

MAE 5753 - TEORIA DA DECISÃO

2º semestre/2015

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PROGRAMA

- 1) Introdução: elementos de um problema de decisão
- 2) Probabilidade e utilidade: construção (coerência)
- 3) Maximização de utilidade esperada
- 4) Formas normal e extensiva de um problema de decisão (resultados importantes)
- 5) Exemplos em Inferencia Estatística
- 6)* Tópicos adicionais: Teoria da Decisão Coletiva, aplicações em Controle de Qualidade, testes simultâneos e decisões seqüenciais.

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Pasta 04 - Material interessante, listas de exercícios, ...

Teoria da Decisão

Aula 01

10/08/2015

- Como chegar a um procedimento ótimo de decisão (umas 4 semanas)
- Rebaixar paradigma de maximização de utilidade esperada
- Estudar três problemas () pela teoria da decisão.
- Talvez cont. qualidade, teor. da dec. coletiva

Contexto:

Você deve escolher uma alternativa (ação) dentro de um conjunto de decisões disponíveis \mathcal{D} .

Exemplos:

1. ~~•~~ Cursar Teoria da Decisão
2. Escolher as disciplinas do semestre
3. Decidir se vai ao jogo em Florianópolis
4. $\Theta = (\mu, \sigma^2)$ Estimar (obter uma estimativa para Θ)
5. Testar hipóteses

$$H_0: \mu = 0$$

$$H^1: \mu = 0, \sigma^2 = 1$$

$$H^2: \mu \neq 0$$

6. Estimar (por intervalo) μ

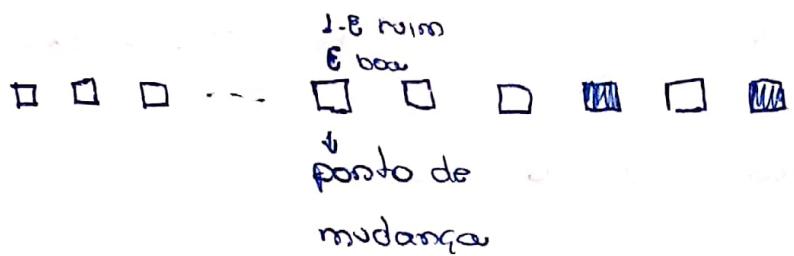
(A, B), A e B dependem dos dados

Serão desejáveis $B - A$ pequeno e A e B bem conectados

aos dados

7. Controle de qualidade

- Máquina que produz peças, uma ou vma;
- No inicio do proc. de prod., a máq. está bem regulada;
- Com o passar do tempo, em algum momento (ponto de mudança), essa máquina começa a alternar entre peças boas e ruins.



No fim de 90, Taguchi propôs um ~~exemplo~~ processo de monitoramento online.

- Tomo MEN e averiguo os peças de m em mr. Quando encontrar algo def., pare o processo.

→ intervalo entre inspeções

Problema: Qual deve ser m ?

, custos: inspeção, prod. defeituosa, manutenção

C_i (operador) C_d (defeito) C_r (reparo)

Até 50, apenas as Cartas de Controle. Taguchi foi original ao considerar custos operacionais.

8. Escolher a composição dos investimentos

Aspectos Importantes:

- (1) Potenciais aplicações
- (2) Aspectos matemáticos e relações com outras áreas (Análise convexa, Análise, Teoria dos Jogos, Pesquisa Operacional).

Teoria das Cons., etc.)

(3) Relação com Inferência estatística
(principalmente Inferência Bayesiana)

A incerteza, em geral, aparece em um problema de decisão.

Probabilidade \rightarrow Representações numéricas de incerteza (int. subjetivas)

Do ponto de vista mat., (Ω, \mathcal{F}, P) , $P: \mathcal{F} \rightarrow \mathbb{R}_+$ é uma prob. se

$$\text{i. } P(\Omega) = 1$$

$$\text{ii. } P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n), \text{ se } A_n's \text{ diss. 2 a 2.}$$

Exemplo 1:

$P: \mathcal{F} \rightarrow \mathbb{R}_+$, $\omega_0 \in \Omega$; $P(A) = I_A(\omega_0)$ é uma med. de prob. (medida de prob. degenerada em ω_0).

$P(\Omega) = I_{\Omega}(\omega_0) = 1$, uma vez que $\omega_0 \in \Omega$; Assim diss. $P\left(\bigcup A_n\right) = I_{\bigcup A_n}(\omega_0) = \sum_{n=1}^{\infty} I_{A_n}(\omega_0) = 1$.

Exemplo 2:

$P_1, P_2: \mathcal{F} \rightarrow [0, 1]$ prob.; $\omega \in [0, 1]$.

$P': \mathcal{F} \rightarrow \mathbb{R}_+$

$$A \mapsto P'(A) = \alpha P_1(A) + (1-\alpha) P_2(A) \quad (\text{medida mista})$$

mistura de P_1 e P_2 .

$$P'\left(\bigcup_{n=1}^{\infty} A_n\right) = \alpha P_1\left(\bigcup_{n=1}^{\infty} A_n\right) + (1-\alpha) P_2\left(\bigcup_{n=1}^{\infty} A_n\right) \stackrel{*}{=}$$

$$= \alpha \sum_{n=1}^{\infty} P_1(A_n) + (1-\alpha) \sum_{n=1}^{\infty} P_2(A_n)$$

$$= \sum_{n=1}^{\infty} \alpha P_1(A_n) + (1-\alpha) P_2(A_n)$$

* Uma vez que $P\left(\bigcup A_n\right) < 1$, segue que $\sum P(A_n)$ é conv.

$$\text{Dai, } \alpha \sum P(A_n) = \sum \alpha P(A_n)$$

$$= \sum_{n=1}^{\infty} P'(A_n)$$

Corint pessimista.

- V : $V \leq E$ (mais fraco no empate do que na vitória)
- E : $V \lesssim D$ (" " na derrota " " ")
- D. $V \sim E$ (V é equivalente à E)
- $V \ll E$ (acredito mais em V do que em E)

Algumas suposições. $V \ll E$, então $VUD \leq EUD$

↓ Teorema.

$$\exists P \text{ t.q. } A \ll B \Leftrightarrow P(A) \leq P(B)$$

Teoria das Decisões, Aula 01, Exercícios

1. Mostre que cada uma das medidas abaixo é uma medida de probabilidade sobre um dado espaço mensurável (Ω, \mathcal{F}) .

a) $P: \mathcal{F} \rightarrow \mathbb{R}_+$, $\omega_0 \in \Omega$
 $A \mapsto P(A) = \mathbb{I}_{\omega_0}(A)$

b) $P_1, P_2: \mathcal{F} \rightarrow [0, 1]$ prob. e $\alpha \in [0, 1]$. Definir

$$P': \mathcal{F} \rightarrow \mathbb{R}_+ \\ A \mapsto P'(A) = \alpha P_1(A) + (1-\alpha) P_2(A)$$

Por definição, dado um espaço $(\Omega, \mathcal{F}, \mu)$, dizemos que μ é uma medida de probabilidade se

1. $\mu(\Omega) = 1$;

2. Dada uma seq. $(A_n)_{n \in \mathbb{N}}$ de eventos disjuntos dois a dois de \mathcal{F} ,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Logo, basta verificar que P e P' satisfazem (1) e (2).

a) 1) $P(\Omega) = \mathbb{I}_{\omega_0}(\omega_0) = 1$, uma vez que $\omega_0 \in \Omega$

2) Dados $A_n \in \mathcal{F}$, $n \neq l$ com $A_n \cap A_l = \emptyset$ p/ $n \neq m$,

ou

$$\omega_0 \notin \bigcup_{n \neq l} A_n \Rightarrow P\left(\bigcup_{n \neq l} A_n\right) = \sum_{n \neq l} \mathbb{I}_{A_n}(\omega_0) = 0 \Rightarrow \omega_0 \in \bigcap_{n \neq l} A_n^c$$

$$\Rightarrow \omega_0 \in A_l^c \text{ e } n \neq l \Rightarrow \omega_0 \notin A_n \forall n \Rightarrow \mathbb{I}_{A_n}(\omega_0) = 0, \forall n \neq l$$

$$\Rightarrow \sum_{n \neq l} P(A_n) = \sum_{n \neq l} \mathbb{I}_{A_n}(\omega_0) = 0$$

ou $\omega_0 \in \bigcup_{m \neq l} A_m \Rightarrow P\left(\bigcup_{m \neq l} A_m\right) = 1$

* Além disso, \exists no t.q. $w_0 \in A_{n_0}$ e $w_0 \notin A_n \forall n \neq n_0$. Logo

$$\sum_{n \geq 1} P(A_n) = P(A_{n_0}) = 1$$

Logo, $P(\bigcup_{n \geq 1} A_n) = \sum_{n \geq 1} P(A_n)$.

* Sabemos que exi. ao menos um n.t.q. $w_0 \in A_n$. É porque, pois se $w_0 \in A_{n_0}$ e $w_0 \in A_n$ com $n \neq n_0$,

b). $P^*(\Omega) = \alpha P_1(\Omega) + (1-\alpha) P_2(\Omega)$ $w \in A_{n_0} \cap A_n = \emptyset$. (\Leftrightarrow)

$$= \alpha \cdot 1 + (1-\alpha) \cdot 1 = \alpha + 1 - \alpha = 1. \quad \checkmark$$

• Dada $(A_n)_{n \geq 1}$ com $A_n \cap A_m = \emptyset \forall n \neq m$, $n, m \in \mathbb{N}$,

$$\begin{aligned} P^*(\bigcup_{n \geq 1} A_n) &= \alpha P_1(\bigcup_{n \geq 1} A_n) + (1-\alpha) P_2(\bigcup_{n \geq 1} A_n) \\ &= \alpha \sum_{n \geq 1} P_1(A_n) + (1-\alpha) \sum_{n \geq 1} P_2(A_n) \\ &= \sum_{n \geq 1} \alpha P_1(A_n) + (1-\alpha) P_2(A_n) \\ &= \sum_{n \geq 1} P^*(A_n) \quad \checkmark \end{aligned}$$

* Como $P_i(\bigcup_{n \geq 1} A_n) \leq 1$, $\sum_{n \geq 1} P_i(A_n)$ é convergente, o que faz com que $\sum_{n \geq 1} P_i(A_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n P_i(A_k) = \sum_{n \geq 1} \alpha P_1(A_n)$, $i=1, 2$.
 pode ser 0 (Oximórfico prob.)

Teoria da Decisão

12/08/2015

Aula 02

Construção das prob. ou partir de ideias primitivas

$\Omega \rightarrow$ conjunto não-vazio representando as possíveis realizações

$\mathcal{F} \rightarrow$ classe de subconjuntos de Ω que satisfaz

$$(1) \emptyset \in \mathcal{F}$$

$$(2) A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$(3) (A_n)_{n \in \mathbb{N}}, A_n \in \mathcal{F}, \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

σ -álgebra de subconjuntos de Ω .

(Roussas, 197?)

(Berry-Samuel, 198?)

Vamos definir \lesssim em $\mathcal{F} \times \mathcal{F}$.

$A \lesssim B \Rightarrow$ "Acredito em B tanto quanto ou mais que A"

Ex: Corinthians

$$\Omega = \{v, e, d\}$$

$$A = \{v\} \quad \{v\} \lesssim \{e\}$$

$$B = \{e\} \quad \{e\} \lesssim \{d\}$$

$$C = \{d\}$$

Nosso objetivo é especificar condições sobre \lesssim de modo a representá-la por uma única medida de probabilidade.

Def: Dizemos que $f: \mathcal{F} \rightarrow \mathbb{R}$ "representa" ("coincide") com \lesssim se

$$\forall A, B \in \mathcal{F}, A \lesssim B \Leftrightarrow f(A) \leq f(B)$$

"Há várias construções desse tipo, inclusive uma em que se consideira simultaneamente prob. e utilidade, de onde também decorre o paradigma da 'otimiz. da utilidade esperada'".

- $A \sim B \Leftrightarrow A \leq B \wedge B \leq A$
- $B \prec A$ representa a negação de $A \leq B$.

Suposições

SP1 $\forall A, B, A \leq B \vee B \leq A$. (comparabilidade ou rel. completa)

SP2 A_1, A_2, B_1, B_2 com $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$

$$A_i \leq B_i, i=1,2 \Rightarrow A_1 \cup A_2 \leq B_1 \cup B_2$$

Se adicionalmente, $A_i \prec B_i$ para algum $i \Rightarrow A_1 \cup A_2 \prec B_1 \cup B_2$

Ex: Corinthians.

$$\begin{array}{ccc} A_1 & & A_2 \\ \{(1,0)\} & \leq & \{(0,1)\} \\ \bigcup_{n=2}^{\infty} \{(n,0)\} & \leq & \bigcup_{n=2}^{\infty} \{(0,n)\} \end{array} \Rightarrow \bigcup_{n=1}^{\infty} \{(n,0)\} \leq \bigcup_{n=1}^{\infty} \{(0,n)\}$$

$$\begin{array}{ccc} B_1 & & B_2 \\ & & \end{array}$$

Consequência:

$$A_1, \dots, A_n, B_1, \dots, B_n, A_i \cap A_j = B_i \cap B_j = \emptyset, i \neq j$$

$$A_i \leq B_i \Rightarrow \bigcup_{i=1}^n A_i \leq \bigcup_{i=1}^n B_i$$

Lemma. $A, B, C \in \mathcal{J}$ tais que $A \cap C = B \cap C = \emptyset$,

$$A \leq B \Leftrightarrow A \cup C \leq B \cup C$$

Dem:

→ Na sup. SP2, tomamos

$$A_1 = A, B_1 = B \Rightarrow A_{11} = B_{11} = C \Rightarrow A \cup C \leq B \cup C$$

$A \leq B$ (por SP2)
 $C \leq C$ (por SP1)

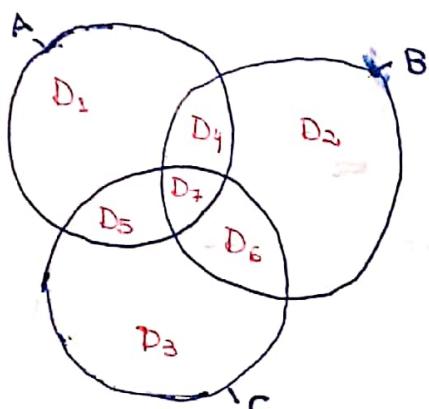
Suponha agora

$$B \subset A \xrightarrow{SP2} B \cup C \subset A \cup C$$

Logo, segue o resultado \square

Teorema. \leqslant é transitiva.

($A, B, C \in \mathcal{F}$ são tais que $A \leqslant B$ e $B \leqslant C$ então $A \leqslant C$)



$$A \leqslant B \Leftrightarrow D_1 \cup D_4 \cup D_5 \cup D_7 \leqslant D_2 \cup D_4 \cup D_6 \cup D_7$$

$$\Leftrightarrow (D_1 \cup D_5) \cup (D_4 \cup D_7) \leqslant (D_2 \cup D_6) \cup (D_4 \cup D_7) \xrightarrow{\text{Lema}} \quad \quad \quad$$

$$\Leftrightarrow (D_3 \cup D_5) \leqslant (D_2 \cup D_6) \quad (\text{I})$$

$$B \leqslant C \Leftrightarrow D_2 \cup D_4 \cup D_6 \cup D_7 \leqslant D_3 \cup D_5 \cup D_6 \cup D_7$$

$$\xrightarrow{\text{Lema}} D_2 \cup D_4 \leqslant D_3 \cup D_5 \quad (\text{II})$$

Pela SP2, segue que de I e II que

$$D_1 \cup D_2 \cup D_4 \cup D_5 \leqslant D_2 \cup D_3 \cup D_5 \cup D_6 \xrightarrow{\text{Lema}}$$

$$\Leftrightarrow D_1 \cup D_4 \cup D_5 \leqslant D_3 \cup D_5 \cup D_6 \quad \quad \quad$$

$$\Leftrightarrow D_1 \cup D_4 \cup D_5 \cup D_7 \leqslant D_3 \cup D_5 \cup D_6 \cup D_7 \quad \quad \quad$$

$$A \leqslant C.$$

Note que com as cond. acé aqui, \mathbb{F} admite uma representação.
Basta def. para cada $A \in \mathbb{F}$,

$$f(A) = |\{C \in \mathbb{F} : C \subseteq A\}|$$

$(A \subseteq B \Leftrightarrow f(A) \leq f(B))$, mas f não é uma prob.

Resultado:

$$A \subseteq B \Leftrightarrow B^c \subseteq A^c$$

$$\begin{aligned} A \subseteq B &\Leftrightarrow (A \cap B) \cup (A \cap B^c) \subseteq (A \cap B) \cup (A^c \cap B) \Leftrightarrow \\ &(A \cap B^c) \subseteq A^c \cap B \Leftrightarrow \\ &(A \cap B^c) \cup (A^c \cap B^c) \subseteq (A^c \cap B) \cup (A^c \cap B^c) \Leftrightarrow \\ &B^c \subseteq A^c \end{aligned}$$

SP3. $\forall A \in \mathbb{F}, \emptyset \subseteq A. (\emptyset \tau \Omega)$

Consequências:

1. $\forall A \in \mathbb{F}, A \subseteq \Omega$

Dem: De SP3, $\emptyset \subseteq A^c$. Como $\emptyset \cap A = \emptyset$ e $A^c \cap A = \emptyset$, segue do lema que $\emptyset \cup A \subseteq A^c \cup A \Leftrightarrow A \subseteq \Omega$

2. $A, B \in \mathbb{F} \& A \subseteq B, A \neq B$

Dem: Como $A \subseteq B$, podemos escrever

B = A \cup (A^c \cap B)

$$B = A \cup (A^c \cap B)$$

Da sup. SP3, temos

$$\emptyset \subseteq A^c \cap B \stackrel{\text{Lema}}{\Leftrightarrow} \emptyset \cup A \subseteq (A^c \cap B) \cup A \Leftrightarrow A \subseteq B.$$

SP4. $(A_n)_{n \geq 1}$ sequência de elementos de \mathbb{F} com $A_n \supseteq A_{n+1}$.

Seja $B \in \mathbb{F}$

$$B \subset A_n \Rightarrow B \subset \bigcap_{n=1}^{\infty} A_n$$

Consequência:

$(A_n)_{n \in \mathbb{N}}$ seq. com $A_n \subseteq A_{n+1}$, $A_n \subsetneq B$, $\forall n \in \mathbb{N}$. Logo

$$\bigcup_{n=1}^{\infty} A_n \subsetneq B.$$

Dem: É fácil ver que $(A_n^c)_{n \in \mathbb{N}}$ é tal que $A_n^c \supseteq A_{n+1}^c$. Além disso, $A_n \subsetneq B$, $\forall n \in \mathbb{N}$, segue do resultado anterior que $B^c \not\subseteq A_n^c$, $\forall n \in \mathbb{N}$.

Da SP4,

$$B^c \subsetneq \bigcap_{n \in \mathbb{N}} A_n^c$$

Usando o último resultado por uma vez, segue pelas Leis de De Morgan que $(\bigcap_{n \in \mathbb{N}} A_n^c)^c \subsetneq (B^c)^c \Leftrightarrow$

$$\bigcup_{n \in \mathbb{N}} A_n \subsetneq B$$

Resultado.

1. $(A_n)_{n \in \mathbb{N}}$ e $(B_n)_{n \in \mathbb{N}}$ sequências em \mathcal{F} tais que $A_i \cap A_j = B_i \cap B_j = \emptyset$, $i \neq j$.
Se $A_n \subsetneq B_n$, $\forall n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \subsetneq \bigcup_{n=1}^{\infty} B_n$

2. Se, adicionalmente, $\exists n_0 \in \mathbb{N}$ t.q.

$$A_{n_0} \subset B_{n_0}$$

então $\bigcup_{n=1}^{\infty} A_n \subsetneq \bigcup_{n=1}^{\infty} B_n$.

Dem.

(1) Da extensão de SP2, segue que, $\forall n \in \mathbb{N}$,

$$\bigcup_{i=1}^n A_i \subsetneq \bigcup_{i=1}^n B_i \subsetneq \bigcup_{n=1}^{\infty} B_n, \text{ pois } \forall n \in \mathbb{N} \quad \bigcup_{i=1}^n B_i \subset \bigcup_{n=1}^{\infty} B_n.$$

Logo, pela transitividade, $\bigcup_{i=1}^{\infty} A_i \subsetneq \bigcup_{i=1}^{\infty} B_i \subsetneq \bigcup_{n=1}^{\infty} B_n \subsetneq B$.

$$\bigcup_{i=1}^{\infty} A_i \subsetneq \bigcup_{i=1}^{\infty} B_i, \quad \forall n \in \mathbb{N}.$$

Da conseq. da SP4, segue que

$$\bigcup_{n \in \omega} A_n \lesssim \bigcup_{n \in \omega} B_n$$

(2) Suponhamos $A_{n_0} \subset B_{n_0}$, $n_0 \in \mathbb{N}^*$.

Da suposição SP2

$$\bigcup_{i=1}^{n_0-1} A_i \lesssim \bigcup_{i=1}^{n_0-1} B_i \Rightarrow \bigcup_{i=1}^{n_0} A_i \lesssim \bigcup_{i=1}^{n_0} B_i$$

$$A_n \subset B_{n_0}$$

Da prim. parte do resultado, vale que $\bigcup_{n=n_0+1}^{\infty} A_n \lesssim \bigcup_{n=n_0+1}^{\infty} B_n$. Então de SP2 novamente,

$$\left(\bigcup_{i=1}^{n_0} A_i \right) \cup \left(\bigcup_{i=n_0+1}^{\infty} A_i \right) \subset \left(\bigcup_{i=1}^{n_0} B_i \right) \cup \left(\bigcup_{i=n_0+1}^{\infty} B_i \right)$$

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$$

Exemplos:

1: $\Omega \neq \emptyset$, $\omega_0 \in \Omega$, \exists o-álg. de subconjuntos de Ω .

Para $A, B \in \mathcal{F}$

$$A \lesssim B \Leftrightarrow \begin{cases} \omega_0 \in B, \text{ ou} \\ \omega_0 \notin B \text{ e } \omega_0 \notin A \end{cases} \quad (\text{Atende SP1, 2, 3, 4})$$

2: $\Omega = \mathbb{N}$

$\mathcal{F} = \mathcal{P}(\mathbb{N})$

$A \in \mathcal{F}, B \in \mathcal{F}$

$$A \lesssim B \Leftrightarrow \begin{cases} B \text{ infinito} \\ B \text{ finito, } |A| \leq |B| \end{cases}$$

Teoria da Decisão, Aula 02, Exercícios

1. Construção da prob. a partir de ideias primitivas
2. Estabeleço Ω como conjunto dos resultados possíveis e \mathcal{F} como uma coleção de eventos de Ω satisfazendo 3 condições (as que def. uma σ -álgebra).
3. A partir disso, defino uma relação \leq em $\mathcal{F} \times \mathcal{F}$, tal que $A \leq B$ indica maior crença em A do que em B .
4. Objetivo é especificar condições sobre \leq de modo a representá-la por uma única medida de prob. Para isso, adianta-se a definição

Def: Dizemos que $f: \mathcal{F} \rightarrow \mathbb{R}$ "representa" (\leq) com \leq se $\forall A, B \in \mathcal{F}, A \leq B \Leftrightarrow f(A) \leq f(B)$.

5. Suposições e Consequências

Antes, mais algumas notações

$$A \cap B \leftrightarrow A \leq B \text{ e } B \leq A$$

$B \subset A$ repres. a negação de $A \leq B$.

SP1. $\forall A, B, A \leq B$ ou $B \leq A$ (compatibilidade ou rel. completa)

SP2. A_1, A_2, B_1, B_2 com $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$,

$$A_i \leq B_i, i=1, 2 \Rightarrow A_1 \cup A_2 \leq B_1 \cup B_2$$

Se, adicionalmente, $A_i \subset B_i$ para algum i , $A_1 \cup A_2 \subset B_1 \cup B_2$.

Dúvida: Se $A_i \leq B_i$, $i=1,2$, $A_1 \cup A_2 \leq B_1 \cup B_2$

$A_1 \cup A_2$ ocorre se ocorre A_1
 A_2
 $A_1 \cap A_2$

$B_1 \cup B_2$ ocorre se ocorre B_1
 B_2
 $B_1 \cap B_2$

$A_1 \leq B_1$, $A_2 \leq B_2$, $A_1 \cap A_2 \leq B_1 \cap B_2$

Exemplo: Longo um dado. Def. $A \leq B$ se $|A| \leq |B|$, $\forall A, B \in \mathbb{S}$.

$$A_1 = \{1\}, A_2 = \{2, 3\}$$

$$B_1 = \{3, 6\}, B_2 = \{1, 4, 5\}$$

$$A_1 \leq B_1; A_2 \leq B_2 \Rightarrow A_1 \cup A_2 \leq B_1 \cup B_2$$

$$A_1 \cap A_2 = \{1\}, B_1 \cap B_2 = \emptyset$$

Conseq. SP2. $A_1, \dots, A_n; B_1, \dots, B_n$, $A_i \cap A_j = B_i \cap B_j = \emptyset$, $i \neq j$,

$$A_i \leq B_i \Rightarrow \bigcup_{i=1}^n A_i \leq \bigcup_{i=1}^n B_i$$

Para $n=1$, vale pois $A_1 \leq B_1$ e

$$\bigcup_{i=1}^n A_i = A_1 \text{ e } \bigcup_{i=1}^n B_i = B_1$$

Para $n=2$, também, pois $A_i \leq B_i$, $i=1, 2$, que garante pelo SP2 que

$$A_1 \cup A_2 \leq B_1 \cup B_2$$

Supondo válido para $n=k$, segue que $\bigcup_{i=1}^k A_i \leq \bigcup_{i=1}^k B_i$. Ademais, sabe-se que $A_{k+1} \leq B_{k+1}$. Pela SP2, segue

$$\bigcup_{i=1}^{k+1} A_i \leq \bigcup_{i=1}^{k+1} B_i$$

Por indução, segue o resultado.

Lema (~~sempre verdadeiro~~) $A, B, C \in \mathcal{F}$ t.q. $A \cap C = B \cap C = \emptyset$.

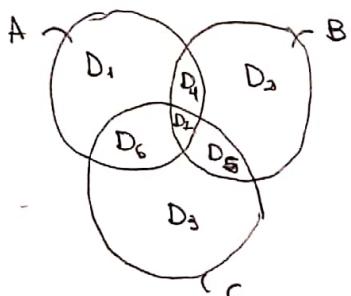
$$A \subsetneq B \Leftrightarrow A \cup C \subsetneq B \cup C.$$

Dem: Na SP2, tome $A_1 = A$, $B_1 = B$, $A_2 = B_2 = C$. Daí, segue o resultado, uma vez que por SP1 $C \subsetneq C$.

Se $B \subsetneq A$, pela SP2, $B \cup C \subsetneq A \cup C$.

Teorema. \subsetneq é transitiva.

$A, B, C \in \mathcal{F}$ t.q. $A \subsetneq B$, $B \subsetneq C$. Então $A \subsetneq C$.



$$D_1 = A \cap B^c \cap C^c$$

$$D_2 = B \cap A^c \cap C^c$$

$$D_3 = C \cap A^c \cap B^c$$

$$D_4 = A \cap B \cap C^c$$

$$D_5 = B \cap C \cap A^c$$

$$D_6 = A \cap C \cap B^c$$

$$D_7 = A \cap B \cap C$$

$$A \subsetneq B \Rightarrow (D_1 \cup D_5) \subsetneq (D_2 \cup D_6)$$

$$B \subsetneq C \Rightarrow D_2 \cup D_4 \subsetneq D_3 \cup D_5$$

↓ SP2 (adiciono eventos corretos até obter o desejado)
retirar

Res: $A \subsetneq B \Leftrightarrow B^c \subsetneq A^c$.

SP3. $\forall A \in \mathcal{F}, \emptyset \subsetneq A \quad (\emptyset \subsetneq \Omega)$

Conseq.

1. $\forall A \in \mathcal{F}, A \subsetneq \Omega$. ($\emptyset \subsetneq A^c \stackrel{\text{SP3}}{\Rightarrow} A \subsetneq A^c \cup A \Rightarrow A \subsetneq \Omega$)

2. $A, B \in \mathcal{F}, A \subseteq B \nRightarrow A \subsetneq B$. ($\emptyset \subsetneq A^c \cap B \stackrel{\text{SP3}}{\Rightarrow} A \subsetneq A^c \cup \overbrace{A^c \cap B}^B$)

SP4. $(A_n)_{n \geq 1}$ seq. el. de \mathcal{F} com $A_n \supseteq A_{n+1}$. Seja $B \in \mathcal{F}$

$$B \subseteq A_n \Rightarrow B \subseteq \bigcap_{n=1}^{\infty} A_n$$

Conseq.

$(A_n)_{n \in \mathbb{N}}$ com $A_n \subseteq A_{n+1}$, $A_n \subseteq B \forall n \in \mathbb{N}$. Segue que $\bigcup_{n \in \mathbb{N}} A_n \subseteq B$.

Dem: $(A_n^c)_{n \in \mathbb{N}}$ é t.q. $A_n^c \supseteq A_{n+1}^c$. Ademais, $A_n^c \subseteq B^c \forall n \in \mathbb{N}$. Segue do Recorr. que $B^c \subseteq A_n^c \forall n \in \mathbb{N}$. Da SP4,

$$B^c \subseteq \bigcap_{n \in \mathbb{N}} A_n^c$$

Usando o ult. res. mais uma vez, segue $(\bigcap_{n \in \mathbb{N}} A_n^c)^c \subseteq (B^c)^c \Leftrightarrow \bigcup_{n \in \mathbb{N}} A_n \subseteq B$.

C...

Exemplos:

1.

SP1. $A \subseteq B$ ou $B \subseteq A$

SP2. $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$ com $A_i \subseteq B_i \Rightarrow A_1 \cup A_2 \subseteq B_1 \cup B_2$

SP3. $\emptyset \subseteq A$ ($\emptyset \subseteq \subseteq$)

SP4. $(A_n)_{n \in \mathbb{N}}$ em \mathcal{F} com $A_n \supseteq A_{n+1} \in \mathcal{F}$.

$$B \subseteq A_n \forall n \Rightarrow B \subseteq \bigcap_{n=1}^{\infty} A_n$$

SP5. Dados A e $B \in \mathcal{F}$,

Se $w_0 \in B$, $A \subseteq B$

Se $w_0 \notin B$, $B \subseteq A$

SP6. Dados $A_i, B_i, i=1, 2$, com $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$ e $A_i \subseteq B_i$,

ou $w_0 \in B_1 \cup B_2 \Rightarrow A_1 \cup A_2 \subseteq B_1 \cup B_2$;

ou $w_0 \notin B_1 \cup B_2 \Rightarrow w_0 \notin B_1 \in w_0 \notin B_2 \Rightarrow w_0 \notin A_1 \in w_0 \notin A_2$
uma vez que $A_i \subseteq B_i, i=1, 2 \Rightarrow w_0 \notin A_1 \cup A_2 \Rightarrow$

$$A_1 \cup A_2 \subseteq B_1 \cup B_2$$

SP3. Como $w_0 \notin \emptyset$, $\forall A \in \mathcal{F}$, segue da def. das relações que

$$\emptyset \subseteq A \quad (\#)$$

pois se $w_0 \in A$, vole $\emptyset \subseteq A$ e se $w_0 \notin A$, temos $w_0 \in \emptyset$, o que ainda garante $(\#)$.

SP4. $(A_n)_{n \in \mathbb{N}}$ com $A_n \supseteq A_{n+1} \in \mathcal{F}$

$$B \cap A_n \neq \emptyset \Rightarrow B \cap \bigcap_{n=1}^{\infty} A_n$$

$$\text{ou } w_0 \notin B \Rightarrow B \cap \bigcap_{n=1}^{\infty} A_n;$$

$$\text{ou } w_0 \notin B \Rightarrow w_0 \in A_n, \forall n \in \mathbb{N} \Rightarrow w_0 \in \bigcap_{n \in \mathbb{N}} A_n \Rightarrow B \cap \bigcap_{n \in \mathbb{N}} A_n.$$

2. $\Omega = \mathbb{N}$, $\mathcal{I} = \mathcal{P}(\mathbb{N})$, $A \in \mathcal{B} \in \mathcal{F}$ $A \subsetneq B \Leftrightarrow \begin{cases} B \text{ inf.} \\ B \text{ fin.}, A \text{ fin.}, |A| \leq |B| \end{cases}$

SP1. Dados $A, B \in \mathcal{P}(\mathbb{N})$, ou

$$\bullet A, B \text{ finitos} \Rightarrow \begin{cases} A \subsetneq B \text{ se } |A| < |B| \\ B \subsetneq A \text{ se } |A| > |B| \end{cases}$$

$$\text{ou, } B \text{ infinito} \Rightarrow A \subsetneq B$$

$$\bullet A \text{ infinito} \Rightarrow B \subsetneq A$$

SP2. Dados $A_i \subsetneq B_i$, $i=1, 2$, com $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$,

$$\bullet B_1 \cup B_2 \text{ infinito} \Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2, \quad \checkmark$$

$$\bullet B_1 \cup B_2 \text{ finito} \Rightarrow B_1 \text{ e } B_2 \text{ são finitos} \stackrel{\text{hipótese}}{\Rightarrow} A_1 \text{ e } A_2 \text{ são finitos} \\ \text{tais que } |A_1| \leq |B_1| \text{ e } |A_2| \leq |B_2| \Rightarrow$$

$$|A_1| + |A_2| \leq |B_1| + |B_2| \Rightarrow |A_1 \cup A_2| \leq |B_1 \cup B_2| \\ \downarrow \\ \text{diss.} \Rightarrow$$

$$\Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2$$

SP3. Uma vez que $|\emptyset| = 0$, segue que

$$|\emptyset| \leq |A|, \forall A \in \mathcal{F} \Rightarrow \emptyset \subsetneq A, \forall A \in \mathcal{F}.$$

SP4. $(A_n)_{n \in \mathbb{N}}$ com $A_0 \supseteq A_{0+1} \supseteq B$ em \mathcal{F} .

$$B \subsetneq A_n \quad \forall n \Rightarrow B \subsetneq \bigcap_{n \in \mathbb{N}} A_n$$

Seja $B = \overline{\mathbb{N}}$ e $A_n = \overline{\mathbb{N}} / \{i_0, \dots, i_{n-1}\}$. Nessas condições, temos

$$B \subsetneq A_n, \quad \forall n \in \mathbb{N}$$

Mas $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} \{i_0, \dots\} = \{\infty\}$. Daí, resulta:

$$B \subsetneq \{\infty\}, \text{ ao que contradiz SP4.}$$

Teoria da Decisão

Aula 08

17/08/15

SP1. $A, B \in \mathcal{F}$, $A \setminus B \text{ ou } B \setminus A$

SP2. $A_1, A_2, B_1, B_2 \in \mathcal{F}$ t.q. $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$ e $A_i \not\subseteq B_i, i=1,2$,

$$A_1 \cup A_2 \not\subseteq B_1 \cup B_2$$

SP3. $\emptyset \not\subseteq A$ ($\emptyset \subsetneq A$)

SP4. $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ com $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}$ e $B \in \mathcal{F}$ t.q.

$$B \not\subseteq A_n, \forall n \in \mathbb{N} \Rightarrow B \not\subseteq \bigcap_{n=1}^{\infty} A_n$$

Ex.

$\Omega \neq \emptyset, \omega_0 \in \Omega$

$$A \setminus B \Leftrightarrow \begin{cases} \omega_0 \in B, \text{ ou} \\ \omega_0 \notin B, \omega_0 \in A. \end{cases}$$

• SP1.

Se $\omega_0 \in A \cup B \Rightarrow A \setminus B \text{ ou } B \setminus A$

$\omega_0 \notin A \cup B \Rightarrow A \setminus B \text{ e } B \setminus A$

• SP2.

(J) $\exists i=1,2$ tal que $\omega_0 \in B_i \Rightarrow \omega_0 \in B_1 \cup B_2 \Rightarrow A_1 \cup A_2 \not\subseteq B_1 \cup B_2$

$\omega_0 \in B_i, i=1,2 \Rightarrow \omega_0 \notin B_1, \omega_0 \notin B_2, \omega_0 \notin A_1 \text{ e } \omega_0 \notin A_2$

$\Rightarrow \omega_0 \notin B_1 \cup B_2 \text{ e } \omega_0 \notin A_1 \cup A_2$

$\Rightarrow A_1 \cup A_2 \not\subseteq B_1 \cup B_2$

SP3

se $w_0 \in A \Rightarrow \emptyset \subsetneq A$

se $w_0 \in A \Rightarrow \emptyset \subsetneq A$; e se $w_0 \notin A$, segue que $\emptyset \subsetneq A$ uma vez que $w_0 \notin \emptyset$. (por conct.)

SP4

- $w_0 \in A_n, \forall n \in \mathbb{N} \Rightarrow w_0 \in \bigcap_{n \in \mathbb{N}} A_n \Rightarrow B \subsetneq \bigcap_{n \in \mathbb{N}} A_n$
- $\exists n_0 \in \mathbb{N}$ t.q. $w_0 \notin A_{n_0} \Rightarrow w_0 \notin \bigcap_{n \in \mathbb{N}} A_n \in w_0 \notin B \Rightarrow B \subsetneq \bigcap_{n \in \mathbb{N}} A_n$

Ex. 2.

$$\Omega = \mathbb{N}, \mathcal{I} = \mathfrak{P}(\mathbb{N})$$

$$A \subsetneq B \Leftrightarrow \begin{cases} B \text{ infinito} \\ A \text{ finito e } |A| \leq |B| \end{cases}$$

SP1

(1) Se A ou B é infinito $\Rightarrow A \subsetneq B$ ou $B \subsetneq A$.

(2) A e B finitos $\Rightarrow |A| \leq |B|$ ou $|B| \leq |A| \Rightarrow A \subsetneq B$ ou $B \subsetneq A$.

SP2

(1) $B_1 \cup B_2$ infinito $\Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2$,

(2) B_1 e B_2 finitos $\Rightarrow \begin{cases} B_1 \cup B_2 \text{ é finito,} \\ A_1 \text{ é finito, } |A_1| \leq |B_1| \\ A_2 \text{ é finito, } |A_2| \leq |B_2| \end{cases} \Rightarrow$

$\Rightarrow B_1 \cup B_2$ é finito, $A_1 \cup A_2$ é finito e $|A_1 \cup A_2| \leq |B_1 \cup B_2|$

$\Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2$.

SP3. A é infinito $\Rightarrow \emptyset \subsetneq A$

A é finito $\Rightarrow 0 \leq |A| < \infty \Rightarrow \emptyset \subsetneq A$.

SP4

$$B = \{2\}$$

$$A_n = \{n, n+1, n+2, \dots\}$$

Claro que $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}$.

Como A_n é infinito para todo $n \in \mathbb{N}$, segue que $B \subsetneq A_n, \forall n \in \mathbb{N}$.

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset.$$

Note que $\bigcap_{n=1}^{\infty} A_n$ é finito, B é finito e $|\bigcap_{n=1}^{\infty} A_n| = 0 < L = |B|$.

Como $\bigcap_{n=1}^{\infty} A_n \subset B$, não vale SP4!

Pergunta. Se valem SP1 e SP2, então

$$A \subset B \Leftrightarrow B^c \subset A^c, \text{ isto é,}$$

$$A \subset B \Rightarrow B^c \supset A^c.$$

Tomando $B = \mathbb{N}$ e $A = \{2n+1, n \geq 0\}$, segue

$$B \subset A \Rightarrow A^c \subset B^c \Rightarrow \{2n, n \geq 0\} \neq \emptyset (?)$$

(Se aceito SP1, SP2, devo aceitar SP3?)

SP5

Existe $X: \mathbb{I}_1 \rightarrow \mathbb{R}$.

$$(1) X(\omega) \in [0, 1], \forall \omega \in \mathbb{I}_1$$

(2) I_1 e I_2 são intervalos subconjugados de $[0, 1]$,

$$\{X \in I_1\} \cap \{X \in I_2\} \Leftrightarrow \lambda(I_1) \leq \lambda(I_2)$$

Tal transformação X é denominada UNIFORME.

Observações.

(1) $\{x \in [a_1, b_1]\} \cap \{x \in [a_2, b_2]\} \Leftrightarrow b_1 - a_1 \leq b_2 - a_2$

(2) $a, b \in [0, 1]$ com $a \leq b$.

$$\{x \in [a, b]\} \cap \{x \in (a, b]\} \cap \{x \in [a, b)\} \cap \{x \in (a, b)\}$$

Teorema. Suponha que \mathcal{F} atende SP1 ou SP5. Então, para todo $A \in \mathcal{F}$, existe um único $a^* \in [0, 1]$ t.q.

$$A \cap \{x \in [0, a^*]\}$$

Dem: Seja $A \in \mathcal{F}$. Definimos $U(A) = \{a \in [0, 1] : A \cap \{x \in [0, a]\} \neq \emptyset\}$.

Note que $U(A) \neq \emptyset$, pois $\exists \in U(A)$ uma vez que $A \cap \Omega = \{x \in [0, 1]\}$.

por um lado,

Seja $a^* = \inf U(A)$. \checkmark Existe $(a_n)_{n \in \mathbb{N}}$ tal que $a_n > a_{n+1}$, $\forall n \in \mathbb{N}$ e $a_n \in U(A)$, $\forall n \in \mathbb{N}$ e $\lim_{n \rightarrow \infty} a_n = a^*$ ($a_n \downarrow a^*$).

Se $a_n \in U(A)$, então $A \cap \{x \in [0, a_n]\} \neq \emptyset$.

Logo, $\forall n \in \mathbb{N}$, $A \cap \{x \in [0, a_n]\} \neq \emptyset$ e, além disso,

$$\{x \in [0, a_n]\} \supseteq \{x \in [0, a_{n+1}]\}$$

Refa SP4,

$$A \cap \bigcap_{n=1}^{\infty} \{x \in [0, a_n]\} \Rightarrow$$

$A \cap \bigcap_{n=1}^{\infty} \{x \in [0, a_n]\} \neq \emptyset$, ou ainda

$$A \cap \{x \in [0, a^*]\}. \quad (\text{I})$$

Supon. $a^* > 0$

Por outro lado, seja $(b_n)_{n \in \mathbb{N}}$ seq. tal que $b_n < b_{n+1}$, $\forall n \in \mathbb{N}$, $b_n \subset a^*$, $\forall n$

$$\text{e } \lim_{n \rightarrow \infty} b_n = a^*.$$

$$\{x \in [0, b_n]\} \subset A, \forall n \geq 1$$

$$\{x \in [0, b_n]\} \not\subset A, \forall n \geq 1$$

$$\{x \in [0, b_n]\} \subseteq \{x \in [0, b_{n+1}]\}.$$

Da conseq. da SP4 para seq. "crecentes", segue que

$$\bigcup_{n=1}^{\infty} \{x \in [0, b_n]\} \subset A \Rightarrow \xrightarrow{\text{b}_n \text{ bairro de } a^*} a^*$$

$$\Rightarrow \{x \in \bigcup_{n=1}^{\infty} [0, b_n]\} \subset A \Rightarrow$$

$$\Rightarrow \{x \in [0, a^*]\} \subset A \Rightarrow \{x \in [0, a^*]\} \not\subset A \quad (\text{II})$$

Uniformidade

De (I) e (II), $A \cap \{x \in [0, a^*]\}$.

Unicidade de a^* . Sejam α_1 e α_2 tais que

$$A \cap \{x \in [0, \alpha_1]\} = A \cap \{x \in [0, \alpha_2]\} \Rightarrow$$

$$\{x \in [0, \alpha_1]\} \sim \{x \in [0, \alpha_2]\} \Leftrightarrow$$

$$\begin{aligned} & \alpha_1 - 0 \leq \alpha_2 - 0 \\ \text{e} \quad & \alpha_1 - 0 \neq \alpha_2 - 0 \end{aligned} \Rightarrow \alpha_1 = \alpha_2.$$

□

Vamos, finalmente, definir uma transformação

$P: \mathcal{F} \rightarrow \mathbb{R}_+$ que "representa" (coincide com) \subseteq

$P: \mathcal{F} \rightarrow \mathbb{R}_+$

$A \in \mathcal{F} \mapsto P(A) = a^*$, onde a^* é tal que $A \cap \{x \in [0, a^*]\}$.

Dease modo,

$$A \cap \{x \in [0, P(A)]\}.$$

Teorema. $P : \mathcal{F} \rightarrow \mathbb{R}_+$ definida acima representa a \mathbb{P} .

Para $A, B \in \mathcal{F}$, $\exists P(A), P(B) \neq 0$ tais que

$$A \cap \{x \in [0, P(A)]\}$$

$$B \cap \{x \in [0, P(B)]\}$$

$$A \neq B \Leftrightarrow \{x \in [0, P(A)]\} \neq \{x \in [0, P(B)]\} \Leftrightarrow P(A) \neq P(B).$$

Teoria da Decisão

Aula 04

19/08/15

SP2 $A_1, A_2, B_1, B_2 \in \mathcal{F}$ com $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$.

$A_i \cap B_i, i=1,2$, $A_1 \cup A_2 \subsetneq B_1 \cup B_2$

caso $A_1 \cap B_2 \Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2$

Resultado

$\forall A \in \mathcal{F}, \exists 1 \alpha^* \in [0,1] \text{ tal que}$

$$A \cap \{X \in [0, \alpha^*]\}$$

Vemos que $A \subsetneq B \Leftrightarrow P(A) < P(B)$.

$$\perp) A \subsetneq B \Leftrightarrow \begin{cases} w_0 \in B \\ w_0 \notin B \in w_0 \notin A \end{cases}$$

$$A \subsetneq B \Leftrightarrow \begin{cases} B \text{ inf.} \\ B \text{ fini., } A \text{ finito, } |A| \leq |B| \end{cases}$$

1. $A_2 \subsetneq B_2 \Leftrightarrow w_0 \notin A_2, w_0 \in B_2 \Rightarrow w_0 \notin B_2 \Rightarrow w_0 \notin A_2$

$$B_1 \cap B_2 = \emptyset$$

$$A_1 \subsetneq B_1$$

$\Rightarrow w_0 \notin A_1 \cup A_2 \Rightarrow A_1 \cup A_2 \subsetneq B_1 \cup B_2$

$$w_0 \in B_1 \cup A_2$$

$A_1 \cup A_2 \subsetneq B_1 \cup B_2$ não

$B_1 \cup B_2 \subsetneq A_1 \cup A_2$

2. No exemplo 2, prov. que vale a primeira parte de SP2. No entanto, não vale a parte (2).

$$A_1 = \{2, 4, 6, \dots\}$$

$$B_1 = \{1, 3, 5, \dots\}$$

$$A_\infty = \{\}\}$$

$$B_\infty = \{2, 4, 6, \dots\}$$

$$A_1 \cap B_1$$

$$A_\infty \cap B_2$$

$$e \quad A_1 \cup A_\infty \subset B_1 \cup B_2$$

$$B_1 \cup B_2 \subset A_1 \cup A_\infty$$

$$A_1 \cup A_\infty \approx B_1 \cup B_2$$

•

Teorema: $P: \mathcal{F} \rightarrow \mathbb{R}_+$

$A \in \mathcal{F} \mapsto P(A) = \alpha^*$ tal que $A \cap \{x \in [0, \alpha^*]\}$

é uma medida de probabilidade.

Demonstração:

Por construção, $P(A) \geq 0$. Além disso, $\{x \in [0, 1]\} = \Omega$.

$$\Rightarrow P(\Omega) = 1.$$

Aditividade finita de P

$A_1, A_2 \in \mathcal{F}$ com $A_1 \cap A_2 = \emptyset$. Da definição de P , temos:

$$A_1 \cap \{x \in [0, P(A_1)]\}, A_2 \cap \{x \in [0, P(A_2)]\} \\ \approx \{x \in [0, P(A_1)]\}$$

$$A_1 \cup A_2 \cap \{x \in [0, P(A_1 \cup A_2)]\}$$

Vamos verificar que $A_1 \cap \{x \in [P(A_1), P(A_1 \cup A_2)]\}$. Suponhamos, ad T A. L

$$A_2 \cap \{x \in [P(A_1), P(A_1 \cup A_2)]\} \stackrel{\text{EP2}}{\Rightarrow} A_1 \cup A_2 \cap \{x \in [0, P(A_1 \cup A_2)]\}$$

$$A_1 \cap \{x \in [0, P(A_1)]\} \quad (\Rightarrow \Leftarrow)$$

• Fazendo sup. $A_2 \cap \{x \in [P(A_1), P(A_1 \cup A_2)]\}$.

Logo, $A_1 \cap A_2 \cap \{x \in [P(A_1), P(A_1 \cup A_2)]\}$

Logo,

$$P(A_2) = P(A_1 \cup A_2) - P(A_1)$$

Γ

Exercício.

$A_1, \dots, A_n \in \mathcal{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$

$$\Rightarrow P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$$

↓

Continuidade "no" varia.

$(A_n)_{n \geq 1}$, com $A_n \in \mathcal{F}$.

$A_n \supseteq A_{n+1}$, $\forall n \geq 1$ tal que

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n = \emptyset$$

$A_{m+1} \subseteq A_m \Rightarrow A_{m+1} \setminus A_m \Leftrightarrow P(A_{m+1}) \leq P(A_m)$.

A seq. $(P(A_n))_{n \geq 1}$ é não-crescente (decrecente) e limitada inferiormente,
Logo,

$$\exists \lim_{n \rightarrow \infty} P(A_n) = c.$$

Clara que $P(A_n) \nearrow c$, $\forall n \geq 1$.

Mas $P(A_n) \nearrow c \Rightarrow \{x \in [0, c]\} \setminus A_n, \forall n \geq 1$

Segue de SF4 que

$$\{x \in [c, c]\} \setminus \bigcap_{n=1}^{\infty} A_n = \emptyset \Rightarrow c \leq 0$$

Uma vez que $c \geq 0$, segue que

$$c = \lim_{n \rightarrow \infty} P(A_n) = 0.$$

De aditividade finita e continuidade no zero, segue que P é σ -aditiva, isto é,

$$\forall (A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}, A_i \cap A_j = \emptyset, i \neq j.$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Vamos então agora que existe P que representa \mathbb{T} (\mathbb{T} satisfaça SP3 e SP5), onde P é uma medida de prob.

Seja $P': \mathcal{F} \rightarrow \mathbb{R}_+$ prob. que representa \mathbb{T} .

Como P' é med. de prob.,

$$P'(\Omega) = P'\left(\{X \in [0,1]\}\right) = 1$$

e

$$P'(\emptyset) = P'\left(\{X \in [0,0]\}\right) = 0.$$

Vamos verificar que $\forall a \in [0,1]$, $P'\left(X \in [0,a]\right) = a$.

Cbs:

$$\forall n \geq 1, \forall j = 1 \dots n,$$

$\{X \in [\frac{j-1}{n}, \frac{j}{n}]\} \cap \{X \in [0, \frac{j}{n}]\}$. Disso, decorre que

$$P'\left(\{X \in (\frac{j-1}{n}, \frac{j}{n})\}\right) = \frac{1}{n}.$$

segue da uniformidade de P' em $[0,1]$

Por construção
de $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$ deve valer que

$$P([0,1]) = 1/n.$$

Pela aditividade de P' ,

$$P'\left(\{X \in [0, \frac{j}{n}]\}\right) = \frac{j}{n}, \quad \forall n \geq 1, j = 1 \dots n.$$

$\forall n \in \mathbb{N}$

$$\lfloor \frac{a}{n} \rfloor \leq \frac{a}{n} \leq \lfloor \frac{a}{n} \rfloor + \frac{1}{n}$$

$\lfloor \cdot \rfloor$ "função piso"

"parte inteira"

$$\frac{\lfloor a \rfloor}{n} \leq \frac{a}{n} \leq \frac{\lfloor a \rfloor + 1}{n}$$

Conseq.

$$\{x \in [0, \lfloor \frac{a}{n} \rfloor / n]\} \subseteq \{x \in [0, a]\} \subseteq \{x \in [0, \lfloor \frac{a}{n} \rfloor / n + 1/n]\}$$

Conseq.

$$P'(\{x \in [0, \lfloor \frac{a}{n} \rfloor / n]\}) \leq P'(\{x \in [0, a]\}) \leq P'(\{x \in [0, \lfloor \frac{a}{n} \rfloor / n + 1/n]\}) \Rightarrow$$

$$\frac{\lfloor a \rfloor}{n} \leq P'(\{x \in [0, a]\}) \leq \frac{\lfloor a \rfloor + 1}{n}$$

Tomando limites ($n \rightarrow \infty$), resulta que

$$P'(\{x \in [0, a]\}) = a.$$

Finalmente, $\forall A \in \mathcal{F}$, temos

$$A \subseteq \{x \in [0, P(A)]\}$$

$$P'(A) = P'(\{x \in [0, P(A)]\}) = P(A). \text{ Logo, } P'(\cdot) = P(\cdot) \text{ e, port., } P \text{ é suscav.}$$

\square

$A, B, D \in \mathcal{F}$.

$A, B, D \in \mathcal{F}$

$A \mathcal{T}^D B$.

Dados

$$A = \{3, 4, 5, 6\}$$

$$B \mathcal{T} A$$

$$B = \{1, 2\}$$

$$A \mathcal{T}^D B$$

$$D = \{1, 2\}$$

Ex 6

$$A \mathcal{T}^D B \Leftrightarrow A \cap D \subseteq B \cap D$$

Ex. (Se $D = \emptyset$? = tudo equivalente)

Consequências.

(1) $A \mathcal{T}^D B \Leftrightarrow B \mathcal{T}^D A$

(2) \mathcal{T}^D é transitiva

$$A \mathcal{T}^D B \Rightarrow A \cap D \subseteq B \cap D \Rightarrow A \cap D \subseteq C \cap D \Rightarrow A \mathcal{T}^D C.$$

$$B \mathcal{T}^D C \Rightarrow B \cap D \subseteq C \cap D$$

(3) $A \mathcal{T}^D B \Leftrightarrow A \cap D \mathcal{T}^B B \cap D \Leftrightarrow$

$$(A \cap B \cap D) \cup (A \cap B^c \cap D) \mathcal{T} (A \cap B \cap D) \cup (A^c \cap B \cap D) \Leftrightarrow$$

$$A \cap B^c \cap D \mathcal{T} A^c \cap B \cap D \Leftrightarrow$$

$$(A^c \cap B^c \cap D) \cup (A \cap B^c \cap D) \mathcal{T} (A^c \cap B \cap D) \cup (A^c \cap B^c \cap D) \Leftrightarrow$$

$$B^c \cap D \mathcal{T} A^c \cap D \Leftrightarrow$$

$$B^c \mathcal{T}^D A^c.$$

Exercícios:

4) $(A_n)_{n \in \mathbb{N}}$, ~~seja~~ $A_n \in \mathcal{F}$, $A_n \supseteq A_{n+1}$, $\forall n \in \mathbb{N}$ e $B \in \mathcal{F}$ t.q.

$$B \mathcal{T}^D A_n, \forall n \in \mathbb{N} \Rightarrow B \mathcal{T}^D \bigcap_{n=1}^{\infty} A_n.$$

Note que

$$P_D : \mathcal{F} \rightarrow \mathbb{R}$$

$$\forall A \in \mathcal{F} \quad P_D(A) = \frac{P(A \cap D)}{P(D)}$$

representa \mathbb{P}^D , se $P(D) \neq 0$.

Com efeito, seja D t.q. $P(D) \neq 0$, $A, B \in \mathcal{F}$.

$$A \mathbb{P}^D B \Leftrightarrow A \cap D \subseteq B \cap D \Leftrightarrow$$

$$P(A \cap D) \leq P(B \cap D) \Leftrightarrow$$

$$\underline{P}(A \cap D) \leq \underline{P}(B \cap D) \Leftrightarrow$$

$$\underline{P}(D) \leq \underline{P}(D)$$

$$P_D(A) \leq P_D(B) \quad \text{e medida de probabilidade.}$$

"Chamamos isso de Construção de Probabilidade Subjetiva". Partimos de alguns axiomas e chegamos a uma representação.

: fruir conjugadas!

P_1 e P_2 são duas expressões de incerteza sobre ~~consequências futuras~~ - a recompensa que se tem a partir de um certo exp.

A partir das, escolher a melhor a partir da utilidade (a ser construída!).

Teoria da Decisão

Aula 05

24/08/15

A & B . Condições sobre $\mathcal{F} \Rightarrow$ 1) $\exists P$ med. de prob. que representa (coincide) com \mathcal{F} e $P(D) = 1$, $D \in \mathcal{F}$

2) $H \in \mathcal{F}$ com $P(H) > 0$,

$$P_D(\cdot) = \frac{P(\cdot \cap D)}{P(D)} \text{ rep. } \mathcal{F}^D$$

A_1, \dots, A_K .

$$P(A_i | D) = \frac{P(D | A_i) P(A_i)}{P(D)}$$

$A_i \rightarrow$ evento associado à qtd de interesse, Θ → parâmetro, $D \rightarrow$ dados

$D \rightarrow$ dados

$$P(A_i | \text{Dados}) \propto P(\text{Dados} | A_i) P(A_i)$$

$$f(\theta | \omega) \propto f(\omega | \theta) \pi(\theta)$$

Ex: X_1, \dots, X_n , dado θ , são c. i. i. d. Ber(θ)

$$P(X_1=x_1, X_2=x_2, \dots, X_n=x_n | \theta) = \prod_{i=1}^n P(X_i=x_i | \theta)$$

$$= \prod_{i=1}^n \theta^{x_i} (1-\theta)^{1-x_i} = \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}$$

$\theta \sim \text{Beta}(a, b)$

$f(\theta | x_1, \dots, x_n)$

Permutabilidade

Dizemos que as variáveis X_1, \dots, X_n são permutáveis se

$$i = 1, \dots, n, \forall x_i, x_{\pi(i)}$$

$$P(X_1=x_1, \dots, X_n=x_n) = P(X_{\pi(1)}=x_{\pi(1)}, \dots, X_{\pi(n)}=x_{\pi(n)}),$$

$\forall \pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ bijetora.

Consequências

1. X_1, \dots, X_n são ident. dist.

$$P(X_1=x) = \sum_{x_1, x_2, \dots, x_n} P(X_1=x_1, \underline{X_2=x_2}, X_3=x_3, \dots, X_{n-1}=x_{n-1}, X_n=x_n)$$

$$P(X_2=x) = \sum_{x_1, x_2, \dots, x_n} P(X_1=x_1, X_2=x, X_3=x_3, \dots, X_n=x_n)$$

$$= \sum_{x_1, x_2, \dots, x_n} P(X_1=x, X_2=x_1, X_3=x_3, \dots, X_n=x_n) = P(X_1=x).$$

E_X $k=2$

x_1	x_1	x_2	x_3
x_1	p_{11}	p_{12}	p_{13}
x_2	p_{21}		p_{23}
x_3	p_{31}	p_{32}	

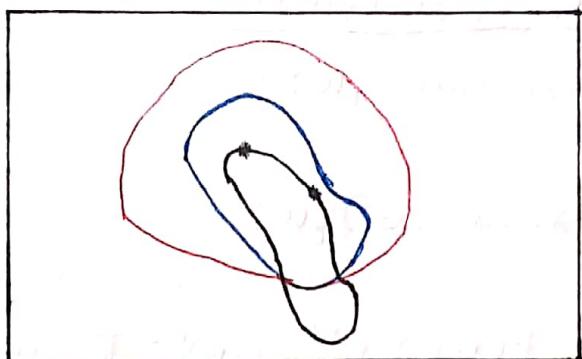
Permutabilidade $\not\Rightarrow$ Independência

$x_1 \backslash x_2$	0	1	
0	0	$1/3$	$1/3$
1	$1/3$	$1/3$	$1/3$

$$X_1 \stackrel{d}{=} X_2 \sim \text{Ber}(1/3)$$

$$P(X_1=1, X_2=1) = 1/3$$

$$P(X_1=1)P(X_2=1) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$



Permutáveis

Identicamente distribuídas

Independentes

Definição. $(X_n)_{n \in \mathbb{N}}$ é uma sequência de v.a. permutáveis se $\forall K \in \mathbb{N}$, X_1, \dots, X_K são permutáveis.

Teorema da Representação de De Finetti.

Digamos $(X_n)_{n \in \mathbb{N}}$ uma sequência permutável de v.a. de Bernoulli. Seja μ a medida de probabilidade do processo. Então, existe $\mu: \mathcal{B}([0,1]) \rightarrow \mathbb{R}_+$ medida de probabilidade, tal que

$\forall K \in \mathbb{N}^*$ e para $x_1, \dots, x_K \in \{0,1\}$, $i=1, \dots, K$,

$$\Pr(X_1=x_1, \dots, X_K=x_K) = \int_0^1 \theta^{x_1} (1-\theta)^{K-x_1} d\mu(\theta)$$

Além disso,

$$\bar{X}_K = \frac{X_1 + \dots + X_K}{K} \xrightarrow{\text{q.c.}} \Theta, \text{ onde } \Theta \text{ é distribuído segundo } \mu.$$

Alguns resultados conhecidos

1) X_1, \dots, X_m dado Θ , são c.i.i.d. $\text{Ber}(\Theta)$.

$\Theta \sim \text{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$.

$$f(\Theta | x_1, \dots, x_n) = \frac{f(x_1, \dots, x_n | \Theta) f(\Theta)}{P(x_1=x_1, \dots, x_n=x_n)}$$

$$* \frac{P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta)}{\int_0^1 P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta) d\theta}$$

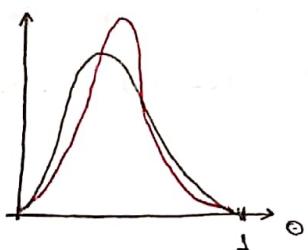
$$\propto P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta)$$

$$\propto \theta^{\sum x_i} (1-\theta)^{n-\sum x_i} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} I_{(0,1)}(\theta)$$

$$\Rightarrow f(\theta | x_1, \dots, x_n) \propto \theta^{a+\sum x_i - 1} (1-\theta)^{b+n-\sum x_i - 1} I_{(0,1)}(\theta)$$

Logo,

$$\theta | x_1, \dots, x_n \sim \text{Beta}(a + \sum_{i=1}^n x_i, b + n - \sum_{i=1}^n x_i)$$



$$\text{Var}(\theta) = \frac{ab}{(a+b)^2 (a+b+1)}$$

"A formação de distribuições beta é conjugada pelo modelo de Bernoulli."

2) X_1, \dots, X_n dado θ , são c.i.i.d. Poisson(θ)

$$\theta \sim \text{Gamma}(a, b)$$

$$f(\theta | x_1, \dots, x_n) = \frac{P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta)}{P(X_1=x_1, \dots, X_n=x_n)}$$

$$\propto P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta)$$

$$\propto \left(\prod_{i=1}^n P(X_i=x_i | \theta) \right) f(\theta) \propto \left(\prod_{i=1}^n \frac{\theta^{x_i} e^{-\theta}}{x_i!} \right) \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} I_{(0,\infty)}(\theta) \frac{\Gamma(x_i)}{\Gamma(x_i)}$$

$$\propto e^{-n\theta} \theta^{\sum_{i=1}^n x_i} \theta^{a-1} e^{-b\theta} I_{(0, \infty)}(\theta)$$

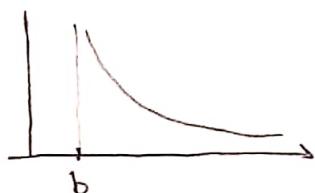
$$\Rightarrow f(\theta | x_1, \dots, x_n) \propto \theta^{a + \sum_{i=1}^n x_i - 1} e^{-(n+b)\theta} I_{(0, \infty)}(\theta)$$

$$\theta | x_1, \dots, x_n \sim \text{Gamma}(a + \sum_{i=1}^n x_i, b/n)$$

a) x_1, \dots, x_n , dado θ , c.i.i.d. Uniforme $(0, \theta)$.

Θ Pareto (a, b)

$$f(\theta) = \begin{cases} C(a, b) / \theta^{a+1} I_{(b, \infty)}(\theta) \\ 0, \text{o. o.} \end{cases}$$



$$f(\theta | x_1, \dots, x_n) \propto f(x_1, \dots, x_n | \theta) f(\theta) \\ = \prod_{i=1}^n f(x_i | \theta) \cdot f(\theta)$$

$$\propto \prod_{i=1}^n \frac{I_{(0, \infty)}(x_i)}{\theta} C(a, b) \frac{1}{\theta^{a+1}} I_{(b, \infty)}(\theta) \\ = \frac{1}{\theta^n} \prod_{i=1}^n I_{(0, \infty)}(x_i) C(a, b) \frac{1}{\theta^{a+1}} I_{(b, \infty)}(\theta) \\ \propto \frac{1}{\theta^{a+n+1}} I_{(0, \infty)}(x_{(n)}) I_{(0, \infty)}(\theta) I_{(\max\{x_{(n)}, b\}, \infty)}(\theta) \Rightarrow$$

$$f(\theta | x_1, \dots, x_n) \propto \frac{1}{\theta^{a+n+1}} I_{(\max\{x_{(n)}, b\}, \infty)}(\theta)$$

$\theta | x_1, \dots, x_n \sim \text{Pareto}(a+n, \max\{x_{(n)}, b\})$

"A fmm. da dens. da Pareto é conjug. pelo modelo uniforme."

Teoria da Decisão

Aula 06

26/08/2023

Ex. 4.

X_1, \dots, X_n dado $\Theta = (\Theta_1, \Theta_2)$ são i.i.d. $N(\Theta_1, \frac{1}{\Theta_2})$, $K \geq 0$.

$$\Theta = R_1 \times R_2$$

$$\Theta_2 \sim \text{Gamma}(a, b), a, b > 0 \quad \Theta_1 | \Theta_2 \sim N(c, d/\Theta_2)$$

$$f(\Theta_1, \Theta_2) = f(\Theta_2) f(\Theta_1 | \Theta_2)$$

$$f(\Theta_2) f(\Theta_1 | \Theta_2)$$

$$f(\Theta | x_1, \dots, x_n) \propto f(x_1, \dots, x_n | \Theta_1, \Theta_2) f(\Theta_1, \Theta_2)$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi^k/\Theta_2}} e^{-\frac{(x_i - \Theta_1)^2}{2\Theta_2}} \frac{b^a}{\Gamma(a)} \Theta_2^{a-1} e^{-b\Theta_2} I_{R_2}(\Theta_2) \frac{1}{\sqrt{d\pi^d/\Theta_2}} e^{-\frac{(\Theta_1 - c)^2}{2d/\Theta_2}}$$

$$\propto \Theta_2^{n/2} e^{-\frac{\Theta_2}{2K} \sum_{i=1}^n (x_i - \Theta_1)^2} \Theta_2^{a-1} e^{-b\Theta_2} \Theta_2^{d/2} e^{-\frac{\Theta_2}{2d} (\Theta_1 - c)^2} I_{R_2}(\Theta_2)$$

$$\propto \Theta_2^{n+1+a-2} e^{-b\Theta_2} I_{R_2}(\Theta_2) e^{-\frac{\Theta_2}{2} \left\{ \Theta_1^2 \left[\frac{1}{d} + \frac{n}{K} \right] - 2\Theta_1 \left[\frac{c}{d} + \frac{n\bar{x}}{K} \right] \right\}} e^{-\frac{\Theta_2}{2K} \sum x_i^2 - \frac{d}{2d} c^2}$$

$$\propto \Theta_2^{n/2+a-1} e^{-\Theta_2 \left[b + \frac{\sum x_i^2}{2K} + \frac{c^2}{2d} \right]} I_{R_2}(\Theta_2) e^{-\frac{\Theta_2}{2} \left(\frac{K+n\bar{x}}{dK} \right) \left(\Theta_1 - \frac{cK+d\bar{x}}{K+n\bar{x}} \right)^2}$$

$$e^{\frac{\Theta_2}{2} \left(\frac{K+n\bar{x}}{dK} \right) \left(\frac{cK+d\bar{x}}{K+n\bar{x}} \right)^2}$$

$$\propto \Theta_2^{n/2+a-2} e^{-\Theta_2 \left[b + \frac{\sum x_i^2}{2K} + \frac{c^2}{2d} - \frac{1}{2} \left(\frac{K+n\bar{x}}{dK} \right) \left(\frac{cK+d\bar{x}}{K+n\bar{x}} \right)^2 \right]} I_{R_2}(\Theta_2)$$

$$\times e^{-\frac{1}{2} \left(\frac{dK}{K+n\bar{x}} \right) \frac{1}{\Theta_2} \left(\Theta_1 - \frac{cK+d\bar{x}}{K+n\bar{x}} \right)^2} \Theta_2^{n/2}$$

$\Theta_2 | x_1, \dots, x_n \sim N(\alpha^*, \beta^*)$

$\Theta_2 | \Theta_2, x_1, \dots, x_n \sim N\left(\frac{c}{\theta_2}, \frac{d}{\theta_2^2}\right)$

Livro De Groot, Cap. 9 → mais conjugadas

Populações Finitas

$P = \{1, \dots, N\}$ uma população.

Seja $X = (x_1, \dots, x_N)$, um vetor de caract. populacionais, ou melhor vetor populacional de características.

$$T(X) = \frac{\sum_{i=1}^N x_i}{N}, \quad P(T(X) = j/N) = \sum_{\substack{x_1, \dots, x_N \\ \sum x_i = j}} P(x_1 = x_1, \dots, x_N = x_N)$$

Amostra $x_1, \dots, x_n, n \leq N$

$$P(T(X) = j/N | X_1 = x_1, \dots, X_n = x_n).$$

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Prefeitura (Utilidade)

$R_i \rightarrow$ conjunto de recompensas (consequências)

1. Vendedor de pamonhas

Preço: v $R' = \{v - c, v - 2c, \dots, 30v - c, \dots\}$

Custo do lote: c

$$iv - c \geq jv - c \quad i \leq j.$$

2. Jogo do Corinthians

$$R = \{\emptyset, \oplus, \ominus, \otimes, U\} \quad \oplus v^+ \ominus v^-$$

Para cada $r \in R$,

$$A(r) = \{r' \in R : r' \geq r\}$$

$$U : R \rightarrow \mathbb{R}_+ \\ r \in R \mapsto U(r) = |A(r)|$$

Sob certas condições, para $r_1, r_2 \in R$,

$$r_1 \geq r_2 \Leftrightarrow U(r_1) \geq U(r_2)$$

30 pamonhas

$$P_{20}(\{jv - 30c\}) = P(\text{"demanda } j"}) \quad j = 0, \dots, 30$$

$$P_{10}(\{jv - 10c\}) = P(\text{"demanda } j"}) \quad j \leq 10$$

$$P_{10}(\{10(v - c)\}) = P(\text{"demanda } 10"})$$

Cenário de Incerteza

1. Parceria

Produção de "i" parcerias, $i=0, \dots, 30$

$$P_i(\{r_j, j=i\}) = P(\text{"demanda } r_i") , j \leq i$$

$$P_i(\{r_{i+1}\}) = P(\text{"demanda } r_{i+1}")$$

$$R = \{(r_j, c_j), i, j \in \{0, 1, \dots, 30\}\}$$

$$P_1 \neq P_2$$

Formulações.

$R \rightarrow$ conjunto de recompensas (consequências)

$\mathcal{A} \rightarrow \sigma\text{-álgebra de subconjuntos de } R$

$\mathcal{P} \rightarrow$ conjunto de medidas de prob. em (R, \mathcal{A})

Vamos considerar τ^* em $R \times R$, $r_1, r_2 \in R$

$$r_1 \tau^* r_2 \quad r_2 \tau^* r_1 \quad r_1 \tau^* r_2$$

para $r_1, r_2 \in R$

$$[r_1, r_2] = \{r \in R : r_1 \tau^* r \tau^* r_2\}$$

para todo $r_1, r_2 \in R$, vamos supor que

$$[r_1, r_2] \in \mathcal{A}$$

Dizemos que $P \in \mathcal{P}$ é limitada se existem $r_1, r_2 \in R$ tais que

$$P([r_1, r_2]) = 1$$

Seja $P_i = \{P \in \mathcal{P} : P \text{ é limitada}\}$.

Vamos estender \mathcal{T}^* para $\mathcal{P}_L \times \mathcal{P}_L$.

Suponemos \mathcal{T}^* completa e transitiva, isto é,

$$P_1 \mathcal{T}^* P_2 \text{ ou } P_2 \mathcal{T}^* P_1 \quad \text{e}$$

$\forall P_1, P_2, P_3 \in \mathcal{P}_L$,

$$P_1 \mathcal{T}^* P_2 \text{ e } P_2 \mathcal{T}^* P_3 \Rightarrow P_1 \mathcal{T}^* P_3$$

Do mesmo modo que \mathcal{T}^* restrita à $R \times R$,

$$\begin{aligned} P_1 N^* P_2 &\Leftrightarrow P_1 \mathcal{T}^* P_2 \text{ e } P_2 \mathcal{T}^* P_1 \\ P_1 T^* P_2 \end{aligned}$$

Para cada $r \in \mathbb{R}$, vamos considerar r equivalente a $P_{(r)} \in \mathcal{P}_L$ onde

$$P_{(r)}(A) = r A.$$

Se $P_1, P_2 \in \mathcal{P}_L$ e $\alpha \in [0, 1]$, então $\alpha P_1 + (1-\alpha) P_2 \in \mathcal{P}_L$.

$$\alpha P_1(A) + (1-\alpha) P_2(A).$$

Em particular, se $r_1, r_2 \in \mathbb{R}$,

$$\alpha r_1 + (1-\alpha) r_2.$$

Suponemos que existem $s_0, t_0 \in \mathbb{R}$ tais que $s_0 \mathcal{T}^* t_0$.

Suposição Adicional:

SU₁: Sejam $P_1, P_2, P \in \mathcal{P}_L$ e $\omega \in (0, 1)$.

$$P_1 \leq^* P_2 \Leftrightarrow \omega P_1 + (1-\omega)P \leq^* \omega P_2 + (1-\omega)P$$

Teoria da Decisão

Aula 07

31/08

Suposições Adicionais:

SU1: $P_1, P_2 \in \mathcal{P}_L$ e $\alpha \in (0,1)$.

$$P_1 \leq^* P_2 \Leftrightarrow \alpha P_1 + (1-\alpha) P_2 \leq \alpha P_2 + (1-\alpha) P_1$$

Implicações:

Lema 1: $P_1, P_2, Q_1, Q_2 \in \mathcal{P}_L$ tais que

$$P_i \leq Q_i, i=1,2, \alpha \in (0,1)$$

Então, por SU1,

$$\alpha P_1 + (1-\alpha) P_2 \leq \alpha Q_1 + (1-\alpha) Q_2$$

Demonstração: Por SU1,

$$\alpha P_1 + (1-\alpha) P_2 \leq^* \alpha Q_1 + (1-\alpha) P_2 \leq^* \alpha Q_1 + (1-\alpha) Q_2$$

Lema 2: $r_1, r_2 \in \mathbb{R}$ tais que $r_1 \leq^* r_2, \alpha \in (0,1)$.

$$r_1 \sim \alpha r_1 + (1-\alpha) r_2 \leq^* \alpha r_2 + (1-\alpha) r_2 \leq^* \alpha r_2 + (1-\alpha) r_2 \sim r_2$$

$$\text{Logo, } r_1 \leq^* r_2 + (1-\alpha)r_1 \leq^* r_2.$$

Lema 3. $r_1, r_2 \in \mathbb{R}$ tais que

$$r_1 \leq^* r_2$$

Sejam $\alpha, \beta \in [0,1]$,

$$\alpha r_2 + (1-\alpha) r_1 \leq^* \beta r_2 + (1-\beta) r_1 \Leftrightarrow \alpha \leq \beta.$$

(*)
Demonstração: Supondo que $\alpha < \beta$ ($0 < \alpha < \beta < 1$)

$$\beta r_2 + (1-\beta)r_1 \sim \frac{1-\beta}{1-\alpha} [\alpha r_2 + (1-\alpha)r_1] + \left(1 - \frac{1-\beta}{1-\alpha}\right) r_1$$

$$\alpha r_2 + (1-\alpha)r_1 \leq \frac{1-\beta}{1-\alpha} [\alpha r_2 + (1-\alpha)r_1] \quad \leftarrow^*$$

$$+ \left(1 - \frac{1-\beta}{1-\alpha}\right) [\alpha r_2 + (1-\alpha)r_1] \quad \begin{matrix} \text{Lema 1} \\ \leftarrow^* r_1 \\ \text{Lema 2} \end{matrix}$$

SUL $\Rightarrow \tau^* \underbrace{\frac{1-\beta}{1-\alpha} [\alpha r_2 + (1-\alpha)r_1] + \left(1 - \frac{1-\beta}{1-\alpha}\right) r_1}_{\sim^* \beta r_2 + (1-\beta)r_1}$

Então,

$$\alpha < \beta \Rightarrow \alpha r_2 + (1-\alpha)r_1 \stackrel{*}{<} \beta r_2 + (1-\beta)r_1$$

(**)

- $\alpha > \beta$

$$\alpha > \beta \Rightarrow \beta r_2 + (1-\beta)r_1 \stackrel{*}{<} \alpha r_2 + (1-\alpha)r_1$$

$$\alpha = \beta$$

(***):

SUL: Segom $P_1, P_2, P \in \mathcal{P}_L$ tais que

$$P_1 \stackrel{*}{<} P \stackrel{*}{<} P_2$$

$$\exists \alpha, \beta \in (0,1) \text{ t.q. } \alpha P_1 + (1-\alpha) P_2 \stackrel{*}{<} P \in P \stackrel{*}{<} \beta P_1 + (1-\beta) P_2$$

Teorema 1. Sejam $r_1, r, r_2 \in \mathbb{R}$ tais que

$$r_1 \leq^* r_2 \text{ e } r_2 \leq^* r \leq^* r_2,$$

, algum α

existe um $\gamma \in [0,1]$ tal que

$$r \sim^\circ r_2 \Rightarrow r \sim^\circ \gamma r_2 + (1-\gamma)r_1$$

Caso trivial. $r \sim^\circ r_2 \Rightarrow \gamma = 1$

$$r \sim^\circ r_1 \Rightarrow \gamma = 0$$

Suponha agora $r_1 \leq^* r \leq^* r_2$. Sejam

$$U = \{\alpha \in [0,1] : r \leq^* \alpha r_2 + (1-\alpha)r_1\}$$

$$D = \{\alpha \in [0,1] : \alpha r_2 + (1-\alpha)r_1 \leq^* r\}$$

Observemos que $1 \in U$ e $0 \in D$ ($U \neq \emptyset$ e $D \neq \emptyset$).

Sejam $\mu^* = \inf\{U\}$ e $d^* = \sup\{D\}$.

Se $\mu \in U$ e $\mu > \mu^*$, então $\mu \in U$. $\text{I} \Rightarrow \text{II}$

$$r \leq^* \mu r_2 + (1-\mu)r_1 \leq^* \mu r_2 + (1-\mu)r_2 \Rightarrow$$

Lemma

$$r \leq^* \mu r_2 + (1-\mu)r_1.$$

Ademais, $\mu^* \notin U$, (II).

Se $\mu^* \in D \Rightarrow r \leq^* \mu^* r_2 + (1-\mu^*)r_1$ e sabemos que $r_1 \leq^* r$

Por SU2, há $\gamma \in (0,1)$ t.q.

$$\begin{aligned} r &\leq \gamma [\mu^* r_2 + (1-\mu^*) r_1] \\ &+ (1-\gamma) r_1 \approx \gamma \mu^* r_2 + (1-\gamma)\mu^* r_1 \end{aligned}$$

$$\Rightarrow \gamma \mu^* \in U.$$

De (I) e (II), segue que: $U = [\mu^*, 1]$

Do mesmo modo, verificamos que

$$d \in D \Rightarrow \forall d' \leq d, d' \in D \text{ e que } d' \notin D. \quad (\text{Exercício!})$$

Assim $D = [0, d^*]$. Como $U \cap D = \emptyset$, temos que $d^* \leq \mu^*$.

Seja $\tilde{\gamma} \in [d^*, \mu^*]$. $\tilde{\gamma} \notin U$ e $\tilde{\gamma} \notin D$. Portanto,

$$\begin{cases} \tilde{\gamma} \notin U \Rightarrow \tilde{\gamma} r_2 + (1-\tilde{\gamma}) r_1 \leq r \\ \tilde{\gamma} \notin D \Rightarrow \exists r \leq \tilde{\gamma} r_2 + (1-\tilde{\gamma}) r_1 \end{cases} \Rightarrow r \approx \tilde{\gamma} r_2 + (1-\tilde{\gamma}) r_1$$

Ademais, $\tilde{\gamma}$ é único. Seja $\tilde{\gamma}' \in [0,1]$ t.q. $\tilde{\gamma}' r_2 + (1-\tilde{\gamma}') r_1 \approx r$.

Assim, $\tilde{\gamma} r_2 + (1-\tilde{\gamma}) r_1 \approx \tilde{\gamma}' r_2 + (1-\tilde{\gamma}') r_1$. Do Lema 3, segue que $\tilde{\gamma}' = \tilde{\gamma}$!

Definição da função U.

Lembre que $s_0, t_0 \in \mathbb{Q}$ t.q.

$$\exists \gamma \in [s_0, t_0]$$

Seja $r \in \mathbb{R}$.

1. Se $s_0 \leq^* r \leq^* t_0 \stackrel{\text{Teo.}}{\Rightarrow} \exists \forall \in [0,1] \text{ t.q. } t_0 =$

$$r \approx \gamma t_0 + (1-\gamma)s_0$$

Seja $U: \mathbb{R} \rightarrow \mathbb{R}_+$. Definimos $U(r) = \underline{\gamma} \nu$,

Assim, $r \approx U(r)t_0 + (1-U(r))s_0$.

Consequência: $U(t_0) = \frac{1}{\underline{\gamma}}$ e $U(s_0) = 0$.

2. $r <^* s_0$. Assim, $r <^* s_0 <^* t_0$. Daí, $\exists \gamma \in (0,1)$ t.q.

$$s_0 \approx \gamma t_0 + (1-\gamma)r$$

Nesse caso, vamos definir $U(r) = \frac{\gamma}{1-\gamma}$.

Finalmente,

3. $t_0 <^* r$.

Neste caso, $s_0 <^* t_0 <^* r$. Pelo Teorema X, $\exists \beta \in (0,1)$ t.q

$$t_0 \approx \beta r + (1-\beta)s_0 \quad (U(t_0) = \beta U(r) + (1-\beta)U(s_0))$$

Definimos nesse caso, $U(r) = \frac{1}{\beta}$.

Notemos que:

i) $r \in [s_0, t_0]$, $r \approx V(r)t_0 + (1 - V(r))s_0$

ii) $r \leq s_0$, $s_0 \approx \frac{V(r)}{1 - V(r)}t_0 + \frac{1}{1 - V(r)}r$.

iii) $t_0 < r$, $t_0 \approx \frac{1}{V(r)}r + \left(1 - \frac{1}{V(r)}\right)s_0$

Teorema 2. Segom $r_1, r_2, r_3 \in \mathbb{R}$ tq. $r_1 < r_2 < r_3$, temos que

$r_2 \approx r_3 + (1 - \alpha)r_1$ para algum $\alpha \in [0, 1]$. Então

$$V(r_2) = \alpha V(r_1) + (1 - \alpha)V(r_3)$$

$$\Rightarrow V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$$

Então se $V(r)$ é contínua em r_1 , temos

$\lim_{r \rightarrow r_1} V(r) = V(r_1)$

Portanto $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$ e assim $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$

Então $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$ e assim $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$

Então $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$ e assim $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$

Então $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$ e assim $V(r_2) - V(r_1) = \alpha(V(r_3) - V(r_1))$

Teoria da Decisão

Aula 08

02/09

$\forall r \in [r_1, r_2]$, onde $r_1, r_2 \in \mathbb{R}$ são tais que $r_1 \leq r_2$, $\exists \alpha = \alpha(r) \in [0, 1]$ tal que

$$r \sim^* \alpha(r)r_2 + (1-\alpha(r))r_1.$$

Definimos $U: \mathbb{R} \rightarrow \mathbb{R}$ t.q.

1) $r \in [s_0, t_0]$

$$r \sim U(r)t_0 + (1-U(r))s_0$$

2) $r \leq s_0$

$$s_0 \sim^* \frac{-U(r)}{1-U(r)}t_0 + \frac{1}{1-U(r)}r$$

$$3) t_0 \leq r \leq s_0 \sim^* \left(1 - \frac{1}{U(r)}\right)s_0$$

Teorema 2. $r_1, r_2, r_3 \in \mathbb{R}$ tais que

$$r_2 \sim^* \alpha r_3 + (1-\alpha)r_1, \text{ para algum } \alpha \in [0, 1]$$

Então

$$U(r_2) = \alpha U(r_3) + (1-\alpha)U(r_1).$$

$\exists s_1, t_1 \in \mathbb{R}$ tales que

$$s_1, t_1 \in \mathbb{R}$$

$$s_1, t_1 \in \mathbb{R}^*, r = \bar{r}^* t_1, \forall r \in \mathbb{R}^* = \{r_3, r_2, r_1, s_0, t_0\}$$

$\forall r \in \mathbb{R}^*, \exists \gamma \in [0,1]$ tal que

$$r = \gamma t_1 + (1-\gamma) s_1$$

Definimos U^* de modo que

$$r = U^*(r) t_1 + (1-U(r)) s_1 \quad (\text{iii})$$

Então,

$$r_2 = U^*(r_2) t_1 + (1-U(r_2)) s_1 \quad (\#)$$

Por outro lado,

$$\begin{aligned} r_2 &= \alpha r_3 + (1-\alpha) r_1 = \alpha [U^*(r_3) t_1 + (1-U^*(r_3)) s_1] + \\ &\quad (1-\alpha) [U^*(r_1) t_1 + (1-U^*(r_1)) s_1] \Rightarrow \end{aligned}$$

$$r_2 = \alpha U^*(r_3) t_1 + (1-\alpha) U^*(r_3) t_1 + [1 - (\alpha U^*(r_3) + (1-\alpha) U^*(r_3))] s_1 \quad (\# \#)$$

Do Lema 3,

$$U^*(r_2) = \alpha U^*(r_3) + (1-\alpha) U^*(r_3) \quad r \in [s_0, t_0]$$

$$r = U(r) t_0 + (1-U(r)) s_0 \Rightarrow$$

$$r = U(r) [U^*(t_0) t_1 + (1-U^*(t_0)) s_1] + (1-U(r)) [U^*(s_0) t_1 + (1-U^*(s_0)) s_1] \Rightarrow$$

$$r = \underbrace{[U(r) U^*(t_0) + (1-U(r)) U^*(s_0)] t_1}_{(i)} + (1-U(r)) s_1$$

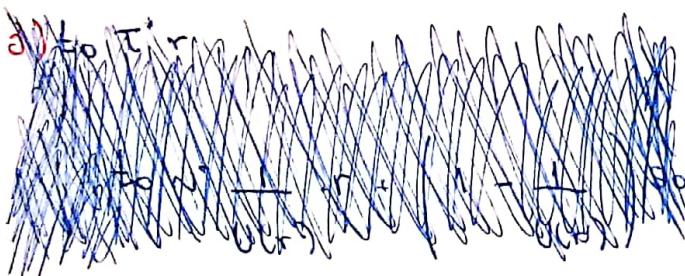
De (i) e (ii)

$$U^*(r) = U(r)U^*(t_0) + (1-U(r))U^*(s_0) \Rightarrow U^*(r) = U(r)[U^*(t_0) - U^*(s_0)] + U^*(s_0) \quad (\text{D})$$

2) $r \approx s_0$

$$s_0 \approx \frac{-U(r)}{1-U(r)} t_0 + \frac{1}{1-U(r)} r \Rightarrow$$

$$\Rightarrow s_0 \approx \frac{-U(r)}{1-U(r)} [U^*(t_0)t_1 + (1-U^*(t_0))s_1] \quad (\Rightarrow)$$
$$+ \frac{1}{1-U(r)} [U^*(r)t_1 + (1-U^*(r))s_1]$$



$$(\Rightarrow) s_0 \approx \underbrace{\left[-\frac{U(r)}{1-U(r)} U^*(t_0) + \frac{1}{1-U(r)} U^*(r) \right] t_1}_{\text{esta parte é constante}} + (1-U(r))s_1 \quad (\text{C})$$

Po outro lado,

$$s_0 \approx U^*(s_0) + (1-U(s_0))s_1 \quad (\text{C})$$

De (o) e (oo)

$$U^*(s_0) = \frac{U(r)}{1-U(r)} U^*(t_0) + \frac{1}{1-U(r)} U^*(r) \Rightarrow$$

$$\Rightarrow U^*(r) = U^*(s_0) + U(r) [U^*(t_0) - U^*(s_0)]. \quad (\square \square)$$

3) $t_0 \in \tau^* r$

$$t_0 \in \frac{1}{U(r)} r + (1 - \frac{1}{U(r)}) s_0 \in N^*$$

$$N^* \frac{1}{U(r)} \left[U^*(r) t_2 + (1 - U^*(r)) s_1 \right] + \left(1 - \frac{1}{U(r)} \right) \left[U^*(s_0) t_2 + (1 - U^*(s_0)) s_1 \right] \Rightarrow$$

$$\Rightarrow t_0 \in N^* \left(\frac{U^*(r)}{U(r)} + \left(\frac{U(r)-1}{U(r)} \right) U^*(s_0) \right) U^*(s_0) t_2 + (1 - \frac{1}{U(r)}) s_1 \quad (\Delta)$$

$$\text{Por outro lado, } t_0 \in U^*(t_0) t_2 + (1 - U^*(t_0)) s_1 \quad (\Delta \Delta)$$

De (Δ) e (ΔΔ), temos

$$U^*(t_0) = \frac{U^*(r)}{U(r)} + \left(\frac{U(r)-1}{U(r)} \right) U^*(s_0) \Rightarrow$$

$$\Rightarrow U^*(r) = U^*(s_0) + U(r) (U^*(t_0) - U^*(s_0)) \quad (\square \square \square)$$

Assim, de (o), (oo) e (ooo), $\forall r \in \mathbb{R}^*$,

$$U^*(r) = U^*(s_0) + U(r) (U^*(t_0) - U^*(s_0))$$

Como

$$U^*(r_2) = \alpha U^*(r_3) + (1-\alpha) U^*(r_1)$$

$$U^*(s_0) + U(r_2)(U^*(t_0) - U^*(s_0)) =$$

$$= \alpha [U^*(s_0) + U(r_3)(U^*(t_0) - U^*(s_0))] + (1-\alpha)[U^*(s_0) + U(r_1)(U^*(t_0) - U^*(s_0))] \rightarrow$$

$$\Rightarrow U(r_2) = \alpha U(r_3) + (1-\alpha) U(r_1)$$

$$\Rightarrow P_1 \cap P_2 \Rightarrow \int U dP_2 \leq \int U dP_1.$$

SV3: $\forall r_1, r_2, r_3 \in \mathbb{R}, \forall \alpha, \beta \in [0,1]$

$$\{r \in \mathbb{R} : \alpha r + (1-\alpha)r_1 \cap \beta r_2 + (1-\beta)r_3 \in A_0\} \in \mathcal{A}_0.$$

Reaultado. $U: \mathbb{R} \rightarrow \mathbb{R}$ como def. no anexo anterior é uma m.a. (σ -mensurável)

Vamos verificar que $H \subset \mathbb{R}$,

$$\{r \in \mathbb{R} : U(r) \leq \infty\} \in \mathcal{A}_0.$$

$$\{r\} \times \{0\} \rightarrow r \cap s_0$$

Logo

$$s_0 \in \frac{-U(r)}{1-U(r)} t_0 + \frac{1}{1-U(r)} r.$$

$$U(r) \leq \infty \Leftrightarrow \frac{-\infty}{1-\infty} \leq \frac{-U(r)}{1-U(r)} \stackrel{\text{Lema 3}}{\Leftrightarrow}$$

$$= \frac{\alpha}{1-\alpha} t_0 + \frac{1-\alpha}{1-\alpha} r \mathbb{E}^* \underbrace{\frac{-U(r)}{1-U(r)} t_0}_{s_0} + \frac{1}{1-U(r)} r$$

Logo,

$$U(r) \leq \alpha \Leftrightarrow \frac{1}{1-\alpha} r + \frac{-\alpha}{1-\alpha} t_0 \mathbb{E}^* s_0$$

Logo,

$$\{r \in \mathbb{R} : U(r) \leq \alpha\} \in \mathcal{A}$$

1). $\alpha \in [0, 1]$

$$\{r \in \mathbb{R} : U(r) \leq \alpha\} = \{r \in \mathbb{R} : U(r) \leq 0\} \cup \{r \in \mathbb{R} : 0 \leq U(r) \leq \alpha\}$$

$$\in \mathcal{A}, \text{ por } \bigcup_{n=1}^{\infty} \{r \in \mathbb{R} : U(r) \leq -1/n\}$$

$\alpha \in [0, 1]$

$$\{r \in \mathbb{R} : 0 \leq U(r) \leq \alpha\}$$

Se $0 \leq U(r) \leq \alpha$, então

$$r \leq U(r)t_0 + (1-U(r))s_0$$

$$r \mathbb{E}^* \alpha t_0 + (1-\alpha) s_0$$

Assim, $0 \leq U(r) \leq \alpha \Leftrightarrow r \mathbb{E}^* \alpha t_0 + (1-\alpha) s_0$.

Logo, $\{r \in \mathbb{R} : 0 \leq U(r) \leq \alpha\} \in \mathcal{A}$.

3) $x > 1$

$\{r \in \mathbb{R} : U(r) \leq x\} = \{r \in \mathbb{R} : U(r) \leq 1\} \cup \{r \in \mathbb{R} : 1 < U(r) \leq x\}$

Se $1 \in U(r)_{\mathbb{F}^\infty}$, então $\tau^r r$ é, assim,

$$t_0 \approx \frac{1}{V(r)} + \left(1 - \frac{1}{V(r)} \right) s_0$$

Como $U(r) \leq x \Leftrightarrow \frac{1}{x} \leq 1/U(r)$, vale que

$$\frac{1}{\alpha} r + \left(1 - \frac{1}{\alpha}\right) s_0 \stackrel{\mathcal{T}^*}{\sim} r + \left(1 - \frac{1}{U(r)}\right) s_0.$$

t_0

Assim.

$$1 < U(r) \leq \infty \iff \frac{1}{r} + \left(1 - \frac{1}{\alpha}\right) \leq \pi^* \text{ to}$$

Logo

$$\{r \in \mathbb{R} : 1 \leq U(r) \leq x\} \in \mathcal{A}$$

Logo

$$\{r \in \mathbb{R} : U(r) \leq x\} \in \mathcal{A}$$

Assim, $\forall c \in \mathbb{R}$, $\exists r \in \mathbb{R}$: $U(r) \leq x \} \subseteq A_c$, portanto, U é variável aleatória

Suponha $r_1, r_2 \in \mathbb{R}$ com $r_1 < r_2$. Para $r \in [r_1, r_2]$, $\exists \alpha = \alpha(r) \in [0, 1]$ tal que

$$r \approx \alpha(r)r_2 + (1-\alpha(r))r_1. \quad (\text{Teorema 1})$$

Do Teorema 3, segue que

$$U(r) = \alpha(r) U(r_2) + (1-\alpha(r)) U(r_1) \Rightarrow$$

$$\alpha(r) = \frac{U(r) - U(r_1)}{U(r_2) - U(r_1)}$$

Seja $P \in \mathcal{P}_L$ ($\exists r^*, r'' \in \mathbb{R}$ t.q. $P([r^*, r'']) = 1$).

$\forall r \in [r^*, r'']$, $\exists \alpha = \alpha(r) \in [0, 1]$

$$r \approx \alpha(r)r'' + (1-\alpha(r))r^*, \text{ onde } \alpha(r) = \frac{U(r) - U(r^*)}{U(r'') - U(r^*)}$$

$$\text{Seja } \beta = \int_{[r^*, r'']} \alpha(r) dP(r) \quad (1)$$

$$\sum_{r \in [r^*, r'']} \alpha(r) P(R=r)$$

SU4. Seja $P \in \mathcal{P}_L$ de modo que $P([r_1, r_2]) = 1$. Para cada $r \in [r^*, r'']$, seja $\alpha(r)$ como dado em (1). Seja β como dado em (2). Então

$$P \approx \beta r'' + (1-\beta)r^*$$

Teorema. Sejam $P_1, P_2 \in \mathcal{P}_L$. ~~Seja~~ π^* definida sobre $\mathcal{P}_L \times \mathcal{P}_L$ atende SU₁ a SU₄. Então existe $U: \mathbb{R} \rightarrow \mathbb{R}$ tal que

$$P_1 \pi^* P_2 \Leftrightarrow E(U|P_1) \leq E(U|P_2)$$

$$(E(U|P_i) = \int_{\mathbb{R}} U dP_i = \int_{\mathbb{R}} U(r) dP_i(r), i=1,2)$$

Démontrer:

Si on a $P_1, P_2 \in \mathcal{P}_L$, il existe des $r_*, r^* \in \mathbb{R}$ tels que

$$P_1([r_*, r^*]) = P_2([r_*, r^*]) = 1.$$

$$\beta_1 = \int_{[r_*, r^*]} \omega(r) dP_1 \text{ et } \beta_2 = \int_{[r_*, r^*]} \omega(r) dP_2.$$

De suite,

$$P_1 \propto \beta_1 r^* + (1-\beta_1) r_* \text{ et } P_2 \propto \beta_2 r^* + (1-\beta_2) r_*$$

Dès lors,

$$P_1 \not\propto P_2 \stackrel{\text{Lemma 3}}{\iff} \beta_1 \leq \beta_2 \iff \int_{[r_*, r^*]} \omega(r) dP_1 \leq \int_{[r_*, r^*]} \omega(r) dP_2$$

$$\begin{aligned} &\iff \int_{[r_*, r^*]} \frac{U(r) - U(r_*)}{U(r^*) - U(r_*)} dP_1 \leq \int_{[r_*, r^*]} \frac{U(r) - U(r_*)}{U(r^*) - U(r_*)} dP_2 \\ &\iff \frac{\int_{[r_*, r^*]} U(r) dP_1 - U(r_*)}{U(r^*) - U(r_*)} \leq \frac{\int_{[r_*, r^*]} U(r) dP_2 - U(r_*)}{U(r^*) - U(r_*)} \Rightarrow \end{aligned}$$

$$P_1 \not\propto P_2 \iff E(U|P_1) \leq E(U|P_2).$$

Teoria da Decisão

Aula 09

Teorema 3.

\mathbb{P}^* em $\mathcal{P}_1 \times \mathcal{P}_2$

\mathbb{P}^* atende SU_2 e SU_4 .

$\exists U: \mathbb{R} \rightarrow \mathbb{R}$ tal que $P_1 \leq^* P_2 \Leftrightarrow E(U|P_1) \leq E(U|P_2)$.

$r_1, r_2 \in \mathbb{R}$

$r_1 \leq^* r_2 \Leftrightarrow I.U(r_1) \leq I.U(r_2)$.

$(P_{fr_1}, \leq^* P_{fr_2})$

Resultado. Sejam U_1 e U_2 funções de $\mathbb{R} \rightarrow \mathbb{R}$ satisfazendo as condições do Teorema 3. Então, $\exists a > 0$ e $b \in \mathbb{R}$ tais que $\forall r \in \mathbb{R}$,

$$U_2(r) = aU_1(r) + b.$$

Dem:

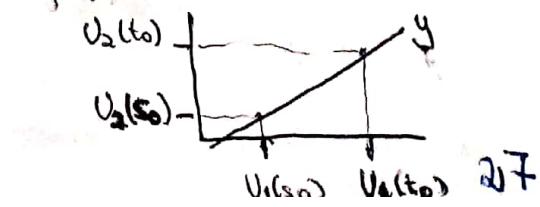
Sejam $U_1, U_2: \mathbb{R} \rightarrow \mathbb{R}$ tais que $\forall P_1, P_2 \in \mathcal{P}_1$,

$P_1 \leq^* P_2 \Leftrightarrow E(U_i|P_1) \leq E(U_i|P_2), i=1,2$.

Como existem $s_0, t_0 \in \mathbb{R}$ com $s_0 \neq t_0$, vale que

$$U_1(s_0) \leq U_1(t_0) \text{ e } U_2(s_0) \leq U_2(t_0)$$

$$y = U_2(s_0) + \frac{U_2(t_0) - U_2(s_0)}{U_1(t_0) - U_1(s_0)} (x - U_1(s_0))$$



- Para $r \in [s_0, t_0]$, a relação linear é verificada.
- Para $r \in \mathbb{R}$ tal que $s_0 \leq r \leq t_0$, $\exists \gamma \in (0,1)$ t.q.

$$r \approx \gamma t_0 + (1-\gamma)s_0. \text{ Assim,}$$

$$U_2(r) = \gamma U_2(t_0) + (1-\gamma)U_2(s_0) \text{ e}$$

$$U_{\alpha}(r) = \gamma U_{\alpha}(t_0) + (1-\gamma)U_{\alpha}(s_0)$$

$$U_{\alpha}(s_0) + \frac{U_{\alpha}(t_0) - U_{\alpha}(s_0)}{U_2(t_0) - U_2(s_0)} [\gamma U_2(t_0) + (1-\gamma)U_2(s_0) - U_{\alpha}(s_0)] =$$

$$= U_{\alpha}(s_0) + \frac{U_{\alpha}(t_0) - U_{\alpha}(s_0)}{U_2(t_0) - U_2(s_0)} \cdot \gamma (U_2(t_0) - U_2(s_0))$$

$$= \gamma U_{\alpha}(t_0) + (1-\gamma)U_{\alpha}(s_0) = U_{\alpha}(r).$$

- Para $r < s_0$, $\exists \delta \in (0,1)$ t.q.

$$s_0 \leq \gamma t_0 + (1-\gamma)r. \text{ Assim,}$$

$$U_2(s_0) = \gamma U_2(t_0) + (1-\gamma)U_2(r) \text{ e}$$

$$U_{\alpha}(s_0) = \gamma U_{\alpha}(t_0) + (1-\gamma)U_{\alpha}(r), \text{ isto é,}$$

$$U_2(r) = \frac{U_2(s_0) - \gamma U_2(t_0)}{1-\gamma} \text{ e } U_{\alpha}(r) = \frac{U_{\alpha}(s_0) - \gamma U_{\alpha}(t_0)}{1-\gamma}.$$

$$U_2(s_0) + \frac{U_2(t_0) - U_2(s_0)}{U_2(t_0) - U_2(s_0)} \left[\frac{U_1(s_0) - \gamma U_1(t_0)}{1 - \gamma} - U_1(s_0) \right] =$$

$$= U_2(s_0) + \frac{U_2(t_0) - U_2(s_0)}{U_2(t_0) - U_2(s_0)} \frac{(-\gamma)}{1 - \gamma} [U_1(t_0) - U_1(s_0)]$$

$$= \frac{U_2(s_0) - \gamma U_2(t_0)}{1 - \gamma} = U_2(r).$$

Exercício

- (Terceiro caso). $r \neq t_0$. Procedendo do mesmo modo, obtemos para $r \in \mathbb{R}$ t.q. $t_0 < r$, que

$$U_2(r) = U_2(s_0) + \frac{U_2(t_0) - U_2(s_0)}{U_2(t_0) - U_2(s_0)} (U_2(r) - U_2(s_0)). \quad (*)$$

Assim, (*) vale para todo $r \in \mathbb{R}$ e basta tomar

$$a = \frac{U_2(t_0) - U_2(s_0)}{U_2(t_0) - U_2(s_0)} \quad \text{e} \quad b = U_2(s_0) - U_2(s_0) \frac{U_2(t_0) - U_2(s_0)}{U_2(t_0) - U_2(s_0)}$$

Comentários.

- (1) O desenvolvimento feito é restrito a medidas de probabilidade finitas. Extensões podem ser obtidas (segundo F.10 De Groot discute a repres. para medidas de probabilidade para os quais $U: \mathcal{R} \rightarrow \mathbb{R}$ e integrável).

"A weaker system of axioms"
(H. Rubin)

- (2) Construção dá sustentação ao paradigma de maximização de utilidade

esperada.

(3) Representação de preferência por números (ou existência de função utilidade).

Paradoxo de ALLAIS (1953)

Problema 1:

P_1 : Prêmio U\$ 500 000,00 com probabilidade 1

P_2 : $\begin{cases} \text{U\$ } 2500000,00, p=0,1 \\ \text{U\$ } 500000,00, p=0,89 \\ \text{U\$ } 0, p=0,01 \end{cases}$

Problema 2:

P_3 : $\begin{cases} \text{U\$ } 500000,00, p=0,41 \\ \text{U\$ } 0, p=0,89 \end{cases}$

P_4 : $\begin{cases} \text{U\$ } 2500000,00, p=0,1 \\ \text{U\$ } 0, p=0,9 \end{cases}$

		P_3	P_4
P_1	0	+	+
P_2	0	10	10
	0	17	17

$$P_2 \succ P_1 \Leftrightarrow E(U|P_2) \leq E(U|P_1) \Leftrightarrow$$

$$U(2500000) \frac{1}{10} + U(500000) \frac{89}{100} < U(500000)$$

$$+ U(0) \frac{1}{100} < U(500000)$$

$$\Leftrightarrow \frac{1}{100} U(500000) > \frac{10}{100} U(2500000) + \frac{1}{1000} U(0) \quad (\star)$$

$$P_3 \subset P_4 \Leftrightarrow E(U|P_3) < E(U|P_4) \Leftrightarrow$$

$$\Leftrightarrow \frac{U(500000)}{100} + U(0) \frac{89}{1000} < \frac{U(2500000)}{100} + U(0) \frac{90}{1000}$$

$$\Leftrightarrow \frac{U(500000)}{100} < \frac{U(2500000)}{100} + U(0) \quad (\star\star)$$

(4) Equivaléncia entre transformações lineares de utilidade.
(ultimo resultado visto)

(5) Suponhamos que $R = \{r_1, r_2, r_3\}$ onde $r_1 \leq r_2 \leq r_3$.

$$U'(r_1) \quad U'(r_2) \quad U'(r_3) \quad U(r) = \underline{U'(r)} - U'(r_1) \\ U'(r_3) - U'(r_1)$$

$$U(r_1) = 0$$

$$U(r_3) = 1$$

$$U(r_2) \in (0,1)$$

Sejam P_1 e P_2 medidas sobre R .

$$P_1 \subset P_2 \Leftrightarrow E(U|P_1) \leq E(U|P_2), \quad P_i(r_i) = p_i, i=1,2,3$$

$$\Leftrightarrow 0p_1 + U(\frac{p_2}{1-p_1})p_2 + p_3 \leq q_0q_1 + U(\frac{q_2}{1-q_1})q_2 + q_3. \quad P_2(r_i) = q_i, i=1,2,3$$

Então

$$\Leftrightarrow P_1 \subset P_2 \Leftrightarrow (p_3 - q_3) + U(r_2)(p_2 - q_2) \leq 0.$$

No paradoxo de Allais (1953),

$$R = \{0, 500000, 2500000\}$$

No prob. 1,

$$P_1, P_2 \rightarrow p_3 - q_3 = (0 - 0,1) = -0,1$$

$$p_2 - q_2 = 1 - 0,89 = 0,11$$

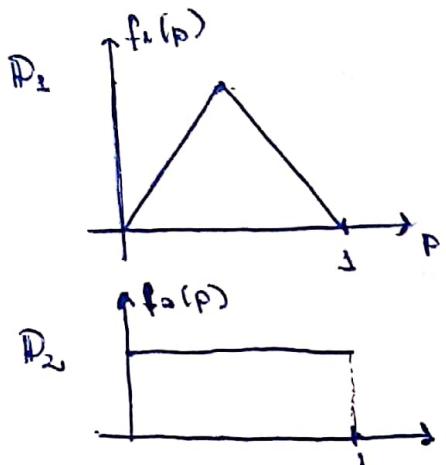
No prob. 2

$$P_3, P_4 \rightarrow p_3 - q_3 = (0 - 0,1) = -0,1$$

$$p_4 - q_4 = (0,11 - 0) = 0,11$$

Versão contínua do paradoxo de Allais

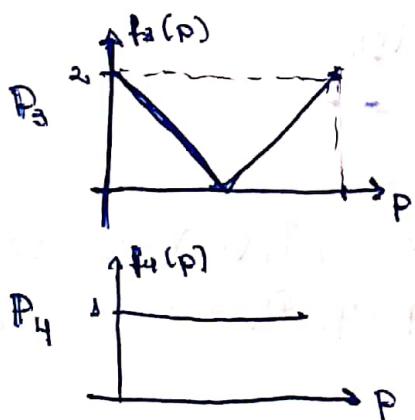
Problema 1



$$f_1(p) = \begin{cases} 4p, & p \in [0, 1/2] \\ 4(1-p), & p \in (1/2, 1) \end{cases}$$

$$f_2(p) = 2, \quad 0 \leq p \leq 1.$$

Problema 2.



$$f_3(p) = 4 |p - 1/2|, \quad p \in [0, 1]$$

$$f_4(p) = 1, \quad 0 \leq p \leq 1.$$

Note que $P_3 \leq^* P_2 \Leftrightarrow P_4 \leq^* P_3$ (Suponha $U: (0, 1) \rightarrow \mathbb{R}$ limitada)

Pode-se verificar que $\forall p \in (0, 1)$,

$$f_1(p) - f_2(p) = f_4(p) \cdot f_3(p)$$

Assim,

$$\mathbb{E}(U|P_3) - \mathbb{E}(U|P_2) = \cancel{\int_0^1 u(p) f_1(p) dp} - \cancel{\int_0^1 u(p) f_2(p) dp} = \cancel{\int_0^1 u(p) (f_1(p) - f_2(p)) dp} =$$

$$= \int_0^1 u(p) f_4(p) f_3(p) dp = \int_0^1 u(p) (f_4(p) \cdot f_3(p)) dp =$$

$$= \int_0^1 w(p) (f_u(p) - f_o(p)) dp = E(U|P_u) - E(U|P_o)$$

Assim,

$$E(U|P_u) - E(U|P_o) = E(U|P_4) - E(U|P_2)$$

Lista 2 no scerox!

Teoria da Decisão, Aula 11

$$L(d, \theta) = -U(r(d, \theta))$$

↳ perda (penalidade) na qual o indivíduo incorre ao escolher $d \in D$ quando o estado da natureza é $\theta \in \Theta$.

Para $d \in D$,

$$\rho(d, P) = \int_{\Theta} L(d, \theta) dP(\theta) \Rightarrow \text{risco da decisão } d.$$

$$p^*(P) = \inf \{ \rho(d, P) : d \in D \} \Rightarrow \text{risco de Bayes}$$

$d^* \in D$ é uma decisão de Bayes se

$$p^*(d^*, P) = p^*(P).$$

Exemplo: $\Theta = \{0, 1\}$, $P_1(\theta=0) = 1/4$

$$D = [0, 1]$$

$$L(d, \theta) = |d - \theta|$$

$$\begin{aligned} \rho(d, P_1) &= \int_{\Theta} L(d, \theta) dP_1(\theta) = L(d, 0)P_1(\theta=0) + L(d, 1)P_1(\theta=1) \\ &= |d - 0| \cdot \frac{3}{4} + |d - 1| \cdot \frac{1}{4} \\ &= \frac{3d}{4} + (1-d) \frac{1}{4} = \frac{1}{4} + \frac{d}{2}. \end{aligned}$$

$$\rho(d, P_1) = \frac{1}{4} + \frac{d}{2}.$$

$$\therefore p^*(P_1) = \frac{1}{4}. \text{ Como } \rho(0, P_1) = 1/4, 0 \text{ é decisão de Bayes.}$$

Suponhamos ainda nesse exemplo P_2 tal que $P_2(\theta=1) = 4/5$.

$$\rho(d, P_2) = \frac{1}{5}d + \frac{4}{5}(1-d) = \frac{4}{5} - \frac{3}{5}d$$

$$\rho^*(P_2) = 1/5.$$

Como $\rho(1, P_2) = \rho^*(P_2)$, 1 é decisão de Bayes contra P_2 .

$$P_1(\theta=1) - (P_1(\theta=0) - P_1(\theta=1))d = \rho(d, P).$$

Se $D = [0, 1]$, então

$$\rho^*(P_1) = 1/4, \text{ no entanto não existe decisão de Bayes.}$$

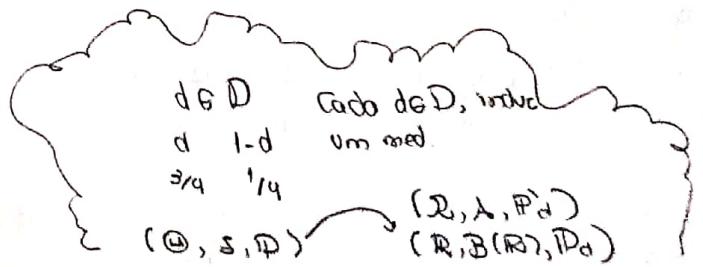
$$\rho^*(P_2) = 1/5, \dots \dots \dots \dots \dots \dots$$

Ainda no Exemplo 1, considere $L(d, \theta) = (d-\theta)^2$, $D = [0, 1]$, P_1 . Para $d \in D$,

$$\begin{aligned} \rho(d, P_1) &= (d-0)^2 P_1(\theta=0) + (d-1)^2 P_1(\theta=1) \\ &= d^2 \cdot \frac{3}{4} + (1-d)^2 \frac{1}{4} \end{aligned}$$

$$\rho(d, P_1) = \frac{d^2}{2} - \frac{d}{2} + \frac{1}{4}, \text{ que atinge valor mínimo em } d = 1/4.$$

Logo, $d^* = 1/4$ é a decisão de Bayes.



Comentários

(1) $D = \{\text{ENCAPAR}, \text{NÃO ENCAPAR}\}$

$$\begin{array}{c} \downarrow \\ d_1 \\ \downarrow \\ d_2 \end{array}$$

$U(d, \theta)$

	$\theta = 1$ (vende)	$\theta = 0$ (não vende)
d_1	$v - c$	$-c$
d_2	v	0

V : preço máximo de venda

c : preço do custo p/ encapar

$$P_{d_1}(\theta=1) = \frac{9}{10}$$

$$E(U|P_1) = (v-c) \frac{9}{10} + (-c) \frac{1}{10} = \frac{9v-10c}{10}$$

$$P_{d_2}(\theta=1) = \frac{1}{10}$$

$$E(U|P_2) = v \cdot \frac{2}{10} + 0 \cdot \frac{8}{10} = \frac{2v}{10}$$

Encapar é o melhor ~~desenvolvimento~~ que não encapar se, se os se

$$\frac{9v-10c}{10} > \frac{2v}{10} \Leftrightarrow c < \frac{7}{8}v.$$

(2) D, P, L

Seja $\lambda: \Theta \rightarrow \mathbb{R}$ tal que $\int_{\Theta} \lambda(\theta) dP(\theta) < \infty$.

$L': D \times \Theta \rightarrow \mathbb{R}$

$$(d, \theta) \mapsto L'(d, \theta) = L(d, \theta) + \lambda(\theta)$$

$$\int_{\Theta} L'(d, \theta) dP = \int_{\Theta} [L(d, \theta) + \lambda(\theta)] dP(\theta) = \int_{\Theta} L(d, \theta) dP(\theta) + \int_{\Theta} \lambda(\theta) dP(\theta)$$

Para todo $d \in D$ e $\lambda: \Theta \rightarrow \mathbb{R}$ tal que $\int_{\Theta} \lambda(\theta) dP(\theta) < \infty$, temos

$\forall d_1, d_2 \in D$, que

$$\int_{\Theta} L(d_1, \theta) dP(\theta) \leq \int_{\Theta} L(d_2, \theta) dP(\theta) \Leftrightarrow L^*(d_2, \theta)$$

$$\int_{\Theta} (\alpha L(d, \theta) + \lambda(\theta)) dP(\theta) \leq \int_{\Theta} (\alpha L(d_2, \theta) + \lambda(\theta)) dP(\theta)$$

Em particular, se para $\theta \in \Theta$, $L(\cdot, \theta)$ é limitada inferiormente, podemos considerar

$$\lambda_0(\theta) = \inf \{L(d, \theta) : d \in D\}$$

Assim, definindo

$$L_0(d, \theta) = L(d, \theta) - \lambda_0(\theta), \text{ temos } L_0(d, \theta) \geq 0, \forall d, \forall \theta.$$

Do comentário anterior,

$$L_0(d, \theta) \geq 0, \forall (d, \theta) \in D \times \Theta \text{ e}$$

$$\int_{\Theta} L(d_1, \theta) dP(\theta) \leq \int_{\Theta} L(d_2, \theta) dP(\theta) \Leftrightarrow$$

$$\int_{\Theta} L_0(d_1, \theta) dP(\theta) \leq \int_{\Theta} L_0(d_2, \theta) dP(\theta), \forall d_1, d_2 \in D.$$

Exemplo: $D = \{d_1, d_2, d_3, d_4\}$, $\Theta = \{\theta_1, \theta_2, \theta_3\}$

$L(d, \theta)$	θ_1	θ_2	θ_3
d_1	2	3	9
d_2	-2	7	-3
d_3	10	7	-9
d_4	-3	8	0

$$\begin{aligned}\lambda_0(\theta_1) &= -3 \\ \lambda_0(\theta_2) &= 3 \\ \lambda_0(\theta_3) &= -4\end{aligned}$$

$L_0(d, \theta)$	θ_1	θ_2	θ_3
d_1	5	0	13
d_2	1	4	1
d_3	13	4	0
d_4	0	5	4

(3) Vimos que, no Exemplo 1, que $p^*(P_1) = 3/4$ ($P_1(\theta=1) = 1/4$) e $p^*(P_2) = 1/5$ ($P_2(\theta=1) = 4/5$).

Fixado Θ , \mathcal{F} e L , consideremos \mathcal{P}_L o conjunto de todas as medidas de probabilidade em (Θ, \mathcal{F}) .

\mathcal{P}_L é um conjunto convexo, pois se $\forall P_1, P_2 \in \mathcal{P}_L$ e $\alpha \in [0,1]$,

$$\alpha P_1 + (1-\alpha) P_2 \in \mathcal{P}_L.$$

Vamos olhar a transformação

$$p^* : \mathcal{P}_L \rightarrow \mathbb{R}$$

$$P \in \mathcal{P}_L \mapsto p^*(P) = \inf \{ \rho(d, P) : d \in D \},$$

resultado. p^* é concava.

Sejam $P_1, P_2 \in \mathcal{P}_L$ e $\alpha \in [0,1]$. Para $d \in D$,

$$\begin{aligned} \rho(d, \alpha P_1 + (1-\alpha) P_2) &= \int_{\Theta} L(d, \theta) d(\alpha P_1 + (1-\alpha) P_2) \\ &= \alpha \int_{\Theta} L(d, \theta) dP_1 + (1-\alpha) \int_{\Theta} L(d, \theta) dP_2 \\ &= \alpha \rho(d, P_1) + (1-\alpha) \rho(d, P_2) \end{aligned}$$

$$\begin{aligned} p^*(\alpha P_1 + (1-\alpha) P_2) &= \inf \{ \rho(d, \alpha P_1 + (1-\alpha) P_2) : d \in D \} \\ &= \inf \{ \alpha \rho(d, P_1) + (1-\alpha) \rho(d, P_2) : d \in D \} \\ &\geq \inf \{ \alpha \rho(d, P_1) \} + \inf \{ (1-\alpha) \rho(d, P_2) : d \in D \} \end{aligned}$$

$$= \alpha p^*(P_1) + (1-\alpha) p^*(P_2)$$

Logo,

$$p^*(\alpha P_1 + (1-\alpha) P_2) \geq \alpha p^*(P_1) + (1-\alpha) p^*(P_2).$$

Exemplo: $D = [0,1]$

$$\Theta = \{0,1\}, \mathcal{F} = \mathcal{P}(\{0,1\})$$

$$P_p = \{ \text{Ber}(p); p \in [0,1] \}$$

$$L(d, \theta) = (d - \theta)^2$$

Seja $d \in D$ e $p \in [0,1]$. P_p

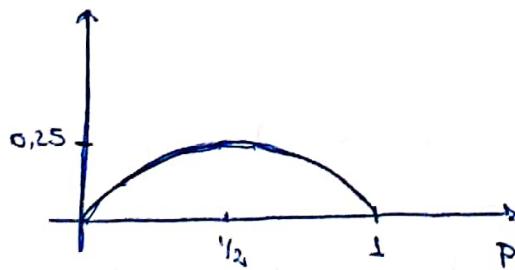
$$p(d, P_p) = (d-0)^2 P_p(\theta=0), (d-1)^2 P_p(\theta=1) =$$

$$= d^2(1-p) + (1-d)^2 p \Rightarrow$$

$$d^2 - pd^2 + p - 2pd + pd^2 = d^2 - 2pd + p$$

$$p(d, P_p) = d^2 - 2pd + p \Rightarrow d^* = p \text{ é decisão de Bayes contra } P_p.$$

$$p^*(P_p) = p^2 - 2p^2 + p = p - p^2 = p(1-p).$$



p^* é estritamente côncava.

No exemplo anterior, considera agora $L(d, \theta) = 1|d-\theta|$.

Para $d \in D$,

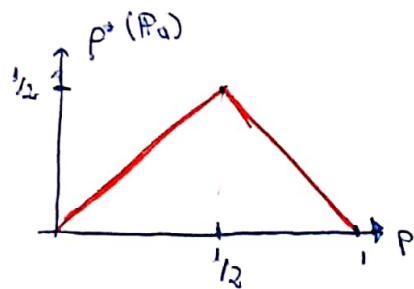
$$\begin{aligned} p(d, P_p) &= 1|d-0| P_p(\theta=0) + 1|d-1| P_p(\theta=1) \\ &= d(1-p) + (1-d)p \\ &= p + d(1-2p) \end{aligned}$$

$p < 1/2$ $\xrightarrow{\text{reto crescente}}$ $d^* = 0$ é decisão de Bayes contra \overline{P}_D
 $p > 1/2$ $\xrightarrow{\text{decrese}}$ $d^* = 1$ " " " "
 $p = 1/2$ $\xrightarrow{\text{const.}}$ toda ds D é decisão de Bayes

$$p < 1/2, \quad p^*(\overline{P}_D) = p(0, \overline{P}_D) = p$$

$$p > 1/2, \quad p^*(\overline{P}_D) = p(1, \overline{P}_D) = p + 1 - 2p = 1 - p$$

$$p = 1/2, \quad p^*(\overline{P}_D) = p^{1/2}$$



Teoria da Decisão

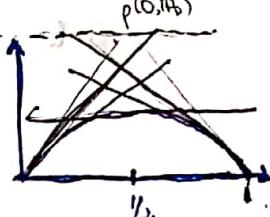
Aula 12

23/09/15

Ex. 1: $\Theta = \{0, 1\}$ (A) $L(d, \theta) = (d - \theta)^2$

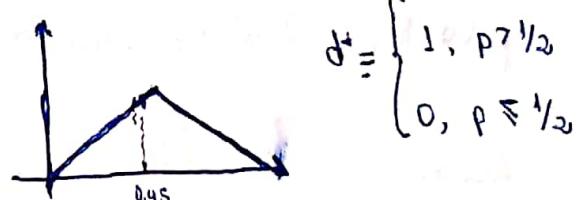
$$P_p(\theta=0) = p \quad P_p(\theta=1) = 1-p \quad \rho^*(P_p) = p(1-p)$$

$$\mathcal{D} = [0, 1]$$



$$(B) L(d, \theta) = |d - \theta|$$

Suponha que em (A), você acha que $\theta=1$ com probabilidade 0,3. Você decidirá $p=0,2$.



$$\text{Risco 'Verdadeiro': } \rho^*(0, 2) = 0,3 \times 0,7 = 0,21$$

$$\text{Risco Incerto: } \rho(0, 2, 0, 3) = (0,2)^2 + (0,3)[1 - 2(0,2)] = 0,02$$

Em (B), $\rho(d, P_\theta) = d(1-p) + (1-d)p = d + p(1-2d)$ e

$$\rho^*(0, 2) = 0,3$$

$$\rho(0, 0, 3) = 0,3$$

Suponha agora que você acha que $\theta=1$ com probabilidade 0,45.

Você decidirá $p=0,55$. Nesse caso

$$\rho^*(0, 45) = 0,45$$

$$\rho(1, 0, 45) = 0,55$$

Exemplo 2. $\Theta = \{0, 1, 2\}$

$$P_{(p_1, p_2)}(\theta=0) = p_1 \quad P_{(p_1, p_2)}(\theta=1) = p_2 \quad P_{(p_1, p_2)}(\theta=2) = p_3$$

$$P_{(p_1, p_2)}(\theta=0) = p_1 \quad P_{(p_1, p_2)}(\theta=1) = p_2 \quad P_{(p_1, p_2)}(\theta=2) = p_3 \quad \mathcal{D} = R_+$$

$$L(d, \theta) = (d - \theta)^2$$

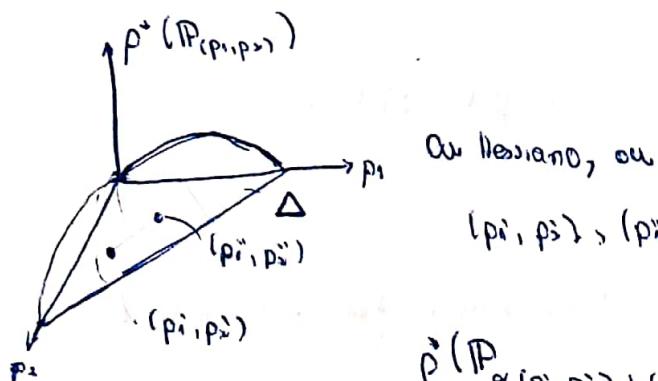
$$\begin{aligned}
 p(d, P_{(p_1, p_2)}) &= E((d - \theta)^2 | P_{(p_1, p_2)}) = \\
 &= E((d - E(\theta | P_{(p_1, p_2)}))^2 | P_{(p_1, p_2)}) \\
 &= (d - E(\theta | P_{(p_1, p_2)}))^2 + E((\theta - E(\theta | P_{(p_1, p_2)}))^2 | P_{(p_1, p_2)}) \\
 &\quad + 2 \cancel{(d - E(\theta | P_{(p_1, p_2)}))} E((\theta - E(\theta | P_{(p_1, p_2)}))^2 | P_{(p_1, p_2)}) \\
 &\quad \cancel{0}
 \end{aligned}$$

$$\Rightarrow p(d, P_{(p_1, p_2)}) = \text{Var}(\theta | P_{(p_1, p_2)}) + (d - E(\theta | P_{(p_1, p_2)}))^2$$

$p(d, P_{(p_1, p_2)})$ atinge valor mínimo em $d^* = E(\theta | P_{(p_1, p_2)})$.

Assim,

$$\begin{aligned}
 p^*(P_{(p_1, p_2)}) &= p(E(\theta | P_{(p_1, p_2)}), P_{(p_1, p_2)}) = \text{Var}(\theta | P_{(p_1, p_2)}) \\
 &= 1p_1 + 4p_2 - (1p_1 + 2p_2)^2
 \end{aligned}$$



Ou Hessiano, ou

$$(p_1, p_2), (p_1'', p_2'') \in \Delta f(p_1, q_2) \subset \mathbb{R}_+^2 : q_1 + q_2 \leq 1 \}$$

$$\begin{aligned}
 p^*(P_{\alpha(p_1, p_2) + (1-\alpha)(p_1'', p_2'')}) &\geq \alpha p^*(P_{(p_1, p_2)}) + \\
 &\quad + (1-\alpha) p^*(P_{(p_1'', p_2'')}) \\
 &= \alpha(1-\epsilon) [(p_1 + 2p_2) - (p_1'' + 2p_2'')]^2
 \end{aligned}$$

Livros de Análise Convexa: Rockafellar, Análise Convexa

Scherovich, Ferguson (Teoria da Decisão)
(Teoria da Eq.)

$$\mathcal{D} = \{d_1, d_2, \dots\}$$

Definição: Ao procedimento que consiste em tomar a decisão $d \in \mathcal{D}$ com probabilidade α_i , i.e., damos o nome de decisão ALEATORIZADA ou MISTA associada à medida $(\alpha_n)_{n \geq 1}$.

Notação: ~~definição~~

$d \in \mathcal{D}$ é "aleatorizada" segundo uma medida degenerada em $\{d\}$. \rightarrow Decisão Pura ou " $d_{(\alpha_n)_{n \geq 1}} = \sum_{i=1}^{\infty} \alpha_i d_i$ " Não Aleatorizada

$$M_p = \{d_{(\alpha)} : \alpha \text{ é medida de prob. em } (\mathcal{D}, \mathcal{P}(\mathcal{D}))\}$$

Claro que $\mathcal{D} \subseteq M_p$.

Fixada \mathbb{P} sobre $(\mathcal{Q}, \mathcal{F})$, $\inf \{\rho(d, \mathbb{P}) : d \in \mathcal{M}\} \leq \inf \{\rho(d, \mathbb{P}) : d \in \mathcal{D}\}$, onde para $d \in M_p$, definimos

$$L(d_{(\alpha)}, \theta) = \sum_{n=1}^{\infty} \alpha_n L(d_n, \theta) \quad (L \text{ estendida a } M_p \times \mathcal{Q})$$

e

$$\rho(d_{(\alpha)}, \mathbb{P}) = \int_{\mathcal{Q}} L(d_{(\alpha)}, \theta) d\mathbb{P}(\theta)$$

Por outro lado, para $d \in \mathcal{M}_p$,

$$\begin{aligned} \rho(d_{(\alpha)}, \mathbb{P}) &= \int_{\mathcal{Q}} L(d_{(\alpha)}, \theta) d\mathbb{P}(\theta) \\ &= \int_{\mathcal{Q}} \left(\sum_{n=1}^{\infty} \alpha_n L(d_n, \theta) \right) d\mathbb{P}(\theta) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \alpha_n \int_{\Theta} L(d_n, \theta) dP(\theta)$$

$$= \sum_{n=1}^{\infty} \alpha_n p(d_n, P) \geq \sum_{n=1}^{\infty} \alpha_n \inf \{ p(d, P), d \in D \} = \\ = \inf \{ p(d, P) : d \in D \}$$

Como para toda $d \in \mathcal{M}_0$,

$$p(d, P) \geq \inf \{ p(d, P) : d \in D \}$$

segue que

$$\inf \{ p(d, P) : d \in \mathcal{M}_0 \} \geq \inf \{ p(d, P) : d \in D \}$$

Vamos considerar ① e ④ finitos.

$$D = \{d_1, \dots, d_m\}, \quad \Theta = \{\theta_1, \dots, \theta_n\}$$

Para $d \in D$, vamos considerar

$$l(d) = (L(d, \theta_1), L(d, \theta_2), \dots, L(d, \theta_n))$$

Do mesmo modo, para $d_{m+1} \in \mathcal{M}_0$,

$$l(d_{m+1}) = \left(\sum_{j=1}^m \alpha_j L(d_j, \theta_1), \sum_{j=1}^m \alpha_j L(d_j, \theta_2), \dots, \sum_{j=1}^m \alpha_j L(d_j, \theta_n) \right)$$

$$l(\mathcal{M}_0) = \{l(d) : d \in \mathcal{M}_0\}$$

Exemplo 1:

$$k=2 \quad \Theta = \{\theta_1, \theta_2\}$$

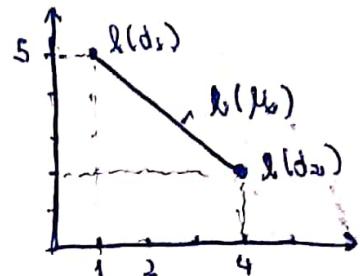
$$m=2 \quad D = \{d_1, d_2\}$$

$$L(d, \theta)$$

$$\begin{matrix} d_1 & d_2 \\ \theta_1 & 1 & 4 \\ \theta_2 & 5 & 2 \end{matrix}$$

$$l(d_1) = (1, 5)$$

$$l(d_2) = (4, 2)$$



$$L(d_{\text{mín}}, \theta_1) = \alpha L(d_1, \theta_1) + (1-\alpha)L(d_2, \theta_2)$$

Exemplo 2:

$$k=2, m=4, \Theta = \{\theta_1, \theta_2\}, D = \{d_1, d_2, d_3, d_4\}$$

$$L(d, \theta)$$

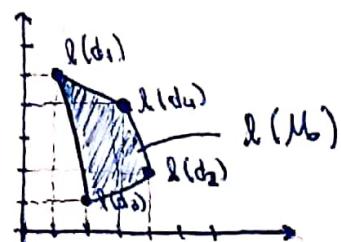
$$\begin{matrix} d_1 & d_2 & d_3 & d_4 \\ \theta_1 & 1 & 4 & 2 & 3 \\ \theta_2 & 5 & 2 & 1 & 4 \end{matrix}$$

$$l(d_1) = (1, 5)$$

$$l(d_2) = (4, 2)$$

$$l(d_3) = (2, 1)$$

$$l(d_4) = (3, 4)$$



vértices do polígono são

chamado de pontos extremais
envoltório convexo

Teoria da Decisão, Aula 14

Alguns outros resultados.

Resultado: d^* é a única decisão de Bayes contra $P \Rightarrow d^*$ é admissível

Suponha d , decisão de Bayes contra P e \emptyset inadmissível.

Então, $\exists d' \in D$ tal que

$$L(d', \theta) \leq L(d, \theta), \forall \theta \in \Theta \text{ e } \exists \theta_0 \in \Theta \text{ tal que}$$

$$L(d', \theta_0) < L(d, \theta_0)$$

$$\rho(d', P) = \int_{\Theta} L(d', \theta) dP \leq \int_{\Theta} L(d, \theta) dP$$

$$\rho(d', P) = \rho^*(P)$$

Logo, d' é decisão de Bayes e, portanto,

$$\{d, d'\} \subseteq \{d^* \in D : \rho(d^*, P) = \rho^*(P)\}$$

Resultado: Seja $D_P = \{d \in D : \rho(d, P) = \rho^*(P)\} \neq \emptyset$.

$\forall d_1, d_2 \in D_P$,

$$L(d_1, \cdot) = L(d_2, \cdot) \Rightarrow \forall d \in D_P, d \text{ é admissível}$$

$\exists d' \in D$ tal que

$$L(d', \theta) \leq L(d^*, \theta), \forall \theta \in \Theta \text{ e}$$

$$L(d', \theta_0) < L(d^*, \theta_0), \text{ para } \theta_0 \in \Theta$$

Do argumento exposto no resultado anterior, segue que $d^* \in D_p$.
Logo, d^* e $d^* \in D_p$ são tais que

$$L(d^*, \theta_0) \neq L(d^*, \theta_0)$$

Resultado: Suponha que $L(d^*, \theta)$ é contínua para todo $\theta \in \Theta$. Se $d^* \in D$ é de bayes contra algum P tal que $P(A) > 0$, $\forall A \in \mathcal{F}$ dado, então d^* é admissível.

Suponhamos $d^* \in D$ inadmissível. Então $\exists d' \in D$ tal que

$$L(d', \theta) \leq L(d^*, \theta) \quad \forall \theta \in \Theta$$

$$L(d', \theta_0) \leq L(d^*, \theta_0), \text{ para algum } \theta_0 \in \Theta.$$

Seja $\epsilon = L(d^*, \theta_0) - L(d', \theta_0) > 0$. Como $L(d', \theta)$ é contínua, $\exists \delta_1 > 0$ tal que

$$|\theta - \theta_0| \leq \delta_1 \Rightarrow |L(d', \theta) - L(d', \theta_0)| \leq \epsilon/4 \Rightarrow$$

$$L(d', \theta) \leq L(d', \theta_0) + \epsilon/4. \quad (\text{I})$$

De mesmo modo, $L(d^*, \theta)$ é contínua, e $\exists \delta_2 > 0$ tal que
 $|\theta - \theta_0| \leq \delta_2 \Rightarrow |L(d^*, \theta) - L(d^*, \theta_0)| \leq \epsilon/4 \Rightarrow$

$$L(d^*, \theta) \geq L(d^*, \theta_0) - \epsilon/4 \quad (\text{II})$$

De (I) e (II)

$$\begin{aligned} L(d^*, \theta_0) + \epsilon/4 &\leq L(d', \theta_0) \leq L(d^*, \theta_0) - \epsilon/4 \\ \epsilon/4 &\leq L(d', \theta_0) - L(d^*, \theta_0) \end{aligned}$$

$$|\theta - \theta_0| \leq \min\{\delta_1, \delta_2\} \Rightarrow L(d^*, \theta) - L(d', \theta) \leq \frac{\epsilon}{2}$$

Seja P uma probabilidade tal como no enunciado. Assum,

$$\rho(d^*, P) - \rho(d^*, P) = \int_{\Theta} L(d^*, \theta) dP - \int_{\Theta} L(d^*, \theta) dP$$

$$= \int_{\Theta} [L(d^*, \theta) - L(d^*, \theta)] dP = \int_{\Theta} [L(d^*, \theta) - L(d^*, \theta)] dP$$

$$+ \int_{\Theta - \{\theta_0\} \cap \min\{\delta_1, \delta_2\}} [L(d^*, \theta) - L(d^*, \theta)] dP \geq \frac{1}{2}$$

$$\geq \frac{1}{2} P(\Theta - \{\theta_0\} \cap \min\{\delta_1, \delta_2\})$$

Logo, d^* não é decisão de Bayes contra P .

(D, Θ, L, P) , prob de decisões

incerteza
perda
param.

decisões

$$\rho(d, P) = \int_{\Theta} L(d, \theta) dP$$

e temos d^* tal que $\rho(d^*, P) = \inf_{d \in D} \rho(d, P)$

Exemplo 3.

$d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6 \quad d^*$

$\theta_1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

$\theta_2 \quad 10 \quad 7 \quad 4 \quad 5 \quad 2 \quad 3 \quad 6$

$\theta_3 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

$\theta_4 \quad P(\theta = \theta_1) = p, P(\theta = \theta_2) = 1-p$

$\theta_5 \quad f(d, \theta) = 1/d$

$\theta_6 \quad$

Decisão de Bayes

$$p \in [1/2, 1]$$

$$\frac{1}{2} \leq p \leq \frac{3}{5} \rightarrow d_3$$

$$p > \frac{3}{5} \rightarrow d_2$$

Problemas de Decisão Com Dados (Observações).

No exemplo 3, considere o experimento que consiste em observar X .

Suponhamos que $X | \Theta = \Theta_1 \sim \text{Bernoulli}(4/5)$

$X | \Theta = \Theta_2 \sim \text{Bernoulli}(1/3)$

Vamos considerar dois cenários.

- Pré-Experimentação

- Pós-Experimentação

CASO PÓS-EXPERIMENTAÇÃO:

$X=0$. Temos então o problema de decisão $(D, \Theta, L, P_{X=0})$, onde $P_{X=0}$ é a medida de probabilidade sobre Θ dado $X=0$.

Pelo Teorema de Bayes,

$$P(\Theta = \Theta_1 | X=0) = \frac{P(X=0 | \Theta = \Theta_1) P(\Theta = \Theta_1)}{\sum_{i=1}^k P(X=0 | \Theta = \Theta_i) P(\Theta = \Theta_i)}$$

$$= \frac{P(X=0 | \Theta = \Theta_1) P(\Theta = \Theta_1)}{P(X=0 | \Theta = \Theta_1) + P(X=0 | \Theta = \Theta_2) P(\Theta = \Theta_2)}$$

$$= \frac{\frac{1}{5} \cdot p}{\frac{1}{5} \cdot p + \frac{2}{3} (1-p)} = \frac{3p}{3p + 10(1-p)}$$

Para d₂ e d₃, temos

$$\begin{aligned} p(d_2, P_{x=0}) &= \int_{\Theta} L(d_2, \theta) dP_{x=0}(\theta) = \frac{1}{10} (1-\theta)^2 - (1-\theta)(1-p) \\ &\Rightarrow L(d_2, \theta_1) P(\theta=\theta_1 | x=0) + L(d_2, \theta_2) P(\theta=\theta_2 | x=0) \\ &\Rightarrow 0 \cdot \frac{3p}{3p+10(1-p)} + \frac{10(1-p)}{3p+10(1-p)} \end{aligned}$$

$$P_{x=0}(\theta)$$

Como $P(\theta=\theta_1 | x=0) = \frac{3p}{10-7p}$, temos que d₂ será decisão

de Bayes contra $P_{x=0}$ se

$$P(\theta=\theta_1 | x=0) = \frac{3p}{10-7p} \leq \frac{1}{2}$$

$$\Leftrightarrow 6p \leq 10-7p \Leftrightarrow p \leq \frac{10}{13}$$

Do mesmo modo, d₃ será decisão de Bayes contra $P_{x=0}$ se

$$\frac{3p}{10-7p} > \frac{3}{5} \Leftrightarrow 15p > 30-21p \Leftrightarrow p > \frac{5}{12}.$$

Assim, temos

$p \leq \frac{10}{13}$, a decisão de Bayes contra $P_{x=0}$ será d₃:

$$\frac{10}{13} \leq p \leq \frac{5}{6}, d_3$$

$$p > \frac{5}{6}, d_2 \text{ será } d_1$$

$$X = \{P_{f(x)=y}\}$$

$$P(\theta = \theta_1 | X=1) = \frac{P(X=1 | \theta = \theta_1) P(\theta = \theta_1)}{\sum_{i=1}^k P(X=1 | \theta = \theta_i) P(\theta = \theta_i)}$$

$$\frac{\frac{4}{5}P}{\frac{4}{5}P + \frac{1}{8}(1-P)} = \frac{12P}{12P + 8(1-P)}$$

$$= \frac{13p}{5+7p}$$

Como $P(S=0_1 | X=1) = \frac{12p}{5+7p}$, temos que

de ser a decisão de Bayes contra $P_{q|x=1}$ se

$$P(\Theta=0,1|X=1) = \frac{12p}{5+7p} \leq \frac{1}{2} \Leftrightarrow 24p \leq 5+7p \Leftrightarrow p \leq \frac{5}{17}$$

Do mesmo modo, a decisão de Bayes contradiz $P_{S|x=1}$, se

$$\frac{12p}{5+7p} > \frac{3}{5} \Leftrightarrow 60p > 15 + 21p \Leftrightarrow p > \frac{5}{13}$$

Assim, tempo

$p \in S$, ds. será a decisão de Bayes contra $P_{\{x=1\}}$

$$\frac{5}{17} \leq p \leq \frac{5}{12}, d_3 = "no" \quad "no" \quad "no" \quad "no" \quad "no" \quad P_{\lambda x + 19}$$

$$P \geq \frac{5}{13}, \quad d_{23} \rightarrow \infty \quad \dots \quad \dots \quad P_{\{x=1\}}$$

Caso Pré-Experimentação.

Suponha que o exp. não foi conduzido e que X é também incerto (desconhecido).

Nesse caso, há incerteza sobre $(\theta, x), \theta \in \Theta$, se possivei observação do experimento. Além disso, devemos especificar agora uma regra de decisão que associa a cada x uma decisão de D .

Seja \mathcal{X} o coto dos possíveis resultados do experimento.

(No exemplo 3, $\mathcal{X} = \{0,1\}$)

Temos assim um novo problema de decisão. Seja

$$\Delta = \{\delta : \mathcal{X} \rightarrow D\}$$

o conjunto de regra de decisões. Seja

$$\Omega = \Theta \times \mathcal{X}, \text{ munido de } \sigma(\Theta, \mathcal{X})$$

Seja $P' : \sigma(\Theta, \mathcal{X}) \rightarrow [0,1]$ uma probabilidade.

Como antes, seja $L : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ e definamos

$$L' : \Delta \times \Omega \rightarrow \mathbb{R} \text{ por.}$$

Para $\delta \in \Delta$ e $(\theta, x) \in \Omega$, $L'(\delta, (\theta, x)) = L(\delta(x), \theta)$.

$$(\Delta, \Omega, L', P')$$

Para cada $\delta \in \Delta$, seja

$$\bar{p}(\delta, P) := \int_{\Omega} l^*(\delta, (\omega, x)) dP(\omega, x) \quad \text{o Risco da decisão}$$

Função de Decisão δ contra P' .

Do mesmo modo,

$$\bar{p}^*(P) = \inf \{\bar{p}(\delta, P'): \delta \in \Delta\} \quad \text{é chamado Risco de Bayes}$$

Teoria da Decisão

Aula 15

07/10/2015

• Problema Pós-Experimentação

(Π, \mathcal{D}, Δ)

Após observar X_{∞} , você resolve o problema $(\mathcal{D}, \Theta, L, P_{X_{\infty}})$

• Problema Pré-Experimentação

$$\Delta = \{\delta: \mathcal{X} \rightarrow \mathcal{D}\}$$

↳ frangues regra de decisão

$$\Omega = \Theta \times \mathcal{X}$$

$$P' \text{ sobre } \sigma(\Theta \times \mathcal{X})$$

$$L': \Delta \times \Omega \rightarrow \mathbb{R}$$

$$(\delta, (\theta, x)) \mapsto L'((\delta, (\theta, x))) = L(\delta(\theta), \theta)$$

Para cada $\delta \in \Delta$,

$$\rho'(\delta, P') = \int_{\Theta \times \mathcal{X}} L'((\delta, (\theta, x))) dP'(\theta, x) \rightarrow \text{risco da função de decisão } \delta \text{ contra } P'$$

$$\rho''(P') = \inf \{\rho'(\delta, P'): \delta \in \Delta\} \rightarrow \text{Risco de Bayes}$$

Uma regra $\delta^* \in \Delta$ tal que $\rho'(\delta^*, P') = \rho''(P')$ é chamada Regra (ou função) de decisão de Bayes.

O procedimento de avaliar $p^*(\delta, P')$ para cada $\delta \in \Delta$ e então tomar $\delta^* \in \Delta$ e então tomar $\delta \in \Delta$ que atinge o Risco de Bayes.

$p^*(P')$ é a chamada FORMA NORMAL do problema de decisão (Δ, Ω, L, P') .

(Δ, Ω, L, P')

Para $\delta \in \Delta$,

$$p^*(\delta, P') = \int_{\Omega \times \Xi} L(\delta, (\theta, \omega)) dP' = \int_{\Omega \times \Xi} L(\delta(\omega), \theta) dP'$$

$$= \int_{\Omega \times \Xi} L(\delta(\omega), \theta) f(\theta, \omega) d\theta d\omega$$

$$= \int_{\Omega} \left[\int_{\Xi} L(\delta(\omega), \theta) f(\omega|\theta) f(\theta) d\omega \right] d\theta$$

$$= \int_{\Omega} \underbrace{\left[\int_{\Xi} L(\delta(\omega), \theta) f(\omega|\theta) d\omega \right]}_{\text{Risco da função de decisão } \delta \in \Delta} f(\theta) d\theta$$

Risco da função de decisão $\delta \in \Delta$

quando o estado de natureza é θ , $R(\delta, \theta)$

$$= \int_{\Xi} \left[\int_{\Omega} L(\delta(\omega), \theta) f(\omega|\theta) \frac{f(\theta)}{f(\omega)} f(\theta) d\theta \right] d\omega \Rightarrow$$

$$p^*(\delta, P') = \int_{\Xi} f(\omega) \left[\int_{\Omega} L(\delta(\omega), \theta) f(\theta|\omega) d\theta \right] d\omega$$

$p^*(\delta(\omega), P_{X=\omega})$

$$7) \int_{\mathcal{X}} f(x) \left[\int_{\Theta} L(d_x^*, \theta) f(\theta|x) d\theta \right] dx, \text{ onde } (D, \Theta, L, P_{x|x})$$

d_x^* é decisão de Bayes do problema $(D, \Theta, L, P_{x|x})$.

Definindo $\delta^* : \mathcal{X} \rightarrow D$ por $\delta^*(x) = d_x^*$, resulta que $\forall \delta \in \Delta$,

$$p^*(\delta, P) \geq p^*(\delta^*, P),$$

Logo, δ^* definido acima é regra de decisão de Bayes contra P .

Ao procedimento que consiste em resolver o problema $(D, \Theta, L, P_{x|x})$ para cada $x \in \mathcal{X}$ e então construir a função de decisão do problema (D, Ω, L', P') como descrito ao lado, dá-se o nome de Forma Extensiva do problema (D, Ω, L', P') .

Retomando o exemplo da aula passada.

$$\Theta = \{\Theta_1, \Theta_2\}$$

$$P(G_1, G_2) \rightarrow P$$

	d_1	d_2	d_3	d_4	d_5	d_6	d_7
Θ_1	0	0	2	3	4	10	6
Θ_2	10	7	4	5	2	3	6

$$\mathcal{B} = \{0,1\} \quad X | \Theta_1 \sim \text{Ber}(4/5)$$

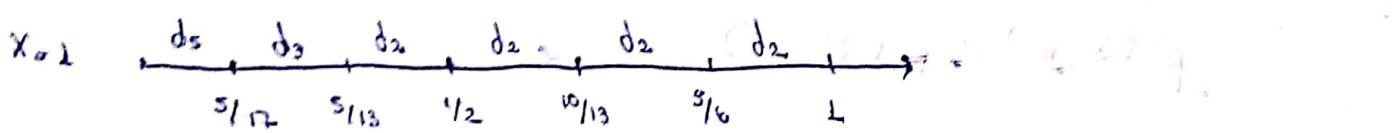
$$X | \Theta_2 \sim \text{Ber}(1/3)$$

$(D, \oplus, L, P_{X=0})$

$(D, \oplus, L, P_{X=1})$

$$d_0^* = \begin{cases} d_2, & se \quad p \leq 3/6 \\ d_3, & se \quad 3/6 < p < 4/6 \\ d_5, & se \quad p \geq 4/6 \end{cases}$$

$$d_1^* = \begin{cases} d_2, & p \leq 5/13 \\ d_3, & 5/13 < p < 6/13 \\ d_5, & p \geq 6/13 \end{cases}$$



Suponhamos $P(G_1 \cap G_2) = p = 4/10$. A sua regra de decisão de Bayes é

$$\delta^*(x) = \begin{cases} d_5, & x=0 \\ d_2, & x=1 \end{cases}$$

Se $P(G_1 \cap G_2) = 8/10$, então

$$\delta^*(x) = \begin{cases} d_3, & x=0 \\ d_2, & x=1 \end{cases}$$

Se $P(G_1 \cap G_2) = 9/10$, então

$$\delta^*(x) = d_2, \quad \forall x \in \{0, 1\}$$

Analogamente, se $p = 3/60$,

$$\delta^*(x) = d_2, \quad \forall x \in \{0, 1\}$$

$$p^*(P_p) = p^*(\delta^*, P_p)$$

$$\sum_{x=0}^1 \sum_{i=1}^2 L(\delta^*(\infty), \Theta_i) P(\Theta=\Theta_i, X=x)$$

$$\sum_{x=0}^1 \sum_{i=1}^2 L(\delta^*(\infty; G_i)) P(\Theta=\Theta_i) P(X=x | G=G_i)$$

$$= L(\delta^*(0), \Theta_1) p \frac{1}{5} + L(\delta^*(0), \Theta_2) (1-p) \frac{2}{3} + L(\delta^*(1), \Theta_1) p \frac{4}{5} \\ + L(\delta^*(1), \Theta_2) (1-p) \frac{1}{3}$$

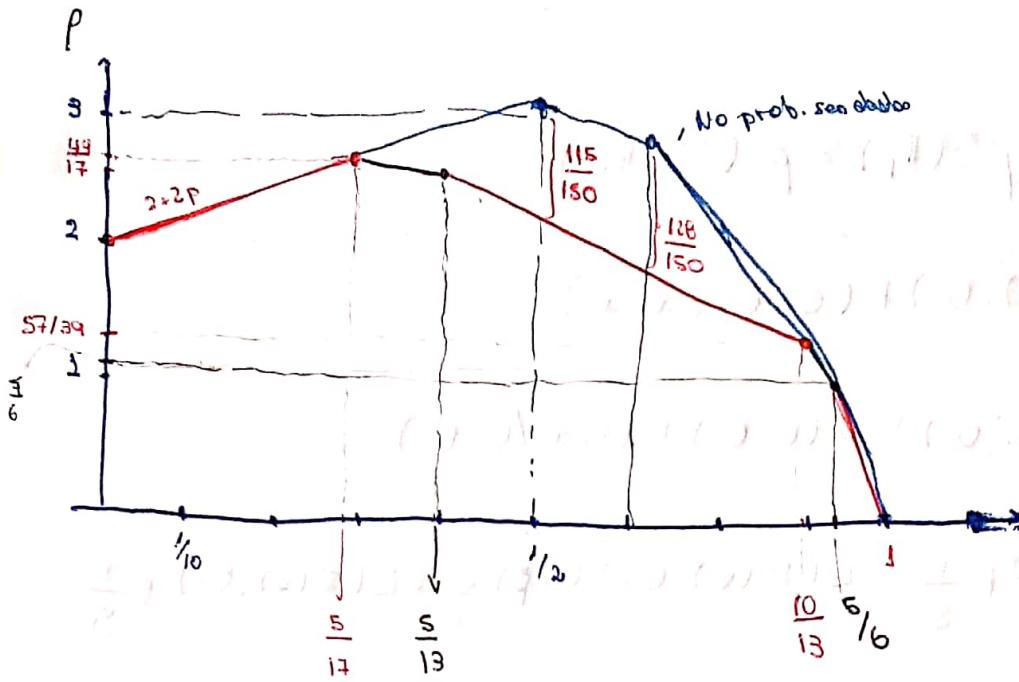
$$p \cap \frac{5}{17} \Rightarrow \frac{4p}{5} + 2(1-p) \frac{2}{3} + 4 \frac{4p}{5} + 2(1-p) \frac{1}{3} = 4p + 2(1-p) = 2 + 2p$$

$$\frac{5}{17} \cap p \cap \frac{5}{13} \Rightarrow \frac{4p}{5} + 2(1-p) \frac{2}{3} + 12 \frac{4p}{5} + 4 \frac{(1-p)}{3} = \frac{40 - 4p}{15}$$

$$\frac{5}{13} \cap p \cap \frac{10}{5} \Rightarrow \frac{4p}{5} + 2(1-p) + 0 \frac{4p}{5} + 7 \frac{(1-p)}{3} = \frac{4p}{5} + \frac{11(1-p)}{5} \\ = \frac{55 - 43p}{5}$$

$$\frac{10}{13} \cap p \cap \frac{5}{6} \Rightarrow 2p + 4 \cdot 2(1-p) + 0 \cdot \frac{4p}{5} + 7 \cdot \frac{(1-p)}{3} = \frac{2p}{5} + \frac{15(1-p)}{3} \\ = \frac{25 - 23p}{5}$$

$$p > \frac{5}{6} \Rightarrow 0 \frac{p}{5} + 7 \frac{2(1-p)}{3} + 0 \frac{4p}{5} + 7 \frac{(1-p)}{3} = 7(1-p)$$



No problema, temos de:

$$P(\cdot) = \begin{cases} 2+2p & , p \leq 1/2 \\ 4-2p & , 1/2 \leq p \leq 3/5 \\ 2(1-p) & , p \geq 3/5 \end{cases}$$

Problema de Determinação do Tamanho Amostral e Da Regra da Decisão

espaco a ser observar
Ω = Ω × ℜ, ℜ = $\prod_{i=1}^{\infty} \mathcal{E}_i$, tudo que desse nexo sobre

$$\Delta_n = \{\delta : \prod_{i=1}^n \mathcal{E}_i \rightarrow D\}$$

$$\Delta_0 = D$$

$$\bar{\Delta} = \bigcup_{n=0}^{\infty} \Delta_n \times \Delta_n \rightarrow \text{espaço de ações.}$$

Π sobre σ(Ω × ℜ), L : $\bar{\Delta} \times \bar{\Omega} \rightarrow \mathbb{R}$

$$((n, \delta^{(n)}), (\theta, x)) \rightarrow L((n, \delta^{(n)}), (\theta, x)) = L(\delta^{(n)}(x), \theta)$$

Muito simplificadamente, vamos considerar

$$c(n, \delta, G, \omega) = C_0 - n, \quad C_0 \text{ fixado!}$$

Para $(n, \delta) \in \bar{\Delta}$, temos

$$\begin{aligned}\bar{\rho}((n, \delta), \bar{P}) &= \int_{G \times \Xi} L((n, \delta), (G, \omega)) d\bar{P} \\ &= \int_{G \times \Xi} [L(\delta(\omega), \theta) + n C_0] d\bar{P} = n C_0 + \int_{G \times \Xi} L(\delta^{(n)}(\omega), \theta) d\bar{P}\end{aligned}$$

Teoria da Decisão

14/10/15

Aula 16

Problema com custos:

$$\mathfrak{X} = \prod_{i=1}^{\infty} \mathfrak{X}_i$$

$$\bar{\Delta}_n = \bigcup_{\Theta} \times \mathfrak{X}$$

$$\Delta_n = \{\delta : \mathfrak{X}_1 \times \dots \times \mathfrak{X}_n \rightarrow D\}, \Delta_0 = D$$

$$\bar{\Delta} = \bigcup_{n=0}^{\infty} \bar{\Delta}_n \times \Delta_n$$

$$L((n, \delta), (\Theta, \infty)) = L(\delta(\omega), \Theta) + \underbrace{C(n, \delta, \Theta, \infty)}_{nC_0, C_0 \geq 0}$$

Avaliando $\bar{\rho}((n, \delta), \bar{\Delta})$.

$$\bar{\rho}((n, \delta), \bar{\Delta}) = \int_{\bigcup_{\Theta} \times \mathfrak{X}} L(\delta(\omega), \Theta) d\bar{P} + nC_0$$

Exemplo:

$$D = \{1, 2, 3\} = \bigcup_{\Theta}$$

L

$d \setminus \Theta$	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

$$L(d, \Theta) = |d - \Theta|$$

$$P(\Theta = i) = p_i, i=1, 2, 3$$

Considerando o prob. sem dados (D, Θ, L, P) .

Decisão ótima d_1 :

$$E[L(d_1, \theta)] \leq E[L(d_2, \theta)] \Leftrightarrow 1p_2 + 2(1-p_1-p_2) \leq p_1 + 1-p_1-p_2$$

$$E[L(d_1, \theta)] \leq E[L(d_3, \theta)] \Leftrightarrow 1p_2 + 2(1-p_1-p_2) \leq 2p_1 + p_2$$

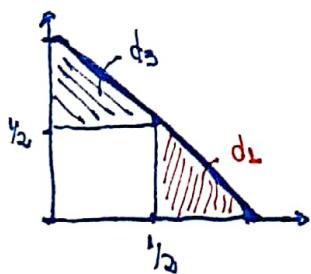
$$\begin{aligned} &\Leftrightarrow 2p_1 \geq 1 \Leftrightarrow p_1 \geq \frac{1}{2}. \\ &\Leftrightarrow 2 - 2p_2 \leq 4p_1 \Leftrightarrow p_1 \geq \frac{1-p_2}{2} \end{aligned} \quad \left. \begin{array}{l} p_1 \geq \frac{1}{2} \\ p_1 \geq \frac{1-p_2}{2} \end{array} \right\}$$

Do mesmo modo, d_3 é decisão ótima se

$$E[L(d_3, \theta)] \leq E[L(d_1, \theta)] \Leftrightarrow p_3 \geq p_1$$

$$E[L(d_3, \theta)] \leq E[L(d_2, \theta)] \Leftrightarrow 2p_1 + p_2 \leq p_1 + p_3$$

$$\Leftrightarrow p_1 + p_2 \leq p_3 \Leftrightarrow p_3 \geq \frac{1}{2}.$$



Suponhamos que estão disponíveis X_1, X_2, X_3, \dots , que dado θ , são c.i.i.d. $\text{Exp}(\theta)$
 Para $n \in \mathbb{N}$ e $(x_1, \dots, x_n) \in \mathbb{R}_+^n$

$$\begin{aligned} P(\theta=1 \mid X_1=x_1, \dots, X_n=x_n) &= \frac{f(x_1, \dots, x_n \mid \theta=1) P(\theta=1)}{\sum_{j=1}^3 f(x_1, \dots, x_n \mid \theta=j) P(\theta=j)} \\ &= \frac{\prod_{i=1}^n f(x_i \mid \theta=1) p_1}{\sum_{j=1}^3 \prod_{i=1}^n f(x_i \mid \theta=j) p_j} \\ &= \frac{\prod_{i=1}^n f(x_i \mid \theta=1) p_1}{\sum_{j=1}^3 \prod_{i=1}^n f(x_i \mid \theta=j) p_j} \end{aligned}$$

$f(x_i \mid \theta=j) = j e^{-jx_i}$

$$\Rightarrow P(\theta=1 / X_1=x_1, \dots, X_n=x_n) = \frac{\prod_{i=1}^n e^{-\sum_{i=1}^n x_i} p_i}{\sum_{j=1}^3 \prod_{i=1}^n e^{-\sum_{i=1}^n x_i} p_j}$$

Suponhamos $p_1=p_2=p_3=1/3$

- Após observar $X_1=x_1, \dots, X_n=x_n$, a decisão será por d₂ se

$$P(\theta=1 / X_1=x_1, \dots, X_n=x_n) > 1/2 \Leftrightarrow e^{-\sum_{i=1}^n x_i} > 2^n e^{-2\sum x_i} + 3^n e^{-3\sum x_i}$$

$$\Leftrightarrow e^{2\sum x_i} - 2^n e^{\sum x_i} - 3^n > 0$$

$$(*) t = e^{\sum x_i}$$

$$t^2 - 2^n t - 3^n > 0 \Leftrightarrow t > \frac{2^n + \sqrt{4^n + 4 \cdot 3^n}}{2}$$

$x_1 < x_2$ ou $t > x_2$, com

$$x_2 = \frac{3^n + \sqrt{4^n + 4 \cdot 3^n}}{2}$$

$$\Leftrightarrow \sum_{i=1}^n x_i > \underbrace{\log \frac{2^n + \sqrt{4^n + 4 \cdot 3^n}}{2}}_b$$

x_1 é neg. Sobra $t > x_2$.

- Decisão por d₃ se

$$P(\theta=0 / X_1=x_1, \dots, X_n=x_n) > 1/2 \Leftrightarrow 3^n e^{-3\sum x_i} > e^{-2\sum x_i} + 2^n e^{-2\sum x_i}$$

$$\Leftrightarrow e^{2\sum x_i} + 2^n e^{2\sum x_i} - 3^n < 0$$

$$\Leftrightarrow 0 < e^{\sum x_i} < \frac{-2^n + \sqrt{4^n + 4 \cdot 3^n}}{2}$$

$$\Leftrightarrow \sum_{i=1}^n x_i < \log \frac{-2^n + \sqrt{4^n + 4 \cdot 3^n}}{2}$$

$$\underbrace{}_a$$

Assim,

$$\delta^{(n)}(\omega_1, \dots, \omega_n) = \begin{cases} 1, & \sum_{i=1}^n \omega_i > b \\ 2, & a < \sum_{i=1}^n \omega_i < b \\ 3, & \sum_{i=1}^n \omega_i \leq a. \end{cases}$$

Por fim, devemos avaliar para cada $(n, \delta^{(n)})$,

$$\bar{\rho}((n, \delta^{(n)}), \bar{P}) = \int_{\Omega^n} L(\delta^{(n)}(\omega), \theta) d\bar{P} + n C_0$$

$$\int_{\Omega^n} L(\delta^{(n)}(\omega), \theta) d\bar{P} = \sum_{i=1}^3 \int_{\mathbb{R}_+^n} L(\delta^{(n)}(\omega_1, \dots, \omega_n), i) f(\omega_1, \dots, \omega_n / \theta=i) P(\theta=i)$$

$$= \frac{1}{3} \left\{ \int_{\{\sum \omega_i \leq a\}} \overset{1}{L}(3,1) f(\omega / \theta=1) d\omega + \int_{\{a < \sum \omega_i \leq b\}} \overset{2}{L}(2,1) f(\omega / \theta=1) d\omega + \int_{\{\sum \omega_i > b\}} \overset{3}{L}(1,1) f(\omega / \theta=1) d\omega \right. \\ + \int_{\{\sum \omega_i \leq a\}} \overset{1}{L}(3,2) f(\omega / \theta=2) d\omega + \int_{\{a < \sum \omega_i \leq b\}} \overset{2}{L}(2,2) f(\omega / \theta=2) d\omega + \int_{\{\sum \omega_i > b\}} \overset{3}{L}(1,2) f(\omega / \theta=2) d\omega \\ \left. + \int_{\{\sum \omega_i \leq a\}} \overset{1}{L}(3,3) f(\omega / \theta=3) d\omega + \int_{\{a < \sum \omega_i \leq b\}} \overset{2}{L}(2,3) f(\omega / \theta=3) d\omega + \int_{\{\sum \omega_i > b\}} \overset{3}{L}(1,3) f(\omega / \theta=3) d\omega \right\}$$

\Rightarrow

$$\int_{\Theta \times \mathcal{E}} L(\delta^{(n)}(\omega), \theta) d\bar{P} =$$

$$\frac{1}{3} \left\{ P(a < \sum_{i=1}^n X_i \leq b | \theta=1) + P(\sum_{i=1}^n X_i > a | \theta=2) + P(\sum_{i=1}^n X_i > b | \theta=2) \right. \\ \left. + P(a < \sum_{i=1}^n X_i \leq b | \theta=3) + 2 \cdot P(\sum_{i=1}^n X_i < a | \theta=3) + 2P(\sum_{i=1}^n X_i > b | \theta=3) \right\}$$

Assim,

$$\bar{p}(c_n, \delta^{(n)}), \bar{P}) = \int_{\Theta \times \mathcal{E}} L(\delta^{(n)}, \theta) d\bar{P} + \underbrace{0,01}_n$$

$$\theta_0 = 0,01$$

Por fim, avaliando para cada $(n, \delta^{(n)})$, $\bar{p}(c_n, \delta^{(n)}), \bar{P})$, obtemos, como decisão ótima em $\bar{\Delta}$,

$$n=15 \in \delta^{(15)}(x_1, \dots, x_{15}) = \begin{cases} 1, & \sum_{i=1}^{15} x_i > 10,4103 \\ 3, & \sum_{i=1}^{15} x_i \leq 6,0689 \\ 2, & 6,0689 \leq \sum_{i=1}^{15} x_i \leq 10,4103 \end{cases}$$

No caso do problema sem dados e como $p_1 = p_2 = p_3 = 1/3$, a dec. ótima é de.

$$\rho(d_2, \bar{P}) = L(d_2, 1) P(\theta=1) + L(d_2, 2) P(\theta=2) + L(d_2, 3) P(\theta=3) \\ = \frac{2}{3}$$

Escolha de experimentos

$\mathbb{E} \rightarrow$ conjunto de experimentos

Para cada $E \in \mathbb{E}$, \mathcal{X}_E é o espaço amostral do experimento E .

E_0 : não conduzir amostragem alguma

$$\Delta_E = \{\delta: \mathcal{X}_E \rightarrow D\}$$

$$\Delta_{E_0} = D$$

$$\tilde{\Delta} = \bigcup_{E \in \mathcal{E}} \{E\} \times \Delta_E$$

$$\text{Custo: } C_E = C_E(-, -, -, -, -)$$

Exemplo: (Exercício)

$$\Theta = \{\theta_1, \theta_2\}$$

Experimento 1:

$$D = \{d_1, d_2\}$$

Obs. $X \perp q. X | \theta_1 \sim \text{Ber}(2/3)$

$X | \theta_2 \sim \text{Ber}(1/3)$

		d	
		d ₁	d ₂
θ ₁	0	a ₁	
	a ₂	0	

Custo: C_x

Experimento 2:

Obs. $Y \perp q. Y | \theta_1 \sim \text{Ber}(3/4)$

$Y | \theta_2 \sim \text{Ber}(1/2)$

Custo: C_y

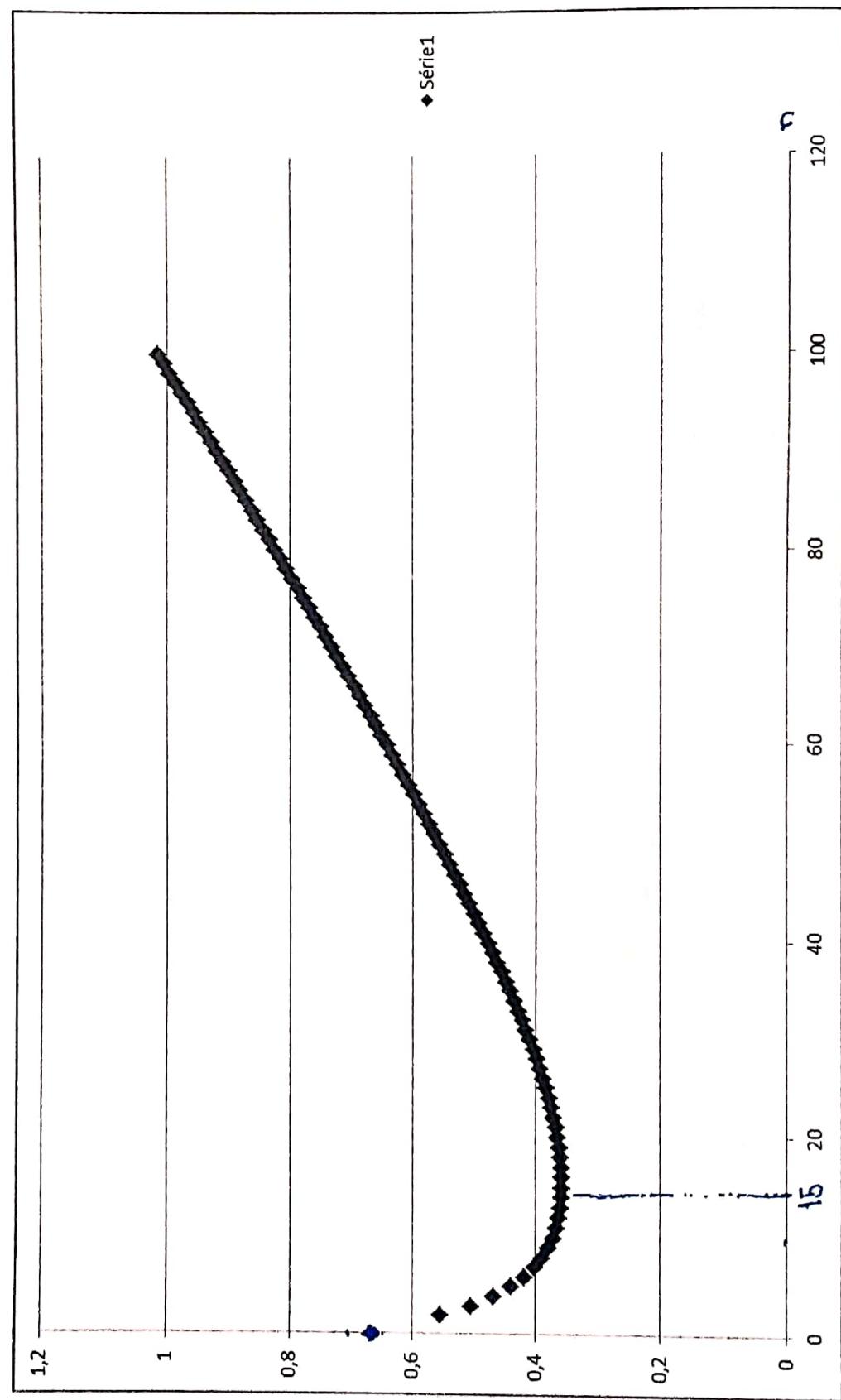
X, Y

(1) Você é informado simultaneamente de X e Y.

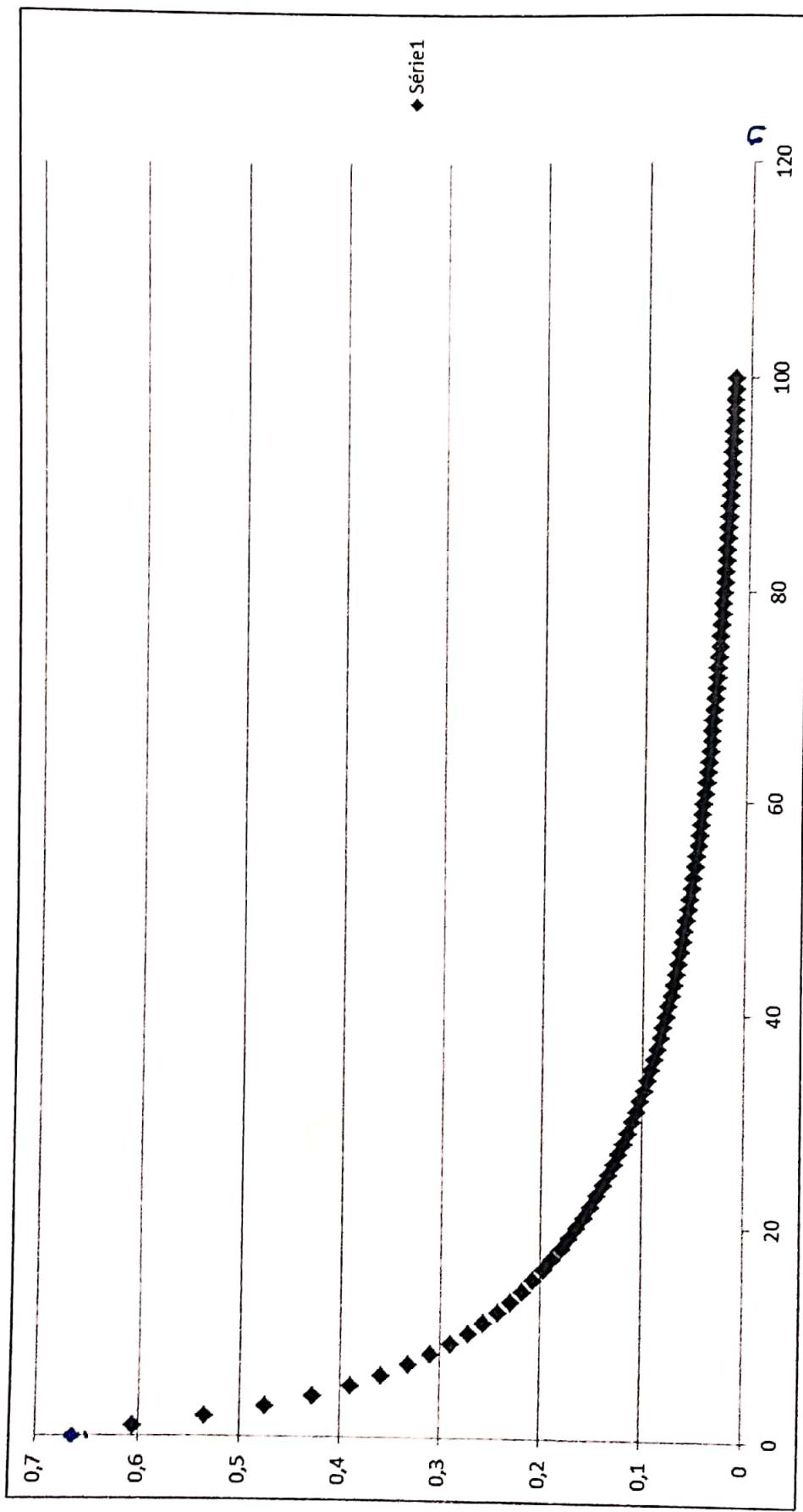
$$f^{(1)}(\theta | x, y) \propto f(x, y | \theta) f(\theta)$$

$$f(\cdot) \xrightarrow{x \in Y} f^{(1)}(\cdot | x, y)$$

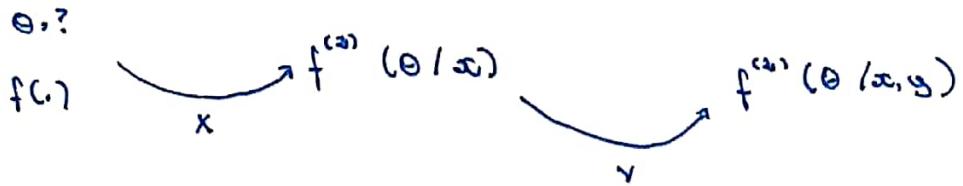
$\bar{p}((n, \delta^m), \bar{P})$



$$\int_{\Omega} \phi^m(\omega) \phi(\omega) d\mu$$



(2) Você é informado primeiro de X e depois de Y .



Se não houver regras de coerências no tempo, supercondicionamento ou regra de Jeffreys.

$$f^{(2)}(\theta/x,y) \propto f(y/\theta,x) f(\theta/x)$$

$$\propto \frac{f(x,y/\theta)}{f(x/\theta)} f(x/\theta) f(\theta)$$

$$\propto f(x,y/\theta) f(\theta) \propto f''(\theta/x,y)$$

:

$$(\Delta, \oplus_{\infty}, \sqcup, \mathbb{P})$$

↓
estimadores
do parâmetro

$$\begin{aligned} L(\delta_1(\theta, x)) &= L(\delta(x), \theta) = |\delta(x) - \theta| \\ &= (\delta(x) - \theta)^3 \end{aligned}$$

ESTIMAÇÃO $(\mathcal{D}, \Theta, L, R)$ A priori obtemos $x \in \mathcal{X}$, (D, Θ, L, R) com $D = \Theta$.

Problema: Pre-experimentação

$$\Delta = \{\delta : \mathcal{X} \rightarrow \mathcal{D}\}$$

No problema de estimativa $\hat{D} = \Theta$, de modo que
 $\Delta = \{\delta : \mathcal{X} \rightarrow \Theta\}$ conjuntos de estimativas

* $(\Delta, \Theta, \mathcal{X}, L, R)$

$$L' : \Delta \times (\Theta, \mathcal{X}) \rightarrow \mathbb{R}_+$$

$$(\delta, (\vartheta, x)) \mapsto L'(\delta, (\vartheta, x)) = L(\delta(x), \vartheta)$$

ou seja, é expresso a discrepância (distância) entre a estimativa $\delta(x)$ e o parâmetro (descritivo) ϑ .

 $L(D, \vartheta) = h(\vartheta) \text{ g}(\text{dist}(d, \vartheta))$, g não-decrescenteAlguns exemplos: $(\Theta \subseteq \mathbb{R})$

(1) $L(d, \vartheta) = |d - \vartheta|$, perda absoluta.

(2) $L(d, \vartheta) = (d - \vartheta)^2$, Perda Quadrática

$$(3) L(d, \theta) = \frac{(d - \theta)^2}{1\theta} \quad (\hat{\theta}(d=0) = 0)$$

* Pseudo distribution: $\Theta \subseteq \mathbb{R}$
 $L(d, \theta) = (d - \theta)^2$, $E(\theta) < \infty$

$$E[L(d, \theta)] =$$

$$\begin{aligned} E[(d - E(\theta))^2] &= E[(d - E(\theta))^2] + 2E[(d - E(\theta))(E(\theta) - \theta)] \\ &= 2/E[(d - E(\theta))(E(\theta) - \theta)] + E[(E(\theta) - \theta)^2] \Rightarrow \end{aligned}$$

$$E[L(d, \theta)] = (d - E(\theta))^2 + Var(\theta)$$

$$\begin{aligned} \text{Assim, } E(\theta) \in \Theta &\Rightarrow d^* = E(\theta) \\ E(\theta) \notin \Theta &\Rightarrow d^* \in \mathbb{D} \text{ total free / } d^* = E(\theta) \\ &= \min \{ |d - E(\theta)| : d \in \mathbb{D} \} \end{aligned}$$

$E(\theta) \in \Theta \Rightarrow$

$$\begin{aligned} P^*(P) &= P(E(\theta), P) = Var(\theta) \\ E(\theta) \notin \Theta &\Rightarrow P^*(P) = P(d^*, P) = (d^* - E(\theta))^2 + Var(\theta) \end{aligned}$$

Após o observatório de $X = \infty$.

Dissertação de mestranato
 Resumão Hobo local:
 $g_2 \approx 35$. . .

Combining Probability
 Distribution

Generalization (1976)
 Statistical Science

* Nos problemas pré-experimentais

$$\delta^* \rightarrow \Theta$$

$$\begin{aligned} \delta^*(x) &= E(\theta | X=x) \\ P^*(P_{X=x}) &= P(E(\theta | X=x), P_{X=x}) = \sqrt{Var(\theta)} \end{aligned}$$

Em geral, para $\delta \in \Delta$,

$$P(\delta^*) = \int_{\Theta} L(\delta(x), \theta) f(\theta | x) d\theta = \int_{\mathbb{R}} f(x) \left[\int_{\Theta} L(\delta(x), \theta) f(\theta | x) d\theta \right] d\theta$$

Agora, estamos considerando o pseudo problema,
 tais,

$$\begin{aligned} P^*(\delta^*) &= \int_{\mathbb{R}} f(x) \left[\int_{\Theta} L(E(\theta | x), \theta) f(\theta | x) d\theta \right] d\theta \\ &= \int_{\mathbb{R}} f(x) \left[\int_{\Theta} (E - E(\theta | x))^2 f(\theta | x) d\theta \right] d\theta \xrightarrow{\text{Var}} \end{aligned}$$

$$P^*(\delta^*) = E(V_{\theta}(\delta/x))$$

Exemplo: X_1, \dots, X_n obedece Θ , $\Sigma_{ii} = \text{Var}(\delta/x)$

Suponhamos, a priori $\Theta \sim N(\theta_0, \sigma_{\theta}^2)$

Vamos provar

$$\Theta / X_i = x_1, \dots, X_n = x_n \sim N\left(\frac{\sigma_0^2 \theta_0 + n\bar{x}\sigma_{\theta}^2}{\sigma_0^2 + n\sigma_{\theta}^2}, \frac{\sigma_0^2}{\sigma_0^2 + n\sigma_{\theta}^2}\right)$$

$$A \text{ p/ } \text{obter } x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$E(\theta / X=x) = \frac{\sigma_0^2 \theta_0 + \sigma_{\theta}^2 \cdot n\bar{x}}{\sigma_0^2 + n\sigma_{\theta}^2}$$

$$P^*(E(\theta/x), P_{x=x}) = V_{\theta}(x/x) = \frac{\sigma_{\theta}^2}{\sigma_0^2 + n\sigma_{\theta}^2}, \forall x \in \mathcal{X}$$

No problema pre-experimental

$$\delta^*(x) = E(\theta/x) = \frac{\sigma_0^2 \theta_0 + n\sigma_{\theta}^2 \bar{X}}{\sigma_0^2 + n\sigma_{\theta}^2}$$

$$P^*(\theta') = P^*(J^*(x), \theta') = E\left(V_{\theta}(\theta/x)\right) = \frac{\sigma_{\theta}^2}{\sigma_0^2 + n\sigma_{\theta}^2} \delta^*(x)$$

$$\frac{\sigma_{\theta}^2}{\sigma_0^2 + n\sigma_{\theta}^2}$$

Ex2. X_1, \dots, X_n obedece Θ , $\text{Var}(\delta/x)$

$$\Theta \sim \text{Beta}(\alpha, \beta), \alpha, \beta > 0$$

Vamos provar

$$\Theta / X_i = x_1, \dots, X_n = x_n \sim \text{Beta}\left(\alpha + \sum_{i=1}^n x_i, \beta + n - \sum_{i=1}^n x_i\right)$$

$$P_{\theta=x} x = (x_1, \dots, x_n) \in \{\theta, 1\}^n,$$

$$E(\theta/x) = \frac{\alpha + \bar{x}n}{\alpha + \beta + n}$$

$$P^*(P_{x=x}) = P\left(E(\theta/x), P_{x=x}\right) = V_{\theta}(x/x) = \frac{(\alpha + n\bar{x})(\beta + n - n\bar{x})}{(\alpha + \beta + n)^2 (\alpha + \beta + n + 1)}$$

* No problema pre-experimental, o estimador de Bayes com relocação à pseudo queleto é

$$E(\theta/x) = \frac{\alpha + n\bar{x}}{\alpha + \beta + n} = \frac{\alpha + b}{\alpha + b + n} \underbrace{\frac{\alpha}{\alpha + b}}_{(E(\theta))} + \frac{n}{\alpha + b + n} \bar{X}$$

$$P^*(P') = P^*(E(\theta/x), P') = E\left(V_{\theta}(x/x)\right) = V_{\theta}(x) - V_{\theta}(E(\theta/x))$$

$\Theta \sim \text{Beta}(\alpha, \beta)$, $1 \leq \omega \leq N$

$$\begin{aligned} &= \frac{\alpha b}{(\alpha+b)^2(\alpha+b+n)} - \text{Var}\left(\frac{\alpha + n\bar{x}}{\alpha+b+n}\right) = \\ &= \text{Var}(\omega) - \left(\frac{n}{\alpha+b+n}\right)^2 \cdot \text{Var}(\bar{x}) = \text{Var}(\omega) \left(\frac{n^2}{\alpha+b+n}\right) \end{aligned}$$

$$= \text{Var}(\omega) \left(\frac{n}{\alpha+b+n}\right)^2 \left[\text{E}(\text{Var}(\bar{x}/\omega)) + \text{Var}\left(\text{E}(\bar{x}/\omega)\right) \right] =$$

$$= \text{Var}(\omega) - \left(\frac{n}{\alpha+b+n}\right)^2 \left[\text{E}\left(\frac{\omega(1-\omega)}{n}\right) + \text{Var}(\omega) \right]$$

$$\Rightarrow \underset{b \rightarrow \infty}{\lim} P^*(P') = \frac{\alpha b}{(\alpha+b)^2(\alpha+b+n)} \left[\frac{\omega b}{n(\alpha+b)(\alpha+b+n)} + \frac{\alpha b}{(\alpha+b)^2(\alpha+b+n)} \right]$$

$$\text{Var}(\omega) = \text{E}\left(\text{Var}(\omega/\bar{x})\right) + \text{Var}\left(\text{E}(\omega/\bar{x})\right)$$

$$\text{Var}(\omega) = \underbrace{\text{E}\left(\text{Var}(\omega/\bar{x})\right)}_{P^*(P')} + \underbrace{\text{Var}\left(\text{E}(\omega/\bar{x})\right)}_{\text{Var}(\delta^*)}$$

Ex 3:

$$\left| \begin{array}{l} \Theta \text{ binom} \\ 10-\text{variables} \end{array} \right| \quad \left| \begin{array}{l} \Theta = ? \\ \Theta = \{0, 1, \dots, 10\} \end{array} \right|$$

$\Theta \sim \text{Bin}(10, 1/3)$

$P(\bar{x}) = \frac{1}{3}$ dient der Θ , und der

$$P^*(P') = P(3, \bar{x}) = \left(3 - E(\omega)\right)^2 + \text{Var}(\omega) = \left(3 - \frac{10}{3}\right)^2 + 10 \cdot \frac{1}{3} \cdot \frac{2}{3} = P^*(P) = \frac{21}{9}$$

$$\begin{aligned} &E\left(\frac{\omega}{\theta} (1-\omega)^5\right) = \int_0^1 \omega^{\alpha-1} (1-\omega)^5 \frac{\Gamma(\alpha, \theta)}{\Gamma(\alpha) \Gamma(b)} \theta^{\alpha-1} (1-\omega)^{b-1} d\omega = \\ &= \frac{\sqrt{(\alpha+b)}}{\Gamma(\alpha) \Gamma(b)} \left[\frac{\Gamma(\alpha+m+b+1)}{\Gamma(\alpha+m) \Gamma(b+1)} \right] = \end{aligned}$$

$$E\left(\frac{\omega^m (1-\omega)^5}{\theta} \right) = \frac{\sqrt{\alpha+b}}{\Gamma(\alpha) \Gamma(b)} \cdot \frac{\Gamma(\alpha+m) \Gamma(b+5)}{\Gamma(\alpha+m+b+1)}$$

$$\text{Aber: } E\left(\frac{\omega}{\theta} (1-\omega)^5\right) = \frac{\omega \theta}{\alpha+b+1} \cdot \frac{\Gamma(\alpha+b)}{\Gamma(\alpha+b+1)} \xrightarrow{\text{Ergebnis } 3} \frac{\theta}{\alpha+b+1}$$

* Superpositions für $X | \Theta$ nach parametrischen $(\Theta, \omega \sim \text{Beta}(\alpha, \beta))$

Pro $X = 0, 1, 2, \dots, n \in \{0, 1, 2\}$

$$P(\Theta = \omega | X=x) \propto P(X=x | \Theta=\omega) P(\omega) \propto$$

$$\propto \frac{(\theta)^{(x)} (1-\theta)^{n-x}}{(x) (1-\theta)^x} \left(\frac{1}{\theta}\right)^{\theta} \left(\frac{1}{1-\theta}\right)^{1-\theta} \frac{P^*(P')}{P(P')}$$

$$\alpha \cdot \frac{\theta!}{(\theta-x)!} \frac{(\lambda_0-\theta)!}{[\lambda_0-(\theta-x)]!} \frac{(\lambda_3)^{\theta-\theta}}{\lambda_3!} \frac{\pi(\theta)}{\pi(x+1, \dots, \lambda_0-\theta)} \frac{\pi(\theta)}{\pi(0, 1, \dots, \lambda_0-\theta)} =$$

$$= \binom{8}{\theta-x} \left(\frac{1}{3}\right)^{\theta-x} \left(\frac{2}{3}\right)^{\theta-(\theta-x)} \frac{\pi(\theta)}{\pi(x, \dots, 8+x)}$$

$$= \theta - x / \chi = x \sim \text{Bin}(8, 1/3), \text{ logo}$$

$$E((\theta - x)/\chi) = \frac{8}{3} \Rightarrow E(\theta/\chi = x) = \theta + \frac{8}{3} \approx 2.667 \approx 3$$

Anzim, tornamor com estimador de Bayes

$$\boxed{2x+3}$$

$$P(x+3, P_{X=x}) = (x+3 - E(\theta/x))^2 + \text{Var}(\theta/x) =$$

$$= (x+3 - (x + \frac{8}{3}))^2 + \frac{8}{3} \cdot \frac{2}{3} \Rightarrow$$

$$P(x+3, P_{X=x}) = \frac{1}{9} + \frac{16}{9} = \frac{17}{9}, \forall x \in \mathbb{N}, \text{ mas}$$

problema Pre-experimental,

$$\tilde{\sigma}^*(x) = x+3$$

$$P^*(P') = \sum_{x=0}^2 P(X=x) \cdot \underbrace{P(x+3, P_{X=x})}_{\tilde{\sigma}^*(x)} =$$

$$\sum_{x=0}^2 L(x+3, \theta) P(x/x)$$

$$= \sum_{x=0}^2 P(X=x) \cdot \frac{17}{9} \Rightarrow P^*(P') = \frac{9}{2}.$$

$$P^*(P') = \int_{\mathbb{R}} L(\delta(x), \theta) dP' = \int_{\mathbb{R}} L(\delta(x), \theta) dP : \text{núco do estimador S}$$

$$P^*(P') = \inf \left\{ P'(\delta, P) : \delta \in \Delta \right\} : \text{núco de Bayes}$$

$$L(\delta, \theta) = (d-\delta)^2 (P \leq R)$$

$$\delta^*(x) = E(\theta/x)$$

$$(2) \text{ PERDA ABSOLUTA } (H \subset \mathbb{R})$$

$$L(d, \theta) = |d - \theta| = \begin{cases} (d - \theta) & \text{if } \theta < d \\ (\theta - d) & \text{if } \theta \geq d \end{cases}$$

Digemos que $m \in \mathbb{R}$ é uma mediana de P se

$$P([-d, m]) = \frac{1}{2} = P([m, \infty)) \Rightarrow \frac{1}{2}$$

$$\begin{aligned} (\text{esse } m \text{ é mediana de } V, \text{ se } P(X \leq m) \geq \frac{1}{2} \text{ e} \\ P(X \geq m) \geq \frac{1}{2}) \end{aligned}$$

Teoria da Decisão

Aula 18

21/10/15

$$p^*(\delta, P) = \int_{\Theta \times \mathbb{R}} L(\delta(\omega), \theta) dP \quad \text{risco do estimador } \delta$$

$$p^*(\delta, P) = \inf \{ p^*(\delta, P') : \delta \in \Delta \} : \text{risco de Bayes}$$

$$L(d, \theta) = (d - \theta)^2, \quad (\Theta = \mathbb{R})$$

$$\delta^*(x) = E(\theta|x)$$

2. Perda Absoluta ($\Theta \subseteq \mathbb{R}$)

$$L(d, \theta) = |d - \theta| = \begin{cases} (d - \theta), & \theta \leq d \\ (\theta - d), & \theta > d \end{cases}$$

Dizemos que $m \in \mathbb{R}$ é uma mediana de P se

$$P((-\infty, m]) \geq \frac{1}{2} \quad e \quad P([m, \infty)) \geq \frac{1}{2}$$

(m é mediana da v.a. X se $P(X \leq m) \geq \frac{1}{2}$ e $P(X \geq m) \geq \frac{1}{2}$)

A priori. ($E(1|\theta) = \infty$)

Seja m_0 uma mediana da distribuição a priori de θ . Vamos avaliar

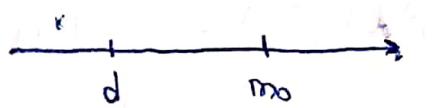
$$E[L(d, \theta)] - E[L(m_0, \theta)], \quad \forall d \in \mathbb{R}$$

Primeiro, olhando para $d < m_0$,

$$E[L(d, \theta)] - E[L(m_0, \theta)] = E[|d - \theta|] - E[|m_0 - \theta|]$$

$$= E[|d - \theta| - |m_0 - \theta|]$$

$$= \int_{\{\theta \leq d\}} (d - \theta) dP + \int_{\{d \leq \theta \leq m_0\}} [(\theta - d) - (m_0 - \theta)] dP$$



$$+ \int_{\{\theta > m_0\}} (m_0 - \theta) dP$$

$$\pi - (m_0 - d) P(\theta \leq d) - (m_0 - d) P(d \leq \theta \leq m_0) + (m_0 - d) P(\theta > m_0)$$

$$= (m_0 - d) [P(\theta \leq m_0) - P(\theta \leq m_0)]$$

$$= (m_0 - d) [2P(\theta \leq m_0) - 1] \quad \text{e, portanto,}$$

$$E[L(m_0, \theta)] \leq E[L(d, \theta)], \quad \forall d < m_0$$

Do mesmo modo, verificamos (Exercício) que

$$E[L(m_0, \theta)] \leq E[L(d, \theta)], \quad \forall d > m_0$$

Logo, m_0 é uma decisão de Bayes no problema a priori.

Após a observação $X = \infty$, $M_d(\theta | \infty)$, a mediana da dist. a posteriori

é decisão de Bayes do problema $(\Theta, \Omega, L, P_{X=\infty})$.

No problema pré-experimentação,

$\delta^*(x) = \text{Md}(\Theta|x)$ é estimador de Bayes com relação à perda absoluta.

Risco de Bayes.

$$P^{**}(P') = \int_{\Theta} f(\theta) \left[\int_{\Omega} |\theta - \text{Md}(\theta|x)| f(x|\theta) dx \right] d\theta$$

Suponha $L(d, \theta) = \begin{cases} p_1(d-\theta), & \theta \leq d \\ p_2(\theta-d), & \theta > d \end{cases}$

$\delta^*(x)$: perc. de ordem $\frac{p_1}{p_1 + p_2}$ no caso subestima.
 $\delta^*(x)$: perc. de ordem $\frac{p_2}{p_1 + p_2}$ no caso superest.

Exemplo 1: X_1, \dots, X_n , dado Θ , c.i.i.d. $N(\theta, \sigma^2)$ e $\Theta \sim N(\theta_0, \sigma_0^2)$

$$\Theta | X_1 = x_1, \dots, X_n = x_n \sim N \left(\frac{\sigma_0^2 \theta_0 + \sigma^2 n \bar{x}}{\sigma_0^2 + n \sigma^2}, \frac{\sigma_0^2 \sigma^2}{\sigma_0^2 + n \sigma^2} \right)$$

$$\text{Md}(\theta|x) = \frac{\sigma_0^2 \theta_0 + \sigma^2 n \bar{x}}{\sigma_0^2 + n \sigma^2}$$

$$\begin{aligned} P(\text{Md}(\theta|x), P_{X=x}) &= \int_{\Theta} |\theta - \text{Md}(\theta|x)| dP_{X=x} \\ &= \sigma_x \int_{\Theta} \left| \frac{\theta - \text{Md}(\theta|x)}{\sigma_x} \right| dP_{X=x} = \sigma_x E(|Z|), Z \sim N(0, 1) \end{aligned}$$

Como $E(|z|) = \sqrt{\frac{2}{\pi}}$, então

$$p(Md(\theta|x), P_{x=x}) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{\sigma_0^2 \cdot v_0^2}{v_0^2 + n\sigma_0^2}}, \quad \forall x \in \mathbb{R}^n$$

No problema pré-experimental,

$$\hat{\theta}^*(x) = Md(\theta|x) = \frac{v_0^2 \sigma_0^2 + \sigma_0^2 n \bar{x}}{v_0^2 + n\sigma_0^2}$$

$$p^*(\bar{x}) = \int_{-\infty}^{\infty} f(x) \sqrt{\frac{2\sigma_0^2 v_0^2}{\pi(v_0^2 + n\sigma_0^2)}} dx = \sqrt{\frac{2\sigma_0^2 v_0^2}{\pi(\sigma_0^2 + n\sigma_0^2)}}$$

Exemplo 2: X_1, \dots, X_n dado é, c.i., d. $\mathcal{U}(0, \theta)$, $\theta > 0$. Suponha

Suponha $\Theta \sim \text{Pareto}(a, b)$, $a, b > 0$, $a \neq 1$.

Vimos que $\Theta | x_1 = x_1, \dots, x_n = x_n \sim \text{Pareto}(a+n, \max\{b, x_1, \dots, x_n\})$

A priori, queremos encontrar m_0 tal que $P(\Theta > m_0) = 1/2$.

$$P(\Theta > m_0) = \int_{m_0}^{\infty} \frac{ab^\theta}{\theta^{a+1}} d\theta = ab^\theta \frac{\theta^{-a}}{-a} \Big|_{m_0}^{\infty} = ab^\theta \frac{1}{a} \frac{1}{m_0^a} = \frac{1}{2} \Rightarrow$$

$$m_0^a = 2b^\theta \Rightarrow m_0 = \sqrt[a]{2} b, \text{ i.e., } m_0 = 2^{1/a} b.$$

Após a observação $x = \infty$, a estimativa de Bayes é

$$Md(\theta|x) = 2 \frac{1}{a+n} \times \max\{b, x_1, \dots, x_n\}$$

No problema pré-experimentação,

$$\hat{\theta}^*(X) = \text{MD}(\Theta | X) = 2^{\frac{1}{\alpha+n}} \max\{b, x_1, \dots, x_n\}$$

é o estimador de Bayes em relação à perda absoluta.

No problema pré-experimentação,

$$\begin{aligned}\hat{\theta}^*(P) &= \int_{\Theta} f(\omega) \left[\int_{\Theta} \left| \Theta - 2^{\frac{1}{\alpha+n}} \max\{b, x_1, \dots, x_n\} \right| f(\omega | \omega) d\omega \right] d\omega \\ &= \int_{\Theta} \dots \int_{\Theta} \dots \text{dede} + \int_{\Theta} \dots \text{dede} \\ &\quad \downarrow \max\{b, x_1, \dots, x_n\} \underline{\alpha+n} \quad (2^{\frac{1}{\alpha+n}} - 1) \\ &= \frac{\alpha+n}{\alpha+n-1} (2^{\frac{1}{\alpha+n}} - 1) E[\max\{b, x_1, \dots, x_n\}] \\ &= \frac{\alpha+n}{\alpha+n-1} (2^{\frac{1}{\alpha+n}} - 1) E[E[\max\{b, x_1, \dots, x_n\} | \Theta]]\end{aligned}$$

Depois de longas jornadas, temos

$$\hat{\theta}^*(P) = \frac{ab[n(\alpha+n) + (\alpha-1)]}{(n+1)(\alpha+n-1)(\alpha-1)} (2^{\frac{1}{\alpha+n}} - 1).$$

Ex. 3:

Θ brancas
10.9 verdes

$$\begin{aligned}Q &\sim \text{Bin}(10, 1/3) & X | \Theta &\sim \text{HG}(0, 1\Theta - \Theta, 2) \\ \Theta - \infty | X = x &\sim \text{Bin}(8, 1/3)\end{aligned}$$

Para o modelo Binomial $(8, 1/3)$,

Se $Y \sim \text{Bin}(8, 1/3)$, temos

$$P(Y \leq 2) = \frac{1024}{2187}$$

$$P(Y \leq 3) = \frac{4864}{6561} \approx \frac{1}{2} \quad . \quad 3 \text{ é mediana}$$

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - \frac{1024}{2187} \approx \frac{1}{2} \text{ e,}$$

consequentemente 3 é mediana de Y .

Assim,

$$\text{Md}(\theta | x) = x + 3.$$

Antes do experimento,

$\text{Md}(\theta | x) = x + 3$ é o estimador de Bayes com relação à perda absoluta.

Nesse caso, o risco de Bayes é dado por

$$\begin{aligned} p^*(R) &= \sum_{x=0}^{\infty} \sum_{j=0}^{10} I(x \geq 3) \cdot j \mid P(\theta=j, X=x) = \\ &= \sum_{x=0}^{\infty} \sum_{j=0}^{10} I(x \geq 3) \cdot j \mid \frac{\binom{x}{j} \binom{10-j}{2-x}}{\binom{10}{2}} \left(\frac{1}{3} \right)^j \left(\frac{2}{3} \right)^{10-j} \end{aligned}$$

(3) Perda 0-1

Θ é discreto: $\Theta = \{Q_1, Q_2, \dots\}$

$$L(d, \theta) = \begin{cases} 0, & d = \theta \\ 1, & d \neq \theta. \end{cases}$$

Para $d \in D = \Theta$,

$$\mathbb{E}[L(d, \theta)] = \sum_{\theta \in \Theta} L(d, \theta) P(\theta)$$

$$= \underbrace{\sum_{\theta=d} L(d, \theta)}_{0} P(d) + \sum_{\theta \neq d} L(d, \theta) P(\theta) =$$

$$\Rightarrow \mathbb{E}[L(d, \theta)] = 1 - P(d)$$

Assim, a depreciação de Bayes é $d^* \in D = \Theta$ s.t.

$$P(d^*) \geq P(d), \forall d \in D = \Theta.$$

Todo é uma medida da distribuição de Θ e depreciação de Bayes.

Após observação $X = \infty$, a estimativa de Bayes com relação à probabilidade $\Theta = 1$ é $M_0(\Theta / \infty)$.

Pré-experimentação: $\delta^*(X) = M_0(\Theta / X)$ é estimador de Bayes.

No ex. 3,

$$\Theta = 1 | X = \infty \sim \text{Bin}(8, 1/3). \quad \text{medida prox. } np = 8/3 = 2,6 \sim$$

$$P\left(\frac{8}{3}\right) \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^6 = \frac{28 \cdot 2^6}{3^8} \quad \text{Assim } x=2 \text{ e } x=3 \text{ são medidas da}$$

$$P\left(\frac{8}{3}\right) \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^5 = \frac{56 \cdot 2^5}{3^8} \quad \text{distribuição de } \Theta / \infty$$

$\delta_1^*(X) = X+2$ e $\delta_2^*(X) = X+3$ são est. de Bayes com relação à perda 0-1.

$L(d, \theta) = \begin{cases} 1, & |d-\theta| \leq c \\ 0, & |d-\theta| > c \end{cases}$

Em geral, podemos considerar a seguinte perda:

$$L(d, \theta) = \begin{cases} 1, & |d-\theta| \leq c \\ 0, & |d-\theta| > c \end{cases}$$

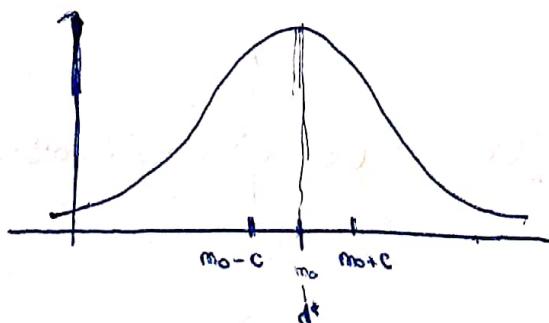
$$\begin{aligned} E[L(d, \theta)] &= \int_{\Theta} L(d, \theta) dP = \int_{\{\theta \in \Theta : |d-\theta| \leq c\}} 0 \cdot dP + \int_{\{\theta \in \Theta : |d-\theta| > c\}} 1 \cdot dP \\ &= P(|d-\theta| > c) \\ &= 1 - P(|d-\theta| \leq c) \Rightarrow \end{aligned}$$

$$E[L(d, \theta)] = 1 - P(d-c \leq \theta \leq d+c)$$

Assim, a decisão de Bayes será d^* tal que

$$P(d^*-c \leq \theta \leq d^*+c) \geq P(d-c \leq \theta \leq d+c), \forall d \in D, \text{ i.e.}$$

d^* é o ponto central do intervalo modal de comprimento $2c$.



com seq. decrescente f.q. em V0 então (em geral).

$$d^*(\text{con}) \geq m_0$$

Um exemplo no caso multivariado

$$\Theta = (\theta_1, \theta_2, \dots, \theta_k) \quad (\Theta \subseteq \mathbb{R}^k)$$

$$d = (d_1, d_2, \dots, d_k)$$

$L(d, \Theta) = (d - \Theta) A (d - \Theta)^T$, $A_{k \times k}$ simétrica e tal que $\forall A \in \mathbb{R}^{k \times k}, \forall c \in \mathbb{R}^k$
 $(A \text{ é semi-definida positiva})$.

Supõe ainda que

$$\mathbb{E}[\theta_i^2] < \infty, \forall i = 1, \dots, k$$

Seja $\mu = (\mathbb{E}(\theta_1), \mathbb{E}(\theta_2), \dots, \mathbb{E}(\theta_k))$ vetor de média de Θ e

$\Sigma_{k \times k}$ a matriz de covariância de Θ ($\Sigma_{ij} = \text{Cov}(\theta_i, \theta_j)$)

$$\mathbb{E}[L(d, \Theta)] = \mathbb{E}[(d - \Theta) A (d - \Theta)^T] = \mathbb{E}[(d - \mu + \mu - \Theta) A (d - \mu + \mu - \Theta)^T]$$

$$= \mathbb{E}[(d - \mu) A (d - \mu)^T + (d - \mu) A (\mu - \Theta)^T + (\mu - \Theta) A (d - \mu) + (\mu - \Theta) A (\mu - \Theta)^T]$$

$$= (d - \mu) A (d - \mu)^T + \mathbb{E}[(\mu - \Theta) A (\mu - \Theta)^T]$$

$$\text{Por } A \text{ é semi-def.} \rightarrow \text{dep. de } d$$

$\approx d - \mu$

Logo, como A é semi-definida positiva, μ é a decisão de Bayes contra P .

A priori,

$$\rho^*(P) = E[L(\mu, \theta)] \equiv E[(\theta - \mu) A(\theta - \mu)^T] \Rightarrow$$

$$= E \left[\sum_{i=1}^k \sum_{j=1}^k \alpha_{ij} (\theta_i - \mu_i)(\theta_j - \mu_j) \right] \quad A(\beta^* - \mu) = (A, \mu) - \Sigma \\ = \sum_{i=1}^k \sum_{j=1}^k \alpha_{ij} \underbrace{E[(\theta_i - E[\theta_i])(\theta_j - E[\theta_j])]}_{\Sigma_{ij} = \text{Cov}(\theta_i, \theta_j)} \Rightarrow$$

$$\rho^*(P) = \text{tr}(A\Sigma)$$

Após observação $X = \alpha$, o a.pt. de Bayes é

$$\mu_{\alpha} = (E(\theta_1 | \alpha), E(\theta_2 | \alpha), \dots, E(\theta_k | \alpha))$$

$$\rho^*(\mu_{\alpha}, P_{k=\alpha}) = \text{tr}(A\Sigma_{\alpha}), \text{ onde } \Sigma_{\alpha} = \text{Cov}(\theta_1, \theta_2 | \alpha) \\ = E(\theta_i \theta_j | \alpha) - E(\theta_i | \alpha) E(\theta_j | \alpha) \\ = E(\theta_i | \alpha) E(\theta_j | \alpha),$$

Finalmente, pré-exp.,

$$\delta^*(x) = g(x) = (E(\theta_1/x), \dots, E(\theta_k/x)) \text{ é estim. de Bayes, contral.}$$

Risco do Bayes:

$$\Sigma_{\alpha} = \text{Cov}(\theta_i, \theta_j | X)$$

$$\rho^*(P') = \int_{\alpha} f(\alpha) \left[\int_{\Theta} L(g(\alpha), \theta) f(\theta | \alpha) d\theta \right] d\alpha = E[\text{tr}(A\Sigma_x)]$$

Aula 19, Apr.

Teoria da Decisão, Palestra Professor da FCI.

Risco em nanotecnologia!

H d.C., vidros com nanopart. de ouro/cobre

Faraday - tamanho da part. alterar corret.

Feynman - Partículas pequenas devem ser est.

Einstein - Calor, tamanho partículas nanossacarídeos

1980, IBM desenv. tecnol. pl. lidar com átomos

Átomo é golpeado pelos nanocubos

Tol. da torrade geram nanomateriais.

- Alterar estrutura das partículas,

Termo de Energia Lívra possa atingir impeto maior



Temperatura Fusão

Nanotecnologia

Risco - riscante - navegar sobre rochas perigosas

Risco } {

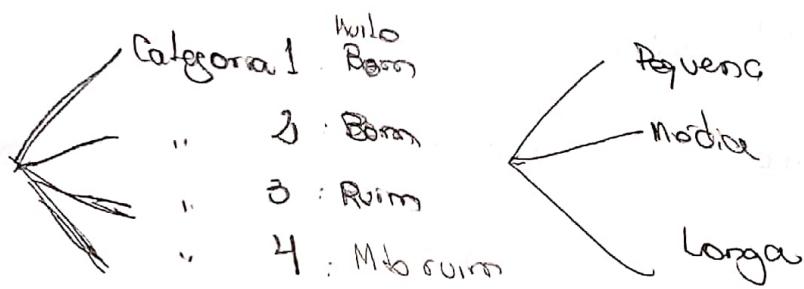
- Severidade
- Probabilidade

www.hysenano.com → relatório de agências de risco

+ 100 000 000 ag. quimi registrados

Aula 19, Apresentação

Centro de: Por bandas



- Seja $\Theta \in \mathbb{N} = \{0, 1, 2, \dots\}$, onde Θ é o nível próprio da massa número de nanop. em kg/m^3 o coda mm^3 .

- 1 iguais
- 3 dif peg
- 5 dif grande
- 7 dif mui. grande
- 9 absurda

limite fragilidade: 15 monomícios.
dissolve o material

ou não se fica encapsulado

$$\begin{matrix} & A & B & C & D \\ A & 1/a & 1 & d & e \\ B & 1/b & 1/d & 1 & f \\ C & 1/c & 1/e & 1/f & 1 \end{matrix}$$

Índice
de conc.
 $\frac{\lambda_{\max} - \lambda}{n-1}$

Bab.: pessoas
certas pdar os
notas corretas

$$CR = \frac{CI}{RI(n)}$$

Teoria da Decisão, 04/11/15 na parte 4

Aula 20

Função

04/11/15

$$\Theta = (\Theta_1, \dots, \Theta_n)$$

$$d = (d_1, \dots, d_n)$$

$$L(d, \Theta) = (d - \Theta) A (d - \Theta)^T$$

$$\delta^*(x) = (E(\Theta_1/x), E(\Theta_2/x), \dots, E(\Theta_n/x))$$

$\delta^*(x) = \text{Função de Risco} = \text{Função de Risco}$

$$\text{Risco de Bayes} = E(\text{Função de Risco})$$

Ex: X_1, \dots, X_n , dado $\Theta = (\Theta_1, \Theta_2)$, c.i.i.d. $N(\Theta_1, \sigma_0^2/\Theta_2)$, $\sigma_0 > 0$.

$\Theta_2 \sim \text{Gama}(a, b)$, $a, b > 0$

$\Theta_1/\Theta_2 \sim N\left(\mu, \frac{\sigma_0^2}{\Theta_2}\right)$, $\mu \in \mathbb{R}$, $\sigma_0 > 0$

Vamos que

$$\Theta_2/x \sim \text{Gama}\left(a + n/2, b + \frac{(x - \bar{x})^2}{2\sigma_0^2} + \frac{1}{2\sigma_0^2} \sum_{i=1}^n (x_i - \bar{x})^2\right)$$

$$\Theta_1/\Theta_2, x = \bar{x} \sim N\left(\frac{\sigma_0\mu_0 + n\bar{x}}{\sigma_0^2 + n\bar{x}}, \frac{\sigma_0^2}{(\sigma_0^2 + n\bar{x})\Theta_2}\right)$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pelo resultado,

$$\delta^*(x) = (E(\theta_1/x), E(\theta_2/x))$$

$$\delta^*(x) = \left(\frac{v_0 \mu_0 + n \sigma_0 \bar{x}}{v_0 + n \sigma_0}, \frac{a + n/2}{b + n/2 \cdot \frac{(x - \bar{x})^2}{v_0 + n \sigma_0} + \frac{1}{2} \sum_{i=1}^n (x_i - \bar{x})^2} \right)$$

$$E(\theta_1/x=x) = E(E(\theta_1/\theta_2, x=x)) = \frac{v_0 \mu_0 + n \sigma_0 \bar{x}}{v_0 + n \sigma_0}$$

Risco de Bayes

$$= E(\text{Tr } \Sigma_x) = E[\text{Var}(\theta_1/x) + \text{Var}(\theta_2/x)] \\ = \text{Var} \theta_1 - \text{Var}(E(\theta_1/x)), \text{Var} \theta_2 - \text{Var}(E(\theta_2/x))$$

Estimação Por Intervalo

$\Theta \subseteq \mathbb{R}$,

$D = \{[a, b], a, b \in \mathbb{R}, a < b\}$

$$L([a, b], \theta) = h(b-a) + \mathbb{I}_{[a, b]^c}(\theta)$$

h não-decrescente, $\mathbb{I}_{[a, b]^c}(\theta)$ é 1 se $\theta \notin [a, b]$

Dentre as penalidades dessa "natureza", vamos considerar a seguinte:

$$L([a, b], \theta) = \begin{cases} b-a, & \theta \in [a, b] \\ b-a + c_1(a-\theta), & \theta < a \\ b-a + c_2(\theta-b), & \theta > b \end{cases}$$

Reescrevemos L da seguinte maneira:

$$L([a, b], \theta) = -a + c_1(a - \theta) \mathbb{I}_{(-\infty, a]}(\theta) + b + c_2(\theta - b) \mathbb{I}_{(b, \infty)}(\theta)$$

$$= (\theta - a) + c_1(a - \theta) \mathbb{I}_{(-\infty, a]}(\theta) + (b - \theta) + c_2(\theta - b) \mathbb{I}_{(b, \infty)}(\theta) \Rightarrow$$

$$L_1(a, \theta)$$

$$L([a, b], \theta) = \underbrace{(\theta - a) \mathbb{I}_{[a, \infty)}(\theta)}_{L_1(a, \theta)} + \underbrace{(c_1 - 1)(a - \theta) \mathbb{I}_{(-\infty, a]}(\theta)}_{L_2(a, \theta)} + \underbrace{(b - \theta) \mathbb{I}_{(-\infty, b]}}_{L_1(b, \theta)}$$

$$+ \underbrace{(c_2 - 1)(\theta - b) \mathbb{I}_{(b, \infty)}}_{L_2(b, \theta)}$$

$$L([a, b], \theta) = L_1(a, \theta) + L_2(b, \theta)$$

Supondo $E(\theta) < \infty$, temos

$$E[L([a, b], \theta)] = E[L_1(a, \theta)] + E[L_2(b, \theta)]$$

$$L_1(a, \theta) = \begin{cases} \theta - a, & a \leq \theta \\ (c_1 - 1)(a - \theta), & a > \theta \end{cases}$$

$$\alpha^* \rightarrow \text{percentil de ordem } \frac{1}{1 + (c_1 - 1)} = \frac{1}{c_1}$$

Poder-se provar (Ex. 1.4) que $E[L_1(a, \theta)]$ atinge valor mínimo em

$$\alpha^* = \text{percentil de ordem } \frac{1}{1 + (c_1 - 1)} = 1/c_1 \text{ de P.}$$

$c_1 \geq 1$ p/ est. bem def.

Do mesmo modo, escrevemos

$$L_2(b, \theta) = \begin{cases} \frac{c_{2-1}}{c_2} (\theta - b), & \theta > b \\ b - \theta, & \theta \leq b \end{cases}$$

Como no caso anterior, $E[L_2(b, \theta)]$ é minímo em

$$b^* = \text{percentil de ordem } \frac{c_{2-1}}{c_2} \text{ de } P$$

Assim, se $\frac{1}{\alpha} \leq \frac{c_{2-1}}{c_2}$, $[a^*, b^*]$ é decisão de Bayes contra P .

Em particular, se $c_1 = \frac{\alpha}{1-\alpha} = c_2$, $\alpha \in \mathbb{R} \setminus \{1\}$, temos

$$\frac{1}{\alpha} = \frac{c_2}{2} \quad \text{e} \quad \frac{c_{2-1}}{c_2} = \frac{\frac{2}{\alpha}-1}{\frac{2}{\alpha}} = 1 - \frac{1}{\alpha}.$$

Exemplo 1: X_1, \dots, X_n , dado θ , c.i.i.d. $N(\theta, v_0^2)$

$$\Theta \sim N(\theta_0, \sigma_0^2)$$

Vimos que

$$\theta | X_1 = x_1, \dots, X_n = x_n \sim N \left(\frac{v_0^2 / \theta_0 + n \sigma_0^2 \bar{x}}{v_0^2 + n \sigma_0^2}, \frac{\sigma_0^2 v_0^2}{v_0^2 + n \sigma_0^2} \right)$$

$$\frac{\theta - \mu_{\theta|x}}{\sigma_{\theta|x}} \Bigg| X = x \sim N(0, 1)$$

$\Theta \subseteq \mathbb{R}$

Em muitos casos, optaremos por um conjunto (uma região) para Θ .
 Suponha que a priori, a distribuição de Θ , $\mathcal{P} : \mathcal{F} \rightarrow [0,1]$, absolutamente contínua (Lebesgue) ($\mathcal{P} \ll \lambda_1$).

Seja f a densidade de Θ .

Definição: Dizemos que $A \in \mathcal{F}$ é uma região (conjunto) HIGHEST DENSITY de probabilidade π , $0 < \pi \leq 1$, se

$$1. \mathcal{P}(\Theta \in A) \geq \pi$$

$$2. f(\theta') \geq f(\theta''), \forall \theta' \in A, \forall \theta'' \in A^c$$

Se para todo $x \in \mathbb{R}$, $\mathcal{P}_x \ll \lambda_1$, com densidade $f(\cdot | x)$, dizemos que $A \in \mathcal{F}$ atendendo (1) e (2) é "uma região HIGHEST POSTERIOR DENSITY" de probabilidade π , ou simplesmente, região HPD.

Vamos considerar então um novo prob. de decisão em que

$$D = \mathcal{F}$$

$$\text{Para } A \in D \text{ e } \Theta \in \Theta$$

$$L(A, \Theta) = \lambda_1(A) + c \mathbb{I}_{A^c}(\Theta)$$

$$L(A, \Theta) = \lambda_1(A) + c \mathbb{I}_{A^c}(\Theta)$$

Para $A \in D$,

$$\begin{aligned} E[L(A, \Theta)] &= E[\lambda_1(A) + c \mathbb{I}_{A^c}(\Theta)] \\ &= \lambda_1(A) + c E[\mathbb{I}_{A^c}(\Theta)] \\ &= \lambda_1(A) + c P(\Theta \in A^c) \\ &= \lambda_1(A) + c - c P(\Theta \in A) \end{aligned}$$

$$= c + \int_A 1 - c f d\lambda_2 = c - c \int_A f d\lambda_2$$

(já que $\int_A 1 d\lambda_2 = \lambda_2(A) = 1$)

$$= c + \int_A 1 - c f d\lambda_2$$

(já que $c \in \mathbb{R}$, é escalar)

$$\mathbb{E}[L(A, \theta)] = c + \int_A [1 - c f] d\lambda_2$$

(já que $\int_A 1 d\lambda_2 = 1$)

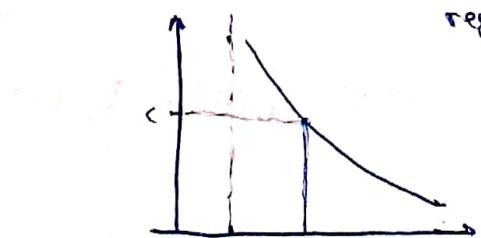
$\mathbb{E}[L(A, \theta)]$ atinge valor mínimo em

$$A^* = \{\theta \in \Theta : 1 - c f(\theta) \leq 0\}, \text{ isso é,}$$

$$A^* = \{\theta \in \Theta : f(\theta) \geq 1/c\}$$

Voltando ao Ex 2.

Fixado $\sigma \in (0, 1)$, $A^* = [\max\{b, x_m\}, \max\{b, x_m\}/(1-\sigma)^{1/\alpha+n}\}$ é
região HPD de prob. σ .



Exemplo 2: X_1, \dots, X_n , dado θ , c.i.i.d. $\text{Ber}(\min\{\theta, 1-\theta\})$. (Não há identificabilidade!)

$\Theta \sim \text{Uniforme}(0, 1)$. Para $x_1, \dots, x_n \in \{0, 1\}$, temos

Rob: (não há exist. $\hat{\theta}_{ML}$,
curv., entre outras.)

$$f(\theta | x_1, \dots, x_n) \propto P(X_1=x_1, \dots, X_n=x_n | \theta) f(\theta) = \prod_{i=1}^n (\max\{\theta, 1-\theta\})^{x_i} [1 - \max\{\theta, 1-\theta\}]^{1-x_i}$$

$$= (\max\{\theta, 1-\theta\})^{\sum x_i} (1 - \max\{\theta, 1-\theta\})^{\sum 1-x_i} = \text{constante}$$

$$f(\theta | \infty, \dots, \infty) \propto \theta^{\sum x_i} (1-\theta)^{n-\sum x_i} \mathbb{I}_{[0, 1]} + (1-\theta)^{\sum x_i} \theta^{n-\sum x_i} \mathbb{I}_{[0, 1/2]}(\theta) \quad \blacksquare$$

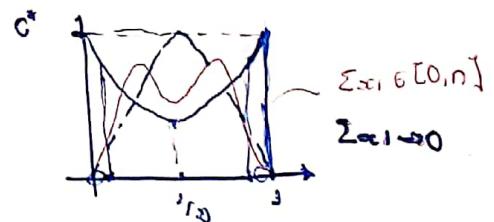
Assum ,

$$f(\theta | \infty, \dots, \infty) = \begin{cases} C^* \theta^{\sum x_i} (1-\theta)^{n-\sum x_i}, & \theta \in [1/2, 1] \\ C' \theta^{n-\sum x_i} (1-\theta)^{\sum x_i}, & \theta \in [0, 1/2] \end{cases}$$

$$C^* = \left(\int_0^1 \theta^{\sum x_i} (1-\theta)^{n-\sum x_i} d\theta \right)^{-1}$$

Para $x_i = 1$, $i = 1, \dots, n$, demos

$$f(\theta | \infty, \dots, \infty) = \begin{cases} C^* \theta^n, & \theta \in [1/2, 1] \\ C' (1-\theta)^n, & \theta \in [0, 1/2] \end{cases}$$



Testes de Hipóteses Aula ~~2~~, 09/11/11

2

Dependence of composition of mixture
on observation -
X-ray declination per th re

Varian $\rho_{\text{gas}}(H_2, H_1) \in T$ & $H = H_0 \cup H_1 \in \mathcal{C}_0$. $N(H_1) = \emptyset$

H : O e H

Note case 2

do: Nægefan Hø; dñ: Rejefan,

Comparisons as respects penicillase.

Op

vr
O
C
m

1

C' "mean simple" fungo

$$\text{Voces} \quad \text{plascerán} \quad X = x \quad \left(\begin{array}{l} \{d_0, d_1\}, H, L, P_{x=x} \end{array} \right) \quad = 0.19800112$$

$$\rho(d_1, R_{\text{ext}}) = \int_{\mathbb{H}} L(d_1, \theta) dR_x = \int_0^{\infty} a_1 dR_x + \int_{\mathbb{H}_1} a_2 dR_x =$$

~~$\int_{\mathbb{H}_1}$~~

$$\rho(d_1, R_{\text{ext}}) = \int_{\mathbb{H}} L(d_1, \theta) dR_x = \int_{\mathbb{H}_0} a_1 dR_x = a_1 \rho(\theta \in \mathbb{A}_0 | \omega).$$

28

No problema para experimentações a níveis de diversidade (ou heterogeneidade) de Bento do Rio Branco

$$q^*(x) = \begin{cases} 1, & P(\theta \in \Theta_1 | x) > \frac{a_1}{a_1 + a_2}, \\ 0, & P(\theta \in \Theta_1 | x) \leq \frac{a_1}{a_1 + a_2}. \end{cases}$$

$$\text{Ex. 1: } (\text{H}) = \left\{ \frac{1}{14}, \frac{1}{12} \right\}$$

Mr. H. J. Doolittle, City Surveyor

$$P(\theta = \frac{\pi}{4}) = y = 1 - P(\theta = \frac{\pi}{2}), y > 0$$

$$\text{Given } \sin \theta = \frac{3}{5} \text{ and } \theta \text{ is an angle in the second quadrant.}$$

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D-133

$$P(\theta = \frac{1}{2} | x_i = 1) = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(+1) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-1)$$

3' $\text{P}(\text{A}-\text{N})\text{B}-3$ W_2

卷之三

1

$$(m - \sum x_i) \log_3 < \log \frac{2^m (1-\theta)}{\theta^{x_1}} \Leftrightarrow$$

$$\sum_{i=1}^m x_i > m \log_3 \left(\frac{1}{\theta} \right) = \log \frac{2^m (1-\theta)}{\theta^m}$$

\log_3

para interpretar

$$= m - \log \left(\frac{a_n (1-\theta)}{\theta^n} \cdot 2^m \right) = n$$

$\log(\theta)$

Logo o teste é Jack

$$f(x_1, \dots, x_n) = \begin{cases} 1, & \sum x_i > k \\ 0, & \text{c.c.} \end{cases}$$

$$\text{Ex. 2: } \Theta = [0, 1], \Theta_0 = [0, \frac{1}{2}], \Theta_1 = [\frac{1}{2}, 1]$$

X_1, \dots, X_n Jacks θ , c.i.d. Bern(θ)

$\Theta \sim \text{Beta}(a, b)$ com $a, b > 0$

Hipóteses: $H_0: \theta \leq \frac{1}{2}$, $H_1: \theta > \frac{1}{2}$

Resultado: $\theta | X = x \sim \text{Beta}(a + \sum x_i, b + n - \sum x_i)$

Rejeição: H_0 , se

$$P\left(\theta > \frac{1}{2} \mid x\right) > \frac{a_1}{a_1 + a_2} \quad \left(\frac{1}{\Gamma(a+b+n)} \frac{\Gamma(a+b+n)}{\Gamma(a+\sum x_i) \Gamma(b+n-\sum x_i)} \right)^n$$

Obs: Para $\theta_0: \theta \leq \theta_0$, th: $\theta > \theta_0$.

μ

Logo $P(\theta > \theta_0 \mid x) > \frac{1}{2} \Leftrightarrow H_0(\theta \mid x) > \theta_0$

Tipos de funções de perda:

U-shaped function (quadrática) ou teste F仇tchilea

ou mediana a posteriori

$$\text{Ex 3: } \Theta = [0, \infty), \theta = (\theta_1, \theta_2), H_0: \theta_1 > \theta_2, H_1: \theta_1 < \theta_2$$

X_1, \dots, X_n Jacks θ_1 , θ_2 e fui para $X_1 \sim \text{Poisson}(\theta_1)$, $i=1, \dots, n$ e $X_j \sim \text{Poisson}(\theta_2)$, $j=1, \dots, n$

Abja $\theta_i \sim \text{Gamo}(\mu_i, \lambda_i)$, $i=1, 2$ com $\theta_1 \perp \theta_2$

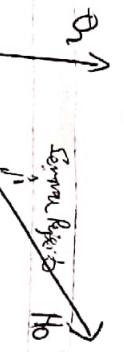
$$f(\theta | x, y) \propto f(x, y | \theta) \propto \prod_{i=1}^n \frac{1}{\Gamma(x_i + y_i)} \theta^{x_i + y_i} \prod_{i=1}^n \lambda_i^{x_i} \mu_i^{y_i} \theta^{x_i} \propto \theta^{x_1 + \dots + x_n} \theta^{y_1 + \dots + y_n} \propto \theta^{x_1 + \dots + x_n + y_1 + \dots + y_n}$$

ou θ_1, θ_2 e (índice para θ)

Int: $C: \theta_1 \perp \theta_2 \mid x, y$ com θ_1, θ_2 i.i.d. $\text{Gamo}(\mu_1 + \sum x_i, \lambda_1 + \lambda_2)$

Rejeição: $\theta_1 | x, y \sim \text{Gamo}(\mu_1 + \sum x_i, \lambda_1 + \lambda_2)$

$$P(\theta_1 > \theta_2 \mid x, y) > \frac{a_1}{a_1 + a_2}$$



Another more example

$$H_0: \theta_1 = \theta_2 \quad H_1: \theta_1 \neq \theta_2$$

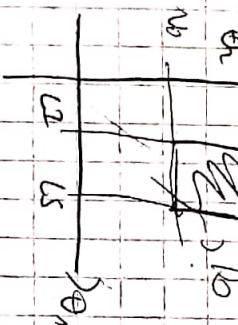
Extreme values alternatives.

$$P(\theta_1 = \theta_2 | x, y) = 0, \quad P(\theta_1 \neq \theta_2) = 1.$$

$$\text{Ex 4: } X_1, \dots, X_n \text{ dado, } \theta = (\theta_1, \theta_2) \in [0, \theta_0] \times [0, \theta_0]$$

$$H_0: \left(\frac{a}{b}, \frac{b}{b} \right)^c$$

$$\theta_1 \sim \mathcal{U}(a, b), \quad \theta_2 \sim N\left(\theta_0, \frac{\sigma^2}{\theta_0}\right)$$



$$\text{Já: } \theta_2 \sim \mathcal{U}(a, b), \quad \theta_1 \sim \prod_{i=1}^n \left[a + \frac{\theta_1}{2}, b + \frac{\theta_1}{2} \right] \frac{(b-a)}{2b}$$

$$\theta_1 | \theta_2, \theta_0 \sim \mathcal{N}\left(\frac{2\theta_0 + \theta_2}{b+a}, \frac{\sigma^2}{b+a}\right)$$

$$\text{Usando que: } \theta_1 | \theta_2, \theta_0 \sim \prod_{i=1}^n \left[a + \frac{\theta_1}{2}, b + \frac{\theta_1}{2} \right] \frac{(b-a)}{2b}$$

que

Desseja por def: α

Assumindo problema unidimensional

$$\Phi(\alpha) = \begin{cases} 1, & E(\theta|\alpha) > \theta_0 \\ 0, & E(\theta|\alpha) \leq \theta_0 \end{cases}$$

11/11/2015

Otros posibilidades

$$H_0: \theta \leq \theta_0 \quad L(d_1, \theta) = (\theta - \theta_0) I(\theta) \quad [\theta_0, \infty)$$

$$H_1: \theta > \theta_0.$$

$$P(d_1, R) = \int_{\Theta} L(d_1, \theta) dP = \int_{\Theta} \theta dP + \int_{\Theta} (\theta - \theta_0) dP$$

$$= \int_{\Theta} \theta dP - \theta_0 P(\theta < \theta_0).$$

Analogamente

$$P(d_2, R) = \int_{\Theta} L(d_2, \theta) dP = \int_{\Theta} (\theta - \theta_0) dP + \int_{\Theta} \theta dP - \theta_0 P(\theta < \theta_0) - \int_{\Theta} \theta dP$$

R

Definido por def: α

$$P(d_1, R) = P(d_1, \theta) \Leftrightarrow \theta P(\theta < \theta_0) = \int_{\Theta} \theta dP \leq \int_{\Theta} \theta dP - \theta_0 P(\theta < \theta_0)$$

$$\Leftrightarrow \boxed{\theta < E(\theta)}.$$

Teoria da Decisão

11/12/15

Aula 2

Observações lista 4.

$$(1) L(A, \theta) = |A| + K \mathbb{I}_{A^c}(\theta)$$

(2) Estudar HPD em função de y .

Outras penalidades possíveis:

$$\Theta = \mathbb{R}$$

$$\Theta_0 \subset \mathbb{R} \quad L(d_0, \theta) = (\theta - \theta_0) \mathbb{I}_{(-\infty, \theta_0]}(\theta)$$

$$H_0: \theta \leq \theta_0 \quad (\theta \in \Theta_0)$$

$$H_1: \theta > \theta_0 \quad (\theta \in \Theta_1) \quad L(d_1, \theta) = (\theta_0 - \theta) \mathbb{I}_{[\theta_0, \infty)}(\theta)$$

$$\rho(d_0, P) = \int_{\Theta} L(d_0, \theta) dP = \int_{\Theta_0} 0 dP + \int_{\Theta_1} (\theta - \theta_0) dP$$

$$= \int_{\Theta_1} \theta dP - \theta_0 P(\theta \in \Theta_1)$$

$$\rho(d_1, P) = \int_{\Theta} L(d_1, \theta) dP = \int_{\Theta_0} (\theta_0 - \theta) dP + \int_{\Theta_1} 0 dP = \theta_0 P(\theta \in \Theta_0) - \int_{[\theta_0, \infty)} \theta dP$$

Decisão por d_1 se

$$\rho(d_1, P) < \rho(d_0, P) \Leftrightarrow \theta_0 P(\theta \in \Theta_0) - \int_{[\theta_0, \infty)} \theta dP < \int_{\Theta_1} \theta dP - \theta_0 P(\theta \in \Theta_1) \Leftrightarrow$$

$$\theta_0 P(\theta \in \Theta_0) < \int_{[\theta_0, \infty)} \theta dP \Leftrightarrow \theta_0 < \mathbb{E}(\theta)$$

Assim, no problema com dados,

$$\Psi^*(\omega) = \begin{cases} 1, & E(\Theta|\omega) \not\subset \Theta_0 \\ 0, & E(\Theta|\omega) \subset \Theta_0. \end{cases}$$

Ex 1. X_1, \dots, X_n , dado Θ , c.i.i.d. $N(\theta, \sigma^2)$

$\Theta \sim N(\theta_0, \sigma_0^2)$ $H_0: \theta \leq \theta^*$ $H_1: \theta > \theta^*$

$$\Theta | x_1, \dots, x_n \sim N \left(\frac{\sigma_0^2 \theta_0 + n \sigma^2 \bar{x}}{\sigma_0^2 + n \sigma^2}, \frac{\sigma_0^2 \cdot \sigma^2}{\sigma_0^2 + n \sigma^2} \right)$$

$$\Psi^*(\omega) = 1 \text{ se } E(\Theta|\omega) \not\subset \Theta^* \Leftrightarrow$$

$$\frac{\sigma_0^2 \theta_0 + n \sigma^2 \bar{x}}{\sigma_0^2 + n \sigma^2} \not> \Theta^* \Leftrightarrow \bar{x} > \frac{\Theta^*(\sigma_0^2 + n \sigma^2) - \sigma_0^2 \theta_0}{n \sigma^2}$$

Ex. 2. X_1, \dots, X_n , dado Θ , c.i.i.d. $Exp(\theta)$

$\Theta \sim \text{Gomai}(a, b)$, $a, b > 0$ $H_0: \theta \leq \theta_0$ $\theta_0 > 0$

$\Theta | x_1, \dots, x_n \sim \text{Goma}(a+n, b+\sum x_i)$ $H_1: \theta > \theta_0$

$$\Psi^*(\omega) = 1 \text{ se } E(\Theta|\omega) \not\subset \Theta_0 \Leftrightarrow \frac{a+n}{b+\sum x_i} \not> \theta_0 \Leftrightarrow \sum x_i < (a+n)\theta_0 - b.$$

Vimos também que para a perda 0-1-1,

$$\Psi^*(\omega) = 1 \text{ se } M_d(\Theta|\omega) \not\subset \Theta_0 \text{ é teste de Bayes}$$

para $H_0: \theta \leq \theta_0$ versus $H_1: \theta > \theta_0$.

Hipóteses Precisas

Suponhamos que $\Theta \subseteq \mathbb{R}^n$ e seja λ_n a medida de Lebesgue em $\mathcal{B}(\mathbb{R}^n)$.

Dizemos que a hipótese nula

$$H_0: \Theta \in A, \quad A \in \Theta \cap \mathcal{B}(\mathbb{R}^n)$$

é precisa se $\lambda_n(A) = 0$, e $\lambda_n(\Theta) > 0$.

Exemplo.

(1) $\Theta = \mathbb{R}$

$$\Theta_0 = \{\theta\}$$

$$H_0: \Theta \in \Theta_0 (\theta = 3)$$

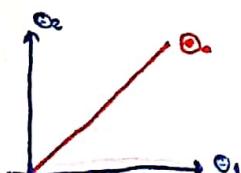
$$H_1: \Theta \notin \Theta_0$$

Claro que $\lambda_1(\{\theta\}) = 0$. $\{\theta\}$ é hipótese precisa.

(2)

$$\Theta = \mathbb{R}_+^2$$

$$\Theta_0 = \{(\theta_1, \theta_2) \in \mathbb{R}_+^2 : \theta_1 = \theta_2\}$$



$$\lambda_2(\Theta_0) = 0.$$

Dois abordagens para o problema de testar hipóteses nulas precisas.

(1) Teste de Jeffreys

(2) FBST (Full Bayesian Significance Test)

Voltando ao Ex 1.

X_1, \dots, X_n , dado θ , c.i.i.d. $N(\theta, \sigma^2)$

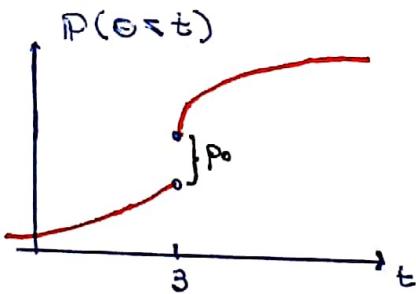
$$H_0: \theta = 3$$

$$H_1: \theta \neq 3$$

Vamos superar, a priori, uma medida de probabilidade mista P^* : $\mathcal{B}(R) \rightarrow [0,1]$

$$P^*(A) = p_0 \mathbb{I}_A(3) + (1-p_0) \int_A \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta-3)^2}{2\sigma^2}} d\theta$$

Alternativamente,



$$P^*(\theta = 3) = p_0$$

$$P^*(G=3 | x_1, \dots, x_n) = \frac{\int_{\mathcal{B}(x_1, \dots, x_n)} f(x_1, \dots, x_n | \theta=3) P^*(\theta=3)}{\int_{\mathcal{B}(x_1, \dots, x_n)} f(x_1, \dots, x_n | \theta=3) P^*(\theta=3) + \int_{\mathcal{B}(x_1, \dots, x_n)} f(x_1, \dots, x_n | \theta) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta-3)^2}{2\sigma^2}} d\theta (1-p_0)}$$

$$= \frac{p_0}{e^{-\sum_{i=1}^n (x_i - 3)^2 / 2\sigma^2} + (1-p_0) \int_R e^{-\sum_{i=1}^n (x_i - \theta)^2 / 2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\theta-3)^2}{2\sigma^2}} d\theta}$$

$$= \frac{p_0}{p_0 + (1-p_0) \int_R e^{\frac{\sum_{i=1}^n (x_i - 3)^2}{2\sigma^2} - \frac{\sum_{i=1}^n (x_i - \theta)^2}{2\sigma^2} - \frac{(\theta-3)^2}{2\sigma^2}} d\theta}$$

$$P_0 + \frac{(1-p_0)}{\sqrt{2\pi\sigma_0^2}} e^{\frac{q_n - 6n\bar{x} - \theta_0^2/2\sigma_0^2}{2\sigma_0^2}}$$

Você decide por H_0 se

$$P^*(\theta = 3/\infty) > \frac{\alpha_2}{\alpha_1 + \alpha_2}$$

Exemplo 2.

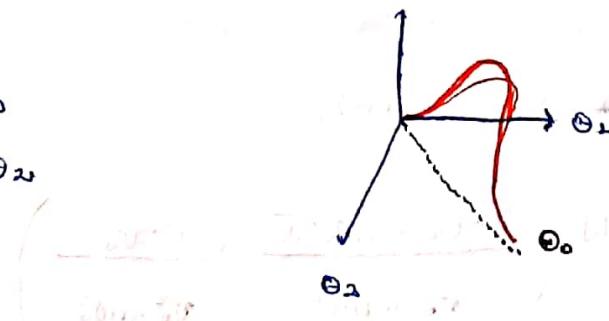
$X_1, X_2, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}$, dado $\Theta = (\theta_1, \theta_2)$, c. independentes tais que

$X_i / \Theta \sim \text{Poisson } (\theta_1), i=1, \dots, n_1$

$Y_j / \Theta \sim \text{Poisson } (\theta_2), j=1, \dots, n_2$

$$H_0: \theta_1 = \theta_2$$

$$H_1: \theta_1 \neq \theta_2$$



Vamos adotar a priori P^* dada por

$$P^*(A) = P_0 \int_{A \cap \Theta_0} \frac{b_0^{\theta_0}}{\Gamma(\theta_0)} \theta_0^{\theta_0-1} e^{-b_0\theta_0} d\theta_0 + (1-p_0) \int_A \frac{\mu_1^{\theta_1-1}}{P(\mu_1)} \theta_1^{\theta_1-1} e^{-\tau_1\theta_1} d\theta_1$$

$$\int_A \frac{\mu_2^{\theta_2-1}}{P(\mu_2)} \theta_2^{\theta_2-1} e^{-\tau_2\theta_2} d\theta_2$$

$$P^*(\Theta_0) = p_0$$

$$P^*(\theta_1 = \theta_2 / (x, y)) = \frac{p_0 \int_{\mathbb{R}_+^2} f(x, y / \theta) f_1(\theta) d\theta}{p_0 \int_{\mathbb{R}_+^2} f(x, y / \theta) f_1(\theta) d\theta + (1-p_0) \int_{\mathbb{R}_+^2} f(x, y / \theta_1, \theta_2) f_2(\theta_1, \theta_2) d\theta_1 d\theta_2}$$

$$\begin{aligned}
 &= \frac{p_0 \int_{R^2} e^{-(\mu_0 + \mu_1) \Theta} \cdot \Theta^{\sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j} \cdot \frac{b_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta_1^{\alpha_1-1} e^{-b_1 \theta_1} d\Theta}{+ (1-p) \int_{R^2} e^{-\mu_1 \theta_1} \theta_1^{\sum_{i=1}^{n_1} x_i} e^{-\mu_2 \theta_2} \theta_2^{\sum_{j=1}^{n_2} y_j} \frac{v_1^{\mu_1}}{\Gamma(\mu_1)} \theta_1^{\mu_1-1} e^{-v_1 \theta_1} \frac{v_2^{\mu_2}}{\Gamma(\mu_2)} \theta_2^{\mu_2-1} e^{-v_2 \theta_2} d\theta_1 d\theta_2} \\
 P(\theta_1 = \theta_2 / (\alpha, y)) &= \frac{p_0 \frac{b_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{\Gamma(\alpha_0 + \sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j)}{(b_0 + n_1 + n_2)^{\alpha_0 + \sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j}}}{\frac{p_0 b_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{\Gamma(\alpha_0 + \sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j)}{(b_0 + n_1 + n_2)^{\alpha_0 + \sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j}} + \frac{(1-p)}{\Gamma(\mu_1) \Gamma(\mu_2) (v_1 + n_1)^{\mu_1 - \sum_{i=1}^{n_1} x_i} (v_2 + n_2)^{\mu_2 - \sum_{j=1}^{n_2} y_j}} \frac{v_1^{\mu_1}}{\Gamma(\mu_1)} \frac{v_2^{\mu_2}}{\Gamma(\mu_2)}} \\
 &\stackrel{?}{=} \frac{\alpha_2}{\alpha_1 + \alpha_2}
 \end{aligned}$$

2. F BST

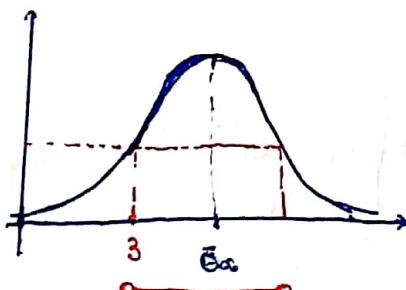
Volvendo ao exemplo 1.

$$H_0: \theta = 3$$

Após observar $\alpha = (x_1, \dots, x_n)$,

$$H_1: \theta \neq 3$$

$$\theta / \alpha \sim N \left(\frac{v_0^2 \theta_0 + n \theta_0^2 \bar{x}}{v_0^2 + n \theta_0^2}, \frac{\theta_0^2 v_0^2}{v_0^2 + n \theta_0^2} \right)$$



$$T_\alpha = \{\theta \in \Theta : f(\theta | \alpha) > f(3 | \alpha)\}$$

conjunto tangente. Em geral, um IPD.

Evidência de Pereira-Stern. $E_V(\sqrt{3}, \alpha) = 1 - P(\theta \in T_\alpha | \alpha)$

Ex. 2.

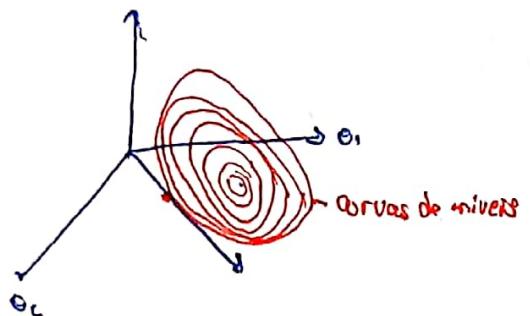
$\Theta_1 \perp\!\!\!\perp \Theta_2 \in \Theta_{1,2} \sim \text{Gamma}(\mu_i, v_i)$, $i=1,2$.

Então

$\Theta_1 \perp\!\!\!\perp \Theta_2 | (x, y)$

$\Theta_1 | (x, y) \sim \text{Gamma}(\mu_1 + \sum x_i, v_1 + n_1)$

$\Theta_2 | (x, y) \sim \text{Gamma}(\mu_2 + \sum y_j, v_2 + n_2)$



$$T_{(x,y)} = \left\{ \Theta \in \Theta : f(\Theta | (x, y)) \geq \sup_{\Theta \in \Theta_0} f(\Theta | (x, y)) \right\}$$

Nesse caso,

$$\text{Ev}(\{\Theta_1 = \Theta_2\}, (x, y)) = 1 - P(\Theta \in T_{(x,y)} | (x, y))$$

Entropy (1999). "Evidence and credibility: a full bayesian sig. test"

Teoria da Decisão

Aula 25

23/11/15

Agrupamentos de Probabilidades

(Θ, \mathcal{F})

$\mathbb{P}^{(G)}(\Theta, \mathcal{F})$

Ind. 1 $\rightarrow P_1 : \mathcal{F} \rightarrow [0,1]$

Ind. 2 $\rightarrow P_2 : \mathcal{F} \rightarrow [0,1]$

:

Ind. n $\rightarrow P_n : \mathcal{F} \rightarrow [0,1]$

$\mu_{\theta}(\Theta, \mathcal{F})$: Conjunto de medidas de probabilidade sobre (Θ, \mathcal{F}) .

Definição: Um agrupamento de probabilidades ou opiniões ("opinion pool") é qualquer transformação de $(\mu_{\theta}(\Theta, \mathcal{F}))^n$ em $\mu_{\theta}(\Theta, \mathcal{F})$.

$P^{(G)} : (\mu_{\theta}(\Theta, \mathcal{F}))^n \rightarrow \mu_{\theta}(\Theta, \mathcal{F})$

$(P_1, \dots, P_n) \mapsto P^{(G)}(P_1, \dots, P_n)$

Exemplos:

1. $P^{(G)}(P_1, \dots, P_n) = P_1, \forall (P_1, \dots, P_n) \in (\mu_{\theta}(\Theta, \mathcal{F}))^n$.

2. Agrupamento linear

$P^{(G)}(P_1, \dots, P_n) = \sum_{i=1}^n \alpha_i P_i, \quad \sum_{i=1}^n \alpha_i = 1.$

3. Agrupamento Logarítmico

$P_i \ll \mu$, $\mu = f \rightarrow \mathbb{R}_+$, $i=1, \dots, n$, e tais que $\mu(\mathcal{F}) = T$

$$\text{então } f_i = \frac{dP_i}{d\mu}, \quad i=1, \dots, n$$

Nessas condições,

$$f^{(G)}(\theta) = \frac{\prod_{i=1}^n (f_i(\theta))^{\alpha_i}}{\int_{\mathbb{R}} \prod_{i=1}^n (f_i(\theta))^{\alpha_i} d\mu(\theta)} \text{ se,}$$

para $A \in \mathcal{F}$,

$$P^{(G)}(A) = \int_A f^{(G)}(\theta) d\mu(\theta).$$

Suponha que, para cada $A \in \mathcal{F}$, $i \in \{1, 2, \dots, n\}$,

$$P_i(A) = \int_A \lambda_i e^{-\lambda_i \theta} d\theta, \quad \lambda_i > 0.$$

No agrupamento linear, $P^{(G)}$ possui densidade de probab. dada por

$$f^{(G)}(\theta) = \sum_{i=1}^n \alpha_i \lambda_i e^{-\lambda_i \theta} I_{R_i}(\theta)$$

No agrupamento logarítmico

$$\begin{aligned} f^{(G)}(\theta) &\propto \prod_{i=1}^n (f_i(\theta))^{\alpha_i} = \prod_{i=1}^n (\lambda_i e^{-\lambda_i \theta} I_{R_i}(\theta))^{\alpha_i} \\ &\propto e^{-(\sum_{i=1}^n \alpha_i \lambda_i) \theta} I_{R_i}(\theta) \end{aligned}$$

$$\Rightarrow f^{(G)}(\theta) = (\sum_{i=1}^n \alpha_i \lambda_i) e^{-(\sum_{i=1}^n \alpha_i \lambda_i) \theta} I_{R_i}(\theta)$$

Propriedades Desejáveis Para Agrupamentos

Unanimidade

$$1. P_1(\cdot) = P_2(\cdot) = \dots = P_n(\cdot) \Rightarrow P^{(e)}(\cdot) = P(\cdot)$$

$$= P(\cdot)$$

$$P^{(e)}(A) = \sum_{i=1}^n \alpha_i P_i(A) = P(A) \Rightarrow \text{Agrupamento Linear satisf. Unanimidade}$$

No agrupamento logarítmico, há unanimidade se $\sum_{i=1}^n \alpha_i = 1$,

2. Bayesianidade Externa

"Agrupar e, então, revisar incerteza é equivalente a revisar incerteza e, então, agrupar opiniões"

Ex: $n=2$, $\Theta = [0,1]$.

$$P_1 \sim \text{Beta}(2,1) \quad \alpha_1 = \alpha_2 = 1/2 \quad (\text{Agr. linear})$$

$$P_2 \sim \text{Beta}(1,2)$$

Suponhamos que os indivíduos concordam que X , dado Θ , é cond. Bern(Θ).

AGRUPANDO PRIMEIRO.

$$f^{(e)}(\theta) = \frac{1}{2} \theta \mathbb{I}_{[0,1]} + \frac{1}{2} (1-\theta) \mathbb{I}_{[0,1]} = \mathbb{I}_{[0,1]}^{(e)}$$

Atualizando $f^{(e)}$ após observar $X=1$, temos

$$f^{(e)}(\theta | X=1) = 2\theta \mathbb{I}_{[0,1]}$$

Por outro lado:

ATUALIZANDO AS INCERTEZAS DOS 2 INDIVÍDUOS:

$$f_1(\theta | X=1) = 3\theta^2 \mathbb{I}_{[0,1]}(\theta)$$

$$f_2(\theta | X=1) = 6\theta(1-\theta) \mathbb{I}_{[0,1]}(\theta)$$

Agrupando as opiniões revisadas:

$$f^{(e)}(\theta | X=1) = \frac{1}{2} 3\theta^2 + \frac{1}{2} 6\theta(1-\theta) \mathbb{I}_{[0,1]}(\theta) = \left(3\theta - \frac{3\theta^2}{2}\right) \mathbb{I}_{[0,1]}(\theta) = \frac{3\theta}{2}(1-\theta) \mathbb{I}_{[0,1]}(\theta)$$

Assim, o agrupamento linear não atende, em geral, Bayesianidade Externa.

Tal propriedade é sobrefeita pelo agrupamento logarítmico f_1, f_2, \dots, f_n .

AGRUPANDO PRIMEIRO.

$$f^{(e)}(\theta) \propto \prod_{i=1}^n (f_i(\theta))^{\alpha_i}$$

Para cada $x \in \mathcal{X}$ e $\theta \in \Theta$, $f(x|\theta)$ é comum aos indivíduos e também é "verossímil" para o grupo. Atualizando $f^{(e)}(\cdot)$ ao observar $X=x$, temos

$$f^{(e)}(\theta|x) \propto f(x|\theta) f^{(e)}(\theta) = f(x|\theta) \prod_{i=1}^n (f_i(\theta))^{\alpha_i}$$

Por outro lado, cada indivíduo atualiza sua incerteza sobre θ primeiro.

$$f_i(\theta|x) \propto f(x|\theta) f_i(\theta)$$

$$\Rightarrow f_i(\theta|x) = k_i f(x|\theta) f_i(\theta), \quad k_i > 0, \quad i=1, \dots, n$$

Em seguida, agrupando as opiniões atualizadas, obtemos

$$f^{(e)}(\theta|x) \propto \prod_{i=1}^n (f_i(\theta|x))^{\alpha_i} = \prod_{i=1}^n (k_i f(x|\theta) f_i(\theta))^{\alpha_i}$$

$$\propto (f(x|\theta))^{\sum_{i=1}^n \alpha_i} \cdot \prod_{i=1}^n (f_i(\theta))^{\alpha_i}$$

Se $\sum_{i=1}^n \alpha_i = 1$, o agrupamento logarítmico atende Bayesianidade Externa.

3. PRESERVAÇÃO DE INDEPENDÊNCIA

Sejam $A, B \in \mathcal{F}$,

$$P_1(A \cap B) = P_1(A)P_1(B), \forall i=1, \dots, n \Rightarrow P^{(e)}(A \cap B) = P^{(e)}(A)P^{(e)}(B).$$

4. UNANIMIDADE FRACA

Se $A \in \mathcal{F}$ é tal que

$$P_1(A) = p, \forall i=1, \dots, n \Rightarrow P^{(e)}(A) = p.$$

CAMPOS DE COINCIDÊNCIA DE OPNIÕES DE DE FINETTI

$\Theta \subseteq \mathbb{R}$

Ind 1 $\rightarrow F_1$ } Por ora, foco em 2 indivíduos . F_i são contínuas! , $i=1,2, \dots$

Ind 2 $\rightarrow F_2$ } F_1 é a probabilidade de aprovação de $a \in \Theta$

Ind. n $\rightarrow F_n$

$\exists (a, b) \subseteq \mathbb{R}$ t. q.

$$P_1((a, b]) = F_1(b) - F_1(a) = F_2(b) - F_2(a) = P_2((a, b])$$

Ideia da prova:

$$m_1 = m_2 \vee (m_1 = \text{med. onda sob. } P_1)$$

$$m_1 \neq m_2 \quad \mathbb{P} = \{ (a, b) \subseteq \mathbb{R} : P_1((a, b]) = 1/2 \}$$

$$(-\infty, m_1], (1^{\circ} \text{ quartil } P_1, 3^{\circ} \text{ quartil } P_1), (m_1, \infty) \in \mathbb{P}$$

$$P_2((-\infty, m_1]) \leq 1/2 \quad P_2((m_1, \infty)) \geq 1/2$$

→ cond. da trans.). $\Rightarrow \exists (a, b) \text{ t. q. } P_2((a, b]) = 1/2$

$\exists I_1 = [a, b]$ d.q. $P_1([a, b]) = P_2([a, b]) = 1/2$

$$\bar{F}_1(t) = \begin{cases} 2F_1(t), & t \leq a \\ 2[F_1(t+b-a) - 1/2], & t > a \end{cases}$$

$$\exists (a', b') \in \mathbb{R}, \bar{F}_1(b') - \bar{F}_1(a') = \bar{F}_2(b') - \bar{F}_2(a') = 1/2$$

$$b' \leq a$$

$$\bar{F}_1(b') - \bar{F}_1(a') = 2F_1(b') - 2F_1(a') = 1/2 \Rightarrow F_1(b') - F_1(a') = 1/4$$

$$\bar{F}_2(b') - \bar{F}_2(a') = 2F_2(b') - 2F_2(a') = 1/2 \Rightarrow F_2(b') - F_2(a') = 1/4.$$

Distribuição. Consenso de M. H. DeGroot.

$$\text{Ind. } 1 \rightarrow P_1^{(0)}$$

w.i.s. o peso que o ind. i dá à opinião do ind. j sobre o.

$$\text{Ind. } 2 \rightarrow P_2^{(0)}$$

Para todo i,j, $w_{ij} \geq 0 \quad \sum w_{ij} = 1$.

$$\vdots$$

$$\text{Ind. } n \rightarrow P_n^{(0)}$$

$n=1$.

$$P_1^{(1)} = \sum_{j=1}^n w_{1j} P_j^{(0)} \quad \text{Cálculo da opinião do ind. 1}$$

$$P_2^{(1)} = \sum_{j=1}^n w_{2j} P_j^{(0)}$$

$$\vdots$$

$$P_n^{(1)} = \sum_{j=1}^n w_{nj} P_j^{(0)}$$

$n=2$

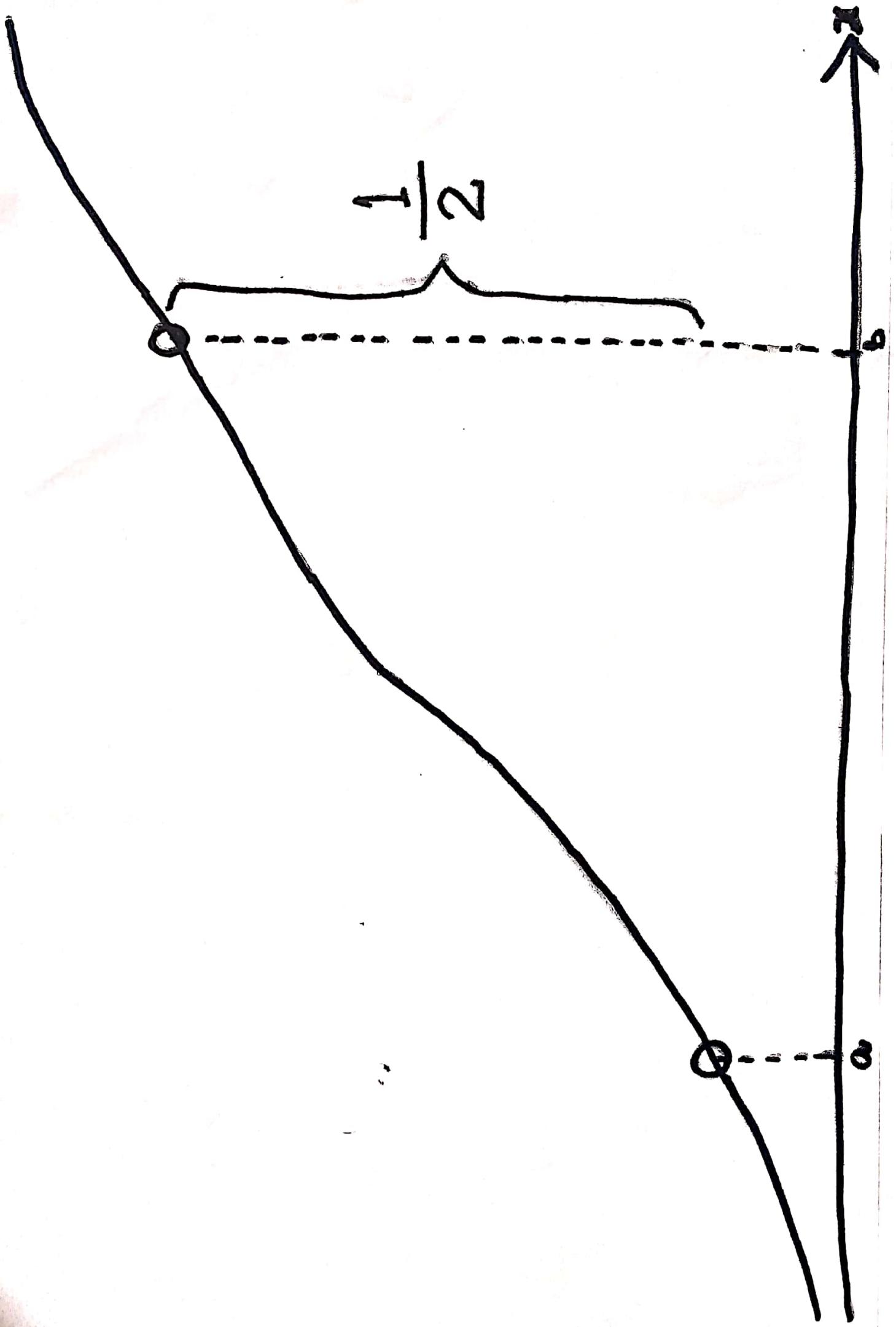
$$P_1^{(2)} = \sum_{j=1}^n w_{1j} P_j^{(1)}, \dots, P_n^{(2)} = \sum_{j=1}^n w_{nj} P_j^{(1)}$$

$$\sum_{s=0}^n w_s^i P_s \rightarrow \sum_{s=0}^{\infty} w_s^i P_s$$

w^* recorrente e aperiodica.

$$\sum_{s=0}^n w_s^i P_s$$

23/11/15



Teoria da Decisão

Aula

25/11/2015

Algoritmos ABC em Environmental Stress Screening

Algoritmo ABC: gerar pontos de uma dist. a posteriori.

Approximate Bayesian Computation

. A B C

. Alg. da Regração 3

Conheço a priori

Conheço a dist. das dadas

Prob. na posteriori

6ero

١٦٨٣

ESS : pecos back on rising

Lote N pecos

Prop. p de peças de qualidade ruim

Taxa de folha das peças de qualidade boa : λ_b

ruimder

Proc. de Entrega

1. Am. das N pegas
 2. Submissão ao endereço
 3. Verif. das quedas rev.
 4. Veri. de qual das que faltaram

Funções c.s.

C_{12} : custo de estressar ruim e sobre
 C_{22} : " " " " bom e falhar

C₂: Custo operar de estressar qq e sobrevi.

1ª Avaliação 05/10/2015

MAE 5753 - 2º semestre de 2015

1) Prove o seguinte resultado admitindo as suposições SP1 a SP4: se $(A_n)_{n \geq 1}$ é uma sequência decrescente de eventos e B é um evento tal que $\cap A_n \subsetneq B$, então $B \subsetneq A_i$ apenas para um número finito de valores de i. Enuncie e prove o resultado análogo para uma sequência crescente de eventos $(A_n)_{n \geq 1}$.

2) Sejam p e m números fixados tais que $0 < p < 1$ e $m > 0$, e considere a seguinte situação: uma pessoa receberá uma quantia de m dólares que deverá alocar entre a ocorrência de um evento A, de probabilidade p, e seu complementar A^c . Sua recompensa será a quantia que foi alocada em A ou A^c , dependendo de qual evento efetivamente ocorrer. Em outras palavras, para todos os valores de x no intervalo $[0, m]$, a pessoa poderá escolher uma entre todas as loterias do tipo $\langle x, p ; m - x, 1 - p \rangle$, isto é, uma loteria que premia com a recompensa x com probabilidade p e com a recompensa $m - x$ com probabilidade $1 - p$. Considerando todos os possíveis de valores de p e m, encontre a melhor alocação dos m dólares considerando que a função de utilidade U da pessoa para o ganho monetário é definida no intervalo $[0, m]$ por

$$U(r) = 2(r/m_0) - (r/m_0)^2, \text{ onde } m_0 \geq m.$$

3) Prove o seguinte resultado a partir das suposições SP1 a SP4 e da suposição CP:
seja D um evento qualquer. Então, $D \sim \emptyset$ se, e somente se, $(A | D) \sim (B | D)$ para todos os eventos A e B.

4) Suponha que um indivíduo irá vender sorvete num jogo de futebol e deve decidir de antemão quanto deve encomendar dessa produto. Suponha que seu ganho é de m reais por cada litro de sorvete vendido, enquanto que sua perda é de c reais por cada litro encomendado mas não vendido. Admitindo que a demanda por esse tipo de sorvete (em litros) é uma variável aleatória absolutamente contínua Θ com f.d.p f e f.d. F , mostre que o lucro esperado desse indivíduo será maximizado encomendando uma quantidade α tal que $F(\alpha) = m/(m + c)$.

5) Suponha que o intervalo de tempo que um homem deve esperar a cada manhã para apanhar o ônibus que o leva para o trabalho é distribuído uniformemente no intervalo $(0, \square)$, onde o valor de \square é desconhecido. Suponha, a priori, que \square seja distribuído segundo o modelo Pareto de parâmetros $a = 1$ e $b > 0$. Por quantas manhãs esse homem deve registrar o tempo de espera pelo ônibus que o leva ao trabalho para que ele seja capaz de especificar um intervalo de comprimento 0.01 unidades tal que a probabilidade de que $\log(\square)$ esteja nesse intervalo seja pelo menos 0,95 ? 279

2ª Avaliação 30/11/2015

MAE 5753 - 2º semestre de 2015

- 1) Suponha que X_1, \dots, X_n , dado θ , sejam condicionalmente independentes e identicamente distribuídas segundo o modelo Poisson de média θ . Suponha, a priori, que θ seja distribuído segundo o modelo Gama de parâmetros a e b , $a, b > 0$. Para estimar θ , considere a seguinte função de perda:

$$L(d, \theta) = (d - \theta)^2 / \theta = (d - \theta)^2 \theta^{-1} .$$

Obtenha o estimador de Bayes para θ nesse caso e mostre que o risco de Bayes é dado por $1/(b + n)$.

- 2) Suponha que X_1, \dots, X_n , dado θ , sejam condicionalmente independentes e identicamente distribuídas segundo o modelo Normal de média θ e variância 9. Considere, a priori, que $P(\theta = -1) = P(\theta = 1) = 1/2$. Para testar as hipóteses $H_0: \theta = -1$ versus $H_1: \theta = 1$, considere a função de perda 0 – 1 – 1. Determine o teste de Bayes nesse caso.
- 3) Suponha que um parâmetro de interesse, θ , é distribuído, a priori, segundo $P(\theta = \theta_1) = \xi$ e $P(\theta = \theta_2) = 1 - \xi$, $0 < \xi < 1$. Para cada ponto amostral $x \in \mathbb{X}$, seja $\xi(x)$ a probabilidade a posteriori de $\{\theta = \theta_1\}$ dado x . Mostre que $E[\xi(x)] = \xi$.

$$\begin{aligned} & (\mathcal{X}, \mathcal{B}(x), P) \\ & \text{se } \theta \\ & \quad \theta_1 \quad \theta_2 \\ & \xi(x) \\ & \theta_1 \theta_2 \end{aligned}$$

$\theta_1 \theta_2 \text{ prior } \rightarrow$

- 4) Suponha que o parâmetro real θ deve ser estimado segundo a seguinte função de perda: $L(d, \theta) = k_1(\theta - d)$, se $d < \theta$, e $L(d, \theta) = k_2(d - \theta)$, se $d \geq \theta$, $k_1, k_2 > 0$. Suponha que $E[|\theta|] < \infty$. Mostre que a decisão de Bayes, d^* , deve satisfazer

$$P(\theta \leq d^*) \geq k_1/(k_1 + k_2)^{-1} \quad \text{e} \quad P(\theta \geq d^*) \geq k_2/(k_1 + k_2)^{-1}.$$

- 5) Suponha que em um problema de decisão arbitrário, a decisão d^* é uma decisão de Bayes contra duas distribuições distintas P_1 e P_2 para o parâmetro θ . Mostre que para qualquer número $\alpha \in (0, 1)$, d^* é também decisão de Bayes contra a distribuição $\alpha P_1 + (1-\alpha) P_2$ para θ .

MAE 5753 - 2º semestre /2015

LISTA DE EXERCÍCIOS 1

25/08

Pasta

4

Nº cópias

4

Exercícios extraídos do livro "Optimal Statistical Decisions" de
M. H. DeGroot.

Exercícios do Capítulo 6 (página 83)

2, 3a, 3b, 3d, 4, 5, 6a, 6c

Exercícios do Capítulo 9 (páginas 186, 187 e 188)

21, 23, 30, 32, 34

this inequality is equivalent to the relation $AD \preceq BD$. These remarks naturally lead to the next assumption.

Assumption CP For any three events A , B , and D , $(A|D) \preceq (B|D)$ if, and only if, $AD \preceq BD$.

As we remarked above, when Assumptions SP_1 to SP_5 are made and there exists a probability distribution P which can be uniquely specified, it is sufficient for most purposes to apply Assumption CP only to events D such that $P(D) > 0$. However, in order to keep the assumptions independent of each other, we have imposed the slightly stronger requirement that Assumption CP applies to all events D . Furthermore, it now follows from Assumptions SP_1 and CP that for any three events A , B , and D , either $(A|D) \preceq (B|D)$ or $(B|D) \preceq (A|D)$. It is possible, of course, that both relations will be correct. In fact, both relations will always be correct when D is an event such that $P(D) = 0$ (see Exercise 6a).

By combining Assumption CP with Theorem 4 of Sec. 6.5, we obtain the following general theorem.

Theorem 1 If the relation \preceq satisfies Assumptions SP_1 to SP_5 and Assumption CP , then the function P defined by Eq. (3) of Sec. 6.4 is the unique probability distribution which has the following property: For any three events A , B , and D such that $P(D) > 0$, $(A|D) \preceq (B|D)$ if, and only if, $P(A|D) \leq P(B|D)$.

FURTHER REMARKS AND REFERENCES

Luce and Suppes (1965) give a good review of subjective probability, including the relatively small amount of work that has been done on the experimental measurement of subjective probabilities. An interesting and controversial group of papers dealing with the existence and measurement of subjective probabilities are those by Ellsberg (1961), Fellner (1961, 1963), Raiffa (1961), Brewer (1963), and Roberts (1963). The book by Fellner (1965) is also of interest here.

Exercises 7 to 9 at the end of this chapter require the actual evaluation of the reader's subjective probabilities of some specific events.

EXERCISES

- Prove that the following results follow from Assumptions SP_1 to SP_5 :
 - If A , B , and D are any three events such that $A \preceq B$ and $B \prec D$, then $A \prec D$.
 - If A and B are any two events, then $A \preceq B$ if, and only if, $A^c \gtrsim B^c$.

(c) If A and B are events such that $A \subset B$, then $A \preceq B$.

(d) If A_1, \dots, A_n and B_1, \dots, B_n are events such that $B_i B_j = \emptyset$ for $i \neq j$ and $A_i \preceq B_i$ for $i = 1, \dots, n$, then $\cup_{i=1}^n A_i \preceq \cup_{i=1}^n B_i$. If, in addition, $A_i \prec B_i$ for at least one value of i ($i = 1, \dots, n$), then $\cup_{i=1}^n A_i \prec \cup_{i=1}^n B_i$. (This is an extension of Theorem 2 of Sec. 6.2 since it is not assumed here that A_1, \dots, A_n are disjoint.) Furthermore, the statements remain true if the condition that $B_i B_j = \emptyset$ for $i \neq j$ is replaced by the weaker condition that $B_i B_j \sim \emptyset$ for $i \neq j$.

2. Suppose that Assumptions SP_1 to SP_5 are made. Let A_1, \dots, A_m be disjoint events, and let B_1, \dots, B_n also be disjoint events such that $\cup_{i=1}^m A_i = S$, $A_1 \preceq \dots \preceq A_m$ and $B_1 \preceq \dots \preceq B_n$, and $m \leq n$. Prove that $B_1 \preceq A_m$.

3. Prove that the following results follow from Assumptions SP_1 to SP_4 :

(a) If $A_1 \supset A_2 \supset \dots$ is a decreasing sequence of events and if B is an event such that $\cap_{i=1}^{\infty} A_i \prec B$, then $A_i \gtrsim B$ for only a finite number of values of i ($i = 1, 2, \dots$). State and prove the analogous result for an increasing sequence of events $A_1 \subset A_2 \subset \dots$.

(b) If $A_1 \subset A_2 \subset \dots$ is an increasing sequence of events and $B_1 \supset B_2 \supset \dots$ is a decreasing sequence of events such that $A_i \preceq B_i$ for $i = 1, 2, \dots$, then $\cup_{i=1}^{\infty} A_i \preceq \cap_{i=1}^{\infty} B_i$. Furthermore, if $A_i \sim B_i$ for $i = 1, 2, \dots$, then $\cap_{i=1}^{\infty} A_i \sim \cap_{i=1}^{\infty} B_i$. State and prove the analogous results for increasing sequences of events $A_1 \subset A_2 \subset \dots$ and $B_1 \subset B_2 \subset \dots$.

(c) If $A_1 \supset A_2 \supset \dots$ is a decreasing sequence of events such that $A_i \preceq B_i$ for $i = 1, 2, \dots$, then $\cap_{i=1}^{\infty} A_i \preceq \cap_{i=1}^{\infty} B_i$. Hence, every subset A of S can be expressed in the form $A = (AS_1) \cup (AS_2)$. Suppose that a relation \preceq is defined between subsets of S as follows: If A and B are any two subsets of S , then $A \preceq B$ if either $P^*(AS_1) < P^*(BS_1)$ and $P^*(AS_2) \leq P^*(BS_2)$. Show that the relation \preceq satisfies Assumptions SP_1 to SP_3 but not Assumption SP_4 .

4. Consider the problem described in Exercise 4, but suppose now that the relation \preceq between subsets of S is defined as follows: If A and B are any two subsets of S , then $A \preceq B$ if either $P^*(A) < P^*(B)$ or $P^*(A) = P^*(B)$ and $P^*(AS) \leq P^*(BS)$. Show that the relation \preceq satisfies Assumptions SP_1 to SP_3 but not Assumption SP_4 .

5. Prove that the following results can be derived from Assumptions SP_1 to SP_4 and Assumption CP :

(a) If D is any event, then $D \sim \emptyset$ if, and only if, $(A|D) \sim (B|D)$ for all events A and B .

(b) If A and D are any events, then $(\emptyset|D) \preceq (A|D) \preceq (D|D)$.

(c) Let A , B , D , and E be events such that $A \preceq B$ and $D \preceq E$. Then $(A|E) \preceq (B|E)$ if, and only if, $[(A \cup D)|E] \preceq [(B \cup D)|E]$.

EXERCISES

CHAP. 9 CONJUGATE PRIOR DISTRIBUTIONS

186. Show that for any given positive numbers μ and σ^2 , there is a unique gamma distribution which has mean μ and variance σ^2 .
187. Suppose that when magnetic recording tape is manufactured by a certain gamma distribution, the mean number W of defects on a 1,200-ft roll of tape is unknown, and the mean number W of defects on any roll of tape is 2 per process, the mean number W is a gamma distribution whose mean is 2 and whose variance is 1. Suppose also that the number of defects on any roll of tape when $W = w$ has a Poisson distribution with mean w . Suppose further that after a random sample of rolls of tape has been taken and the number of defects on each roll has been counted, the mean of the posterior distribution of W is 1.6 and the variance is 0.16. Show that eight rolls of tape were included in the random sample and that the average number of defects per roll in the sample was 1.5.
188. An unknown proportion W of the items produced by a certain machine are defective. Suppose that the prior distribution of W is a beta distribution with parameters $\alpha = 1$ and $\beta = 99$. Suppose also that items produced by the machine are selected at random and observed one at a time until exactly five defective items have been found. If, when sampling terminates, the mean of the posterior distribution of W is 0.02, show that 195 nondetective items were observed during the sampling process.
189. Suppose that in a large population of voters, the proportion W who belong to the Liberal Party is unknown, and suppose that the prior distribution of W is a beta distribution with parameters $\alpha = 1$ and $\beta = 10$. In a random sample of 1,000 voters, it is found that 123 belong to the Liberal Party, what is the posterior distribution of W ?
- (a) If, instead of taking a random sample as in part (a), voters are selected one at a time until exactly 123 have been found who belong to the Liberal Party. Suppose that a total of 1,000 voters had to be selected in order to accomplish this. What is the posterior distribution of W ?
190. Prove Theorem 3 of Sec. 9.4.
191. The length of life of a lamp manufactured by a certain process has an exponential distribution with an unknown value of the parameter W . Suppose that the prior distribution of W is a gamma distribution for which the coefficient of variation is 0.5. A random sample of lamps is to be tested, and the length of life of each of the lamps is to be noted. If the coefficient of variation of the posterior distribution of W must be reduced to the value 0.1, show that 96 lamps should be tested.
192. Extend Theorem 3 of Sec. 9.4 so that it covers the case of a random sample from a gamma distribution with parameters α and W , where the value of α is known, and the value of W is unknown.
193. Consider a normal distribution for which the value of the mean W is unknown, and the variance is 4, and suppose that the prior distribution of W is a normal distribution whose variance is 9. How large a random sample must be taken from the given normal distribution in order to be able to specify an interval having a length of 1 unit such that the probability that W lies in this interval is at least 0.95? (Answer: $n = 62$.)
194. Prove Theorem 2 of Sec. 9.5.
195. Suppose that the value of the precision W of a normal distribution is unknown, and suppose that the distribution of W is a gamma distribution with parameters α and β . Let V denote the variance of the given normal distribution.
- (a) Find the p.d.f. of V .
- (b) Show that if $\alpha > 1$, $E(V) = \beta / (\alpha - 1)$.
- (c) Show that if $\alpha > 2$, $\text{Var}(V) = \beta^2 / [(\alpha - 1)^2(\alpha - 2)]$.

26. Suppose that a random sample is to be taken from a normal distribution with a specified value of the mean and an unknown value of the precision W . Suppose also that the prior distribution of W is a gamma distribution and that the coefficient of variation of the posterior distribution of W must be reduced to the value 0.1. Show that this requirement will be satisfied, regardless of the value of the coefficient of variation of the prior distribution, if a sample of size $n = 200$ is taken.
27. Consider a normal distribution with an unknown value of the mean M and an unknown value of the precision R , and suppose that the prior joint distribution of M and R is as specified in Theorem 1 of Sec. 9.6. Find the conditional distribution of R when $M = m$.
28. Consider the conditions specified in Exercise 27. Suppose that the coefficient of variation of the prior distribution of R has the value 0.5. How large a random sample must be taken from the given normal distribution in order that the coefficient of variation of the posterior distribution of R will be reduced to the value 0.1? (Answer: $n = 192$.)

29. Consider the conditions specified in Exercise 27. Suppose that under the conditional posterior distribution of M when $R = 3$, the variance of M must be reduced to the value 0.01. Show that this requirement will be satisfied, regardless of the values of the parameters of the prior distribution, if a random sample of size $n = 34$ is taken from the given normal distribution.
30. The length of time for which a certain man must wait each morning for a bus taking him to work is uniformly distributed on the interval $(0, W)$, where the value of W is unknown and the prior distribution of W is a Pareto distribution with parameters $w_0 > 0$ and $\alpha = 1$. On how many mornings must the man observe his waiting time before he will be able to specify an interval having a length of 0.01 unit such that the probability that the unknown value of $\log W$ lies in this interval is at least 0.95? (Answer: $n = 299$.)

31. Consider the prior distribution of W specified in Exercise 30. On how many mornings must the man observe his waiting time in order that the coefficient of variation of the posterior distribution of W will be reduced to the value 0.01? (Answer: $n = 101$.)
32. Prove Theorem 2 of Sec. 9.7.
33. Consider a uniform distribution on the interval (W_1, W_2) , where the values of W_1 and W_2 are unknown, and suppose that the prior joint distribution of W_1 and W_2 is a bilateral bivariate Pareto distribution with parameters $r_1 > 0$, $r_2 > 0$, and $\alpha = 2$. How large a random sample must be taken from the uniform distribution in order that the coefficient of variation of the posterior distribution of the random variable $W_2 - W_1$ will be reduced to the value 0.01? (Answer: $n = 140$.)
34. Suppose that a box contains N balls, of which an unknown number W are red and the rest are blue. Suppose also that the prior distribution of W is a hypergeometric distribution with parameters A , B , and N , where A and B are positive integers such that $A + B \geq N$.

- (a) Now suppose that although the exact value of W is unknown, the statistician knows that $r \leq W \leq s$, where r and s are integers such that $0 < r \leq s < N$. Show that there are unique values of A and B such that, under the prior hypergeometric distribution,
- $$\Pr\{r \leq W \leq s\} = 1, \quad \Pr\{W = r\} > 0, \quad \text{and} \quad \Pr\{W = s\} > 0.$$

- (b) Next, suppose that the statistician knows only that $0 \leq W \leq N$. Show that any prior hypergeometric distribution such that $A \geq N$ and $B \geq N$ will assign positive probability to each integer $0, 1, 2, \dots, N$.

CHAP. 9 CONJUGATE PRIOR DISTRIBUTIONS

EXERCISES

188. Suppose that one ball is selected from the box at random. What is the probability, under the prior distribution, that it will be red?

(c) Suppose that n balls ($1 \leq n < N$) are selected at random from the box without replacement and that x of these balls are red. Show that the posterior distribution of the number of red balls among the $N - n$ balls remaining in the box is a hypergeometric distribution with parameters $A = x$, $B = (n - x)$, and $N - n$.

35. Suppose that there are k different types of items in a very large population and let W_i be the unknown proportion of the population that includes items of type i ($i = 1, 2, \dots, k$). Suppose also that the prior distribution of $\mathbf{W} = (W_1, \dots, W_k)'$ is a Dirichlet distribution with parametric vector $\alpha = (\alpha_1, \dots, \alpha_k)$ such that $\alpha_1 + \dots + \alpha_k = 6$. How large a random sample of items must be taken in order to be sure that no matter what the values of the individual components of the vector α are and no matter what the observed outcomes are, the posterior variance of each proportion W_i ($i = 1, \dots, k$) will be at most 0.005? (Answer: $n = 43$.)

36. Consider a bivariate normal distribution with an unknown mean vector $\mathbf{M} = (M_1, M_2)'$ and a precision matrix \mathbf{r} which is known to be

$$\mathbf{r} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Suppose that the prior distribution of \mathbf{M} is a bivariate normal distribution for which the precision matrix is

$$\tau = \begin{bmatrix} 1 & -1 \\ -1 & 6 \end{bmatrix}.$$

How large a random sample must be taken in order that the variance of the posterior distribution of the random variable $M_1 - M_2$ will be reduced to the value 0.01? (Answer: $n = 297$.)

37. Suppose that a random $k \times k$ symmetric, positive definite matrix \mathbf{R} has a Wishart distribution with α degrees of freedom ($\alpha > k - 1$) and precision matrix τ . Show that the coefficient of variation of the determinant $|\mathbf{R}|$ is

$$\sqrt{\frac{k(2\alpha - k + 3)}{(\alpha - k + 2)(\alpha - k + 1)}}.$$

Hint: See Exercise 14 of Chap. 5.

38. Consider a bivariate normal distribution with a specified mean vector and an unknown precision matrix \mathbf{R} . Suppose that the prior distribution of \mathbf{R} is a Wishart distribution with 3 degrees of freedom and precision matrix τ . How large a random sample must be taken in order that the coefficient of variation of the posterior distribution of the determinant $|\mathbf{R}|$ will be reduced to the value 0.1? (Answer: $n = 399$)

39. Consider again the conditions described in Exercise 38, and suppose that the elements of the 2×2 matrix \mathbf{R} are

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

How large a random sample must be taken in order that the coefficient of variation of the posterior distribution of the random variable R_{11} will be reduced to the value 0.1? (Answer: $n = 197$.)

40. Prove that Eq. (3) of Sec. 9.11 is correct.

41. Consider a multivariate normal distribution with an unknown value of the mean vector \mathbf{M} and an unknown value of the precision matrix \mathbf{R} , and suppose that the prior joint distribution of \mathbf{M} and \mathbf{R} is a multivariate normal-Wishart distribution as specified in Theorem 1 of Sec. 9.10. What is the conditional distribution of \mathbf{R} when $\mathbf{M} = \mathbf{m}$?

42. Consider a bivariate normal distribution with an unknown value of the mean vector \mathbf{M} and an unknown value of the precision matrix \mathbf{R} . Suppose that the prior joint distribution of \mathbf{M} and \mathbf{R} is a bivariate normal-Wishart distribution as specified in Theorem 1 of Sec. 9.10, and suppose that $\alpha = 3$ in this prior distribution. How large a random sample must be taken in order that the coefficient of variation of the posterior distribution of the determinant $|\mathbf{R}|$ will be reduced to the value 0.1? The posterior distribution of the determinant $|\mathbf{R}|$ will be the same as the answer in Exercise 38.

43. Consider the conditions specified in Exercise 42, and suppose again that $\alpha = 3$ in the prior joint distribution of \mathbf{M} and \mathbf{R} . Suppose that a random sample is taken, and let \mathbf{u}^* be the location vector and \mathbf{T}^* the precision matrix of the posterior t distribution of the two-dimensional vector \mathbf{M} . Determine the size of a random sample that must be taken in order for the posterior distribution of \mathbf{M} to satisfy the following equation:

$$\Pr[(\mathbf{M} - \mathbf{u}^*)^T \mathbf{T}^* (\mathbf{M} - \mathbf{u}^*) \leq 10] \geq 0.95.$$

Hint: See expression (16) of Sec. 5.6. (Answer: $n = 5$.)

44. Prove Theorem 1 of Sec. 9.13.

45. Prove that if the joint distribution of the random vector \mathbf{M} and the random variable W is a multivariate normal-gamma distribution as specified in Theorem 1 of Sec. 9.13, then the marginal distribution of \mathbf{M} is a multivariate t distribution with 2α degrees of freedom, location vector \mathbf{u} , and precision matrix $(\alpha/\beta)\tau$.

46. Consider five different normal distributions with unknown means M_1, \dots, M_5 and with a common unknown precision W . Suppose that the prior joint distribution of M_1, \dots, M_5 and W is as follows: For any given value $W = w$, the random variables M_1, \dots, M_5 are independent and have the same normal distribution with mean 3 and precision $2w$. Furthermore, the marginal distribution of W is a gamma distribution with parameters $\alpha = 10$ and $\beta = 5$. Suppose also that a random sample of eight observations is taken from each of the five normal distributions. Let x_{ij} denote the value of the j th observation from the i th distribution, and for $i = 1, \dots, 5$, let \bar{x}_i denote the average of the eight observations from the i th distribution. Finally, let the values μ_1, \dots, μ_5 and c be defined as follows:

$$\mu_i = \frac{3 + 4\bar{x}_i}{5}, \quad i = 1, \dots, 5,$$

and

$$c = \frac{2.37}{600} \left[50 + 5 \sum_{i=1}^5 \sum_{j=1}^8 (x_{ij} - \bar{x}_i)^2 + 8 \sum_{i=1}^5 (\bar{x}_i - 3)^2 \right].$$

Show that under the posterior joint distribution of M_1, \dots, M_5 ,

$$\Pr \left[\sum_{i=1}^5 (M_i - \mu_i)^2 \leq c \right] = 0.95.$$

Hint: See expression (16) of Sec. 5.6.

47. Suppose that $\mathbf{X}_1, \dots, \mathbf{X}_n$ is a random sample of symmetric $k \times k$ matrices from a Wishart distribution with m degrees of freedom ($m > k - 1$) and an unknown value of the precision matrix \mathbf{R} . Suppose also that the prior distribution of \mathbf{R} is a Wishart distribution with α degrees of freedom and precision matrix τ such that $\alpha > k - 1$ and τ is a $k \times k$ positive definite matrix. Show that the posterior distribution of \mathbf{R} when $\mathbf{X}_i = \mathbf{x}_i$ ($i = 1, \dots, n$) is a Wishart distribution with $\alpha + mn$ degrees of freedom and precision matrix $\tau + \sum_{i=1}^n \mathbf{x}_i$.

12^a LISTA DE Exercícios

14/09

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Pasta 4

Nº cópias 4

Exercícios extraídos do livro "Optimal Statistical Decisions", de M.H. DeGroot

Exercícios 2, 3 e 4, página 115.

Exercício 7, página 116.

Exercícios 8(e) e 9, página 117.

Exercício 13, página 118.

114 Distributions P_n and Q_n are bounded from below, they are subject to the conclusions of Lemma 3.

Now suppose that $P <^* Q$. By Assumption U_6 , there must exist a positive integer n_0 such that $P_{n_0} <^* Q$. Also, by Lemma 2, $Q \precsim^* Q_{n_0}$. Therefore, by Assumption U_2 and Lemma 2, there exists a number α ($0 < \alpha < 1$) such that for $n = 1, 2, \dots$,

$$P_{n_0} <^* \alpha Q_{n_0} + (1 - \alpha) P_{n_0} \precsim^* Q \precsim^* Q_{n_0}. \quad (15)$$

Hence, by Lemma 3, for $n = 1, 2, \dots$,

$$c_{n_0} < \alpha d_{n_0} + (1 - \alpha) c_{n_0} \leq d_n. \quad (16)$$

It now follows from Lemmas 1 and 2 that

$$E(U|P) \leq c_{n_0} < \alpha d_{n_0} + (1 - \alpha) c_{n_0} \leq E(U|Q). \quad (17)$$

The relation (17) establishes that $E(U|P) < E(U|Q)$.

Conversely, suppose that $E(U|P) < E(U|Q)$. Then there is a positive integer n_0 such that $c_{n_0} < E(U|Q) \leq d_{n_0}$. Hence, there must exist a number α ($0 < \alpha < 1$) such that for $n = 1, 2, \dots$,

$$c_{n_0} < \alpha d_{n_0} + (1 - \alpha) c_{n_0} < E(U|Q) \leq d_n. \quad (18)$$

It follows from Lemma 3 that for $n = 1, 2, \dots$,

$$P_{n_0} <^* \alpha Q_{n_0} + (1 - \alpha) P_{n_0} \precsim^* Q_n. \quad (19)$$

By Assumption U_6 and Lemma 2, we can now obtain the following relation:

$$P \precsim^* P_{n_0} <^* \alpha Q_{n_0} + (1 - \alpha) P_{n_0} \precsim^* Q. \quad (20)$$

Hence, $P <^* Q$. ■

It was established in Theorem 2 of Sec 7.9 that the function U which has been constructed in this chapter and increasing linear transformations of U are the only functions which have the property described in Theorem 1.

Further Remarks and References

It should be remarked that no conditions have been imposed on the distributions in the class \mathcal{P} which do not belong to \mathcal{P}_E . For some distributions $P \in \mathcal{P}$, the value of $E(U|P)$ can be regarded as $+\infty$. For such a distribution P , it is natural to assume that $P \gtrsim^* Q$ for any distribution $Q \in \mathcal{P}_E$, but it is not obvious how two such distributions should be compared with each other. Similar remarks apply to those distributions $P \in \mathcal{P}$ such that the value of $E(U|P)$ can be regarded as $-\infty$. However,

there may still exist distributions $P \in \mathcal{P}$ such that $E(U|P)$ does not exist, either as a finite or as an infinite number. Although it could be assumed that the relation \precsim^* specifies a complete ordering of all distributions in \mathcal{P} , and further assumptions could be introduced which would assign a position in this ordering to each distribution outside of \mathcal{P}_E , it does not appear worthwhile to do so, and this subject will not be explored further. It should be emphasized that when the function U is bounded, then the classes \mathcal{P} , \mathcal{P}_E , and \mathcal{P}_B are identical and the above questions do not arise.

The first axiomatic development of utility was given by Von Neumann and Morgenstern (1947). Other developments are due to Marschak (1950), Herstein and Milnor (1953), and Debreu (1960). Derivations are also given in the texts of Blackwell and Girshick (1954) and Chernoff and Moses (1959). Other interesting discussions and developments are given by Fishburn (1964, 1967a), Luce and Raiffa (1957), Luce and Suppes (1965), Pratt, Raiffa, and Schlaifer (1964), and Savage (1954). An indication of research in the theory of utility which emphasizes various aspects that have not been considered here is given in several of the papers in the collection edited by Thrall, Coombs, and Davis (1954) and in the papers of Aumann (1964), Koopmans, Diamond, and Williamson (1964), and Radner (1964).

Fishburn (1968) has published a bibliography on utility theory containing 315 items.

EXERCISES

1. Consider the possibility of your winning or losing any amount of money up to \$1,000. By using a technique of the type described in Sec. 7.5, determine several different points on your utility function and sketch that function as a curve over the interval from $-1,000$ to $+1,000$.

2. Consider an experiment in which a person first specifies a number of dollars x and then observes the value of a random variable Y . Suppose that if $Y \geq x$, he receives Y dollars as his reward. If $Y < x$, he receives a random reward X having some given probability distribution. It is assumed that the random variables X and Y are independent. Show that in order to maximize the expected utility of his reward, the person should specify a number x such that $U(x) = E[U(X)]$.

3. Suppose that two people, J and K , desire to make a bet. Person J will pay \$1 to person K if a specific event A occurs, and person K will pay x dollars to person J if the event A does not occur. Suppose that both J and K agree that $P(A) = p$ ($0 < p < 1$) and that the utility functions U_J and U_K of both persons are strictly increasing functions of monetary gain. Show that there is an amount x which is mutually agreeable to both J and K if, and only if,

$$U_J^{-1} \left[\frac{U_J(0) - p U_J(-1)}{1 - p} \right] < U_K^{-1} \left[\frac{U_K(0) - p U_K(-1)}{1 - p} \right].$$

Show also that when this relation is satisfied, any value of x which is between these two numbers will result in a mutually satisfactory bet.

4. Let $R = [r_1, r_2, \dots]$ be a countable set of rewards, and let U be a utility function on R . Let P_1, P_2, \dots be a sequence of probability distributions on

For each distribution P_i in this sequence and each reward $r_j \in R$, let $p_{ij} = P_i[r_j]$. Also, let $\alpha_1, \alpha_2, \dots$ be a sequence of numbers such that $\alpha_i \geq 0$ for $i = 1, 2, \dots$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, and let

$$p_j = \sum_{i=1}^{\infty} \alpha_i p_{ij} \quad j = 1, 2, \dots$$

Finally, let \tilde{P} denote the probability distribution on R such that $p_j = \tilde{P}\{\text{receiving } r_j\}$ for $j = 1, 2, \dots$

Verify that \tilde{P} is a probability distribution. Also, show that if U is a bounded function, then the distribution \tilde{P} cannot be preferred to each of the distributions P_i ($i = 1, 2, \dots$). Give a simple example to show that if U is not bounded, then it might be true that $E(U|P_i)$ is finite for $i = 1, 2, \dots$ but $E(U|P)$ is infinite.

(5) For any real numbers r_1 and r_2 and for any probability p ($0 \leq p \leq 1$), let $(r_1, p; r_2, 1 - p)$ denote a lottery which yields a reward of r_1 dollars with probability p and a reward of r_2 dollars with probability $1 - p$. Without referring to your utility function as sketched in Exercise 1, decide which lottery you prefer in each of the following pairs:

- (a) $(250, \frac{1}{2}; 0, \frac{1}{2})$ or $(40, \frac{1}{2}; 70, \frac{1}{2})$;
- (b) $(400, \frac{1}{2}; -100, \frac{1}{2})$ or $(150, \frac{2}{3}; 0, \frac{1}{3})$;
- (c) $(1,000, \frac{1}{2}; -1,000, \frac{1}{2})$ or $(50, \frac{1}{2}; -50, \frac{1}{2})$.

Now find which lottery would be preferred in parts a to c, as determined by the sketch of your utility function made in Exercise 1. If these preferences do not agree with the decisions you have just made, revise your sketch.

6. Suppose that g is a real-valued function over some interval of the real line and that g can be differentiated twice throughout the interval. Prove that g is concave if, and only if, $g''(x) \leq 0$ at every point x in the interval.

(7) Consider two boxes each of which contains both red balls and green balls. It is known that one-half the balls in box 1 are red and the other half are green. In box 2, the proportion X of red balls is not known with certainty, but this proportion has a probability distribution over the interval $0 \leq X \leq 1$.

(a) Suppose that a person is to select one ball at random from either box 1 or box 2. If that ball is red, he wins \$1; if it is green, he wins nothing. Show that under any utility function which is an increasing function of monetary gain, the person should prefer selecting the ball from box 1 if, and only if, $E(X) < \frac{1}{2}$.

(b) Suppose that a person can select n balls ($n \geq 2$) at random from either of the boxes but that all n balls must be selected from the same box; suppose that each selected ball will be put back in the box before the next ball is selected; and suppose that he will receive \$1 for each red ball selected and nothing for each green ball. Also, suppose that his utility function U of monetary gain is strictly concave over the interval $[0, n]$, and suppose that $E(X) = \frac{1}{2}$. Show that the person should prefer to select the balls from box 1.

Hint: Show that if the balls are selected from box 2, then for any given value $X = x$, $E(U|x)$ is a concave function of x on the interval $0 \leq x \leq 1$. This can be done by showing that

$$\frac{d^2}{dx^2} E(U|x) \stackrel{?}{=} n(n-1) \sum_{i=0}^{n-2} [U(i) - 2U(i+1) + U(i+2)] \binom{n-2}{i} x^i (1-x)^{n-2-i} < 0.$$

Then apply Jensen's inequality to $E(U|X)$.

(c) Consider again the problem in part (b), with the same utility function and the same distribution of X . Suppose now, however, that instead of receiving \$1 for each red ball and nothing for each green ball, the person will receive \$1 for each green ball and nothing for each red ball. Show that he should still prefer to select the balls from box 1.

8. Let p and m be fixed numbers such that $0 < p < 1$ and $m > 0$, and consider the following situation: A person is given a stake of m dollars which he can allocate between an event A of probability p and its complement A^c . He keeps as his reward the amount which he has allocated to either A or A^c , whichever actually occurs. In other words, for all values of x in the interval $0 \leq x \leq m$, he can choose among all lotteries of the form $(x, p; m - x, 1 - p)$, as defined in Exercise 5. Being careful in each case to consider every possible pair of values of p and m , find the preferred allocation of the m dollars when the person's utility function U is defined on the interval $[0, m]$ of monetary gains as follows:

- (a) $U(r) = r^\alpha$, where $\alpha < 1$.
- (b) $U(r) = r$.
- (c) $U(r) = r^\alpha$, where $0 < \alpha < 1$.
- (d) $U(r) = \log r$.
- (e) $U(r) = 2(r/m_0) - (r/m_0)^2$, where $m_0 \geq m$.

9. Consider a roulette wheel which is partitioned into k disjoint events A_1, \dots, A_k , such that $P(A_i) = p_i$, for $i = 1, \dots, k$, and $\sum_{i=1}^k p_i = 1$. Suppose that a person is given a stake of m dollars which he can allocate among the k events A_1, \dots, A_k and that he receives as his reward the amount x_i which he has allocated to the event A_i that actually occurs. In other words, he can choose among all lottery tickets which yield a reward x_i with probability p_i ($i = 1, \dots, k$), subject to the restriction that $x_i \geq 0$ for $i = 1, \dots, k$ and $\sum_{i=1}^k x_i = m$. Suppose also that the person's utility function U , as a function of positive monetary rewards r , is defined as $U(r) = \log r$. Show that his preferred allocation is given by $x_i = mp_i$ for $i = 1, \dots, k$. (Note that this exercise is a generalization of Exercise 8d.)

10. Suppose that a person's current fortune is x_0 dollars and that he has an opportunity to purchase a lottery ticket which yields either a reward of r dollars ($r > 0$) or a reward of 0 dollars with equal probabilities. Suppose also that for any value of x ($x > 0$), his utility $U(x) = \log x$. Show that he should be willing to purchase the lottery ticket for any amount b such that

$$b < x_0 + \frac{1}{2}[r - (r^4 + 4x_0^2)\frac{1}{4}]$$

11. Suppose that Mr. A and Mr. B are equally wealthy, each having a current fortune of x_0 dollars. Suppose also that both men have the same utility function U and that for any number x , their utility of having a fortune of x dollars is $U(x) = (x - x_0)^\frac{1}{2}$.

Suppose now that one of the two men receives, as a gift, a lottery ticket which yields either a reward of r dollars ($r > 0$) or a reward of 0 dollars with equal probabilities. Show that there exists a number $b > 0$ having the following property: Regardless of which man receives the lottery ticket, he can sell it to the other man for b dollars and the sale will be advantageous to both men.

12. Let R be an abstract set of rewards on which a utility function U is defined which has the property stated in Theorem 1 of Sec. 7.10. For any reward $r \in R$, let the set $[r, \infty)$ be defined by Eq. (1) of Sec. 7.10. Suppose that P and Q

are distributions in the class Ω_F such that for every reward $r \in R$,

$$P\{[r, \infty)\} \leq Q\{[r, \infty)\}.$$

Show that $P \precsim^* Q$. This exercise can be rephrased as follows: Let X and Y be random variables with d.f.'s F and G , respectively. Suppose that both $E(X)$ and $E(Y)$ exist, and suppose that $F(x) \leq G(x)$ for every number x . Show that $E(X) \geq E(Y)$. [See Lehmann (1959), p. 73.]

13. Suppose that a person is going to sell Fizzy Cola at a football game and must decide in advance how much to order. Suppose that he makes a gain of m cents on each quart that he sells at the game but suffers a loss of c cents on each quart that he has ordered but does not sell. If it is assumed that the demand for Fizzy Cola at the game, as measured in quarts, is an absolutely continuous random variable X with p.d.f. f and d.f. F , show that his expected profit will be maximized if he orders an amount a such that $F(a) = m/(m + c)$.

statistical decision problems

LISTA DE EXERCÍCIOS 3

MAE 5753 - 2º semestre / 2015

14/00

Pasta 4

Nº cópias 5

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Exercício 27, página 154

15 exercícios

the posterior g.p.d.f. $\xi(\cdot | x)$ of W , before Y is observed, as follows:

$$\xi(w|x) = \frac{g(x|w)\xi(w)}{\int_{\Omega} g(x|w')\xi(w') d\nu(w')}.$$

Furthermore, the conditional g.p.d.f. $h(\cdot | w, x)$ of Y when $W = w$ and $X = x$ is

$$h(y|w, x) = \frac{f(x, y|w)}{g(x|w)}. \quad (3)$$

Hence, for the second stage of the experiment, which involves the observation of Y , the g.p.d.f. given by Eq. (2) can be regarded as the prior g.p.d.f. of W , and the conditional g.p.d.f.'s given by Eq. (3) for each value $w \in \Omega$ form the appropriate family of distributions of Y . The posterior g.p.d.f. $\xi(\cdot | x, y)$ of W when $Y = y$ can now be computed as follows:

$$\xi(w|x, y) = \frac{h(y|w, x)\xi(w|x)}{\int_{\Omega} h(y|w', x)\xi(w'|x) d\nu(w')}.$$

If the g.p.d.f.'s given by Eqs. (2) and (3) are substituted in Eq. (4), the result is Eq. (1). This means that if the observations are made in more than one stage, then the posterior distribution can be computed in different stages by letting the posterior distribution after each stage serve as the prior distribution for the next stage. It also follows from this derivation that if the posterior distribution of W when $X = x$ and $Y = y$ is computed in two stages, the final result is the same regardless of whether X or Y is observed first.

The decision-making process can now be described in the following highly simplified, but helpful, way. At any given time, the statistician has a probability distribution for the parameter W . As time progresses, the statistician gains information about W from various sources and uses this information to revise his distribution for W . From time to time, when he must choose a decision whose consequences are related to W , he will select a decision that is optimal against his current distribution for W .

Some aspects of this description of decision making are realistic. As we live our lives, we revise our beliefs about various parameters as we learn about them, and when a decision must be chosen, we obviously make the choice on the basis of our current beliefs. However, in some situations, the choice of a decision at a certain time may affect the information that will become available and, hence, may affect the decisions that the statistician can choose at future times. Problems in which the statistician must consider the future and plan ahead appropriately are called *sequential decision problems*. Such problems will be treated in Chaps. 12 to 14.

EXERCISES

1. Consider a decision problem in which the parameter space $\Omega = \{w_1, w_2, w_3, w_4\}$, the decision space $D = \{d_1, d_2, d_3\}$, and the loss function L is specified by the accompanying table. Suppose that the p.f. ξ of the parameter W is such that $\xi(w_1) = \frac{1}{6}$, $\xi(w_2) = \frac{3}{8}$, $\xi(w_3) = \frac{1}{4}$, and $\xi(w_4) = \frac{1}{4}$. Show that decision d_3 is the Bayes decision against ξ .

Table for Exercise 1

	d_1	d_2	d_3
w_1	0	2	3
w_2	1	0	2
w_3	3	4	0
w_4	1	2	0

Table for Exercise 2

	d_1	d_2	d_3
w_1	0	10	4
w_2	8	0	3

2. Consider a decision problem in which the parameter space $\Omega = \{w_1, w_2\}$, the decision space $D = \{d_1, d_2, d_3\}$, and the loss function L is specified by the accompanying table. Prove that decision d_3 is a Bayes decision against a given distribution of the parameter W if, and only if, $\frac{1}{3} \leq \Pr(W = w_1) \leq \frac{6}{7}$.

$$L(w, d) = 100(w - d)^2.$$

- Suppose that the p.d.f. ξ of the parameter W is specified by the equation

$$\xi(w) = 2w \quad 0 \leq w \leq 1.$$

Show that the value $d = \frac{2}{3}$ is the Bayes decision against ξ and that the Bayes risk is $\frac{50}{7}$.

3. Consider the decision problem described in Exercise 1, but suppose that the loss function L specified in that exercise is replaced by the new loss function L_0 specified by the table on page 150. Prove that against any distribution of the parameter W , the Bayes decisions when the loss function is L_0 are the same as the Bayes decisions when the loss function is L .

$$L_0(w, d) = 100(w - d)^2.$$

ble decision in the set D but that d^* is not a Bayes decision against any distribution of the parameter W .

Table for Exercise 4

	d_1	d_2	d_3
w_1	4	6	7
w_2	0	-1	1
w_3	0	1	-3
w_4	1	2	0

✓ 5. Suppose that in an arbitrary decision problem, a certain decision d^* is a Bayes decision against two distinct distributions P_1 and P_2 of the parameter W . For any number α such that $0 < \alpha < 1$, show that d^* must also be a Bayes decision against the distribution $\alpha P_1 + (1 - \alpha)P_2$.

✓ 6. Consider the decision problem presented in Exercise 3. Suppose that statistician A believes that the p.d.f. of W is ξ , as specified in Exercise 3, but that statistician B believes that the p.d.f. of W is ξ_B , as specified by the equation $\xi_B(w) = 3w^2$.

How much additional risk does B think that A will incur because of A's incorrect belief about the distribution of W ?

7. Consider a decision problem in which $\Omega = \{w_1, w_2\}$, $D = \{d_1, \dots, d_5\}$, and the loss function L is given by the accompanying table. Against what distributions of the parameter W is the Bayes decision not unique?

Table for Exercise 7

	d_1	d_2	d_3	d_4	d_5
w_1	0	4	2	1	5
w_2	4	5	0	1	0

8. Consider a decision problem in which both $\Omega = \{w_1, w_2\}$, $D = \{d_1, \dots, d_7\}$, and the loss function L is given by the accompanying table. For each distribution ξ of the parameter W , find all Bayes decisions against ξ .

Table for Exercise 8

	d_1	d_2	d_3	d_4	d_5	d_6	d_7
w_1	1	6	0	2	7	3	4
w_2	10	1	13	8	0	5	4

✓ 9. Consider a decision problem in which both Ω and D have an infinite number of elements. Suppose that $\Omega = \{w_1, w_2, \dots\}$, $D = \{d^*, d_1, d_2, \dots\}$, and the loss function L is given by the accompanying table. Prove that d^* is the only admissible decision.

Table for Exercise 9

	d^*	d_1	d_2	d_3	d_4	\dots
w_1	$\frac{1}{2}$	0	0	0	0	\dots
w_2	$\frac{1}{2}$	1	0	0	0	\dots
w_3	$\frac{1}{3}$	1	1	0	0	\dots
w_4	$\frac{1}{3}$	1	1	1	0	\dots
w_5	$\frac{1}{3}$	1	1	1	1	\dots
\dots	\dots	\dots	\dots	\dots	\dots	\dots

10. Suppose that there is probability $\frac{1}{10}$ that a signal is present in a certain system at any given time and probability $\frac{9}{10}$ that no signal is present in the system. Suppose that a measurement made on the system when a signal is present is normally distributed with mean 50 and precision 1 and a measurement made on the system distributed with mean 52 and precision 1. Suppose that a measurement made on the system at a certain time has the value x . Show that the posterior probability that a signal is greater than the posterior probability that no signal is present if, and only if, $x < 51 - \frac{1}{2} \log 9$.

✓ 11. Consider a decision problem in which $\Omega = \{w_1, w_2\}$, $D = \{d_1, d_2\}$, and the loss function L is given by Table 8.2. Suppose that the statistician can observe a random variable Z whose conditional distribution when $W = w_1$ is normal with mean 0 and variance 1 and whose conditional distribution when $W = w_2$ is normal with mean 1 and variance 1. For any given prior probability $\xi = \Pr(W = w_1)$, let δ be the decision function such that $\delta(z) = d_1$ if

$$z \leq \frac{1}{2} + \log \frac{\xi}{2(1 - \xi)}$$

and $\delta(z) = d_2$ otherwise. (a) Show that δ is a Bayes decision function against ξ .

(b) Sketch the Bayes risk $\rho^*(\xi)$ as a function of ξ over the interval $0 \leq \xi \leq 1$.

12. Suppose that a statistician must decide whether a certain random variable has a uniform distribution on the interval $(0, 1)$ or a uniform distribution on the interval $(0, \frac{1}{2})$. Each of these two distributions has probability $\frac{1}{2}$ of being the correct distribution. Suppose that the loss is 0 if a correct decision is made and is $a > 0$ if an incorrect decision is made. Suppose that the statistician can select the number of observations which will be made on the random variable but that the cost of each observation is $c > 0$. Show that the statistician should make n^* observations, where n^* is the nonnegative integer which minimizes the value $a2^{-(n+1)} + nc$.

13. Consider a decision problem in which $\Omega = \{w_1, w_2\}$, $D = \{d_1, d_2, d_3\}$, and the loss function L is given by the table on page 152. Suppose that an observation X is available with the following conditional distributions:

$$\begin{aligned} \Pr(X = 1|W = w_1) &= \frac{3}{4}, \\ \Pr(X = 1|W = w_2) &= \frac{1}{4}, \\ \Pr(X = 0|W = w_1) &= \frac{1}{4}, \\ \Pr(X = 0|W = w_2) &= \frac{3}{4}. \end{aligned}$$

Suppose that $\xi = \Pr(W = w_1)$. Find a Bayes decision function against each value of ξ ($0 \leq \xi \leq 1$), and sketch the Bayes risk $\rho^*(\xi)$ as a function of ξ .

Table for Exercise 13

	d_1	d_2	d_3
w_1	0	10	3
w_2	10	0	3

14. Suppose that for the conditions in Exercise 13, before the statistician chooses a decision, he can observe the values of the random variables X_1, \dots, X_n which, for any given value $W = w_i$ ($i = 1, 2$), are a random sample from the conditional distribution of X . For any given sample size n , find a Bayes decision function against each value of ξ .
15. Suppose that for the conditions in Exercise 14, $\xi = \frac{1}{3}$ and the cost c of each observation is $\frac{1}{10}$. Show that the optimal sample size n is 8 and that the minimum total risk which can be attained is 1.33.

16. Suppose that for the conditions in Exercises 13 and 14, the cost of each observation is not constant but that any observation whose value is 1 costs 0.15 and any observation whose value is 0 costs 0.05. Show that when $\xi = \frac{1}{2}$, the optimal sample size and minimum total risk will be precisely the same as those in Exercise 15.
17. Consider a decision problem in which $\Omega = \{w_1, w_2\}$, $D = \{d_1, d_2\}$, and the loss function L is given by the accompanying table. Suppose that $\Pr(W = w_1) = \Pr(W = w_2) = \frac{1}{2}$. Suppose also that the conditional distribution of an observation X when $W = w_1$ is normal with mean -1 and variance 9 and that the conditional distribution of X when $W = w_2$ is normal with mean 1 and variance 9. Suppose, further, that before the statistician chooses a decision, he can observe values of the random variables X_1, \dots, X_n which, for any given value of W , are a random sample from the conditional distribution of X just given. If each observation in the sample costs 1 unit, show that the optimal number of observations n is 42 and that the minimum total risk is 57.4.

18. Consider k random variables X_1, \dots, X_k . Suppose that the p.d.f. of exactly one of these random variables is g and the p.d.f. of each of the other $k - 1$ random variables is h but that it is not known which one of the random variables has p.d.f. g . For $i = 1, \dots, k$, let ξ_i be the prior probability that X_i is the random variable whose p.d.f. is g . Here $\xi_i > 0$ for $i = 1, \dots, k$ and $\sum_{i=1}^k \xi_i = 1$. (a) Suppose that the random variable X_1 is observed and found to have the value x . Find the posterior probability that the p.d.f. of X_1 is g . (b) Suppose that the random variable X_1 is observed and found to have the value x . Find the posterior probability that the p.d.f. of X_1 is g .
19. Consider two boxes A and B each of which contains both red balls and

green balls. It is known that in one of the boxes, $\frac{1}{2}$ of the balls are red and $\frac{1}{2}$ of the balls are green and that in the other box, $\frac{1}{4}$ of the balls are red and $\frac{3}{4}$ of the balls are green. Let the box in which $\frac{1}{2}$ of the balls are red be denoted as box W , and suppose that it is not known with certainty whether $W = A$ or $W = B$. Assume that $\Pr(W = A) = \xi$ and $\Pr(W = B) = 1 - \xi$, where ξ is a given number such that $0 < \xi < 1$.

Suppose that the statistician may select one ball at random from either box A or box B and that after observing its color, he must decide whether $W = A$ or $W = B$. Prove that if $\frac{1}{2} < \xi < \frac{2}{3}$, then in order to maximize the probability of making a correct decision, he should select the ball from box B . Prove also that if $\frac{2}{3} \leq \xi \leq 1$, then it does not matter from which box the ball is selected.

20. Consider a decision problem in which $\Omega = \{w_1, w_2\}$, $D = \{d_1, d_2\}$, and the loss function L is given by Table 8.4. Suppose that the statistician can observe either a random variable X or a random variable Y whose conditional distributions are as follows:

$$\begin{aligned} \Pr(X = 1|W = w_1) &= \frac{2}{3}, & \Pr(X = 0|W = w_1) &= \frac{1}{3}, \\ \Pr(X = 1|W = w_2) &= \frac{1}{2}, & \Pr(X = 0|W = w_2) &= \frac{1}{2}, \\ \text{and} \\ \Pr(Y = 1|W = w_1) &= \frac{3}{4}, & \Pr(Y = 0|W = w_1) &= \frac{1}{4}, \\ \Pr(Y = 1|W = w_2) &= \frac{1}{2}, & \Pr(Y = 0|W = w_2) &= \frac{1}{2}. \end{aligned}$$

Suppose also that the cost of observing X is the same as the cost of observing Y . Show that for any prior distribution of W and any values of the losses a_1 and a_2 , the statistician should observe Y rather than X .

21. Let W be a parameter which takes the values w_1 and w_2 with prior probabilities specified by the equations $\Pr(W = w_1) = \xi$ and $\Pr(W = w_2) = 1 - \xi$. Suppose that an observation X is to be taken with conditional g.p.d.f.'s $f(\cdot|w_i)$ for $i = 1, 2$. Let $\xi(x)$ denote the posterior probability that $W = w_1$ when $X = x$. Prove that $E[\xi(X)] = \xi$, where the expectation is computed by assuming that W has the specified prior distribution.

22. For the conditions specified in Exercise 21, if it is assumed that $W = w_1$, prove that $E[\xi(X)] \geq \xi$. Note: This exercise can be interpreted as stating that, on the average, the posterior distribution will assign greater probability to the correct value of W than the prior distribution assigned to that value.

23. For the conditions specified in Exercise 21, suppose that the prior distribution of W is such that $\xi = 1 - \xi = \frac{1}{2}$. If it is assumed that $W = w_1$, prove that for any number ϵ ($0 < \epsilon < 1$),

$$\Pr[X] \leq \epsilon \leq \frac{\epsilon}{1 - \epsilon}.$$

Note: This exercise can be interpreted as stating that there is only a small probability that the posterior distribution will assign a small probability to the correct value of W .

24. Consider a set $\Omega = [w_1, w_2, w_3]$ containing three points, and let \mathcal{C} be the set of all probability distributions (p_1, p_2, p_3) such that $p_i \geq 0$ ($i = 1, 2, 3$) and $p_1 + p_2 + p_3 = 1$. Let \mathcal{C} be the set of points either inside or on the boundary of an equilateral triangle which has a unit height. Also, let v_1, v_2 , and v_3 denote the vertices of that triangle, and for $i = 1, 2, 3$, let S_i denote the side of the triangle that is opposite the vertex v_i . Show that the sum of the distances from any point in the triangle to the three sides of the triangle is 1. Then show that there is a one-to-one correspondence between the sets \mathcal{C} and \mathcal{C}_e , in which any point $(p_1, p_2, p_3) \in \mathcal{C}$ corresponds to the point $\mathbf{x} \in \mathcal{C}$ whose distance from side S_i is p_i ($i = 1, 2, 3$).

25. Suppose that in Exercise 24, corresponding to any fixed point $(p_1, p_2, p_3) \in \Omega$, a mass p_i is placed at the vertex v_i ($i = 1, 2, 3$). Find the center of gravity \mathbf{x} of this system of three mass points, and show that the position of \mathbf{x} defines a one-to-one correspondence between Ω and \mathcal{C} . [For any point $\mathbf{x} \in \mathcal{C}$, the corresponding values (p_1, p_2, p_3) are called the *barycentric coordinates* of \mathbf{x} .]

26. Show that the correspondence between Ω and \mathcal{C} defined in Exercise 25 is the same as the correspondence defined in Exercise 24.

27. Suppose that each of k statisticians has his own prior distribution for a certain parameter W , and let ξ_i be the g.p.d.f. which statistician i assigns to W from $(i = 1, \dots, k)$. Suppose also that an executive forms his opinion about W from the opinions of the k statisticians and that he assigns to W the g.p.d.f. ξ^* defined, at each point $w \in \Omega$, as follows:

$$\xi^*(w) = \alpha_1 \xi_1(w) + \dots + \alpha_k \xi_k(w).$$

Here $\alpha_1, \dots, \alpha_k$ are weights such that $\alpha_i \geq 0$ ($i = 1, \dots, k$) and $\alpha_1 + \dots + \alpha_k = 1$. The value of α_i reflects the relative weight that the executive gives to the opinion of statistician i . Suppose further that the k statisticians and the executive observe together the value of a random variable X whose conditional g.p.d.f. when $W = w$ is $f(\cdot | w)$. Show that the posterior g.p.d.f. of the executive will again be a linear combination of the posterior g.p.d.f.'s of the k statisticians, with new weights β_1, \dots, β_k which will depend on the observed value of X . Also, discuss the conditions under which the weight β_1 in the posterior g.p.d.f. will be greater than the weight α_1 in the prior g.p.d.f.

CHAPTER 9 *conjugate prior distributions*

9.1 SUFFICIENT STATISTICS

Consider a statistical problem in which a large amount of experimental data has been collected. The treatment of the data is often simplified if the statistician computes a few numerical values, or statistics, and considers these values as summaries of the relevant information in the data. In some problems, a statistical analysis that is based on these few summary values can be just as effective as any analysis that could be based on all the observed values. In this chapter we shall consider problems for which fully informative summaries of this type are available. Such summaries are known as *sufficient statistics*.

Suppose that W is a parameter which takes values in the space Ω . Also, suppose that X is a random variable, or random vector, which takes values in the sample space S . We shall let $f(\cdot | w)$ denote the conditional g.p.d.f. of X when $W = w$ ($w \in \Omega$). It is assumed that the observed value of X will be available for making inferences and decisions relating to the parameter W . In this context, any function T of the observation X , whether or not T is a real-valued function, is called a *statistic*.

Loosely speaking, a statistic T is called a sufficient statistic if, for any prior distribution of W , its posterior distribution depends on the

Exercícios do livro "Optimal Statistical Decisions", de M. H. DeGroot.

Exercícios 1, 4, 6, 7, 8, 9, 10, 12, 13 e 19, páginas 260 a 263.

EXERCÍCIOS COMPLEMENTARES

Exercícios do livro "Theory of Statistics", de M. J. Schervish

Exercícios 4 e 5 (página 286), 27 e 31 (página 342) e 40 (página 343).

(1) $\Theta = \{\theta_1, \theta_2, \dots\} = \{\theta_n\}_{n \geq 1}$. $f = P(\Theta)$. $P: f \rightarrow [0,1]$: priori para Θ .

X : espaço amostral. $X|\theta$ com distribuição $P(\cdot|\theta)$.

(a) Defina região HPD neste caso.

(b) Considere o problema (Δ, Ω, L', P') , onde $\Delta = \{f: X \rightarrow P(\Theta)\}$, $\Omega = \Theta \times X$,

P' como acima e $L': \Delta \times \Omega \rightarrow \mathbb{R}$ dado por

$$L'(\delta, (\theta, x)) = |\delta(x)| + k \cdot \sum_{\substack{\epsilon \\ \delta(x)}} I_A(\theta), \text{ onde } |A| \text{ é o número de elementos de } A, A \subseteq \Theta$$

Mostre que a decisão de Bayes associa a cada $x \in X$ uma região HPD da posterior de θ dado x :

(2) $\Theta = [0, 1]$

$f(\theta) = 4|\theta - \frac{1}{2}| \cdot I_{[0,1]}(\theta)$. $X|\theta \sim \text{Ber}(\theta)$. Para $X=1$, avaliar HPD em função de α .

(3) $\Theta = \mathbb{R}$

$$f(\theta) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-(\theta-2)^2}{2}}. X|\theta \sim N(\theta, \frac{1}{2}).$$

Obter posterior de θ dado $X=x$.

Avaliar HPD em função de $x \in \mathbb{R}$ e $\alpha \in (0,1)$.

EXERCISES

1. Suppose that X_1, \dots, X_n is a random sample from a Poisson distribution for which the value of the mean W is unknown. Suppose also that the prior distribution of W is a gamma distribution with parameters α and β ($\alpha > 1, \beta > 0$). Finally, suppose that it is desired to estimate the value of W and that the loss function L is specified by the equation of Sec. 11.2. Prove that the Bayes estimator of W is specified by Eq. (14) of Sec. 11.2, and prove that the Bayes risk is $1/(\beta + n)$.
2. Suppose that X_1, \dots, X_n is a random sample from a Bernoulli distribution for which the parameter W is unknown, and suppose that the prior distribution of W is a beta distribution with parameters α and β . If the value of W must be estimated when the loss L is specified by the equation $L(w, d) = (w - d)^2$ and if the sampling cost is c per observation, show that the optimal number of observations n is specified by the following equation:

$$n = \left[\frac{\alpha\beta}{c(\alpha + \beta)(\alpha + \beta + 1)} \right]^{\frac{1}{2}} - (\alpha + \beta).$$

3. Suppose that X_1, \dots, X_n is a random sample from a normal distribution with an unknown value of the mean W and a specified value of the precision r . Suppose also that the prior distribution of W is a normal distribution with mean μ and precision r . If the value of W must be estimated when the loss L is specified by the equation $L(w, d) = (w - d)^2$ and if the sampling cost is c per observation, show that the optimal number of observations n is specified by the following equation:

$$n = \left(\frac{1}{cr} \right)^{\frac{1}{2}} - \frac{\tau}{r}.$$

4. Suppose that X_1, \dots, X_n is a random sample from a uniform distribution on the interval $(0, W)$, where the value of W is unknown and the prior distribution of W is a Pareto distribution with parameters w_0 and α ($\alpha > 2$). If the value of W must be estimated when the loss L is specified by the equation $L(w, d) = (w - d)^2$ and if the sampling cost is c per observation, show that the optimal number of observations n is specified by the following equation:

$$n = \left[\frac{2\alpha w_0^2}{c(\alpha - 2)} \right]^{\frac{1}{2}} - (\alpha - 1).$$

5. Let W be a random variable whose d.f. is G , and suppose that $E(|W|^\alpha) < \infty$, where $\alpha > 1$. Prove that the expectation $E(|W - d|^\alpha)$ is minimized when d is the unique number such that

$$\int_{w < d} (d - w)^{\alpha-1} dG(w) = \int_{w > d} (w - d)^{\alpha-1} dG(w).$$

6. Let W be a random variable with g.p.d.f. ξ , and suppose that ξ is symmetric with respect to the value y , so that $\xi(y + y) = \xi(y - y)$ for all values of y ($-\infty < y < \infty$). Suppose that Δ is a nonnegative convex function on the real line that is symmetric with respect to the value 0, and suppose also that for all values of d ($-\infty < d < \infty$),

$$\frac{n}{R(d)} = \int_{-\infty}^{\infty} \Delta(w - d)\xi(w) d\nu(w) < \infty.$$

Prove that R is a convex function which is also symmetric with respect to the value v and that $R(d)$ is minimized at the value $d = v$.

7. Suppose that W is a parameter whose value must be estimated when the loss function L is specified by the following equation:

$$L(w, d) = \left(\frac{w - d}{d} \right)^2.$$

It is assumed that $E(W) < \infty$. If $E(W) \neq 0$, show that a Bayes estimate d is specified by the equation

$$d = \frac{E(W^2)}{E(W)}.$$

If $E(W) = 0$ and $E(W^2) > 0$, show that there is no Bayes estimate. Finally, show that in either case, the Bayes risk ρ^* is

$$\rho^* = \frac{\text{Var}(W)}{E(W^2)}.$$

8. Suppose that W is a parameter whose value must be estimated when the loss function L is specified as follows:

$$L(w, d) = \begin{cases} k_1(w - d) & \text{for } d \leq w, \\ k_2(d - w) & \text{for } d \geq w. \end{cases}$$

Here k_1 and k_2 are positive constants and it is assumed that $E(|W|) < \infty$. Show that a number d will be a Bayes estimate if, and only if, the following relations are satisfied:

$$\Pr(W \leq d) \geq \frac{k_1}{k_1 + k_2} \quad \text{and} \quad \Pr(W \geq d) \geq \frac{k_2}{k_1 + k_2}.$$

9. Let X_1, \dots, X_n be a random sample from a Bernoulli distribution for which the parameter W is unknown. Suppose that the value of W must be estimated when the loss function L is

$$L(w, d) = \frac{(w - d)^2}{w(1 - w)}.$$

Suppose also that the prior distribution of W is the uniform distribution on the interval $(0, 1)$. Show that the Bayes estimator $\delta^*(X_1, \dots, X_n)$ is specified by the equation $\delta^*(X_1, \dots, X_n) = \bar{X}$ and show that the Bayes risk is $1/n$.

10. Let X_1, \dots, X_n be a random sample from an exponential distribution for which the parameter W is unknown. Suppose that the value of W must be estimated when the loss function L is

$$L(w, d) = \left(\frac{w - d}{w(1 - w)} \right)^2.$$

Suppose also that the prior distribution of W is a gamma distribution with parameters α and β such that $\alpha > 2$. (a) If the number of observations n is fixed, find a Bayes decision function and compute the Bayes risk. (b) If the cost per observation is c , show that the optimal number of observations n is specified by the following equation:

$$n = \frac{\beta}{[\bar{c}(\alpha - 1)(\alpha - 2)]^{\frac{1}{2}}} - (\alpha - 1).$$

11. Suppose that $\mathbf{W} = (W_1, \dots, W_k)'$ is a random vector for which the mean vector is $\boldsymbol{\psi}$ and the covariance matrix is $\boldsymbol{\Sigma}$. Show that for any fixed $k \times k$

EXERCISES

Theorem 1 of Sec. 9.6. If μ' and β' are as defined in that theorem, show that generalized maximum likelihood estimators $\hat{m}(X_1, \dots, X_n)$ and $\hat{\beta}(X_1, \dots, X_n)$ are specified by the equations $\hat{m}(X_1, \dots, X_n) = \mu'$ and

$$\hat{\beta}(X_1, \dots, X_n) = \frac{\alpha + (n-1)/2}{\beta'}.$$

12. Suppose that the random vector $\mathbf{X} = (X_1, \dots, X_k)'$ has a multinomial distribution for which the parameters are n and $\mathbf{W} = (W_1, \dots, W_k)'$. Suppose that the value of \mathbf{W} must be estimated when the loss function L is
- $$L(\mathbf{w}, \mathbf{d}) = \sum_{i=1}^k (w_i - d_i)^2.$$
- Finally, suppose that the prior distribution of \mathbf{W} is a Dirichlet distribution for which the parametric vector is $\alpha = (\alpha_1, \dots, \alpha_k)'$. Find the Bayes estimator, and show that the value of the Bayes risk is
- $$\frac{\alpha_0 \alpha_0^2}{\alpha_0(\alpha_0 + 1)(\alpha_0 + n)} - \sum_{i=1}^k \alpha_i \alpha_i^2.$$

where $\alpha_0 = \sum_{i=1}^k \alpha_i$. Let X_1, \dots, X_n be a random sample from a normal distribution for which both the mean M and the precision R are unknown. Suppose that the value of M must be estimated and that when $M = m$, $R = r$, and the value of the estimate is d , the loss is

$$L(m, r, d) = (m - d)^2.$$

Suppose also that the prior joint distribution of M and R is a normal-gamma distribution as specified in Theorem 1 of Sec. 9.6, and assume that $\alpha > 1$ in this prior distribution. For any fixed number of observations n , find a Bayes decision function and show that the Bayes risk is $\beta/[(\alpha - 1)(r + n)]$.

14. Suppose that X_1, \dots, X_n is a random sample from a Bernoulli distribution for which the parameter W is unknown, and suppose also that the prior distribution of W is a beta distribution with parameters α and β . Show that when

$$1 - \alpha < \sum_{i=1}^n X_i < n + \beta - 1,$$

a generalized maximum likelihood estimator $\hat{w}(X_1, \dots, X_n)$ of W is specified by the equation

$$\hat{w}(X_1, \dots, X_n) = \frac{\sum_{i=1}^n X_i + \alpha - 1}{n + \alpha + \beta - 2}.$$

15. Suppose that X_1, \dots, X_n is a random sample from a Poisson distribution for which the mean W is unknown, and suppose also that the prior distribution of W is a gamma distribution with parameters α and β . Show that when $\alpha + \sum_{i=1}^n X_i > 1$, a generalized maximum likelihood estimator $\hat{w}(X_1, \dots, X_n)$ of W is specified by the equation

$$\hat{w}(X_1, \dots, X_n) = \frac{\sum_{i=1}^n X_i + \alpha - 1}{n + \beta}.$$

16. Suppose that X_1, \dots, X_n is a random sample from a normal distribution for which both the mean M and the precision R are unknown. Suppose also that the prior joint distribution of M and R is a normal-gamma distribution as specified in

Theorem 1 of Sec. 9.6. If μ' and β' are as defined in that theorem, show that generalized maximum likelihood estimators $\hat{m}(X_1, \dots, X_n)$ and $\hat{\beta}(X_1, \dots, X_n)$ are specified by the equations $\hat{m}(X_1, \dots, X_n) = \mu'$ and

$$\hat{\beta}(X_1, \dots, X_n) = \frac{\alpha + (n-1)/2}{\beta'}.$$

17. Consider the problem presented in Sec. 11.6. Prove that the posterior probability that $W = w$, will be greater than the prior probability p if, and only if, the magnitude of the difference between the observed value of \bar{x} and the value specified in Eq. (5) of Sec. 11.6 is less than the following number:

$$\frac{1}{n} \left\{ (\tau + n) \left[\tau(w_i - \mu)^2 + \log \frac{\tau + n}{\tau} \right] \right\}^{\frac{1}{2}}.$$

18. Verify that Eq. (2) of Sec. 11.7 is correct.

19. Let X_1, \dots, X_n be a random sample from a Bernoulli distribution for which the value of the parameter W is unknown. Suppose that under the prior distribution of W , $\Pr(W = \frac{1}{2}) = p > 0$ and the remaining probability $1 - p$ is uniformly distributed over the interval $0 < W < 1$. Find the posterior probability that $W = \frac{1}{2}$ when $\sum_{i=1}^n X_i = y$, and show that this posterior probability is greater than the prior probability p if, and only if, the following inequality is satisfied:

$$\binom{n}{y} \left(\frac{1}{2} \right)^n > \frac{1}{n+1}.$$

20. Suppose that the random vector $\mathbf{X} = (X_1, \dots, X_k)'$ has a multinomial distribution with parameters n and $\mathbf{W} = (W_1, \dots, W_k)'$, where n is a specified positive integer and the values of the components of the vector \mathbf{W} are unknown. Suppose also that the prior distribution of \mathbf{W} satisfies the following three conditions:
(a) $\Pr(W_1 = \frac{1}{2}) = p > 0$. (b) When $W_1 = \frac{1}{2}$, the joint distribution of W_2, \dots, W_k is a uniform distribution on the set of points (w_2, \dots, w_k) such that $w_i > 0$ ($i = 2, \dots, k$) and $\sum_{i=2}^k w_i = \frac{1}{2}$. (c) When $W_1 \neq \frac{1}{2}$, the joint distribution of W_2, \dots, W_k is a uniform distribution on the set of points (w_2, \dots, w_k) such that $w_i > 0$ ($i = 1, \dots, k$), and $w_1 \neq \frac{1}{2}$. Find the posterior distribution of the vector \mathbf{W} when $X_i = x_i$ ($i = 1, \dots, k$), and show that

$$\frac{\Pr(W_1 = \frac{1}{2} | X_1 = x_1, \dots, X_k = x_k)}{\Pr(W_1 \neq \frac{1}{2} | X_1 = x_1, \dots, X_k = x_k)} = \frac{p}{1 - p} \frac{n+k-1}{2^n(k-1)} \binom{n+k-1}{x_1}.$$

21. Let F be a distribution function on the real line for which the mean μ exists, and let the function T_F on the real line be defined by Eq. (9) of Sec. 11.8. (a) Prove that T_F is a nonnegative convex function which is strictly decreasing at any point s such that $T_F(s) > 0$. (b) Prove that T_F satisfies the relations (10) of Sec. 11.8.
22. If F is the d.f. of a normal distribution for which the mean is μ and the precision is τ , prove that the function T_F satisfies Eq. (3) of Sec. 11.9.
23. Suppose that the joint distribution of two random variables X and Y is as follows: The conditional distribution of X when $Y = y$ ($-\infty < y < \infty$) is a normal distribution with mean μ and variance σ_1^2 , and the marginal distribution of Y is a normal distribution with mean μ and variance σ_2^2 . Show that the marginal distribution of X is a normal distribution with mean μ and variance $\sigma_1^2 + \sigma_2^2$.

24. The mileage Y that can be obtained from a certain gasoline is a random variable which depends on the amount x of a certain chemical in the gasoline. For any given value of x , the mileage Y has a normal distribution for which the mean is $m_1 + m_2 x + m_3 x^2$ and the precision is w . Here m_1, m_2, m_3 , and w are the unknown

Chapter 4. Hypothesis Testing

3. Let $X = (X_1, \dots, X_n)$ be such that the X_i are conditionally IID with $N(0, \sigma^2)$ distribution given $\Sigma = \sigma$ under the hypothesis H . Let $T(x) = \sqrt{n\bar{x}}/s$, where $\bar{x} = \sum_{i=1}^n x_i/n$ and $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/(n-1)$. Define $x \leq y$ if $T(x) \leq T(y)$. Find $p_H(x)$, showing that it is the same for all σ .

Section 4.2:

4. Suppose that $X \sim N(\theta, 1)$ given $\Theta = \theta$. Suppose that $L(\theta, 0) < L(\theta, 1)$ for all $\theta < \theta_0$ and that $L(\theta, 1) > L(\theta, 0)$ for all $\theta > \theta_0$. Prove that, for every prior there exists k such that the formal Bayes rule will be to choose action $a = 1$ if $X < k$.
5. Suppose that X_1, \dots, X_n are conditionally IID with $N(\mu, \sigma^2)$ distribution given $\Theta = (\mu, \sigma)$. Use the improper prior having Radon-Nikodym derivative $1/\sigma$ with respect to Lebesgue measure on $(0, \infty) \times \mathbb{R}$. Let μ_0 and d be known values, and suppose that the loss function is

$$L(\theta, a) = \begin{cases} c & \text{if } a = 1 \text{ and } |\mu - \mu_0| \leq d\sigma, \\ 1 & \text{if } a = 0 \text{ and } |\mu - \mu_0| > d\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the formal Bayes rule will be of the following form: Choose $a = 1$ if $|T| > k$ for some constant k , where $T = \sqrt{n}(\bar{X}_n - \mu_0)/S_n$ and

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i, \quad S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

6. Suppose that $X \sim N(\theta, 1)$ given $\Theta = \theta$ and Θ has $\Pr(\Theta = \theta_0) = p_0$ and given $\Theta \neq \theta_0$, $\Theta \sim N(\theta_0, \tau^2)$. Prove that the posterior density of Θ with respect to the measure $\nu(A) = I_A(\theta_0) + \lambda(A)$, where λ is Lebesgue measure, is given by

$$f_{\Theta|X}(\theta|x) = \begin{cases} \frac{p_1}{\sqrt{2\pi(1+\tau^2)}} \exp\left[-\frac{\tau^2}{2(1+\tau^2)}(\theta - \theta_1)^2\right] & \text{if } \theta \neq \theta_0, \\ 0 & \text{if } \theta = \theta_0, \end{cases}$$

where

$$\begin{aligned} \theta_1 &= \frac{x\tau^2 + \theta_0}{1 + \tau^2}, \\ \frac{p_1}{1 - p_1} &= \frac{p_0}{(1 - p_0)} \sqrt{1 + \tau^2} \exp\left\{-\frac{1}{2}\left[\frac{\tau^2}{1 + \tau^2}\right](x - \theta_0)^2\right\}. \end{aligned}$$

7. Suppose that $X \sim N(\theta, 1)$ given $\Theta = \theta$. Let $H : \Theta = \theta_0$ and $A : \Theta \neq \theta_0$. Let the conditional prior given $\Theta \neq \theta_0$ be $N(\theta_0, \tau^2)$.

- (a) Prove that the Bayes factor is minimized if
- $$\tau^2 = \begin{cases} (x - \theta_0)^2 - 1 & \text{if } |x - \theta_0| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Section 4.3.1:

- (b) Show that the minimum Bayes factor is $|x - \theta_0| \exp(\{-[x - \theta_0]^2 + 1\}/2)$

if $|x - \theta_0| > 1$, and is 1 if $|x - \theta_0| \leq 1$.

8. In Example 4.21 on page 224, prove that

$$k = \lim_{\tau \rightarrow \infty} \frac{p_0}{\Phi\left(\frac{\sqrt{\pi/2}}{\tau}\right) - \Phi\left(\frac{-\sqrt{\pi/2}}{\tau}\right)}.$$

Section 4.2:

9. Let (α_0, α_1) be a point on the lower boundary of the risk set for a simple–simple hypothesis-testing problem. Prove that $\alpha_0 + \alpha_1 \leq 1$.

10. In a simple–simple hypothesis-testing problem, prove that the minimax rule for a 0–1 loss function is any test that corresponds to the point where ∂_L intersects the line $y = x$.

11. Let $\Omega = [0, 1]$. Suppose that P_0 says that $X \sim U(-\sqrt{3}, \sqrt{3})$ and P_1 says that $X \sim N(0, 1)$. Let $\Omega_H = \{0\}$. Draw the risk set for a 0–1 loss function, and find the minimax rule.

12. In a simple–simple hypothesis-testing problem with 0–1 loss, show that a MP level α test has size α unless all tests with size α are inadmissible.

13. Return to the situation in Problem 28 on page 212. Consider the hypothesis $H : X \sim f_0$ versus $A : X \sim f_1$. Find all α such that the MP level α test is of the form “Reject H if $-d \leq X \leq d$,” and write d as a function of α .

14. *Prove Proposition 4.46 on page 235.

15. In Example 4.49 on page 236, prove that the Bayes rule with size α has higher power than the unconditional size α test.

Section 4.3.2:

16. Suppose that the loss function is 0–1– c , that $\Omega_H = \{\theta_0\}$, and that $\Omega = \{\theta_0\} \cup \Omega_A$. Prove that a UMP level α test has no larger Bayes risk than any other size α test, no matter what prior we use.

Section 4.3.3:

17. Let $Y = |X|$ where $f_{X|\Theta}(x|\theta) = 1/(\pi\theta(1 + (x/\theta)^2))$. Suppose that $\Theta > 0$ for sure. Prove that the family of distributions for Y has MLR.

18. Let X have Cauchy distribution $Cau(\theta, 1)$ given $\Theta = \theta$.

- (a) Prove that the MP level α test of $H : \Theta = \theta_0$ versus $A : \Theta = \theta_1$ for $\theta_1 > \theta_0$ is essentially unique. That is, if ϕ and ψ are both MP level α tests, then $F_\theta(\phi(X)) = \psi(X)) = 1$ for all θ .

- (b) Prove that there is no UMP level α test of $H : \Theta = \theta_0$ versus $A : \Theta > \theta_0$ for $0 < \alpha < 1$.

19. Let the parameter space be the open interval $\Omega = (0, 100)$. Let X_1 and X_2 be conditionally independent given $\Theta = \theta$ with $X_1 \sim Poi(\theta)$ and $X_2 \sim Poi(100 - \theta)$. We are interested in the hypothesis $H : \Theta \leq c$ versus $A : \Theta > c$.

Section 5.5. Estimation

26. Consider the situation in Example 5.30 on page 308. Let squared error be the loss, that is, $L(\theta, a) = (\mu^2 - a)^2$. Show that the UMVUE dominates the MLE if $n \geq 2$. Find a formula for the difference in the risk functions. Also, find an estimator that dominates both the MLE and the UMVUE.

Section 5.1.4:

27. Consider the situation in Problem 6 on page 339. Find the likelihood function and show that a Bayesian would take the data at face value (that is, a Bayesian would calculate the same posterior as if the sample size had been fixed in advance at whatever value N turns out to be, no matter what the prior is).
28. Consider the situation in Problem 7 on page 339. If Λ and M are independent a priori with $\Lambda \sim \Gamma(a, b)$ and $M \sim \Gamma(c, d)$, find the posterior mean of Λ given the data.

29. In Example 5.10 on page 299, find the posterior mean of Θ given $X = x$ for all x assuming that the prior for Θ is $Beta(\alpha_0, \beta_0)$.
30. Return to the situation of Problem 16 on page 140. If Θ has a prior density $f_\Theta(\theta) = ac^\alpha I_{(c, \infty)}(\theta)/\theta^{\alpha+1}$, find the posterior mean of Θ .

- 31.* Suppose that, conditional on N , $\{X_i\}_{i=1}^{\infty}$ are independent with the first N of them having $Ber(1/3)$ distribution and the rest having $Ber(2/3)$ distribution. The prior for N is $f_N(n) = 2^{-n}$ for $n = 1, 2, \dots$

- (a) Find the posterior distribution of N given a finite sample X_1, \dots, X_n , for known n .
- (b) If $X_1 = 0, \dots, X_n = 0$ is the observed finite sample, find the posterior mean of N .

Section 5.1.5:

32. Let \mathcal{P}_0 be the set of distributions on $(\mathbb{R}, \mathcal{B}^1)$ with finite variance. Let $T(P)$ be the standard deviation of the distribution P . Show that $IF(x; T, P) = (\bar{x} - \mu)^2/[2\sigma] - \sigma/2$, where μ is the mean of P .

33. Let \mathcal{P}_0 be the class of distributions on $(\mathbb{R}, \mathcal{B}^1)$ with bounded support, and let $T(P)$ be the supremum of the support. Prove that the influence function for T is $IF(x; T, P) = 0$ if $x \leq T(P)$ and ∞ if $x > T(p)$.

34. Find the influence function for the 100 $\alpha\%$ trimmed mean at a continuous distribution P .

Section 5.2:

35. Prove Proposition 5.48 on page 316.

36. Let the parameter space be Ω with σ -field of subsets τ . Let X be a random quantity taking values in a set \mathcal{X} , and let X have conditional density $f_{X|\Theta}$ given Θ . Let $v : \mathcal{X} \rightarrow [0, \infty)$ be a measurable function such that the set $\{x \in \Omega : f_{X|\Theta}(x|\theta) \geq v(x)\}$

is in τ for all x . Let Θ have a prior distribution μ_Θ , and let μ_X denote the prior predictive distribution of X . Let $C : \mathcal{X} \rightarrow \tau$ be another set function such that $\mu_\Theta(C(x)) \leq \mu_\Theta(L(x))$, a.s. $[\mu_X]$. Prove that $\Pr(\Theta \in C(x)|X = x) \leq \Pr(\Theta \in L(x)|X = x)$, a.s. $[\mu_X]$.

37. Prove Proposition 5.56 on page 319.

38. Prove Proposition 5.61 on page 321.

39. Prove Proposition 5.79 on page 329.

40. Suppose that $\Omega = \mathbb{R}$ and that the posterior density of Θ given X is strongly unimodal. Let the action space be the set of all closed and bounded intervals $[a_1, a_2]$ in \mathbb{R} .

(a) Let the loss function be $L_t(\theta, [a_1, a_2]) = a_2 - a_1 + c(1 - I_{[a_1, a_2]}(\theta))$. Prove that the formal Bayes rule is an HPD region.

(b) Let the loss function be $L_q(\theta, [a_1, a_2]) = (a_2 - a_1)^2 + c(1 - I_{[a_1, a_2]}(\theta))$. Find the formal Bayes rule.

Section 5.3:

41. Suppose that one wished to construct a parametric bootstrap estimate in the second part of Example 5.80 on page 330.

(a) Explain how to construct the parametric bootstrap estimate using the $U(0, \theta)$ parametric family.

(b) Find the distribution of $R(X^*, \hat{F}_n)$ for the parametric bootstrap estimate.

(c) Will the parametric bootstrap estimate have the same problem that the nonparametric bootstrap estimate has?

42. How would one use the nonparametric bootstrap to find the bias and standard deviation for the sample correlation coefficient from a sample of n pairs $(X_1, Y_1), \dots, (X_n, Y_n)$?

43. Let $(x_1, Y_1), \dots, (x_n, Y_n)$ be data pairs, and suppose that we entertain a regression model in which $E(Y_i|B_0 = \beta_0, B_1 = \beta_1) = \beta_0 + \beta_1 x_i$. The y -intercept of the regression line is $x = -\beta_0/\beta_1$. Let (\hat{B}_0, \hat{B}_1) be the usual least-squares regression estimator.

(a) How would one use the bootstrap to find the bias and standard deviation of the ratio $-\hat{B}_0/\hat{B}_1$?

(b) Suppose that one used the following formula for the approximate variance of the ratio of two random variables Z_0/Z_1 :

$$\frac{\text{Var}(Z_0)}{Z_1^2} + \frac{Z_0^2 \text{Var}(Z_1)}{Z_1^4} - 2 \frac{Z_0 \text{Cov}(Z_0, Z_1)}{Z_1^3}.$$

Show how you would use this to find bootstrap confidence intervals for $-\hat{B}_0/\hat{B}_1$.

Aula 11

$$L(d, \theta) = -U(r(d, \theta))$$

\rightarrow perda incorrida ao dec. $\theta \in D$ quando est. not. é θ .

Fun de D.

$$p(d, P) = \int_{\Theta} L(d, \theta) dP(\theta) \Rightarrow \text{risco do decisão } d.$$

$$p^*(P) = \inf \{ p(d, P) : d \in D \} \Rightarrow r. \text{ de Bayes}.$$

$d^* \in D$ é dec. de Bayes contra P se $p^*(d^*, P) = p^*(P)$

Comentário

2. D, P, L

$$\text{Seja } \lambda : \Theta \rightarrow \mathbb{R} \text{ t.q. } \int_{\Theta} \lambda(\theta) dP(\theta) < \infty$$

$$L' : D \times \Theta \rightarrow \mathbb{R}$$

$$(d, \theta) \mapsto L'(d, \theta) = L(d, \theta) + \lambda(\theta)$$

$$\begin{aligned} \int_{\Theta} L'(d, \theta) dP &= \int_{\Theta} [L(d, \theta) + \lambda(\theta)] dP(\theta) \\ &= \int_{\Theta} L(d, \theta) dP(\theta) + \int_{\Theta} \lambda(\theta) dP(\theta) \end{aligned}$$

Para todos $a > 0$ e $\lambda : \Theta \rightarrow \mathbb{R}$ t.q. $\int_{\Theta} \lambda(\theta) dP(\theta) < \infty$, temos $\forall d_1, d_2 \in D$, que

$$\int_{\Theta} L(d_1, \theta) dP(\theta) \leq \int_{\Theta} L(d_2, \theta) dP(\theta) \Leftrightarrow$$

$$\int_{\Theta} [\underline{L}(d_1, \theta) + \lambda(\theta)] dP(\theta) \leq \int_{\Theta} [\underline{L}(d_2, \theta) + \lambda(\theta)] dP(\theta)$$

3. Fixado Θ , \mathcal{F} e L (não negativo), seja \mathcal{P}_L o conj. de todos os med. de prob. em (Θ, \mathcal{F}) .

\mathcal{P}_L é convexo, pois $\forall P_1, P_2 \in \mathcal{P}_L$, se $P_1 + (1-\lambda)P_2 \in \mathcal{P}_L$, $\lambda \in [0, 1]$. Vamos olhar a transf.

$$p^* : \mathcal{P}_L \rightarrow \mathbb{R}$$

$$\{P \in \mathcal{P}_L \mapsto p^*(P) = \inf \{ p(d, P) : d \in D \}\}$$

Resul-todo: p^* é convexa, i.e., $p^*(\lambda P_1 + (1-\lambda)P_2) \geq \lambda p^*(P_1) + (1-\lambda)p^*(P_2)$.

Aula 12

Risco incorrido por A sob o ponto de vista de B:

$$\rho(d_A^*, P_B), \text{ onde}$$

$d_A^* = \rho^*(P_A)$. Sobre B, o risco adicional é

$$\rho(d_A^*, P_B) - \rho^*(P_B).$$

Se $L(d, \theta) = (d - \theta)^2$, então

$$\rho(d, P) = \text{Var}(\theta | P) + (d - E(\theta | P))^2,$$

de modo que, $d^* = E(\theta | P)$ e,

$$\rho^*(P) = \text{Var}(\theta | P).$$

$$D = \{d_1, \dots, d_n\}$$

Def. A proc. que assoc. $d_i \in D$ com prob. π_i , i.e., dá-nos o nome de decisão.

$$L(d_k, \theta) = \sum \pi_i L(d_i, \theta) \quad \text{"L est. à } \mu \times \theta \text{"}$$

Aula 14.

d é admissível se

$$L(d, \theta) \leq L(d', \theta) \quad \forall \theta \in \Theta, \quad d \neq d' \Rightarrow$$

$$\exists \theta_0 \in \Theta \text{ s.t. } L(d, \theta_0) \leq L(d', \theta_0) \quad \forall d' \neq d \in D.$$

Se d é única dec. do Bayes, d é admissível.

Problema da Decisão

$$(D, \Theta, L, P)$$

D: decisões
 Θ : parâmetros
L: perdas
P: prob.

$\forall d \in D$, calc.

$$\rho(d, P) = \int_{\Theta} L(d, \theta) d\theta$$

e temos $d^* = \arg \min_{d \in D} \rho(d, P) = \inf_{d \in D} \rho(d, P).$

Aula 14

Problema de Decisão com Dados

- Pré - Experimentações

- Pós - Experimentações

~~Pós~~ Pós Experimentações

$\rightarrow (\mathcal{D}, \Theta, L, P_{x=\omega})$, onde $P_{x=\omega}$ é a med. do prob. sob o todo $\omega \in \Omega$.

Para cada realização $\omega = \omega$, um problema de decisão distinto.

$X = x$ induz $P_{x=\omega}$ distintas pf. ω dist., o que leva à ris. dist.

Caso Pré-Experimentações

Suponha que o exp. não foi conduzido e que X é um incerto (desconhecido).

Nesse caso, há incertezas sobre (θ, ω) , $\theta \in \Theta$, ou possível obs. do exp.

Além disso, devemos agora especificar uma regra (função) de decisão que associa a cada ω uma decisão $d \in \mathcal{D}$.

Se \mathfrak{X} o cjo das poss. res. de exp. (No ex. 3 $\mathfrak{X} = \{0, 1\}$)

Temos um novo prob. de decisão.

$$\Delta : \{\delta : \mathfrak{X} \rightarrow \mathcal{D}\} \text{ cjo de regras de decisão}$$

$$\Delta = \Theta \times \mathfrak{X}, \text{ munido de } \sigma(\Theta, \mathfrak{X})$$

$$P : \sigma(\Theta \times \mathfrak{X}) \rightarrow [0, 1] \text{ um prob.}$$

Caro ontes, $L : \mathcal{D} \times \Theta \rightarrow \mathbb{R}$ é dcf.

$$L : \Delta \times \Delta \rightarrow \mathbb{R} \text{ por}$$

$$\forall \delta \in \Delta = (\theta, \omega) \in \Delta, L(\delta, (\theta, \omega)) = L(\delta(\omega), \theta).$$

$$(\Delta, \Delta, L, P)$$

\forall cada $\delta \in \Delta$, $p^{(\delta, P)}$:

D. Pré. Exp.

$$(\Delta, \Omega, L, P)$$

|

$$\Delta = \{ \delta: \mathcal{X} \rightarrow D \} \quad P': \mathcal{O}(\mathcal{X} \times \Theta) \rightarrow [0,1]$$

$$\Omega = \mathcal{X} \times \Theta$$

$$L'(\delta, (\theta, x)) = L(\delta(x), \theta)$$

Para cada $\delta \in \Delta$,

$$\bar{\rho}(\delta, P) = \int_{\Omega} L'(\delta, (\theta, x)) dP'(\theta, x)$$

|

row da função decisiva δ contra P'

Do mesmo modo

$$\bar{\rho}^*(P) = \inf \{ \bar{\rho}(\delta, P) : \delta \in \Delta \} \text{ é chamado R. de Bayes}$$

Aula 15

Pós:

Observa $X=x$, você resolve o prob. $(D, \Theta, L, P_{X=x})$

Pré:

$$\Delta = \{ \delta: \Xi \rightarrow D \}$$

| funções regra de decisão

$$\Omega = \Theta \times \Xi$$

D' sobre $\sigma(\Theta \times \Xi)$

$$L: \Delta \times \Omega \rightarrow \mathbb{R}$$

$$(\delta, (\theta, \omega)) \mapsto L(\delta, (\theta, \omega)) = L(\delta(\omega), \theta)$$

Para cada $\delta \in \Delta$,

$$p^*(\delta, P) = \int_{\Theta \times \Xi} L(\delta, (\theta, \omega)) dP(\theta, \omega) \rightarrow \text{risco da f. decisão } \delta \text{ contra } P$$

$$p^{**}(P) = \inf \{ p^*(\delta, P) \mid \delta \in \Delta \} \rightarrow \text{Risco de Bayes}$$

$\delta^* \in \Delta$ t.q. $p^*(\delta^*, P) = p^{**}(P)$ é regra de decisão de Bayes.

Prob. Det. Tomando Am. e da Regra de Decisão

$$\bar{\Omega} = \Theta \times \Xi, \quad \Xi = \prod_{i=1}^{\infty} \Xi_i : \text{tudo que secon sobre o qual}\newline \text{há incerteza}$$

$$\Delta_n = \{ \delta: \prod_{i=1}^n \Xi_i \rightarrow D \}$$

$$\Delta_0 = D$$

$$\bar{\Delta} = \bigcup_{n=0}^{\infty} \{ \delta \} \times \Delta_n \Rightarrow \text{espaço de ações}$$

$$\bar{P} \text{ sobre } \sigma(\Theta \times \Xi), \bar{L}, \bar{\Delta} \times \bar{\Omega} \rightarrow \mathbb{R}$$

$$((n, \delta^{(n)}), (\theta, \omega)) \mapsto \bar{L}((n, \delta^{(n)}), (\theta, \omega))$$

$$= L(\delta^{(n)}(\omega), \theta) + n C_0 \cdot 1_{\{ \theta = 0 \}}$$

$$L(\delta^{(n)}(\omega), \theta) + n C_0 \cdot 1_{\{ \theta = 0 \}}$$

$$\bar{\rho}((n, \delta), \bar{P})$$

$$= \int_{\Theta \times \mathfrak{X}} [(n, \delta), (\Theta, \omega)] d\bar{P}$$

$$= \int_{\Theta \times \mathfrak{X}} [L(\delta(\omega), \Theta) + nC_0] d\bar{P}$$

$$= nC_0 + \int_{\Theta \times \mathfrak{X}} L(\delta^{(n)}(\omega), \Theta) d\bar{P}$$

$$\bar{\Delta}_n = \Theta \times \mathfrak{X}, \quad \mathfrak{X} = \prod_{i=1}^n \mathfrak{X}_i$$

$$\Delta_n = \left\{ \delta : \prod_{i=1}^n \mathfrak{X}_i \rightarrow \mathbb{D} \right\}$$

$$\bar{\Delta} = \bigcup_{n=1}^{\infty} \{ n \} \times \Delta_n$$

~~D(Am)~~

$$L: \bar{\Delta} \times \Theta \rightarrow \mathbb{R}$$

$$(n, \delta), (\Theta, \omega) \mapsto L((n, \delta), (\Theta, \omega))$$

$$\bar{\rho}((n, \delta), \bar{P}) = \int_{\Theta \times \mathfrak{X}} L((n, \delta), (\Theta, \omega)) d\bar{P}$$

$$= nC_0 + \int_{\Theta \times \mathfrak{X}} L(\delta^{(n)}(\omega), \Theta) d\bar{P}$$

$$= nC_0 + \int_{\Theta \times \mathfrak{X}} L(\delta^{(n)}(\omega), \Theta) d\bar{P}$$

Aula 16

am costos

$$\mathcal{E} = \prod_i \mathcal{E}_i$$

$$\bar{\Omega} = \Theta \times \mathcal{E}$$

$$\Delta_n : \{ \delta : \mathcal{E}_n \rightarrow \mathcal{D}_n \rightarrow D \}, \Delta_0 = D$$

$$\bar{D} = \bigcup_{n=0}^{\infty} \mathcal{E}_n \times \Delta_n$$

$$L((c_n, \delta), (\theta, \pi)) = L(\delta^{(n)}, \theta) + \text{inf}_{\pi} \quad \text{for } n \in \mathbb{N}$$

Aula 17

Estimação

(D, Θ, L, P)

Após obs. $x \in \mathcal{X}$, $(D, \Theta, L, P_{x=x})$ com $D = \Theta$

Pr. Pré-Exp.

$$\Delta = \{\delta : \mathcal{X} \rightarrow D\}$$

No de ej. $D = \Theta$, de modo que

$$\Delta = \{\delta : \mathcal{X} \rightarrow \Theta\} \text{ conj. dos ej. possíveis}$$

$(\Delta, \oplus \times \mathcal{X}, L, P')$

$$L' : \Delta \times (\Theta \times \mathcal{X}) \rightarrow \mathbb{R}_+$$

$$(\delta, (\theta, x)) \mapsto L'(\delta(\theta, x)) = L(\delta(x), \theta)$$

Usualmente, L expressa o desacordo entre $\delta(x)$ e θ .

$$L(d, \theta) = h(\theta) g(\text{dist}(d, \theta)), \text{ g não decresc.}$$

Perda Quadrática $L(d, \theta) = (d - \theta)^2$

$$\text{Se } E(\theta) \in \Theta \Rightarrow d^2 = E(\theta)$$

$$\text{Se } E(\theta) \notin \Theta \Rightarrow d^2 \in D \text{ s.t. } |d^2 - E(\theta)| = \min \{|d - E(\theta)| : d \in D\}.$$

$$P'(P) = \text{Var}(\theta). \text{ Se } E(\theta) \in \Theta \text{ e}$$

$$(d^2 - E(\theta))^2 + \text{Var}(\theta) \text{ c.c.}$$

Após = obs. de $X=x$,

$(\Theta, \Theta, L, P_{x=x})$

$$d_x^2 = E(\theta | X=x)$$

$$P'(P_{x=x}) = \text{Var}(\theta | X)$$

No prob. Pré-exp.

$$\delta^*: \mathcal{E} \rightarrow \mathbb{R}$$
$$\theta \in \mathcal{E} \mapsto \delta^*(\theta) = E(\theta | X=x), \text{ e}$$

$$\delta^*(x) = E(\theta | X=x) - \text{com rel. per la quadratica}$$

Em geral, pl $\delta \in \Delta$

$$P(\delta, P') = \int_{\Theta \times \mathcal{E}} L(\delta(x), \theta) dP'$$
$$= \int_{\mathcal{E}} f(x) \left[\int_{\Theta} L(\delta(x), \theta) f(\theta | x) d\theta \right] dx$$

Par:

$$P^*(\delta^*) = \underbrace{\int_{\mathcal{E}} f(x) \int_{\Theta} (\theta - E(\theta | X=x))^2 f(\theta | x) d\theta}_{\text{Var } (\theta | X=x)} dx$$
$$= E(\text{Var}(\theta | X))$$

Testes de Hipóteses

\mathbb{F} , \exists - σ -álgebra de Ω

Vamos tomar $\Theta_0, \Theta_1 \in \mathbb{F}$. e $\Theta = \Theta_0 \cup \Theta_1$, $\Theta_0 \cap \Theta_1 = \emptyset$

$$H_0: \Theta \in \Theta_0$$

$$H_1: \Theta \in \Theta_1$$

Neste caso

$$D = \{d_0, d_1\}$$

d_0 : N reg. H_0 ; d_1 : reg.

$$\begin{array}{ccc} \Theta \in \Theta_0 & \Theta \in \Theta_1 \\ d_0 & \Theta & \Theta_2 \\ & a_1 & 0 \end{array} \quad (\text{mais simpl. f. de perda})$$

Obs $X = \omega$, $(\{d_0, d_1\}, \Theta, L, P_{X=\omega})$

$$p(d_0, P_{X=\omega}) = \int_{\Theta} L(d_0, \theta) dP_X = \int_{\Theta_1} a_2 dP_X = a_2 P_{X=\omega}(\Theta \in \Theta_1)$$

$$p(d_1, P_{X=\omega}) = \int_{\Theta_0} a_1 dP_{\omega} = a_1 P(\Theta \in \Theta_0 | \omega)$$

$$p(d_1, P_X) \leq p(d_0, P_X) \Rightarrow a_1 P(\Theta \in \Theta_0 | \omega) \leq a_2 P(\Theta \in \Theta_1 | \omega)$$

$$\Leftrightarrow a_1 P(\Theta \in \Theta_0 | \omega) \leq a_2 - a_1 P(\Theta \in \Theta_0 | \omega)$$

$$\Leftrightarrow P(\Theta \in \Theta_0 | \omega) \leq \frac{a_1}{a_1 + a_2}$$

$$P(\Theta \in \Theta_0 | \omega) \leq \frac{a_1^2}{a_1 + a_2}$$

No prob. Prés., a regra de dec. de Bayes (ou teste de Bayes) é do

$$\psi^*(x) = \begin{cases} 1 & P(\Theta \in \Theta_1 | x) > \frac{\alpha}{\alpha_1 + \alpha_2} \\ 0 & P(\Theta \in \Theta_0 | x) < \frac{\alpha}{\alpha_1 + \alpha_2} \end{cases}$$

Esf. por intervalo

④

$$D = \{[a, b], a, b \in \mathbb{R}, a \leq b\}$$

$$L([a, b], \theta) = h(b - a) + K \prod_{\theta \in [a, b]} g(\theta)$$

$$D = J = \{\theta\} \text{ não fechado}$$

$$L(A, \theta) = h(A) + C \prod_{\theta \in A} g(\theta)$$

$$\text{nosesse } \sum_{i=1}^n x_i = m$$

$$f(\theta/x) = C \left\{ \theta^n \frac{\pi(\theta)}{[x_1, s]} + (1-\theta)^n \frac{\pi(\theta)}{[0, x_1]} \right\}$$

$$C = \frac{1}{\int_{x_1}^s \theta^n d\theta + \int_0^{x_1} (1-\theta)^n d\theta} = \frac{1}{2 \int_{x_1}^s \theta^n d\theta} = \frac{1}{2 \frac{s - (x_1)}{n+1}} =$$

$$\frac{n+1}{s - (x_1)} \Rightarrow$$

? \Rightarrow

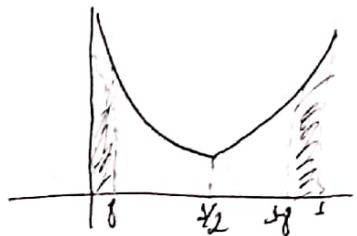
$$P(0 \leq \theta \leq q/x = x) = \frac{q}{s} \Rightarrow$$

$$\int_0^q (1-\theta)^n d\theta = \frac{1}{n+1} \Rightarrow \frac{-(1-\theta)^{n+1}}{n+1} \Big|_0^q = \frac{q}{n+1} \Rightarrow$$

$$\Rightarrow \frac{1}{n+1} \left\{ s - (s-q)^{n+1} \right\} = \frac{q}{n+1} \Rightarrow (s-q)^{n+1} = s - \frac{q(n+1)}{n+1} \Rightarrow$$

$$(s-q)^{n+1} = \frac{q}{n+1} [2 - (x_1)^n]$$

$[0, q] \cup [s-q, s]$ é a região HD de probabilidade
 $P, 0 \leq P \leq s.$



Testes de Hipóteses

$H_0, f,$

$\Theta_0, \Theta_1 \in f$ tais que $\Theta = \Theta_0 \cup \Theta_1$, e
 $\Theta_0 \cap \Theta_1 = \emptyset$

$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1$

$D = \{d_0, d_1\}$

$L(d, \theta)$		$\theta \in \Theta_0$		$\theta \in \Theta_1$	
		d_0	α_2	d_1	α_1
d_0		0		α_2	α_1
d_1		α_1		0	α_2

Normalizar
o L dado por

é dividido pelo
 $0 - \alpha_1 - \alpha_2$ (ou
 $0 - 1 - \frac{\alpha_2}{\alpha_1}$) ou
 $0 - 1 - c.$

Vai obter $X = x$.

$(\{d_0, d_1\}, \Theta, L, P_{X=x})$

$$P(d_0, P_{X=x}) = \int_{\Theta} L(d_0, \theta) dP_x = \int_{\Theta_0} \alpha_2 dP_x + \int_{\Theta_1} \alpha_2 dP_x = \alpha_2 P(\theta \in \Theta_0)$$

$$P(d_1, P_{X=x}) = \int_{\Theta} L(d_1, \theta) dP_x = \int_{\Theta_1} \alpha_1 dP_x + \int_{\Theta_0} \alpha_1 dP_x = \alpha_1 P(\theta \in \Theta_1)$$

A. obter $X=x$, decidimos por H_1 se

$$P(d_s, R_2) < P(d_0, R_1) \Leftrightarrow \alpha_1 P(\theta \in \Theta_0 | x) < \alpha_2 P(\theta \in \Theta_1 | x)$$

$$\Leftrightarrow \alpha_1 P(\theta \in \Theta_0 | x) < \alpha_2 - \alpha_2 P(\theta \in \Theta_1 | x) \Leftrightarrow$$

$$P(\theta \in \Theta_0 | x) < \frac{\alpha_2}{\alpha_1 + \alpha_2} \quad (\text{ou, alternativamente,})$$

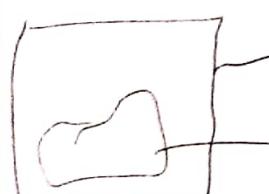
$$P(\theta \in \Theta_1 | x) > \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

* No problema pré-experimentação, a Regra de decisão

Decisão de Bayes (ou teste de Bayes) é dada por

$$\psi^*(x) = \begin{cases} 1, & P(\theta \in \Theta_1 | x) > \frac{\alpha_1}{\alpha_1 + \alpha_2} \\ 0, & P(\theta \in \Theta_1 | x) < \frac{\alpha_1}{\alpha_1 + \alpha_2} \end{cases}$$

$$H_0: \theta \in \Theta_0 \times H_1: \theta \in \Theta_1$$



$$\Psi = \{\psi: \mathbb{X} \rightarrow \{0, 1\}\}$$

$$\Psi' = \{\psi \in \Psi: \text{também } \psi \leq \infty\}$$

$$EX3: \Theta = \{\frac{1}{4}, \frac{1}{2}\}$$

X_1, \dots, X_n , dado Θ , c.c.i.d. Bern(θ)

$$P(\theta = \frac{1}{4}) = p = 1 - P(\theta = \frac{1}{2}), p \in (0, 1)$$

$$\text{se } H_0: \theta = \frac{1}{4} \quad P(\theta = \frac{1}{2} | X_1 = x_1, \dots, X_n = x_n) =$$

$$H_1: \theta = \frac{1}{2}$$

$$q^n = \frac{(\frac{1}{2})^{x_1} (\frac{1}{2})^{n-x_1}}{(\frac{1}{4})^{x_1} (\frac{1}{2})^{n-x_1} (\frac{1}{2}+p) (\frac{3}{4})^{n-x_1}} p$$

$$\Rightarrow \frac{2^n (1-p)}{2^n (1-p) + 3^{n-x_1}} > \frac{\alpha_1}{\alpha_1 + \alpha_2} \Leftrightarrow 2^n (1-p) \alpha_2 > 3^{n-x_1} \alpha_1 \Leftrightarrow$$

$$(n - x_1) \log 3 \leq \log \frac{2^n (1-p) \alpha_2}{\alpha_1} \Leftrightarrow$$

$$\sum_{i=1}^n x_i > n \log 3 - \frac{\log \frac{2^n (1-p) \alpha_2}{\alpha_1}}{\log 3} \Leftrightarrow$$

$$\sum_{i=1}^n x_i > n - \frac{\log \frac{\alpha_2 (1-p)}{\alpha_1} 2^n}{\log 3}$$

$$\psi^*(x) = \overline{I}(x_1 + \dots + x_n) \frac{\log 3}{\log 3}$$

* O teste de Bayes nesse caso é dado por

$$\psi^*(x_1, \dots, x_n) = \begin{cases} 1, & \sum_{i=1}^n x_i > k \\ 0, & \text{C.C.} \end{cases}$$

$$EX2: \Theta = [0, 1], \Theta_0 = [0, 1/2], \Theta_1 = [1/2, 1]$$

X_1, \dots, X_m , dada θ , c.i.i.d. $Ber(\theta)$

$$\theta \sim Beta(a, b), a, b > 0.$$

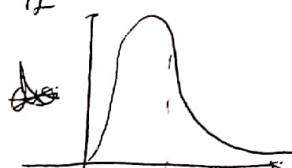
$H_0: \theta \leq 1/2$, Do resultado visto,

$$H_1: \theta > 1/2 \quad \theta | X \sim Beta(a + \sum_{i=1}^m x_i, b + m - \sum_{i=1}^m x_i)$$

Rejetamos H_0 se.

$$P(\theta > 1/2 | x) > \frac{a_1}{a_1 + a_2} \iff$$

$$\int_{1/2}^1 \frac{\Gamma(a+b+m)}{\Gamma(a+\sum x_i)\Gamma(b+m-\sum x_i)} \theta^{a+\sum x_i-1} (1-\theta)^{b+m-\sum x_i-1} d\theta > \frac{a_1}{a_1 + a_2}$$



Se $a_1 = a_2 = 1$, decidimos por

H_1 se e se não.

$$P(\theta > 1/2 | x) > 1/2 \iff \text{Md}(\theta | x) > 1/2$$

$$\Theta = [0, 1], \Theta_0 = [0, 1/2], \Theta_1 = [1/2, 1]$$

$$P(\theta > 1/2 | x) > 1/2 \iff \text{Md}(\theta | x) > 1/2$$

Anotações

$$EX3: \Theta = \mathbb{R}_+^3, \theta = (\theta_1, \theta_2)$$

$X_1, \dots, X_n, Y_1, \dots, Y_m$, dada θ , São cond. independentes tal que $X_i | \theta \sim Poisson(\theta_1)$, $i = 1, \dots, n$, e $Y_j | \theta \sim Poisson(\theta_2)$, $j = 1, \dots, m$.

$$\theta_i \sim Gama(\mu_i, v_i), i = 1, 2 \quad \theta_1 \perp\!\!\!\perp \theta_2$$

$$f(\theta | x, y) \propto f(x, y | \theta) f(\theta) \propto \prod_{i=1}^n f(x_i | \theta_i) f(\theta_i) f(\theta_2)$$

$$\prod_{i=1}^n f(x_i | \theta_i) \prod_{j=1}^m f(y_j | \theta_2) f(\theta_1) f(\theta_2) \propto$$

$$e^{-n_1 \theta_1} \theta_1^{\sum x_i} e^{-n_2 \theta_2} \theta_2^{\sum y_j} \theta_1^{v_1 - 1} e^{-v_1 \theta_1} \cdot \theta_2^{v_2 - 1} e^{-v_2 \theta_2} \Rightarrow$$

$$\Rightarrow f(\theta | x, y) \propto \theta_1^{\sum x_i - (v_1 + n_1)} \theta_2^{\sum y_j - (v_2 + n_2)} e^{-\frac{n_1 \theta_1}{\theta_1} - \frac{n_2 \theta_2}{\theta_2}}$$

Isto é,

$$\theta_1 \perp\!\!\!\perp \theta_2 | x, y, \text{ com } \theta_1 | x, y \sim Gama(\nu_1 + \sum x_i, \nu_1 + n_1)$$

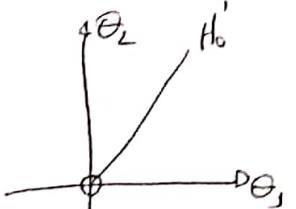
$$\theta_2 | x, y \sim Gama(\nu_2 + \sum y_j, \nu_2 + n_2)$$

\Rightarrow Rejetamos H_0 se.

$$P(\theta_1 \geq \theta_2 | x, y) \geq \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

ainda nesse exemplo, seym.

$$H_0: \theta_1 = \theta_2$$



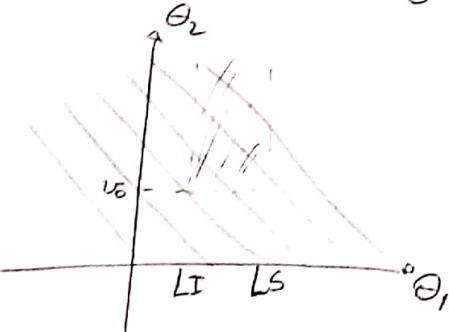
$$H_1: \theta_1 \neq \theta_2$$

$$P(\theta_1 = \theta_2 | x, y) = 0, \forall (x, y) !!!$$

$E(X_i)$: X_1, \dots, X_n dada $\theta = (\theta_1, \theta_2)$, c.i.i.d. $N(\theta_1, \frac{1}{\theta_2})$

$$H_0: [L_I, L_S] \times [U_0, \infty)$$

$$H_1: ([L_I \times L_S] \times [U_0, \infty))^\complement$$



$$\theta_2 \sim \text{Gama}(a', b')$$

Vimos que

$$\theta_1 | \theta_2 \sim N(\theta_0, \frac{\sigma_0^2}{\theta_2})$$

$$\theta_2 | x_1, \dots, x_n \sim \text{Gama}$$

$$\theta_2 | x_1, \dots, x_n \sim \text{Gama}\left(a + \frac{n}{2}, b + \frac{n}{2} \frac{(\bar{x} - \theta_0)^2 + \sum (x_i - \bar{x})^2}{m\sigma_0^2 + n\theta_0^2}\right)$$

$$\theta_1 | \theta_2, x_1, \dots, x_n \sim N\left(\frac{U_0 \theta_0 + n\bar{x}}{U_0 + n\theta_0}, \frac{\sigma_0^2 U_0}{(U_0 + n\theta_0)\theta_2}\right)$$

Vou decidir por H_1 se ero se.

$$P(\theta \in H_1 | x) < \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

θ_1, θ_2 gerar em f.

EX5: X_1, \dots, X_n dada θ , c.c.i.d. $\text{Exp}(\theta)$

$$H_0: 1 \leq \theta \leq 2$$

$$H_1: \theta \in \bigcup_{n=3}^{\infty} [3n, 3n+3]$$

$$H_0: \theta < 3 \text{ ou } \theta > 2$$

$$H_1: \theta \notin \bigcup_{n=3}^{\infty} [3n, 3n+3]$$

Supondo prior, $\theta \sim \text{Gama}(a', b')$

Para os exercícios complementares.

$$L'(S, (\theta, x)) = -|\delta(x)| + \dots \quad \text{EXERCÍCIO 1.}$$

$$L(A, \theta) = (A) + K \prod_{A_i} I_{A_i}(\theta)$$

Para os exercícios 2 e 3

I) estudar HPD em função f.

Mesma coisa a

$$EV(\{\theta_1 = \theta_2\}, (x, y)) = 1 - P(\theta \in T_{(x,y)} | (x, y))$$

Hipóteses Precisas

$$\Theta \subseteq \mathbb{R}^k$$

$$\lambda_k(\Theta) = 0 \Leftrightarrow \lambda_k(\Theta) > 0.$$

(1) Teste de Jeffreys.

$$A \in \mathcal{F}$$

$$P^*(A) = P_0 P_1(A \cap \Theta_0) + (1-P_0) P_2(A)$$

de modo que

$$P_1(\Theta_0) = 1, \quad P_2(\Theta_0) = 1 \Leftrightarrow P_2(\Theta_0) = 0.$$

(2) F BST.

$$T_{\theta x} = \left\{ \theta \in \Theta : f(\theta/x) > \sup_{\theta \in \Theta} f(\theta/x) \right\}$$

$$EV(\Theta_0, x) = 1 - P(\theta \in T_x | x)$$

Teorema ($\Theta, \mathcal{I} - \Theta$) ~ Dir(a_0, b_0)

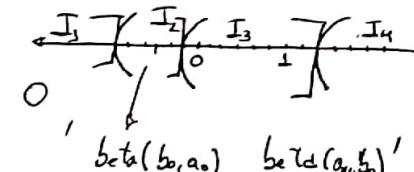
\Leftrightarrow
V $m \geq 3$, V partição (I_1, I_2, \dots, I_m) da Reta.

Prob($X \in [I_1, I_2, \dots, I_m]$) ~ Dirichlet($\alpha_1, \alpha_2, \dots, \alpha_m$),

com $\alpha_i = \sum_k^r \alpha_{ijk}$ sempre que $I_j = [I_1, I_2, \dots, I_{j-1}]$

Prova (ideia)

\Rightarrow V $m \geq 3$, V $I = (I_1, I_2, \dots, I_m)$, i.e., $m=4$



\Rightarrow Para todos valores de m ,
e outras partições,
idem, ...
idem, ...

$$\Leftarrow \Theta = \text{Prob}\{X = 1\} = \text{Prob}\{X \in I_3\} = \text{Hip.} \sim \text{Beta}(\alpha_3, \sum \alpha - \alpha_3)$$

* Supondo, que X tem valores em

$$\mathcal{S} = \{0, 1, 2\} \subset \mathbb{R}^1.$$

$$G \in \mathcal{G} = \left\{ \begin{array}{l} \text{Triangular } (\theta). \quad \theta \in \mathbb{R} \\ \text{Triangular } (\theta). \quad \theta \in \mathbb{R} \end{array} \right\}$$



$$H \leftarrow H = \left\{ \text{Prioris sobre } G = \right\} = \left\{ \text{Prioris sobre } S_3 \right\}$$

$$H_* = \left\{ \text{Dirichlet}(a, b, c) : a > 0, b > 0 \right\} \subset H, \quad c > 0$$

$$\Theta \sim \text{Dir}(a_0, b_0, c_0) \iff H \in \mathcal{H} = \left\{ \text{Dir}_i: a_i, b_i, c_i \right\}$$

$$G \sim \text{Dirichlet}(a_0, b_0, c_0)$$

Suspongo agora que X toma valores em
 $\mathbb{Z} = \{0, 1, 2, \dots\}$

$$G = \left\{ \text{funções de distribuição para } X \right\} =$$

$$G = \left\{ (\theta_0, \theta_1, \theta_2, \dots) : \theta_i \geq 0, \sum_{i=0}^{\infty} \theta_i = 1 \right\} \cong S_\infty$$

$$G \hookrightarrow S_\infty$$

$$H = \left\{ \text{Probabilidades } \sum_{i=0}^{\infty} \theta_i \right\} \supseteq \mathcal{H}$$

A assim definido:

DEF: H^* é tal que:

$\forall m \geq 1, \forall$ ponto $(I_1, I_2, I_3, \dots, I_m)$ da reta,

$$(P(X \in I_1), P(X \in I_2), \dots, P(X \in I_m)) \sim D(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m)$$

$$\text{com } \alpha_j = \sum_{k=1}^r \alpha_{jk} \text{ sempre que } (J_1, I_{j2}, I_{j3}, \dots, I_{jr}) \text{ for}$$

sub-partição de I .

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{tal que } f(x) \geq 0, \text{ ou } \int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^{+\infty} f(x) dx, \quad \int_{J_K} f(x) dx$$

$$\forall J_j \subset \mathbb{R}$$

Fim (mento): 1

~~\mathcal{H}^*~~

$$\mathcal{H}^*$$

\mathcal{H}^* se chama "Processo de Dirichlet".

A função f é um (ímpar) parâmetro do processo Dirichlet.

Notação: $G \sim \text{Processo Dirichlet}(f) = \text{Processo Dirichlet}(f(\cdot))$

H.
H. Rubin (1987)
 $a, b \in \mathbb{C} > 0$.

θ	$\theta \in T_x$	$\theta \notin T_x$
do	$a+b$	a
de	0	c

Uma nova função
Perda.

Decidir por de se:

$$T_x = \left\{ \theta \in \Theta : f(\theta/x) > \inf_{\theta \in \Theta} f(\theta/x) \right\}$$

$$P(d_s, R) < P(d_0, R_x) \Leftrightarrow \int_a dR_x + \int_c dR_x < \int_{T_x}^{a+b} dR_x + \int_{T_x^c}^a dR_x \quad \leftrightarrow$$

$$E_v$$

$$E_v(\theta, x) \leq (a+b)(1 - E_v(\theta, x)) + a E_v(\theta_0, x) \Leftrightarrow$$

$$(c+b) E_v(\theta_0, x) \leq a+b \Leftrightarrow E_v(\theta_0, x) \leq \frac{a+b}{c+b}$$

II, f

\Leftarrow Duas $f \times f$ $\xrightarrow{\text{SPs}}$ Representação de incerteza
por probabilidades. (I)

$R, I \Leftarrow^* R \times R \xrightarrow{\text{SPs}}$ Representações preferências
($D \times D$) por utilidade. (II) \oplus

$(R' \times R')$ $I \rightarrow \Pi \rightarrow$ maximização de
UMA PESSOA (UM AGENTE DECISOR) Utilidade esperada

GRUPO:

Individuos $\{1, 2, \dots, n\}$

$D = \{d_1, \dots, d_m\} \rightarrow$ espacos de scões.

* Como deve proceder o grupo de individuos para descrever as "preferencias" do grupo?

$$\leq_1^*, \leq_2^*, \dots, \leq_n^*$$

→ como definir \leq_g^* a partir de $\leq_1^*, \dots, \leq_n^*$?

→ Algunas proposta para construir \leq_g^*

(1) Ordenação por maioria simples.

$$a, b \in D$$

$$a \leq_g^* b \Leftrightarrow |\{i \in \{1, \dots, n\} : a \leq_i^* b\}| \geq |\{i \in \{1, \dots, n\} : b \leq_i^* a\}|$$

EX: $m=3$, $m=3$

$$d_1 \leq_1^* d_2 \leq_1^* d_3$$

$$d_2 \leq_2^* d_3 \leq_2^* d_1 \quad \left. \right\} d_3 \leq_g^* d_2 \leq_g^* d_1$$

$$d_3 \leq_3^* d_1 \leq_3^* d_2 \quad \left. \right\} d_1 \leq_g^* d_3 \leq_g^* d_2$$

French, S.
- Dissertação de
mestrado: ROSENBERG,
H. Loschi,
(1992)



Ex2:

$$\left. \begin{array}{l} d_1 \leq_1^* d_2 \leq_1^* d_3 \\ d_3 \leq_1^* d_1 \leq_2^* d_2 \\ d_2 \leq_2^* d_3 \leq_3^* d_1 \end{array} \right\} \Rightarrow \begin{array}{l} d_1 \not\leq_g^* d_2 \\ d_2 \not\leq_g^* d_3 \\ d_3 \not\leq_g^* d_1 \end{array}$$

não
há
transitividade

(2) Escolha do "mais preferido"

$$a, b \in D$$

$$a \leq_g^* b \Leftrightarrow \left| \left\{ i \in \{1, \dots, n\} : c \leq_i^* b, \forall c \in D \right\} \right| \geq \left| \left\{ i \in \{1, \dots, n\} : c \leq_i^* a \right\} \right|$$

No EXEMPLOS:

$$d_1 \sim_g^* d_2 \leq_g^* d_3$$

No EXEMPLO 2:

$$d_1 \not\sim_g^* d_2 \sim_g^* d_3$$

$$\leq_i^* \in P(D \times D)$$

Uma transformação de $(P(D \times D))^n$ em $P(D \times D)$ que associa a cada $(\leq_1^*, \dots, \leq_n^*) \in P(D \times D)^n$ uma relação $\leq_g^* \in P(D \times D)$ é chamada uma CONSTITUIÇÃO do GRUPO $\{1, \dots, n\}$

$$\leq_g^*: (P(D \times D))^n \longrightarrow P(D \times D), (\leq_1^*, \dots, \leq_n^*) \mapsto \leq_g^* = \bigcap_{i=1}^n \leq_i^*$$

Pergunta:

Formas de Ben
esta social

Quais seriam "bases" constitutivas para um grupo?

* A.1 - Anou (1953) provou que, sob certas condições, não há constituição que atenda a "desiderata" mínimas.

- Anou considera as seguintes condições "desejadas" para uma constituição:

A.1 - $\leq_1^*, \leq_2^*, \dots, \leq_n^*$ e \leq_g^* sejam completos.

A.2 - $n \geq 2$, $m = 3$.

A.3 - \leq_g^* este definido para todo $(\leq_1^*, \leq_2^*, \dots, \leq_n^*)$
de relações completas em $D \times D$
(UNIVERSALIDADE)

A.4 - $a \leq_g^* b, \forall i = 1, \dots, n \Rightarrow a \leq_g^* b$.
(UNANIMIDADE/OTIMALIDADE DE PARETO)

A.5 - Relações binárias.

A.6 - "Non-Dictatorship"
 $\exists i \in \{1, \dots, m\}$ tal que $\leq_i^* = \leq_i$

TEOREMA

NÃO EXISTE CONSTITUIÇÃO DO GRUPO \leq_g^* ,
a partir de $\leq_1^*, \dots, \leq_n^*$, CONSISTENTE COM
A.1 a A.6.

DEFINIÇÃO: Sejam $a, b \in \mathbb{D}$

Dizemos que $V \subseteq \{1, \dots, n\}$ é decisiva para a
em relação a b , se sempre que $b \leq_i^* a, i \in V$,
 $a \leq_g^* b, i \in V, b \leq_g^* a$.

Nota que $V(a, b \in \mathbb{D}), \{1, 2, \dots, n\}$ é decisiva para a
em relação a b . (Por A.4)

DEFINIÇÃO 2:

V é decisiva mínimal se V é decisiva para
algum a em relação a algum $b, a, b \in \mathbb{D}$,
mas nenhum subconjunto próprio de V é decisiva
para algum a em relação a algum b .

Seja $a, b \in \mathbb{D} \rightarrow$ se V é conjunto decisivo
mínimal para a em relação a b .

Se superjuntos $|V| > 1$. Seja $j \in V$.

Consideremos a seguinte "Relação" em
 $(P(D \times D))$ Seja $c \neq a, c \neq b$. ~~$c \leq_j^* a \wedge c \leq_j^* b$~~

$c \leq_j^* b \leq_j^* a$

$b \leq_i^* a \leq_i^* c, i \in V_0 \setminus \{j\}$

$a \leq_i^* c \leq_i^* b, i \notin V_0$

$b \leq_j^* a \quad (\text{I})$

Como V_0 é decisiva minimal, então $c \leq^* b$
(caso contrário, $V_0 \setminus \{j\} \subsetneq V_0$ seria decisiva
decisiva para \leq em relação a b) (II)

D.e (I) e (II) - (A.S), segue que $c \leq_j^* a$.

Como $c \leq_j^* a, a \leq_i^* c, \forall i \neq j \in \mathbb{N} \Rightarrow c \leq_j^* a$,

Resulta que $\{j\}$ é decisiva para \leq em
relação a \leq . (ABSURDO)

Logo é absurdo supor $|V_0| > 1$.

Portanto, $|V_0| = 1$.

AGRUPAMENTO DE PROBABILIDADE

(Θ, \mathcal{F})

Ind 1: $\rightarrow P_1: \mathcal{F} \rightarrow [0,1]$

Ind 2: $\rightarrow P_2: \mathcal{F} \rightarrow [0,1]$

Ind n: $\rightarrow P_n: \mathcal{F} \rightarrow [0,1]$

$M(\Theta, \mathcal{F})$ (conjunto de medidas de probabilidade
sobre (Θ, \mathcal{F})).

DEF: Um agrupamento de probabilidades ou
opiniões ("opinião puro") é qualquer
transformação de $(N(\Theta, \mathcal{F}))^n$ em $M(\Theta, \mathcal{F})$

$P^{(G)}: (N(\Theta, \mathcal{F}))^n \rightarrow M(\Theta, \mathcal{F})$

$(P_1, \dots, P_n) \mapsto P^{(G)} = P^{(G)}(P_1, \dots, P_n)$

Exemplos:

(1) $P^{(G)}(P_1, \dots, P_n) = P_1, \forall (P_1, \dots, P_n) \in (M(\Theta, \mathcal{F}))^n$.

(2) Agrupamento Linear

$P^{(G)}(P_1, \dots, P_n) = \sum_{i=1}^n d_i P_i, \quad d_i \geq 0, \quad \sum d_i = 1$.