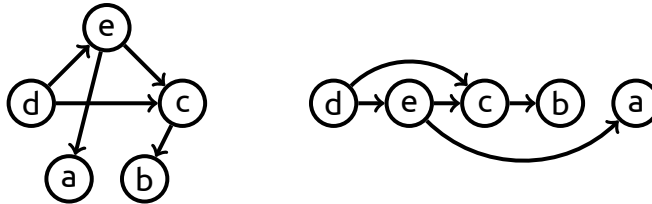


Week 1

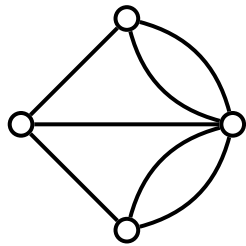
- A **graph** $G = (V, E)$ consists of the set of **vertices** V and the set of edges E .
- For an edge $e = \{u, v\}$, we say:
 - e **connects** u and v ;
 - u and v are **end points** of e ;
 - u and e are **incident** (v and e are **incident**);
 - u and v are **adjacent** or **neighbors**.
- The **degree** $\deg(v)$ of a vertex v is the number of edges incident to it. A vertex of degree 0 is called **isolated**.
- In a directed graph, the **indegree** (**outdegree**) of a vertex v is the number of edges ending at v (leaving v).
- The **degree of a graph** is the maximum degree of its vertex. A k -regular graph is a graph where each vertex has degree k .
- The **complement** of a graph $G = (V, E)$ is a graph $\bar{G} = (V, \bar{E})$ s.t. $(u, v) \in \bar{E}$ if and only if $(u, v) \notin E$.
- A **walk** in a graph is a sequence of edges, where each edge (except for the 1st one) starts with a vertex where the previous edge ended. The **length** of a walk is the number of edges in it.
- A **path** is a walk where all edges are distinct.
- A **simple path** is a walk where all vertices are distinct.
- A **cycle** in a graph is a path whose 1st vertex is the same as the last one.
- A **simple cycle** is a cycle where all vertices except for the 1st one are distinct. (And there 1st vertex is taken twice.)
- A graph is called **connected** if there is a path between every pair of its vertices.
- A **connected component** of a graph G is a maximal connected subgraph of G .
- The **path graph** P_n consists of n vertices v_1, \dots, v_n and $n - 1$ edges $\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}$.
- The **cycle graph** C_n consists of n vertices v_1, \dots, v_n and n edges $\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
- The **complete graph (clique)** K_n contains n vertices v_1, \dots, v_n and all $n(n - 1)/2$ edges between them.
- Three equivalent definitions of a **tree**:
 - a connected graph without cycles;
 - a connected graph on n vertices with $n - 1$ edges;
 - a graph with a unique simple path between any pair of its vertices.
- A graph G is **bipartite** if its vertices can be partitioned into two disjoint sets L and R s.t. every edge of G connects a vertex in L with a vertex in R .

Week 2

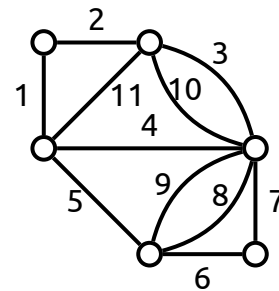
- **Degree sum formula:** for any undirected graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges: $\sum_{v \in V} \text{degree}(v) = 2 \cdot |E|$
- **Lower bound on the number of connected components:** an undirected graph $G(V, E)$ has at least $|V| - |E|$ connected components.
- A **directed acyclic graph (DAG)** is a directed graph without cycles.
- A **topological ordering** of a directed graph is an ordering of its vertices such that, for each edge (u, v) , u comes before v . Such an ordering exists, if and only if the graph is acyclic.



- An **Eulerian cycle (or path)** visits every edge exactly once.



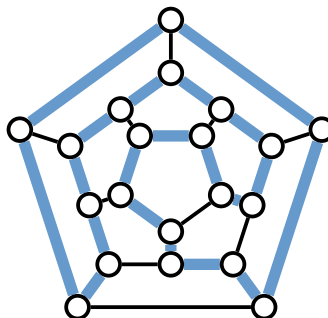
non-Eulerian graph



Eulerian graph

Criteria:

- A connected *undirected* graph contains an Eulerian cycle, if and only if the degree of every node is even.
- A strongly connected *directed* graph contains an Eulerian cycle, if and only if, for every node, its in-degree is equal to its out-degree.
- A **Hamiltonian cycle** visits every node exactly once.



Week 3

- A **spanning tree** of a graph G , is a subgraph of G which is a tree and contains all vertices of G .
- A **minimum spanning tree** of a weighted graph is a spanning tree of the smallest weight.
- **Kruskal's minimum spanning tree algorithm**:
 - Start with an empty graph T .
 - Repeat $n - 1$ times:
 - Add to T an edge of the smallest weight which doesn't create a cycle in T .
- A graph is **bipartite** if and only if it has no cycles of odd length.
- A **matching** in a graph is a set of edges without common vertices.
- A **maximal matching** is a matching which cannot be extended to a larger matching.
- A **maximum matching** is a matching of the largest size.
- If $G = (V, E)$ is a graph, and $S \subseteq V$ is its subset of vertices, then the **neighborhood** $N(S)$ of S is the set of all vertices connected to at least one vertex in S .
- **Hall's theorem**: In a bipartite graph $G = (L \cup R, E)$, there is a matching which covers all vertices from L if and only if for every subset of vertices $S \subseteq L$, $|S| \leq |N(S)|$.
- A graph is **planar** if it can be drawn in the plane such that its edges do not meet except at their end points.
- A **face** of a planar drawing of a graph is a region bounded by the edges of the graph. (There is always one infinitely large **outer** face.)
- **Euler's formula**: for a planar drawing of a connected planar graph: $v - e + f = 2$.
- Every planar graph has a vertex of degree ≤ 5 .
- In every planar graph on $n \geq 3$ vertices: $e \leq 3v - 6$.
- In every bipartite planar graph on $n \geq 4$ vertices: $e \leq 2v - 4$.

Week 4

- A **graph coloring** is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.
- The **chromatic number** $\chi(G)$ of a graph G is the smallest number of colors needed to color the graph.
- **Four color theorem**: every planar graph can be colored in 4 colors.
- **Brooks' theorem**: A graph G of maximum degree Δ can be colored with Δ colors, unless G is complete (K_n) or a cycle of odd length (C_{2k+1}).
- A **clique** of a graph is a set of vertices such that every two vertices are connected by an edge.
- A **maximal clique** is a clique which is not contained in any other clique.
- A **maximum clique** is a clique such that there are no cliques with more vertices.
- The **clique number** $\omega(G)$ of a graph G is the number of vertices in its maximum clique.
- An **independent set (IS)** of a graph is a set of vertices such that no two vertices are connected by an edge.
- A **maximal independent set** is an IS which is not contained in any other IS (i.e., cannot be extended to a larger IS).
- A **maximum independent set** is an IS such that there are no IS's with more vertices.
- The **independence number** $\alpha(G)$ of a graph G is the number of vertices in its maximum IS.
- $\omega(G) = \alpha(\bar{G})$.
- $\chi(G) \cdot \alpha(G) \geq n$.
- **Mantel's theorem**: A graph on n vertices without triangles has at most $\lfloor n^2/4 \rfloor$ edges.
- **Turán's theorem**: If a graph G on n vertices contains no K_{r+1} , then it has at most $(1 - \frac{1}{r}) \frac{n^2}{2}$ edges.
- For two integers k, ℓ , the **Ramsey number** $R(k, \ell)$ is the minimum number, s.t. every graph with at least $R(k, \ell)$ vertices must have either a clique of size k or an independent set of size ℓ .
- $R(3, 3) = 6$; $R(4, 4) = 18$; $43 \leq R(5, 5) \leq 48$.
- **Ramsey's theorem**: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$.
- A **vertex cover** of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C .
- A **minimal vertex cover** is a vertex cover which does not contain other vertex covers.
- A **minimum vertex cover** is a vertex cover of the smallest size.
- The size of a minimum vertex cover is denoted by $\beta(G)$.
- $\beta(G) + \alpha(G) = n$.
- **König's theorem**: in a bipartite graph, the number of edges in a **maximum matching** equals the number of vertices in a **minimum vertex cover**.