Hyperbolic Graph Neural Networks: A Review of Methods and Applications

 $\begin{array}{c} \textbf{Menglin Yang}^1 \,, \,\, \textbf{Min Zhou}^{2*} \,, \,\, \textbf{Zhihao Li}^3 \,, \,\, \textbf{Jiahong Liu}^3 \,, \\ \textbf{Lujia Pan}^2 \,, \,\, \textbf{Hui Xiong}^4 \,\, \text{and Irwin King}^{1*} \end{array}$

¹The Chinese University of Hong Kong, Hong Kong, China ²Huawei Noah's Ark Lab, Shenzhen, China ³Harbin Institute of Technology, Shenzhen, China ⁴Hong Kong University of Science and Technology, Guangzhou, China

Abstract

Graph neural networks generalize conventional neural networks to graph-structured data and have received widespread attention due to their impressive representation ability. In spite of the remarkable achievements, the performance of Euclidean models in graph-related learning is still bounded and limited by the representation ability of Euclidean geometry, especially for datasets with highly non-Euclidean latent anatomy. Recently, hyperbolic spaces have gained increasing popularity in processing graph data with tree-like structure or power-law distribution, owing to its exponential growth property. In this survey, we comprehensively revisit the technical details of the current hyperbolic graph neural networks¹, unifying them into a general framework and summarizing the variants of each component. Besides, we summarize a series of related applications in a variety of fields. More importantly, we identify several challenges, which potentially serve as guidelines for further flourishing the achievements of graph learning in hyperbolic spaces.

1 Introduction

Graphs are data structures that extensively exist in real-world complex systems, varying from social networks [Grover and Leskovec, 2016; Yang et al., 2020], protein interaction networks [Vázquez et al., 2003], recommender systems [Chen et al., 2022], knowledge graphs [Wang et al., 2017], to the financial transaction systems [Sawhney et al., 2021]. They form the basis of innumerable systems owing to their widespread utilization, allowing relational knowledge about interacting entities to be stored and accessible rapidly. As a result, graph-related learning tasks gain an increasing attention in machine learning and network science research. Many researchers have applied Graph Neural Networks (GNNs) for a variety of tasks, including node classification [Kipf and Welling, 2017; Veličković et al., 2018; Wu et al., 2019], link prediction [Kipf and Welling, 2016;

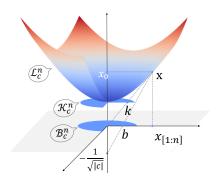


Figure 1: Illustration of three prevalent and isomorphic hyperbolic models: Lorentz model, Kelin model, and Poincaré ball model.

Zhang and Chen, 2018], and graph classification [Xu et al., 2019; Errica et al., 2019] by embedding nodes in low-dimensional vector spaces, encoding topological and semantic information simultaneously. Many GNNs are built in Euclidean space in that it features a vectorial structure, closed-form distance and inner-product formulae, and is a natural extension of our intuitively appealing visual three-dimensional space [Ganea et al., 2018].

Despite the effectiveness of Euclidean space for graphrelated learning tasks, its ability to encode complex patterns is intrinsically limited by its polynomially expanding capacity. Although nonlinear techniques [Bouchard et al., 2015] assist to mitigate this issue, complex graph patterns may still need an embedding dimensionality that is computationally intractable. As revealed by recent research [Bronstein et al., 2017] that many complex data shows non-Euclidean underlying anatomy, for example, the tree-like structure extensively exists in many real-world networks, such as the hypernym structure in natural languages, the subordinate structure of entities in the knowledge graph, the organizational structure for financial fraud, and the power-law distribution in recommender systems.² In these situations, Euclidean space fails to make the most powerful or adequate geometrical representations.

Recently, hyperbolic space has gained increasing popularity in processing tree-like graph data. Figure 1 depicts three prevalent models for hyperbolic space and they are isomor-

^{*}Corresponding Author

¹https://github.com/marlin-codes/HGNNs

²The power-law distribution can be traced back to the hierarchical structures [Ravasz and Barabási, 2003].

Table 1: Summary of operations in the Poincaré ball model and the Lorentz model (c < 0)

	Poincaré Ball Model	Lorentz Model
Manifold	$\mathcal{B}_c^n = \{x \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x} \rangle_2 < -1/c\}$	$ \begin{vmatrix} \mathcal{L}_{c}^{n} = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathcal{L}} = 1/c \right\} \\ g_{\mathbf{x}^{c}}^{\mathcal{L}^{n}} = \eta, \text{ where } \eta \text{ is } I \text{ except } \eta_{0,0} = -1 \end{vmatrix} $
Metric	$ \begin{vmatrix} \mathcal{B}_{c}^{n} = \{x \in \mathbb{R}^{n} : \langle \mathbf{x}, \mathbf{x} \rangle_{2} < -1/c \} \\ g_{\mathbf{x}^{c}}^{\mathcal{B}^{n}} = (\lambda_{\mathbf{x}}^{c})^{2} g^{\mathbf{E}^{n}} \text{ where } \lambda_{\mathbf{x}}^{c} = \frac{2}{1+c\ \mathbf{x}\ _{2}^{2}} \text{ and } g^{\mathbf{E}^{n}} = \mathbf{I}_{n} $	$g_{\mathbf{x}^c}^{\mathcal{L}_c^n} = \eta$, where η is I except $\eta_{0,0} = -1$
Induced distance	$d_{\mathcal{B}}^{c}(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{ c }} \cosh^{-1} \left(1 - \frac{2c\ \mathbf{x} - \mathbf{y}\ _{2}^{2}}{(1 + c\ \mathbf{x}\ _{2}^{2})(1 + c\ \mathbf{y}\ _{2}^{2})} \right)$	$d_{\mathcal{L}}^{c}(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{ c }} \cosh^{-1} \left(c \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{L}} \right)$
Logarithmic map	$\log_{\mathbf{x}}^{c}(\mathbf{y}) = \frac{2}{\sqrt{ c \lambda_{\mathbf{x}}^{c}}} \tanh^{-1} \left(\sqrt{ c } \left\ -\mathbf{x} \oplus_{c} \mathbf{y} \right\ _{2} \right) \frac{-\mathbf{x} \oplus_{c} \mathbf{y}}{\left\ -\mathbf{x} \oplus_{c} \mathbf{y} \right\ _{2}}$	$\log_{\mathbf{x}}^{c}(\mathbf{y}) = \frac{\cosh^{-1}(c\langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{L}})}{\sinh(\cosh^{-1}(c\langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{L}}))} \left(\mathbf{y} - c\langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{L}}\mathbf{x}\right)$
exponential map	$\exp_{\mathbf{x}}^{c}(\mathbf{v}) = \mathbf{x} \oplus_{c} \left(\tanh\left(\sqrt{ c } \frac{\lambda_{\mathbf{x}}^{c} \ \mathbf{v}\ _{2}}{2}\right) \frac{\mathbf{v}}{\sqrt{ c \ \mathbf{v}\ _{2}}} \right)$	$\exp_{\mathbf{x}}^{c}(\mathbf{v}) = \cosh\left(\sqrt{ c }\ \mathbf{v}\ _{\mathcal{L}}\right)\mathbf{x} + \mathbf{v}\frac{\sinh\left(\sqrt{ c }\ \mathbf{v}\ _{\mathcal{L}}\right)}{\sqrt{ c }\ \mathbf{v}\ _{\mathcal{L}}}$
Parallel transport	$PT_{\mathbf{x} \to \mathbf{y}}^c(\mathbf{v}) = \frac{\lambda_{\mathbf{x}}^c}{\lambda_{\mathbf{y}}^c} \operatorname{gyr}[\mathbf{y}, -\mathbf{x}]\mathbf{v}$	$PT_{\mathbf{x} \to \mathbf{y}}^c(\mathbf{v}) = \mathbf{v} - \frac{c\langle \mathbf{y}, \mathbf{v} \rangle_{\mathcal{L}}}{1 + c\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{L}}} (x + \mathbf{y})$

phic. The typical geometric property of hyperbolic space is that its volume increases exponentially in proportion to its radius, whereas the Euclidean space grows polynomially. Such a geometric trait brings two benefits enabling it to well deal with tree-like graph data. The *first* one is that hyperbolic space exhibits minimal distortion and fits the hierarchies particularly well since the space closely matches the growth rate of tree-like data while the Euclidean space cannot. The *second* is that even though with a low-embedding dimension space, hyperbolic models are surprisingly able to produce high-quality representation, which makes it to be especially favorable in low-memory and low-storage scenarios.

Scope and Structure of the Survey. To the best of our knowledge, no surveys have been conducted especially on the methods and applications of Hyperbolic Graph Neural Networks (HGNNs), and the most recent relevant study [Peng et al., 2021] mainly sketches the progress of hyperbolic neural network without focusing on the latest methods and applications in graph field. In this work, we attempt to fill this gap by scoping to the latest research efforts on HGNNs. The main contributions of this work are summarized in below:

- We provide a detailed technique review over existing HGNN models, unifying them by a general framework and outlining the variants of each module. Additionally, we discuss recent studies on theoretical and empirical analysis of HGNN models.
- We systematically categorize the applications and divide them into numerous scenarios. For each case, we present several major applications and their corresponding methods.
- We summarize several challenges and opportunities for future research, providing insights for further flourishing the achievements of graph learning built with hyperbolic spaces.

2 Preliminaries and Notation

In this section, we briefly introduce a list of some of the most helpful Riemannian geometry concepts, definitions, and operations in hyperbolic geometry. For a more detailed introduction, please refer to [Lee, 2006] and [Lee, 2013].

Manifold and Tangent Space. Riemannian geometry is a sub-field of differential geometry in which a smooth manifold \mathcal{M} is associated with a Riemannian metric $g^{\mathcal{M}}$. An

n-dimensional manifold $(\mathcal{M},g^{\mathcal{M}})$ is a topological space, a generalization of a 2-dimensional surface with high dimensions. For each point \mathbf{x} in \mathcal{M} , a tangent space $\mathcal{T}_{\mathbf{x}}\mathcal{M}$ is defined as the first-order approximation of \mathcal{M} around \mathbf{x} , which is an n-dimension vector space and isomorphic to \mathbb{R}^n . The Riemannian manifold metric $g^{\mathcal{M}}$ assigns a smoothly varying positive definite inner product $<\cdot,\cdot>:\mathcal{T}_{\mathbf{x}}\mathcal{M}\times\mathcal{T}_{\mathbf{x}}\mathcal{M}\to\mathbb{R}$ on the tangent space, which allows us to define several geometric properties, such as geodesic distances, angles, and curvature.

Geodesics and Induced Distance Function. For a curve $\gamma: [\alpha, \beta] \to \mathcal{M}$, the shortest length of γ , i.e., geodesics, is defined as $L(\gamma) = \int_{\alpha}^{\beta} \|\gamma'(t)\|_g dt$. Then the distance of $\mathbf{u}, \mathbf{v} \in \mathcal{M}$, $d_{\mathcal{M}}(\mathbf{u}, \mathbf{v}) = \inf L(\gamma)$ where γ is a curve that $\gamma(\alpha) = \mathbf{u}, \gamma(\beta) = \mathbf{v}$.

Maps and Parallel Transport. The maps define the relationship between the hyperbolic space and the corresponding tangent space. For a point $\mathbf{x} \in \mathcal{M}$ and vector $\mathbf{v} \in \mathcal{T}_{\mathbf{x}}\mathcal{M}$, there exists a unique geodesic $\gamma:[0,1] \to \mathcal{M}$ where $\gamma(0)=\mathbf{x},\gamma'(0)=\mathbf{v}$. The exponential map $\exp_{\mathbf{x}}:\mathcal{T}_{\mathbf{x}}\mathcal{M} \to \mathcal{M}$ is defined as $\exp_{\mathbf{x}}(\mathbf{v})=\gamma(1)$ and logarithmic map $\log_{\mathbf{x}}$ is the inverse of $\exp_{\mathbf{x}}$. The parallel transport $PT_{\mathbf{x}\to\mathbf{y}}:\mathcal{T}_{\mathbf{x}}\mathcal{M} \to \mathcal{T}_{\mathbf{y}}\mathcal{M}$ achieves the transportation from point \mathbf{x} to \mathbf{y} that preserves the metric tensors.

Hyperbolic Models. Hyperbolic geometry is a Riemannian manifold with a constant negative curvature. There exist multiple equivalent hyperbolic models, like the Poincaré ball model, Lorentz model and Klein model, which show different characteristics but are mathematically equivalent. We here mainly introduce two widely studied and adopted hyperbolic models in HGNNs, i.e., the Poincaré ball model and the Lorentz model. Let $\|.\|$ be the Euclidean norm and $\langle .,. \rangle_{\mathcal{L}}$ denote the Minkowski inner product, respectively. The two models are given by Definition 2.1 and Definition 2.2, respectively. The related formulas and operations, e.g., distances, maps, and parallel transports are further summarized in Table 1, where \oplus_c and $\operatorname{gyr}[.,.]v$ are Möbius addition [Ungar and others, 2007] and gyration operator [Ungar and others, 2007], respectively.

Definition 2.1 (Poincaré Ball Model). With negative curvature c (c < 0), the Poincaré ball model is defined as a Riemannian manifold ($\mathcal{B}_c^n, g_{\mathbf{x}}^{\mathcal{B}}$), where $\mathcal{B}_c^n = \left\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2 < -1/c\right\}$ is an open n-dimensional ball with radius $1/\sqrt{|c|}$. Its metric tensor $g_{\mathbf{x}}^{\mathcal{B}} = (\lambda_{\mathbf{x}}^c)^2 g^E$, where

 $\lambda_{\mathbf{x}}^c = 2/(1+c\|\mathbf{x}\|_2^2)$ is the conformal factor and g^E is the Euclidean metric, i.e., \mathbf{I}_n .

Definition 2.2 (Lorentz Model). With negative curvature $c \quad (c < 0)$, the Lorentz model (also named hyperboloid model) is defined as the Riemannian manifold $(\mathcal{L}_c^n, g_{\mathcal{L}}^{\mathcal{L}})$, where $\mathcal{L}_c^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathcal{L}} = 1/c\}$ and $g_{\mathbf{x}}^{\mathcal{L}} = 1/c$ $diag([-1, 1, ..., 1])_n$.

In the following, we use \mathcal{H} to denote the case that is both applicable to \mathcal{B} and \mathcal{L} . However, in certain circumstances, the precise notation will be used to describe a specific model. We do not differentiate the notations of the hyperbolic origin, logarithmic maps, exponential maps and parallel transports in \mathcal{B} from that in \mathcal{H} since they are identifiable by context.

HGNN Methodologies

Recently, GNNs have shown significant superiority in graphrelated tasks and applications, owing to their capability to explicitly encode node attributes and its interaction simultaneously, as well as implicitly learn high-order dependencies. Due to the properties of hyperbolic spaces, hyperbolic GNNs (i.e., HGNNs) have further made remarkable achievements in the study of graph data, especially for those with a treelike structure. The key to implementing HGNNs lies in three fundamental steps, namely feature transformation, neighborhood aggregation and non-linear activation. Before that, we need to initialize the feature by projecting the input Euclidean feature to the hyperbolic manifold. In the following, we will first introduce the initialization layer and then present the details of key components in HGNNs.

Hyperbolic Initialization Layer

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set \mathcal{V} and edge set \mathcal{E} , let $(\mathbf{x}_i^{\bar{E}})_{i \in \mathcal{V}}$ be *n*-dimensional input node feature where the superscript E denotes that the node feature lies in Euclidean space. The node feature can be pre-trained embedding [Zhang and Gao, 2021] or node attributes. In the following, we use the superscripts $\mathcal{T}_{\mathbf{x}}$, $\mathcal{H}(\mathcal{B}/\mathcal{L})$ to denote tangent features at x and hyperbolic (Poincaré ball model/Lorentz model) features, respectively.

Initialization layer in Poincaré ball model. To map the Euclidean feature \mathbf{x}^E to the Poincaré ball model \mathcal{B} , the exponential map is applied, i.e.,

 $\mathbf{x}^{\mathcal{B}} = \exp_{\mathbf{o}}^{c}(\mathbf{x}^{E}).$

Initialization layer in Lorentz model. To project \mathbf{x}^E to the Lorentz model \mathcal{L} , the element 0 is required to added in general, i.e., $\mathbf{x}^{T_0} = (0, \mathbf{x}^E)$ which then is projected to the Lorentz model with exponential map, i.e.,

 $\mathbf{x}^{\mathcal{L}} = \exp_{\mathbf{o}}^{c}(\mathbf{x}^{\mathcal{T}_{\mathbf{o}}}) = \exp_{\mathbf{o}}^{c}((0, \mathbf{x}^{E})),$ (2) where $\mathbf{x}^{\mathcal{T}_0}$ is in the Lorentzian tangent space at origin since

we have $\langle \mathbf{o}, \mathbf{x}^{\mathcal{T}_{\mathbf{o}}} \rangle_{\mathcal{L}} = 0$.

In the case of no node feature available, there are two common manners to produce the initial feature. One is based on sampling from multivariate Gaussian [Bose et al., 2020; Sun et al., 2021]. The other is to utilize the one-hot vector [Qu and Zou, 2022; Yang et al., 2021] with a trainable embedding matrix $\mathbf{W} \in \mathbb{R}^{n \times d}$. After getting the initial value \mathbf{x}^{E} , it can be projected in \mathcal{B} or \mathcal{L} by Equation (1) or (2).

Hyperbolic Feature Transformation

Hyperbolic feature transformation usually involves matrixvector multiplication and bias addition.

Matrix-vector Multiplication. To implement matrixvector multiplication, the tangential method is applicable for both \mathcal{B} and \mathcal{L} . Specifically, in \mathcal{B} , the matrix-vector multiplication [Ganea et al., 2018; Liu et al., 2019] is formulated by

$$\mathbf{M} \otimes_{c}^{\mathcal{B}} \mathbf{x}^{\mathcal{B}} = \exp_{\mathbf{o}}^{c}(\mathbf{M} \log_{\mathbf{o}}^{c}(\mathbf{x}^{\mathcal{B}})), \tag{3}$$

where $\mathbf{x}^{\mathcal{B}} \in \mathcal{B}_c^n, \mathbf{M} \in \mathbb{R}^{n \times d}$. In \mathcal{L} , implementing matrixvector multiplication is the same idea. It is worth noting that we need to carefully keep the node feature always on the Lorentz model. In particular, to ensure that node features are still in tangent space at origin after multiplying matrix M, we only need to transform the value in last n coordinates as the proposal from [Zhang et al., 2021c], i.e.,

 $\mathbf{M} \otimes_{c}^{\mathcal{L}} \mathbf{x}^{\mathcal{L}} = \exp_{\mathbf{o}}^{c}(0, \mathbf{M} \log_{\mathbf{o}}^{c}(\mathbf{x}^{\mathcal{L}})_{[1:n]}),$ (4) where $\mathbf{x}^{\mathcal{L}} \in \mathcal{L}_{c}^{n}, \mathbf{M} \in \mathbb{R}^{n \times d}$. In this way, the first coordinate can always be 0 indicating that the transformed result is guaranteed in the tangent space at o. Recent study [Chen et al., 2021] shows that the above linear transformation can be considered as a Lorentz transformation with only a special rotation but no boost. They then derived a more expressive transformation method, which is equipped with both rotation and boost operations, i.e.,

$$\mathbf{M} \otimes_{c}^{\mathcal{L}} \mathbf{x}^{\mathcal{L}} = \left(\begin{bmatrix} \mathbf{v}^{\mathsf{T}} \\ \mathbf{W} \end{bmatrix} \right) \otimes_{c}^{\mathcal{L}} \mathbf{x}^{\mathcal{L}} = \begin{bmatrix} \frac{\sqrt{\|\mathbf{W}\mathbf{x}^{\mathcal{L}}\|^{2} - 1/c}}{\mathbf{v}^{\mathsf{T}}\mathbf{x}^{\mathcal{L}}} \mathbf{v}^{\mathsf{T}} \\ \mathbf{W} \end{bmatrix},$$
(5)

where $\mathbf{v} \in \mathbb{R}^{n+1}$, and $\mathbf{W} \in \mathbb{R}^{d \times (n+1)}$ are trainable parameters.

Bias Addition. To implement bias addition, tangent space at origin is still a useful medium and the formula in \mathcal{B} and \mathcal{L}

can be uniformly expressed as: $\mathbf{x}^{\mathcal{H}} \oplus_{c}^{\mathcal{H}} \mathbf{b}^{\mathcal{H}} = \exp_{\mathbf{x}^{\mathcal{H}}}^{c} \left(P_{\mathbf{o} \to \mathbf{x}^{\mathcal{H}}}^{c} \left(\log_{\mathbf{o}}^{c} (\mathbf{b}^{\mathcal{H}}) \right) \right), \qquad (6)$ where $\mathbf{b}^{\mathcal{H}}$ is the bias in \mathcal{H}_{c}^{n} , and the equations of parallel transport $P_{\mathbf{x}^{\mathcal{H}} \to \mathbf{y}^{\mathcal{H}}}^{c}(\cdot)$ in \mathcal{B} and \mathcal{L} are summarized in Table 1.

3.3 Hyperbolic Neighborhood Aggregation

Hyperbolic neighborhood aggregation can be divided into two procedures: computation of neighborhood weights and computation of mean aggregation.

Computation of neighborhood weights. According to current research, neighborhood weights may be computed using structural information, node distance, feature attention, or a combination of the aforementioned approaches.

(1) By structure information [Liu et al., 2019],

$$\alpha_{ij} = 1/\sqrt{\tilde{d}_i \tilde{d}_j},\tag{7}$$

where $\tilde{d}_i = d_i + 1$ and d_i is the degree of node i. (2) By feature attention [Chami et al., 2019],

 $\alpha_{ij} = \operatorname{softmax}(\operatorname{LeakReLU}(\mathbf{W}^T[\mathbf{x}_i^{\mathcal{T}_o}||\mathbf{x}_j^{\mathcal{T}_o}])).$ (3) By node distance [Zhang *et al.*, 2021b], (8)

$$\alpha_{ij} = \operatorname{softmax}(-d_{\mathcal{H}}^c(\mathbf{x}_i^{\mathcal{H}}, \mathbf{x}_i^{\mathcal{H}})), \tag{9}$$

the squared form in [Zhang et al., 2021c]

$$\alpha_{ij} = \operatorname{softmax}(-d_{\mathcal{H}}^{c}(\mathbf{x}_{i}^{\mathcal{H}}, \mathbf{x}_{j}^{\mathcal{H}})^{2}), \tag{10}$$

or a more complex form [Gulcehre et al., 2019]

$$\alpha_{ij} = f(-\beta d_{\mathcal{H}}^c(\mathbf{x}_i^{\mathcal{H}}, \mathbf{x}_j^{\mathcal{H}}) - \gamma), \tag{11}$$

where β and γ are parameters that can be set manually or learned along with the training process, and f can be $\operatorname{softmax}(\cdot)$.

(4) By the integration of feature attention and node distance [Zhu et al., 2020],

$$\alpha_{ij} = \operatorname{softmax} \left(\operatorname{LeakReLU}(\mathbf{W}^{T}[\mathbf{x}_{i}^{T_{\mathbf{o}}} || \mathbf{x}_{j}^{T_{\mathbf{o}}}]) \times d_{\mathcal{H}}^{c}(\mathbf{x}_{i}^{\mathcal{H}}, \mathbf{x}_{j}^{\mathcal{H}}) \right),$$
(12)

where the distance $d_{\mathcal{H}}^{c}(\mathbf{x}_{i}^{\mathcal{H}}, \mathbf{x}_{i}^{\mathcal{H}})$ makes feature attention entangled with hyperbolic geometry.

(5) By the combination of feature attention and structure information³,

$$\alpha_{ij} = \text{sigmoid}\left(\text{LeakReLU}(\mathbf{W}^T[\mathbf{x}_i^{\mathcal{T}_o}||\mathbf{x}_j^{\mathcal{T}_o}])\right) \times 1/\sqrt{\tilde{d}_i \tilde{d}_j}.$$
(13)

Computation of Mean Aggregation. For mean aggregation, or weighted mean pooling, it cannot be computed by simply averaging the inputs, which may lead a deviation out of the hyperbolic manifold. Currently, there are three typical ways to implement mean aggregation: tangential method, Einstein midpoint, and Lorentzian centroid.

(1) Tangential method [Liu et al., 2019; Chami et al., 2019] is defined by,

$$AGG(\mathbf{x}_{i}^{\mathcal{H}}) := \exp_{\mathbf{o}}^{c} \left(\sum_{j \in \mathcal{N}_{i}} \alpha_{ij} \left(\log_{\mathbf{o}}^{c}(\mathbf{x}_{i}^{\mathcal{H}}) \right) \right). \tag{14}$$

(2) Einstein midpoint method [Gulcehre et al., 2019; Dai et al., 2021] is formulated as,

$$AGG(\mathbf{x}_{i}^{\mathcal{H}}) := \begin{cases} \overline{\mathbf{x}}_{j}^{\mathcal{K}} = p_{\mathcal{H} \to \mathcal{K}} \left(\mathbf{x}_{j}^{\mathcal{H}} \right) \\ \mathbf{m}_{i}^{\mathcal{K}} = \sum_{j \in \mathcal{N}_{i}} \gamma_{j} \overline{\mathbf{x}}_{j}^{\mathcal{K}} / \sum_{j \in \mathcal{N}_{i}} \gamma_{j}, \\ \mathbf{x}_{i}^{\mathcal{H}} = p_{\mathcal{K} \to \mathcal{H}} \left(\mathbf{m}_{i}^{\mathcal{K}} \right) \end{cases}$$
where $p_{\mathcal{M}_{1} \to \mathcal{M}_{2}}$ denotes the projection from \mathcal{M}_{1} to \mathcal{M}_{2} and $\gamma_{j} = (1 - \|\overline{\mathbf{x}}_{j}^{\mathcal{K}}\|^{2})^{-1/2}$ is the Lorentz factor.

(3) Lorentzian centroid method [Zhang et al., 2021c; Chen et al., 2021; Qu and Zou, 2022] is expressed as,

$$AGG(\mathbf{x}_{i}^{\mathcal{L}}) := \frac{\sum_{j \in \mathcal{N}_{i}} \alpha_{ij} \mathbf{x}_{j}^{\mathcal{L}}}{\sqrt{c} |\| \sum_{j \in \mathcal{N}_{i}} \alpha_{ij} \mathbf{x}_{j}^{\mathcal{L}} \|_{\mathcal{L}}|}.$$
 (16)

The tangential mean computation is one of the most straightforward methods. It is applicable to the Poincaré ball and Lorentz models. However, directly executing the weighted mean in the tangent space need extra caution to guarantee the results still live in the manifold. On the other hand, it lacks of a differentiable Fréchet mean operation [Lou et al., 2020]. The Einstein midpoint is based on the Klein coordinates and applicable to Poincaré ball and Lorentz models by the isomorphic bijections. The Lorentzian centroid is designed for Lorentz model. Besides, there is another equivalent centroid for Poincaré ball model as shown in [Shimizu et al., 2020], i.e., Möbius gyromidpoint [Ungar, 2008]. These three centroids can be characterized as a minimizer of the weighted sum of calibrated squared distance [Shimizu et al., 2020].

3.4 Non-linear Activation

The non-linear activation can be achieved in the tangent space at origin with the same idea of matrix-vector multiplication [Chami *et al.*, 2019], i.e.,

 $\sigma^{\otimes^{c_{\ell-1},c_{\ell}}}(\mathbf{x}^{\mathcal{H}}) = \exp_{\mathbf{o}}^{c_{\ell}}\left(\sigma(\log_{\mathbf{o}}^{c_{\ell-1}}(\mathbf{x}^{\mathcal{H}}))\right), \quad (17)$ where ℓ denotes the $\ell-th$ layer. For the Lorentz model, [Zhang et al., 2021c] proposed a more accurate form,

$$\sigma^{\otimes^{c_{\ell-1},c_{\ell}}}\left(\mathbf{x}^{\mathcal{H}}\right) = \exp_{\mathbf{o}}^{c_{\ell}}\left(0, \sigma(\log_{\mathbf{o}}^{c_{\ell-1}}(\mathbf{x}_{[1:n]}^{\mathcal{H}}))\right), \quad (18)$$

which ensures that the result still lives in the Lorentz manifold. Besides, according to the manifold-preserving properties between Lorentz model and Poincaré ball model, another way is to convert the Lorentzian feature to the Poincaré feature and implement the non-linerity in Poincaré ball model [Dai et al., 2021]

3.5 Overall View

In general, a unified hyperbolic graph convolutional layer can be formulated as,

$$\mathbf{h}_{i}^{\ell,\mathcal{H}} = \left(\mathbf{W}^{\ell} \otimes^{c_{\ell-1}} \mathbf{x}_{i}^{\ell-1,\mathcal{H}}\right) \oplus^{c_{\ell-1}} \mathbf{b}^{\ell},$$

$$\mathbf{y}_{i}^{\ell,\mathcal{H}} = \mathrm{AGG}^{c_{\ell-1}} \left(\mathbf{h}^{\ell,\mathcal{H}}\right)_{i},$$

$$\mathbf{x}_{i}^{\ell,\mathcal{H}} = \sigma^{\otimes^{c_{\ell-1}c_{\ell}}} \left(\mathbf{y}_{i}^{\ell,\mathcal{H}}\right).$$
(19)

To reduce the maps between tangent space and hyperbolic manifold, some research works, for example [Liu et al., 2019], achieve all three steps in the tangent space. Although this simplification can reduce some computation burden, its performance decreases as well according to the experimental results [Zhu et al., 2020].

HGNNs have achieved great breakthroughs in a variety of graph tasks, ranging from node classification, link prediction to graph classification. Node classification is a fundamental task on graphs, which is to distinguish the node category. There are three common methods to get the category probability prediction in the node-level embeddings: tangential method [Chami et al., 2019]; centroid-based method [Liu et al., 2019]; hyperplane-based method [Zhu et al., 2020]. For hyperbolic link prediction, the probability of an edge is generally computed via the Fermi-Dirac distribution [Chami et al., 2019]. For graph-level predictions [Qu and Zou, 2022], the pooling operations (e.g., summation, average, or maximization) are performed on the node embeddings of each graph over all nodes to get the graph-level representations. In addition, HGNNs have made great breakthroughs in a variety of real-world applications. In the following, we will introduce them in detail.

4 Applications

Hyperbolic spaces have many successful applications in a variety of fields, including NLP, CV, etc., while the applications of HGNNs are mainly on recommendation systems, knowledge graphs, and drug molecules, where the dataset is graphstructured in natural and have the tree-like characteristic.

4.1 HGNNs for Recommender Systems

The recommender systems can be simplified as a bipartite graph, in which vertices represent users or items, and edges denote their interactions. Considering the prevalence of

³From https://github.com/HazyResearch/hgcn

the power-law distribution in user-item networks, hyperbolic space has attracted considerable attention. In the following, we review the recommender systems built upon hyperbolic space from four aspects.

Graph Neural Collaborative Filtering. In the settings of graph neural collaborative filtering, the features of items and users are often not available. [Sun *et al.*, 2021; Wang *et al.*, 2021b; Yang *et al.*, 2022] built the feature via Gaussian sampling. Further, they incorporate multiple layers of HGNN to gather higher-order information for explicitly modeling useritem interactions.

Social Network Enhanced Recommender System. Apart from user-item interactions, users' preferences are strongly tied to their social relationships (e.g., friends, followers). [Wang *et al.*, 2021a] designed a multi-aspect perceiving HGNN in hyperbolic space to capture multi-aspect interactions of users on item, via defining a learnable interactive relation for each specific user-item pair.

Knowledge Graph Enhanced Recommender System. Recommender systems incorporating the knowledge graph as side information can not only address the issues of data sparsity and cold start issues, but also provide explanations for recommended items. [Chen et al., 2022] proposed a Lorentzian knowledge-enhanced graph convolutional network for recommendation, which extracts high-order interaction in user-item bipartite graphs and knowledge graph for recommendation.

Session-based Recommender System. Session-based recommendation learns the user preferences by analyzing the short-term and long-term patterns based on the user behavior. [Guo *et al.*, 2021] proposed a contrastive learning manner for session-based recommendation and [Li *et al.*, 2022] presented a hyperbolic space-based hypergraph convolutional neural network to learn for session-based recommender systems.

4.2 HGNNs for Knowledge Graph

Knowledge graph is a graph-structured network representing the real-world facts with triplets to store a large number of entity and relation information. Given that in large-scale knowledge graphs, the number of entities are scale-free and can be organized to the underlying hierarchical structure, hyperbolic geometry thus provides a powerful alternative to learn low-dimensional embedding while preserving the underlying hierarchy.

Knowledge graph completion and associations are two important applications. [Wang et al., 2021c] proposed an attentive neural context aggregation to adaptively integrate the relational context for enhancing the ability to preserve the hierarchical relations. Moreover, [Wang et al., 2021d] explored the mixed-curvature for knowledge graph completion. For HGNN-based knowledge graph associations, [Sun et al., 2020] developed a HyperKA model which employs an HGNN for KG embedding and utilizes a hyperbolic transformation across embedding spaces to capture multi-granular knowledge associations.

Compared with hyperbolic transnational model for knowledge graph, e.g., KyperKG [Kolyvakis *et al.*, 2019] MuRP [Balazevic *et al.*, 2019], AttH [Chami *et al.*, 2020],

and HERCULES [Montella *et al.*, 2021], HGNN-based models show more competitive performance with further extracting and incorporating high-order relationships.

4.3 HGNNs for Molecular

Molecules are also naturally represented as graphs, with nodes representing atoms and edges representing chemical bonds. Recently, many studies used GNNs and some of their variations to predict chemical properties. For molecular applications, the research mainly focuses on molecular representation and generation. The basic motivation to apply hyperbolic spaces on molecules is to model the underlying hierarchical structure in it.

Molecular Representation. [Wu *et al.*, 2021] developed a hyperbolic relational graph convolution network plus (HRGCN+) by combining molecular graphs and molecular descriptors for drug discovery. HRGCN+ allows medicinal chemists to understand models at both the atom and descriptor levels, which can also aid in the extraction of hidden information. [Yu *et al.*, 2020] proposed to learn molecular embedding through the hyperbolic VAE framework, to discover new side-effects and re-position drugs.

Molecular Generation. Learning the implicit structure of molecules using hyperbolic space is an important method to extract molecular latent hierarchical structure. Hyperbolic molecule generation is generally via graph generation models, including normalized flow [Bose *et al.*, 2020], GAN [Qu and Zou, 2022], etc.

4.4 HGNNs for Other Applications

In this part, we will discuss more successful applications of HGNNs from different fields. For skeleton-based action recognition, [Peng et al., 2020] designed a hyperbolic spatial-temporal GCN that combines several dimensions on the manifold and provides an effective technique to explore the dimension for each ST-GCN layer. For quantitative trading and investment decision making, [Sawhney et al., 2021] modeled the inter-stock relations by HGNN and developed a stock model HyperStockGAT, which constructs the stock graph by the relation in Wikidata and their industry information. For medical ontology matching, [Hao et al., 2021] proposed MEDTO framework which is built upon HGNN and a heterogeneous graph layer.

5 Challenges and Opportunities

Although we have witnessed the rapid development and achievements of HGNNs in recent years, there are still issues and challenges that need better solutions. [Peng *et al.*, 2021] discussed several open problems which are also shared by HGNNs. In this section, we further summarize several challenges in the HGNN community which also provides opportunities for future study.

5.1 Challenges I: Complex Structures

In graph representation learning, hyperbolic space is emerging as a potential alternative. The most noticeable advantages are credit to its exponential volume growth property, which makes this space to be much more embeddable than

Euclidean space, especially for datasets with implicit tree-like layouts. However, the structure of a real-world network is always complex and complicated, in which some areas are tree-like, some are flat and some are circular. Directly embedding a graph with intricate layouts into the hyperbolic manifold inevitably leads to structural inductive biases and distortions.

Opportunities: To this point, there are some preliminary attempts, but there is still much potential for improvements. An intuitive approach is to combine Euclidean and/or spherical spaces to complement the hyperbolic space. For example, GIL [Zhu et al., 2020] utilizes hyperbolic space and Euclidean space interactively, and places different weights on two branches to cope with intricate complex graph structure. M²GNN [Wang et al., 2021d] incorporates three different spaces, namely hyperbolic, Euclidean and spherical spaces for knowledge graph completion. SelfMGNN [Sun et al., 2022] resorts to a mixed-curvature space via the Cartesian product. On the other hand, ACE-HGNN [Fu et al., 2021] attempts to learn an optimal curvature to model the tree-like graph with different hyperbolicity via a multi-agent reinforcement learning framework. HGCL [Liu et al., 2022] enhances the modeling of HGNN by contrastive learning. These methods generally need to create multiple branches, which unavoidably increases the computational complexity to a certain extent. Besides, most of the above methods solve this issue in a global manner. A local identifier, like Ricci curvature [Ni et al., 2019; Topping et al., 2022] can be more effective and efficient.

5.2 Challenge II: Geometry-aware Learning

Though HGNNs have made noticeable achievements, most of the efforts mainly focus on how to generalize GNNs into hyperbolic space by properly designing the transformation operations among the spaces. In many of HGNNs, the optimization target (like cross-entropy in [Chami *et al.*, 2019] and [Zhang *et al.*, 2021c]) are generally similar with the Euclidean counterparts which are geometrically irreverent with hyperbolic properties. On the other hand, hyperbolic space is curved, that is, locations closer to the origin are flatter and relatively narrow, whereas regions further from the origin are broader, and this property is seldom considered when designing HGNNs.

Opportunities: From the optimization target, how to integrate the hyperbolic geometry with the learning objective is the *first* valuable problem to explore. [Yang et al., 2022] presented a geometrically aware regularized learning for hyperbolic recommender systems. The second opportunity is to make the learning process geometry-aware, keeping the awareness of node position rather than being equipped with a geometry-unconscious optimization target, like the constant margin in metric learning. Last, based on the hyperbolic geometric trait, the exponentially hyperbolic space provides spacious room for the samples to be well arranged especially in the area far away from the origin. Intuitively, by encouraging the model to preserve the inherent data structure and pushing the overall embedding far away from the origin, the model representation ability could be then largely improved.

5.3 Challenges III: Trustworthy HGNNs

HGNNs have been proven to produce better representations of hierarchical graphs. However, there are still a few trustworthy issues to be solved at present: (1) Where is the superiority of the hyperbolic space? For example, the better performance of the hyperbolic model originates from where? the better fitting of high-level nodes, tail nodes, or both? (2) It is unclear in what conditions the HGNNs are guaranteed to be better than Euclidean GNNs. In other words, the generalization error and the robustness of HGNNs has not been well studied and analyzed.

Opportunities: Currently, [Zhang et al., 2021a] made an initial study using hyperbolic metric learning for recommender systems. To figure out the superiority of HGNNs, further investigations are required. For the generalization error, [Suzuki et al., 2021a] pointed that in hyperbolic ordinal embedding (HOE), the generalization error bound is at most exponential with respect to the embedding radius and HOE can represent a tree better than linear graph embedding [Suzuki et al., 2021b]. The analysis facilitates the exploration of the generalization error of HGNNs and the robustness of HGNNs.

5.4 Challenges IV: Scalable HGNNs

Despite the tremendous success in the performance improvement, HGNNs also bring in more computational requirements compared with the Euclidean counterparts, e.g., the frequent exponential and logarithmic maps [Yu and De Sa, 2022]. In real-world scenarios, graphs often on the scale of millions of nodes and edges and computational resources are limited. The difficulty of extending HGNNs to large-scale graphs remains one of the main challenges.

Opportunities: A large body of research work have been proposed to improve the scalability of GNNs by decoupling the graph process, simplifying the feature transformation or aggregation operations [Wu et al., 2019; Frasca et al., 2020], or via a series sampling techniques [Zeng et al., 2020; Hamilton et al., 2017; Huang et al., 2021], which could be further integrated with the property of HGNNs.

6 Conclusion

Hyperbolic space can be regarded as a continuous tree, making it well-suited for modeling datasets with latent hierarchy layouts. HGNNs extend GNNs into hyperbolic space and have achieved great success in graph data, especially with tree-like structure. In this survey, we present the technical details about HGNNs, including the methodologies, applications, challenges, opportunities, and current state-of-play in models and algorithms of HGNNs networks. More specifically, we unify them by a general framework and summarize the variants of each module. We also identified several challenges that need to be overcome. To some extent, these challenges serve as guidelines for flourishing the achievements of hyperbolic graph learning. Although many researchers are actively engaged in addressing these problems, we point to the numerous opportunities that still exist to contribute to the development of this important ever-evolving field.

References

- [Balazevic *et al.*, 2019] Ivana Balazevic, Carl Allen, and Timothy M. Hospedales. Multi-relational poincaré graph embeddings. In *NeurIPS*, pages 4465–4475, 2019.
- [Bose *et al.*, 2020] Avishek Joey Bose, Ariella Smofsky, Renjie Liao, Prakash Panangaden, and William L Hamilton. Latent variable modelling with hyperbolic normalizing flows. In *ICML*, pages 1045–1055, 2020.
- [Bouchard *et al.*, 2015] Guillaume Bouchard, Sameer Singh, and Theo Trouillon. On approximate reasoning capabilities of low-rank vector spaces. In *AAAI spring symposia*, 2015.
- [Bronstein *et al.*, 2017] Michael M Bronstein, Joan Bruna, Yann LeCun, Arthur Szlam, and Pierre Vandergheynst. Geometric deep learning: going beyond euclidean data. *IEEE Signal Processing Magazine*, 34(4):18–42, 2017.
- [Chami *et al.*, 2019] Ines Chami, Zhitao Ying, Christopher Ré, and Jure Leskovec. Hyperbolic graph convolutional neural networks. In *NeurIPS*, pages 4868–4879, 2019.
- [Chami et al., 2020] Ines Chami, Adva Wolf, Da-Cheng Juan, Frederic Sala, Sujith Ravi, and Christopher Ré. Low-dimensional hyperbolic knowledge graph embeddings. In ACL, pages 6901– 6914. ACL, 2020.
- [Chen et al., 2021] Weize Chen, Xu Han, Yankai Lin, Hexu Zhao, Zhiyuan Liu, Peng Li, Maosong Sun, and Jie Zhou. Fully hyperbolic neural networks. arXiv preprint arXiv:2105.14686, 2021.
- [Chen et al., 2022] Yankai Chen, Menglin Yang, Yingxue Zhang, Mengchen Zhao, Ziqiao Meng, Jianye Hao, and Irwin King. Modeling scale-free graphs for knowledge-aware recommendation. WSDM, 2022.
- [Dai et al., 2021] Jindou Dai, Yuwei Wu, Zhi Gao, and Yunde Jia. A hyperbolic-to-hyperbolic graph convolutional network. In CVPR, pages 154–163, 2021.
- [Errica *et al.*, 2019] Federico Errica, Marco Podda, Davide Bacciu, and Alessio Micheli. A fair comparison of graph neural networks for graph classification. In *ICLR*, 2019.
- [Frasca *et al.*, 2020] Fabrizio Frasca, Emanuele Rossi, Davide Eynard, Benjamin Chamberlain, Michael Bronstein, and Federico Monti. SIGN: Scalable inception graph neural networks. In *ICML GRL worshop*, 2020.
- [Fu et al., 2021] Xingcheng Fu, Jianxin Li, Jia Wu, Qingyun Sun, Cheng Ji, Senzhang Wang, Jiajun Tan, Hao Peng, and S Yu Philip. ACE-HGNN: Adaptive curvature exploration hyperbolic graph neural network. In *ICDM*, pages 111–120, 2021.
- [Ganea *et al.*, 2018] Octavian Ganea, Gary Bécigneul, and Thomas Hofmann. Hyperbolic neural networks. In *NeurIPS*, pages 5345–5355, 2018.
- [Grover and Leskovec, 2016] Aditya Grover and Jure Leskovec. node2vec: Scalable feature learning for networks. In *KDD*, pages 855–864, 2016.
- [Gulcehre *et al.*, 2019] Caglar Gulcehre, Misha Denil, Mateusz Malinowski, Ali Razavi, Razvan Pascanu, Karl Moritz Hermann, Peter Battaglia, Victor Bapst, David Raposo, Adam Santoro, et al. Hyperbolic attention networks. In *ICLR*, 2019.
- [Guo et al., 2021] Naicheng Guo, Xiaolei Liu, Shaoshuai Li, Qiongxu Ma, Yunan Zhao, Bing Han, Lin Zheng, Kaixin Gao, and Xiaobo Guo. HCGR: Hyperbolic contrastive graph representation learning for session-based recommendation. arXiv preprint arXiv:2107.05366, 2021.

- [Hamilton et al., 2017] William L Hamilton, Rex Ying, and Jure Leskovec. Inductive representation learning on large graphs. In NeurIPS, pages 1025–1035, 2017.
- [Hao et al., 2021] Junheng Hao, Chuan Lei, Vasilis Efthymiou, Abdul Quamar, Fatma Özcan, Yizhou Sun, and Wei Wang. Medto: Medical data to ontology matching using hybrid graph neural networks. In KDD, pages 2946–2954, 2021.
- [Huang *et al.*, 2021] Zengfeng Huang, Shengzhong Zhang, Chong Xi, Tang Liu, and Min Zhou. Scaling up graph neural networks via graph coarsening. In *KDD*, pages 675–684, 2021.
- [Kipf and Welling, 2016] Thomas N Kipf and Max Welling. Variational graph auto-encoders. *Bayesian Deep Learning Workshop* (NIPS 2016), 2016.
- [Kipf and Welling, 2017] Thomas N Kipf and Max Welling. Semisupervised classification with graph convolutional networks. In *ICLR*, 2017.
- [Kolyvakis et al., 2019] Prodromos Kolyvakis, Alexandros Kalousis, and Dimitris Kiritsis. HyperKG: Hyperbolic knowledge graph embeddings for knowledge base completion. arXiv preprint arXiv:1908.04895, 2019.
- [Lee, 2006] John M Lee. Riemannian manifolds: an introduction to curvature, volume 176. Springer Science & Business Media, 2006.
- [Lee, 2013] John M Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013.
- [Li et al., 2022] Yicong Li, Hongxu Chen, Xiangguo Sun, Zhenchao Sun, Lin Li, Lizhen Cui, Philip S Yu, and Guandong Xu. Hyperbolic hypergraphs for sequential recommendation. In CIKM, 2022.
- [Liu *et al.*, 2019] Qi Liu, Maximilian Nickel, and Douwe Kiela. Hyperbolic graph neural networks. In *NeurIPS*, pages 8230–8241, 2019.
- [Liu et al., 2022] Jiahong Liu, Menglin Yang, Min Zhou, Shanshan Feng, and Philippe Fournier-Viger. Enhancing hyperbolic graph embeddings via contrastive learning. In NeurIPS 2nd SSL workshop, 2022.
- [Lou et al., 2020] Aaron Lou, Isay Katsman, Qingxuan Jiang, Serge Belongie, Ser-Nam Lim, and Christopher De Sa. Differentiating through the fréchet mean. In ICML, pages 6393–6403. PMLR, 2020.
- [Montella *et al.*, 2021] Sebastien Montella, Lina Maria Rojas-Barahona, and Johannes Heinecke. Hyperbolic temporal knowledge graph embeddings with relational and time curvatures. In *ACL/IJCNLP* (*Findings*), volume ACL/IJCNLP 2021 of *Findings* of *ACL*, pages 3296–3308, 2021.
- [Ni et al., 2019] Chien-Chun Ni, Yu-Yao Lin, Feng Luo, and Jie Gao. Community detection on networks with ricci flow. Scientific reports, 9(1):1–12, 2019.
- [Peng et al., 2020] Wei Peng, Jingang Shi, Zhaoqiang Xia, and Guoying Zhao. Mix dimension in poincaré geometry for 3d skeleton-based action recognition. In ACM MM, pages 1432– 1440, 2020.
- [Peng et al., 2021] Wei Peng, Tuomas Varanka, Abdelrahman Mostafa, Henglin Shi, and Guoying Zhao. Hyperbolic deep neural networks: A survey. TPAMI, 2021.
- [Qu and Zou, 2022] Eric Qu and Dongmian Zou. Hyperbolic neural networks for molecular generation. arXiv preprint arXiv:2201.12825, 2022.

- [Ravasz and Barabási, 2003] Erzsébet Ravasz and Albert-László Barabási. Hierarchical organization in complex networks. *Physical review E*, 67(2):026112, 2003.
- [Sawhney *et al.*, 2021] Ramit Sawhney, Shivam Agarwal, Arnav Wadhwa, and Rajiv Shah. Exploring the scale-free nature of stock markets: Hyperbolic graph learning for algorithmic trading. In *WWW*, pages 11–22, 2021.
- [Shimizu *et al.*, 2020] Ryohei Shimizu, Yusuke Mukuta, and Tatsuya Harada. Hyperbolic neural networks++. In *ICLR*, 2020.
- [Sun et al., 2020] Zequn Sun, Muhao Chen, Wei Hu, Chengming Wang, Jian Dai, and Wei Zhang. Knowledge association with hyperbolic knowledge graph embeddings. arXiv preprint arXiv:2010.02162, 2020.
- [Sun et al., 2021] Jianing Sun, Zhaoyue Cheng, Saba Zuberi, Felipe Pérez, and Maksims Volkovs. HGCF: Hyperbolic graph convolution networks for collaborative filtering. In WWW, pages 593–601, 2021.
- [Sun et al., 2022] Li Sun, Zhongbao Zhang, Junda Ye, Hao Peng, Jiawei Zhang, Sen Su, and Philip S Yu. A self-supervised mixedcurvature graph neural network. AAAI, 2022.
- [Suzuki et al., 2021a] Atsushi Suzuki, Atsushi Nitanda, Jing Wang, Linchuan Xu, Kenji Yamanishi, and Marc Cavazza. Generalization error bound for hyperbolic ordinal embedding. In *ICML*, pages 10011–10021, 2021.
- [Suzuki et al., 2021b] Atsushi Suzuki, Atsushi Nitanda, Linchuan Xu, Kenji Yamanishi, Marc Cavazza, et al. Generalization bounds for graph embedding using negative sampling: Linear vs hyperbolic. In NeurIPS, 2021.
- [Topping et al., 2022] Jake Topping, Francesco Di Giovanni, Benjamin Paul Chamberlain, Xiaowen Dong, and Michael M Bronstein. Understanding over-squashing and bottlenecks on graphs via curvature. In ICLR, 2022.
- [Ungar and others, 2007] Abraham A Ungar et al. The hyperbolic square and mobius transformations. *Banach Journal of Mathematical Analysis*, 1(1):101–116, 2007.
- [Ungar, 2008] Abraham Albert Ungar. A gyrovector space approach to hyperbolic geometry. Synthesis Lectures on Mathematics and Statistics, 1(1):1–194, 2008.
- [Vázquez et al., 2003] Alexei Vázquez, Alessandro Flammini, Amos Maritan, and Alessandro Vespignani. Modeling of protein interaction networks. Complexus, 1(1):38–44, 2003.
- [Veličković *et al.*, 2018] Petar Veličković, Guillem Cucurull, Arantxa Casanova, Adriana Romero, Pietro Liò, and Yoshua Bengio. Graph attention networks. In *ICLR*, 2018.
- [Wang *et al.*, 2017] Quan Wang, Zhendong Mao, Bin Wang, and Li Guo. Knowledge graph embedding: A survey of approaches and applications. *TKDE*, 29(12):2724–2743, 2017.
- [Wang et al., 2021a] Hao Wang, Defu Lian, Hanghang Tong, Qi Liu, Zhenya Huang, and Enhong Chen. Hypersorec: Exploiting hyperbolic user and item representations with multiple aspects for social-aware recommendation. *TOIS*, page 1–28, 2021.
- [Wang *et al.*, 2021b] Liping Wang, Fenyu Hu, Shu Wu, and Liang Wang. Fully hyperbolic graph convolution network for recommendation. *arXiv* preprint arXiv:2108.04607, 2021.
- [Wang *et al.*, 2021c] Shen Wang, Xiaokai Wei, Cicero Nogueira Dos Santos, Zhiguo Wang, Ramesh Nallapati, Andrew Arnold, and S Yu Philip. Knowledge graph representation via hierarchical hyperbolic neural graph embedding. In *IEEE Big Data*, pages 540–549. IEEE, 2021.

- [Wang et al., 2021d] Shen Wang, Xiaokai Wei, Cicero Nogueira Nogueira dos Santos, Zhiguo Wang, Ramesh Nallapati, Andrew Arnold, Bing Xiang, Philip S Yu, and Isabel F Cruz. Mixedcurvature multi-relational graph neural network for knowledge graph completion. In WWW, pages 1761–1771, 2021.
- [Wu et al., 2019] Felix Wu, Amauri Souza, Tianyi Zhang, Christopher Fifty, Tao Yu, and Kilian Weinberger. Simplifying graph convolutional networks. In *ICML*, pages 6861–6871. PMLR, 2019.
- [Wu et al., 2021] Zhenxing Wu, Dejun Jiang, Chang-Yu Hsieh, Guangyong Chen, Ben Liao, Dongsheng Cao, and Tingjun Hou. Hyperbolic relational graph convolution networks plus: a simple but highly efficient qsar-modeling method. Briefings in Bioinformatics, 22(5):bbab112, 2021.
- [Xu et al., 2019] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How powerful are graph neural networks? In ICLR, 2019.
- [Yang et al., 2020] Menglin Yang, Ziqiao Meng, and Irwin King. Featurenorm: L2 feature normalization for dynamic graph embedding. In *ICDM*, pages 731–740, 2020.
- [Yang et al., 2021] Menglin Yang, Min Zhou, Marcus Kalander, Zengfeng Huang, and Irwin King. Discrete-time temporal network embedding via implicit hierarchical learning in hyperbolic space. In KDD, pages 1975–1985, 2021.
- [Yang et al., 2022] Menglin Yang, Min Zhou, Jiahong Liu, Defu Lian, and Irwin King. HRCF: Enhancing collaborative filtering via hyperbolic geometric regularization. In WWW, 2022.
- [Yu and De Sa, 2022] Tao Yu and Christopher De Sa. Hyla: Hyperbolic laplacian features for graph learning. *arXiv preprint* arXiv:2202.06854, 2022.
- [Yu et al., 2020] Ke Yu, Shyam Visweswaran, and Kayhan Bat-manghelich. Semi-supervised hierarchical drug embedding in hyperbolic space. *Journal of chemical information and modeling*, 60(12):5647–5657, 2020.
- [Zeng et al., 2020] Hanqing Zeng, Hongkuan Zhou, Ajitesh Srivastava, Rajgopal Kannan, and Viktor Prasanna. GraphSAINT: Graph sampling based inductive learning method. In ICLR, 2020.
- [Zhang and Chen, 2018] Muhan Zhang and Yixin Chen. Link prediction based on graph neural networks. *NeurIPS*, 31, 2018.
- [Zhang and Gao, 2021] Chengkun Zhang and Junbin Gao. Hypehan: Hyperbolic hierarchical attention network for semantic embedding. In *IJCAI*, pages 3990–3996, 2021.
- [Zhang et al., 2021a] Sixiao Zhang, Hongxu Chen, Xiao Ming, Lizhen Cui, Hongzhi Yin, and Guandong Xu. Where are we in embedding spaces? a comprehensive analysis on network embedding approaches for recommender systems. In KDD, 2021.
- [Zhang et al., 2021b] Yiding Zhang, Xiao Wang, Chuan Shi, Xunqiang Jiang, and Yanfang Fanny Ye. Hyperbolic graph attention network. TBD, 2021.
- [Zhang et al., 2021c] Yiding Zhang, Xiao Wang, Chuan Shi, Nian Liu, and Guojie Song. Lorentzian graph convolutional networks. In WWW, pages 1249–1261, 2021.
- [Zhu et al., 2020] Shichao Zhu, Shirui Pan, Chuan Zhou, Jia Wu, Yanan Cao, and Bin Wang. Graph geometry interaction learning. NeurIPS, 2020.