Lista 01 - P.E

Monitoria

2. Por definição

$$P(X_i = k) = (1-p)^{k-1}\, p \quad , \quad k \geq 1$$

Nessas condições, para $k \in \{1, \ldots, n-1\}$,

$$P(X_1 = k / X_1 + X_2 = n) = \frac{P(X_1 = k, X_1 + X_2 = n)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_1 = k, X_2 = n-k)}{P(X_1 + X_2 = n)}$$

$$= \frac{P(X_1 = k)\, P(X_2 = n-k)}{\sum_{i=1}^{n-1} P(X_1 = i)\, P(X_2 = n-i)}$$

$$= \frac{(1-p)^{k-1} p\, (1-p)^{n-k-1} p}{\sum_{i=1}^{n-1} (1-p)^{i-1} p\, (1-p)^{n-i-1} p}$$

$$= \frac{p^2 (1-p)^{n-2}}{\sum_{i=1}^{n-1} p^2 (1-p)^{n-2}} = \frac{1}{n-1}.$$

Logo, $X_1 / X_1 + X_2 \sim \mathcal{U}\{1, 2, \ldots, n-1\}$.

1.

$X \setminus Y$	-1	0	2	
-1	0	$1/6$	$1/12$	$3/12$
0	$1/6$	0	$1/12$	$3/12$
1	$3/12$	$1/12$	$1/6$	$6/12$
	$5/12$	$3/12$	$4/12$	

a)

$Z = 2X + Y$

z	-2	-1	0	1	2	4
$P(Z=z)$	$1/6$	$1/6$	$1/12$	$3/12$	$2/12$	$1/6$

b)

x	-1	0	1
$p_X(x)$	$3/12$	$3/12$	$6/12$

y	-1	0	2
$p_Y(y)$	$5/12$	$3/12$	$4/12$

c) Para $Y=-1$, $E(X/Y=-1) = -1 \cdot 0 + 0 \cdot 1/5 + 1 \cdot 3/5 = 3/5$

Para $Y=0$, $E(X/Y=0) = -1 \cdot 2/3 + 0 \cdot 0 + 1 \cdot 1/3 = -1/3$

Para $Y=2$, $E(X/Y=2) = -1 \cdot 1/4 + 0 \cdot 1/4 + 1 \cdot 1/2 = 1/4$

Logo,

w	$-1/3$	$3/5$	$1/4$
$P(E(X/Y)=w)$	$3/12$	$5/12$	$4/12$

• $E(X \mid Y=y) = \sum_x x\, p_{X|Y}(x/y)$

1.d) Dado $Y = 2$,

$$P(X=-1/Y=2) = \frac{P(X=-1, Y=2)}{P(Y=2)} = \frac{1/12}{4/12} = \frac{1}{4}$$

$$P(X=0/Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{1/12}{4/12} = \frac{1}{4}$$

$$P(X=1/Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{1/6}{4/12} = \frac{1}{2}$$

x	-1	0	1
$P_{X/Y}(X=x/Y=2)$	$1/4$	$1/4$	$1/2$

e) Por def., a var. cond. de X dado $Y=y$ é dada por

$$Var(X/Y=y) = E((X - E[X/Y=y])^2/Y=y)$$

$$= E(X^2/Y=y) - (E(X/Y=y))^2$$

Temos

$$E(X^2/Y=-1) = 1.0 + 0.\frac{2}{5} + 1.\frac{3}{5} = \frac{3}{5}$$

$$E(X^2/Y=0) = 1.\frac{2}{3} + 0.0 + 1.\frac{1}{3} = 1$$

$$E(X^2/Y=2) = 1.\frac{1}{4} + 0.\frac{1}{4} + 1.\frac{1}{2} = \frac{3}{4}$$

Daí,

$$Var(X/Y=-1) = 3/5 - (3/5)^2 = \frac{3}{5}\left(1-\frac{3}{5}\right) = \frac{6}{25}$$

$$Var(X/Y=0) = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$Var(X/Y=2) = \frac{3}{4} - \left(\frac{1}{4}\right)^2 = \frac{1}{4}\left(3 - \frac{1}{4}\right) = \frac{11}{16}$$

logo

w	$6/25$	$11/16$	$8/9$
$P(Var(X/Y)=w)$	$5/12$	$4/12$	$3/12$

3. Por definição

$$E(X/Y=2) = \sum_{k=1}^{2} k\, P(X=k / Y=2)$$

$$= \sum_{k=1}^{2} k\, \frac{P(X=k, Y=2)}{P(Y=2)}$$

$$= 1.\frac{P(X=1,Y=2)}{P(Y=2)} + 2.\frac{P(X=2,Y=2)}{P(Y=2)}$$

Mas, $P(X=1,Y=2) = \sum_{l=1}^{2} P(X=1, Y=2, Z=l) = P(X=1,Y=2,Z=1) + P(X=1,Y=2,Z=2) = \frac{1}{16} + 0 = \frac{1}{16}$ $= 0 + 1/4 + 1/4$

e $P(Y=2) = \sum_{k=1}^{2}\sum_{l=1}^{2} P(X=k, Y=2, Z=l) = \frac{1}{16} + 0 + 0 + \frac{1}{4} = \frac{5}{16}$. Logo

$$E(X/Y=2) = 1.\frac{1/16}{5/16} + 2.\frac{1/4}{5/16} = \frac{1}{5} + \frac{8}{5} = \frac{9}{5}$$

Agora,

$$E(X/Y=2, Z=1) = \sum_{k=1}^{2} k\, P(X=k / Y=2, Z=1) = \sum_{k=1}^{2} k\, \frac{P(X=k, Y=2, Z=1)}{P(Y=2, Z=1)}$$

Como $P(X=1, Y=2, Z=1) = 1/16$, $P(X=2, Y=2, Z=1) = 0$ e $P(Y=2, Z=1) = \sum_{k=1}^{2} P(X=k, Y=2, Z=1) = 1/16$

segue que

$$E(X/Y=2, Z=1) = 1.\frac{1/16}{1/16} + 2 \times \frac{0}{1/16} = 1.$$

4) $\forall y$,

$$E(X/Y=y) \overset{\text{def}}{=} \sum_x x\, P(X=x / Y=y) \overset{\text{def}}{=} \sum_x x \frac{P(X=x, Y=y)}{P(Y=y)} \overset{\text{ind.}}{=} \sum_x x \frac{P(X=x) P(Y=y)}{P(Y=y)}$$

$$= \sum_x x\, P(X=x) \underset{\text{def}}{=} EX$$

5)

a) Primeiro, note que após n passos teremos $r+b+n$ bolas na urna, dos quais X_n serão vermelhas. Na etapa $n+1$, teremos

$$X_{n+1}+1 \quad \text{com probabilidade} \quad X_n / r+b+n$$

$$X_n \qquad X_{n-1} \quad \text{com probabilidade} \quad \frac{r+b+n-X_n}{r+b+n}$$

Daí,

$$E(X_{n+1}/X_n) = (X_n+1)\frac{X_n}{r+b+n} + \frac{X_n\,(r+b+n-X_n)}{r+b+n}$$

Assim,

$$E(X_{n+1}) = E(E(X_{n+1}/X_n)) = \frac{E(X_n^2) + E(X_n) + (r+b+n)E(X_n) - E(X_n^2)}{r+b+n}$$

$$= \frac{(1+r+b+n)}{r+b+n} EX_n$$

$$= \frac{(1+r+b+n)}{r+b+n}\,\frac{(1+r+b+n-1)}{r+b+n-1} EX_{n-1}$$

$$= \frac{(1+r+b+n)}{r+b+n-1}\,\frac{(1+r+b+n-2)}{r+b+n-2} EX_{n-2}$$

$$= \frac{(1+r+b+n)}{r+b+} EX_0 = \frac{(1+r+b+n)}{r+b} r$$

$$EX_n = \frac{r}{r+b}(r+b+n) = r + n\frac{r}{r+b}$$

Lista 2

Para y t.q $f_Y(y) > 0$,

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f(y)},$$

$$E[X/Y=y] = \int_{-\infty}^{+\infty} x\, f_{X/Y}(x/y)\,dx$$

Ex X e Y ind.

$$P(X<Y) = \int_{-\infty}^{+\infty} P(X<Y/Y=y)\, f_Y(y)\,dy = \int_{-\infty}^{+\infty} P(X<y/Y=y)\, f_Y(y)\,dy$$

$$= \int_{-\infty}^{+\infty} P(X<y) f_Y(y)\,dy = \int_{-\infty}^{+\infty} F_X(y)\, f_Y(y)\,dy$$

Ex

$$P(X+Y<z) = \int_{-\infty}^{+\infty} P(X+Y<z/Y=y)\, f_Y(y)\,dy = \int_{-\infty}^{+\infty} P(X<z-y/Y=y)\, f_Y(y)\,dy$$

$$= \int_{-\infty}^{+\infty} P(X<z-y)\, f_Y(y)\,dy = \int_{-\infty}^{+\infty} F_X(z-y)\, f_Y(y)\,dy$$

1. $f(x,y) = \dfrac{e^{-x/y}\, e^{-y}}{y}$, $0<x<\infty$, $0<y<\infty$. Mostre que $E[X/Y=y] = y$.

Temos

$$f_Y(y) = \int_{-\infty}^{+\infty} \frac{e^{-x/y}\, e^{-y}}{y}\,dx = e^{-y} \underbrace{\int_0^{+\infty} \frac{1}{y} e^{-\frac{1}{y}x}\,dx}_{1} = e^{-y}$$

Assim,

$$f_{X/Y}(x/y) = \frac{1}{y} e^{-x/y}, \quad x \in \mathbb{R}_+, \text{ i.e. } X/Y=y \sim Exp(1/y)$$

Daí, $E(X/Y=y) = y$.

3. $f(x,y) = \dfrac{(y^2-x^2)}{8} e^{-y}$, $0<y<\infty$, $-y \leq x \leq y$. Mostrar que $E[X/Y=y] = 0$.

$$f_Y(y) = \int_{-y}^{y} \frac{y^2 e^{-y}}{8} - \frac{x^2 e^{-y}}{8}\,dx = \frac{y^2 e^{-y}}{\cancel{4}} \cancel{y} - \frac{e^{-y}}{\cancel{4}} \frac{\cancel{2} y^3}{3} = \frac{\cancel{4}\, y^3 e^{-y}}{6}$$

Assim

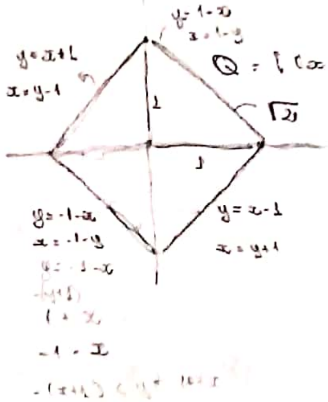
$$f_{X|Y}(x/y) = \frac{(y^2-x^2)e^{-y}/8}{y^3 e^{-y}/6} = \frac{6(y^2-x^2)}{8\,y^3} = \frac{3y^2-3x^2}{4y^3}$$

Daí

$$E[X/Y=y] = \int_{-y}^{+y} x\left(\frac{3y^2-3x^2}{4y^3}\right)dx = \frac{3y^2}{4y^3}\left(\frac{y^2}{2}-\frac{y^2}{2}\right) - \frac{3}{4y^3}\left(\frac{y^4}{4}-\frac{y^4}{4}\right) = 0$$

4.

Como a escolha é ao acaso, temos



$Q = \{(x,y) : |x|+|y| \leq 1\}$

$$f_{X,Y}(x,y) = \frac{1}{2}, \quad (x,y)\in Q$$

Ademais,

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy$$

Para $-1 \leq x \leq 0$, $-(x+1) \leq y \leq x+1$

$$f_X(x) = \int_{-(x+1)}^{(x+1)} \frac{1}{2}\,dy = \frac{2(x+1)}{2} = x+1,$$

Para $0 < x \leq 1$, $x-1 \leq y \leq -x+1$

$$f_X(x) = \int_{x-1}^{1-x} \frac{1}{2}\,dy = 1-x, \quad 0 < x \leq 1$$

isto é,

$$f_X(x) = \begin{cases} 1+x, & -1 \leq x \leq 0 \\ 1-x, & 0 < x \leq 1 \end{cases}$$

De modo análogo, para $-1 \leq y \leq 0$, $-(1+y) \leq x \leq y+1$ e

$$f_Y(y) = \int_{-(1+y)}^{1+y} \frac{1}{2}\,dy = 1+y, \quad -1 \leq y \leq 0$$

e, para $0 < y \leq 1$, $y-1 \leq x \leq 1-y$.

$$f_Y(y) = \int_{y-1}^{1-y} \frac{1}{2}\,dy = 1-y, \quad 0 < y \leq 1$$

isto é,

$$f_Y(y) = \begin{cases} 1+y, & -1 \leq y \leq 0; \\ 1-y, & 0 < y \leq 1 \end{cases}$$

Como para $(x,y) \in (-1,0)^2$,

$f(x,y) \neq (1+x)(1+y)$

Segue que X e Y não são independentes.

Lista 2

b) Para $y \in (-1, 0]$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)} = \frac{1/2}{1+y} = \frac{1}{2(1+y)}, \quad -1 \leq x \leq 0.$$

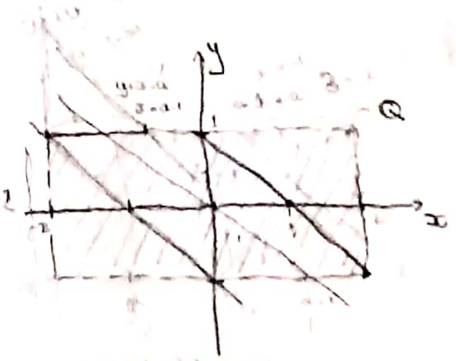
Para $y \in [0, 1)$,

$$f_{X|Y}(x|y) = \frac{1/2}{1-y} = \frac{1}{2(1-y)}, \quad -1 \leq x \leq 0.$$

i.e.

$$f_{X|Y}(x|y) = \begin{cases} 1/(2+2y), & -1 \leq y \leq 0, \\ 1/(2-2y), & 0 \leq y \leq 1. \end{cases}$$

8.



a) Temos que

$$f(x,y) = \frac{1}{8}, \quad (x,y) \in Q$$

$$f_Y(y) = \int_{-2}^{2} \frac{1}{8}\, dx = \frac{1}{2}, \quad -1 \leq y \leq 1$$

Daí,

$$f_{X|Y}(x|y) = \frac{1/8}{1/2} = \frac{1}{4}, \quad -2 \leq x \leq 2. \quad \text{Em part. vale p/ } y = 1/2.$$

b) Note que para cada $y \in (-1, 1)$, $X|Y=y \sim \mathcal{U}(-2, 2)$, de modo que

$$E(X|Y=y) = 0.$$

Daí, $E(X|Y) = 0$ com prob., $\forall y \in (-1,1)$

c) Calculo a média condicional $E[X | X+Y=a]$

Por def, $E[X|X+Y=a] = \int_{-2}^{2} x\, f_{X|X+Y=a}(x)\, dx$

Mas

$$f_{X|X+Y=a}(x) = \frac{f_{X,X+Y}(x,a)}{f_{X+Y}(a)}$$

Agora, note para

• $-3 \le a \le -1$,

$$P(X+Y \le a) = \frac{(a+3)^2}{16} = \frac{a^2+6a+9}{16}$$

• $-1 \le a \le 1$

$$P(X+Y \le a) = \frac{(a+3)^2}{16} - \frac{(a+1)^2}{16} = \frac{a^2+6a+9-a^2-2a-1}{16} = \frac{4a+8}{16} = \frac{a}{4}+\frac{1}{2}$$

• $1 \le a \le 3$

$$P(X+Y \le a) = 1 - \frac{(3-a)^2}{16} = \frac{16-(9-6a+a^2)}{16} = \frac{-a^2+6a+7}{16} = \frac{7}{16} + \frac{a(6-a)}{16}$$

Daí,

$$f_{X+Y}(a) = \begin{cases} \frac{2a+6}{16}, & -3 \le a \le -1 \\ \frac{1}{4}, & -1 \le a \le 1 \\ \frac{6-2a}{16}, & 1 \le a \le 3 \end{cases}$$

Ademais

$$f_{X,X+Y}(x,a) = f_{X,Y}(x,a-x) = 1/8$$

$(x, a-x) \in Q$

$P(X+Y=a, X=x) = P(X+Y=a \mid X=x)P(Y=x)$

$Y=a-x$

$= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8}, \quad x \in (-2,2),\ a-x \in (-1,1)$

$a \in (-3,3)$

Daí, para

• $a \in (-3,-1)$

$$f_{X|X+Y=a}(x) = \frac{1/8}{2a+6/16} = \frac{2}{2a+6}$$

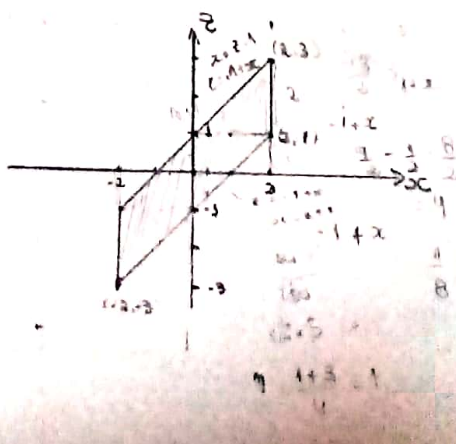
• $a \in [-1,1]$

$$f_{X|Z=3}(x) = \frac{1/8}{1/4} = \frac{1}{2}$$

• $a \in [1,3]$

$$f_{X|Z=3}(x) = \frac{2}{6-2a}$$

$X \sim U(-2,2)$, $Y \sim U(-1,1)$



Logo,

$$E(X/Z=z) = \int_{z-1}^{1+z} x\, f_{X|Z}(x/z)\,dx = \frac{1}{8f_Z(z)} \left.\frac{x^2}{2}\right|_{z-1}^{1+z}$$

$$= \frac{1}{8f_Z(z)} \frac{[1+2z+z^2-(1-2z+z^2)]}{2}$$

$-1-z < -x < 1+z$

$1+z > x > z-1$

$$= \frac{z}{4f_Z(z)} \quad \text{se} \quad -1 \leq z \leq 1$$

$$= \begin{cases} & , -3 \leq z \leq -1 \\ \frac{z/4}{1/4} & , -1 < z \leq 1 \\ & , 1 < z < 3 \end{cases}$$

$$= \begin{cases} \frac{z-1}{2} & , -3 \leq z \leq -1, \\ z & , -1 < z \leq 1, \\ \frac{z+1}{2} & , 1 < z < 3. \end{cases}$$

$f_{X|Z=a}(x/a)$ $f(x/a-x)$

$x+y=a$ $= \frac{f(x,a-x)}{f(a-x)}$

$y=a-x$

Se $-3 \leq z \leq -1$

$$E(Y/Z=z) = \int_{-2}^{1+z} x\, f_{X|Z}(x/z)\,dx = \frac{1}{8f_Z(z)} \left.\frac{x^2}{2}\right|_{-2}^{1+z} = \frac{1}{2\times 8f_Z(z)}[1+2z+z^2-4]$$

$z^2+2z-3 \;|\; z+3$

$-z^2-3z \quad z-1$

$-z$

$+z+3$

$$= \frac{z^2+2z+3}{2\times 8\left(\frac{z+3}{8}\right)} = \frac{z^2+2z+3}{2(z+3)} = z-1$$

$z^2-2z-3 \;|\; z-3$

$-z^2+3z \quad z+1$

z

Se $1 \leq z < 3$

$$E(Y/Z=z) = \frac{1}{2\times 8f_Z(z)} \left. x^2 \right|_{z-1}^{2} = \frac{4-z^2+2z-1}{2(3-z)} = \frac{z^2-2z-3}{2(z-3)}$$

$=$

Lista 3

Exercício 2. Seja R a v.a. representando o tempo de vida do rádio. Do enunciado,

$$R \sim Exp(1/10)$$

A probabilidade de interesse é dada por

$$P(R \geq 20 / R \geq 10) = P(R \geq 10) = e^{-\frac{1}{10}\cdot 10} = e^{-1}$$

em que na segunda passagem usamos a propriedade de perda de memória da exponencial, isto é, dados $s>0$ e $t>0$,

$$P(R \geq s+t / R \geq t) = \frac{P(R \geq s+t, R \geq t)}{P(R \geq t)} = \frac{P(R \geq s+t)}{P(R \geq t)} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = P(R \geq s)$$

No nosso caso, $t = 10$ e $s = 10$.

Exercício 3. Sejam

T_1: v.a. rep. o tempo de vida do microfone

T_2: " DVD

T: " aparelho

Pelo enunciado, $T_1 \sim Exp(1/500)$, $T_2 \sim Exp(1/1000)$. Assim,

$$T = \min(T_1, T_2) \sim Exp\left(\frac{1}{500} + \frac{1}{1000}\right)$$

A probabilidade de interesse é dada por

$$P(T = T_2) = P(T_2 \leq T_1) = \int_0^{+\infty}\int_0^{t_1} f_{T_1,T_2}(t_1,t_2)\, dt_2\, dt_1$$

$$= \int_0^{+\infty}\int_0^{t_1} f_{T_1}(t_1) f_{T_2}(t_2)\, dt_2\, dt_1 = \int_0^{+\infty} f_{T_1}(t_1)\left(1 - e^{-\frac{t_1}{1000}}\right) dt_1$$

$$= 1 - \int_0^{+\infty} \lambda_1 e^{-(\lambda_1+\lambda_2)t_1}\, dt_1 = \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{1/1000}{3/1000} = 1/3$$

Exercício 4.

a)

$$P(\max\{X_1, X_2, X_3\} \leq x) = P(X_1 \leq x, X_2 \leq x, X_3 \leq x)$$
$$\overset{ind}{=} \prod_{i=1}^{3} P(X_i \leq x)$$
$$= \prod_{i=1}^{3} [1 - e^{-\lambda_i x}]$$
$$= (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x})(1 - e^{-\lambda_3 x})$$

b) $$P(\min\{X_1, X_2, X_3\} \leq x) = 1 - P(\min\{X_1, X_2, X_3\} \geq x)$$
$$= 1 - P(X_1 \geq x, X_2 \geq x, X_3 \geq x)$$
$$= 1 - \prod_{i=1}^{3} e^{-\lambda_i x}$$
$$= 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3) x},$$

donde segue que $\min\{X_1, X_2, X_3\} \sim Exp(\lambda_1 + \lambda_2 + \lambda_3)$.

c) Do item a)

$$P(\max\{X_1, X_2, X_3\} \leq x) = (1 - e^{-\lambda_2 x} - e^{-\lambda_1 x} + e^{-(\lambda_1 + \lambda_2) x})(1 - e^{-\lambda_3 x})$$
$$= (1 - e^{-\lambda_2 x} - e^{-\lambda_1 x} + e^{-(\lambda_1+\lambda_2)x} - e^{-\lambda_3 x} + e^{-(\lambda_2+\lambda_3)x} + e^{-(\lambda_1+\lambda_3)x} - e^{-(\lambda_1+\lambda_2+\lambda_3)x})$$

Daí,

$$f_{\max\{X_1,\dots,X_3\}}(x) = \left(\sum_{i=1}^{3} \lambda_i e^{-\lambda_i x}\right) - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)x} - (\lambda_1+\lambda_3) e^{-(\lambda_1+\lambda_3)x} - (\lambda_2+\lambda_3) e^{-(\lambda_2+\lambda_3)x} + (\lambda_1+\lambda_2+\lambda_3) e^{-(\lambda_1+\lambda_2+\lambda_3)x}$$

Assim, é fácil ver que

$$E(\max\{X_1, X_2, X_3\}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1+\lambda_2} - \frac{1}{\lambda_1+\lambda_3} - \frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_1+\lambda_2+\lambda_3} \blacksquare$$

Lista 3

Exercício 5. Seja T_i a v.a. rep o tempo de vida do componente i, $i=1,2,\dots,n$. Queremos determinar

$$P(T_1 < \min(T_2,\dots,T_n))$$

Pelo Exercício 4.b), sabemos que

$$\min(T_2,\dots,T_n) \sim Exp(\lambda_2+\dots+\lambda_n)$$

Daí, pelos mesmos arg. do Ex. 3, segue que

$$P(T_1 < \min(T_2,\dots,T_n)) = \frac{\lambda_1}{\lambda_1+(\lambda_2+\dots+\lambda_n)} = \frac{\lambda_1}{\lambda_1+\dots+\lambda_n} \quad \blacksquare$$

Lista 04

2. Os instantes $t_1 < t_2$ são dois números reais. Em particular,

$$N_2(t_i) = N_1(t_i) + 1 \quad , i = 1, 2$$

Queremos determinar a dist. de $Y = N_2(t_2) - N_2(t_1)$. Mas $N_2(t_2) \sim Poi(\lambda_2 t_2)$ e $N_2(t_1) \sim Poisson(\lambda_2 t_1)$. Daí, pelo Ex. 1, $Y \sim Poisson(\lambda_2(t_2 - t_1))$. Outra forma segue ao perceber que Y indica o nº de ocorrências em (t_1, t_2), o que pela perda de memória do PPP uma $Poisson(\lambda(t_2 - t_1))$

Na verdade, t_1 pode ser fixado e $t_2 = t_1 + X$, $X \sim Exp(\lambda_2)$. Condicionando, segue $Y \sim Geo(p_2)$.

3. Sejam $t_1 < t_2$ os dois instantes genéricos entre duas chamadas ñ comerciais $\frac{\lambda_2}{\lambda_1}$

Como as comercios chegam conf. um processo de Poisson com taxa λ_1 indep. do Processo para as ñ comercias, a prob. de que entre t_1 e t_2 ocorram 5 chamadas comerciais é dado por

$$P(N_1(t_2) - N_1(t_1) = 5) = \frac{e^{-\lambda_1(t_2 - t_1)} (\lambda_1(t_2 - t_1))^5}{5!}$$

em que $N(t)$ é um proc. de Poisson com intensidade λ_1.

Na vdd, a intenção era tomar T_1 e T_2 como aleatório. Daí, surge a geométrica.

4. Seja $(N'(t))_{t \geq 0}$ o processo que conta o número de carros entrevistados. Note que

$$\{N'(t) \geq k\} = \{N(t) \geq 15k\} \quad , k \in \mathbb{N}$$

em que $(N(t))_{t \geq 0}$ é o PPP(λ) que conta as passagens dos carros. Assim

$$P(N'(t) = k) = P(N(t) \geq 15k) - P(N(t) \geq 15(k+1))$$

$$= \sum_{m=15k}^{+\infty} P(N(t) = m) - \sum_{m=15k+15}^{+\infty} P(N(t) = m)$$

$$= \sum_{m=15k}^{15k+14} \frac{e^{-\lambda t} (\lambda t)^m}{m!} = e^{-\lambda t} \sum_{l=0}^{14} \frac{(\lambda t)^{15k+l}}{(15k+l)!}$$

$$= e^{-\lambda t} (\lambda t)^{15k} \sum_{l=0}^{14} (\lambda t)^l / (15k+l)!$$

6. Sabemos que $N(1) \sim$ ...

Queremos determinar $E(N(T))$ e $Var(N(T))$. Mas

$$0 \qquad 9 \qquad 11$$

$$N(9) + N(T)$$

$$E(N(T)) = E(\underbrace{E(N(T)\mid T)}_{N(T)\mid T \sim Poisson\left(\frac{1}{3}\cdot T\right)}) = E(3T) = E3T = 30.$$

$$\lambda = \frac{1}{\mu} = \frac{1}{3}$$

$N(1) \sim Pois(\lambda)$

$E(N(1)) = \lambda = 3.$

e

$$Var(N(T)) = E(Var(N(T)\mid T)) + Var(E(N(T)\mid T))$$

$$= E(3T) + Var(3T)$$

$$= 30 + 9\, Var(T) = 30 + 9\, \frac{(11-9)^2}{12} = 30 + 3 = 33$$

12. É dado por

$$P(N_\mu(0, 1/4) = 0) = e^{-25 \times \frac{1}{4} \cdot \frac{1}{5}} = e^{-5/4}.$$

02 e 03

Pode-se fixar t_1, uma vez que a prob. ind. do mesmo.

$$P(N(t_2) - N(t_1) = m) = E(P(N(t_1 + X) - N(t_1) = m) / X = s)$$

$$= \int_0^{+\infty} P(N(t_1 + s) - N(t_1) = m)\, \lambda_1 e^{-\lambda_1 s}\, ds$$

$$= \int_0^{+\infty} \frac{(\lambda_2 s)^m e^{-\lambda_2 s}}{m!}\, \lambda_1 e^{-\lambda_1 s}\, ds$$

$$= \frac{\lambda_2^m \lambda_1}{m!\,\lambda} \int_0^{+\infty} s^m \lambda e^{-\lambda s}\, ds = \frac{\lambda_2^m \lambda_1}{m!\,\lambda} \cdot \frac{m!}{\lambda^m}$$

$$= \left(\frac{\lambda_2}{\lambda}\right)^m \left(\frac{\lambda_1}{\lambda}\right)$$

Lista 5

1. Como o número médio de eventos do tipo I até o tempo t é tal que

$$\lambda m_I(t) = \lambda \int_0^t e^{-2s}\, ds = \frac{\lambda}{2}\left[1 - e^{-2t}\right] \to \frac{\lambda}{2}$$

quando $t \geq +\infty$, segue que o número de eventos do tipo I ocorridos em $[0, \infty)$ não será finito

b) $N_I(\infty) \overset{D}{:=} \lim_{t \to \infty} N_I(t) \sim \text{Poisson}(\lambda/2)$

c) $\lambda/2$.

2. a) Seja Y a v.a. rep. o nº de emails pesados chegados até t.

$$P(Y = k \mid N(t) = n) = \binom{n}{k} e^{-k} (1 - e^{-1})^{n-k}, \quad k = 0, \dots, n$$

$P(N_1(t) = k \mid N(t) = n) = P(N_1(t) = k, N_2(t) = n-k) / P(N(t) = k)$

b) A prob. de ser classificado como pesado no inst. t é dado por

$$p(t) = P(X_t \geq 1) = e^{-1}, \quad \text{para todo } t \geq 0$$

Daí,

$$(N_p(t), t \geq 0) \sim PPP(\lambda e^{-1})$$

Assim

$$N_p(t+s) - N_p(t) \sim \text{Pois}(\lambda e^{-1} s)$$

c) Pelas propriedades de incrementos indep. do PPP, a resposta sendo dada por b)

3)

$$E(Y) = E(E(Y \mid U_1))$$
$$= E(Y \mid U_1 = 1)P(U_1 = 1) + E(Y \mid U_1 = 2)P(U_1 = 2)$$
$$= E(X_1)\frac{1}{2} + E(X_2)\frac{1}{2}$$
$$= \frac{1}{\lambda}\cdot\frac{1}{2} + \frac{1}{\lambda}\cdot\frac{1}{2} = \frac{1}{\lambda}$$

4)

a) $E(S_4) = \dfrac{4}{\lambda}$

b) Note que

$$S_{(4)} \mid N(1) = 2 \;=\; 1 + S'_{(2)}$$

Assim,

$$E(S_{(4)} \mid N(1)) = 1 + \frac{2}{\lambda}$$

c) $E(N(4) - N(2) \mid N(1) = 3)$
$\overset{ind}{=} E(N(4) - N(2)) = \lambda . 2$

d) $E(S_1 \mid N(2) = 1)$
$= 1$, uma vez que $S_1 / N(2) = 1 \sim \mathcal{U}(0,2)$

e) $E(S_1 \mid N(3) = k)$, $k \geq 2$

Nesse caso,

$S_1 / N(3) = k \overset{d}{=} \min\{U_1, \dots, U_k\}$ $\{U_i\}_{i=1}^{k}$ unif em $(0,3)$ ind

Logo

$$P(S_1 \geq x \mid N(3) = k) = P(U_i \geq x,\ i = 1, \dots k)$$
$$= \left(\frac{3 - x}{3}\right)^k, \quad 0 \leq x \leq 3 \ ;\ 1 \text{ se } x \leq 0 \text{ e } 0 \text{ c.c.}$$

Daí,

$$E(S_1 / N(3) = k) = \int_0^3 \frac{(3 - x)^k}{3^k}\, dx = \int_0^3 \frac{u^k}{3^k}\, du = \left.\frac{u^{k+1}}{3^k (k+1)}\right|_0^3 = \frac{3}{k+1}$$

8) $P(N(s) = m \mid N(t) = n) = \binom{n}{m}\left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$, $m = 0, \dots, n$
$s \leq t$

Lista 06

3. Seja $M(t)$ o processo de saídas. De fato,

1. $M(0)=0$ (Da relação $M(t)\leq N(t)$, segue $0\leq M(0)\leq N(0)=0$)
2. $\{M(t): t\geq 0\}$ tem incrementos independentes

Vamos mostrar que $M(t+s)-M(t)$ é ind. de $M(t)$, $t, s \in \mathbb{R}_+$. Para tal, classifique cada cliente que chega como do tipo I se ele sai em $[0,t]$ e do tipo II se sai em $(t, t+s]$, de modo que $N(t)=N_I(t)+N_{II}(t)$, N_I e N_{II} indep por Teo já visto em aula.

Agora, basta notar que

$$M(t) = N_I(t) = N_I(t+s) \text{ e } M(t+s)-M(t) = N_{II}(t+s), \quad \square$$

3. $\mathbb{P}(M(t,t+h)=1) = \lambda G(t)h + o(h)$

Cada cliente que chega em $s\in(0,t+h)$ será classificado como do tipo II com probabilidade $G(t+h-s)-G(t-s)$ e $M(t,t+h)$ será uma Poisson com média

$$\lambda\int_0^{t+h} G(t+h-s)-G(t-s)\,ds = \lambda\int_0^{t+h}[G'(t-s)h+o(h)]\,ds$$

$$= \lambda h\int_0^t G'(y)dy + o(h)$$

$$= \lambda h G(t) + o(h)$$

Daí,

$$\mathbb{P}(M(t,t+h)=1) = (\lambda h G(t)+o(h))\,e^{-\lambda h G(t)-o(h)} = \lambda h G(t)+o(h)$$

4. $\mathbb{P}(M(t,t+h)\geq 2) = o(h)$ (Segue do item anterior.

5. Sabemos que

$$N(t) \sim \text{Poisson}\,(t^2 + 2t)\,, \quad t \geq 0$$

Assim, $\{N(t) : t \geq 0\}$ é um PP não homogêneo com intensidade $2t+2$.
O número de eventos em $[4,5]$ é uma Poisson com média

$$\int_4^5 2t + 2\, dt = t^2 + 2t \Big|_4^5 = 25 + 10 - 16 - 8 = 11$$

Logo,

$$P(N(4,5) = n) = \frac{(11)^n e^{-11}}{n!}\,, \quad n \in \mathbb{N}.$$

13. Basta ver que

$$X / N(t) = \sum_{k=1}^{N(t)} (t - U_k),$$

com a conv. $\sum_{k=1}^{0} = 0$ e $U_1, U_2, \ldots$ são v.a unif. ind em $(0,t)$

Assim,

a) $$E(X/N(t)) = E\left(\sum_{k=1}^{N(t)} (t - U_k)\right) = N(t)\,(t - t/2)$$
$$= N(t)\,\frac{t}{2}$$

b) $$\text{Var}(X/N(t)) = \sum_{k=1}^{N(t)} \text{Var}(t - U_k) = N(t)\,\frac{t^2}{12}$$

c) $$\text{Var}(X) = E(\text{Var}(X/N(t))) + \text{Var}(E(X/N(t)))$$
$$= E\left(\frac{N(t)\,t^2}{12}\right) + \text{Var}\left(\frac{N(t)\,t}{2}\right)$$
$$= \frac{t^2}{12}\lambda t + \frac{t^2}{4}\lambda t = \frac{4}{12}\lambda t^3 = \frac{\lambda t^3}{3}$$

Lista 06

14 Uma Poisson com média

$$(10-8)\times 4 + (12-10)\times 8 + (17-12)\times 6 = 8+16+30 = 54 \text{ clientes}$$

17. Sabemos que $N(t) \sim PPP(2,5)$. Nas condições do enunciado, o tempo de vida de um indivíduo é dado por S_{196}. Como

$$S_{196} = \sum_{k=1}^{196} T_k \,, \quad (T_k : k \geq 1) \text{ exponenc. ind. com taxa } 2,5 \text{ plano}$$

segue que

a) $E(S_{196}) = 196 \cdot \dfrac{1}{2,5} = 78,4$ anos

b) $Var(S_{196}) = \dfrac{196}{6,25} = 31,36$ anos^2

Lista 07

3.5, 6, 7, 8

3 É óbvio que vale para $n=1$. Suponha que vale para k. Nessas condições,

$$P^{(k+1)} = P^{(k)} P = \begin{vmatrix} 1/2 + 1/2(2p-1)^k & 1/2 - 1/2(2p-1)^k \\ 1/2 - 1/2(2p-1)^k & 1/2 + 1/2(2p-1)^k \end{vmatrix} \begin{vmatrix} p & 1-p \\ 1-p & p \end{vmatrix}$$

de modo que,

$$(P^{(k+1)})_{1,1} = \cancel{\frac{p}{2}} + \frac{p}{2}(2p-1)^k + \frac{1}{2} - \cancel{\frac{p}{2}} - \frac{1}{2}(2p-1)^k + \frac{p}{2}(2p-1)^k$$

$$= \frac{1}{2} + \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1},$$

$$(P^{(k+1)})_{1,2} = \frac{1}{2} + \frac{1}{2}(2p-1)^k - \cancel{\frac{p}{2}} - \frac{p}{2}(2p-1)^k + \cancel{\frac{p}{2}} - \frac{p}{2}(2p-1)^k,$$

$$= \frac{1}{2} - \frac{1}{2}(2p-1)^k(2p-1) = \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1}$$

$$(P^{(k+1)})_{2,1} = (P^{(k+1)})_{1,2} = \frac{1}{2} - \frac{1}{2}(2p-1)^{k+1}$$

$$(P^{(k+1)})_{2,2} = (P^{(k+1)})_{1,1} = \frac{1}{2} + \frac{1}{2}(2p-1)^{k+1}$$

Logo, pelo princípio da indução, a relação

$$P^{(n)} = \begin{vmatrix} 1/2 + 1/2(2p-1)^n & 1/2 - 1/2(2p-1)^n \\ 1/2 - 1/2(2p-1)^n & 1/2 + 1/2(2p-1)^n \end{vmatrix}$$

vale para todo natural n.

5.

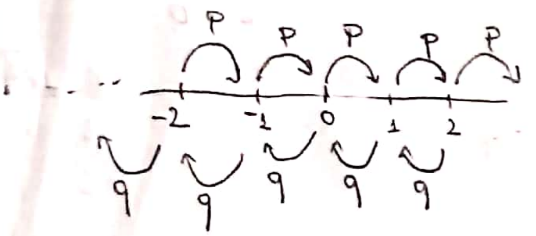
a) Como $\eta_0 = \xi_0$, segue que $(\eta_n : n \geq 0)$ é uma C.M. com medida inicial

$$\alpha(\cdot) = \mathbb{P}(\xi_0 = \cdot)$$

e, para $i_0, \dots, i_n \in \mathbb{N}$, com prob. de trans dadas por

$$\mathbb{P}(\eta_n = i_n / \eta_{n-1} = i_{n-1}, \dots, \eta_0 = i_0) = \mathbb{P}(\eta_{n-1} + \varepsilon_n = i_n / \eta_{n-1} = i_n)$$

$$= \mathbb{P}(\varepsilon_n = i_n - i_{n-1})$$



$$= \begin{cases} 0, & \text{se } |i_n - i_{n-1}| > 1 \text{ ou } i_n = i_{n-1} \\ p, & \text{se } i_n = i_{n-1} + 1 \\ q, & \text{se } i_n = i_{n-1} - 1 \end{cases}$$

b) Dist. inicial: a dist. de ξ_0.

Para $n \geq 1$,

$$\mathbb{P}(\eta_n = i_n / \eta_{n-1} = i_{n-1}, \dots, \eta_0 = i_0) = \mathbb{P}(\max\{\eta_{n-1}, \varepsilon_n\} = i_n / \eta_{n-1} = i_{n-1})$$

$$= \mathbb{P}(\max\{i_{n-1}, \varepsilon_n\} = i_n)$$

$$P = \begin{array}{c|cc|} & -1 & 1 \\ -1 & q & p \\ 1 & 0 & 1 \end{array}$$

$$= \begin{cases} 1 & i_n = 1, \; i_{n-1} = 1 \\ 0 & i_n = -1, \; i_{n-1} = 1 \\ p & i_n = 1, \; i_{n-1} = -1 \\ q & i_n = -1, \; i_{n-1} = -1 \end{cases}$$

Lista 07

c) (η_n) assume valores em $S = \{-1, 1\}$.

Para $i_0, i_1, \dots, i_n \in S$,

$$P(\eta_n = i_n / \eta_{n-1} = i_{n-1}, \dots, \eta_0 = i_0) =$$

$$= P(\eta_{n-1} \times \xi_n = i_n / \eta_{n-1} = i_{n-1}, \dots, \eta_0 = i_0)$$

$$= P(\eta_{n-1} \times \xi_n = i_n / \eta_{n-1} = i_{n-1})$$

$$= P(\xi_n \cdot i_{n-1} = i_n / \eta_{n-1} = i_{n-1}) =$$

ind ξ_n de η_{n-1}

$$= P(\xi_n = i_n / i_{n-1})$$

$$P = \begin{array}{c|cc|} & -1 & 1 \\ -1 & p & q \\ 1 & q & p \end{array}$$

$$= \begin{cases} p & i_n = 1, \ i_{n-1} = 1 \\ q & i_n = -1 \\ q & i_n = 1, \ i_{n-1} = -1 \\ p & i_n = -1 \end{cases}$$

6. Note que $(X_n : n \geq 0)$ assume valores em $\{0, 1, \dots, d\}$ e

$$P(X_n = j / X_{n-1} = i) = \begin{cases} 0, & \text{se } |j-i| > 1 \\ \frac{i}{d} \cdot \frac{d-i}{d} + \frac{d-i}{d} \cdot \frac{i}{d}, & \text{se } i = j \\ \frac{d-i}{d} \cdot \frac{d-i}{d}, & \text{se } j = i+1, \\ i^2 / d^2, & \text{se } j = i-1, \ i > 1 \end{cases}$$

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ d \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots\ d-1 & d \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{d^2} & \frac{2(d-1)}{d^2} & \frac{(d-1)^2}{d^2} & 0 & 0 & 0 \\
0 & 4/d^2 & \frac{4(d-2)}{d^2} & \frac{(d-2)^2}{d^2} & 0 & 0 \\
\vdots & \vdots & & \vdots & & \\
\vdots & & & & & \\
\cdot & \cdots & & \cdots & 1 & 0
\end{bmatrix} \end{array}
$$

Ademais

$$
\mathbb{P}(X_0 = k) = \frac{\binom{d}{k}\binom{d}{d-k}}{\binom{d}{k}}, \quad k=0,\dots,d
$$

7. Seja X_n o quarto visitado por Alfredinho na n-ésima tentativa. Note que $(X_n, n \geq 1)$ é um C.M. com $X_0 = 1$ e matriz de transição

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6
\end{bmatrix} \end{array}
$$

Note que a cada escolha, pode ocorrer sucesso com prob. $1/6$ e fracasso com prob. $5/6$. Ademais, dada a independência entre as escolhas, o número de quartos visitados por Alfredinho antes de chegar ao seu é uma v.a. geométrica. Assim

$$
\mathbb{P}(X_{10} = 5, X_9 \neq 5, \dots, X_1 \neq 5, X_0 = 1)
$$
$$
= \mathbb{P}(X_{10} = 5 / X_9 \neq 5)\, \mathbb{P}(X_9 \neq 5 / X_8 \neq 5) \dots \mathbb{P}(X_1 \neq 5 / X_0 = 1)
$$
$$
= \frac{1}{6} \cdot \frac{5}{6} \cdots \frac{5}{6} = \left(\frac{5}{6}\right)^9 \frac{1}{6}
$$

Lista 07

8. Nesse caso,

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}
\end{array}
$$

Apenas para exibir os detalhes, note que

$$P(X_2 = 1 / X_0 = 2) = \frac{1}{6} + \frac{1}{2}\cdot\frac{1}{6} + \left(\frac{1}{2}\right)^2 \frac{1}{6} + \dots$$

$$= \frac{1}{6}\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k = \frac{1}{6}\,\frac{1}{1-1/2} = \frac{1}{6}\cdot 2 = \frac{1}{3}.$$

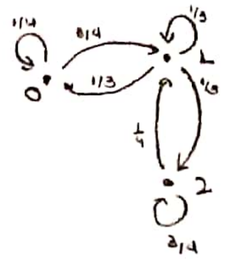
Lista 08

2.

a) $P(X_0=0, X_1=1, X_2=1) = P(X_2=1 / X_1=1)\, P(X_1=1 / X_0=0)\, P(X_0=0)$

$= p_{11}\, p_{01}\, p^{(0)}(0)$ $\qquad p^{(0)} = (1/4, 1/2, 1/4)$

$= \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{1}{16}$



b) $P(X_2=1, X_1=1 / X_0=0) = \frac{P(X_2=1, X_1=1, X_0=0)}{P(X_0=0)}$

$= P(X_2=1 / X_1=1)\, \frac{P(X_1=1, X_0=0)}{P(X_0=0)}$

$= P(X_2=1 / X_1=1)\, P(X_1=1 / X_0=0)$

$= p_{11}\, p_{01} = p_{01}\, p_{11}$

c) $p_{01}^{(2)} = P(X_2=1 / X_0=0) = \sum_{i=0}^{2} P(X_2=1, X_1=i / X_0=0)$

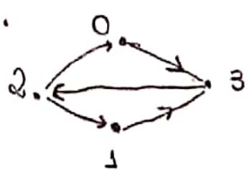
$\overset{(b)}{=} \sum_{i=0}^{2} p_{0i}\, p_{i1} = p_{00}\, p_{01} + p_{01}\, p_{11}$

$= \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{7}{16}$

4. 1.

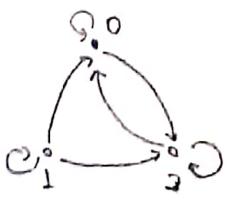
{0, 1, 2} recorr.

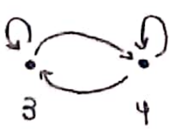
2.



{0, 1, 2, 3} recorr.

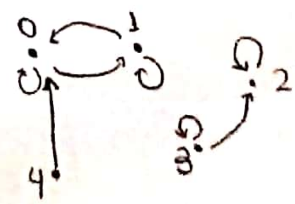
3.





{1}, {0, 2}, {3, 4}
trans. rec rec

4.



{0, 1}, {2}, {3}, {4}
rec rec trans trans.

10. No modelo de Ehrenfest.

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & d-1 & d \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1/d & 0 & \frac{d-1}{d} & & & \\ \vdots & & & & & & \\ d & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}$$

Se $P(X_0 = x) = \binom{d}{x}/2^d$, segue que

$$P(X_1 = x) = \sum_{i=0}^{d} P(X_1 = x / X_0 = i)\, P(X_0 = i)$$

$$= P(X_1 = x / X_0 = x-1) \frac{\binom{d}{x-1}}{2^d} + P(X_1 = x / X_0 = x+1) \frac{\binom{d}{x+1}}{2^d}, \quad x = 1, \ldots, d-1$$

$$= \frac{d+1-x}{d} \frac{d!}{(x-1)!\,(d+1-x)!} \frac{1}{2^d} + \frac{x+1}{d} \frac{d!}{(x+1)!\,(d-1-x)!} \frac{1}{2^d}$$

$$= \frac{(d-1)!}{(x-1)!\,(d-x)!} \frac{1}{2^d} + \frac{(d-1)!}{x!\,(d-1-x)!} \frac{1}{2^d}$$

$$= \underbrace{\left[\binom{d-1}{x-1} + \binom{d-1}{x}\right]}_{\binom{d}{x}} \frac{1}{2^d} = \binom{d}{x} \frac{1}{2^d}$$

Se $x = 0$,

$$P(X_1 = 0) = P(X_1 = 0 / X_0 = 1) \frac{\binom{d}{1}}{2^d}$$

$$= \frac{1}{d} \cdot d \cdot \frac{1}{2^d} = \frac{\binom{d}{0}}{2^d}$$

Se $x = d$,

$$P(X_1 = d) = P(X_1 = d / X_0 = d-1) \frac{\binom{d}{d-1}}{2^d}$$

$$= \frac{1}{d} \cdot d \cdot \frac{1}{2^d} = \binom{d}{d} 2^{-d}$$

Logo, $X_1 \sim \text{Binomial}(d, 1/2)$ [inv. p/ o proc. de Ehrenfest].

Lista 08

13. É uma cadeia irredutível com S finito, donde segue a ex. de π.

$$(\pi_0, \pi_1, \pi_2)\begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = (\pi_0, \pi_1, \pi_2) \quad , \sum_{i=0}^{2} \pi_i = 1$$

Temos

$$\begin{cases} 0.4\pi_0 + 0.3\pi_1 + 0.2\pi_2 = \pi_0 \\ 0.4\pi_0 + 0.4\pi_1 + 0.4\pi_2 = \pi_1 \\ 0.2\pi_0 + 0.3\pi_1 + 0.4\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \sim \begin{cases} 0.6\pi_0 - 0.3\pi_1 - 0.2\pi_2 = 0 \\ 0.4\pi_0 - 0.6\pi_1 + 0.4\pi_2 = 0 \\ 0.2\pi_0 + 0.3\pi_1 - 0.6\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\sim \begin{cases} 0.6\pi_0 - 0.3\pi_1 - 0.2\pi_2 = 0 \\ 1.6\pi_0 - 1.2\pi_1 = 0 \Rightarrow \pi_1 = \frac{4}{3}\pi_0 \\ 0.8\pi_0 - 0.8\pi_2 = 0 \Rightarrow \pi_2 = \pi_0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\pi_0 + \frac{4}{3}\pi_0 + \pi_0 = \pi_0\left(2 + \frac{4}{3}\right) = 1 \Rightarrow \pi_0 = \frac{3}{10} \Rightarrow \pi_2 = \frac{3}{10} \Rightarrow \pi_1 = \frac{4}{10}$$

$\therefore$

$$\pi = \left(\frac{3}{10} \ \frac{4}{10} \ \frac{3}{10}\right)$$

15.

a) $\mu = \frac{1}{4} + \frac{2}{4} = \frac{3}{4} < 1 \Rightarrow \pi_0 = 1$

b) $\mu = \frac{6}{4} > 1 \Rightarrow \pi_0$ é a menor sol. de $\pi_0 = 1/4 + \frac{3}{4}\pi_0^2$, i.e., a men. sol. de

$$3\pi_0^2 - 4\pi_0 + 1 = 0 \Rightarrow \pi_0 = \frac{4 \pm \sqrt{16-12}}{6} \begin{cases} 1 \\ 1/3 \end{cases}$$

Logo, $\pi_0 = 1/3$

c) $\mu = 1/2 + 3/3 = 3/2 > 1$. Como

$$\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3 \Leftrightarrow 2\pi_0^3 - 3\pi_0 + 1 = 0$$

$$\Leftrightarrow (\pi_0 - 1)(2\pi_0^2 + 2\pi_0 - 1) = 0 \Rightarrow \pi_0 = \frac{\sqrt{3}-1}{2}$$

$$\Leftrightarrow \pi_0 = 1 \quad \pi_0 = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

18. Se $(X_n, n \geq 1)$ a quantia de Maria após a n-ésima jogada. Sabemos que

$$p = P(X_n = n+1 / X_0 = n) = 2/3 \quad , n \in \{1, \ldots, 14\}$$

$$q = \mathbb{P}(X_1 = n-1 / X_0 = n) = 1/3$$

$$\mathbb{P}(X_1 = 15 / X_0 = 15) = 1 \quad e \quad \mathbb{P}(X_1 = 0 / X_0 = 0) = 1.$$

Para Maria, a chance de interesse é dada por (ruína do jogador)

$$P_5^m = \frac{1 - (1/2)^5}{1 - (1/2)^{15}} = \frac{2^5 - 1}{2^{15} - 1} \cdot 2^{10}$$ (ela parte com 5 pontos)

Para João,

$$P_{10}^J = \frac{1 - 2^{10}}{1 - 2^{15}} = \frac{(1 - 2^5)(1 + 2^5)}{(1 - 2^{15})}$$ (ele parte com 10)

$$\frac{P_{10}^J}{P_5^m} = \frac{1 + 2^5}{2^{10}} = \frac{1}{2^{10}} + \frac{1}{2^5} \neq 1 \Rightarrow \text{as chances } \nexists \text{ são iguais.}$$

Lista 09

2.

$$P(X(T)=n) = \int_0^{+\infty} P(X(T)=n \mid T=s)\, f_T(s)\, ds$$

$$= \int_0^{+\infty} P(X(s)=n)\, v e^{-vs}\, ds$$

$$= \int_0^{+\infty} \frac{(\lambda s)^n e^{-\lambda s}}{n!}\, v e^{-vs}\, ds$$

$$= \frac{\lambda^n v}{n!(\lambda+v)} \int_0^{+\infty} s^n (\lambda+v) e^{-(\lambda+v)s}\, ds$$

$$= \frac{\lambda^n v}{\cancel{n!}(\lambda+v)} \frac{\cancel{n!}}{(\lambda+v)^n} = \left(\frac{\lambda}{\lambda+v}\right)^n \left(\frac{v}{\lambda+v}\right), \quad n \geq 0$$

3.

a) Sim, $\lambda_n = \lambda$ e $\mu_n = 0$, $n \geq 0$.

b) $PP(\lambda) \to +\infty$ q.c. qdo $t \to \infty$.

4. a) $v_0 = 0$ e $v_i = \lambda(1-q^i)$, $i \geq 1$ e $P_{i,j} = P(X = j-i \mid X > 0)$, $j \in \{1, \dots, i\}$

$X \sim \text{Binomial}(i, p)$

b) Não, uma vez que taxa positiva de salto para vizinhos + dist.

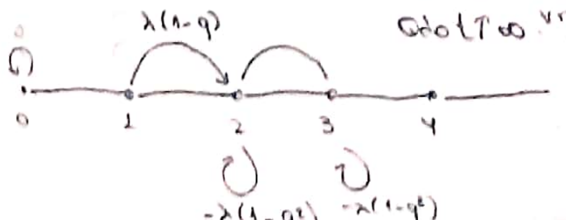
c) Vai a inf, uma vez que domina um $PP(\lambda(1-q))$ que $(\cancel{X(t)})\to\infty$ q.c. $X(t)$ $\forall t$.

d)

$\forall n \geq 0$ $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$ $\forall t \geq 0$

Qdo $t \to \infty$ $\forall n \geq 0$ $P(N(t) \geq n) = P(S_n \leq t) = 1$

$q = 0$



$$\frac{n p q^n}{1-q^n} \to 1 \qquad \frac{np(1-p)^n}{1-(1-p)^n} \to 1$$

Lista 10

2. For the general Birth-Death process,

- Backwards

$$P_{0j}'(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t)$$

$$P_{ij}'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t)$$

$$P_{ij}(0) = \delta_{ij}$$

- Forwards

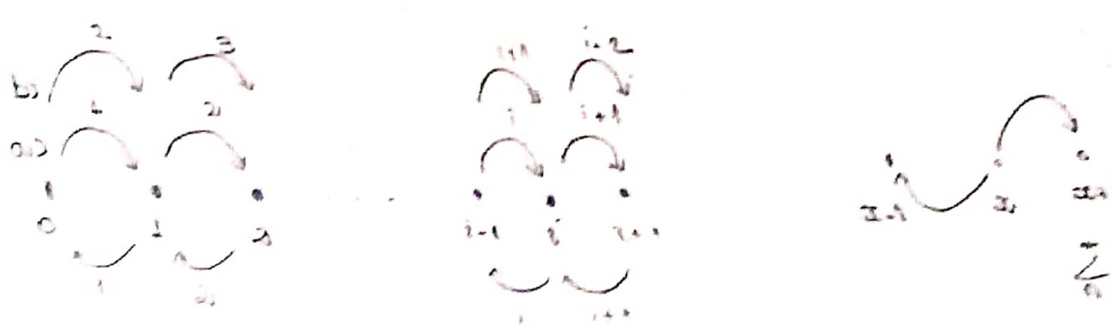
$$P_{i0}'(t) = -P_{i0}(t)\lambda_0 + P_{i1}(t)\mu_1$$

$$P_{ij}'(t) = P_{i,j-1}(t)\lambda_{j-1} - P_{ij}(t)(\lambda_j + \mu_j) + P_{i,j+1}(t)\mu_{j+1}$$

$$P_{ij}(0) = \delta_{ij}$$

For the Poisson Process, consider $\lambda_n = \lambda$ e $\mu_n = 0$ $\forall n \geq 0$ in above equations.

5. b)



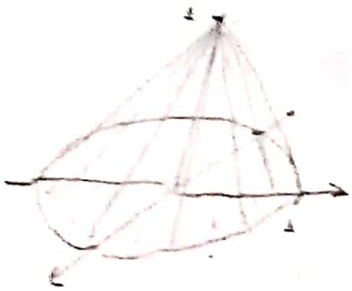
$$\sum_n \frac{1}{2n+1} \geq \frac{1}{2}\sum \frac{1}{n} = +\infty$$

a) $$\sum_{n=1}^{+\infty} \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\cdots\frac{\lambda_{n-1}}{\mu_n} = \sum_{n=1}^{+\infty}\left(\frac{1}{1}\right)\cdot\left(\frac{2}{2}\right)\cdot\left(\frac{n}{n}\right) = \sum_{n=1} 1 = +\infty$$

b) $$\sum_{n=1}^{+\infty}\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = \sum_{n=1}^{+\infty}(n+1) = +\infty$$

$$\sum_{n=1}^{+\infty} \frac{1}{n+1} = +\infty$$

Seja Y uma v.a unif distribuída em $Q = \{(x,y) \in \mathbb{R} : x^2 + y^2 < R^2\}$, com $R \sim \mathcal{U}(\frac{1}{2}, 1)$



Obtenha

$f_{R|Y}(r)$.

$$f_{RY}(r,y) = f_R(r)\, f_{Y|R=r}(y|r)$$

$$= \frac{2}{\pi r^2}$$

$$f_Y(y) = \int_{1/2}^{1} \frac{2}{\pi} r^{-2}\, dr$$

$$= \frac{2}{\pi}\left(-\frac{1}{r}\right)\Big|_{1/2}^{1}$$

$$= \frac{2}{\pi}$$

2

2 21

3 17

3

6

2

0

2

3

4

2

4

4

6

6

$$(1-e^{a})^2 = (1-e^{a})(1-e^{a})$$

$$= 1 - 2e^{a} + e^{2a}$$

$$P_{n,n-1}(t) = \int_0^t f_{T_n}(s)\, P(T_{n-1} > t-s)\, ds$$

$$= \int_0^t \mu_n e^{-\mu_n s}\, e^{-\mu_{n-1}(t-s)}\, ds$$

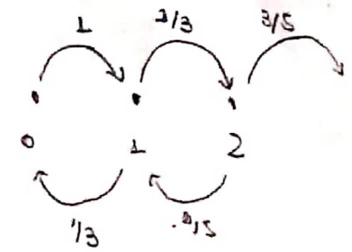
$$= \mu_n e^{-\mu_{n-1} t} \int_0^t e^{-(\mu_n - \mu_{n-1}) s}\, ds$$

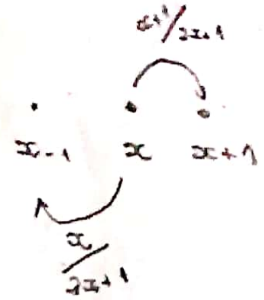
$$= \mu_n e^{-\mu_{n-1} t} \left[-\frac{e^{-(\mu_n - \mu_{n-1}) s}}{(\mu_n - \mu_{n-1})} \Bigg|_0^t \right]$$

$$= \mu_n e^{-\mu_{n-1} t} \left[\frac{1 - e^{-(\mu_n - \mu_{n-1}) t}}{\mu_n - \mu_{n-1}} \right]$$

Para determinar se rec. nula ou trans., vamos olhar a cadeia imersa

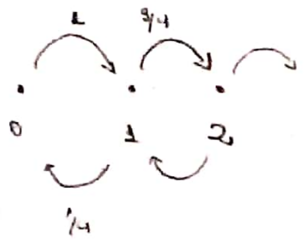
a)



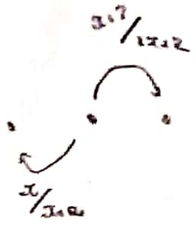


$$\sum_{k=1}^{+\infty} \frac{q_1 \cdots q_k}{p_1 \cdots p_k} < +\infty$$

b)



$$\sum_{k=1}^{+\infty} \frac{2}{(k+1)(k+2)} < +\infty$$



$$= \sum_{k=1}^{+\infty} \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{k}{k+1}$$

$$= \sum_{k=1}^{+\infty} \frac{1}{k+1} = +\infty$$

Logo, rec. nulo em a) e transitório em b).

4 a) Basta toma $\lambda_x = 0 \ \forall x \in \mathbb{N}$ em a.

b) $P_{xx}(t) = \mathbb{P}(T_x > t) = e^{-\mu_x t}$

c) $P_{x,}$

d) 0. 1. 2. n-1. n — μ, 2μ, $n\mu$

0 é absorv.

Os demais são transientes

$$\frac{d}{dt} P_{n}(t) = -n\mu \cdot P_{n}(t) \Rightarrow P_n(t) = e^{-n\mu t}$$

$$P'_{n,i}(t) = (i+1)\mu P_{n,i+1}(t) - i\mu P_{n,i}(t), \quad i = 0, \ldots, n-1$$