

Quiz-1 answers and solutions

Coursera. Stochastic Processes

June 7, 2019

1 week quiz

1. Let η be a random variable with distribution function F_η . Define a stochastic process $X_t = \eta + t$. Compute the distribution function of a finite-dimensional distribution $(X_{t_1}, \dots, X_{t_n})$, where $t_1, \dots, t_n \in \mathbb{R}_+$:

Answer: $\mathbb{F}_\eta\{\min(x_1 - t_1, \dots, x_n - t_n)\}$

Solution: $\mathbb{F}_X(\vec{x}) = \mathbb{P}(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n) = \mathbb{P}(\eta + t_1 \leq x_1, \dots, \eta + t_n \leq x_n) = \mathbb{P}(\eta \leq x_1 - t_1, \dots, \eta \leq x_n - t_n) = \mathbb{F}_\eta\{\min(x_1 - t_1, \dots, x_n - t_n)\}$

2. Let S_n be a renewal process such that $\xi_n = S_n - S_{n-1}$ takes the values 1 or 2 with equal probabilities $p = 1/2$. Find the mathematical expectation of the counting process N_t at $t=3$:

Answer: 15/8

Solution:

We can calculate it without using the Laplace transform in this case:

$$\begin{aligned}\mathbb{E}(N_3) &= \sum_{k=0}^{\infty} k \cdot \mathbb{P}(N_3 = k) \\&= 0 + 1 \cdot \mathbb{P}(N_3 = 1) + 2 \cdot \mathbb{P}(N_3 = 2) + 3 \cdot \mathbb{P}(N_3 = 3) + 4 \cdot \mathbb{P}(N_3 = 4) \\&= 1 \cdot \mathbb{P}(\xi_1 = 2; \xi_2 = 2) + 2 \cdot (\mathbb{P}(\xi_1 = 1; \xi_2 = 2) + \mathbb{P}(\xi_1 = 2; \xi_2 = 1) + \\&\quad + \mathbb{P}(\xi_1 = 1; \xi_2 = 1, \xi_3 = 2)) + 3 \cdot \mathbb{P}(\xi_1 = 1; \xi_2 = 1; \xi_3 = 2) \\&= 1 \cdot \frac{1}{4} + 2 \cdot \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8}\right) + 3 \cdot \frac{1}{8} = \frac{1}{4} + 1\frac{1}{4} + \frac{3}{8} \\&= 15/8\end{aligned}$$

3. Let $S_n = S_{n-1} + \xi_n$ be a renewal process and $p_\xi(x) = \lambda e^{-\lambda x}$. Find the mathematical expectation of the corresponding counting process N_t :

Answer: none of the above

Solution:

step 1. $p \rightarrow \mathcal{L}_p$

$$\mathcal{L}_p(s) = \int_0^\infty e^{-sx} \cdot \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(s+\lambda)x} dx = -\frac{\lambda}{s+\lambda} (0 - 1) = \frac{\lambda}{s+\lambda}$$

step 2. $\mathcal{L}_p \rightarrow \mathcal{L}_u$

$$\mathcal{L}_u = \frac{\frac{\lambda}{s+\lambda}}{s \left(1 - \frac{\lambda}{s+\lambda}\right)} = \frac{\lambda}{s^2}$$

step 3. $\mathcal{L}_u \rightarrow u(t)$

Since $\mathcal{L}_u = \lambda \cdot \frac{1!}{s^{1+1}}$, then $u(t) = \lambda t$.

4. Let η be a random variable with distribution function F_η . Define a stochastic process $X_t = e^\eta t^2$. What is the distribution function of $(X_{t_1}, \dots, X_{t_n})$ for positive t_1, \dots, t_n ?

Answer: $\mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \dots, \ln(x_n/t_n^2))\}$

Solution:

$$\begin{aligned}\mathbb{F}_{\vec{X}}(\vec{x}) &= \mathbb{P}(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n) \\ &= \mathbb{P}(e^\eta t_1^2 \leq x_1, \dots, e^\eta t_n^2 \leq x_n) \\ &= \mathbb{P}(\eta \leq \ln(x_1/t_1^2), \dots, \eta \leq \ln(x_n/t_n^2)) \\ &= \mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \dots, \ln(x_n/t_n^2))\}.\end{aligned}$$

5. Let N_t be a counting process of a renewal process $S_n = S_{n-1} + \xi_n$ such that the i.i.d. random variables ξ_1, ξ_2, \dots have a probability density function

$$p_\xi(x) = \begin{cases} \frac{1}{2}e^{-x}(x+1), & x \geq 0, \\ x < 0. \end{cases}$$

Find the mean of N_t .

Answer: $-\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t}$

Solution:

Step 1. $p \rightarrow \mathcal{L}_p(s)$

$$\begin{aligned}\mathcal{L}_p(s) &= \int_0^\infty e^{-sx} \cdot \frac{1}{2}e^{-x}(x+1)dx \\ &= \frac{1}{2} \int_0^\infty e^{-(s+1)x} \cdot x dx + \frac{1}{2} \int_0^\infty e^{-(s+1)x} dx \\ &= -\frac{1}{2(s+1)} \int_0^\infty x d(e^{-(s+1)x}) - \frac{1}{2(s+1)}(0-1) \\ &= -\frac{1}{2(s+1)}(xe^{-(s+1)x}|_0^\infty - \int_0^\infty e^{-(s+1)x} dx) + \frac{1}{2(s+1)} \\ &= -\frac{1}{2(s+1)}(0 + \frac{e^{-(s+1)x}}{s+1}|_0^\infty) + \frac{1}{2(s+1)} \\ &= -\frac{1}{2(s+1)}(0 + \frac{0-1}{s+1}) + \frac{1}{2(s+1)} \\ &= \frac{1}{2(s+1)^2} + \frac{1}{2(s+1)} \\ &= \frac{s+2}{2(s+1)^2}.\end{aligned}$$

Step 2. $\mathcal{L}_p(s) \rightarrow \mathcal{L}_u(s)$

$$\begin{aligned}\mathcal{L}_u &= \frac{\frac{s+2}{2(s+1)^2}}{s \left(1 - \frac{s+2}{2(s+1)^2}\right)} \\ &= \frac{s+2}{s(2(s+1)^2 - s - 2)} \\ &= \frac{s+2}{s^2(2s+3)}.\end{aligned}$$

Step 3. $\mathcal{L}_u \rightarrow u$

$$\begin{aligned}\frac{s+2}{s^2(2s+3)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{2s+3} \\ &= \frac{A \cdot s(2s+3) + B \cdot (2s+3) + C \cdot s^2}{s^2(2s+3)} \\ &= \frac{s^2(2A+C) + s(3A+2B) + 3B}{s^2(2s+3)}.\end{aligned}$$

$$\begin{cases} 2A+C=0 \\ 3A+2B=1 \\ 3B=2 \end{cases} \rightarrow \begin{cases} C=2/9 \\ A=-1/9 \\ B=2/3 \end{cases}$$

$$L_u = -\frac{1}{9s} + \frac{2}{3s^2} + \frac{1}{9(s+\frac{3}{2})}$$

Therefore,

$$\mathbb{E}N_t = u(t) = -\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t}.$$

6. Let ξ and η be 2 random variables. It is known that the distribution of η is symmetric, that is, $\mathbb{P}\{\eta > x\} = \mathbb{P}\{\eta < -x\}$ for any $x > 0$, and moreover $\mathbb{P}\{\eta = 0\} = 0$. Find the probability of the event that the trajectories of stochastic process $X_t = \xi^2 + t(\eta + t)$, $t \geq 0$ increase:

Answer: $\frac{1}{2}$

Solution:

$$X_t = t^2 + \eta t + \xi^2$$

$$\mathbb{P}\left(\frac{d}{dt}X_t > 0 \forall t \geq 0\right) = \mathbb{P}(2t + \eta > 0 \forall t \geq 0) = \mathbb{P}(\eta > 0) = \frac{1}{2}$$