# Differential Equations

MATH 308 at Texas A&M

Using *Elementary Differential Equations, 11$^{th}$ Edition*

Brandon Nguyen

Spring 2020

# Contents

# Lecture 1

# 14 January 2020

## 1.1   First Order Differential Equations

**Definition 1.1 First Order Differential Equation**

The generic form of a **first order differential equation** is

$$y' = f(x, y) \tag{1.1}$$

Sometimes, $t$ is substituted for $x$, especially if the function relates to time.

**Definition 1.2 General Solution**

A solution to a differential equation is considered to be **general** if there is an arbitrary constant present in the final answer, i.e. a problem without initial values.

**Example 1.1**

$$y' = 1 \tag{1.2}$$
$$y = \int y' \, dx \tag{1.3}$$
$$= 1 \, dx \tag{1.4}$$
$$= x + C \tag{1.5}$$

> **Definition 1.3 Open Differential Equations**
>
> Equations without solutions are considered to be **open**. Many differential equations are without solutions.

> **Example 1.2 Open Differential Equation**
>
> $$y' = x'y - x^3 \tag{1.6}$$
>
> This differential equation does not have a solution; thusly open.

> **Example 1.3**
>
> $$y' = y \tag{1.7}$$
>
> $$\int y' \, dy = \int y \, dy \nLeftrightarrow y' = y \tag{1.8}$$
>
> Notice above that the integration of both sides is not the same as the differential equation.
>
> $$y' = e^x \implies y \int y' \, dx = \int e^x \, dx \tag{1.9}$$
>
> Using the above, the general solution can be found
>
> $$y = Ce^x \tag{1.10}$$

> **Remark 1.1 Regarding Example 1.3**
>
> If both sides of a differential equation are dependent on the same variable — i.e. the same variable appearing on both sides of the equation, then taking the intergal of both sides is not a valid method to solve the equation.

> **Definition 1.4 Initial Value Problems**
>
> An **initial value problem** (IVP), or **initial condition problem**, is a problem where an initial condition of the equation is defined which leads to a **unique solution** to the equation.

**Example 1.4 Initial Value Problem**

$$y' = x, \ y(0) = 1 \tag{1.11}$$

Notice that this is an **initial value problem**, because $y(0) = 1$. Also notice that $y$ is an anti-derivative w.r.t. $x$; because each side of the equation is independent of one another (unlike *Example 1.3*).

$$\int y' \, dx = \int x \, dx \tag{1.12}$$

$$\implies \quad y = \frac{1}{2}x^2 + C \tag{1.13}$$

$$y(0) = 1 \tag{1.14}$$

$$\implies \quad = \frac{1}{2}(0^2) + C \tag{1.15}$$

$$\implies \quad C = 1 \tag{1.16}$$

$$\implies \quad y = \frac{1}{2}x^2 + 1 \tag{1.17}$$

## 1.2   Differentiable Functions

**Definition 1.5 Differentiability**

Given $f : \mathbb{R} \to \mathbb{R}$ and point $a$, $\exists f'(a) \iff \exists T_1(a)$, where $T_1$ is a tangent line (Taylor polynomial of degree one).
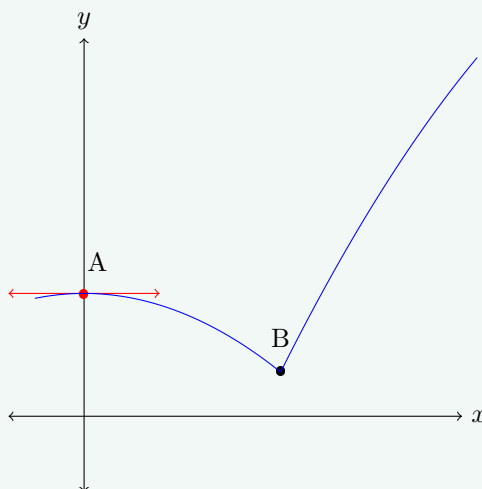


In the example, point A has a singular tangent line and is therefore differentiable. Point B has infinitely many tangent lines, and is therefore both undefined and not differentiable.

## 1.3   Kinematics

**Example 1.5 Kinematics with Differential Equations**

Given an object with a velocity $v_0$, and acceleration $a$, find the position $s$ at any time $t$.

$$\frac{d}{dt}v(t) = a \tag{1.18}$$

$$\implies \quad v(t) = \int a\,dt \tag{1.19}$$

$$= at + C \tag{1.20}$$

$$\because \quad v(0) = v_0 \tag{1.21}$$

$$v_0 = a(0) + C \tag{1.22}$$

$$C = v_0 \tag{1.23}$$

$$\frac{d}{dt}s(t) = \int v\,dt \tag{1.24}$$

$$\implies \quad s(t) = \int v\,dt \tag{1.25}$$

$$= \int (at + v_0)\,dt \tag{1.26}$$

$$= \frac{1}{2}at^2 + v_0 t \tag{1.27}$$

# Lecture 2

# 16 January 2020

## 2.1 Linear Differential Equations

> **Definition 2.1 First Order Linear Differential Equations**
>
> $$\underbrace{y' + p(t)y = g(t)}_{\text{Usual form}} \iff y' = g(t) - p(t)y \qquad (2.1)$$
>
> A **first order linear differential equation** (LDE) is linear due to $y$ being dependent on only one variable, $t$.

Notice that $t$ is typically used in place of $x$ as most differential equations are used in models dependent on time; as such, most differential equations are in the form $y' = f(t, y)$ as opposed to $y' = f(x, y)$.

> **Example 2.1**
>
> $$\text{Solve} \quad (4 + t^2)y' + 2ty = 4t \qquad (2.2)$$
>
> $$\text{Notice:} \qquad (4y + t^2 y)' = \frac{d}{dt}(4y + t^2 y) = 4t \qquad (2.3)$$
>
> $$= 4y' + (t^2 y)^2 \qquad (2.4)$$
>
> $$= 4y' + (2ty + t^2 y') \qquad (2.5)$$
>
> $$= (4 + t^2)y' + 2ty \qquad (2.6)$$

### Example 2.1

The original problem can now be reduced to:

$$\frac{d}{dt}(4y + t^2 y) = 4t \tag{2.7}$$

$$\text{let} \quad z(t) = 4y + t^2 y \tag{2.8}$$

$$= 2t^2 + C \tag{2.9}$$

$$\implies \quad 4y + t^2 y = 2t^2 + C \tag{2.10}$$

$$\therefore \quad y = \frac{1}{4 + t^2}(2t^2 + C) \tag{2.11}$$

### Remark 2.1 Constants

Notice in the above example that the constant, $C$, is being multiplied by $\frac{1}{4+t^2}$. When expanding the answer, it now becomes $y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$. Notice how the constant is dependent on the variable $t$, and is therefore not the same as just $C$.

### Definition 2.2 Integrating Factors with LDEs

An **integrating factor**, $\mu(t)$ is a function $\mu(t) : \mathbb{R} \to \mathbb{R}$, that satisfies $\frac{d}{dt}\mu(t) = \mu(t)y' + \mu(t)p(t)y$.

### Remark 2.2

There are infinitely many integrating factors due to the arbitrary constant $C$ from indefinite integration, see **Method 2.1** and **Example 2.2** on the following page.

**Method 2.1 Solution of the General LDE Case**

Solve $y' + p(t)y = g(t)$.

1. Multiply the LDE by $\mu(t)$ results in:

$$\mu(t)(y' + p(t)y) = \mu(t)g(t) \tag{2.12}$$

2. Letting $z(t) = \mu(t)y$, and $z' = \mu(t)g(t)$ yields:

$$z(t) = \int \mu(t)g(t)\, dt \tag{2.13}$$

$$\implies y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\, dt \tag{2.14}$$

$$\implies \mu(t) = \exp\left(\int p(t)\, dt\right) \tag{2.15}$$

3. Therefore the solution of the general case is

$$y(t) = \left(\exp\left(\int p(t)\, dt\right)\right)^{-1} \cdot \int \exp\left(\int p(t)\, dt\right)g(t)\, dt \tag{2.16}$$

**Example 2.2 Solving an IVP involving LDEs**

Working with example 2.1.4 from the textbook:

$$ty' + 42y = 4t^2, \ y(1) = 2 \tag{2.17}$$

1. Compute the integrating factor $(\mu(t))$

$$\mu(t) = \exp\left(\int p(t)\, dt\right) \tag{2.18}$$

$$= \exp\left(\int 2t^{-1}\, dt\right) \tag{2.19}$$

$$= \exp(2\ln(t) + C) \Leftrightarrow e^{2\ln(t)+C} \tag{2.20}$$

2. Find the general case
   When solving, 0 can be subsituted in for $C$ to simplify calculations; for $C \neq 0$ it is trivially shown that the constant will cancel out in

**Example 2.2 Solving an IVP involving LDEs**

computing the solution.

$$y_c(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\,dt \tag{2.21}$$

$$= \frac{1}{t^2}\left(\int t^2 \cdot 4t\,dt\right) \tag{2.22}$$

$$= \frac{1}{t^2}(t^4 + C) \tag{2.23}$$

**Note:** $y_c(t)$ is used to denote the general case.

3. Find formula w.r.t. intial value

$$y(1) = 2 \tag{2.24}$$

$$\implies y(1) = (1)^2 + \frac{C}{(1)^2} \tag{2.25}$$

$$\implies \quad C = 1 \tag{2.26}$$

$$\therefore \quad y(t) = t^2 + t^{-2} \tag{2.27}$$

# Lecture 3

# 21 January 2020

## 3.1 Linear Differential Equations (cont).

**Example 3.1**

Given $y' - 2y = t^2 e^{2t}$ find:

1. The general solution

$$p(t) = -2,\ g(t) = t^2 e^{2t} \tag{3.1}$$

$$\mu(t) = \exp\left(\int -2\,dt\right) \tag{3.2}$$

$$= e^{-2t+C} \tag{3.3}$$

$$y_c(t) = e^{2t} \int t^2\,dt \tag{3.4}$$

$$= e^{2t}\left(\frac{1}{3}t^3 + C\right) \tag{3.5}$$

2. What is $\lim_{t\to\infty} y_c(t)$?
   There are infinitely many $y_c(t)$; the answer may vary with the value of $C$. In this case, the value of $C$ does not matter.

$$\lim_{t\to\infty} y_c(t) = +\infty$$

## 3.2   Separable Differential Equations

**Definition 3.1 Separable Differential Equations**

A **separable differential equation** (SDE) can be defined by

$$\frac{dy}{dx} = y' = f(x, y) = -\frac{M(x, y)}{N(x, y)} \tag{3.6}$$

where

$$M(x, y) = -f(x, y) \tag{3.7}$$
$$N(x, y) = 1 \tag{3.8}$$

it is **separable** because it can be written in the **differential form**

$$M(x)\, dx + N(y)\, dy = 0 \tag{3.9}$$

**Theorem 3.1**

If $\frac{dy}{dx} = \frac{M(x)}{N(y)}$, then $\int N(y)\, dy = \int M(x)\, dx$

**Proof 3.1**

Choose $\widetilde{N}$ such that $\frac{d\widetilde{N}(y)}{dx} = M(x)$:

$$\frac{d\widetilde{N}(y)}{dy} = \frac{d\widetilde{N}(y)}{dx}\frac{dx}{dy} = \frac{d\widetilde{N}(y)}{dy}\frac{dy}{dx} = \frac{d\widetilde{N}(x)}{dx} \tag{3.10}$$

$$\frac{d\widetilde{N}(y)}{dy} = \frac{dy}{dx} \tag{3.11}$$

$$\implies \frac{d\widetilde{N}(y)}{dx} = M(x) \tag{3.12}$$

**Example 3.2**

Find a particular solution that passes through the point $(0, 1)$.

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y} \tag{3.13}$$

$$\implies \int (4 + y)\, dy = \int (4x - x^3)\, dx \tag{3.14}$$

$$4y + \frac{1}{2}y^2 + C_1 = 2x^2 - \frac{1}{4}x^4 + C_2 \tag{3.15}$$

$$4y + \frac{1}{2}y^2 = 2x^2 - \frac{1}{4}x^4 + (C_2 - C_1) \tag{3.16}$$

$$\implies 2y + 16y + x^4 - 8x^2 + C = 0 \tag{3.17}$$

$$(0, 1) \implies 2(1) + 16(1) + 0^4 - 8(0)^2 + C = 0 \tag{3.18}$$

$$C = -18 \tag{3.19}$$

$$\therefore 2y + 16y + x^4 - 8x^2 = 18 \tag{3.20}$$

**Homework 3.1**

$$y' = \frac{dy}{dx} = \frac{x^2}{y} \tag{3.21}$$

$$y\, dy = x^2\, dx \tag{3.22}$$

$$\int y\, dy = \int x^2\, dx \tag{3.23}$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C \tag{3.24}$$

$$y(x) = \pm\sqrt{\frac{2}{3}x^3 + C} \tag{3.25}$$

# Lecture 4

# 23 January 2020

## 4.1 Separable Equations (cont.)

**Example 4.1**

From the textbook, 2.2, ex. 2.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad y(0) = -1 \tag{4.1}$$

Given the above, determine the interval in which the solution exists.

$$\int 2(y-1)\,dy = \int (3x^2 + 4x + 2)\,dx \tag{4.2}$$

$$\implies y^2 - 2y + C_1 = x^3 + 2x^2 + 2x + C_2 \tag{4.3}$$

The solution above is the **general implicit solution**. The constants, $C_1$ and $C_2$ can be combined into one constant, $C$, because they are independent.

Next, use the initial value to solve for C

$$y(0) = -1 \tag{4.4}$$

$$\implies (-1)^2 - 2(-1) = 0^3 + 2(0)^2 + 2(0) + C \tag{4.5}$$

$$\implies \qquad C = 3 \tag{4.6}$$

**Example 4.1 (cont.)**

Then complete the square on the left hand side to get the **explicit solution**.

$$(y^2 - 2y + 1) - 1 = x^3 + 2x^2 + 2x + 3 \tag{4.7}$$
$$\implies \quad (y - 1)^2 = x^3 + 2x^2 + 2x + 4 \tag{4.8}$$
$$\implies \quad y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4} \tag{4.9}$$
$$\implies \quad y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \tag{4.10}$$
$$\implies \quad y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{4.11}$$
$$\because \quad y(0) = -1 \tag{4.12}$$

**Note:** It is also possible to use the quadratic formula in order to convert this instance of an implicit into an explicit solution.

**Observation**: Because the unique solution involves a square root, a function defined for $x \in [0, \infty)$, it is possible to reduce the original question to findinding when the radicand is non-negative.

$$x^3 + 2x^2 + 2x + 4 = 0 \tag{4.13}$$
$$(x^2 + 2)(x + 2) = 0 \tag{4.14}$$
$$\implies \quad x \geq -2 \tag{4.15}$$

The factor $x^2 + 2$ will always be positive, so now the question is further reduced to when $x + 2$ will be non-negative, which is $x \in [-2, \infty)$.

Therefore, the interval of which the solution exists is $(-2, \infty)$

**Remark 4.1 Solutions to Differential Equations**

In **Example 4.1**, notice the final answer was an open interval, $(-2, \infty)$, rather than a half closed interval, $[-2, \infty)$, even if the solution would be defined if $x = -2$. The reason for this is that **solutions to differential equations must also be differentiable**.

At point $x = -2$, the unique solution is defined, however, it is not differentiable as $\lim_{x \to -2^-}$ does not exist, because the function is not defined for $x < -2$.

## 4.2  Mathematical Modelling

**Example 4.2 Modelling**

Consider a pond fille with 10 million gallons of fresh water. A flow of 5 million gallons per year with water that is contaminated wiht a chemical enters the pond. There is also an outflow of this mixture on the order of 5 million gallons per year.

Let $\gamma(t)$ be the concentration of the fluid entering the pod at time $t$, and let $Q(t)$ be the quantity of chemicals in the pod at time $t$.

It is determined that

$$\gamma(t) = 2 + \sin(2t) \text{ g} \cdot \text{gal}^{-1}$$

Find $Q(t)$ using the given information.

We can infer that $Q(0) = 0$ because the water starts off fresh at $t = 0$. We know that $\frac{dQ}{dt}$ is equal to the rate at which chemicals are entering minus the rate at which they leave, leading us to

$$\frac{dQ}{dt} = I(t)\gamma(t) - \frac{O(t)}{V(t)}\big[Q(t)\big]$$

Where $I(t)$ describes the rate at which the contaminated water enters, $O(t)$ describes the rate at which the water mixture leaves the pond, and $V(t)$ describes the total volume of the pond at any given time.
In this case,

$$I(t) = 5 \times 10^6 \text{ gal year}^{-1} \tag{4.16}$$

$$O(t) = 5 \times 10^6 \text{ gal year}^{-1} \tag{4.17}$$

$$V(t) = 10^7 \text{ gal} \tag{4.18}$$

$$\tag{4.19}$$

Plugging in the values yields the following,

$$\frac{dQ}{dt} = 5 \times 10^6 \gamma(t) - \frac{1}{2}Q(t) \tag{4.20}$$

$$\tag{4.21}$$

**Example 4.2 Modelling**

Solving the linear differential equation,

$$\frac{dQ}{dt} + \frac{1}{2}Q(t) = 5 \times 10^6 \gamma(t) \tag{4.22}$$

$$\implies \qquad Q_c(t) = 5 \times 10^6 e^{-\frac{1}{2}t} \int e^{\frac{1}{2}t}(2 + \sin(2t)) \, dt \tag{4.23}$$

$$\implies \qquad Q_c(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17}cos(2t) + Ce^{-\frac{1}{2}t} \tag{4.24}$$

$$Q_c(0) = 2 \times 10^7 - \frac{4 \times 10^7}{17} + C = 0 \tag{4.25}$$

$$\implies \qquad C = \frac{-3 \cdot 10^8}{17} \tag{4.26}$$

$$Q(t) = 2 \times 10^7 + \frac{2 \times 10^7}{17}\sin(2t) - \frac{4 \times 10^7}{17}cos(2t) - \frac{3 \cdot 10^8}{17}e^{-\frac{1}{2}t} \tag{4.27}$$

**Remark 4.2 Behavior of Example 4.2**

When graphing this equation, it can be seen that in the long term the equation becomes periodic despite beginning with an irregular pattern. This is due to the fact that the term $-\frac{3 \cdot 10^8}{17}e^{-\frac{1}{2}t}$ is able to affect the behavior in the short term, however, it is decaying exponentially and tends towards 0. The sin and cos functions are periodic which cause the sinusoidial shape of the graph as $t \to \infty$.

# Lecture 5

# 28 January 2020

## 5.1    Mathematical Modelling (cont.)

**Example 5.1**

Example 2.3.1 from the textbook.

1. Find the amount of salt in the tank at a time $t$ $(Q(t))$.
   Inference: $Q(0) = Q_0$

$$\frac{dQ}{dt} = \frac{1}{4}r - \frac{rQ}{100} \tag{5.1}$$

$$\implies Q' + \frac{r}{100}Q = \frac{1}{4}r \tag{5.2}$$

$$\implies \qquad Q_c = \exp\left(-\frac{r}{100}t\right) \int \left(\exp\left(\frac{r}{100}t\right)\frac{1}{4}r\right)dt \tag{5.3}$$

$$= \frac{r}{4}\exp\left(\frac{-r}{100}t\right)\left(\frac{100}{r}\exp\left(\frac{r}{100}t\right) + C\right) \tag{5.4}$$

$$= 25 + \frac{r}{4}\exp\left(-\frac{r}{100}t\right)C \tag{5.5}$$

$$= 25 + C\exp\left(-\frac{r}{100}t\right) \tag{5.6}$$

$$Q(0) = Q_0 \tag{5.7}$$

$$\implies \qquad C = (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.8}$$

$$\implies \qquad Q(t) = 25 + (Q_0 - 25)\exp\left(-\frac{r}{100}t\right) \tag{5.9}$$

**Example 5.1**

2. Find the limiting amount, $Q_l$, after a long time.

$$\lim_{t \to \infty} (Q_c(t)) = Q_c = 25 \tag{5.10}$$

**Remark 5.1 Regarding Example 5.1**

Notice that no matter the amount of salt that the system starts with, it will always tend towards 25 lbs of salt in the tank.

## 5.2   Exact Differential Equations

**Definition 5.1 Exact Differential Equations**

A differential equation is exact iff

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \Leftrightarrow N(x,y)y' + M(x,y) = 0 \tag{5.11}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{5.12}$$

Given $\psi(x,y)$, parameterize by using $\delta(t) = \psi(f_1(t), f_2(t))$.

$$\frac{d\psi(x,y)}{dt} = \frac{d\delta}{dt} \tag{5.13}$$

$$= \frac{\partial \psi(x,y)}{\partial x} \frac{df_1}{dt} + \frac{\partial \psi(x,y)}{\partial y} \frac{df_2}{dt} \tag{5.14}$$

**Example 5.2**

$$\psi(x,y) = x^2 y + xy \tag{5.15}$$

$$f_1(t) = t, \quad f_2(t) = t^2 \tag{5.16}$$

$$\delta(t) = \psi(f_1, f_2) \tag{5.17}$$

$$= t^2 t^2 + t t^2 \tag{5.18}$$

$$\delta'(t) = 4t^3 + 3t^2 \tag{5.19}$$

$$\frac{\partial \psi(x,y)}{\partial x} \cdot 1 + \frac{\partial \psi(x,y)}{\partial y} \cdot 2t \tag{5.20}$$

$$= (2f_1 f_2 + f_2) \cdot 1 + (f_1^2 + f_1) \cdot 2ty \tag{5.21}$$

$$= 4t^3 + 3t^2 \tag{5.22}$$

**Example 5.2**

Notice how **Equation 5.19** and **Equation 5.22** are the same, but derived via different methods.

**Example 5.3**

1. $y' = \frac{\frac{1}{y}}{x}$ is an exact differential equation.
   Let $M(x,y) = \frac{1}{x}$, and $N(x,y) = y$.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial \frac{1}{x}}{\partial y} = 0$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial y}{\partial x} = 0$$

   Because both partial derivatives are equal, they are exact.

2. $y' = x$ is exact.

3. $y' = \frac{xy}{x+y} \iff (x+y)\,dy + xy\,dx = 0$ is exact.

4. $y' = \frac{xy+x}{\frac{1}{2}x^2+y}$ is exact.

**Theorem 5.1 Exactness**

The equation
$$M(x,y)\,dx + N(x,y)\,dy = 0$$
is exact if, and only if, $\exists\,\psi(x,y)$ s.t.

$$\frac{\partial \psi(x,y)}{\partial x} = M(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = N(x,y)$$

**Remark 5.2 Relationship**

Exact differential equations are a superset of the separable differential equations, i.e. all separable differential equations are exact differential equations.

# Lecture 6

# 30 January 2020

## 6.1   Exact Differential Equations (cont.)

**Example 6.1**

Solve

$$(y \cos x + 2xe^y) + (\sin x + x^2 + x^2 e^y - 1)y' = 0 \tag{6.1}$$

Checking if **Equation 6.1** is exact,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(y \cos x + 2xe^y)}{\partial y} = \cos x + 2xe^y \tag{6.2}$$

$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(\sin x + x^2 e^y - 1)}{\partial y} = \cos x + 2xe^y \tag{6.3}$$

From the above, this is an exact differential equation.

$$\psi(x,y) = \int M(x,y)\, dx + h(y) \tag{6.4}$$

$$= \int (y \cos x + 2xe^y)\, dx + h(y) \tag{6.5}$$

$$= h(y) + y \sin x + x^2 e^y + C \tag{6.6}$$

$$= h(y) + y \sin x + x^2 e^y \tag{6.7}$$

Notice that the constant can be neglected as it can be contained in $h(y)$.

**Example 6.1**

Now solving for $h(y)$,

$$\psi_y(x, y) = N(x, y) \tag{6.8}$$

$$\implies \frac{dh}{dy} + \frac{\partial(y \sin x + e^y x^2)}{\partial y} = \sin x + x^2 e^y - 1 \tag{6.9}$$

$$\frac{dh}{dy} + \sin x + x^2 e^y = \sin x + x^2 e^y - 1 \tag{6.10}$$

$$\frac{dh}{dy} = -1 \tag{6.11}$$

$$h = -y + C \tag{6.12}$$

Then,

$$\psi(x, y) = y \sin x + x^2 e^y - y + C \tag{6.13}$$

Finally, $y(x)$ is given by the implicit expression

$$y \sin x + x^2 e^y - y = C \tag{6.14}$$

**Example 6.2**

Solve
$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{6.15}$$

Checking if the equation is exact,

$$\frac{\partial(3xy + y^2)}{\partial y} = 3x + 2y \tag{6.16}$$

$$\frac{\partial(x^2 + xy)}{\partial x} = 2x + y \tag{6.17}$$

Notice that they are not equal; however,

$$\mu(x)(3xy + y^2) + \mu(x)(x^2 + xy)y' = 0 \tag{6.18}$$

is an exact differential equation if

$$-\frac{N_x(x, y) + M_y)x, y}{N(x, y)} \tag{6.19}$$

is a function dependent only on $x$. However, $\forall M(x, y), N(x, y) \nexists \mu(x)$. $\mu(x)$

**Example 6.2**

can be found by solving the differential equation,

$$\frac{d\mu}{dx} = \frac{-N_x(x,y) + M_y(x,y)}{N(x,y)}\mu \tag{6.20}$$

$$\mu(x) = \exp\left(\int \frac{N_x - M_y}{N}\,dx\right) \tag{6.21}$$

In this problem,

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{1}{x} \tag{6.22}$$

$$\mu(x) = \exp\left(\int \frac{dx}{x}\right) = x + C \tag{6.23}$$

Multiplying **Equation 6.15** by $\mu(x)$ yields,

$$(3x^2y + y^2x) + (x^3 + x^2y)y' = 0 \tag{6.24}$$

Checking if the equation is exact yields the following,

$$\frac{\partial 3x^2y + xy^2}{\partial y} = 3x^2 + 2xy \tag{6.25}$$

$$\frac{\partial x^3 + x^2y}{\partial x} = 3x^2 + 2xy \tag{6.26}$$

and is therefore exact.

$$\psi(x,y) = \int (3x^2y + xy^2)\,dx + h(y) \tag{6.27}$$

$$= x^3y + \frac{1}{2}x^2y^2 + h(y) \tag{6.28}$$

$$\frac{\partial \psi(x,y)}{\partial y} = x^3 + x^2y + \frac{dh}{dy} \tag{6.29}$$

$$= N(x,y) \tag{6.30}$$

$$\frac{dh}{dy} = x^3 + x^2y = x^3 + x^2y \tag{6.31}$$

$$h = 0 \tag{6.32}$$

Finally, $y(x)$ can be expressed as,

$$x^3y + \frac{1}{2}x^2y^2 = C \tag{6.33}$$

**Method 6.1 Solving Exact Differential Equations**

1. Step 1: Determine if the equation is exact

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{6.34}$$

2. Step 2: Find $\psi(x,y)$ such that $\psi_x(x,y) = M(x,y)$, and $\psi_y(x,y) = N(x,y)$. Generally,

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.35}$$

this works because

$$\frac{\partial \psi(x,y)}{\partial x} = \frac{\partial \int M(x,y)\,dx}{\partial x} + \frac{\partial h(y)}{\partial x} \tag{6.36}$$

$$= M(x,y) + 0 \tag{6.37}$$

Then find $h(y)$ such that $\psi_y(x,y) = N(x,y)$.

**Remark 6.1**

Note in step 2 of **Method 6.1**

$$\psi(x,y) = \int M(x,y)\,dx + h(y) \tag{6.38}$$

can also be defined as

$$\psi(x,y) = \int N(x,y)\,dy + h(x) \tag{6.39}$$

$$\frac{\partial \psi(x,y)}{\partial y} = \psi_y(x,y) = \frac{\partial \int N(x,y)\,dy}{\partial y} + \frac{\partial h(x)}{\partial y} \tag{6.40}$$

$$= N(x,y) + 0 \tag{6.41}$$

**Remark 6.2**

$y(x)$ is a solution for $M(x,y)\,dx + N(x,y)\,dy = 0$ iff $\psi(x,y(x)) = c$. Consider the following,

$$\frac{d\psi(f_1,f_2)}{dt} = \frac{\partial \psi(x,y)}{\partial x}\frac{df_1}{dt} + \frac{\partial \psi(x,y)}{\partial y}\frac{df_2}{dt}$$

**Remark 6.2**

we can replace $t$ with $x$, let $f_1 \equiv x$ and $f_2 \equiv y(x)$, then

$$\frac{d\psi(f_1(x), f_2(x))}{dx} = \frac{\partial \psi(x, y)}{\partial x} \frac{df_1}{dx} + \frac{\partial \psi(x, y)}{\partial y} \frac{df_2}{dx} \tag{6.42}$$

$$= \frac{\partial \psi(x, y)}{\partial x} + \frac{\partial \psi(x, y)}{\partial y} \frac{dy}{dx} \tag{6.43}$$

finally,

$$N(x, y)\frac{dy}{dx} = \frac{\partial \psi(x, y)}{\partial y} \frac{dy}{dx} = \frac{d\psi(x, y(x))}{dx} - \frac{\partial \psi(x, y)}{\partial x}$$

**Remark 6.3**

Notice in **Equation 6.31** has 3 variables: $h, x, y$; however, the terms with $x$ cancel, leaving just $h$ and $y$. This occurs due to the equation being exact.

# Lecture 7

# 4 February 2020

Recall in the last lecture:

$$M_y(x, y) = N_x(x, y) \tag{7.1}$$
$$\implies \exists\, \psi(x, y(x)) := \psi_x = M(x, y); \psi_y = N(x, y) \tag{7.2}$$
$$\psi(x) = \psi(x, y) \equiv C \tag{7.3}$$

For example,

$$\psi(x, y) = x + y \tag{7.4}$$
$$x + y(x) = C \tag{7.5}$$
$$y = C - x \tag{7.6}$$

And if $\frac{M_y(x,y) - N_x(x,y)}{N(x,y)}$ depends only on $x$, then $\exists\, \mu(x) : \frac{d\mu}{dx} = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}\mu$.
Thus, the differential equation $\mu M + \mu N y' = 0$ is an exact differential equation.

## 7.1  Uniqueness and Exactness

**Theorem 7.1 Uniqueness of Linear Differential Equations**

Consider the linear first order differential equation,

$$y' + p(t)y = g(t); \quad y(t_0) = y_0 \tag{7.7}$$

such that in some open interval, $I = (\alpha; \beta)$, $p(t)$ and $g(t)$ are continuous
and $t_0 \in I$.
Then,

$$\exists!\, y(t) : y(t_0) = y_0 \wedge y' + p(t)y = g(t) \tag{7.8}$$

> **Theorem 7.2 Uniqueness of Non-linear Differential Equations**
>
> Consider the following,
>
> $$y' = f(t, y) \land y(t_0) = y_0 \tag{7.9}$$
>
> such that $f(t, y)$ and $\frac{\partial f(t,y)}{\partial y}$ are continuous over the domains $t \in (\alpha; \beta)$, and $y \in (\gamma; \delta)$.
> Then, $h > 0, I = (t_0 - h, t_0 + h) : \exists t_0 \in I, y(t_0) = y_0$.

> **Example 7.1**
>
> $$ty' + 2y = 4t^2; \quad y(1) = 2 \tag{7.10}$$
>
> Use **Theorem 7.1** to find an interval $\exists! y(t)$.
>
> $$y' + \frac{2}{t}y = 4t \tag{7.11}$$
>
> $$p(t) = \frac{2}{t}, \quad g(t) = 4t \tag{7.12}$$
>
> In the interval $I := (\alpha, \beta)$, $\exists t \in I : p(t), g(t) \implies \exists! y(t)$
>
> 1. $\forall\, t \in (-\infty, 0) \cup (0, \infty), p(t)$
>
> 2. $\forall\, t \in (-\infty, \infty), g(t)$
>
> 3. $1 \in (\alpha, \beta)$
>
> 4. Therefore, $\alpha = 0, \beta = \infty \implies I = (0, \infty) = \mathbb{R}^+$

# Lecture 8

# 6 February 2020

## 8.1  First Order Differential Equation Review

Topics covered in First Order Differential Equations.

- $y' = f(x, y)$   $y(x_0) = y_0$

- First Order LDE, $y' + p(t)y = g(t)$

- Separable, $y' = \frac{M(x,y)}{N(x,y)}$

- Exact, $M(x, y) + N(x, y)y' = 0$; $M_y(x, y) = N_x(x, y)$

- Uniqueness and Existance Theorems

- Modelling

## 8.2  Second Order Differential Equations

**Definition 8.1 Second Order Differential Equations**

The general form of a **second order differential equation** (SODE) is

$$y'' = f(x, y, y'); \quad y(x_0) = y_0; \quad y'(x_0) = y_1 \qquad (8.1)$$

**Example 8.1**

The following are SODEs,

$$y'' = 1 \tag{8.2}$$
$$y'' = 1 + y' \tag{8.3}$$
$$y'' = \frac{x}{t} \tag{8.4}$$

An example of a SODE IVP,

$$y'' = x + y + y'; \quad y(0) = 1; \quad y'(0) = -3 \tag{8.5}$$

**Definition 8.2 Second Order Linear Differential Equations**

A **second order linear differential equation** (SOLDE) has the general form

$$y'' + p(t)y' + q(t)y = g(t) \tag{8.6}$$

where $p(t)$, $q(t)$, $g(t)$ are continuous over some interval $I$.

**Theorem 8.1 SOLDE Uniqueness Theorem**

If $p(t), q(t), g(t)$ are continuous in some interval $I : (\alpha, \beta)$
Then, for any $t_0 \in I$, the IVP defined by

$$y'' + p(t)y' + q(t)y = g(t), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{8.7}$$

has a unique solution.

**Definition 8.3 Cases of SOLDEs**

**Homogeneous SOLDEs** (HSOLDE) are of the following form

$$y'' + p(t)y' + q(t)y = 0 \tag{8.8}$$

If a Homogeneous SODE is defined where $p(t)$, and $q(t)$ are constants, it is considered as a **homogeneous SOLDE with Constant Coefficients** (CHSOLDE).

## 8.3 Homogeneous Second Order Linear Differential Equations with Constant Coefficients

**Example 8.2 CHSOLDE**

Find the general solution of

$$L[y] = y'' + 5y' + 6y = 0 \qquad (8.9)$$

Consider the following quadratic (characteristic function, or characteristic polynomial).

$$f(r) = r^2 + 5r + 6 = 0 \qquad (8.10)$$

There are 2 different roots to the characteristic function,

$$r_1 = -3; \quad r_2 = -2 \qquad (8.11)$$

Now consider the equations,

$$y_1(t) = e^{r_1 t} = e^{-2t} \qquad (8.12)$$

$$y_2(t) = e^{r_1 t} = e^{-3t} \qquad (8.13)$$

Then, $y_1(t)$ and $y_2(t)$ are solutions of **Equation 8.9**.
**Proof:**

$$y_1'(t) = -3e^{-3t}; \quad y_1'' = 9e^{-3t} \qquad (8.14)$$
$$L[y_1] = 9e^{-3t} + 5(-3)e^{-3t} + 6e^{-3t} = 0 \qquad (8.15)$$
$$0 = (9 - 15 + 6)e^{-3t} \qquad (8.16)$$

Therefore, the general solution to **Equation 8.9**

$$y_c = C_1 e^{-3t} + C_2 e^{-2t} \qquad (8.17)$$

where $C_1$ and $C_2$ are constants.

**Example 8.3**

$$L[y] = y'' + ay' + by = 0 \tag{8.18}$$

$$f(r) = r^2 + ar + b = 0 \tag{8.19}$$

Suppose that $r_0$ is a root of $f(r) = 0$

Consider

$$y_0(t) = e^{r_0 t} \tag{8.20}$$

$$y_0'(t) = r_0 e^{r_0 t} \tag{8.21}$$

$$y_0''(t) = r_0^2 e^{r_0 t} \tag{8.22}$$

$$\implies L(y_0) = r_0^2 e^{r_0 t} + a r_0 e^{r_0 t} + b e^{r_0 t} \tag{8.23}$$

$$= e^{r_0 t}(r_0^2 + a r_0 + b) \tag{8.24}$$

Things to consider, what if $r_0 \in \mathbb{C}$ or $r_0 = r_1$?

**Example 8.4 CHSOLDE IVP**

Find the solution of the CHSOLDE IVP,

$$L[y] = y'' + 5y' + 6y = 0; \quad y(0) = 2; \quad y'(0) = 3 \tag{8.25}$$

1. Find the general solution

$$y_c(t) = C_1 y_1 + C_2 y_2 \implies y_c(t) = C_1 e^{-3t} + C_2 e^{-2t} \tag{8.26}$$

2. Find the particular values of $C_1$ and $C_2$ such that $C_1 y_1(0) + C_2 y_2(0) = 2$ and $(C_1 y_1(0) + C_2 y_2(0))' = 3$.

$$\begin{cases} C_1 e^{-3(0)} + C_2 e^{-2(0)} = 2 \\ -3C_1 e^{-3(0)} + -2C_2 e^{-2(0)} = 3 \end{cases} \tag{8.27}$$

Solving the linear combination yields $C_1 = 7$, $C_2 = 9$. Then, the solution to this IVP is

$$y(t) = -7e^{-3t} + 9e^{-2t} \tag{8.28}$$

# Lecture 9

# 11 February 2020

## 9.1 Second Order Linear Differential Equations

**Theorem 9.1 Principle of Superposition**

Suppose that $y_1$ and $y_2$ are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{9.1}$$

Then, $C_1y_1 + C_2y_2$ is another solution for $L[y] = 0$ where $C_1$ and $C_2$ are constants. ($C_1y_1 + C_2y_2$ is the liear combination of $y_1$ and $y_2$)

**Proof 9.1 Principle of Superposition**

Show that
$$L[C_1y_1 + C_2y_2] = 0 \tag{9.2}$$

$$
\begin{align}
&L[C_1y_1 + C_2y_2] \tag{9.3}\\
&= (C_1y_1 + C_2y_2)'' + p(t)(C_1y_1 + C_2y_2)' + q(t)(C_1y_1 + C_2y_2) \tag{9.4}\\
&= C_1y_1'' + C_2y_2'' + p(t)C_1y_1' + p(t)C_2y_2' + q(t)C_1y_1 + q(t)C_2y_2 \tag{9.5}\\
&= (C_1y_1'' + p(t)C_1y_1' + q(t)C_1y_1) + (C_2y_2'' + p(t)C_2y_2' + q(t)C_2y_2) \tag{9.6}\\
&= C_1L[y_1] + C_2L[y_2] = 0 \tag{9.7}
\end{align}
$$

From the above, $C_1L[y_1] = 0$ and $C_2L[y_2] = 0$, therefore

$$L[C_1y_1 + C_2y_2] = 0 \tag{9.8}$$

**Theorem 9.2 Existance and Uniqueness Theorem**

Given

$$L[y] = y'' + p(t)y' + q(t)y = g(t); \quad y(t_0) = z_0; \quad y'(t_0) = z_1 \qquad (9.9)$$

suppose $t_0 \in I$.
Then, this IVP has exactly 1 solution. Moreover, this solution will be defined throughout the interval.

**Example 9.1 Application of Existance and Uniqueness Theorem**

Find the longest interval in which the solution of the IVP is certain to exist.

$$(t^2 - 3t)y'' + ty' - (t + 3)y = 0; \quad y(1) = 2; \quad y(1) = 1 \qquad (9.10)$$

The equation is equivalent to

$$L[y] = y'' + \frac{t}{t^2 - 3t}y' - \frac{t + 3}{t^2 - 3t}y = 0 \qquad (9.11)$$

1. $\forall\ t \in (-\infty, \infty), \lim_{a \to t}(g(a))$

2. $\forall\ t \in (-\infty, 0) \cup (0, 3) \cup (3, \infty), \lim_{a \to t}(q(a))$

3. $\forall\ t \in (-\infty, 3) \cup (3, \infty), \lim_{a \to t}(p(a))$

From the above,
$$I = (0, 3) \qquad (9.12)$$

**Example 9.2**

Find the unique solution of the IVP given by

$$L[y] = y'' + p(t)y' + q(t)y = 0; \quad y(t_0) = 0; \quad y'(t_0) = 0 \qquad (9.13)$$

where $p(t)$ and $q(t)$ are continuous for $t \in (-\infty, \infty)$.
The solution is
$$y(t) = 0 \qquad (9.14)$$

and because of the uniqueness theorem, this is the only answer.

## 9.2    Linear Algebra with 2 Unknowns Detour

**Definition 9.1**

General form of a linear system with 2 Unknowns

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{9.15}$$

where $x$ and $y$ are the two unknowns. The linear combination above can be rewritten as

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9.16}$$

**Definition 9.2 Matricies**

A $n \times m$ **matrix** is a $n \times m$ table filled with numbers or functions. They are written with parenthesis or brackets around the numbers, such as

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \tag{9.17}$$

When $n = m$, the matrix is considered to be a **square matrix**.

> **Definition 9.3 Determinant**
>
> An import concept involved with square matricies is the determinant, in the case of
> $$\det(A) = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = ad - bc \tag{9.18}$$

> **Theorem 9.3**
>
> The solution to
> $$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{9.19}$$
>
> is given by
> $$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.20}$$
>
> where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$. No other solution exists.

## 9.3 Wronskian

> **Definition 9.4 Wronskian**
>
> For two differentiable functions $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$, the **Wronskian** of $y_1$ and $y_2$ is defined by
> $$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{9.21}$$

## 9.4   Miscellaneous Definitions

Additional notes that were either not covered or were missed from previous lectures.

> **Definition 9.5 Differential Operator**
>
> Let $p$ and $q$ are continuous over the open interval $I$, where $t \in (\alpha, \beta)$, where $\alpha = -\infty$ or $\beta = \infty$ are included. Then for any function $\phi$ that is twice differentiable on $I$. The **differential operator** is defined by
>
> $$L[\phi] = \phi'' + p\phi' + q\phi \qquad (9.22)$$
>
> Note that result of the operator is a function itself, so the value of $L[\phi]$ at point $t$ is
>
> $$L[\phi] = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \qquad (9.23)$$

# Lecture 10

# 13 February 2020

## 10.1   Applications of the Wronskian

**Corollary 10.1**

Recall in the last lecture, **Theorem 9.3**. Let

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{10.1}$$

If $|A| = 0$, then for **some** values of $c_1$ and $c_2$ the linear system

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{10.2}$$

does not have a solution (inconsistent).

**Theorem 10.1**

Assume that $y_1$ and $y_2$ are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.3}$$

where $p$ and $q$ are continuous, and $t_0$ is a fixed point.
Then, $\forall z_0 \wedge \forall z_1$, it is possible to find $c_1$ and $c_2$ such that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{10.4}$$

satisfies the IVP

$$L[y], \quad y(t_0) = z_0, \quad y'(t_0) = z_1 \tag{10.5}$$

if and only if

$$W[y_1, y_2](t_0) \neq 0 \tag{10.6}$$

**Proof 10.1**

Suppose that $\forall z_0, z_1 \implies \exists C_1, C_2$ such that

$$\begin{cases} C_1 y_1(t_0) + C_2 y_{2(t_0)} = z_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) = z_1 \end{cases} \tag{10.7}$$

then, $\exists! C_1, C_2$ iff $W[y_1, y_2](t_0) \neq 0$

**Theorem 10.2**

Suppose that $y_1$ and $y_2$ are solutions to

$$L[y] = 0 \tag{10.8}$$

Then, the family of solutions

$$y = C_1 y_1 + C_2 y_2 \tag{10.9}$$

includes all solutions of $L[y] = 0$ iff $\exists t_0 \implies W[y_1, y_2](t_0) \neq 0$

**Example 10.1 Application of 10.2**

The solutions to

$$y'' - 5y' + 6y = 0 \tag{10.10}$$

are

$$y_1 = e^{2t}, \quad y_2 = e^{3t} \tag{10.11}$$

$$y = C_1 e^{2t} + C_2 e^{3t} \tag{10.12}$$

Calculating the Wronskian

$$W[y_1, y_2] = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} \tag{10.13}$$

$$= e^{5t} \tag{10.14}$$

$$e^{5t} \neq 0 \tag{10.15}$$

Therefore, there does not exist other solutions to this CHSOLDE.

**Theorem 10.3 Abel's Theorem**

If $y_1, y_2$ are solutions to a SOLDE,

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.16}$$

where $p, t$ are continuous over an open interval, $I$, then the Wronskian at point $t$ is given by Abel's Formula,

$$W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right) \tag{10.17}$$

where $c$ is some arbitrary constant dependent on $y_1, y_2$, but not on $t$.

$$\forall t \in I, W[y_1, y_2](t) \equiv 0 \iff c = 0 \tag{10.18}$$

$$\forall t \in I, W[y_1, y_2](t) \not\equiv 0 \iff c \neq 0 \tag{10.19}$$

**Example 10.2**

Suppose that

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t} \tag{10.20}$$

are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.21}$$

Show that if $r_1 \neq r_2$, then $C_1 y_1 + C_2 y_2$ includes all solutions of $L[y] = 0$.

$$W[e^{r_1 t}, e^{r_2 t}] = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \tag{10.22}$$

$$= (r_2 - r_1)e^{(r_1 + r_2)t} \tag{10.23}$$

$$\neq 0 \tag{10.24}$$

**Definition 10.1 Fundamental Set of Solutions**

If $y_1$ and $y_2$ are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{10.25}$$

such that $C_1 y_1 + C_2 y_2$ includes all possible solutions of $L[y] = 0$, then $y_1$ and $y_2$ form a **fundamental set of solutions** (FSS).
Alternatively, if and only if

$$W[y_1, y_2] \neq 0 \tag{10.26}$$

then there exists fundamental set containing $y_1$ and $y_2$.

**Example 10.3**

Show that $y_1(t) = t^{\frac{1}{2}}$, $y_2(t) = t^{-1}$ form a FSS of

$$2t^2 y'' + 3t y' - y = 0, \quad t > 0 \tag{10.27}$$

1. Ensure they are solutions of $L[y] = 0$
   i.e. $L[t^{\frac{1}{2}}] = 0$, $L[t^{-1}] = 0$

$$L[t^{\frac{1}{2}}] = 2t^2(-\frac{1}{4})t^{-\frac{3}{4}} + 3t(\frac{1}{2})t^{-\frac{1}{2}} + t^{\frac{1}{2}} \tag{10.28}$$

$$= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \tag{10.29}$$

$$\equiv 0 \tag{10.30}$$

**Example 10.3**

$$L[t^{-1}] = (2t^2)(2t^{-3}) + 3t(-1)t^{-2} - t^{-1} \tag{10.31}$$

$$= 4t^{-1} - 3t^{-1} - t^{-1} \tag{10.32}$$

$$\equiv 0 \tag{10.33}$$

2. Ensure that the Wronskian is not constantly equal to 0

$$W[t^{\frac{1}{2}}, t^{-1}] = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-1} \end{vmatrix} \tag{10.34}$$

$$= -t^{-\frac{1}{2}} - -\frac{1}{2}t^{-\frac{3}{2}} \tag{10.35}$$

$$\not\equiv 0 \tag{10.36}$$

Then,

$$L[y] = y'' + ay' + by = 0; \quad f(r) = r^2 + ar + b = 0 \tag{10.37}$$

has only one solution of degree 2.

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}, \quad r_1 = r_2 \iff \sqrt{a^2 - 4b} \equiv 0 \tag{10.38}$$

**Example 10.4**

1. If $ar^2 + br + c = 0$ has equal roots $r_1$, show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt} \tag{10.39}$$

When $r = r_1$, $L[e^{rt}] = 0$, therefore $e^{rt}$ is a solution to

$$L[y] = ay'' + by' + cy = 0 \tag{10.40}$$

2. Then,

$$\frac{\partial}{\partial r}L[e^{rt}] = L[\frac{\partial}{\partial r}e^{rt}] = L[te^{rt}] \tag{10.41}$$

$$= ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1) \tag{10.42}$$

Because $r = r_1 \implies L[te^{rt}] = 0$, $te^{rt}$ is another solution to $L[y] = 0$. Show that $e^{rt}, te^{rt}$ form a FSS.

a. $L[e^{rt}] = 0$

**Example 10.4**

b. $L[te^{rt}] = 0$

$$(te^{rt})' = e^{rt} + rte^{rt} \tag{10.43}$$

$$(te^{rt})'' = 2re^{rt} + r^2te^{rt} \tag{10.44}$$

$$L[te^{rt}] = (2re^{rt} + r^2te^{rt}) + a(e^{rt} + rte^{rt}) + b \tag{10.45}$$

$$= e^{rt}(2r + a) + te^{rt}(r^2 + ar + b) \tag{10.46}$$

# Lecture 11

# 18 February 2020

## 11.1 Test Corrections

No points were missed, however, to clear a misunderstanding:

7. Show that
$$y_1 = t^{-1}; \quad y_2 = t^{1/2} \tag{11.1}$$

form a fundamental set of solutions for

$$L[y] = 2ty'' + 3y' - \frac{1}{t}y = 0; t > 0 \tag{11.2}$$

Wrong:

$$L[c_1 t^{-1} + c_2 t^{1/2}] = 0 \implies c_1 L[t^{-1}] + c_2 L[t^{1/2}] = 0 \tag{11.3}$$

Correct:
$$L[t^{-1}] = 0; \quad L[t^{1/2}] = 0 \tag{11.4}$$

Then, prove that
$$W[t^{-1}, t^{1/2}] \neq 0 \tag{11.5}$$

The wrong assumption proved the principle of superposition (the linear combination of $y_1, y_2$ are also solutions to $L[y]$), however, it does not prove that $y_1, y_2$ are indeed solutions to $L[y]$. To correctly do the problem, one must compute the linear operator on $y_1$ and $y_2$, and ensure that they equal 0.

# Lecture 12

# 20 February 2020

## 12.1   Imaginary Roots

Now consider a characteristic function,

$$f(r) = r^2 + ar + b = 0 \tag{12.1}$$

where $r \in \mathbb{C}$. The roots of $f(r) = 0$ can then be expressed by

$$r_{1,2} = \lambda \pm i\tau \tag{12.2}$$

Then, what is the general solution of

$$L[y] = y'' + ay' + by = 0 \tag{12.3}$$

where the characteristic function yields non-real roots?

---

**Theorem 12.1 Imaginary Roots**

In the case that the solutions to $f(r) = 0$ are

$$r_{1,2} = \lambda \pm i\tau \tag{12.4}$$

the general solution of the corresponding $L[y] = 0$ is given by

$$y_1 = e^{\lambda t} \cos(\tau t); \quad y_2 = e^{\lambda t} \sin(\tau t) \tag{12.5}$$

The solutions $y_1$ and $y_2$ are a fundamental set of solutions.

---

**Homework 12.1**

Show that $y_1 = e^{\lambda t} \cos(\tau t); \quad y_2 = e^{\lambda t} \sin(\tau t)$ are solutions to $L[y] = 0$, with a characteristic function that has imaginary roots.

---

**Proof 12.1 Fundamental Set**

In **Theorem 12.1**, $y_1 = e^{\lambda t}\cos(\tau t); \quad y_2 = e^{\lambda t}\sin(\tau t)$

$$W[y_1, y_2] \tag{12.6}$$

$$= \begin{vmatrix} e^{\lambda t}\cos(\tau t) & e^{\lambda t}\sin(\tau t) \\ \lambda e^{\lambda t}\cos(\tau t) - \tau e^{\lambda t}\sin(\tau t) & \lambda e^{\lambda t}\sin(\tau t) + \tau e^{\lambda t}\cos(\tau t) \end{vmatrix} \tag{12.7}$$

$$= e^{2\lambda t}\begin{vmatrix} \cos(\tau t) & \sin(\tau t) \\ \lambda\cos(\tau t) - \tau\sin(\tau t) & \lambda\sin(\tau t) + \tau\cos(\tau t) \end{vmatrix} \tag{12.8}$$

$$= e^{2\lambda t}[\lambda\cos(\tau t)\sin(\tau t) + \tau\cos^2(\tau t) - \lambda\sin(\tau t)\cos(\tau t) + \tau\sin^2(\tau t)] \tag{12.9}$$

$$= \tau e^{2\lambda t} \not\equiv 0 \tag{12.10}$$

Therefore, $y_{1,2}$ form a fundamental set of solutions.

**Example 12.1**

Solve

$$L[y] = y'' - 2y' + 6y = 0 \tag{12.11}$$

The characteristic function is given by

$$f(r) = r^2 - 2r + 6 = 0 \tag{12.12}$$

The roots are

$$r_{1,2} = \frac{2 \pm \sqrt{-20}}{2} = 1 \pm i\sqrt{5} \tag{12.13}$$

Therefore,

$$y_1 = e^t\cos\left(\sqrt{5}t\right); \quad y_2 = e^t\sin(\sqrt{5}t) \tag{12.14}$$

Then, the general solution can be given as

$$y_c = c_1 e^t\cos(\sqrt{5}t) + c_2 e^t\sin(\sqrt{5}t) \tag{12.15}$$

**Example 12.2**

Solve
$$L[y] = 9y'' + 6y' + y = 0 \tag{12.16}$$

The characteristic function is given by
$$f(r) = r^2 + 6y' + y = 0 \tag{12.17}$$

The roots are
$$(3r + 1)^2 \tag{12.18}$$
$$r_1 = r_2 = -\frac{1}{3} \tag{12.19}$$

This implies
$$y_1 = \exp(-\frac{1}{3}t); \quad y_2 = t\exp(-\frac{1}{3}t) \tag{12.20}$$

Then, the general solution can be given as
$$y_c = c_1 \exp(-\frac{1}{3}t) + c_2 t \exp(-\frac{1}{3}t) \tag{12.21}$$

# Lecture 13

# 25 February 2020

Note: Lectures from 25 February to 3 March, 2020 are not presented by Dr. Darbinyan

## 13.1    Nonhomogeneous SOLDEs

> **Definition 13.1 Nonhomogeneous SOLDE**
>
> The general form of a Nonhomogeneous SOLDE (NSOLDE) is given by
>
> $$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.1}$$

> **Theorem 13.1**
>
> Suppose that $y_1, y_2$ are solutions of
>
> $$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.2}$$
>
> then,
> $$y_1 - y_2 = 0 \tag{13.3}$$

> **Proof 13.1**
>
> $$L[y_1] = g \tag{13.4}$$
> $$L[y_2] = g \tag{13.5}$$
>
> Then,
> $$L[y_1] - L[y_2] = L[y_1, y_2] = g - g = 0 \tag{13.6}$$

**Theorem 13.2 General Solutions to NSOLDEs**

Given the following NSOLDE

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{13.7}$$

and its corresponding HSOLDE,

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{13.8}$$

The general solution involves a particular solution of **Equation 13.7** consists of a particular solution and a solution of the corresponding HSODLE,

$$y = \phi(t) = C_1 y_1 + C_2 y_2 + Y \tag{13.9}$$

where $y_1, y_2$ are solutions to **Equation 13.8**, and $Y$ is a particular solution of **Equation 13.7**

**Proof 13.2**

From the **Theorem 13.1**, we get

$$\phi - Y = c_1 y_1 + c_2 y_2 \tag{13.10}$$

this is the same as **Equation 13.9**.

From **Theorem 13.2**, solving NSOLDEs involves

1. Finding the general solution of the corresponding HSOLDE. (complementary solution; $y_c$)

2. Find any particular solution, $Y$, to the NSOLDE.

3. Then, the solution to the NSOLDE is the sum of the particular and general solutions.

## 13.2    Method of Undetermined Coefficients

**Example 13.1**

Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} \tag{13.11}$$

The corresponding HSOLDE is given by

$$y'' - 3y' - 4y = 0 \tag{13.12}$$

It is possible to "guess" the particular solution to a NSOLDE based on the form of $g(t)$. In this case, a reasonable guess would be

$$Y = Ae^{2t} \tag{13.13}$$

Substituting $Y$ into the original equation yields

$$L[Ae^{2t}] = (Ae^{2t})'' - 3(Ae^{2t})' - 4(Ae^{2t}) = 3e^{2t} \tag{13.14}$$

Solving for the undetermined coefficient,

$$(Ae^{2t})' = (2Ae^{2t}), (Ae^{2t})'' = (4Ae^{2t}) \tag{13.15}$$

$$4Ae^{2t} - 3(2Ae^{2t}) - 4(Ae^{2t}) = 3e^{2t} \tag{13.16}$$
$$4A - 6A - 4A = 3 \tag{13.17}$$
$$-6A = 3 \tag{13.18}$$
$$A = -\frac{1}{2} \tag{13.19}$$

Therefore,
$$Y = -\frac{1}{2}e^{2t} \tag{13.20}$$

is a particular solution to this NSOLDE.

**Example 13.2**

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin(t) \tag{13.21}$$

Guess,

$$Y = A\sin(t) \tag{13.22}$$
$$\implies 2\sin(t) = -A\sin(t) - 3A\cos(t) - 6A\sin(t) \tag{13.23}$$

As can be seen above, the guess $A\sin(t)$ is incorrect due to the appearance of the $\cos(t)$ term, creating an open subspace. Creating a closed subspace yields

$$Y = A\cos(t) + B\sin(t) \tag{13.24}$$

Then,

$$Y' = -A\sin(t) + B\cos(t) \tag{13.25}$$
$$Y'' = -A\cos(t) + B\sin(t) \tag{13.26}$$
$$L[Y] = (-A\cos(t) + B\sin(t)) - 3(-A\sin(t) + B\cos(t)) \tag{13.27}$$
$$- 4(A\cos(t) + B\sin(t)) = 2\sin(t) \tag{13.28}$$
$$2\sin(t) = \cos(t)(-A - 3B - 4A) + \sin(t)(-B + 3A - 4B) \tag{13.29}$$
$$\begin{cases} -A - 3B - 4A &= 0 \\ -B + 3A - 4B &= 2 \end{cases} \tag{13.30}$$

By solving the linear combination,

$$A = \frac{3}{17}, \quad B = -\frac{5}{17} \tag{13.31}$$

Finally, a particular solution given by this method is

$$y = \frac{3}{17}\cos(t) = \frac{5}{7}\sin(t) \tag{13.32}$$

**Example 13.3**

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos(2t) \tag{13.33}$$

A guess at a particular solution would be

$$Y = Ae^t \cos(2t) + Be^t \sin(2t) \tag{13.34}$$

Then, substitution

$$Y' = Ae^t \cos(2t) - 2Ae^t \sin(2t) + Be^t \sin(2t) + 2Be^t \cos(2t) \tag{13.35}$$

$$Y'' = (-3A + 4B)e^t \cos(2t) - (4A + 3B)\sin(2t) \tag{13.36}$$

$$\begin{cases} 10A + 2B & = 8 \\ 2A - 10B & = 0 \end{cases} \tag{13.37}$$

Solving the linear combination yields

$$A = \frac{10}{13}; \quad B = \frac{2}{13} \tag{13.38}$$

Finally, a particular solution given by this method is

$$y = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t) \tag{13.39}$$

**Example 13.4**

Find a particular solution of

$$L[y] = y'' - 3y' - 4y = 3e^{2t} + 2\sin(t) \tag{13.40}$$

It is possible to separate this NSOLDE into two,

$$L[y] = 3e^{2t}; L[y] = 2\sin(t) \tag{13.41}$$

And a particular solution to **Equation 13.40** is the linear combination of the solutions to the two NSOLDEs in **13.41**. Having solved the two NSOLDEs previously,

$$Y_1 = -\frac{1}{2}e^{2t} \tag{13.42}$$

$$Y_2 = -\frac{5}{17}\cos(t) + \frac{3}{17}\sin(t) \tag{13.43}$$

Then, a particular solution is given by

$$y = -\frac{1}{2}e^{2t} - \frac{5}{17}\cos(t) + \frac{3}{17}\sin(t) \tag{13.44}$$

> **Example 13.5**
>
> Find a particular solution of
>
> $$L[y] = y'' - 3y' - 4y = 2e^{2t} \tag{13.45}$$
>
> if
> $$Y = Ae^{-t}; \quad Y' = -Ae^{-t}; \quad Y'' = Ae^{-t} \tag{13.46}$$
>
> then,
> $$L[Y] = Ae^{-t} - 3(-Ae^{-t}) - 4(Ae^{-t}) = 2e^{-t} \tag{13.47}$$
>
> When solving for A, we get $0 = 2$, which is false, therefore $Ae^{-t}$ is not a form of a particular solution. Then,
>
> $$Y = Ate^{-t}; \quad L[Y] = 2e^{-t} \tag{13.48}$$
>
> $$L[Y] = (Ate^{-t} - 2Ae^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} \tag{13.49}$$
> $$-5Ae^{t} = 2e^{-t} \tag{13.50}$$
> $$A = -\frac{2}{5} \tag{13.51}$$
> $$y = -\frac{2}{5}te^{-t} \tag{13.52}$$

In the previous example, the corresponding homogeneous equation is

$$L[y] = y'' - 3y' - 4y = 0 \tag{13.53}$$

And the solutions to this equation are

$$y_1 = e^{-t}; \quad y_2 = e^{4t} \tag{13.54}$$

As it can be seen, the guess of $Y = Ae^{-t} = Ay_1$,

$$L[Ae^{-t}] = L[Ay_1] = AL[y_1] = 0 \tag{13.55}$$

Therefore,

$$L[Y] \neq 2e^{-t} \tag{13.56}$$

# Lecture 14

# 27 February 2020

## 14.1 Method of Undetermined Coefficients (cont)

From the last lecture, we have considered

$$L[y] = y'' - 3y' + 4y = g(t) \tag{14.1}$$

with the following cases for $g(t)$

1. $g$ = polynomial
2. $g = e^{\alpha t}$
3. $g = \sin(\alpha t)$
4. $g = e^{\alpha t} \sin(\beta t)$
5. $g = \alpha e^{-t}$

---

**Example 14.1**

Given

$$L[y] = y'' - 2y' + y = e^t \tag{14.2}$$

Possible guesses following the previously considered equation and cases,

$$Y = \begin{cases} Ae^t \\ Ate^t \end{cases} \tag{14.3}$$

However, neither one of these cases work, because the corresponding homogeneous equation has solutions $y_1 = e^t, y_2 = te^t$. Therefore, another guess would be $At^2 e^t$.

$$L[At^2 e^t] \implies A = \frac{1}{2} \tag{14.4}$$

---

A general rule of thumb is to increase the order of $t$ until $L[Y] \neq 0$.

## 14.2   Variation of Parameters

What if a NSOLDE is given by

$$L[Y] = \frac{P(t)}{e^{\alpha t}} \tag{14.5}$$

or some other complicated function?

---

**Example 14.2**

Find the general solution of

$$L[y] = y'' + 4y = 8 \tan(t); \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{14.6}$$

Solution:

$$L[Y] = 0 \implies f(r) = r^2 + 4 = 0 \implies r_{1,2} = \pm 2i \tag{14.7}$$

Then, a general solution to the CHSOLDE

$$y = c_1 \cos(2t) + c_2 \sin(2t) \tag{14.8}$$

If the constants $c_{1,2}$ are replaced by some functions $u_{1,2}$ then,

$$y = u_1(t)\cos(2t) + u_2(t)\sin(2t) \tag{14.9}$$

is a solution of the original NSOLDE.

$$y' = -2u_1 \sin(2t) + 2u_2 \cos(2t) + u_1' \cos(2t) + u_2' \sin(2t) \tag{14.10}$$

Then, choosing a restriction and applying it yields

$$u_1' \cos(2t) + u_2' \sin(2t) = 0 \tag{14.11}$$

$$y' = -2u_1 \sin(2t) + 2u_2 \cos(2t) \tag{14.12}$$

$$y'' = -4u_1 \cos(2t) - 4u_2 \sin(2t) - 2u_1' \sin(2t) + 2u_2' \cos(2t) \tag{14.13}$$

Substituting the values into the NSOLDE,

$$\begin{aligned} y'' + 4y = &- 4u_1 \cos(2t) - 4u_2 \sin(2t) - 2u_1' \sin(2t) + 2u_2' \cos(2t) \\ &+ 4u_1 \cos(2t) + 4u_2 \sin(2t) = 8 \tan(t) \end{aligned} \tag{14.14}$$

Solving for $u_2'$

$$u_2' = -u_1' \cot(2t) \tag{14.15}$$

Solving for $u_1'$

$$u_1' = -\frac{8 \tan(t) \sin(2t)}{2} = -8 \sin^2 t \tag{14.16}$$

---

**Example 14.2**

Plugging in $u_{1,2}$

$$u_2' = 4\frac{\sin(t)(2\cos^2(t) - 1)}{\cos(t)} = 4\sin(t)\left(2\cos(t) - \frac{1}{\cos(t)}\right) \qquad (14.17)$$

Then,

$$u_1 = 4\sin(t)\cos(t) - 4t + c_1 \qquad (14.18)$$

$$u_2 = 4\ln(\cos(t)) - 4\cos^2 t + c_2 \qquad (14.19)$$

Finally, a solution can be written as

$$y = -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos(2t))\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \qquad (14.20)$$

**Theorem 14.1**

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \qquad (14.21)$$

If $p, q, g$ are continuous on the open interval $I$; and the solutions to the corresponding HSOLDE satisfy $W[y_1, y_2] \not\equiv 0$. Then, a particular solution is given by

$$Y = y_2 \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\,ds - y_1 \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\,ds \qquad (14.22)$$

$\forall t_0 \in I$ the general solution is

$$y = c_1 y_1 + c_2 y_2 + Y \qquad (14.23)$$

**Proof 14.1**

Given

$$L[y] = y'' + py' + qy = g \qquad (14.24)$$

Where $p, q, g$ are continuous functions, in the case that $g = 0$, gives us a HSOLDE which has a general solution

$$y_c = c_1 y_1 + c_2 y_2 \qquad (14.25)$$

(Keep in mind that only CHSOLDEs, or cases where $p, q$ are constants have been discussed.) Substituting $c_{1,2}$ for $u_{1,2}$

$$y = u_1 y_1 + u_2 y_2 \qquad (14.26)$$

**Proof 14.1**

To ensure that the solution is one of the NSOLDE and not the HSOLDE, we take the derivative

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \tag{14.27}$$

Setting a restriction

$$u_1' y_1 + u_2' y_2 = 0 \tag{14.28}$$

Then, $y'$ can be simplified and be differentiated once more,

$$y' = u_1 y_1' + u_2 y_2' \tag{14.29}$$

$$y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \tag{14.30}$$

Then substituting for $y, y', y''$ yields

$$\begin{aligned} g = {} & u_1 \left( y_1'' + p y_1' + q y_1 \right) \\ & + u_2 \left( y_2'' + p y_2' + q y_2 \right) \\ & + u_1' y_1' + u_2' y_2' \end{aligned} \tag{14.31}$$

In the above equation, the lines with coefficients $u_{1,2}$ are equal to zero, as $y_{1,2}$ are solutions to the HSOLDE, giving

$$u_1' y_1' + u_2' y_2' = g \tag{14.32}$$

Then, solving the system gives

$$u_1' = -\frac{y_2 g}{W[y_1, y_2]}; \quad u_2' = \frac{y_1 g}{W[y_1, y_2]} \tag{14.33}$$

Note that because $y_{1,2}$ form a FSS, $W[y_1, y_2] \not\equiv 0$. Now to solve for $u_{1,2}$

$$u_1 = -\int \frac{y_2 g}{W[y_1, y_2]} \, dt + c_1; \quad u_2 = \int \frac{y_1 g}{W[y_1, y_2]} \, dt + c_2 \tag{14.34}$$

# Lecture 15

# 3 March 2020

## 15.1   Indefinite Integrals

**Definition 15.1 Improper Integrals**

$$\int_a^\infty f(t)\,dt = \lim_{A \to \infty} \int_a^A f(t)\,dt \tag{15.1}$$

If the limit exists then it is convergent, otherwise it is considered to be divergent

**Example 15.1**

Does the following integral converge?

$$\int_1^\infty t^{-1}\,dt \tag{15.2}$$

Solution:

$$\lim_{A\to\infty} \int_1^A (t^{-1})\,dt \tag{15.3}$$

$$\lim_{A\to\infty} \left(\ln(t)|_1^A\right) = \infty \tag{15.4}$$

Therefore, it is divergent.

**Example 15.2**

For what values does the following integral converge?

$$\int_0^\infty e^{ct}\,dt \tag{15.5}$$

$$= \lim_{A\to\infty} \int_0^A e^{ct}\,dt \tag{15.6}$$

$$= \lim_{A\to\infty} \frac{1}{c}e^{ct}\bigg|_0^A \tag{15.7}$$

$$= \lim_{A\to\infty} \left[\frac{1}{c}e^{ct} - \frac{1}{c}\right] \tag{15.8}$$

$$= \frac{1}{c}, c < 0 \tag{15.9}$$

**Example 15.3**

For what values of $p$ does the integral converge?

$$\int_1^\infty t^{-p}\,dt \tag{15.10}$$

$$= \lim_{A\to\infty} \int_1^A t^{-p}\,dt \tag{15.11}$$

$$= \lim_{A\to\infty} \frac{1}{1-p} t^{1-p}\Big|_1^A \tag{15.12}$$

$$= \lim_{A\to\infty} \frac{1}{1-p}\left(A^{1-p}-1\right), p \neq -1 \tag{15.13}$$

$$= \frac{1}{p-1}, p > 1 \tag{15.14}$$

## 15.2   Laplace Transformation

In algebra, we were introduced to the concept of factorization,

$$x^2 + 4x + 3 = 0 \tag{15.15}$$
$$(x+3)(x+1) = 0 \tag{15.16}$$
$$\implies x \in \{-2, -1\} \tag{15.17}$$

Factorization is useful because it was a tool to solve for the roots of a polynomial. The Laplace tranformation can also be thought of as a tool that would help in solving ODEs.

**Definition 15.2**

The **Laplace Transformation** is an integral transformation, given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{15.18}$$

**Theorem 15.1**

Suppose that

1. $f$ is piecewise continuous on the intervals $t \in [0, A], A \in \mathbb{R}^+$

2. $\exists (k, a, M), (K, M) > 0, |f(t)| \leq ke^{at}, t \geq M$

Then, $\forall s > a, F(s)$

> **Example 15.4**
>
> Find
> $$\mathcal{L}\{1\} = \frac{1}{s^2} \tag{15.19}$$

# 5 - 19 March 2020

Lectures on dates 5, 17, 19 March were cancelled.

# Lecture 16

# 24 March 2020

## 16.1 Laplacians

$$\mathcal{L} \circ f(t) \longrightarrow F(s), f : \mathbb{R} \to \mathbb{R} \tag{16.1}$$

Some uses for Laplacians include stochastic processes and probability.

Note: As the inverse Laplace transform $(\mathcal{L}^{-1})$ will not be introduced in this lecture, all steps involving $\mathcal{L}\{y\} \to y$ will be skipped in the examples in this lecture.

As defined previously,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$
$$\|$$
$$\|$$
$$F(s) = \lim_{x \to \infty} \int_0^x e^{-st} f(t)\, dt$$

---

**Example 16.1 Basic Examples**

$$f \equiv 0 \tag{16.2}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty (e^{-st})(0)\, dt \tag{16.3}$$

$$= 0 \tag{16.4}$$

---

**Example 16.1 Basic Examples**

$$f \equiv c \tag{16.5}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty ce^{-st}\,dt \tag{16.6}$$

$$= \lim_{x\to\infty} \int_0^x ce^{-st}\,dt \tag{16.7}$$

$$= \lim_{x\to\infty} -\frac{c}{s}e^{-st}|_0^x \tag{16.8}$$

$$= \lim_{x\to\infty} \left(\frac{1}{s}e^{-st}\right) \tag{16.9}$$

$$= cs^{-1} \tag{16.10}$$

**Theorem 16.1**

The Laplacian of a differential:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{16.11}$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \tag{16.12}$$

**Example 16.2**

Find the solution of the following using Laplacians.

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0 \tag{16.13}$$

Solution:

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\} \tag{16.14}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 \tag{16.15}$$

$$(s^2\mathcal{L}\{y\} - sy(0) - y'(0)) - (s\mathcal{L}\{y\} - y(0)) - 2\mathcal{L}\{y\} = 0 \tag{16.16}$$

$$(s^2 - s - 2)\mathcal{L}\{y\} - s + 1 = 0 \tag{16.17}$$

$$\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2} \tag{16.18}$$

$$= \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \tag{16.19}$$

$$\implies y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \tag{16.20}$$

**Example 16.3**

Solve using Laplacians.

$$y'' + y = \sin(2t), y(0) = 2, y'(0) = 1 \tag{16.21}$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\} \tag{16.22}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{2}{s^2 + 4} \tag{16.23}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2 + 1}\left(\frac{1}{s^2 + 4} + 2s + 1\right) \tag{16.24}$$

$$= \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1} + \frac{2}{3}\left(\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right) \tag{16.25}$$

The full solution would involve the inverse Laplacian of the last equation (**16.25**).

# Lecture 17

# 26 March 2020

## 17.1 Inverse Laplace Transform

> **Definition 17.1**
>
> If
> $$F(s) = \mathcal{L}\{f(t)\} \tag{17.1}$$
> then, the **inverse Laplace transform** is
> $$f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{17.2}$$

Addendum to the previous lecture, when solving for:

$$y'' + y = \sin(2t), y(0) = 2, y'(0) = 1 \tag{17.3}$$

We came to the point,

$$\mathcal{L}\{y\} = \frac{2s+1}{s^2+1} \cdot \frac{2}{(s^2+1)(s^2+4)} \tag{17.4}$$

So then,

$$y = \mathcal{L}^{-1}\{\frac{2s+1}{s^2+1} \cdot \frac{2}{(s^2+1)(s^2+4)}\} \tag{17.5}$$

### 17.1.1 Step Functions

For every function $f : \mathbb{R} \to \mathbb{R}$, there exists a graph. Most graphs we are accustomed to seeing are continuous. However, there exists step functions where there are a finite number of steps.

**Definition 17.2**

The unit step function can be defined as follows:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \tag{17.6}$$

**Example 17.1**

Consider

$$f(t) = \begin{cases} 2 & t \in [0, 4) \\ 5 & t \in [4, 7) \\ -1 & t \in [7, 9) \\ 1 & t \geq 9 \end{cases} \tag{17.7}$$

It can be written as the linear combination of multiple unit step functions:

$$f(t) = 2u_0(t) + 3u_4(t) - 6u_7(t) + 2u_9(t) \tag{17.8}$$

**Theorem 17.1**

$$\mathcal{L}\{u_c(t) \cdot f(t - c)\} = e^{-cs}\mathcal{L}\{f(t)\}, t > c \tag{17.9}$$

**Corollary 17.1**

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t - c) \tag{17.10}$$

**Example 17.2**

Compute the following

$$\mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s}\} \tag{17.11}$$

Note that

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \tag{17.12}$$

$$\mathcal{L}^{-1}\{e^{-2s} \cdot \frac{1}{s}\} = u_c(t) \cdot f(t - 2) \tag{17.13}$$

**Theorem 17.2**

$$\mathcal{L}\{e^{ct}f(t)\} = F(s - c) \tag{17.14}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{ct}\mathcal{L}^{-1}\{F(s + c)\} \tag{17.15}$$

Also, in the last lecture

$$\mathcal{L}\{y\} = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1} \tag{17.16}$$

Notice that in both fractions, it can be expressed as a linear combination of the following function

$$F(s) = \frac{1}{s} \tag{17.17}$$

rewriting the expression yields:

$$\mathcal{L}\{y\} = \frac{1}{3}F(s-2) + \frac{2}{3}F(s+1) \tag{17.18}$$

Now solving for y

$$y = \mathcal{L}^{-1}\left\{\frac{1}{3}F(s-2) + \frac{2}{3}F(s+1)\right\} \tag{17.19}$$

The inverse Laplacian can also be written as a linear combination

$$y = \mathcal{L}^{-1}\left\{\frac{1}{3}F(s-2)\right\} + \mathcal{L}^{-1}\left\{\frac{2}{3}F(s+1)\right\} \tag{17.20}$$

Remember to **Theorem 17.2**, we can adjust $F(s+c)$ back to $F(s)$:

$$y = \frac{1}{3}e^{2t}\mathcal{L}^{-1}\{F(s+2)\} + \frac{2}{3}e^{t}\mathcal{L}^{-1}\{F(s+1)\} \tag{17.21}$$

Now, notice that $\mathcal{L}^{-1}\{F(s)\} = 1$, and rearranging **Theorem 17.2**, we find that $\mathcal{L}^{-1}\{F(s+c)\} = e^{-ct}\mathcal{L}^{-1}\{F(s)\}$. If we apply this to our solution, we get the final solution:

$$y = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-t} \tag{17.22}$$

**Lecture 18**

**31 March 2020**

# Lecture 19

# 2 April 2020