}[-2e^{-s}]^{t} + e^{3t}[-e^{-2s}]^{t} && (1.71) \\ &= e^{t} && (1.72) \end{aligned}$$

Checking,

$$L[Ae^t] = Ae^t - 5Ae^t + 6e^t = 2e^t \implies A = 1 \implies Y = e^t \tag{1.73}$$

Therefore,

$$y = c_1 e^{2t} + c_2 e^{3t} + e^t \tag{1.74}$$

Homework 146.2

Given:

$$y'' - y' - 2y = 2e^{-t} \tag{1.75}$$

Solution:

$$y_1 = e^{2t}, \quad y_2 = e^{-t} \tag{1.76}$$

$$W[e^{2t}, e^{-t}] = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{t} - 2e^{t} = -3e^{t} \tag{1.77}$$

$$\begin{aligned}
Y &= -e^{2t}\int^{t} \frac{(2e^{-s})(e^{-s})}{-3e^{s}}\, ds + e^{-t}\int^{t} \frac{(2e^{-s})(e^{2s})}{-3e^{s}}\, ds && (1.78)\\
&= -e^{2t}\int^{t} -\frac{2}{3}e^{-3s}\, ds + e^{t}\int^{t} -\frac{2}{3}\, ds && (1.79)\\
&= -e^{2t}(-\frac{2}{9}e^{-3t}) + e^{-t}(-\frac{2}{3}t) && (1.80)\\
&= \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} && (1.81)\\
Y &= -\frac{2}{3}te^{-t} && (1.82)
\end{aligned}$$

Remark: As can be seen above, $\frac{2}{9}e^{-t}$ has a constant coefficient, and is of the form y_2, therefore, it is excluded from the final result of Y. Checking:

$$\begin{aligned}
L[Ate^{-t} + Be^{-t}] &= 2Ae^{-t} + Ate^{-t} + Be^{-t} - Ae^{-t} && (1.83)\\
&\quad + Ate^{-t} + Be^{-t} - 2Ate^{-t} - 2Be^{-t} = 2e^{-t} && (1.84)\\
\implies A &= -\frac{2}{3} \therefore Y = -\frac{2}{3}te^{-t} && (1.85)
\end{aligned}$$

Therefore,

$$y = c_1e^{2t} + c_2e^{-t} - \frac{2}{3}te^{-t} \tag{1.86}$$

Homework 146.3

Given:

$$4y'' - 4y' + y = 16e^{t/2} \tag{1.87}$$

Solution:

$$4y'' - 4y' + y = 16e^{t/2} \iff y'' - y' + \frac{1}{4} = 4e^{t/2} \tag{1.88}$$

$$y_1 = e^{t/2}, \quad y_2 = te^{t/2} \tag{1.89}$$

Homework 146.3

$$W[e^{t/2}, te^{t/2}] = \begin{vmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & e^{t/2} + \frac{1}{2}te^{t/2} \end{vmatrix} = e^t \begin{vmatrix} 1 & t \\ \frac{1}{2} & \left(1 + \frac{1}{2}t\right) \end{vmatrix} = e^t \tag{1.90}$$

$$\begin{aligned} Y &= -e^{t/2} \int^t \frac{(se^{s/2})(4e^{s/2})}{e^s}\, ds + te^{t/2} \int^t \frac{(4e^{s/2})(e^{s/2})}{e^s}\, ds && (1.91) \\ &= -e^{t/2}(2t^2) + te^{t/2}(4t) && (1.92) \\ &= -2t^2e^{t/2} + 4te^{t/2} && (1.93) \\ &= 2t^2e^{t/2} && (1.94) \end{aligned}$$

Checking:

$$\begin{aligned} L[At^2e^{t/2}] = 8Ae^{t/2} + 8Ate^{t/2} + At^2e^{t/2} - 8At^{t/2} && (1.95) \\ - 2At^2e^{t/2} + At^2e^{t/2} = 16e^{t/2} && (1.96) \\ \implies A = 2 \implies Y = 2t^2e^{t/2} && (1.97) \end{aligned}$$

Therefore,

$$y = c_1e^{t/2} + c_2te^{t/2} + 2t^2e^{t/2} \tag{1.98}$$

1.3 Laplacians

This section, compute the Laplace transform of the following.

Homework 247.16

Given:

$$f(t) = \begin{cases} 1, & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases} \tag{1.99}$$

Solution: Rewrite piecewise as Heaviside step functions

$$f(t) = u_0(t) - u_\pi(t) \tag{1.100}$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{u_0(t)\} - \mathcal{L}\{u_\pi(t)\} && (1.101) \\ &= \frac{1 - e^{-\pi s}}{s} && (1.102) \end{aligned}$$

Homework 247.17

Given:

$$f(t) = \begin{cases} t, & t \in [0,1) \\ 1, & t \in [1,\infty) \end{cases} \tag{1.103}$$

Solution:

$$f(t) = tu_0(t) + (1-t)u_1(t) \tag{1.104}$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{tu_0(t)\} + \mathcal{L}\{(1-t)u_1(t)\} && (1.105) \\ &= \mathcal{L}\{tu_0(t)\} - \mathcal{L}\{(t-1)u_1(t)\} && (1.106) \\ &= \frac{1}{s^2} - e^{-s} \cdot \frac{1}{s^2} && (1.107) \\ &= \frac{1-e^{-s}}{s^2} && (1.108) \end{aligned}$$

Homework 247.18

Given:

$$f(t) = \begin{cases} t, & t \in [0,1) \\ 2-t, & t \in [1,2) \\ 0, & t \in [2,\infty) \end{cases} \tag{1.109}$$

Solution:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{tu_0(t)\} - 2\mathcal{L}\{(t-1)u_1(t)\} + \mathcal{L}\{(t-2)u_2(t)\} && (1.110) \\ &= \frac{1}{s^2} - 2\left(e^{-s} \cdot \frac{1}{s^2}\right) + e^{-2s} \cdot \frac{1}{s^2} && (1.111) \\ &= \frac{1-2e^{-s}+e^{-2s}}{s^2} && (1.112) \end{aligned}$$

1.4 Inverse Laplacians

Find the inverse Laplace transform of the following.

Homework 255.1

Given:

$$F(s) = \frac{3}{s^2+4} \tag{1.113}$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} &(1.114)\\
&= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} &(1.115)\\
&= \frac{3}{2}\sin(2t) &(1.116)
\end{aligned}$$

Homework 255.3

Given:

$$F(s) = \frac{2}{s^2+3s-4} \tag{1.117}$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\} &(1.118)\\
&= -\mathcal{L}^{-1}\left\{\frac{\frac{2}{5}}{s+4}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{2}{5}}{s-1}\right\} &(1.119)\\
&= \frac{2}{5}\left(e^{t} - e^{-4t}\right) &(1.120)
\end{aligned}$$

Homework 255.5

Given:

$$F(s) = \frac{2s-3}{s^2-4} \tag{1.121}$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{2s}{s^2-4}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s^2-4}\right\} &(1.122)\\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2-4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} &(1.123)\\
&= 2\cosh(2t) - \frac{3}{2}\sinh(2t) &(1.124)
\end{aligned}$$

Remark 1.1

Notice in the previous example, if you plug it into some online solvers or look at other people's solutions, you may get solutions that do not involve hyperbolic trig. Therefore, it is necessary to keep the following in mind:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{1.125}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{1.126}$$

Homework 255.7

Given:

$$F(s) = \frac{1-2s}{s^2+4s+5} \tag{1.127}$$

Solution: Hint: complete the square in the denominator

$$\begin{aligned}
F(s) = F(s) &= \frac{1-2s}{(s^2+4s+4)-4+5} && (1.128)\\
&= \frac{1-2s}{(s+2)^2+1} && (1.129)\\
&= \frac{5}{(s+2)^2+1} - 2\frac{s+2}{(s+2)^2+1} && (1.130)\\
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} && (1.131)\\
&= 5e^{-2t}\sin t - 2e^{-2t}\cos t && (1.132)
\end{aligned}$$

1.5 Solving ODEs with Laplacians

For the following, solve the given ODE via Laplacians.

Homework 255.9

Given:

$$y'' + 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 0 \tag{1.133}$$

Solution:

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{0\} \tag{1.134}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 3\left(s\mathcal{L}\{y\} - y(0)\right) + 2\mathcal{L}\{y\} = 0 \tag{1.135}$$

$$(s^2 + 3s + 2)\mathcal{L}\{y\} - s - 3 = 0 \tag{1.136}$$

$$\tag{1.137}$$

$$\mathcal{L}\{y\} = \frac{s+3}{s^2 + 3s + 2} \tag{1.138}$$

$$= \frac{s+3}{(s+2)(s+2)} = \frac{A}{s+2} + \frac{B}{s+1} \tag{1.139}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \tag{1.140}$$

$$= -e^{-2t} + 2e^{-t} \tag{1.141}$$

Homework 255.11

Given:

$$y'' - 2y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 0 \tag{1.142}$$

Solution:

$$\mathcal{L}\{y'' - 2y' + 4y\} = \mathcal{L}\{0\} \tag{1.143}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2s\mathcal{L}\{y\} - 2y(0) + 4\mathcal{L}\{y\} = 0 \tag{1.144}$$

$$(s^2 - 2s + 4)\mathcal{L}\{y\} - 2s + 4 = 0 \tag{1.145}$$

Homework 255.11

$$\begin{aligned}
\mathcal{L}\{y\} &= \frac{2s-4}{s^2-2s+4} && (1.146)\\
&= \frac{2s-4}{s^2-2s+1-1+4} && (1.147)\\
&= \frac{2s-4}{(s-1)^2+3} && (1.148)\\
&= \frac{2s+2}{(s-1)^2+3} - \frac{2}{(s-1)^2+3} && (1.149)\\
y &= \mathcal{L}^{-1}\left\{\frac{2s+2}{(s-1)^2+3}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-1)^2+3}\right\} && (1.150)\\
&= 2e^t\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)e^t && (1.151)
\end{aligned}$$

Homework 255.13

Given:

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1 \tag{1.152}$$

Solution:

$$\mathcal{L}^{-1}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{1.153}$$

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{y^{(4)} - 4y''' + 6y'' - 4y' + y\right\} && (1.154)\\
&= s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\\
&\quad - 4\left[s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)\right]\\
&\quad + 6\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right]\\
&\quad - 4\left[s\mathcal{L}\{y\} - y(0)\right]\\
&\quad + \mathcal{L}\{y\} && (1.155)\\
&(s^4 - 4s^3 + 6s^2 - 4s + 1)\mathcal{L}\{y\} - s^2 - 1 + 4s - 6 = 0 && (1.156)
\end{aligned}$$

$$\begin{aligned}
L[y] &= \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1} && (1.157)\\
&= \frac{s^2-4s+7}{(s-1)^4} && (1.158)\\
&= \frac{A}{(s-1)^4} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)} && (1.159)
\end{aligned}$$

Homework 255.13

$$\begin{cases} -B+C+A &= 7 \\ B-2C &= -4 \\ C &= 1 \\ D &= 0 \end{cases} \implies A=4, B=-2, C=1, D=0 \tag{1.160}$$

$$\mathcal{L}\{y\} = \frac{4}{6}\frac{6}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} \tag{1.161}$$

Therefore,

$$y = \frac{4}{6}e^t t^3 - e^t t^2 + e^t t \tag{1.162}$$

Homework 255.15

Given:

$$y'' + \omega^2 y = \cos(2t), \quad \omega^2 \neq 4, \quad y(0)=1, \quad y'(0)=0 \tag{1.163}$$

Solution:

$$\mathcal{L}\{y''+\omega^2 y\} = \mathcal{L}\{\cos(2t)\} \tag{1.164}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \omega^2\mathcal{L}\{y\} = \frac{s}{s^2+4} \tag{1.165}$$

$$(s^2+\omega^2)\mathcal{L}\{y\} = \frac{s}{s^2+4} \tag{1.166}$$

$$\mathcal{L}\{y\} = \frac{s}{(s^2+4)(s^2+\omega^2)} + \frac{s}{(s^2+\omega^2)} \tag{1.167}$$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos(\omega t) \tag{1.168}$$

$$\frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+\omega^2} \tag{1.169}$$

$$s = A(s^3-\omega^2 s) + B(s^2+\omega^2) + C(s^3+4s) + D(s^2+4) \tag{1.170}$$

$$\begin{cases} A+C &= 0 \\ B+D &= 0 \\ 4C-A\omega^2 &= 1 \\ \omega^2 B + 4D &= 0 \end{cases} \implies A = \frac{1}{4+\omega^2}, B=0, C=-\frac{1}{4+\omega^2}, D=0 \tag{1.171}$$

Homework 255.15

$$y_2 = \mathcal{L}^{-1}\left\{\frac{-s}{(4+\omega^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{(4+\omega^2}\right\} \quad (1.172)$$

$$= \frac{1}{4+\omega^2}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{-s}{s^2+\omega^2}\right\}\right) \quad (1.173)$$

$$= \frac{1}{4+\omega^2}\left(\cos(2t) - \cos(\omega t)\right) \quad (1.174)$$

$$(1.175)$$

Therefore,

$$y = \frac{1}{4+\omega^2}\left(\cos(2t) - \cos(\omega t)\right) + \cos(\omega t) \quad (1.176)$$

1.6 Heaviside Function

Rewrite the following as a combination of Heaviside functions.

Homework 263.5

Given:

$$f(t) = \begin{cases} 0, & t \in [0,3) \\ -2, & t \in [3,5) \\ 2, & t \in [5,7) \\ 1, & t \in [7,\infty) \end{cases} \quad (1.177)$$

Solution:

$$f(t) = -2u_3(t) + 4u_5(t) - u_7(t) \quad (1.178)$$

Homework 263.7

Given:

$$f(t) = \begin{cases} 1, & t \in [0,2) \\ e^{-(t-2)}, & t \in [2,\infty) \end{cases} \quad (1.179)$$

Solution:

$$f(t) = u_0(t) + (e^{-t+2} - 1)u_2(t) \quad (1.180)$$

Homework 263.13

Given:

$$\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} \quad (1.181)$$

Solution:

$$e^{2t}t^3 \quad (1.182)$$

1.7 Solving Nonhomogeneous ODEs with Heavisides

Homework 268.4

Given:

$$y'' + 3y' + 2y = \begin{cases} 1, & t \in [0, 10) \\ 0, & t \in [10, \infty) \end{cases}, y(0) = 0, y'(0) = 0 \tag{1.183}$$

Solution:

$$g(t) = u_0(t) - u_{10}(t) \tag{1.184}$$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{u_0(t) - u_{10}(t)\} \tag{1.185}$$

$$s^2\mathcal{L}\{y\} + sy(0) + y'(0) + 3\left(s\mathcal{L}\{y\} + y(0)\right) + 2\mathcal{L}\{y\} = \frac{1}{s} - \frac{e^{-10s}}{s} \tag{1.186}$$

$$(s^2 + 3s + 2)\mathcal{L}\{y\} = \frac{1 - e^{-10s}}{s} \tag{1.187}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \tag{1.188}$$

$$= A(s^2 + 3s + 2) + B(s^2 + 2s) + C(s^2 + s) \tag{1.189}$$

$$\begin{cases} A + B + C & = 0 \\ 3A + 2B + C & = 0 \\ 2A & = 1 \end{cases} \implies \begin{cases} B + C & = -\frac{1}{2} \\ 2B + C & = -\frac{3}{2} \\ A & = \frac{1}{2} \end{cases} \implies A = C = \frac{1}{2}, B = -1 \tag{1.190}$$

$$\mathcal{L}\{y\} = \frac{1}{2}\left(\left(\frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}\right) - e^{-10s}\left(\frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}\right)\right) \tag{1.191}$$

$$y = \frac{1}{2}\left(1 - 2e^{-t} + e^{-2t}\right) - \frac{1}{2}u_0(t)\left(1 - 2e^{-t} + e^{-2t}\right) \tag{1.192}$$

Homework 268.6

Given:

$$y'' + y' + \frac{5}{4}y = \begin{cases} \sin(t), & t \in [0, \pi) \\ 0, & t \in [\pi, \infty) \end{cases}, y(0) = 0, y'(0) = 0 \tag{1.193}$$

Solution:

$$\mathcal{L}\{y'' + y' + \frac{5}{4}y\} = \mathcal{L}^{-1}\{\sin(t)u_0(t) - \sin(t)u_\pi(t)\} \tag{1.194}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + s\mathcal{L}\{y\} - y(0) + \frac{5}{4}\mathcal{L}\{y\} = \frac{1}{s^2+1} - e^{-\pi s}\left(\frac{1}{s^2+1}\right) \tag{1.195}$$

$$\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}\{y\} = \frac{1}{s^2+1} - \frac{e^{-\pi s}}{s^2+1} \tag{1.196}$$

$$\mathcal{L}\{y\} = \frac{1}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)} - \frac{e^{-\pi s}}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)} \tag{1.197}$$

$$\frac{1}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{\left(s^2+s+\frac{5}{4}\right)} \tag{1.198}$$

$$1 = (As+B)\left(s^2+s+\frac{5}{4}\right) + (Cs+D)(s^2+1) \tag{1.199}$$

$$\begin{cases} A + C &= 0 \\ A + B + D &= 0 \\ \frac{5}{4}A + B + C &= 0 \\ \frac{5}{4}B + D &= 1 \end{cases} \implies A = -\frac{16}{17}, B = \frac{4}{17}, C = \frac{16}{17}, D = \frac{12}{17} \tag{1.200}$$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{-\frac{16}{17}s + \frac{4}{17}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{16}{17}s + \frac{12}{17}}{s^2+s+\frac{5}{4}}\right\} \tag{1.201}$$

$$y_2 = -e^{-\pi s}\left(\mathcal{L}^{-1}\left\{\frac{-\frac{16}{17}s + \frac{4}{17}}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{16}{17}s + \frac{12}{17}}{s^2+s+\frac{5}{4}}\right\}\right) \tag{1.202}$$

1.8 Convolutions

Homework 278.5

Find the Laplace transform of the following,

$$f(t) = \int_0^t e^{-(t-\tau)} \sin\tau \, d\tau \tag{1.203}$$

Solution:

$$\begin{aligned} F(s) &= \mathcal{L}^{-1}\left\{e^{-t}\right\} \mathcal{L}^{-1}\left\{\sin t\right\} && (1.204)\\ &= \frac{1}{s-1}\frac{1}{s^2+1} && (1.205)\\ &= \frac{1}{(s-1)(s^2+1)} && (1.206) \end{aligned}$$

Homework 278.7

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^4(s^2+1)} \tag{1.207}$$

Solution:

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \frac{1}{6}\frac{6}{s^4}\frac{1}{s^2+1} && (1.208)\\ &= \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} && (1.209)\\ &= \frac{1}{6}t^3 * \sin(t) && (1.210)\\ &= \int_0^t \tau^3 \sin(t-\tau)\, d\tau && (1.211)\\ &= t^3 - 6t + 6\sin(t) && (1.212) \end{aligned}$$

Homework 278.8

Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s+1)(s^2+4)} \tag{1.213}$$

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \frac{1}{s+1} \cdot \frac{1}{s^2+4} && (1.214)\\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} && (1.215)\\
&= e^{-t} * \cos(2t) && (1.216)\\
&= \int_0^t e^{-\tau}\cos(2t-2\tau)\,d\tau && (1.217)
\end{aligned}$$