Multivariable Calculus

Yicheng Wang

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Contents

Introduction

Intro to Class

Class Information

- Mr. Joseph Stern
- jstern6@schools.nyc.gov (official uses only)
- mr.stern.mathematics@gmail.com (often checks, use for mundane matters)
- Office: 351 (is free during pd. 3, 8, and 9)
- Grading policies are the same as the ones listed for the math dept.
- Not a lot of tests and not a lot of homeworks, more focused on learning

Fundamental Theorem of Calculus

This is the culminating point of single variable calculus, it has two parts:
Part I:

$$\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$

Part II:

$$\frac{d}{dx}\int_a^x f(u)du = f(x)$$

Multivariable Calculus

This branch of calculus deals with how functions that takes in multiple independent variables behave. Like the function $f(x, y, z) = (x \sin y)^{e^z}$. However, we often observe multiple-dimension functions with a bound on the variables.

Big Question of Multivariable Calculus

The Big Questions

- What, if anything, is the higher dimensional analogue of the Fundamental Theorem of Calculus.
- Assuming there are higher dimensional analogues, do their form depend on the particular dimensions involved.

Strict Definition of the FTC

When we integrate from a to b of a function $f(x)$, we are integrating over the interval (a, b). This process does not concern the boundry points. However, the FTC establishes a relationship between the integral over the interval and the integral over the boundries. It states a equivalency of the integral of the derivative over the interval and the integral of the function at the boundries.

In the one variable form, we can write the FTC as:

$$\int_a^b f'(x)dx = \int_{\partial[a,b]} f = +f(b) - f(a)$$

The negative in front of $f(a)$ signifies a unintented orientation of the curve, that as x go towards b, their value grow larger.

Note that ∂ is the symbol we use to denote the boundary of something.

Basics of Calculus with 2D Domain

Below is a curve C in 2 dimensions, and the region inside it (gray area) can be denoted as R.

This curve is known as a **simple**, **closed** curve.

Simple means that each point is crossed by the curve at most once.

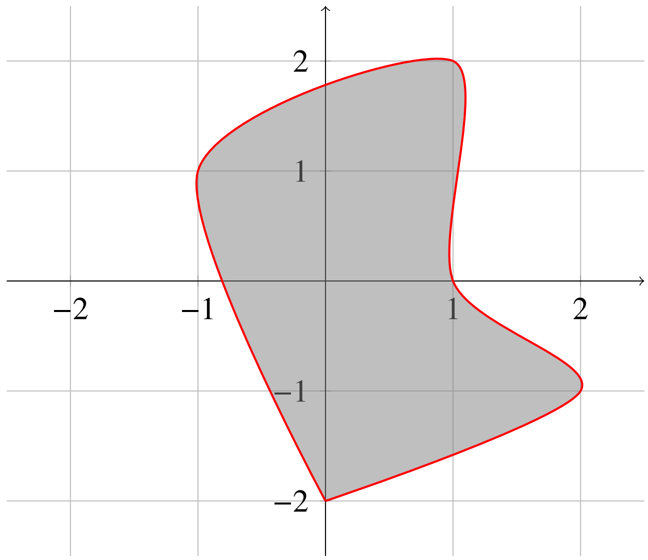
Closed means that the curve does not have a unqiue starting point and end point.

When integrating in 2 dimensions, we also need to pick an "interval." In this case, R, the bounded region, would be an interval (analogous to (a, b) in single variable calculus) and C would be the boundary (analogous to a, b).

If we were to integrate a function $f(x, y)$ over R, it is denoted by:

$$\iint_R f(x, y)dA_{xy}$$

The dA_{xy} is what is known as the **area element**. It is an infinitesimally small piece of area (this is analogous to dx, the length element in single variable calculus).



Note that in single variable calculus, there is an implied orientation, going left to right is the positive "direction," In multivariable calculus, it is accepted that the positive direction for the curve to go in is the *counterclockwise* direction.

Green's Theorem

This is one of the FTC's generalization to higher dimensions. The Green's Theorem works with functions that take in 2 variables.

Suppose there exists $f(x, y)$ and $g(x, y)$, and a region R bounded by a positively oriented, simple closed curve C. Then Green's theorem states that:

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA_{xy} = \int_C f dx + g dy$$

The RHS of the equation is evaluated parametrically.

Generalized FTC

The goal of this course in multivariable calculus is to reach the following conclusion:

For some function ω evaluated over the region M:

$$\int_M d\omega = \int_{\partial M} \omega$$

Chapter 1

Foundations of Calculus

1.1 Foundations - $\mathbb{R}$

1.1.1 Definitions of Different Number Structures

$\mathbb{N}$

We can define the natural number system by sets, like the following:

$$0 = \emptyset$$

And from there we introduce a succession operation:

$$n + 1 = n \cup \{n\}$$

So for example, $1 = \{0\} = \{\emptyset\}$, $1 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, etc.

Set theory is the most basic concept in mathematics, and nothing goes deeper.

$\mathbb{Z}$

Positive integers are defined as an ordered pair of natural numbers, for example, $2 = (2, 0)$, and $-2 = (0, 2)$

$\mathbb{Q}$

Rationals are defined as an infinite set of ordered pairs of integers. A rational number $q = \frac{n}{m}$, then $q = \{(n, m), (2n, 2m), (3n, 3m) \ldots (-2n, -2m), (-3, -3m) \ldots\}$

$\mathbb{R}$

There are several definitions of the real number system

- Each real number can be thouhgt of as an infinite sequence in the following format:

$$(s, N, d_1, d_2, d_3 \ldots)$$

  Where $s = \pm 1$, $N \in \mathbb{N}$, and $d_n \in \{0, 1, 2, 3, \ldots 8, 9\}$. It is also not the case that $d_n = d_{n+1} = \cdots = 0$. If there is a terminal decimal, we express it as 9 repeated.
- Each real number forms a subdivision of $\mathbb{Q}$ into two disjoint sets that cover the entirety of $\mathbb{Q}$, one of which lies entirely to the left of the other.

1.1.2 Basic Structure of $\mathbb{R}$

$\mathbb{R}$ is an instance of many kinds of mathematical structures, such as:

- Field under addition and multiplication.
- Ordered Set
  - the set has an ordering that reflect the operations in the field
  - this structure is what allows for comparisons, like the $<$ function
- Metric Space
  - there exists a standard distance operation between numbers
  - in $\mathbb{R}$, $dist(x, y) := |x - y|$ (:= means "is defined as")
  - the distance operation obeys certain laws such as the triangle inequality
- Vector Space
  - elements can be thought of as vectors
- Geometric Space
  - this structure means that you ca measure both length and angle
  - in $\mathbb{R}$, the angle measure can be either 0 or π
  - in higher dimensions there are more angles

1.1.3 Properties of $\mathbb{R}^n$

In higher dimensions, several of the properties of $\mathbb{R}$ are no longer valid. $\forall n > 1$, $\mathbb{R}^n$ is **NOT** a field, and **NOT** an ordered set.

1.1.4 Basic Axioms for $\mathbb{R}$

$\mathbb{R}$ is a field under addition and multiplication ($x, y \in \mathbb{R}$)

1. Additive closure: $x + y \in \mathbb{R}$
2. Associative Property of Addition: $x + (y + z) = (x + y) + z$
3. Communicative Property of Addition: $x + y = y + x$

4. 0 is the identity element of addition: $x + 0 = x$
5. Every element has an additive inverse: $x + (-x) = 0$
6. Multiplicative closure: $xy \in \mathbb{R}$
7. Associative Property of Multiplication: $x(yz) = (xy)z$
8. Communicative Property of Multiplication: $xy = yx$
9. 1 is the identity element of multiplication: $x(1) = x$
10. Every element (except 0) has a multiplicative inverse: $x \cdot \frac{1}{x} = 1$
    - Theorem: $\forall x \in \mathbb{R}$, $x \cdot 0 = 0$
    - Proof: $x \cdot 0 = x \cdot (0 + 0)$, then we apply the distributive law, and get $x \cdot 0 = x \cdot 0 + x \cdot 0$. Now we add $-x \cdot 0$ to both side, and we get: $0 = x \cdot 0$
11. Distributive Law: $x(y + z) = xy + xz$

$\mathbb{R}$ is an ordered field and has a proper subset (aka not the entire set) $\mathbb{R}^+$ (the **positives**) such that:

1. $\mathbb{R}^+$ is closed under addition and multiplication.
2. $1 \in \mathbb{R}^+, 0 \notin \mathbb{R}^+$
3. **Trichotomy Property**: for any $x \in \mathbb{R}$, x is either 0, $\in \mathbb{R}^+$ or $\notin \mathbb{R}^+$

Definition of $<$ and $>$:

- $x < y$ means $y - x \in \mathbb{R}^+$
- $x > y$ means $y < x$

1.1.5 Separation Axiom

If $\mathcal{A} \subseteq \mathbb{R}$ and $\mathcal{B} \subseteq \mathbb{R}$. Satisfying:

1. $\mathcal{A} \cap \mathcal{B} = \emptyset$
2. $\mathcal{A} \neq \emptyset, \mathcal{B} \neq \emptyset$
3. $\mathcal{A} < \mathcal{B}$ "$\mathcal{A}$ is to the left of $\mathcal{B}$"
    - $\forall a \in \mathcal{A}, \forall b \in \mathcal{B}, a < b$

$\exists$ at least 1 $c \in \mathbb{R}$ such that $\mathcal{A} \leq c \leq \mathcal{B}$

Note that $\mathcal{A}$ and $\mathcal{B}$ do not have cover the entire real number line.

Existence of Irrationals

The main difference between the $\mathbb{Q}$ and the $\mathbb{R}$ is the separation axiom. The rationals does not have that property. So it is possible to have two non-empty, non-overlapping subsets, one entirely to the left of the other, but has no rational number to form a boundary. For example:

$$\mathcal{A} = \mathbb{Q}^- \cup \{0\} \cup \{q \in \mathbb{Q}^+ | q^2 < 2\}$$

$$\mathcal{B} = \{q \in \mathbb{Q}^+ | q^2 \geq 2\}$$

We know that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$ because 0 is in $\mathcal{A}$ and 2 is in $\mathcal{B}$. We also know that $\mathcal{A} \cup \mathcal{B} = \mathbb{Q}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. As well as the fact that $\mathcal{A} < \mathcal{B}$.

Now, if we want to find the boundary element, q_0, which separates $\mathcal{A}$ and $\mathcal{B}$. We know that $\mathcal{A} \leq q_0 \leq \mathcal{B}$. So that must mean $q_0 = \sqrt{2}$. However, $\sqrt{2} \notin \mathbb{Q}$. Therefore, we know that the set of rational numbers do not follow the separation axiom.

This basically means that the rational number system have holes in it, holes that are filled within the real number system.

1.2 Sequences in $\mathbb{R}$

1.2.1 The Least Upper Bound Theorem (LUB Theorem)

Theorem

If $\mathcal{A} \subseteq \mathbb{R}$ is non-empty, and is **bounded above** (So $\exists b_1 \in \mathbb{R}$ such that $\mathcal{A} < b_1$), then $\mathcal{A}$ has a **least upper bound**, i.e. a number $b_0 \in \mathbb{R}$ such that $\mathcal{A} \leq b_0$ and for any b with $\mathcal{A} \leq b$, $b_0 \leq b$

$$\mathcal{A} \subseteq \mathbb{R}, \mathcal{A} \neq \emptyset, (\exists\, b_1 \in \mathbb{R} : \mathcal{A} \leq b_1) \rightarrow [\exists\, b_0 \in \mathbb{R} : \mathcal{A} < b_0, \forall b \in \mathbb{R}(\mathcal{A} \leq b \rightarrow b_0 \leq b)]$$

b_0 is known as the least possible upper bound, or the *supremum* of $\mathcal{A}$, we write $b_0 = \sup \mathcal{A}$.

Similarly, for any non-empty set $\mathcal{A}$ bounded below, it has a **greatest lower bound**, $\inf \mathcal{A}$, called the *infimum* of $\mathcal{A}$

Some other notations on this, $\sup_{x \in D} f(x)$ means the supremum of all the values of $f(x)$ over D.

Proof

Define $\mathcal{B}$ to be the set of all upper bounds of $\mathcal{A}$. Let $C = \mathbb{R} \setminus \mathcal{B}$. Clearly $\mathcal{B}$ is nonempty; also C is non-empty because it contains $x_0 - 1$, where $x_0 \in \mathcal{A}$. By the way in which we defined C, $\mathcal{B} \cap C = \emptyset$. Pick any $c \in C$ and $b \in \mathcal{B}$. By thedefinition of C, $\exists\, x_1 \in \mathcal{A} : c < x_1$. But $x_1 \leq b$ by the definition of $\mathcal{B}$. Therefore $C < \mathcal{B}$. By the separation postulate, $\exists\, b_0 \in \mathbb{R} : C \leq b_0 \leq \mathcal{B}$. Note that $\mathcal{A} \setminus \{b_0\} \subseteq C$. Thus, b_0 is an upper bound for $\mathcal{A}$. Morever, it is the least upper bound because $b_0 \leq \mathcal{B}$.

1.2.2 Bounded Monotone Sequence Theorem

Theorem

For any sequence $\{a_n\}$

1. If $a_n \leq a_{n+1}$ for all $n \geq 1$, and $\exists\, b \in \mathbb{R}$ such that $a_n \leq b$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n$ exists and is less than or equal to b
2. If $a_n \geq a_{n+1}$ for all $n \geq 1$, and $\exists\, b \in \mathbb{R}$ such that $a_n \geq b$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n$ exists and is greater than or equal to b

Proof

We first convert the sequence $\{a_n\}$, which is bounded by b into the set $\mathcal{A} = \{a_n | n \geq 1\}$. We know that $\mathcal{A} \neq \emptyset$ because the sequence has some terms. We also know that $\mathcal{A}$ is bounded above by b; $\mathcal{A} < b$

By the Least Upper Bound Theorem, $\exists\, b_0 = sup\mathcal{A}$. We now show that $b_0 = \lim_{n\to\infty} a_n$. By the definition of limits, to say $b_0 = \lim_{n\to\infty} a_n$ means to say $\forall \varepsilon > 0, \exists\, N > 0, \forall n \geq N, |a_n - b_0| < \varepsilon$

If we look at the number $b_0 - \varepsilon$, it is not an upper bound on $\mathcal{A}$ because b_0 is the least upper bound and $\varepsilon > 0$. Therefore, $\exists\, a_N > b_0 - \varepsilon$. Since $\{a_n\}$ is increasing, $\forall n > N$, $a_n > b_0 - \varepsilon$. If we rearrange the terms, we get $b_0 - a_n < \varepsilon$. Therefore, b_0 (which exists by the least upper bound theorem) is the limit of a_n as $n \to \infty$.

1.2.3 Archimedean Property

Property

For any positive numbers x and y, it is possible to find some $n \in \mathbb{N}$ such that $nx > y$.

$$\forall x, y > 0, \exists\, n \in \mathbb{N} : nx > y$$

Proof

Assume that $\neg\, \exists\, n : nx > y$, this is logically equivalent to $\forall n : nx \leq y$. Let $C = \{nx \mid n \in \mathbb{N}\}$. Then $C \leq y$, let $c = \sup C$. We claim that $\exists\, N : c - \frac{1}{2}x < Nx \leq c$. This is true because if such N does not exist, then $c - \frac{1}{2}x$ would be an upper bound, but c is the least upper bound, so such N must exist. Now we've established the existence of N, let us consider $(N+1)x$. $(N+1)x = Nx + x > (c - \frac{1}{2}x) + x = c + \frac{1}{2}x > c$. But $(N+1)x \in C$, so it should be $< c$. We have a contradiction. This shows that the original assumption is false, so $\forall x, y > 0, \exists\, n \in \mathbb{N} : nx > y$

Consequences

This property can be used to show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

If we consider the definition of limits, the statement is equivalent to saying that $\forall \varepsilon \in \mathbb{R} > 0, \exists\, N \in \mathbb{N}, \forall n > N, n \in \mathbb{R}, \frac{1}{n} < \varepsilon$.

If we rearrange the term, we get that we need to show $1 < \varepsilon N$ for any ε. This is true because of the Archimedean Property. $\forall n > N$, since $\varepsilon > 0$, $1 < \varepsilon N < \varepsilon n$. Therefore we know the limit is truely 0.

1.2.4 Well Ordering Principle

Principle

For any set $\mathcal{A} \in \mathbb{N}, \mathcal{A} \neq \emptyset$, $\min \mathcal{A}$ exists.

Proof

Consider $\mathcal{A} \neq \emptyset \in \mathbb{N}$. Let $\mathcal{J} := \mathbb{N} \setminus \mathcal{A}$. Note that $0 \notin \mathcal{A}$ because $0 = \min \mathbb{N} = \min \mathcal{A}$, which can not happen. Therefore $0 \in \mathcal{J}$. Now by induction we can prove that if $0, 1, \ldots, J \in \mathcal{J}$, then $J + 1 \in \mathcal{J}$. Suppose $J + 1 \in \mathcal{A}$, since $0, 1, \ldots, J \notin \mathcal{A}$, $J + 1 = \min \mathcal{A}$, which cannot happen. Therefore $J + 1 \in \mathcal{J}$. By this argument we prove that $\mathcal{J} = \mathbb{N}$, therefore as long as $\mathcal{A} \neq \emptyset$, it has a minimum.

1.2.5 Sunrise Lemma

Lemma

For any sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, there exists at least one subsequence $(a_{n_k})_{k=1}^{\infty}$ [here, $(n_k)_{k=1}^{\infty}$ is strictly increasing sequence of $\mathbb{N}$, note that $n_k \geq k, \forall k \in \mathbb{N}$] that is monotone.

$$\forall (a_n)_{n=1}^{\infty} \in \mathbb{R}, \exists\ (a_{n_k})_{k=1}^{\infty} : a_{n_{k+1}} \geq a_{n_k}$$

Proof

This proof uses a concept known as **vista**. N is a vista in the sequence (a_n) if $a_N > a_k \forall k > N$. Note that if two points, M and N are both vistas and $M > N$, then $a_N > a_M$, because otherwise a_N would not be a vista.

There are two cases that covers all possible sequences, a sequence (a_n) has either a finite amount of vistas or an infinite amount of vistas.

Let's first look at the case in which a sequence (a_n) has an infinite amount of vistas. Let the set $\mathcal{V}$ be the set of vistas. Now we can create a constaly decrasing sequence recursively:

$$n_1 = \min \mathcal{V}$$
$$n_k = \min (\mathcal{V} \cap (n_{k-1}, \infty))$$

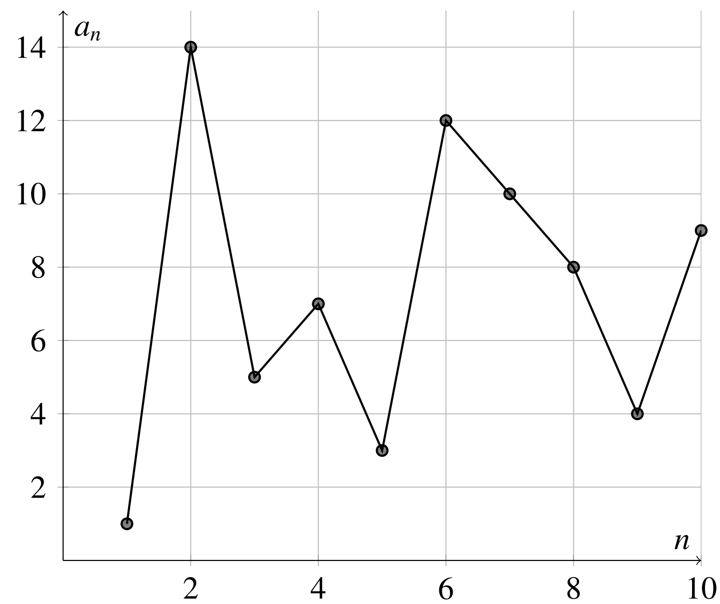


Figure 1.1: a_2, a_6, a_7 and a_10 are vistas

We know n_1 exists because of the Well Ordering Principle. And we know n_k exists because $\mathcal{V}$ is an infinite set.

Thus we've created a sequence (a_{n_k}) that is strictly decreasing.

Now let us consider the case in which the set of vistas is finite. Still, let $\mathcal{V}$ be the set of vistas. Then let's define n_k in the following way:

$$n_1 = \begin{cases} 1, & \text{if } \mathcal{V} = \emptyset \\ 1 + \max \mathcal{V}, & \text{if } \mathcal{V} \neq \emptyset \end{cases}$$

$$n_k = \min \{n > n_{k-1} | a_n > a_{n_{k-1}}\}$$

Note that $\{n > n_{k-1} | a_n > a_{n_{k-1}}\}$ cannot be empty, because the vista set is finite by assumption.

Then (a_{n_k}) is increasing.

1.2.6 Bolzano-Weierstrass Theorem

Theorem

Every bounded sequence $\in \mathbb{R}$ has at least 1 convergent subsequence.

Proof

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence. Take any monotone subsequence $(a_{n_k})_{k=1}^{\infty}$. We know it exists because of the Sunrise Lemma. Then this sequence $(a_{n_k})_{k=1}^{\infty}$ is both monotone and bounded. Therefore it converges by the Bounded Monotone Sequence Theorem.

1.2.7 Extreme Value Theorem in $\mathbb{R}$ (EVT - 1)

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous [for any point x_0, $f(x) = \lim_{x \to x_0} f(x)$ for any $x_0 \in (a, b)$, $f(a) = \lim_{x \to a^+} f(x)$, $f(b) = \lim_{x \to b^-} f(x)$]. Then $\exists\, c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$, $\forall x \in [a, b]$. In other words, the function takes on a minimum and a maximum at some point in its domain.

Proof

First we prove that $f(x)$ is a bounded above: $\exists\, M \geq 0 : f(x) \leq M$, $\forall x \in [a, b]$

We prove this by contradiction, assume that $f(x)$ is not bounded above. Thus, for any $n \in \mathbb{N}$, $\exists\, x_n \in [a, b]$ such that $f(x_n) > n$. The sequence $(x_n)_{n=1}^{\infty}$ is bounded between $[a, b]$, so by the Bolzano-Weierstrass Theorem, it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ converging to some point $\hat{x} = \lim_{k \to \infty} x_{n_k}$.

Claim: $\hat{x} \in [a, b]$.

If not, this means that $\hat{x} < a$ or $\hat{x} > b$. For illustration, say $\hat{x} > b$. Then $\exists\, \varepsilon > 0$ such that $[a, b] \cap (\hat{x} - \varepsilon, \hat{x} + \varepsilon) = \emptyset$. But $\exists\, M$ such that $x_M \in (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$ because $\lim_{n \to \infty} x_n = \hat{x}$. But x_M is also within the closed interval $[a, b]$, because all x_n are chosen from that set. But this is a contradiction because we found an element x_M that is supposedly in two disjoint sets. Therefore, $\hat{x} \in [a, b]$.

But now, $\lim_{k \to \infty} f(x_{n_k}) = f(\hat{x})$. Because of the assumed continuity on $f(x)$. So $\exists\, K$ such that $\forall k \geq K$, $f(x_{n_k}) < f(\hat{x}) + 1 \in \mathbb{R}$. Because $f(x_{n_k})$ is approaching $f(\hat{x})$. On the other hand, we know that $f(x_{n_k}) > n_k > f(\hat{x}) + 1$ when k is sufficiently large. But now we have a contradiction, $f(x_{n_k}) > f(\hat{x}) + 1$ AND $f(x_{n_k}) < f(\hat{x}) + 1$. Therefore, $f(x)$ must be bounded above.

A similar prove can show that the function is bounded below and is left as an exercise to the reader.

Now we can prove that $\sup f(x)$ is taken on by $f(x)$. We now know $\mathcal{R} = f([a, b]) = \{f(x) \mid x \in [a, b]\}$. We know that this set is bounded both above and below. Thus, by the LUB Theorem, we know that $\sup \mathcal{R}$ and $\inf \mathcal{R}$ exists. Let $S := \sup \mathcal{R}$, $I := \inf \mathcal{R}$. $\exists\, (y_n)_{n=1}^{\infty}$ such that $y_n \in \mathcal{R}$ for all n, and $\lim_{n \to \infty} y_n = S$. Since $y_n \in \mathcal{R}$, $\exists\, x_n \in [a, b]$ such that $f(x_n) = y_n$. Now the sequence $(x_n)_{n=1}^{\infty}$ is bounded between $[a, b]$, so it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to $\hat{x} \in [a, b]$. By the same argument as before, the limit must be within the same interval. Also, by continuity, of the function $f(x)$, $\lim_{k \to \infty} f(x_{n_k}) = f(\hat{x})$. But this limit is y_{n_k}. And $\lim_{k \to \infty} = y_{n_k} = S$. Because if an entire sequence approaches a number, any subsequence converges to that number as well. But that means $f(\hat{x}) = S$. This shows that the value S is taken on by the function at some point in $[a, b]$. And the same is for the infimum. ■

1.2.8 Intermediate Value Theorem in $\mathbb{R}$

Theorem

$f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$. $\forall y$ between $f(a)$ and $f(b)$ ($f(a) < y < f(b)$ or $f(a) > y > f(b)$). Then $\exists c \in (a, b) : f(c) = y$.

Proof

WLOG say $f(a) < y < f(b)$. Take the set $S = \{x \in [a, b] \mid f(x) \leq y\}$. Since $a \in S$, $S \neq \emptyset$, we also know that $S \leq b$. Now take $c = \sup S$, $a \leq c \leq b$ ($c \in [a, b]$). There are three possible cases, $f(c)$ is either $> y$, $< y$, or $= y$.

Consider the case in which $f(c) < y$. If this is the case, then we can chose an arbitrarily small $\delta > 0$ such that $f(c + delta) < y$. However, then $c + \delta \in S$. But this causes a contradiction because $c = \sup S$ and there should not be any element of S that's larger than c. Therefore, this case is impossible.

Now consider the case in which $f(c) > y$. Take some arbitrarily small $u \in [0, \delta], \delta > 0$ such that $f(c - u) > y$. However, then $c - \delta$ is therefore an upper bound of the set S, we once again reach the same contradiction.

Therefore, since c exists, $f(c) = y$.

Chapter 2

Basics of Calculus

2.1 Basic Calculus Concepts in $\mathbb{R}^d$

2.1.1 Limits in Higher Dimensions

Definition of "Approaching"

Note that the concept of "approaches" is loosely defined on the plane.

Note that there are infinitely many ways in which the point (x, y) can move to (a, b). It can go in a straight line, a curve, or even a spiral. Therefore it's not a good idea to define "approaching" by drawing lines.

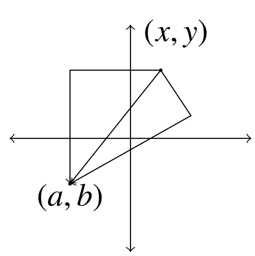
Instead, we say that $(x, y) \to (a, b)$ iff $dist(x, y \ ; \ a, b) \to 0$.

Definition of "Limit"

In single variable calculus, we sometimes describe limits on the border points with:

$$\lim_{x \to a^+} f(x) = L$$

What we mean by this is that:



$$\forall \varepsilon > 0, \exists\, \delta(\varepsilon) : \forall x \in D \cap (a, \infty), |x - a| < \delta(\varepsilon) \to |f(x) - L| < \varepsilon$$

When we take limit in higher dimensions, the path of approach can vary as long as $(x, y) \in D$.

Let $f : D \to \mathbb{R}$ and $D \subseteq \mathbb{R}^2, D \neq \emptyset$, Let $(a, b) \in D \cup \partial D$.

Then we say:

$$L = \lim_{\substack{(x,y)\to(a,b)\\(x,y)\in D}} f(x, y) \text{ if and only if:}$$

$$\forall \varepsilon > 0, \exists\, \delta > 0, \forall (x, y) \in D : dist(x, y\ ;\ a, b) < \delta \to |f(x, y) - L| < \varepsilon$$

Note that this limits the approach when $(a, b) \in \partial D$, it limits the path to inside the domain.

Finding the Limit

In higher dimensions there is no easy ways of finding the limit like the l'hôpital's rule. The way to find the limit is to use the definition. Pick two arbitrary paths of approach, and calculate the limit for each. If they do not equal, then the limit doesn't exist. But if they are the same, then you pick more lines and that value is probably the limit, then we prove it is the limit using its definition.

Examples

Example – Polar Substitution 1

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$$

$$D = \text{dom } f = \mathbb{R}^2 \setminus \{(0, 0)\}, (0, 0) \in \partial D$$

We use polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, then the expression becomes $\frac{2r^2\cos(\theta)\sin(\theta)}{r^2}$. However, this depends on θ. But that is determined by the direction of approach. For example, if $\theta = 0$, we are approching the origin from above in the x direction, and y stays the same, but if $\theta = \frac{\pi}{4}$, we are approaching the origin from above in both the x and y direction. But if we plug in 0 as θ, we get that the limit is 0, but if we plug in $\theta = \frac{\pi}{4}$, we don't get 0.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = DNE$$

Example – Polar Substitution 2

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2 + y^2}$$

We again do our polar substitution, we get that the expression is $\frac{2r^3\cos(\theta)\sin^2(\theta)}{r^2} = 2r\cos(\theta)\sin^2(\theta)$. However, we see that the approach of $(x,y) \to (0,0)$ is equivalent to $r \to 0$. Therefore, our limit becomes

$$\lim_{r\to 0^+} 2r\cos(\theta)\sin^2(\theta)$$

However, we don't know what θ is, but it doesn't matter, since the trig functions are bounded, we know that as r goes to 0, so does the limit.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \boxed{0}$$

2.1.2 Continuity

Definition of Continuity

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, $D \neq \emptyset$. Let $(a,b) \in D$. We say that f is **continuous** at (a,b) if:

$$f(a,b) = \lim_{\substack{(x,y)\to(a,b)\\(x,y)\in D}} f(x,y)$$

Or in other terms:

$$\forall\varepsilon > 0, \exists\,\delta > 0, \forall(x,y) \in D : dist(x,y\ ;\ a,b) < \delta \to |f(x,y) - f(a,b)| < \varepsilon$$

And we say that f is continuous if f is continuous at $(a,b), \forall(a,b) \in D$.

Uniform Continuity

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^d$. We say f is uniformally continuous on D if:

$$\forall\varepsilon > 0, \exists\,\delta > 0 : \forall\vec{x},\vec{y} \in D, ||\ \vec{x} - \vec{y}\ || < \delta \to |f(\vec{x}) - f(\vec{y})| < \varepsilon$$

The difference between this and regular continuity is that the δ in regular continuity is defined by both ε and the specific point we are considering. Uniform continuity, however, the value δ is independent to the point you chose within the domain and is just dependent on ε.

For example, consider $y = \tan x$ where $D = (-\frac{\pi}{2}, \frac{\pi}{2})$. the value required for δ for a fixed ε gets smaller and smaller as x approaches both endpoints. This function is continuous but not uniformally so. If it were uniformally continuous, that δ value would NOT change.

Thm: If f is uniformally continuous on D, then f is continuous for every point in D.

2.2 Concepts in Higher Dimensional Math

2.2.1 Definitions and Terms

Cartesian Product

The Cartesian Plane represents the set

$$\mathbb{R}^2 := \{(x, y) \mid x, y \in R)\}$$

This is known as the **Cartesian Product** of $\mathbb{R}$ with itself. The Cartesian Product of two sets $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \times \mathcal{T} := \{(s, t) \mid s \in \mathcal{S}, t \in \mathcal{T}\}$.

Similarly, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \times \mathbb{R}$.

Shapes

open-ball: $B_r(P) := \{X \mid dist(X, P) < r\}$

closed-ball: $\bar{B}_r := \{X \mid dist(X, P) \leq r\}$

Sphere: $S_r := \{X \mid dist(X, P) = r\}$

Boundary

Given $D \subseteq \mathbb{R}^2, D \neq \emptyset$. We say $(a, b) \in \partial D$, i.e. (a, b) is on a **boundry point** of D if $\forall \varepsilon > 0$, there are points $(x, y) \in D$ and $(u, v) \in D^c$ ($D^c := \mathbb{R}^2 \setminus D$) such that $dist(x, y\ ;\ a, b) < \varepsilon$ and $dist(u, v\ ;\ a, b) < \varepsilon$

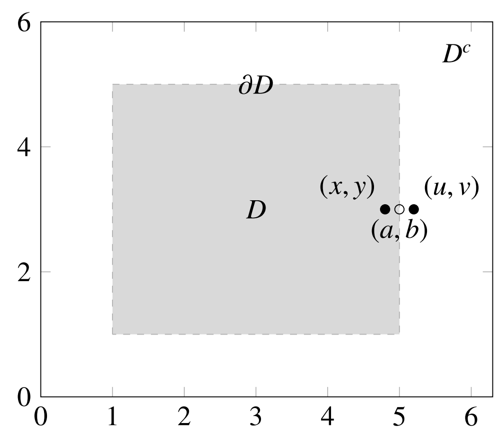


Figure 2.1: $(a, b) \in \partial D$ if we can find (x, y) and (u, v) for all ε

Thm: Because the definition is symmetrical, $\partial D^c = \partial D$.

Interior

Let $D \subseteq \mathbb{R}^2$. We say (a, b) is an **interior point** of D if $\exists\, r > 0 : B_r(a, b) \subseteq D$.

The set of all interior points is called the **interior** of D and is written as int D or D°.

Exterior

Let $D \subseteq \mathbb{R}^2$. We say (a, b) is an **exterior point** for D if it is an interior point of D^c. $\exists\, r > 0 : B_r(a, b) \subseteq D^c$.

The set of all exterior points for D is the **exterior** of D, written as ext D.

Thm: For any $D \subseteq \mathbb{R}^2$, $\mathbb{R}^2 = \text{int } D \cup \partial D \cup \text{ext } D$. And $\text{int } D \cap \partial D = \emptyset$, $\text{int } D \cap \text{ext } D = \emptyset$, $\partial D \cap \text{ext } D = \emptyset$.

Thm: $\text{int } D = \text{ext } D^c$ and $\text{ext } D = \text{int } D^c$

Thm: $\text{ext } D \subseteq D^c$

Closure

The **closure** of $D \subseteq \mathbb{R}^2$:

$$\bar{D} := D \cup \partial D$$

Thm: $\partial \bar{D} = \partial D$, $\text{int } \bar{D} = \text{int } D$, and $\text{ext } \bar{D} = \text{ext } D$

Thm: $\text{int } D \subseteq D \subseteq \bar{D}$.

Ordered Pair

The ordered pair (a, b) can be thought of as a set, but a set is inheritly unordered. To express the order, we can do the following: $(a, b) = \{\{a\}, \{a, b\}\}$. Now we know that a is the first element because it appears in both subsets.

We can then expand this into higher dimensions like the following: $(a, b, c) = ((a, b), c)$. Note that this means that $((a, b), c) \neq (a, (b, c))$. But this does not matter to us.

Fundamental Postulate of Ordered Pairs:

$(a_1, a_2, a_3, \ldots, a_n) = (b_1, b_2, b_3, \ldots, b_n)$ if and only if $a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_n = b_n$.

Vector and Points

Vectors are quantities of directionality and length, its location does not matter. Points are just positions in space. In higher dimensions with no ambiance space (flat space surrounding the surface, i.e. the shortest distance in the ambiant space is the straight line), we define a vector as all the lines with the same direction at a certain point.

However, the nice thing about $\mathbb{R}^d$ is that there is always ambiance space, so we will not make any notational distinction between a point and a vector.

The length of a vector in d space is defined as:

$$||\vec{a}|| := dist(\vec{0}, \vec{a}) = \sqrt{\sum_{i=1}^{d} a_i^2}$$

2.2.2 Distance

Euclidean Distance

The distance function in one space between two points a and b is simply $|a - b|$. However, we can also write it in the following way: $\sqrt{(a-b)^2}$

In $\mathbb{R}^2$, the distance function is:

$$dist(x, y \ ; \ a, b) := \sqrt{(x-a)^2 + (y-b)^2}$$

And in $\mathbb{R}^3$, the distance function is:

$$dist(x, y, z \ ; \ a, b, c) := \sqrt{(x-a)^2 + (y-b)^2 + (c-z)^2}$$

The generalized form of Euclidean Distance in N space is:

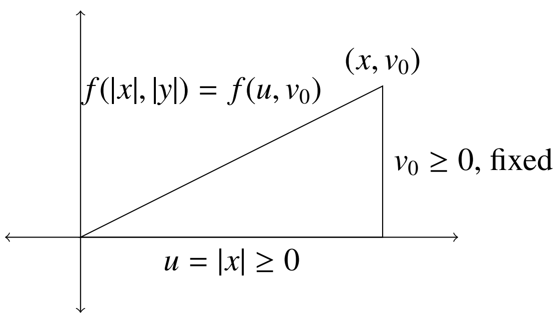
$$dist(\vec{p}, \vec{q}) = \sqrt{\sum_{j=1}^{N} (p_j - q_j)^2}$$

This is known as the **Euclidean Distance**. We use this specific definition of distance because this is preserved under an infinite set of rigid or isometric motions, such as rotation, reflection, translation, etc.

The Euclidean Distance formula is also used because it has its roots within the Pythagorean Theorem. However, that is a very high level formula, and using it from the very basics of geometry is cheating. However, there is a clever way to limit the distance formula down to almost only the Euclidean Distance formula. We set the following restrictions on a distance function:

- Translation-Invariant and symmetric
  $\forall T_{h,k}(x, y) \mapsto (x+h, y+k)$
  $dist(x+h, y+k \ ; \ \hat{x}+h, \hat{y}+h) = dist(x, y \ ; \ \hat{x}, \hat{y})$
  $\therefore dist(x, y \ ; \ \hat{x}, \hat{y}) = f(|x-\hat{x}|, |y-\hat{y}|)$
  where f is a function defined on $[0, \infty) \times [0, \infty)$

- Basic Reflection Symmetry ("isotropy" or direction independent)
  $dist(x, y\ ;\ 0, 0) = dist(y, x\ ;\ 0, 0)$
  $\therefore f(u, v) = f(v, u)\ \forall u \geq 0, v \geq 0$
- Self-Distance of $(0, 0)$
  $dist(0, 0\ ;\ 0, 0) = 0$
  $\therefore f(0, 0) = 0$
- Recreate the Standard Distance Function of Each Axis
  $dist(x, 0\ ;\ \hat{x}, 0) = |x - \hat{x}|$ and $dist(y, 0\ ;\ \hat{y}, 0) = |y - \hat{y}|$
  $\therefore f(u, 0) = u, f(0, v) = v, \forall u \geq 0, v \geq 0$
- Asymptotic Flatness



$\lim_{u\to\infty} \frac{f(u,v_0)}{u} = 1 \quad (v_0 \text{ fixed})$

$\lim_{v_0\to\infty} \frac{f(u,v_0)}{v} = 1 \quad (u \text{ fixed})$

- Continuity
  f should be continuous in its two variables
- Set of Isometry that Fix $(0, 0)$ is an Infinite Set
  An isometry bijective transformation of the plane onto itself that preserves distances.

Given all of these, we can guess a form of the theorem, we claim that the form of the theorem is the following:

$$f(u, v) = F(G(u) + G(v))$$
$$F : [0, \infty] \to \mathbb{R}$$
$$G : [0, \infty] \to \mathbb{R}$$

Thm: $\exists$ only one suitable pair F, G is

$$G(x) = x^2$$
$$F(x) = \sqrt{x}$$

Then we arrive at the Euclidean Formula

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y{-}_0)^2}$$

Properties:

1. $dist(\vec{p}, \vec{q}) = dist(\vec{q}, \vec{p})$
2. $dist(\vec{p}, \vec{q}) \geq 0$
3. $dist(\vec{p}, \vec{q}) = 0 \leftrightarrow \vec{p} = \vec{q}$

Basic Distance Bounds Lemma

$\forall \vec{p}, \vec{q} \in \mathbb{R}^d$, and $\forall j \in \{1, 2, 3, \ldots, d\}$:

$$|p_j - q_j| \leq dist(\vec{p}, \vec{q}) \leq \sqrt{d} \max_{1 \leq k \leq d} |p_k - q_k|$$

Proof

Note that $(p_j - q_j)^2 \leq \sum_{k=1}^{d} (p_k - q_k)^2$ is trivial, because you can only add positive number when you add squares. Now let's take the square root, and we get

$$\sqrt{(p_j - q_j)^2} = |p_j - q_j| \leq \sqrt{\sum_{k=1}^{d} (p_k - q_k)^2} = dist(\vec{p}, \vec{q})$$

To prove the other inequality, it is trivial as well. We can just factor out the length of the vector d and multiply that with the maximum value of the distance vector. Then we get:

$$dist(\vec{p}, \vec{q}) = \sqrt{\sum_{k=1}^{d} (p_k - q_k)^2} \leq \sqrt{d \max_{1 \leq k \leq d} (p_k - q_k)^2} = \sqrt{d} \max_{1 \leq k \leq d} |p_k - q_k|$$

Cor: Componentwise Nature of Convergence

Let $(\vec{p}_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$, and let $\vec{p} \in \mathbb{R}^d$. Then $\vec{p}_n \to \vec{p}$ if and only if $p_{n|j} \to p_j$ ($\vec{p} = (p_1, p_2, p_3, \ldots, p_d)$ and $\vec{p}_n = (p_{n|1}, p_{n|2}, \ldots, p_{n|d})$). Otherwise known as convergence of points can be reduced to conversion of dimensions.

This follows directly from the inequality, because if the total distance goes to 0, then $|p_j - q_j|$ goes to 0. Therefore if the points converge, the corresponding coordinates must converge.

To prove the converse, we prove using the other side of the distance bounds. If all d coordinates are going to 0, then if we take the maximum, that would be going to 0. (the maximum of a sequence is less than the sum of the sequence, but if every term of the sum is going to 0, then the sum is going to 0, then the maximum is going to 0). Therefore the distance must also be going to 0. Thus the two points converges.

Distances Between Sets

We define the distance between a point $\vec{p}$ and a set D as:

$$dist(\vec{p}; D) = \inf_{\vec{d} \in D} dist(\vec{d}; \vec{p})$$

We also define the distance between two sets D_1 and D_2 as:

$$dist(D_1; D_2) = \inf_{\substack{\vec{p}\in D_1 \\ \vec{q}\in D_2}} dist(\vec{p}; \vec{q})$$

2.2.3 Coordinate Space

Space

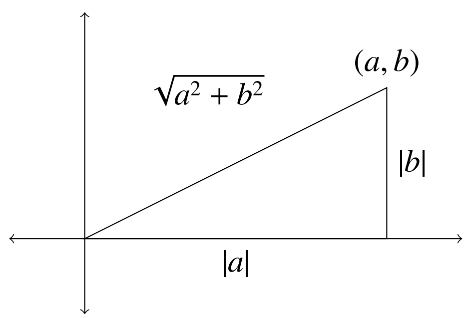
$$\mathbb{R}^d = \{(x_1, x_2, \dots, x_d) \mid x_1, \dots, x_d \in \mathbb{R}\}$$

Line

$$l = \{(a_1 + tb_1, a_2 + tb_2, \dots, a_d + tb_d) \mid t \in \mathbb{R}\}$$

Where $b_1^2 + b_2^2 + \cdots + b_d^2 > 0$

Pythagorean Theorem



We declare the x-axis and y-axis to be $\perp$.

And from the picture we can see that the Pythagorean Theorem is true for all axial triangles. But note that the Euclidean distance is preserved under rotation. We therefore deduce that the Pythagorean Theorem is true for any right triangles.

Perpendicularity

Now we can define perpendicularity. Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $dist(\vec{a}\ ;\ \vec{b})^2 = dist(\vec{0}\ ;\ \vec{a})^2 + dist(\vec{0}\ ;\ \vec{b})^2$, where $\vec{a} \neq 0, \vec{b} \neq 0$.

Note that we call the two orthogonal if one of the vectors is 0.

Dot Product

If we stay in $\mathbb{R}^2$, algaberically we get:

$$(a_2 - b_2)^2 + (a_2 - b_2)^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2)$$

Now if we subtract both side by $a_1^2 + a_2^2 + b_1^2 + b_2^2$, and divide by $-\frac{1}{2}$, we get:

$$a_1 b_2 + a_2 b_2 = 0$$

Thm: $\vec{a} \perp \vec{b}$ iff $a_1 b_1 + a_2 b_2 = 0$ where $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$.

We define this product as the **dot product**.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 \text{ in } \mathbb{R}^2$$

In $\mathbb{R}^d$,

$$\vec{a} \cdot \vec{b} = \sum_{j=1}^{d} a_j b_j$$

Thm (properties):

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})$ (Distributive law)

2.2.4 Functions

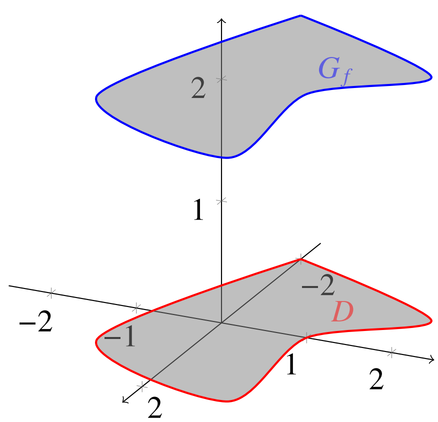


Figure 2.2: The constant function $f(x, y) = 2$ over D

Domain

The domain is a subset of the xy plane ($\mathbb{R}^2$).

Let the domain of the function $f(x, y)$ be an ordered pair within the curve, or $(x, y) \in D$

Range

Let's call the range of function f, $G_f = \{x, y, f(x, y) \mid (x, y) \in D\} \subseteq \mathbb{R}^3$.

One-to-One

We define the one-to-one in the following manner, if $f(x, y)$ is one-to-one:

$$(x, y), (u, v) \in D, f(x, y) = f(u, v) \text{ iff } x = u \wedge y = v$$

There is no vertical line test equivalent in higher dimensions. We can only test one-to-one-ness in this manner. It is very hard to find a one-to-one function in higher dimensions, but one of them is the *interlacing* function ($*$).

Let $x, y \in [0, 1] \times [0, 1]$, with $x = 0.d_1d_2d_3\ldots$ and $y = 0.e_1e_2e_3\ldots$, then the interlacing function $f(x, y) = x * y = 0.d_1e_1d_2e_2d_3e_3\ldots$. Fun fact, this function is continuous and differentiable *nowhere*.

Graphs of Functions

In general, the **graph** of $y = f(x_1, x_2, x_3, \ldots, x_d) = f(\vec{x})$ is the set $G = \{(x_1, x_2, \ldots, x_d, y) \mid y = f(\vec{x})\}$. Note that this exists in $\mathbb{R}^{d+1}$

2.3 Inequalities

Before we go futher, we need to take a detour with inequalities, which will come in very handy later in the curriculum when taking limits.

2.3.1 Level of Operations

Powers/root → Multiplation/division → addition/subtraction → succession/pretrition

2.3.2 Triangle Inequality (for Absolute Values)

The triangle inequality states that:

$$0 \leq |a+b| \leq |a| + |b|$$

With equality iff $ab \geq 0$ or a and b have the same signs.

2.3.3 AM-GM

$$\mu = [x_1, x_2, \ldots, x_n] \text{ and } x_1, x_2, x_3, \ldots, x_n \geq 0$$

"Multiset" $\mu = \{(x, n), (y, m), \ldots\}$

We define the arithmetic mean of a multiset as:

$$A(\mu) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

And the geometric mean as:

$$G(\mu) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

AM-GM Inequality

$A(\mu) \geq G(\mu)$, with equality iff all elements of μ are the same.

Proof

This is done by mathematical induction. Base case is $n = 2$, then $\mu = [x, y]$. Then $A(\mu) = \frac{x+y}{2}, G(\mu) = \sqrt{xy}$

We know by the trivial inequality that $(\sqrt{x} + \sqrt{y})^2 \geq 0$, with equality case happening iff $x = y$. Then we get:

$$\begin{aligned} x - 2\sqrt{xy} + y &\geq 0 \\ \frac{x+y}{2} &\geq \sqrt{xy} \\ A(\mu) &\geq G(\mu) \end{aligned}$$

Now we induce on n, we seek to prove that case n implies case $2n$.

$\mu = [x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$

Then we know that

$$\begin{aligned} A(\mu) &= \frac{A(\mu_x) + A(\mu_y)}{2} \\ &\geq \frac{G(\mu_x) + G(\mu_y)}{2} \\ &\geq \sqrt{G(\mu_x)G(\mu_y)} \\ &= G(\mu) \end{aligned}$$

Note that in all inequalities used, the equality case is always when all x_n and y_n are the same element, therefore the equality case holds in all cases where the length of the list is 2^n.

Now we prove that case n implies $n-1$

$\mu = [x_1, x_2, x_3, \ldots, x_{n-1}]$

Note that we can construct $\mu' = [x_1, x_2, x_3, \ldots, x_{n-1}, A(\mu)]$

Note that $A(\mu') = A(\mu)$, and since the AM-GM inequality is true for μ' by the assumption, we know

$$\begin{aligned} A(\mu') = A(\mu) &\geq \sqrt[n]{x_1 x_2 x_3 \ldots x_{n-1} A(\mu)} \\ &\geq \sqrt[n]{G(\mu)^{n-1} A(\mu)} \\ &\geq \sqrt[n]{G(\mu)^{n-1} G(\mu)} \\ &\geq G(\mu) \end{aligned}$$

2.3.4 Young's Inequality

Hölder Conjugate

q is said to be the Hölder Conjugate of p:

$$q := p^* := \frac{p}{p-1}$$

Note that $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

Young's Inequality

$a, b \geq 0$; $p > 1$; $q = p^*$, then Young's Inequality states that:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

With equality case iff $a^p = b^q$

Proof

We first proof Young's Inequality assuming that $p, q \in \mathbb{Q}$. We can rewrite $p = \frac{n+m}{n}$ and $q = \frac{n+m}{m}$ for some $n, m \in \mathbb{N}$. Now Young's Inequality turns into:

$$ab \leq \frac{na^{\frac{n+m}{n}}}{n+m} + \frac{mb^{\frac{m+n}{m}}}{n+m}$$

If we let $x = a^{\frac{1}{n}}$ and $y = b^{\frac{1}{m}}$. Then the inequality turns into:

$$ab = x^n y^m = \leq \frac{nx^{n+m} + my^{n+m}}{n+m}$$

And that is true by weighted AM-GM, with equality iff $x = y$, which equals to $a^{\frac{1}{n}} = b^{\frac{1}{m}}$, which equals to

We can prove the inequality for irrational by taking limits, because $\forall n \geq n_0 : f(n) \leq g(n)$, and the limits of both $f(x)$ and $g(x)$ as $n \to \infty$ exists and are finite, then we know that $\lim_{n\to\infty} f(n) \leq \lim_{n\to\infty} g(n)$. When p and q are irrational, we construct $\{p_n\}$ and $\{q_n\}$ as two sequences of rationals that approaches p and q, the left hand side of Young's Inequality is unaffected by the limit, and by what we've just said about limits, we know that:

$$ab \leq \lim_{n\to\infty} \frac{a^{p_n}}{p_n} + \frac{b^{q_n}}{q_n} = \frac{a^p}{p} + \frac{b^q}{q}$$

2.3.5 Hölder's Inequality

***p*-norm**

In $\mathbb{R}^2$: let $\|(a, b)\|_p = (|a^p| + |b^p|)^{1/p}$ for any $p \in \mathbb{R} > 1$, this is known as the p-norm of a vector. Note that $\|(a, b)\|_2 = \|(a, b)\| = \sqrt{a^2 + b^2}$

Hölder's Inequality

$\forall (a, b), (c, d) \in \mathbb{R}, p > 1, q = p^*$:

$$0 \leq |ac| + |bd| \leq \|(a, b)\|_p \|(c, d)\|_q$$

Equality happens iff $(\frac{a}{s})^p = (\frac{c}{t})^q$ and $(\frac{b}{s})^p = (\frac{d}{t})^q$ where $s = \|(a, b)\|_p$ and $t = \|(c, d)\|_q$

The whole inequality is only 0 if both points have at least one component on the axis.

Proof

We know that the absolute value of the product is equal to the product of the absolute value. If we apply Young's Inequality to $|\frac{a}{s}||\frac{c}{t}|$ and $\left|\frac{b}{s}\right|\left|\frac{d}{t}\right|$, we get:

$$\left|\frac{a}{s}\right|\left|\frac{c}{t}\right| \leq \frac{|a|^p}{p|s|^p} + \frac{|c|^q}{q|t|^q}$$
$$\left|\frac{b}{s}\right|\left|\frac{d}{t}\right| \leq \frac{|b|^p}{p|s|^p} + \frac{|d|^q}{q|t|^q}$$

Now we add:

$$\frac{1}{st}(|ac| + |bd|) \leq \frac{|a|^p + |b|^p}{p|s|^p} + \frac{|c|^q + |d|^q}{q|t|^q}$$

Note that $|a|^p + |b|^p = |s|^p$ and $|c|^q + |d|^q$, so everything cancels

$$\frac{1}{st}(|ac| + |bd|) \leq \frac{1}{p} + \frac{1}{q}$$

But we know that $q = p^*$, therefore, $\frac{1}{p} + \frac{1}{q} = 1$, and we get:

$$|ac| + |bd| \leq st = ||(a, b)||_p \cdot ||(c, d)||_q$$

The equality case occurs at basically the same way as Young's Inequality's equality case.

2.3.6 Cauchy-Schwarz Inequality

This is a special case of Hölder's Inequality, where $p = q = 2$. (This is very important, 2 is the *only* value that is its own conjugate, this is why Euclidean distance is so special)

If we plug in 2 for p and q and use the Triangle Inequality:

$$|ac + bd| \leq |ac| + |bd| \leq ||(a, b)|| \cdot ||(c, d)||$$

Now if $(a, b) = \vec{u}$ and $(c, d) = \vec{v}$, the inequality becomes:

$$|\,\vec{u} \cdot \vec{v}\,| \leq ||\,\vec{u}\,|| \cdot ||\,\vec{v}\,||$$

With equality iff $\vec{v} \parallel \vec{u}$

2.3.7 Minkowski's Inequality

Take two vectors, $\vec{u}$ and $\vec{v}$, the inequality states that:

$$\| \vec{u} + \vec{v} \|_p \leq \| \vec{u} \|_p + \| \vec{v} \|_q$$

With equality iff $\vec{v} = t\vec{u}$ or $\vec{u} = t\vec{v}$ for some $t \geq 0$.

Proof

The calculation works in any dimension, for simplicity's sake, let's work in $\mathbb{R}^2$, let $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$

$$\| \vec{u} + \vec{v} \|_p^p = | a + c |^p + | b + d |^p = | a + c \| a + c |^{p-1} + | b + d \| b + d |^{p-1}$$

Now we factor and use the Triangle Inequality for Absolute Value:

$$\leq (|a| + |c|) | a + c |^{p-1} + (|b| + |d|) | b + d |^{p-1}$$

Now we rearrange the terms:

$$= (|a||a + c|^{p-1} + |b||b + d|^{p-1}) + (|c||a + c|^{p-1} + |d||b + d|^{p-1})$$

Now we apply Hölder's Inequality, we get:

$$\leq (|a|^p + |b|^p)^{\frac{1}{p}} (|a + c|^{(p-1)q} + |b + d|^{(p-1)q})^{\frac{1}{q}} + (|c|^p + |d|^p)^{\frac{1}{p}} (\ldots)$$

Note that $q = p^*$, therefore $(p - 1)q = p$:

$$= (\| \vec{u} \|_p + \| \vec{v} \|_p) \| \vec{u} + \vec{v} \|_p^{\frac{p}{q}}$$

Since $q = p^*$, we know that $\frac{p}{q} = p - 1$, and if we bring the inequality to the original left hand side:

$$\leq (\| \vec{u} \|_p + \| \vec{v} \|_p) \| \vec{u} + \vec{v} \|_p^{p-1}$$

Now we divide:

$$\| \vec{u} + \vec{v} \|_p \leq \| \vec{u} \|_p + \| \vec{v} \|_p$$

Now let's consider the equality cases. If one of the vectors is 0, then the inequality is trivially true.

If neither vectors are the 0 vector, we see the equality cases of all the inequalities used to prove Minkowski's. First we used the triangle inequality, which only has equality when $ac \geq 0$ and $bd \geq 0$. Next we applied Hölder's, which has equality case when both coordinates are proportional. Therefore, the two vectors must be positive multiples of one another.

2.3.8 Triangle Inequality

$$\| \vec{u} + \vec{v} \| \leq \| \vec{u} \| + \| \vec{v} \|$$

Note that this is just a special case of Minkowski's Inequality where $p = 2$.

This can be generalized by mathematical induction to $dist(\vec{p_0}, \vec{q_n}) \leq \sum_{j=1}^{n} dist(\vec{p}_{j-1}, \vec{p}_j)$ (Otherwise known that the shortest distance between two points is the straight line, or the **Generalized Triangle Inequality** or the "Broken Line Inequality")

2.3.9 Reverse Triangle Inequality

From the Triangle Inequality we know:

$$\| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \|$$

Let $\vec{z} = \vec{x} + \vec{y}$, then we subtract, we get:

$$\| \vec{z} \| \leq \| \vec{x} \| + \| \vec{z} - \vec{x} \|$$
$$\| \vec{z} \| - \| \vec{x} \| \leq \| \vec{z} - \vec{x} \|$$

Because $\vec{x}$ and $\vec{z}$ are just variables, we can switch them, and we get:

$$\| \vec{x} \| - \| \vec{z} \| \leq \| \vec{x} - \vec{z} \| = \| \vec{z} - \vec{x} \|$$

Since the right hand side is greater than both of the above qualities, we can just say it's greater than the absolute value of the difference. Hence we get the Reverse Triangle Inequality:

$$\| \vec{z} - \vec{x} \| \geq |\| \vec{z} \| - \| \vec{x} \||$$

2.4 Linear Algebra

Before we delve deeper into calculus, it is worthwile to take a little detour into the realm of linear algebra.

2.4.1 Linear Mappings/Functions

Consider a function $\vec{l} : \mathbb{R}^d \to \mathbb{R}^e$. $\vec{l}$ is called a **linear mapping** if the image of any k-flat in $\mathbb{R}^d$ $(0 \leq k \leq d)$ is a $\tilde{\text{k}}$-flat in $\mathbb{R}^e$, where $\tilde{k} \leq k$. Basically these definitions "preserve flatness."

2.4.2 Sufficient Conditions for Linear Mapping

For $\vec{l}$ to be a linear function, it must have the following properties:

1. Additivity: $\vec{l}(\vec{x}+\vec{y}) = \vec{l}(\vec{x}) + \vec{l}(\vec{y})$
2. Homogeneity: $\vec{l}(c\vec{x}) = c\vec{l}(\vec{x})$

Proof: Take k-flat in $\mathbb{R}^d$ $F = \{t_1\vec{a}_1 + t_2\vec{a}_2 + \cdots + t_k\vec{a}_k \mid t_1, t_2, \ldots, t_k \in \mathbb{R}\}$. Here, $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k \in \mathbb{R}^d$ such that all of them are independent of each other, i.e. no $\vec{a}_i$ is a linear combination of $\{\vec{a}_j \mid j \neq i\}$. Because otherwise $\vec{a}_i$ can be broken up and the dimension of F will be reduced. This can also be phrased as the set of $\vec{a}_i$ must satisfy the following condition: $F = \vec{0} \implies t_1 = t_2 = \cdots = t_k = 0$. Because if there exists a non-trivial solution, we can subtract all the terms with their coefficient being 0, and divide through the non-zero coefficient, then we would express $\vec{a}_i$ as a linear combination of others. Therefore, if all the vectors are independent, the equation $F = \vec{0}$ only has the trivial solution.

So now consider the linear mapping $\vec{l}$. We can distribute because the mapping is additive, and we can factor out the coefficients because it is homogeneous.

$$\begin{aligned}\vec{l}(F) &= \{\vec{l}(t_1\vec{a}_1 + \cdots + t_k\vec{a}_k) \mid t_1, \ldots, t_k \in \mathbb{R}\} \\ &= \{t_1\vec{l}(\vec{a}_1) + \cdots + t_k\vec{l}(\vec{a}_k)\}\end{aligned}$$

If we denote $\vec{l}(\vec{a}_i) := b_i$, we can rewrite $\vec{l}(F) = \{t_1 b_1 + \cdots + t_k b_k\}$. This is a $\tilde{\text{k}}$-flat for some $\tilde{\text{k}} \leq$ k, because we can always pick a subset of $\vec{b}_i$ such that they are all independent of each other (unless they are all $\vec{0}$, but in that case $\vec{l}(F)$ is just the 0-flat), and then we reduce, and the final result would be a $\tilde{\text{k}}$-flat.

Example: Homogeneity but not additive and therefore non-linear.

Note that if we are dealing with one dimension, all functions that have an unlimit range are linear mappings. However, in $\mathbb{R}^2$, homogeneity is not enough.

Consider the function:

$$\vec{f}(x, y) = \begin{cases} (\frac{x^3+y^3}{x^2+y^2}, \frac{xy^2}{x^2+y^2}) & \text{if } (x, y) \neq (0, 0) \\ (0, 0) & \text{if } (x, y) = (0, 0) \end{cases}$$

$\vec{f}$ is homogeneous as it is a composition of homogeneous functions, but it does not preserve flatness for obvious reasons, and that is because this function is not additive.

2.4.3 Matrices

Definition

Matrices are list of vectors, with each column being a single vector. For example, $((1,2,0),(-1,3,4))$ can be rewritten as

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 4 \end{bmatrix}$$

This is known as a matrix, and an element of a matrix can be denoted with two subscripts with the lower case of the matrix' name, with the first subscript denoting the row number, and the second denoting the column number. If the above matrix is A, then $a_{11} = 1$ and $a_{31} = 0$.

Transposition

$B = A^T$ (B is A transposed), then $b_{ij} = a_{ji}$

Determinant

Geometrically, if you treat the matrix $A \in \mathbb{R}^{d\times n}$ as an d-dimensional strucutre composed of n d-dimensional vectors, then the determinant is the "volume" of said structure.

Algebraically

$$\det A = \sum_{\sigma\in S_n} (sgn\ \sigma) a_{1\sigma_1} a_{2\sigma_2} \dots a_{n\sigma_n}$$
$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$$

where

$$sgn\ \sigma = \frac{\Pi_{i<j}(x_{\sigma_i} - x_{\sigma_j})}{\Pi_{i<j}(x_i - x_j)}$$

where $x_1, x_2, \dots, x_n$ are distant values. This function, in a more explicit/less percise form is:

$$sgn = (-1)^N$$

where N is the number of ordered pairs (i, j) where $i < j$ but $\sigma_j < \sigma_i$.

And S_n is the set of permutations of σ

Properties of the Determinant Function:

1. $\det I = 1$ (unit property)
2. $\det[\vec{a}_1, \ldots, c\vec{a}_j, \ldots, \vec{a}_n] = c \det A$
3. $\det A = \det A^T$

Dot Product

If we have $A = [\vec{a}_1 \vec{a}_2 \ldots \vec{a}_n] \in \mathbb{R}^{d\times n}$ and $B = [\vec{b}_1 \vec{b}_2 \ldots \vec{b}_m] \in \mathbb{R}^{d\times m}$, then we say that the dot product of the two is:

$$A \cdot B = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \ldots & \vec{a}_1 \cdot \vec{b}_m \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \ldots & \vec{a}_2 \cdot \vec{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n \cdot \vec{b}_1 & \vec{a}_n \cdot \vec{b}_2 & \ldots & \vec{a}_n \cdot \vec{b}_m \end{bmatrix}$$

This operation is distributive, which means that $A \cdot (B + C) = A \cdot B + A \cdot C$. However, this product is not associative, i.e. $A \cdot (B \cdot C) \neq (A \cdot B) \cdot C$.

To solve this problem, we have the matrix multiplication operator.

Matrix Multiplation

For $A \in \mathbb{R}^{n\times d}$, $B \in \mathbb{R}^{d\times m}$, we define the matrix multiplication product to be:

$$\boxed{AB := A^T \cdot B} \in \mathbb{R}^{n\times m}$$

If we call $C = AB$, then we can say that

$$c_{ij} = \sum_{k=1}^{d} a_{ik} b_{kj} \; (1 \leq i \leq n, 1 \leq j \leq m)$$

As a whole, the C column would look like this (here we denote $\vec{\alpha}_i$ as the i^{th} row of A and suppose A has p rows). Then:

$$C = \begin{bmatrix} \vec{\alpha}_1 \cdot \vec{b}_1 & \ldots & \vec{\alpha}_1 \cdot \vec{b}_m \\ \vdots & \ddots & \vdots \\ \vec{\alpha}_p \cdot \vec{b}_1 & \ldots & \vec{\alpha}_m \cdot \vec{b}_m \end{bmatrix}$$

Note this operation is both distribut and associative, quick proof:

$$A(BC) \stackrel{?}{=} (AB)C$$

$$[A(BC)]_{ij} = \sum_k a_{ik}[BC]_{kj} = \sum_k a_{ik}(\sum_l b_{kl}c_{lj}) = \sum_{(k,l)} a_{ik}b_{kl}c_{lj}$$

$$[(AB)C]_{ij} = \sum_l [AB]_{il}c_{li} = \sum_l (\sum_k a_{ik}b_{kl})c_{lj} = \sum_{(l,k)} a_{ik}b_{kl}c_{lj}$$

Therefore the values of the two matrices are the same. We also have to prove that they are the same size. If $A \in \mathbb{R}^{n\times d}$, $B \in \mathbb{R}^{d\times e}$ and $C \in \mathbb{R}^{e\times m}$. $BC \in \mathbb{R}^{d\times m}$ and $A(BC) \in \mathbb{R}^{n\times m}$. $AB \in \mathbb{R}^{n\times e}$ and $(AB)C \in \mathbb{R}^{n\times m}$. Therefore matrix multiplication is associative.

Examples

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2} = O$$

Note that in the world of matrices, the product of two non-zero matrices can result in the zero matrix.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Note that the communicative property does not hold for matrix multiplication.

Norm

Given $A \in \mathbb{R}^{e\times d} = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_d]$, we define the norm of A, $||A||$ as:

$$||A|| := \sqrt{\sum_{j=1}^{d} || \vec{a}_j ||^2} = \sqrt{\sum_{j=1}^{d}\sum_{i=1}^{e} a_{ij}^2}$$

In other words, the norm of a matrix is the sqareroot of the sum of the squares of every element in the matrix.

Properties of the Norm:

1. $||A|| > 0, ||A|| = 0 \leftrightarrow A = O$
2. $||cA|| = |C|||A||$

3. $||A + B|| \leq ||A|| + ||B||$ (Triangle Inequality)
4. $||AB|| \leq ||A||||B||$ (Generalized Cauchy-Schwarz Inequality)

To prove these, we first establish a correspondence between any matrix $A \in \mathbb{R}^{e \times d}$ and a vector in $\mathbb{R}^{ed}$, we define the mapping Ψ from $\mathbb{R}^{e \times d}$ to $\mathbb{R}^{ed}$:

$$A \overset{\Psi}{\longleftrightarrow} (a_{11}, \ldots, a_{1d}, a_{21}, \ldots, a_{2d}, \ldots, a_{e1}, \ldots, a_{ed})$$

Note that under this operation, all vector space properties of the matrix are preserved. Note that $||A|| = ||\Psi(A)||$, $\Psi(cA) = cA$, and $\Psi(A + B) = A + B$.

However, note that the product is not exactly preserved with this transformation to vector, since the size of the matrix plays an integral part in matrix multiplication. To prove property 4 of the norm, we need to do a bit of work.

Let $C = AB$, then wwe know that

$$\begin{aligned} ||AB||^2 = \sum_i \sum_j \left(\sum_k a_{ik} b_{kj} \right)^2 &= \sum_i \sum_j (\vec{\alpha}_i \cdot \vec{b}_j)^2 \\ &\leq \sum_i \sum_j || \vec{\alpha}_i ||^2 || \vec{b}_j ||^2 \\ &= (\sum_i || \vec{\alpha}_i ||^2)(\sum_j || \vec{b}_j ||^2) \\ &= ||A^T||^2 ||B||^2 \\ &= ||A||^2 ||B||^2 \end{aligned}$$

Therefore, $||AB|| \leq ||A||||B||$.

Inverses

Suppose there exists $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, we say that B is a **two sided inverse** of A if $AB = BA = I (= I_n)$, where I is the identity matrix, which functions like the number one in real number multiplication.

Thm: If A has a two-sided inverse, then it has exactly one, namely A^{-1}.

Suppose that B and C are both two-sided inverses for A, i.e. $AB = BA = I$ and $AC = CA = I$. We know we can represent $B = BI = B(AC) = BA(C) = IC = C$. Therefore $B = C$.

Thm: A is invertible iff $\det A \neq 0$.

Let $B = A^{-1}$, $AB = I$. Now we take the determinant of both sides, we get:

$$\det(AB) = \det I$$

Since the determinant is distributive, we get:

$$(\det A)(\det B) = 1$$

Now it's clear that $\det A \neq 0$

CONVERSE = TODO

Matrix Valued Functions

It is entirely possible for functions to give out matrices as its output. Suppose $A : D \subseteq \mathbb{R}^d \to \mathbb{R}^{e \times k}$. Similar to how vector valued functions have component functions, matrix valued functions have entry functions. A would look something like this:

$$A(\vec{x}) = \begin{bmatrix} a_{11}(\vec{x}) & \dots & a_{1k}(\vec{x}) \\ \vdots & \vdots & \vdots \\ a_{e1}(\vec{x}) & & a_{ek}(\vec{x}) \end{bmatrix}$$

Continuity for such functions is an entrywise property: A is continuous at $\vec{p}$ iff a_{ij} is continuous at $\vec{p}\ \forall i, j$

2.4.4 Components of Linear Functions

Theorem

Given a linear mapping $\vec{l}(\vec{x})\mathbb{R}^d \to \mathbb{R}^e = (l_1(\vec{x}), l_2(\vec{x}), \dots, l_e(\vec{x}))$, then we know that each l_i is linear ($\mathbb{R}^d \to \mathbb{R}$).

Proof

Let us first check additivity:

$$\vec{l}(\vec{x} + \vec{y}) = (l_1(\vec{x} + \vec{y}), \dots, l_e(\vec{x} + \vec{y}))$$

$$\vec{l}(\vec{x}) + \vec{l}(\vec{y}) = (l_1(\vec{x}) + l_1(\vec{y}), \dots, l_e(\vec{x}) + l_e(\vec{y}))$$

Now if we inspect each element, each l_i is additive. A similar argument can be made to prove homogeneity. Therefore, each l_i is a linear mapping.

2.4.5 Cancellation Law of the Dot Product

Theorem

If $\vec{a} \cdot \vec{x} = \vec{b} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^d$, where $\vec{a}, \vec{b} \in \mathbb{R}^d$, then $\vec{a} = \vec{b}$.

Proof

Take $\vec{x} = \vec{e}_1 = (1, 0, 0, \ldots, 0)$, this yields that $a_1 = b_1$. Then we can take $\vec{x}$ to any of the basic component vectors, we get that $a_2 = b_2, a_3 = b_3, \ldots, a_d = b_d$. Therefore, $\vec{a} = \vec{b}$.

2.4.6 Expression of Linear Functions

Theorem

For any linear map $\vec{l} : \mathbb{R}^d \to \mathbb{R}^e$. There is an unique matrix $A \in \mathbb{R}^{e \times d}$ (d columns and e rows) such that $\vec{l}(\vec{x}) = A\vec{x}$ where the right hand size is a matrix product, where $\vec{x}$ is regarded as a $d \times 1$ column matrix.

Proof

A real valued linear function $\lambda : \mathbb{R}^d \to \mathbb{R}$ has the form $\lambda(\vec{x}) = \vec{a} \cdot \vec{x}$ for some $\vec{a} \in \mathbb{R}^d$.

Note that for any vector $\vec{x}$, we can write it as:

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_d\vec{e}_d$$

Now consider $\lambda(\vec{x})$:

$$\begin{aligned}
\lambda(\vec{x}) &= \lambda(x_1\vec{e}_1 + \cdots + x_d\vec{e}_d) \\
&= \lambda(\vec{e}_1)x_1 + \lambda(\vec{e}_2)x_2 + \cdots + \lambda(\vec{e}_d)x_d
\end{aligned}$$

Note that if we write $a_i = \lambda(\vec{e}_i)$, we get that $\lambda(\vec{x}) = \vec{a} \cdot \vec{x}$.

Now let's consider $\vec{l}$, we can write each of its component function as a dot product between a vector and $\vec{x}$. So we get:

$$\vec{l} = (\vec{\alpha}_1 \cdot \vec{x}, \vec{\alpha}_2 \cdot \vec{x}, \ldots, \vec{\alpha}_d \cdot \vec{x}) = \begin{bmatrix} \vec{\alpha}_1 \cdot \vec{x} \\ \vec{\alpha}_2 \cdot \vec{x} \\ \vdots \\ \vec{\alpha}_d \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{\alpha}_1^T \\ \vec{\alpha}_2^T \\ \vdots \\ \vec{\alpha}_d^T \end{bmatrix} \vec{x} = A\vec{x}$$

2.5 Topology of $\mathbb{R}^d$

2.5.1 Bounded

Definition

Definition: $D \subseteq \mathbb{R}^2$ is bounded if $\exists\, M > 0 : D \subseteq [-M, M] \times [-M, M]$.

For higher dimensions, we simply extend the square to a box, hyperbox, etc.

2.5.2 Closed

Definition

Definition: $D \subseteq \mathbb{R}^2$ is closed if $\forall((x_n, y_n))_{n=1}^{\infty} \in D$ that converges, the limit point (a, b) of the sequence also lies in D.

Theorems

Thm: The intersection of two closed sets is also a closed set.

Proof: Take $C, D \in \mathbb{R}^d$ as the two closed sets. $C \cap D$ either $= \emptyset$ or $\neq \emptyset$, If $C \cap D = \emptyset$, then the statement is true. Otherwise, $\forall((\vec{p}_n))_{n=1}^{\infty}$ within $C \cap D$, we seek to prove that its convergence point $\vec{p} \in C \cap D$. Because $((\vec{p}_n))_{n=1}^{\infty} \in C \cap D \in C$ and the closure of C, we know that $\vec{p} \in C$. By a similar logic we know that $\vec{p} \in D$. Therefore $\vec{p} \in C \cap D$. Therefore $C \cap D$ is closed.

2.5.3 Open

Definition

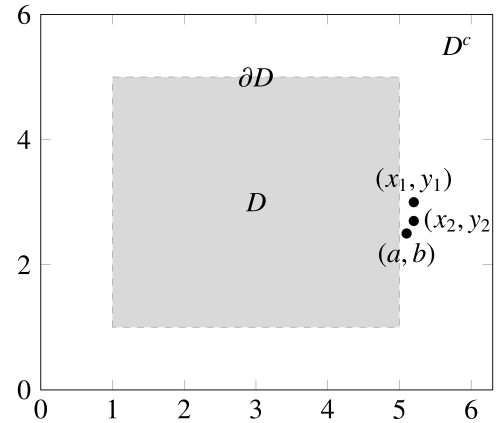
Define $D \subseteq \mathbb{R}^d$ is **open** if $\forall(a, b) \in D, \exists\, r > 0 : B_r(a, b) \subseteq D$

Another definition of is that D is open if $D = \text{int}\, D$. This means that every point of D is also an interior point.

Theorems

Thm: If D is open, then D^c is closed. If D is closed, then D^c is open.

Proof: Assume D is open, we prove D^c is closed. Choose any convergent sequence $((x_n, y_n))_{n=1}^{\infty}$, converging to (a, b), where $(x_n, y_n) \in D^c$ for all $n \geq 1$.



We prove this by contradiction. Assume that $(a, b) \in D$. Since D is open, $\exists\, r > 0 : B_r(a, b) \subseteq D$. $\exists\, N, \forall n \geq \mathbb{N}, (x_n, y_n) \in B_r(a, b)$ since $(x_n, y_n) \to (a, b)$. But we assumed that $(x_n, y_n) \in D^c$, and $(x_n, y_n) \in D$. But $D \cap D^c = \emptyset$. Therefore D^c is closed under taking limits.

Thm: An open-ball $B_r(\vec{p}) = \{\vec{x} \in \mathbb{R} \mid dist(\vec{x}, \vec{p}) < r\}$, $r > 0$, is an open set.

Proof: $\forall \vec{q} \in B_r(\vec{p})$, $B_\varepsilon(\vec{q}) \subseteq B_r(\vec{p})$, $\varepsilon > 0$.

$d = dist(\vec{p}, \vec{q}) < r$. Therefore, $r - d > 0$. Take $\varepsilon = \frac{1}{2}(r - d) > 0$. Let $\vec{x} = B_\varepsilon(\vec{q})$, show $\vec{x} \in B_r(\vec{p})$.

$dist(\vec{x}, \vec{q}) < \varepsilon$

$dist(\vec{x}, \vec{p}) \leq dist(\vec{x}, \vec{q}) + dist(\vec{q}, \vec{p}) < \varepsilon + d = d + \frac{1}{2}r - \frac{1}{2}d = \frac{1}{2}r + \frac{1}{2}d < \frac{1}{2} \cdot 2 \cdot r = r$

Thm: The union of any number of open sets is an open set.

Proof: Let $U = u_1 \cup u_2 \cup u_3 \cup \cdots \cup u_n$ where u_i is an open set. We know that $\forall \vec{p} \in U$, $\vec{p} \in$ some u_i, which means $\exists\, r > 0 : B_r(\vec{p}) \in u_i \in U$. Therefore U is an open set.

2.5.4 Compact

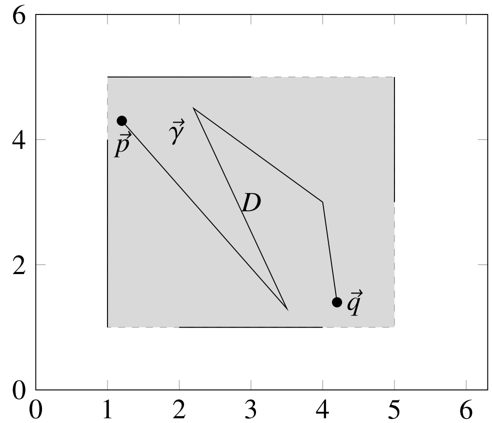
Definition

We say that a set $D \subseteq \mathbb{R}^2$ is **compact** if it is both bounded and closed

2.5.5 Connected

Definition

$D \subseteq \mathbb{R}^d$ is said to be connected if $\forall \vec{p}, \vec{q} \in D$, $\exists$ a continuous function $\vec{\gamma} : [a, b] \to \mathbb{R}^d$ that $\vec{\gamma}(a) = \vec{p}$, $\vec{\gamma}(b) = \vec{q}$, and $\forall t \in [a, b], \vec{\gamma}(t) \in D$



Theorems

Let $D \subseteq \mathbb{R}^d$ be connected, $D \neq \emptyset$. Suppose $D = A \cup B$, where A and B are open. Such that $A \cap B = \emptyset$. Then $A = \emptyset$ or $B = \emptyset$.

Proof:

Assume that D can be broken up into two open, non-empty sets A and B such that $D = A \cup B$. Because $A \neq \emptyset$, we can pick $\vec{p} \in A$ and similarly we can pick $\vec{q} \in B$. Clearly, $\vec{p}, \vec{q} \in D$. Since D is connected, $\exists$ continuous $\vec{\gamma} : [a, b] \to D$ such that $\vec{\gamma}(a) = \vec{p}, \vec{\gamma}(b) = \vec{q}$. Let $S = \{t \in [a, b] \mid \vec{\gamma}(t) \in A\}$. We know that $a \in S$, so $S \neq \emptyset$. Note that $S \leq b$, and since it has a bound, it has a supremum. Let $t_0 = \sup S$, note that $a \leq t_0 \leq b$. Since $\vec{\gamma}(t_0) \in D$, it is either in A or in B (since $A \cap B = \emptyset$). Suppose $\vec{\gamma}(t_0) \in A$. Since A is open, we can draw an open ball (B_ε) around $\vec{\gamma}(t_0)$. Now consider $\vec{\gamma}(t_0 + \delta)$ for some small positive δ. If δ is small enough, $\delta \in B_\varepsilon \in A$. This is a contradiction, as then it means that $t_0 + \delta \in S$. This is a contradiction, as then $t_0 \neq \sup S$. A similar contradiction can be made for the case that $\vec{\gamma}(t_0) \in B$. Therefore the original assumption was false.

2.6 Sequences in $\mathbb{R}^d$

2.6.1 Bounded Sequence Theorem

Theorem

Any bounded sequence in $\mathbb{R}^d$ (bounded within the box of $[-M, M] \times [-M, M] \times [-M, M] \times \ldots$) has a convergent subsequence.

Lemma

If $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$, then every subsequence $(x_{n_k})_{k=1}^{\infty}$ also converges to x

Proof: $\forall \varepsilon > 0, \exists\, N, \forall n \geq N : |\, x_n - x\,| < \varepsilon$.

$\therefore \forall k > K : n_k \geq N \longrightarrow |\, x_{n_k} - x \,| < \varepsilon$

Proof

$P_n = (x_n, y_n)$, consider $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$. Since $-M \leq x_n \leq M$, $(x_n)_{n=1}^{\infty}$ is bounded. Pick some convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Consider $(y_{n_k})_{k=1}^{\infty}$. Since $-M \leq y_{n_k} \leq M$, $(y_{n_k})_{k=1}^{\infty}$ is bounded. Therefore it has a convergent subsequence, $(y_{n_{k_j}})_{j=1}^{\infty}$, which converges to $y \in \mathbb{R}$. Note that $(x_{n_{k_j}})_{j=1}^{\infty} \to x$ because it is a subsequence of $(x_{n_k})_{k=1}^{\infty}$, which converges to x. Thus $P_{n_{k_j}} = (x_{n_{k_j}}, y_{n_{k_j}}) \to (x, y) = P$, which is the result of a corollary of the Basic Distance Bounds Lemma.

2.6.2 Cauchy's Convergence Theorem

Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is a cauchy sequence if

$$\forall \varepsilon, \exists\, n, m \geq N : |\, x_n - x_m \,| < \varepsilon$$

Convergence → Cauchy

Note that any convergent sequence is cauchy, because as terms get together to a limit, they also go very closely together.

"Formal" Proof:

Let $(x_n)_{n=1}^{\infty}$ be convergent, with limit $x \in \mathbb{R}$. Then, by definition, $\forall \varepsilon > 0, \exists\, N_\varepsilon, \forall n \geq N_\varepsilon : |x_n - x| < \varepsilon$. Note that we can replace ε with $\frac{\varepsilon}{2}$, all we have to change is the cutoff point from N_ε to $N_{\frac{\varepsilon}{2}}$. Now if we take two subscripts $n, m \geq N_{\frac{\varepsilon}{2}} \longrightarrow |\, x_n - x_m \,| = |\, (x_n - x) + (x - x_m) \,| \leq |\, x_n - x \,| + |\, x_m - x \,|$ because of the Triangle Inequality for Absolute Values. However, note that $|\, x_n - x \,| \leq \frac{\varepsilon}{2}$ and $|\, x_m - x \,| \leq \frac{\varepsilon}{2}$. Therefore, $|\, x_n - x_m \,| \leq |\, x_n - x \,| + |\, x_m - x \,| \leq \varepsilon$ ■

Cauchy's Convergence Theorem

In $\mathbb{R}$, every cauchy sequence converges to a limit in $\mathbb{R}$.

Lemma #1: Every cauchy sequence is bounded.

Let us take $\varepsilon = 1$, then the definition of "cauchiness" becomes:

$$\exists\, N_1, \forall n, m \geq N_1 : |\, x_n - x_m \,| < 1$$

Let $M := \max\{|x_1|, |x_2|, \ldots, |x_{N_1 - 1}|, |x_{N_1}| + 1\}$. We claim that $|x_n| \leq M$, for all $n \geq 1$. This is true because when $n \in \{1, 2, \ldots, N_1 - 1\}$, the statement is true by definition of M. When $n \geq N_1$, we know that $|x_n| \leq |x_{N_1}| + 1 \leq M$ because we can let $m = N_1$, then by the definition of "cauchiness," we know that $|x_n - x_{N_1}| < 1$.

Now we see that M is a bound on the sequence for all $n \geq 1$. Therefore the sequence is bounded.

Lemma #2: If a subsequence of a cauchy sequence converges to $x \in \mathbb{R}$, the whole sequence must converge to x.

Say $(x_n)_{n=1}^{\infty}$ is cauchy, and $(x_{n_k})_{k=1}^{\infty}$ converges to x. For any arbitrary $\varepsilon > 0$, we try to find N such that $\forall n \geq N : |x_n - x| < \varepsilon$. If we prove the existence of N for all ε, we will have proven that the original sequence converges.

We know that $\forall \varepsilon > 0, \exists\, K_\varepsilon, \forall k \geq K_\varepsilon : |x_{n_k} - x| < \varepsilon$. We add and subtract x_{n_k} and group terms, and use the Triangle Inequality: $|x_n - x| = |(x_n - x_{n_k}) + (x_{n_k} - x)| \leq |x_n - x_{n_k}| + |x_{n_k} - x|$. Note that $|x_{n_k} - x| < \varepsilon$ provided $k \geq K_\varepsilon$ from the convergent subsequence condition. We also know that $|x_n - x_{n_k}| < \varepsilon$ provided that $n, n_k \geq N_\varepsilon$, which we call the "cauchy cutoff." This is true from the "cauchiness" condition.

We know that $k \to \infty$ implies $n_k \to \infty$. This means eventually $n_k > N_\varepsilon$ provided that $k > L_{N_\varepsilon}$. Now let $k = \max\{L_{N_\varepsilon}, K_\varepsilon\}$ and $n \geq N_\varepsilon$, which implies $|x_n - x_{n_k}| < \varepsilon$ and $|x_{n_k} - x| < \varepsilon$. Now we know: $|x_n - x| \leq 2\varepsilon$ provided $n \geq N_\varepsilon$. Therefore the cauchy sequence converges to x.

With these two lemmas, the theorem becomes very easy to prove:

Because of Lemma #1 and the Bolzano-Weierstrass Theorem, we know that for all cauchy sequences, there is a bounded subsequence that converges to some value x. Then by Lemma #2, we know that the entire cauchy sequence converges to x as well, therefore the sequence converges.

Higher Dimensions

$(P_n)_{n=1}^{\infty}$ is cauchy if $\forall \varepsilon > 0, \exists\, N_\varepsilon, \forall n, m \geq N_\varepsilon : dist(P_n, P_m) < \varepsilon$. This is easy to prove due to the coordinate nature of $\mathbb{R}^d$.

2.6.3 Heine-Borel Theorem

Theorem

Let K be a compact set in $\mathbb{R}^d$. Let $\{u_\lambda \mid \lambda \in \Lambda\}$ be a family of open sets in $\mathbb{R}^d$ which covers K in the sense that $K \subseteq \cup_{\lambda \in \Lambda} u_\lambda = \{\vec{p} \mid \exists\, \lambda \in \Lambda : \vec{p} \in u_\lambda\}$. Then $\exists\, \lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ such that $K \subseteq u_{\lambda_1} \cup u_{\lambda_2} \cup \cdots \cup u_{\lambda_n}$ (for some finite n)

Proof

Because K is compact, we can surround K in a box (R) and divide that box into 4 congruent sections, R_1, R_2, R_3 and R_4. Assume that K cannot be covered by a finite amount of u_λ's. That

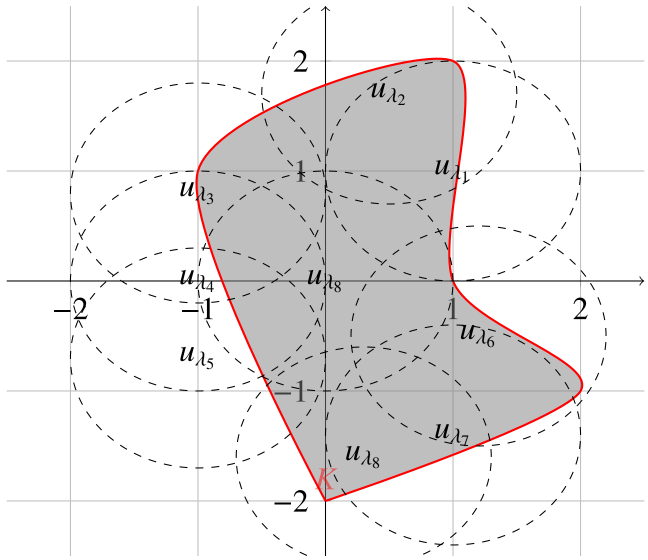


Figure 2.3: Here the closed set K is completed covered by the open sets u_{λ_n}

implies that at least one of the 4 quadrants (R_i) cannot be covered by a finite collection of u_λ. Then we divide that into 4 quadrants, and then one of the further qudrants $R_{i_1 i_2}$ cannot be covered by a finite collection of u_λ's, and this go on forever. Say that $K \cap R_{i_1 i_2 i_3 \ldots i_n}$ cannot be covered by finitely many u_λ's. Note that this cannot be $\emptyset$ for any n.

Let $(\vec{p_n})_{n=1}^{\infty}$ be any sequence of points in $K \cap R_{i_1 i_2 i_3 \ldots i_n}$. This is a bounded sequence because K is bounded. Then we know that there exists $(\vec{p_{n_k}})_{k=1}^{\infty}$ that converges to some point $\vec{p}$. Note that $\vec{p} \in K$ because K is closed. Thus $\vec{p} \in u_{\lambda^*}$ for some $\lambda^* \in \Lambda$. Since u_{λ^*} is open, $\exists\, r > 0$ s.t. $B_r(\vec{p}) \in u_{\lambda^*}$. But when n is large:

$$diam\ R_{i_1 i_2 \ldots i_n} = \frac{diam\ R}{2^n} < \frac{r}{2}$$

Therefore, when n_k is big enough ($n_k \to \infty$ as $n \to \infty$)

$$R_{i_1 i_2 \ldots i_n} \subseteq B_r(\vec{p})$$

Therefore, we know that $dist(\vec{p}, \vec{p_{n_k}}) < \frac{r}{2}$. Therefore, no point within $K \cap R_{i_1 i_2 i_3 \ldots i_n}$ is greater than r away from $\vec{p}$.

Now $K \cap R_{i_1 i_2 i_3 \ldots i_n} \subseteq u_{\lambda^*}$ because $K \cap R_{i_1 i_2 i_3 \ldots i_n}$ is within a ball. Therefore we can reject this entire process and we have just proved that K can be covered by a finite collection of u_{λ_n} for any given Λ.

2.7 Real Valued Functions

2.7.1 Extreme Value Theorem in $\mathbb{R}^d$

Theorem

Let $f : D \to \mathbb{R}$ be a continuous function mapping from the compact set D to the reals. Then $\exists\, P, Q \in D$, not necessarily distinct, such that $\exists\, x \in D : f(P) \leq f(x) \leq f(Q)$.

Proof

First we prove that f must be bounded from above. Assume that it is not, take a sequence $(\vec{P_n})_{n=1}^{\infty} \in D$, the assumption implies that implies for every positive integer n, $f(\vec{P_n}) > n$. Because it is bounded by D, we can pick a convergent subsequence $\vec{P_{n_k}}$, which converges to $\vec{P}$. However, since D is closed, we know that $\vec{P} \in D$. However, now we have a contradiction. Because f is continuous, $\lim_{n\to\infty} f(\vec{P_n}) \to \vec{P}$. This is a contradiction, because the right hand side $\vec{P}$ is finite (it's within D), but the left hand side goes to infinity by the assumption. Therefore the assumption is false, thus the function f is bounded from above.

Now we know that f is bounded from above, we know $M := \sup(f(D))$ exists where $0 \leq M < \infty$. We can chose a sequence $(P_n)_{n=1}^{\infty} \in D$ such that $f(P_n) \to M$. The sequence is bounded, so it has a convergent subsequence, $\vec{P_{n_k}} \to \vec{P}$. By the closure of D, $\vec{P} \in D$. Note that $f(\vec{P}) = \lim_{k\to\infty} f(\vec{P_{n_k}}) = M$ by the continuity of f. Therefore, the function actually takes on its maximum value at that point.

2.7.2 Uniform Continuity Theorem

Theorem

If $f : K \to \mathbb{R}$, where $K \subseteq \mathbb{R}^d$ is compact, and f is continuous at each $\vec{x} \in K$. Then f is uniformally continuous on K.

Proof

Fix $\varepsilon > 0$. For each $\vec{x} \in K$, let $u_{\vec{x}}$ be an open ball, centered at $\vec{x}$ such that for any $\vec{y} \in K \cap 2u_{\vec{x}}$ [$2u_{\vec{x}}$ is an open ball centered at $\vec{x}$ with twice the radius of $u_{\vec{x}}$], $|f(\vec{x}) - f(\vec{y})| < \frac{\varepsilon}{2}$. Because the function is continuous at $\vec{x}$, there is a radius 2δ around $\vec{x}$ such that $|f(\vec{x}) - f(\vec{y})| < \frac{\varepsilon}{2}$. Now we see that every $\vec{x} \in K$ is covered by at least one such open ball, namely $u_{\vec{x}}$. The collection $\{u_{\vec{x}} \mid \vec{x} \in K\}$ is an open covering of K. By Heine-Borel, we can select a finite set of points $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ such that K is covered by $u_{\vec{x}_1} \cup u_{\vec{x}_2} \cup \cdots \cup u_{\vec{x}_n}$. Take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\} > 0$ where δ_j is the radius of $u_{\vec{x}_j}$ for $j = 1, 2, 3, \ldots, n$.

Let $\vec{p}, \vec{q} \in K$ such that $\| \vec{p} - \vec{q} \| < \delta$. We need to prove that $|f(\vec{p}) - f(\vec{q})| < \varepsilon$. $\vec{p} \in K \to \exists\, j : \vec{p} \in u_{\vec{x}_j} \to \| \vec{p} - \vec{x}_j \| < \delta_j$. But we know that $\| \vec{p} - \vec{q} \| < \delta_j$. Now by the Triangle Inequality, we get:

$$\| \vec{q} - \vec{x}_j \| \leq \| \vec{q} - \vec{p} \| + \| \vec{p} - \vec{x}_j \| \leq 2\delta_{\vec{x}_j}$$

This means that $\vec{q} \in 2u_{\vec{x}_j}$. By the way we picked our δ, we know that $|f(\vec{q}) - f(\vec{x}_j)| < \frac{\varepsilon}{2}$. Similarly, we also have $|f(\vec{p}) - f(\vec{x}_j)| < \frac{\varepsilon}{2}$. Now if we apply the triangle inequality again, we get:

$$|f(\vec{p}) - f(\vec{q})| = |f(\vec{p}) - f(\vec{x}_j) + f(\vec{x}_j) - f(\vec{q})| \leq |f(\vec{p}) - f(\vec{x}_j)| + |f(\vec{q}) - f(\vec{x}_j)| < \varepsilon$$

Therefore, f is uniformally continuous over K.

2.7.3 Intermediate Value Theorem

Theorem

If $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^d$ is connected and continuous. $\forall \vec{p}, \vec{q} \in D$ such that $f(\vec{p}) \neq f(\vec{q})$ and if $y \in \mathbb{R}$ is between $f(\vec{p})$ and $\vec{q}$, then $\exists \vec{r} \in D$ such that $f(\vec{r}) = y$.

Proof

WLOG, assume $f(\vec{p}) < f(\vec{q})$

Because D is connected, there is some path $\vec{\gamma}(t) = (x(t), y(t)) \in D$, $a \leq t \leq b$ such that $\vec{\gamma}(a) = \vec{p}$ and $\vec{\gamma}(b) = \vec{q}$ and is continuous over $[a, b]$. Now we construct $g(t) = f(\vec{r}(t)) = (f \cdot \vec{r})(t) \in varmathbbR$. Because $g(t)$ is a composition of continuous functions, $g(t)$ is also continuous. $g(a) = f(\vec{\gamma}(a)) = f(\vec{p})$ and $g(b) = f(\vec{\gamma}(b)) = f(\vec{q})$. Now we see that $g(a) < y < g(b)$. Therefore, by the IVT for single-variable functions, $\exists\, t_0$ such that $g(t_0) = y$. Now we plug t_0 into $\vec{\gamma}$, $\vec{r} = \vec{\gamma}(t_0) \in D$. Then $f(\vec{r}) = f(\vec{\gamma}(t_0)) = g(t_0) = y$.

2.8 Vector Valued Functions

2.8.1 Definition

$\vec{f} : D \to \mathbb{R}^e$, $D \subseteq \mathbb{R}^d$ is known as a **vector-valued/point-valued** function.

$\vec{f}(\vec{p}) = (f_1(\vec{p}), f_2(\vec{p}), \ldots, f_e(\vec{p}))$ where $f_1, f_2, \ldots, f_e$ are real-valued and are called the **component functions** of f.

2.8.2 Continuity

$f : \mathbb{R}^d \to \mathbb{R}^e$ is continuous at $\vec{a} = (a_1, a_2, a_3, \ldots, a_d) \in D$ iff $\forall \varepsilon, \exists\, \delta > 0 : \forall \vec{x} \in D$:

$$\| \vec{x} - \vec{a} \|_d < \delta \to \| f(\vec{x}) - f(\vec{a}) \|_e < \varepsilon$$

Component-wise Nature of Continuity

$f : D \to \mathbb{R}^e$ is continuous at a point $\vec{a} \in D$ iff $f_1, f_2, \ldots, f_e$ are all continuous at $\vec{a}$.

Proof: First fix an ε, then by the basic distances bound lemma, we get a bunch of inequalities:

$$\begin{aligned}
|f_1(\vec{x}) - f_1(\vec{p})| &< \frac{\varepsilon}{\sqrt{e}}, \forall \vec{x} \in D \cap B_{\delta_1}(\vec{p}) \\
|f_2(\vec{x}) - f_2(\vec{p})| &< \frac{\varepsilon}{\sqrt{e}}, \forall \vec{x} \in D \cap B_{\delta_2}(\vec{p}) \\
&\cdots\cdots\ \ \cdots\cdots \\
|f_e(\vec{x}) - f_e(\vec{p})| &< \frac{\varepsilon}{\sqrt{e}}, \forall \vec{x} \in D \cap B_{\delta_e}(\vec{p})
\end{aligned}$$

Then let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_e\} > 0$, which must exist and satisfy all the distance inequalities, specifically $\max_{1 \le j \le e}(|f_j(\vec{x}) - f_j(\vec{p})|)$. Then again, by the basic distance bounds lemma, we know that $\| \vec{f}(\vec{x}) - \vec{f}(\vec{p}) \|_e \le \max_{1 \le j \le e}(|f_j(\vec{x}) - f_j(\vec{p})|)$. Therefore we get that for any fixed ε, we can find a δ such that $\| \vec{x} - \vec{p} \|_d < \delta \to \| f(\vec{x}) - f(\vec{p}) \|_e < \varepsilon$

Composition of Continuous Functions

If $\vec{f} : D \to E$, where $D \subseteq \mathbb{R}^d$, $E \subseteq \mathbb{R}^e$ and $\vec{g} : E \to \mathbb{R}^k$ are both continuous on their respective domains. Then $\vec{h} = \vec{g} \circ \vec{f}$ is continuous on D

Proof: To prove this, fix $\varepsilon > 0$. There's a $\eta > 0$ such that $\forall \vec{y} \in E \cap B_\eta(\vec{f}(\vec{p}))$, because we know that $\vec{g}$ is continuous at $\vec{f}(\vec{p})$, we get:

$$\| \vec{g}(\vec{y}) - \vec{g}(\vec{f}(\vec{p})) \| < \varepsilon$$

To guarantee that $\vec{y} = \vec{f}(\vec{x})$ lies within η units of $\vec{f}(\vec{p})$ i.e. $\| \vec{f}(\vec{x}) - \vec{f}(\vec{p}) + \| < \eta$, we can take $\vec{x} \in D \cap B_\delta(\vec{p})$ where $\delta > 0$ corresponding to η [using the continuity of $\vec{f}$ at $vecp \in D$].

Now, as long as $\vec{x} \in D \cap B_\delta(\vec{p})$, we have $\vec{f}(\vec{x}) \in E \cap B_\eta(\vec{f}(\vec{p}))$. Thus, if we take $\vec{y} = \vec{f}(\vec{p})$, we get: $\| \vec{g}(\vec{f}(\vec{x})) - \vec{g}(\vec{f}(\vec{p})) \| < \varepsilon$, or $\| \vec{h}(\vec{x}) - \vec{h}(\vec{p}) \| < \varepsilon$. So $\vec{h}$ is continuous at $\vec{p}$.

2.8.3 Compactness Theorem

Theorem

Let $\vec{f} : D \to \mathbb{R}^e$ be a contnuous function, where $D \subseteq \mathbb{R}^d$ is compact. Then its *range* $\vec{f}(D) := \{f(\vec{p}) \mid \vec{p} \in D\}$ is also compact. In other words: compactness is preserved under continuous mappings. Note that this is the generalization of the Extreme Value Theorem.

Proof

To prove this, write $R := f(D)$. We need to show that R is closed and bounded in $\mathbb{R}^e$. Boundedness is easy. Since each component function f_j of f is real valued, by EVT each component function f_j has an absolute bound M_j, so that $\mid f_j(\vec{p}) \mid \leq M_j$ for all $\vec{p} \in D$. Take $M := \max\{M_1, M_2, \ldots, M_e\}$. Then for all $\vec{p} \in D$, we have

$$\parallel \vec{f}(\vec{p}) \parallel = \sqrt{f_1(\vec{p})^2 + \cdots + f_e(\vec{p})^2} \leq \sqrt{e \times M^2} = M\sqrt{e}$$

This says that the range R lies within the closed ball of radius $M\sqrt{e}$ centered at $\vec{0}$ in $\mathbb{R}^e$. It therefore certainly lies within some closed cube centered at $\vec{0}$, and hence is bounded.

To prove closedness, let $(\vec{y}_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^e$ with limit $\vec{y}$, such that $\vec{y}_n \in R$ for each $n \geq 1$. We need to prove that $\vec{y} \in R$. Since R is the range of f, we must have $\vec{y} = \vec{f}(\vec{x}_n)$, where $\vec{x}_n \in D$. Because D is bounded, we can pick a convergent subsequence $\vec{x}_{n_k} \to \vec{x}$. But because D is closed, we know that $\vec{x} \in D$. Because $\vec{f}$ is continuous, we get that $\vec{y}_{n_k} = \vec{f}(\vec{x}_{n_k}) \to \vec{f}(\vec{x})$. However, we know that $\vec{y}_{n_k} \to \vec{y}$. But since each sequence converges to one point, we know that $\vec{y} = \vec{f}(\vec{x})$, where $\vec{x} \in D$. Therefore, $\vec{y} \in R$.

Therefore, R is closed and bounded.

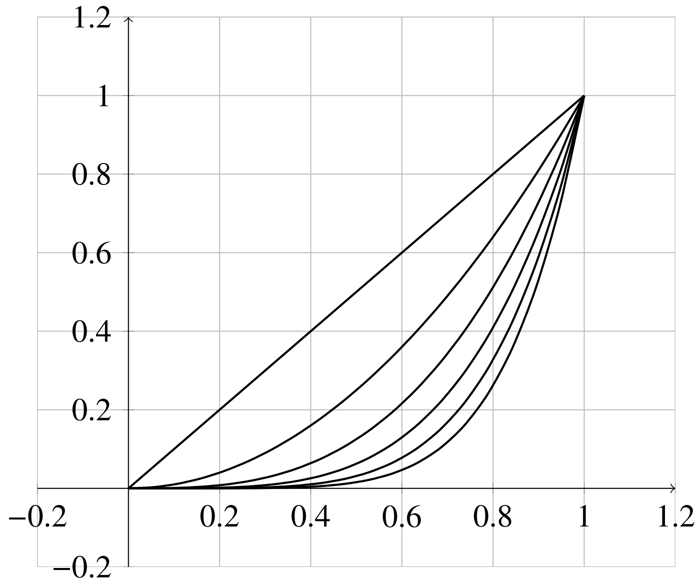
2.8.4 Connectedness Theorem

Theorem

$\vec{f} : D \to \mathbb{R}^e$, $D \in \mathbb{R}^d$ is continuous on D. If D is connected, $E = \vec{f}(D)$ (the range of the domain), is also connected. Note that this is the generalization of the Intermediate Value Theorem.

Proof

$\forall \vec{u}, \vec{v} \in E$, we can find two points $\vec{p}, \vec{q}$ such that $\vec{f}(\vec{p}) = \vec{u}$ and $\vec{f}(\vec{q}) = \vec{v}$. Now because D is connected, $\exists\, \vec{\gamma} : [a, b] \to \mathbb{R}^d$, $\vec{\gamma}([a, b]) \subseteq D$. Now we consider $\vec{\delta} = \vec{f}(\vec{\gamma}(t))$, $\vec{\delta} : [a, b] \to \mathbb{R}^e$. Since it's a composition of continuous functions, $\vec{\delta}$ is also continuous. And $\forall t \in [a, b]$, $\vec{\delta}(t) = \vec{f}(\vec{\gamma}(t)) \in E$. We also know that $\vec{\delta}(a) = \vec{f}(\vec{\gamma}(a)) = \vec{f}(\vec{p}) = \vec{u}$, and $\vec{\delta}(b) = \vec{f}(\vec{\gamma}(b)) = \vec{f}(\vec{q}) = \vec{v}$. Therefore,



$\forall \vec{u}, \vec{v} \in E$, there is a continuous path that connects $\vec{u}$ to $\vec{v}$ and stays within E. Therefore, E is connected.

2.9 Sequences of Functions

2.9.1 Infinity Norm

For $f : D \to \mathbb{R}$ that is bounded, we say: $\| f \|_\infty := \sup_{x\in D} |f(x)|$

We can also express this as: $\| f \|_D := \sup_{x\in D} |f(x)|$

2.9.2 Pointwise Convergence

$f_n : [a, b] \to \mathbb{R}$ for $n \geq 1$, and assume that they are all bounded on $[a, b]$. ($\exists\, M_n > 0 : |f_n(x)| \leq M_n$ for all $x \in [a, b]$).

$f : [a, b] \to \mathbb{R}$, f is bounded ($\exists\, M > 0 : |f(x)| \leq M$ for all $x \in [a, b]$).

If $\forall x \in [a, b] : \lim_{n\to\infty} f_n(x) = f(x)$, then we say that $f_n \xrightarrow{p} f(x)$ (f_n converges “pointwise” to f).

Example

$f_n(x) = x^n$, $n \geq 1$ on $[0, 1]$

If the domain is $[0, 1)$, then $f_n \xrightarrow{p} f$ where $f(x) = 0$.

On $[0,1]$, $f_n \xrightarrow{p} \tilde{f}$, where $\tilde{f}$ is 0 for $0 \leq x < 0$, and 1 when $x = 1$

2.9.3 Uniform Convergence

In Single Variable Calculus

$f_n : [a,b] \to \mathbb{R}$ for $n \geq 1$, and assume that they are all bounded on $[a,b]$. ($\exists\, M_n > 0 : |f_n(x)| \leq M_n$ for all $x \in [a,b]$).

$f : [a,b] \to \mathbb{R}$, f is bounded ($\exists\, M > 0 : |f(x)| \leq M$ for all $x \in [a,b]$).

We then claim that $f_n \xrightarrow{u} f$ as $n \to \infty$ if $\forall \varepsilon$, $\exists\, N_\varepsilon$ such that $\forall n \geq N_\varepsilon : \| f_n - f \|_\infty < \varepsilon$. In other words, this forces that the greatest vertical difference between the two functions will be arbitrarily small after N_ε. This forces the two functions to be "close" as a whole.

General Case

Let $D \in \mathbb{R}^d$ be a non-empty set, and let $\vec{f} : D \to \mathbb{R}^e$ and $\vec{f_n} : D \to \mathbb{R}^e$ for $n \geq 1$. We say that $\vec{f_n} \xrightarrow{u} \vec{f}$ on D if:

$$\| f_n - f \|_D \to 0 \text{ as } n \to \infty$$

$f_n \xrightarrow{u} f$ over D implies $f_n \xrightarrow{p} f$ over D

Pick any $x \in D$, $0 \leq \| \vec{f_n}(\vec{x}) - \vec{f}(\vec{x}) \| \leq \| \vec{f_n} - \vec{f} \|_D \to 0$ as $n \to \infty$. Then by the squeeze theorem, $\|\vec{f_n}(\vec{x}) - \vec{f}(\vec{x})| = 0$, therefore $\vec{f_n}(\vec{x}) \to \vec{f}(\vec{x})$ as $n \to \infty$.

Note the the converse is NOT true.

2.9.4 Uniform Convergence Theorem

Theorem

If $\vec{f_n} \xrightarrow{u} \vec{f}$ on D and each $\vec{f_n}$ is continuous on D, then $\vec{f}$ is also continuous on D. In other words, a uniform limit of continuous functions must also be continuous.

Proof

Pick any $\vec{p} \in D$, and fix $\varepsilon > 0$. We need to find δ such that $\forall \vec{x} \in D$ with $\| \vec{x} - \vec{p} \| < \delta$, then $\| \vec{f}(\vec{x}) - \vec{f}(\vec{p}) \| < \varepsilon$. We can apply the triangle inequality and we get:

$$\| \vec{f}(\vec{p}) - \vec{f}(\vec{p}) \| \leq \| \vec{f}(\vec{x}) - \vec{f}_n(\vec{x}) \| + \| \vec{f}_n(\vec{x}) - \vec{f}_n(\vec{p}) \| + \| \vec{f}_n(\vec{p}) - \vec{f}(\vec{p}) \|$$

For large enough n, we know that $\| \vec{f}_n - \vec{f} \|_D < \frac{\varepsilon}{3}$ because $\vec{f}_n \xrightarrow{u} \vec{f}$. Now we know that the first and third term are bounded by $\frac{\varepsilon}{3}$. The second term is bounded by $\frac{\varepsilon}{3}$ because $\vec{f}_n$ is uniformally continuous.

Therefore, we know that $\| \vec{f}(\vec{p}) - \vec{f}(\vec{p}) \| \leq \varepsilon$ for any δ we pick. $\therefore$ $\vec{f}$ is continuous on D.

Chapter 3

Differential Calculus

3.1 Differentiability of Multivariate Functions

3.1.1 In Single Variable Calculus

Definition

$f : [a, b] \to \mathbb{R}$ is said to be **differentiable** at $p \in [a, b]^\circ$ (interior of $[a, b]$) if $\exists\, a \in \mathbb{R}$ such that:

$$\boxed{a = \lim_{h \to 0} \frac{f(p + h) - f(p)}{h}}$$

Now we can try to expand this to higher dimensions.

If we try to apply the same definition, we would supposedly get

$$a = \lim_{\vec{h} \to \vec{0}} \frac{f(\vec{p} + \vec{h}) - f(\vec{p})}{\vec{h}}$$

But this makes no sense! As we are dividing a number by a vector within the limit. Therefore we must reconsider the definition of a derivative.

Let's rearrange some terms in our original equation, we get:

$$\lim_{h \to 0} \left| \frac{f(p + h) - f(p)}{h} - a \right| = 0$$

Let what's inside the limit be $\alpha(p, h)$. Now we multiply both sides by $|h|$. We get:

$$|h| \left| \frac{f(p + h) - f(p)}{h} - a \right| = |f(p + h) - f(p) - ah|$$

Now we reach a new definition of differentiability. We say that f is differentiable at p if $\exists\, a \in \mathbb{R}$ such that:

$$\boxed{\forall \varepsilon > 0, \exists\, \delta : |h| < \delta \implies |f(p+h) - f(p) - ah| < \varepsilon |h| \quad (*)}$$

Then a is known as the **derivative** of f at p, we write it as $a = f'(p) = \frac{df}{dx}(p)$

Geometric Interpretation

Basically we are trying to find a line with slope a that mimics the behavior of function f locally around p, in some open ball $B_r(p)$. Let the difference between p and a point close to p along $f(x)$ as $g(x) = f(p+x) - f(p)$ for $x \in (r, -r)$. Let the function of the line be $l(x) = f(p) + a(x-p)$ be a line that approximates f over $B_r(p)$. Note that this requires that the vertical distance between $g(x)$ and $l(x)$ goes to 0 faster than x grows, in other words the follow condition must be true:

$$\boxed{\frac{|g(x) - l(x)|}{|x|} \to 0 \text{ as } x \to 0}$$

We define local similar behavior as **superlinear decay or approximation**.

Note that this is simply a restatement of condition $(*)$, Note that $f(p+h) - f(p)$ is simply $g(x) - l(x)$, and the difference between that and a has to be arbitarily small ($< \varepsilon$, note that the $|h|$ cancels out if you rearrange the terms this way). Therefore the two definitions are equivalent.

Uniqueness of the Derivative

Thm: Either $\nexists\, a \in \mathbb{R}$ that satisfies condition $(*)$, or $\exists!\, a \in \mathbb{R}$ such that condition $(*)$ is true. In the latter case we denote the unique value of a by $f'(p)$ or $\frac{df}{dx}(p)$

Proof: Say both l and $\tilde{l}$ superlinearly approximate to $f(x)$ at point p, $l(x) = ax$ and $\tilde{l}(x) = bx$. Then by definition we know that $\frac{|g(x)-ax|}{|x|} \to 0$ and $\frac{|g(x)-bx|}{|x|} \to 0$ as $x \to 0$. Now we seek to find the difference between l and $\tilde{l}$:

$$\begin{aligned} 0 \le \frac{|ax - bx|}{|x|} &= |a - b| \\ &= \frac{|(g(x) - bx) - (g(x) - ax)|}{|x|} \\ &\le \frac{|g(x) - bx|}{|x|} + \frac{|g(x) - ax|}{|x|} \to 0 \text{ as } x \to 0 \end{aligned}$$

Therefore by the squeeze theorem $|a - b| = 0 \implies a = b \implies l = \tilde{l}$. Therefore there is only one line l that can superlinearly approach g at $x = 0$.

3.1.2 The Gradient

Differentiability in Real Valued Functions

We can easily generalize the definition into higher dimensions, except we replace the concept of the product with dot product. We say that $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$ is **differentiable** at $\vec{p} \in D^\circ$ if $\exists\, \vec{a}$ such that:

$$\boxed{\forall \varepsilon > 0, \exists\, \delta : 0 < \|\, \vec{h}\, \| < \delta \implies |f(\vec{p} + \vec{h}) - f(\vec{p}) - \vec{a} \cdot \vec{h}| < \varepsilon \|\, \vec{h}\, \| \quad (**)}$$

Here $\vec{a}$ is known as the **gradient** of f at point $\vec{p}$. It is commonly written as $\vec{\nabla} f(\vec{p})$. Note that even though in the definition looks like $\vec{\nabla}$ is an operator applied to $f(\vec{p})$, it is not. It only modifies f to create a different function.

Uniqueness of the Gradient

Thm: For the function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$, either no $\vec{a}$ satisfies $(**)$ at point $\vec{p}$, in which case f is not differentiable at $\vec{p}$. Or $\exists!\, \vec{a}$ that satisfies $(**)$, in which case $\vec{a} = \vec{\nabla} f(\vec{p})$

Proof: This is very similar to how we proved that the derivative is unique if it exists. Similarly, we assume that $\vec{a}$ and $\vec{b}$ are both valid gradients for f around $\vec{p}$. Then we attempt to find the difference between the vertical distances between the two approximations created by the gradients:

$$
\begin{gathered}
|(\Delta f - \vec{b} \cdot \vec{h}) - (\Delta f - \vec{a} \cdot \vec{h})| = |\vec{a} \cdot \vec{h} - \vec{b} \cdot \vec{h}| < 2\varepsilon \|\, \vec{h}\, \| \\
\|\, (\vec{a} - \vec{b}) \cdot \vec{h}\, \| < 2\varepsilon \|\, \vec{h}\, \| \\
\left| (\vec{a} - \vec{b}) \cdot \frac{\vec{h}}{\|\, \vec{h}\, \|} \right| < 2\varepsilon
\end{gathered}
$$

Let $\vec{u} = \frac{\vec{h}}{\|\, \vec{h}\, \|}$. Note, however, that $\|\, \vec{u}\, \| = 1$, and that is the only restriction on $\vec{u}$. Therefore we create a sequences of possible $\vec{u}$: $\vec{e}_i, \forall 0 < i \leq d$, with each $\vec{e}_i$ composed of all 0's except for the i^{th} component. This means that for every single component of $|(\vec{a} - \vec{b})|$ is bounded by 2ε. Therefore, by the componentwise nature of distance, we know that $\vec{a} - \vec{b} = 0$. Which means that then $\vec{a}$ and $\vec{b}$ are not distinct. Therefore $\vec{\nabla} f(\vec{p})$ is unique.

Geometric Interpretation of the Definition

In single variable calculus, we can say that the derivative defines a linear function that approximates the function at a given point $\vec{p}$. In higher dimensions, we need a function that provides us a superlinear approximation of f. In higher dimensions, we find the **tangent plane or hyperplane**.

A hyperplane in $\mathbb{R}^{d+1}$ is a set of the form $\Pi = \{\vec{x} \in \mathbb{R}^{d+1} \mid (\vec{x} - \vec{p}_0) \cdot \vec{n} = 0\}$ where $\vec{p}_0 \in \mathbb{R}^{d+1}$ and $\vec{n} \in \mathbb{R}^{d+1}$ with $\vec{n} \neq 0$. In a more geometric sense, this definition says that we want the set of all $\vec{x}$'s such that $(\vec{x} - \vec{p}_0) \perp \vec{n}$ (recall that the dot product of two perpendicular vectors is 0). In this definition, $\vec{p}_0$ would be the foot of $\vec{n}$ and $\vec{x} - \vec{p}$ would be every point's displacement vector from the "center" of the plane. We call $\vec{n}$ the **normal** to Π, we write it as $\vec{n} \perp \Pi$.

But how do we relate this to the definition of the gradient? For this we go back to definition $(**)$, specifically $|f(\vec{p} + \vec{h}) - f(\vec{p}) - \vec{a} \cdot \vec{h}| < \varepsilon \parallel \vec{h} \parallel$. If we replace $\vec{p} + \vec{h}$ by $\vec{x}$, the definition becomes:

$$| f(\vec{x}) - [f(\vec{p}) + \vec{a} \cdot (\vec{x} - \vec{p})] | < \varepsilon \parallel \vec{x} - \vec{p} \parallel$$

Note that $| f(\vec{x}) - [f(\vec{p}) + \vec{a} \cdot (\vec{x} - \vec{p})] |$ looks like the distance between f and the graph $y = f(\vec{p}) + \vec{a} \cdot (\vec{x} - \vec{p})$ at $\vec{x}$. Now we need to prove that the graph of $y = f(\vec{p}) + \vec{a} \cdot (\vec{x} - \vec{p})$ is indeed a hyperplane. Let us rearrange the terms and we get:

$$-\vec{a} \cdot (\vec{x} - \vec{p}) + (y - f(\vec{p})) = 0$$

Now we define a few vectors in $\mathbb{R}^{d+1}$:

$$\begin{aligned}
\vec{N} &= (-a_1, -a_2, \ldots, -a_d, 1) \\
\vec{P}_0 &= (p_1, p_2, \ldots, p_d, f(\vec{p})) \\
\vec{X} &= (x_1, x_2, \ldots, x_d, y) \\
\vec{X} - \vec{P}_0 &= (x_1 - p_1, x_2 - p_2, \ldots x_d - p_d, y - f(\vec{p}))
\end{aligned}$$

Therefore, we can rearrange the previous equation as:

$$(\vec{X} - \vec{P}_0) \cdot \vec{N} = 0$$

Which is the general form of a hyperplane, and note that $\vec{N} \neq 0$ because its last component is 1.

Now if we go back to the definition of a gradient, we see that all that it is saying is that there exists one hyperplane $f(\vec{p}) + \vec{a} \cdot (\vec{x} - \vec{p})$ that approximates f superlinearly near $\vec{p}$, and by the uniqueness of the gradient, the hyperplane is unique.

We now know that $\vec{\nabla} f(\vec{p})$ defines a hyperplane that is tangent to the graph of $f(\vec{x})$ at point $\vec{p}$, but what does this value represent? If we look back to how we got to the hyperplane, we defined $\vec{N}$, the normal to the tangent hyperplane as $(-\vec{\nabla} f(\vec{p}), 1)$. Therefore we can interpret the gradient as the projection of the negation of the normal vector onto the domain. Its direction is the direction of fastest increase in the function at point $\vec{p}$ and the magnitude of the gradient is the rate of fastest increase at $\vec{p}$. Note that the opposite direction of the gradient (the direction of the normal vector projected onto the domain) is therefore the direction of fastest decrease in the function at point $\vec{p}$ and the magnitude of that vector is the rate of fastest decrease at $\vec{p}$.

Computation of $\vec{\nabla} f$

Take $\vec{h} = h\vec{e}_j$, where $h \to 0$ and the set of $\vec{e}_j$ is known as the *standard basis vectors* in $\mathbb{R}^d$:

$$\vec{e}_j = \begin{cases} \vec{e}_1 & = (1, 0, 0, \ldots, 0) \\ \vec{e}_2 & = (0, 1, 0, \ldots, 0) \\ \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \\ \vec{e}_d & = (0, 0, 0, \ldots, 1) \end{cases}$$

Note that because $\| \vec{e}_j \| = 1$, $\| \vec{h} \| = |h| \, \| \vec{e}_j \| = |h|$. Now if we fix a $j \in \{1, 2, 3, \ldots, d\}$ and apply the definition of differentiability, the unique gradient ($\vec{a}$) must satisfy:

$$\forall \varepsilon > 0, \exists\, \delta > 0 : \forall h \text{ with } |h| < \delta \implies |f(\vec{p} + h\vec{e}_j) - f(\vec{p}) - \vec{a} \cdot h\vec{e}_j| < \varepsilon\, |h|$$

However, note that when we dot $\vec{a}$ with $\vec{e}_j$, the result is the j^{th} component of $\vec{a}$, or a_j. Now if we divide through by $|h|$, we get:

$$\left| \frac{f(\vec{p} + h\vec{e}_j) - f(\vec{p})}{h} - a_j \right| < \varepsilon$$

What this says is that $\left| \frac{f(\vec{p}+h\vec{e}_j)-f(\vec{p})}{h} \right|$ approaches a_j indefinitely, therefore, we can rewrite the relationship as a limit statement:

$$\boxed{a_j = \lim_{h \to 0} \left| \frac{f(\vec{p} + h\vec{e}_j) - f(\vec{p})}{h} \right|}$$

We call this a_j as a **partial derivative** of $f(\vec{x})$ at j^{th} component, which can be written as $\partial_{x_j} f(\vec{p})$ or $\frac{\partial f}{\partial x_j}(\vec{p})$.

Note that:

$$\partial_{x_j} f(\vec{p}) = \frac{d}{dx_j}\bigg|_{x_j = p_j} f(p_1, p_2, \ldots, p_{j-1}, x_j, p_{j+1}, \ldots, p_d)$$

In other words, we can hold all other components of f constant and differentiate based on only one component, and plug in the value p_j after the differentiation.

Now we know how to compute the gradient of f, it is simply the vector of all the partial derivatives:

$$\vec{\nabla} f = (\partial_{x_1} f(\vec{p}), \partial_{x_2} f(\vec{p}), \ldots, \partial_{x_d} f(\vec{p}))$$

Example

Let's compute $\vec{\nabla} f(1,0,2)$ of

$$f(x,y,z) = x^2 \sin(y + xz)$$

Let's calculate the partials first:

$$\begin{aligned}
\partial_x f &= x^2 \cos(y + xz)z + 2x \sin(y + xz) \\
\partial_y f &= x^2 \cos(y + xz) \\
\partial_z f &= x^3 \cos(y + xz)
\end{aligned}$$

And if we plug in the numbers we get:

$$\begin{aligned}
\partial_x f(1,0,2) &= 2\cos(2) + 2\sin(2) \\
\partial_y f(1,0,2) &= \cos(2) \\
\partial_z f(1,0,2) &= \cos(2)
\end{aligned}$$

Therefore, $\boxed{\vec{\nabla} f(1,0,2) = (2\cos(2) + 2\sin(2), \cos(2), \cos(2))}$

Directional Derivative

Take $\vec{u} \in \mathbb{R}^d$, $\vec{u} \neq \vec{0}$ and a function $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$, and take a point on the function f, $\vec{p}$ such that f is continuous at $\vec{p}$, we define the **directional derivative** $\partial_{\vec{u}} f(\vec{p})$ as:

$$\begin{aligned}
\partial_{\vec{u}} f(\vec{p}) &:= \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h} && (3.1) \\
&= \frac{d}{dt}\bigg|_{t=0} f(\vec{p} + t\vec{u}) && (3.2)
\end{aligned}$$

This quality describes how fast the function f is changing at $\vec{p}$ in the direction of $\vec{u}$. The numerator of definition (1) is the difference between the function value at $\vec{p}$ and at $\vec{p}+h\vec{u}$, i.e. a little increment in the direction of $\vec{u}$, and we divide that by h.

Definition (2) gives a single variable definition of a directional derivative. We can write $g(t)$ as the change in function value as we move away $\vec{p}$ in the direction $\vec{u}$. Note that the curve of the form $\vec{\gamma}(t) = \vec{p} + t\vec{u}$ ($\vec{u} \neq \vec{0}$) has a special name, it is called **uniform rectilinear motion**. It is rectilinear because it is a line and it is uniform because the speed of the curve is constant. The speed of a curve is represented as $||\vec{r}'(t)||$

However, the magnitude of $\vec{u}$ also has meaning, its magnitude is the rate of change of the function in the direction of $\vec{u}$.

Thm: Let $\vec{u}$ be any unit vector, then:

$$\partial_{\vec{u}} f(\vec{p}) = \vec{\nabla} f(\vec{p}) \cdot \vec{u}$$

Proof:

Geometric Interpretation of the Gradient

If $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$ is differentiable at $\vec{p} \in D^\circ$, then:

$$\vec{u}_0 = \frac{\vec{\nabla} f(\vec{p})}{\| \vec{\nabla} f(\vec{p}) \|}$$

i.e. the unit vector of the gradient is the direction of steepest ascent for f at $\vec{p}$.

$$\| \vec{\nabla} f(\vec{p}) \| = \max_{\| \vec{u} \| = 1} \partial_{\vec{u}} f(\vec{p})$$

And the magnitude of the gradient is the rate at which the function is ascending at the point.

This is a combination of the Cauchy-Schwarz Inequality and the theorem which states that $\partial_{\vec{u}} f(\vec{p}) = \vec{\nabla} f(\vec{p}) \cdot \vec{u}$. By the Cauchy-Schwarz Inequality, we know that the length of the gradient is an upper bound of the directional derivative. This maximum is also obtained, because in Cauchy-Schwarz inequality, the equality case happens when the two vector have the same direction. Therefore, we know that the gradient is the direction of fastest ascent of a function at any given point $\vec{p}$

However, the Cauchy-Schwarz inequality also states that the dot product is bounded from below by negative of the product of the lengths, which is achieved when the two vector are anti-directional. Therefore we also know that the negative of the gradient points in the direction of steepest descent and the magnitude can be written as:

$$-\| \vec{\nabla} f(\vec{p}) \| = \min_{\| \vec{u} \| = 1} \partial_{\vec{u}} f(\vec{p})$$

3.1.3 Basic Rules of the Gradient Operator

1. $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$ (If f and g are both differentiable, then $f+g$ is differentiable as well)
2. $\vec{\nabla}(cf) = c\vec{\nabla} f$ (floaty constant rule)
3. $\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f$ (product rule)

4. $\vec{\nabla}\frac{f}{g} = \frac{g\vec{\nabla}f - f\vec{\nabla}g}{g^2}$ (quotient rule, as long as $g \neq 0$)

These rules are true because in each component, the partial operator obeys these rules, therefore the gradient also obeys these rules.

3.2 Theorems about Gradient

3.2.1 Rolle's Theorem

Theorem

If f is differentiable on $(a - r, b + r)$ where $r > 0$ and $a < b$, and $f(a) = f(b)$. Then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Proof

f is either constant or it is not on $[a, b]$. If it is constant, then the theorem is trivially true. If not, then there exists $p \in [a, b]$ such that $f(p) \neq f(a) = f(b)$. If $f(p) > f(a)$, then take c to be a global maximum point for the function f on the interval $[a, b]$ (EVT). Note that $c \neq a$, $c \neq b$, so $c \in (a, b)$. Let us then consider $f'(c)$. We are given that

$$\lim_{h\to 0}\frac{f(c+h) - f(c)}{h}$$

But if we consider the limits from two directions, we get:

$$\lim_{h\to 0^+}\frac{f(c+h) - f(c)}{h} \leq 0$$
$$\lim_{h\to 0^-}\frac{f(c+h) - f(c)}{h} \geq 0$$

Therefore, $f'(c) = 0$. A similar argument can be made if $f(p) < f(a$

3.2.2 Mean Value Theorem

Review of the Single Variable Theorem

If $f : [a, b] \to \mathbb{R}$ is differentiable on $(a - r, b + r)$ where $r > 0$ and continuous on $[a, b]$, then $\Delta f = f(b) - f(a) = f'(c)(b - a)$ where $c \in (a, b)$.

Proof

For the given function f, let us create a function $l : y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, or a secant line connecting a to b. Let $g(x) = f(x) - l(x)$. Then, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Note also $g(a) = 0 = g(b)$. Then by Rolle's Theorem, $\exists\, c \in (a,b)$ such that $g'(c) = 0$. This implies that at c, $f(c)$ has the same derivative as $l(c)$, or $\frac{f(b)-f(a)}{b-a}$.

Theorem

Exists $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$, $\vec{p}, \vec{q} \in D^\circ$, $[\vec{p}, \vec{q}] \subseteq D^\circ$ [1]. Also assume that f is differentiable on D. Then $\exists\, \vec{r} \in (\vec{p}, \vec{q})$ such that:

$$\boxed{f(\vec{q}) - f(\vec{p}) = \vec{\nabla} f(\vec{r}) \cdot (\vec{q} - \vec{p})}$$

Proof

Let $g(t) = f((1-t)\vec{p} - t\vec{q})$, where $-\varepsilon \leq t \leq 1 + \varepsilon$ for some $\varepsilon > 0$. We can do this because $\vec{p}, \vec{q} \in D^\circ$. Now we can try to take the derivative of $g(t)$. To do this we use the chain rule: $g'(t) = \vec{\nabla} f((1-t)\vec{p} + t\vec{q}) \cdot (\vec{q} - \vec{p})$. We now apply the Mean Value Theorem to g, we know that $\exists\, c \in (0,1)$ such that $g'(c) =$

1 - 0 = f() - f().$Nowwehave:$

$$f(\vec{q}) - f(\vec{p}) = \vec{\nabla} f((1-c)\vec{p} + c\vec{q}) \cdot (\vec{q} - vecp)$$

Now we simply call $(1-c)\vec{p} + c\vec{q}\ \vec{r}$, and we're done.

For Vector Valued Functions

Sadly there is no generalization of the MVT to vector valued functions. If we apply the MVT to each of its component functions, we see that for each $\vec{r}$ may be different. Therefore there is no easy, consistent generalization.

In order to actually generalize this, we will need to make the statement weaker.

We can at least provide a bound for $|f(\vec{p}) - f(\vec{q})|$. By Cauchy-Schwarz inequality, we get:

$$|f(\vec{q}) - f(\vec{p})| \leq \|\ \vec{\nabla} f(\vec{r})\ \|\ \|\ \vec{q} - \vec{p}\ \|$$

for some $\vec{r} \in (\vec{p}, \vec{q})$. However, we can write the most general case as:

[1] This represents a **closed segment** with the end points of $\vec{p}$ and $\vec{q}$, which can be expressed as $[\vec{p}, \vec{q}] = \{(1-t)\vec{p} + t\vec{q} \mid 0 \leq t \leq 1\}$. An **open segment** is written as $(\vec{p}, \vec{q}) = \{(1-t)\vec{p} + t\vec{q} \mid 0 < t < 1\}$

$$\boxed{|f(\vec{q}) - f(\vec{p})| \leq \left(\sup_{\vec{x} \in [\vec{p}, \vec{q}]} \| \vec{\nabla} f(\vec{x}) \| \right) \| \vec{q} - \vec{p} \|}$$

This is known as the **Mean Value Inequality**. And as a corollary, we can take any B such that if $\| \vec{\nabla} f \| \leq B$ on $[\vec{p}, \vec{q}]$ then:

$$|f(\vec{q}) - f(\vec{p})| \leq B \| \vec{q} - \vec{p} \|$$

To generalize the Mean Value Inequality to vector valued function, let $\vec{f} : D \subseteq \mathbb{R}^d \to \mathbb{R}^e$, and $[\vec{p}, \vec{q}] \subseteq D^\circ$. If $\vec{f}$ is C^1 [2] in some open set U such that $[\vec{p}, \vec{q}] \subseteq U \subseteq D^\circ$.

$$\boxed{\therefore \| \vec{f}(\vec{q}) - \vec{f}(\vec{p}) \| \leq \left(\max_{[\vec{p}, \vec{q}]} \| D\vec{f} \| \right) \| \vec{q} - \vec{p} \|}$$

FIXME moving needed!

Let $M := \max_{[\vec{p}, \vec{q}]} \| D\vec{f} \|$. Fix $\varepsilon > 0$. Let $D = \{t \in [0, 1] \mid \| \vec{f}(\vec{q}_t) - \vec{f}(\vec{p}) \| \leq (M + \varepsilon) \| \vec{q}_t = \vec{p} \| \}$ where $\vec{q}_: = (1 - t)\vec{p} + t\vec{q}$. Note that $\vec{q}_0 = \vec{p}$ and $\vec{q}_1 = \vec{q}$.

Note that $0 \in S$, which means that $S \neq \emptyset$. Also note that $S \leq 1$ because $t \in [0, 1]$. Therefore $\sup S$ exists and finite (bounded by $[0, 1]$), let us call this value T. This means that $\exists\, t_n \in S$ such that $t_n \to T^-$. Therefore, $\forall n, \| \vec{f}(\vec{q}_{t_n}) - \vec{f}(\vec{p}) \| \leq (M + \varepsilon) \| \vec{q}_{t_n} - \vec{p} \|$. Now take the limit as $n \to \infty$ and we get:

$$\| \vec{f}(\vec{q}_T) - \vec{f}(\vec{p}) \| \leq (M + \varepsilon) \| \vec{q}_T - \vec{p} \|$$

This means that $T \in S$ as well. Assume for contradiction that $T < 1$. It will follow that $T = 1$. Because $\vec{q}_1 = \vec{q}$, it follows that:

$$\| \vec{f}(\vec{q}) - \vec{f}(\vec{p}) \| < (M + \varepsilon) \| \vec{q} - \vec{p} \|$$

Now take the limit as $\varepsilon \to 0^+$. We can then conclude:

$$\| \vec{f}(\vec{q}) - \vec{f}(\vec{p}) \| \leq M \| \vec{q} - \vec{p} \|$$

Say $T < 1$. Choose $\delta > 0$ such that $\tau = T + \delta \leq 1$. We now seek to show:

$$\| \vec{f}(\vec{q}_\tau) - \vec{f}(\vec{p}) \| \overset{?}{\leq} (M + \varepsilon) \| \vec{q}_\tau - \vec{p} \|$$

[2] For vector valued function, C^n over U means that every $\frac{\partial^n f_i}{(\partial x_j)^n}(\vec{x})$ is continuous for all $\vec{x} \in U$, $1 \leq i \leq e$ and $q \leq j \leq d$

We first add a bunch of terms to the left hand side and use the triangle inequality.

$$\| \vec{f}(\vec{q}_\tau) - \vec{f}(\vec{p}) \| = \| \vec{f}(\vec{q}_\tau) - \vec{f}(\vec{q}_T) - A_T(\vec{q}_\tau - \vec{q}_T) \|$$

3.2.3 Caution

Before we move on, it is important to note that the mere existence of the partial derivatives is **NOT** sufficient to guarantee differentiability at point $\vec{p}$

In other words, there may be a function f and a point $\vec{p} \in D_f$, such that $\partial_{x_1} f(\vec{p}), \partial_{x_2} f(\vec{p}), \ldots, \partial_{x_d} f(\vec{p})$ all exist in the sense of limits, the function may still not be differentiable at $\vec{p}$. This is because differentiability means that the function can be locally approximated by a hyperplane, and it is entirely possible that a function behaves nicely on every axis but still fails to behave nicely between the axis. In other words, the partials tells you the smoothness of the section curves, and nothing in between.

For example, consider a hemisphere with an octave cut out. TODO later

They don't even imply continuity at $\vec{p}$!

3.2.4 Sufficient Condition for Differentiability

Theorem

If $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$ and $\vec{p} \in D^\circ$ and if $\vec{\nabla} f(\vec{x})$ exists for all points $\vec{x} \in B_r(\vec{p})$ $(r > 0)$ (in that the vector of partial derivatives exists and is defined) and is continuous at $\vec{p}$, then f is differentiable at $\vec{p}$.

In other words, the gradient not only have to exist at $\vec{p}$, it also has to exist around $\vec{p}$. In other words, we need to define each partial as a function of $\vec{x}$ and prove that all those are continuous, we do so by the theorem that states the composition of continuous functions is continuous.

Example

$$\begin{aligned}
f(x, y) &= xy \sin(x + y^2) \\
\partial_x f(x, y) &= xy \cos(x + y^2) + y \sin(x + y^2) \\
\partial_y f(x, y) &= 2xy^2 \cos(x + y^2) + x \sin(x + y^2)
\end{aligned}$$

In this case, take $\vec{p} = (0, 1)$, we know that the partials all exist around $\vec{p}$ and is definitely continuous at $\vec{p}$, since both partials are compositions of continous functions. Therefore we can say that the gradient exists at $\vec{p}$.

Proof

Let $\vec{h} = (h_1, h_2, \ldots, h_d)$ and consider the point $\vec{p} + \vec{h}$. We will construct a path from $\vec{p}$ to $\vec{p} + \vec{h}$ such that in each "step" we move h_d in the d^{th} axial direction, and we call each intermediate point $\vec{p}_d$ in the following manner where $\vec{e}_d$ is the d^{th} standard basis vector.

$$\begin{aligned}
\vec{p}_0 &= \vec{p} \\
\vec{p}_1 &= \vec{p}_0 + h_1\vec{e}_1 \\
\vec{p}_2 &= \vec{p}_1 + h_2\vec{e}_2 \\
\vdots &= \vdots \vdots \vdots \\
\vec{p}_d &= \vec{p}_{d-1} + h_d\vec{e}_d
\end{aligned}$$

Now consider the function $\Delta f = f(\vec{p} - \vec{h}) - f(\vec{p})$, we can write this as a telescoping sum:

$$\Delta f = \sum_{j=1}^{d} \{f(\vec{p}_j) - f(\vec{p}_{j-1})\}$$

Within each term, we can use the mean value theorem from one dimensional calculus. We can do so because $\vec{p}_j$ and $\vec{p}_{j-1}$ only differ in 1 coordinate, like the following:

$$\begin{aligned}
\vec{p}_{j-1} &= (p_1 + h_1, \ldots p_{j-1} + h_{j-1}, \boxed{p_j}, \ldots p_d) \\
\vec{p}_j &= (p_1 + h_1, \ldots p_{j-1} + h_{j-1}, \boxed{p_j + h_j}, \ldots p_d)
\end{aligned}$$

We also know that the partial derivatives of f exists for all points within $B_r(\vec{p})$, therefore we can indeed apply the Mean Value Theorem.

Now we apply the MVT:

$$\Delta f = \sum_{j=1}^{d} \partial_{x_j} f(\vec{q}_j) h_j$$

Where $\vec{q}_j = (p_1 + h_1, \ldots, p_{j-1} + h_{j-1}, p_j + \theta h_j, \ldots, p_d)$ where $0 < \theta < 1$. In other words, $\vec{q}_j$ is somewhere between $\vec{p}_{j-1}$ and $\vec{p}_j$ in the j^{th} coordinate.

Now let us consider the definition of differentiability, we need to prove that

$$|\Delta f - \vec{\nabla} f(\vec{p}) \cdot \vec{h}| = |\sum_{j=1}^{d} \partial_{x_j} f(\vec{q}) h_j - \sum_{j=1}^{d} \partial_{x_j} f(\vec{p}) h_j|$$

Now we can apply the triangle inequality on the summations and get:

$$|\Delta f - \vec{\nabla} f(\vec{p}) \cdot \vec{h}| < \sum_{j=1}^{d} |\partial_{x_j} f(\vec{q}_j) - \partial_{x_j} f(\vec{p})| \, |h_j|$$

Now we divide both sides by the length of $\vec{h}$:

$$\begin{aligned} \frac{|\Delta f - \vec{\nabla} f(\vec{p}) \cdot \vec{h}|}{\| \vec{h} \|} &\leq \sum_{j=1}^{d} |\partial_{x_j} f(\vec{q}_j) - \partial_{x_j} f(\vec{p})| \, \frac{|h_j|}{\| \vec{h} \|} \\ &\leq \sum_{j=1}^{d} |\partial_{x_j}(\vec{q}_j) - \partial_{x_j} f(\vec{p})| \end{aligned}$$

Now let $\vec{h} \to \vec{0}$. Then each $\vec{q}_j$ approaches $\vec{p}$. Then we know:

$$\sum_{j=1}^{d} |\partial_{x_j}(\vec{q}_j) - \partial_{x_j} f(\vec{p})| \to 0$$

Since the number of terms in a sum is fixed and each term within the sum is going to 0, the entire sum is going to 0, which means $\| \vec{h} \| < \delta(\varepsilon)$, we can say that the sum is less than ε, which means that

$$\frac{|\Delta f - \vec{\nabla} f(\vec{p}) \cdot \vec{h}|}{\| \vec{h} \|} \leq \sum_{j=1}^{d} |\partial_{x_j} f(\vec{q}_j) - \partial_{x_j} f(\vec{p})| \, \frac{|h_j|}{\| \vec{h} \|} \leq \varepsilon$$

Which means that a superlinear decay is possible, which means that f is differentiable.

3.2.5 Differentiability $\implies$ Continuity

Theorem

If $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$ is differentiable at $\vec{p} \in D^\circ$ then f is continuous at $\vec{p}$.

Proof

Let us examine our definition of differentiability and continuity. f is said to be differentiable if: $\forall \varepsilon > 0, \exists\, \delta > 0, \forall \vec{h}$ with $\| \vec{h} \|$:

$$|f(\vec{p} + \vec{h}) - f(\vec{p}) - \vec{\nabla} f(\vec{p}) \cdot h| < \varepsilon \| \vec{h} \|$$

To prove continuity, we need to extablish a bound on $|f(\vec{p}+\vec{h}) - f(\vec{p})|$:

$$\begin{aligned}
|f(\vec{p}+\vec{h}) - f(\vec{p})| &= |f(\vec{p}+\vec{h}) - f(\vec{p}) - \vec{\nabla} f(\vec{p}) \cdot \vec{h} + \vec{\nabla} f(\vec{p}) \cdot \vec{h}| \\
&\leq |f(\vec{p}+\vec{h}) - f(\vec{p})| + |\vec{\nabla} f(\vec{p}) \cdot \vec{h}| \\
&\leq \varepsilon \| \vec{h} \| + \| \vec{\nabla} f(\vec{p}) \| \| \vec{h} \| \\
&= (\varepsilon + \||\vec{\nabla} f(\vec{p}) \||) \times \| \vec{h} \|
\end{aligned}$$

Now we are done because $(\varepsilon + \| \vec{\nabla} f(\vec{p}) \|) \times \| \vec{h} \|$ is a constant which we can make arbitarily small. Therefore, f is continuous if it is differentiable.

3.2.6 Equality of Mixed Partials (Clairaut's/Schwarz' Theorem)

Notations – Higher Order of Partials

Classically, we also define second order partial derivatives as follows:

$$\begin{aligned}
\partial_{x_i}\partial_{x_i} &= \partial_{x_i}^2 f = \frac{\partial^2 f}{\partial x_i^2} \\
\partial_{x_i}\partial_{x_j} &= \partial_{x_i}\partial_{x_j} f = \frac{\partial^2 f}{\partial x_i \partial x_j}
\end{aligned}$$

Notation – Composition of Functions

We define a self-application of a function by an "exponent" statement, for example, $f(f(x))$ is sometimes written as $f^2(x)$. To differentiate this from regular exponentiation, $(f \circ f)(x) = f^{\circ 2}(x)$.

Special Set of Functions – C^k

$f \in C^2$ if over some set, $\partial_{x_i}\partial_{x_j} f$ is continuous for any $i, j \in \{1, 2, \ldots, d\}$.

We can also define C^k ($k \geq 2$), which is defined as the set of functions f for which for any $i_1, i_2, \ldots, i_k \in \{1, 2, 3, \ldots, k\}$, $\partial_{x_{i_1}}\partial_{x_{i_2}} \ldots \partial_{x_{i_k}} f$ is continuous over some set.

And we define C^0 on a region if the function itself (or the 0^{th} derivative) is continuous on that region, and C^1 is defined as the set of functions if its partials are continuous over some region.

Note that C^1 implies differentiability because C^1 implies that the partials not only exist, but are also continuous around the point that is being differentiated.

Thm; If f is C^k ($k \geq 1$), then f is also $C^{k-1}, C^{k-2}, \ldots, C^0$. In other words, $C^k(D) \subseteq C^{k-1}(D) \subseteq C^{k-2} \subseteq \cdots \subseteq C^1(D) \subseteq C^0(D)$.

We prove this by induction, with the base case being $k = 1$, which implies C^0. This is trivial because C^1 implies differentiability which implies continuity over the set.

The inductive step is trivial, suppose we have $f \in C^{k+1}$, we wish to prove that f is also $C^k, C^{k-1}, \ldots, C^0$. Note that we actually only have to prove f is C^k, and the inductive assumption will take care of it from there. For every possible combination of partials,$\partial_{x_{i_1}} \partial_{x_{i_2}} \ldots \partial_{x_{i_{k+1}}} f$, consider the function $g := \partial_{x_{i_1}} \partial_{x_{i_2}} \ldots \partial_{x_{i_k}} f$. Note that the combination of partials is simply $\partial_{x_{i_{k+1}}} g$. Now we seek to prove that g is continuous, which is just the base case. Therefore, if $f \in C^{k+1}$, $f \in C^k, C^{k-1}, \ldots, C^0$.

Theorem

$f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$ and $\vec{p} \in D^\circ$, if $\partial_{x_j}\partial_{x_i}$ and $\partial_{x_i}\partial_{x_j}$ both exist and are continuous in the neighborhood of $\vec{p}$ then $\partial_{x_j}\partial_{x_i} f(\vec{p}) = \partial_{x_i}\partial_{x_j} f(\vec{p})$.

In other words, if $f \in C^2(B_r(\vec{p}))$, $r > 0$, for $\vec{p} \in D^\circ$, then $\partial_{x_j}\partial_{x_i} f(\vec{p}) = \partial_{x_i}\partial_{x_j} f(\vec{p})$

Proof

WLOG, assume $i \leq j$. Let $a = p_i$ and $b = p_j$.

Define $g(x, y) = f(p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_{j-1}, y, p_{j+1}, \ldots, p_d)$.

Conside the function $\Delta(h) = g(a + h, b + h) + g(a, b) - g(a + h, b) - g(a, b + h)$. Let us rearrange some terms, $\Delta(h) = \{g(a + h, b + h) - g(a + h, b)\} - \{g(a, b + h) - g(a, b)\}$. If we define $G(x) = g(x, y + h) - g(x, y)$, we can rewrite $\Delta(h) = G(a + h) - G(a)$. Now apply the MVT, and we get:

$$\Delta(h) = G'(a + \theta_h h) \times h \text{ where } 0 < \theta_h < 1$$

Note that $G'(x) = \partial_x g(x, y + h) - \partial_x g(x, y)$. The derivates must exist because C^2 implies differentiability. Now if we expand $\Delta(h)$, we get:

$$\Delta(h) = [\partial_x g(a + \theta_h h, b + h) - \partial_x g(a + \theta_h h, b)]h$$

Note that here only the second argument of g is changing, so let us apply the MVT again (C^2 implies that the partials are differentiable), and we get:

$$\Delta(h) = \partial_y(\partial_x g)(a + \theta_h h, b + \phi_h h)h^2 \text{ where } 0 < \phi_h < 1$$

Now take the limit of $\frac{\Delta(h)}{h^2}$ as $h \to 0$, we get:

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \partial_y \partial_x g(a, b)$$

But note that if we made G a function of y instead of x ($G(y) := g(x+h, y) - g(x, y)$), we could have written $\Delta(h) = \{g(a, b+h) - g(a, b+h)\} - \{g(a+h, b) - g(a, b)\}$, which is just $G(b+h) - G(b)$. Now we can apply the same procedure except we first differentiate with respect to y and then with respect to x, and we get that the limit of $\frac{\Delta(h)}{h^2}$ as $h \to 0$ is $\partial_x \partial_y g(a, b)$. Therefore, $\partial_y \partial_x g(a, b) = \partial_x \partial_y g(a, b)$. Now we can convert $g(x, y)$ back into $f(\vec{f})$:

$$\partial_y \partial_x g(x, y) = \partial_{x_j} \partial_{x_i} f(p_1, \ldots, x, \ldots, y, \ldots p_d)$$
$$\partial_x \partial_y g(x, y) = \partial_{x_i} \partial_{x_j} f(p_1, \ldots, x, \ldots, y, \ldots p_d)$$

If we plug back in the definition of a and b, we get that $\partial_{x_j} \partial x_i f(\vec{p}) = \partial_{x_i} \partial_{x_j} f(\vec{p})$. ■

Corollary and Generalization

If $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$ is C^k on $B_r(\vec{p})$ for some $k \geq 2$, then if $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ are permutations of each other, then $\partial_{x_{i_1}} \partial_{x_{i_2}} \ldots \partial_{x_{i_k}} f(\vec{p}) = \partial_{x_{j_1}} \partial_{x_{j_2}} \ldots \partial_{x_{j_k}} f(\vec{p})$

Proof: We again prove this via induction, base case is the original theorem. We now seek to prove that if the Clairaut theorem applies for all functions that are $C^0, C^2, \ldots, C^k$, then it is also applicable to functions that are C^{k+1}. We do so by proving that we can switch the location of any two partials and the result would not change. Consider $\partial_{x_{i_1}} \partial_{x_{i_2}} \partial_{x_{i_3}} \ldots \partial_{x_{i_k}} \partial_{x_{i_{k+1}}} f(\vec{p})$ and $\partial_{x_{i_1}} \partial_{x_{i_2}} \partial_{x_{i_3}} \ldots \partial_{x_{i_{k+1}}} \partial_{x_{i_k}} f(\vec{p})$. Note that because $f \in C^2$, $\partial_{x_{i_{k+1}}} \partial_{x_{i_k}} f(\vec{p}) = \partial_{x_{i_k}} \partial_{x_{i_{k+1}}} f(\vec{p})$. Therefore, the two original statements are equal. Now we consider the what if a pair in the middle were switched, like the following:

$$\partial_{x_{i_1}} \partial_{x_{i_2}} \partial_{x_{i_3}} \ldots \partial_{x_{i_m}} \partial_{x_{i_n}} \ldots \partial_{x_{i_k}} \partial_{x_{i_{k+1}}} f(\vec{p})$$
$$\partial_{x_{i_1}} \partial_{x_{i_2}} \partial_{x_{i_3}} \ldots \partial_{x_{i_n}} \partial_{x_{i_m}} \ldots \partial_{x_{i_k}} \partial_{x_{i_{k+1}}} f(\vec{p})$$

Note that since $f \in C^k$, therefore, $\partial_{x_{i_{n+1}}} \ldots \partial_{x_{i_{k+1}}} f(\vec{p})$ is C^{k-n}, which means it is C^2 as well as long as $n \leq k-2$. (The case of $n > k-2$ has already been considered). Therefore, denote $g(\vec{p}) = \partial_{x_{i_{n+1}}} \ldots \partial_{x_{i_{k+1}}} f(\vec{p})$ and now we just apply the base case again to get that if we switch two partials in the middle the end result in fact does not change.

3.3 Differentiability of Vector Valued Functions

3.3.1 Definition of Differentiability

Let $\vec{f} : D \subseteq \mathbb{R}^d \to \mathbb{R}^e$. Let $\vec{p} \in D^\circ$. Then $\vec{f}$ is differentiable at $\vec{p}$ if $\exists\, A \in \mathbb{R}^{e \times d}$ such that

$$\boxed{\forall \varepsilon > 0, \exists\, \delta > 0 : 0 < \| \vec{h} \| < \delta \implies \| \vec{f}(\vec{p} + \vec{h}) - \vec{f}(\vec{p}) - A\vec{h} \| < \varepsilon \| \vec{h} \|}$$

If A exists, then we call it the derivative of $\vec{f}$ at point $\vec{p}$, we write it as $D\vec{f}(\vec{p}) \in \mathbb{R}^{e \times d}$. This is called the **Jacobian Derivative** of the function $\vec{f}$ at the point $\vec{p}$.

3.3.2 Uniqueness of the Jacobian Derivative

Theorem

Either there does not exist a matrix A such that satisfies the condition for differentiation, or there exists only one A. In the latter case, $\vec{f}$ is said to be differentiable at $\vec{p}$ and $A = D\vec{f}\vec{p}$.

Proof

First we seek to prove that the inequality decomposes componentwise. We know that the i^{th} component of a vector difference is the difference of the i^{th} component of each vector. If we express $\vec{\alpha}_i^T$ as the i^{th} row of A, then we get that for the i^{th} component, the definition of a derivative reduces to:

$$|f_i(\vec{p} + \vec{h}) - f_i(\vec{p}) - \vec{\alpha}_i \cdot \vec{h}| < \varepsilon || \vec{h} ||$$

Note that here, $\vec{\alpha}_i = \vec{\nabla} f_i(\vec{p})$. Therefore we know that α_i is uniquely determined because of the uniqueness of the gradient. This then means that A is unique because each row is uniquely determined.

3.3.3 Form of the Jacobian Derivative

We also get an expression for $D\vec{f}$:

$$D\vec{f} = \begin{bmatrix} (\vec{\nabla} f_1)^T \\ (\vec{\nabla} f_2)^T \\ \vdots \\ (\vec{\nabla} f_e)^T \end{bmatrix} = \begin{bmatrix} Df_1 \\ Df_2 \\ \vdots \\ Df_e \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_d} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 & \cdots & \partial_{x_d} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_e & \partial_{x_2} f_e & \cdots & \partial_{x_d} f_e \end{bmatrix} = [\partial_{x_1} \vec{f} \ \partial_{x_2} \vec{f} \ \ldots \ \partial_{x_d} \vec{f}]$$

In other words, $[D\vec{f}]_{ij} = \partial_{x_j} f_i = \frac{\partial f_i}{\partial x_j}$

3.4 Theorems about Differentiability

3.4.1 Chain Rule

Theorem

Take $\vec{f} : D \subseteq \mathbb{R}^d_{\vec{x}} \to \mathbb{R}^e_{\vec{y}}$ and $\vec{g} : E \subseteq \mathbb{R}^e_{\vec{y}} \to \mathbb{R}^k_{\vec{z}}$.[3] And suppose $\vec{f}(D) \cap E \neq \emptyset$. Take a point $\vec{p} \in D^\circ$ and $\vec{q} = \vec{f}(\vec{p}) \in E^\circ$. If $\vec{f}$ is differentiable at $\vec{p}$ and $\vec{g}$ is differentiable at $\vec{q}$, then $\vec{g} \circ \vec{f}$ is ferentiable at $\vec{p}$ and $D(\vec{g} \circ \vec{f})(\vec{p}) = D\vec{g}(\vec{f}(\vec{p}))D\vec{f}(\vec{p})$. In other words, $[D(\vec{g} \circ \vec{f})]_{ij} = \partial_{x_j}(g_i \circ \vec{f})(\vec{p})$. Classically, this has also been written as $\frac{\partial z_i}{\partial x_j}(\vec{p})$. Note that $[D\vec{g}(\vec{q})D\vec{f}(\vec{p})]_{ij} = \sum_{l=1}^{e} \partial_{y_l} g_i(\vec{q}) \partial_{x_j} f_l(\vec{p})$. This has been classically been written as:

$$\frac{\partial z_i}{\partial x_j} = \sum_{l=1}^{e} \frac{\partial z_i}{\partial y_l}(\vec{f}(\vec{p}))\frac{\partial y_l}{\partial x_j}(\vec{p})$$

The "legit" definition of the chain rule is:

$$\partial_{x_j}(g_i \circ \vec{f})(\vec{p}) = \sum_{l=1}^{e} \partial_{y_l} g_i(\vec{q}) \partial_{x_j f_l(\vec{p})}$$

This is also clasically written as:

$$\frac{\partial z_i}{\partial x_j} = \sum_{l=1}^{e} \frac{\partial z_i}{\partial y_l}\frac{\partial y_l}{\partial x_j}$$

Proof

For simplicity's sake, let $B := D\vec{g}(\vec{f}(\vec{p}))$ and $A := D\vec{f}(\vec{p})$. We know that A satisfies $\forall \varepsilon > 0, \exists\, \delta(\varepsilon) > 0, \forall \vec{h}$ with $\| \vec{h} \| < \delta(\varepsilon)$:

$$\| \Delta \vec{f}(\vec{p}, \vec{h}) - A\vec{h} \| < \varepsilon \| \vec{h} \|$$

Let us call this condition $*_{\vec{f}}$

We also know that B satisfies $\forall \varepsilon' > 0, \exists\, \delta'(\varepsilon') > 0, \forall \vec{k}$ with $\| \vec{k} \| < \delta'(\varepsilon')$:

$$\| \Delta \vec{g}(\vec{q}, \vec{k}) - B\vec{k} \| < \varepsilon' \| \vec{k} \|$$

[3] This is a naming convention, the subscript is how we denote the axis in the space. For example, dimensions within $\mathbb{R}^d_{\vec{x}}$ are denoted by $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$

Let us call this condition $*_{\vec{g}}$

We seek to prove the following:

$\forall \varepsilon_2 > 0, \exists\, \delta_2(\varepsilon_2) > 0, \forall \vec{h}$ with $\| \vec{h} \| < \delta_2(\varepsilon_2)$:

$$\| (\vec{g} \circ \vec{f})(\vec{p} + \vec{h}) - (\vec{g} \circ \vec{f})(\vec{p}) - (BA)\vec{h} \| \overset{?}{<} \varepsilon_2 \| \vec{h} \|$$

Let's define a vector $\vec{k} := \Delta\vec{f} = \vec{f}(\vec{p} + \vec{h}) - \vec{f}(\vec{p})$. Note that since $\vec{q} = \vec{f}(\vec{p})$, $\vec{q} + \vec{k} = \vec{f}(\vec{p} + \vec{h})$. We can therefore rewrite the equation as:

$$\| \vec{g}(\vec{q} + \vec{k}) - \vec{g}(\vec{q}) - B\vec{k} + B\vec{k} - (BA)\vec{h} \| \overset{?}{<} \varepsilon_2 \| \vec{h} \|$$

Now we left factor B out of the left hand side:

$$\| \vec{g}(\vec{q} + \vec{k}) - \vec{g}(\vec{q}) - B\vec{k} + B(\vec{k} - A\vec{h}) \| \overset{?}{<} \varepsilon_2 \| \vec{h} \|$$

Now we can apply the triangle inequality and the generalized Cauchy-Schwarz Inequality to get the following:

$$\| \vec{g}(\vec{q} + \vec{k}) - \vec{g}(\vec{q}) - B\vec{k} + B(\vec{k} - A\vec{h}) \| \leq \| \vec{g}(\vec{q} + \vec{k}) - \vec{g}(\vec{q}) - B\vec{k} \| + \|B\| \, \| \vec{k} - A\vec{h} \|$$

But note that the first part of the right hand side is condition $*_{\vec{g}}$, so we get:

$$\| \vec{g}(\vec{q}+\vec{k}) - \vec{g}(\vec{q}) - B\vec{k} \| + \|B\| \, \| \vec{k} - A\vec{h} \| < \varepsilon' \| \vec{k} \| + \|B\| \, \| \vec{k} - A\vec{h} \| = \varepsilon' \| \vec{k} - A\vec{h} + A\vec{h} \| + \|B\| \, \| \vec{k} - A\vec{h} \|$$

Now we can apply the same trick, except WLOG we assume that $\| \vec{k} \| < \delta'(\varepsilon')$ and assume $\varepsilon' \leq 1$. we get:

$$\varepsilon' \| \vec{k} - A\vec{h} + A\vec{h} \| + \|B\| \, \| \vec{k} - A\vec{h} \| \leq \varepsilon' \| \vec{k} - A\vec{h} \| + \varepsilon' \|A\| \, \| \vec{h} \| + \|B\| \, \| \vec{k} - A\vec{h} \|$$

Now we factor and apply the bounds on ε':

$$\varepsilon' \| \vec{k} - A\vec{h} \| + \varepsilon' \|A\| \, \| \vec{h} \| + \|B\| \, \| \vec{k} - A\vec{h} \| \leq (\|B\| + 1) \| \vec{k} - A\vec{h} \| + \|A\| \, \| \vec{h} \| \varepsilon'$$

Note that $\| \vec{k} - A\vec{h} \| = \| \vec{f}(\vec{p} + \vec{h}) - \vec{f}(\vec{p}) - A\vec{h} \| < \varepsilon \| \vec{h} \|$. From this we get:

$$(\|B\| + 1) \| \vec{k} - A\vec{h} \| + \|A\| \, \| \vec{h} \| \varepsilon' \leq (\|B\| + 1)\varepsilon \| \vec{h} \| + \|A\| \, \| \vec{h} \| \varepsilon'$$

If we take $\varepsilon = \varepsilon'$, we get:

$$(\|B\| + 1)\varepsilon \| \vec{h} \| + \|A\| \, \| \vec{h} \| \varepsilon' \leq (\|A\| + \|B\| + 1)\varepsilon \| \vec{h} \|$$

Note that $(\|A\| + \|B\| + 1)\varepsilon$ can be arbitarily small because it is a constant times an arbitarily small value. If we call this value ε_2, we get that

$$\| (\vec{g} \circ \vec{f})(\vec{p} + \vec{h}) - (\vec{g} \circ \vec{f})(\vec{p}) - (BA)\vec{h} \| < \varepsilon_2 \| \vec{h} \|$$

Special Cases

Consider the following "chain": $\mathbb{R}^d_{\vec{x}} \supseteq D \xrightarrow{\vec{f}} \mathbb{R}^e_{\vec{y}} \supseteq E \cap \vec{f}(D) \xrightarrow{\vec{g}} \mathbb{R}^k_{\vec{z}}$. The special case we're interested in is if $d = 1$ and $k = 1$. In this case, the original $\vec{x}$-axis can be thought of as the time axis, and $\vec{f}$ can be thought of as a path $\vec{\gamma} : I \to \mathbb{R}^e$ where $D = I$, an interval, $I \subseteq \mathbb{R}$. The difference between this ans the general chain rule is that the domain is connected (because the domain is time). This way, the "chain" reduces to:

$$\mathbb{R} \supseteq D \xrightarrow{(g \circ \vec{\gamma})} \mathbb{R}$$

Therefore the derivative is an ordinary single-calculus derivative. However, to calculate the derivative with the chain rule, we have to do the following;

$$(g \circ \vec{\gamma})'(t) = Dg(\vec{\gamma}(t))D\vec{\gamma}(t)$$

This can be written as:

$$(g \circ \vec{\gamma})'(t) = \vec{\nabla} g(\vec{\gamma}(t))^T \vec{\gamma}'(t) = \boxed{\vec{\nabla} g(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)}$$

Here, $\vec{\gamma}'(t)$ is known as the **velocity vector** of the curve, which is the column matrix (vector) of the derivatives of the component functions of $\vec{\gamma}$ evaluated at t. It turns out that this value is also equal to:

$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

This is because all the operations used here distributes across components. This is the physics-y interpretation of the derivative. The magnitude of this vector is the instataneous speed of the curve.

$$\| \vec{\gamma}'(t) \| = \left\| \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h} \right\| = \lim_{h \to 0^+} \frac{\| \vec{\gamma}(t+h) - \vec{\gamma}(t) \|}{h}$$

However, this physical interpretation is also the same as the Jacobian Derivative of $\vec{\gamma}$. But since there is only one input variable, there is only one partial to compute, $\partial_t \gamma_1$, otherwise known as γ_1'. Therefore, the Jacobian Derivative becomes a column vector of the derivatives of the component functions.

3.4.2 Implicit Function Theorem

Theorem

Let $F : D \subseteq \mathbb{R}^{d+1} \to \mathbb{R}$ be C^1 in D°. Suppose $F(\vec{A}) = 0$, where $\vec{A} = (\vec{a}, \alpha)$, $\vec{a} \in \mathbb{R}^d, \alpha \in \mathbb{R}$. Also, suppose $\partial_y F(\vec{A}) \neq 0$[4]. Then $\exists$ an open set $U \subseteq \mathbb{R}^d$ and an open interval $I \subseteq \mathbb{R}$ such that $\vec{a} \in U$ and $\alpha \in I$ and $U \times I \subseteq D^\circ$, and $\exists$ a function $f : U \to I$ such that $\{\vec{X} \in U \times I \mid F(\vec{X}) = 0\} = G_f := \{(\vec{x}, f(\vec{x})) \mid \vec{x} \in U\}$. Also, f is also C^1 and

$$\vec{\nabla} f(\vec{x}) = -\frac{1}{\partial_y F(\vec{x}, f(\vec{x}))} \vec{\nabla}_{\vec{x}} F(\vec{x}, f(\vec{x}))$$

where $\vec{\nabla}_{\vec{x}} F = (\partial_{x_1} F, \partial_{x_2} F, \ldots, \partial_{x_d} F)$ is called the **partial gradient** of F.

In other words, locally, in some neighborhood of a solution point $\vec{A}$, $F(\vec{x}, y) \leftrightarrow y = f(\vec{x})$.

Proof Part I – Existence of f

Without loss of generality let us assume that $\partial_y F(\vec{A}) > 0$. Since we know that F is C^1, $\partial_y F(\vec{X})$ is continuous. This means that around the solution point $\vec{A}$, we can find a convex set centered at $\vec{A}$ such that $\partial_y F > 0$. The projection of $\vec{A}$ onto the y-axis is α, and the projection "down" onto the $\vec{x}$-space is $\vec{a} \times 0$.

Now let us consider a function $g_{\vec{a}}(y) = F(\vec{a}, y)$, or the value of F as we move A along the y-axis. By definition $g'_{\vec{a}}(y) = \partial_y F(\vec{a}, y)$. Note that this is positive when $y = \alpha$, therefore the function $g_{\vec{a}}$ is increasing at α. Since $g_{\vec{a}}(\alpha) = 0$, it means that there exists a positive δ such that $g_{\vec{a}}(\alpha + \delta) > 0$ and $g_{\vec{a}}(\alpha - \delta) < 0$.

But note that $g_{\vec{a}}(\alpha + \delta) = F(\vec{a}, \alpha + \delta)$, and since F is C^1, F takes on a positive value near $(\vec{a}, \alpha + \delta)$, specifically, we can draw a square centered at this point with constant y coordinate with edge length r^+. A similar square can be drawn around $(\vec{a}, \alpha - \delta)$ that have negative F values with edge length r^-. Now we can draw an open square U around $\vec{a}$, with edge length $r = \min(r^+, r^-)$, and let the interval $(\alpha - \delta, \alpha + \delta)$ be I.

Now pick any point $\vec{X}$ on U and consider the function $g_{\vec{X}}(y) = F(\vec{X}, y)$. We know that $g'_{\vec{X}}(y) = \partial_y F(\vec{X}, y) > 0$, so we know that $g_{\vec{X}}$ is strictly increasing over the interval I. Consider $g_{\vec{X}}(\alpha - \delta)$, it is negative by how we picked $\vec{X}$ and δ. Now consider $g_{\vec{X}}(\alpha + \delta)$. Now by the Intermediate Value Theorem, $g_{\vec{X}}(c) = 0$ for some unique c between $\alpha - \delta$ and $\alpha + \delta$. Note that the value of c is uniquely determined by the location of X, therefore we can write it as a function $f(\vec{X})$. This uniquely defines f because there is exactly one point of the form $(\vec{x}, y)$ such that $g_{\vec{x}}(y) = 0$, $y \in I$.

Note that we now have the following identity:

[4]Here, we write the function F as $F(\vec{x}, y) = F(x_1, x_2, \ldots, x_d, y) = F(\vec{X})$, so we are differentiating with respect to the last variable.

$$\boxed{F(\vec{x}, f(\vec{x})) \equiv 0 \text{ for all } \vec{x} \in U}$$

Proof Part II – Continuity of f

Now pick an arbitrary vector h such that $\vec{x}+\vec{h} \in U$, by the above identity, we get that $F(\vec{x}+\vec{h}, f(\vec{x}+\vec{h})) = 0$. Now we can subtract the two:

$$F(\vec{x}+\vec{h}, f(\vec{x}+\vec{h})) - F(\vec{x}, f(\vec{x})) = 0$$

Now we can apply the Mean Value Theorem, we can rewrite the left hand side as: $\vec{\nabla} F(\vec{X}^*) \cdot \vec{H}$ where $\vec{X}^* \in [(\vec{x}, f(\vec{x})), (\vec{x}+\vec{h}, f(\vec{x}+\vec{h}))]$ and $\vec{H} = (\vec{h}, \Delta f) = (\vec{h}, f(\vec{x}+\vec{h}) - f(\vec{x}))$. We can rewrite this as

$$\vec{\nabla}_{\vec{x}} F(\vec{X}^*) \cdot \vec{h} + \partial_y F(\vec{X}^*) \Delta f = 0$$

$$-\partial_y F(\vec{X}^*) \Delta f = \vec{\nabla}_{\vec{x}} F(\vec{X}^*) \cdot \vec{h}$$

Now we take absolute value of both sides, and apply the Cauchy-Schwarz Inequality to the right hand side

$$\begin{aligned} |-\partial_y F(\vec{X}^*) \Delta f| &= |\vec{\nabla}_{\vec{x}} F(\vec{X}^*) \cdot \vec{h}| \\ &\le \|\vec{\nabla}_{\vec{x}} F(\vec{X}^*)\| \, \|\vec{h}\| \\ &\le \|\vec{\nabla}_{\vec{x}} F(\vec{X}^*)\| \, \|\vec{h}\| \\ &\le M \|\vec{h}\| \end{aligned}$$

where $M := \max \|\vec{\nabla} F\|$ over the closure of $U \times I$. Now we can break the absolute value on the left hand side, and we get

$$\partial_y F(\vec{X}^*) \, |\Delta f| \le M \, \|\vec{h}\|$$

we can drop the absolute value because we picked the convex set such that $\partial_y F > 0$. Now we can get a bound on Δf

$$|\Delta f| \le \frac{M \, \|\vec{h}\|}{\partial_y F(\vec{X}^*)}$$

Now if we shrink the initial convex set such that $\partial_y F > \frac{1}{2} \partial_y F(\vec{A})$ (we can do this due to F's continuity), then we can say

$$0 \leq |\Delta f| \leq \frac{M \parallel \vec{h} \parallel}{\partial_y F(\vec{X}^*)} \leq \frac{2M \parallel \vec{h} \parallel}{\partial_y F(\vec{A})}$$

Now we take the limit as $\vec{h} \to \vec{0}$, note that then the rightmost upper bound tends to 0 as $\vec{h}$ is the only variable present. Therefore $|\Delta f|$ is squeezed between 0 and a value that approaches 0, therefore it is 0. Therefore f is continuous.

Proof Part III – C^1 Nature and Form of f

Let us go back to

$$\vec{\nabla}_{\vec{x}} F(\vec{X}^*) \cdot \vec{h} + \partial_y F(\vec{X}^*)\Delta f$$

Now let us take $\vec{h}$ to be of a specific form, let $\vec{h} = h\vec{e}_j$ where $\vec{e}_j$ is the j^{th} elementary component vector. In this form, we can rewrite the above expression as

$$h\{-\partial_{\vec{x}} F(\vec{X}^*) \cdot \vec{e}_j\} = \partial_y F(\vec{X}^*)\Delta f$$

$$-\partial_{\vec{x}} F(\vec{X}^*) \cdot \vec{e}_j = \partial_y F(\vec{X}^*)\left(\frac{\Delta f}{h}\right)$$

Now let us isolate $\left(\frac{\Delta f}{h}\right)$

$$\frac{\Delta f}{h} = -\frac{\vec{\nabla}_{\vec{x}} F(\vec{X}^*) \cdot \vec{e}_j}{\partial_y F(\vec{X}^*)}$$

Now we can use the C^1 property of F and we get

$$\frac{\Delta f}{h} = -\frac{\vec{\nabla}_{\vec{x}} F(\vec{x}, f(\vec{x})) \cdot \vec{e}_j}{\partial_y F(\vec{x}, f(\vec{x}))}$$

But since when you dot any vector with $\vec{e}_j$, you simply pick out its j^{th} component, so we get:

$$\frac{\Delta f}{h} = -\frac{\partial_{x_j} F(\vec{x}, f(\vec{x}))}{\partial_y F(\vec{x}, f(\vec{x}))}$$

Now if we take the limit as $h \to 0$, we get the j^{th} partial derivative of f, which has the form of

$$\partial_{x_j} f(\vec{x}) = -\frac{\partial_{x_j} F(\vec{x}, f(\vec{x}))}{\partial_y F(\vec{x}, f(\vec{x}))}$$

All the partials exist, but to prove that f is differentiable we also have to prove they are all continuous. This follows directly from the C^1 nature of F and the fact that compositions of continuous functions is continuous. Therefore, f is also differentiable and we can write a form for the gradient of f:

$$\boxed{\vec{\nabla} f = -\frac{1}{\partial_y F \circ (Id \times f)} \vec{\nabla}_{\vec{x}} F \circ (Id \times f)}$$

(where Id is the identity function, so $(Id \times f)(\vec{x}) = (\vec{x}, f(\vec{x}))$)

Generalization

$F(\vec{x}, y) = 0$ is equivalent to $y = f(\vec{x})$ locally in a neighborhood of some solution point, $(\vec{a}, \alpha)$ if F is C^1, $F(\vec{a}, \alpha) = 0$, $(\vec{a}, \alpha) \in D_F^\circ \subseteq \mathbb{R}^{d+1}$ and $\partial_y F(\vec{a}, \alpha)$

However, if $\vec{F}$ is a vector-valued function, we also have this type of equivalency ($\vec{F}(\vec{x}, \vec{y}) = \vec{0}$ is equivalent to $\vec{y} = \vec{f}(\vec{x})$ for $\vec{x} \in U \subseteq \mathbb{R}^d$ and $\vec{y} \in V \subseteq \mathbb{R}^e$) locally around some solution point $\vec{F}(\vec{a}, \vec{b}) = \vec{0} \in \mathbb{R}^e$ if $\vec{F}$ is C^1 on $D_{\vec{F}}^\circ$[5] and $(\vec{a}, \vec{b}) \in D_{\vec{F}}^\circ \subseteq \mathbb{R}^{d+e}$ and $\partial_{\vec{y}} \vec{F}(\vec{a}, \vec{b}) \neq 0$ where $\partial_{\vec{y}} \vec{F} = \det D_{\vec{y}} \vec{F}$ where $D_{\vec{y}}$ (known as the **partial Jacobian**) can be described as:

$$D_{\vec{y}} F = \left[\frac{\partial F_i}{\partial y_j} \right]_{\substack{1 \le i \le e \\ 1 \le j \le e}}$$

Similarly, we know that if all the conditions are met, we know that $\vec{f}$ is C^1. We also have a semi-explicit formula for the Jacobian of f:

$$\boxed{D\vec{f} = -(D_{\vec{y}} \vec{F})^{-1} D_{\vec{x}} \vec{F}}$$

By definition we know $\vec{F}(\vec{x}, \vec{f}(\vec{x})) \equiv \vec{0}$ for all $\vec{x} \in U$. Now we can differentiate:

$$D_{\vec{x}} \vec{F}(\vec{x}, \vec{f}(\vec{x})) \equiv D_{\vec{x}}\, \vec{0} = O \in \mathbb{R}^{e \times d}$$

But we can rewrite the left hand side as $D_{\vec{x}}\, [\vec{F} \circ (Id \times \vec{f})]$, here we can apply the chain rule:

$$(D_{\vec{x}} \vec{F})(\vec{x}, \vec{f}(\vec{x}))\, (D_{\vec{x}}(Id \times \vec{f}))(\vec{x}) = O$$

[5] A vector valued function is C^1 if all of its component functions are C^1

We can express $(Id \times \vec{f})(\vec{x})$ as $(\vec{x}, \vec{f}(\vec{x}))$, which can be written as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_d \\ f_1(\vec{x}) \\ \vdots \\ f_e(\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{x} \\ \vec{f}(\vec{x}) \end{bmatrix}$$

Since the Jacobian Operator works row by row, we can distribute it

$$D_{\vec{x}} \begin{bmatrix} \vec{x} \\ \vec{f}(\vec{x}) \end{bmatrix} = \begin{bmatrix} D_{\vec{x}}\vec{x} \\ D_{\vec{x}}\vec{f}(\vec{x}) \end{bmatrix} = \begin{bmatrix} I_d \\ D\vec{f}(\vec{x}) \end{bmatrix}$$

Example

$$(*) \begin{cases} F(u, v, x, y, z) = 0 \\ G(u, v, x, y, z) = 0 \\ H(u, v, x, y, z) = 0 \end{cases}$$

Suppose (a, b, p, q, r) is a specific solution for (u, v, x, y, z) in $(*)$ where $(a, b, p, q, r) \in D^{\circ}$

We can rewrite this as:

$$\begin{aligned} F &: px + qy + rz = -(au + bv) \\ G &: p'x + q'y + r'z = -(a'u + b'v) \\ H &: p''x + q''y + r''z = -(a''u + b''v) \end{aligned}$$

Here we can rewrite this as the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} p & q & r \\ p' & q' & r' \\ p'' & q'' & r'' \end{bmatrix} \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \vec{b} = \begin{bmatrix} -(au + bv) \\ -(a'u + b'v) \\ -(a''u + b''v) \end{bmatrix}$$

To solve this matrix equation we need that $\det A \neq 0$. We can also write A as:

$$\begin{bmatrix} F \\ G \\ H \end{bmatrix}$$

We know that

$$\partial_{(x,y,z)}(F,G,H) = \begin{vmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x G & \partial_y G & \partial_z G \\ \partial_x H & \partial_y H & \partial_z H \end{vmatrix}$$

The Implicit Function Theorem says that if $\partial_{(x,y,z)}(F,G,H)|_{(a,b,p,q,r)} \neq 0$ then $(*)$ is locally equivalent near (a,b,p,q,r) to:

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

And we call the vector function (f,g,h) $\vec{f}$, we also know that f,g,h are C^1 on their domain. We also have the semi-explicit representation of $D\vec{f}$:

$$D\begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \\ \partial_u h & \partial_v h \end{bmatrix} = -\{D_{(x,y,z)}(F,G,H)\}^{-1} D_{(u,v)}(F,G,H)$$

FIXME - MOVE

To prove this we use an inductive argument, the base case is 1 equation one variable $(F(\vec{x},y) = 0)$, which is done. Suppose that we can solve 2 equations in 2 unknowns locally and with C^1 solutions. We seek to prove that we can solve 3 equations in 3 unknows.

To be definite, suppose $\partial_z H \neq 0$ in a neighborhood of (a,b,p,q,r)

We have the following:

$$H(u,v,x,y,z) = 0 \longleftrightarrow z = \phi(u,v,x,y)$$

where ϕ is some C^1 explicit function in the neighborhood of (a,b,p,q,r).

Let us define $\tilde{F}(u,v,x,y) := F(u,v,x,y,\phi(u,v,x,y))$ and $\tilde{G}(u,v,x,y) := G(u,v,x,y,\phi(u,v,x,y))$. We know that both of these equal to 0, as well as the fact that $\tilde{F}$ and $\tilde{G}$ are C^1.

Then we only need to show the following:

$$\partial_{(x,y)}(\tilde{F},\tilde{G}) = \begin{vmatrix} \partial_x \tilde{F} & \partial_y \tilde{F} \\ \partial_x \tilde{G} & \partial_y \tilde{G} \end{vmatrix} \neq O$$

To do this we use the chain rule:

$$\begin{aligned} \partial_x \tilde{F} &= \partial_x F + (\partial_z F)(\partial_x \phi) = F_x + F_z \phi_x \\ \partial_y \tilde{F} &= \partial_y F + (\partial_z F)(\partial_y \phi) = G_x + G_z \phi_x \end{aligned}$$

The same holds true for $\partial_x\tilde{G}$ and $\partial_y\tilde{G}$. Note also that $\partial_x H = H_x + H_z\phi_x = 0$ and $\partial_y H = 0$ as well. Therefore:

$$\partial_{(x,y)}(\tilde{F},\tilde{G}) = \begin{vmatrix} F_x + F_z\phi_x & F_y + F_z\phi_y \\ G_x + G_z\phi_x & G_y + G_z\phi_y \end{vmatrix} = \begin{vmatrix} F_x + F_z\phi_x & F_y + F_z\phi_y & F_z/H_z \\ G_x + G_z\phi_x & G_y + G_z\phi_y & G_z/H_z \\ H_x + H_z\phi_x & H_y + H_z\phi_y & 1 \end{vmatrix}$$

We are given that

$$\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix} = H_z \begin{vmatrix} F_x & F_y & F_z/H_z \\ G_x & G_y & G_z/H_z \\ H_x & H_y & 1 \end{vmatrix} \neq 0$$

However, note that we can get from here to the original 3×3 determinant by column operations, therefore we know that $\partial_{(x,y)}(\tilde{F},\tilde{G}) \neq 0$. Therefore we have the degeneracy condition.

Then we can write

$$\begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$$

by the assumption. Now we plug back into the formula for z and we get:

$$z = h(u,v) = \phi(u,v,f(u,v),g(u,v))$$

Now we have successfully written all three components as C^1 functions of u and v.

FORM

$$D\vec{f} = -\{D_{\vec{x}}\vec{F}\}^{-1}(D_{\vec{u}}\vec{F})$$

Let $\vec{G}(\vec{u}) = \vec{F}(\vec{u},\vec{f}(\vec{u})) \equiv \vec{0}$

Now we can Jacobian both sides

$$D\vec{G}(\vec{u}) \equiv D\vec{0} = O$$

Note that $\vec{G} = \vec{F} \circ (Id \times \vec{f})$. Now we can apply the chain rule:

$$\begin{aligned} D\vec{G} &= D(\vec{F} \circ (Id \times \vec{f})) \\ &= \{(D\vec{F}) \circ (Id \times \vec{f})\}D(Id \times \vec{f}) \end{aligned}$$

Now we try to write these out in a type of "block decomposition." Let us start with $D(Id \times \vec{f})$

$$D(Id \times \vec{f})(\vec{u}) = \begin{bmatrix} D_{\vec{u}}\vec{u} \\ (D\vec{f})(\vec{u}) \end{bmatrix} = \begin{bmatrix} D(Id) \\ D\vec{f} \end{bmatrix}(\vec{u}) = \begin{bmatrix} I \\ D\vec{f} \end{bmatrix}(\vec{u}) = \begin{bmatrix} I \\ D\vec{f}(\vec{u}) \end{bmatrix}$$

We also have a block decomposition of $D\vec{F}$. We know:

$$\vec{F}(\vec{u}, \vec{x}) = \begin{bmatrix} F_1(\vec{u}, \vec{x}) \\ F_2(\vec{u}, \vec{x}) \\ \vdots \\ F_e(\vec{u}, \vec{x}) \end{bmatrix}$$

Therefore we know:

$$\begin{aligned} D\vec{F} &= \left[\begin{array}{ccc|ccc} \partial_{u_1} F_1 & \dots & \partial_{u_d} F_1 & \partial_{x_1} F_1 & \dots & \partial_{x_e} F_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \partial_{u_1} F_2 & \dots & \partial_{u_d} F_e & \partial_{x_1} F_e & \dots & \partial_{x_e} F_e \end{array}\right] \\ &= [D_{\vec{u}}\vec{F} \mid D_{\vec{x}}\vec{F}] \end{aligned}$$

Now we can bring everything back together and we have

$$\begin{aligned} D\vec{G}(\vec{u}) &= [D_{\vec{u}}\vec{F} \circ (Id \times \vec{f}) \mid D_{\vec{x}}\vec{F} \circ (Id \times \vec{f})] \begin{bmatrix} I \\ D\vec{f}(\vec{u}) \end{bmatrix} \\ &= D_{\vec{u}}\vec{F} \circ (Id \times \vec{f}) + (D_{\vec{x}}\vec{F} \circ (Id \times \vec{f}))(D\vec{f}) \equiv O \end{aligned}$$

Now if we isolate $D\vec{f}$, we get

$$\boxed{D\vec{f} = -(D\vec{F})^{-1}(D_{\vec{u}}F)}$$

We know that the inverse matrix exists because we assumed that

$$\det D_{\vec{x}}F \neq 0$$

3.4.3 Inverse Function Theorem

Theorem

Suppose we have two sets of variables $\vec{x}$ and $\vec{u}$ where

$$\begin{cases} x_1 &= f_1(u_1, u_2, \ldots, u_d) \\ x_2 &= f_2(u_1, u_2, \ldots, u_d) \\ \vdots \\ x_d &= f_d(u_1, u_2, \ldots, u_d) \end{cases}$$

or that $\vec{x} = \vec{f}(\vec{u})$. Assume that we have a particular solution $\vec{b} = \vec{f}(\vec{a})$ where $\vec{f}$ is C^1 and $\vec{a} \in (\text{dom } \vec{f})^\circ$ and $\partial_{\vec{x}}\vec{f}(\vec{a}) \neq 0$. Then there exists an open set U around $\vec{a} \in \mathbb{R}^d_{\vec{u}}$ and there exists an open set V around $\vec{b} \in \mathbb{R}^d_{\vec{x}}$. There also exists a C^1 function $\vec{g} : V \to U$ such that $\vec{f} \circ \vec{g} = Id_V$ and $\vec{g} \circ \vec{f}|_{\vec{g}(V)} = Id_{\vec{g}(V)}$. $\vec{g}$ is known as the **local inverse** of $\vec{f}$ near $\vec{b}$.

Random Fact

If $\vec{f}$ is linear, it can be written as $\vec{x} = \vec{f}(\vec{u}) = A\vec{u}$, and its inverse function would be $\vec{u} = A^{-1}\vec{x}$ (if we assume that $\det A \neq 0$). However, note that $\vec{f}(\vec{u}) = A\vec{u}$ means that $D\vec{f}(\vec{u}) \equiv A$.

If we look at the component functions of $\vec{f}$, we can see a list of partials

$$\begin{aligned} f_1(\vec{u}) = A\vec{u} \cdot \vec{e}_1 &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1d}u_d \\ \partial_{u_1} f_1(\vec{u}) &= a_{11} \\ \partial_{u_2} f_1(\vec{u}) &= a_{12} \end{aligned}$$

$$\boxed{\partial_{u_j} f_i(\vec{u}) = a_{ij}}$$

Therefore, $\det A = \partial\vec{f}(\vec{u}) \neq 0$ at $\vec{a}$.

Proof

Let us create a function $\vec{F}(\vec{u}, \vec{x}) := \vec{x} - \vec{f}(\vec{u})$. By the Implicit Function Theorem, we know that this is equivalent to $\vec{x} = \vec{f}(\vec{u})$. However, there is nothing in the Implicit Function Theorem that says we cannot solve for $\vec{u}$. A couple of conditions must be met beforehand, however, including that $\vec{F}$ has to be C^1, that $(\vec{a}, \vec{b}) \in (\text{dom } \vec{F})^\circ$, and that $\partial_{\vec{u}}\vec{F}(\vec{a}, \vec{b}) \neq 0$.

Let us start with the last condition first. $D_{\vec{u}}\vec{F} = D_{\vec{u}}Id(\vec{x}) - (D_{\vec{u}}\vec{f})(\vec{u}) = -D\vec{f}(\vec{u})$

$$\begin{aligned}\partial_{\vec{u}}\vec{F} = \det D_{\vec{u}}\vec{F} &= \det(-D\vec{f}(u)) \\ &= (-1)^d \det D\vec{f}(\vec{u}) \\ &= (-1)^d \partial\vec{f}(\vec{u}) \neq 0 \text{ at } (\vec{a}, vecb)\end{aligned}$$

We also know that $\vec{F}$ is C^1 by assumption. Now all we have to check is that $(\vec{a}, \vec{b}) \in (\operatorname{dom} \vec{F})^\circ$. $\operatorname{dom} \vec{F} = (\operatorname{dom} \vec{f} \times \mathbb{R}^d_{\vec{x}}) \subseteq \mathbb{R}^{2d}_{(\vec{u},\vec{x})}$. So $\vec{b}$ is not a problem. We also know that $\vec{a} \in \operatorname{dom} \vec{f}$ by assumption.

Therefore we conclude, using the Implicit Function Theorem that $\exists$ open sets U, V around $\vec{a}, \vec{b}$ respectively, and $\exists\, \vec{g} : V$ such that $\vec{F}(\vec{g}(\vec{x}), \vec{x}) \equiv \vec{0}$ for all $\vec{x} \in V$. This implies that $\vec{x} - \vec{f}(\vec{g}(\vec{x})) = \vec{0}$. Therefore $\boxed{\vec{f}(\vec{g}(\vec{x})) \equiv \vec{x} \text{ for all } \vec{x} \in V}$.

To prove the converse, take any point $\vec{u} \in \vec{g}(\vec{V})$, plug in $\vec{f}(\vec{u})$ for $\vec{x}$, then $\vec{f}(\vec{u}) = \vec{f}(\vec{g}(\vec{x}_0)) = \vec{x}_0 \in V$. Now we plug the thing back again and we get $\vec{f}(\vec{g}(\vec{f}(\vec{u}))) = \vec{f}(\vec{u})$. Now since $\vec{f}$ is one-to-one locally, we can cancel out the outmost function, so we get $\vec{g}(\vec{f}(\vec{u})) = \vec{u}$. Therefore, we can say $\vec{f}$ and $\vec{g}$ are local inverses of each other.

To prove that $\vec{f}$ is one-to-one, suppose $\vec{f}(\vec{u}_1) = \vec{f}(\vec{u}_2)$ but $\vec{u}_1 \neq \vec{u}_2$, and $\vec{u}_1, \vec{u}_2 \in \vec{g}(V)$. Then we know, by the continuity of $\vec{g}$ that there exists $\vec{x}_1, \vec{x}_2 \in V$ such that $\vec{u}_1 = \vec{g}(\vec{x}_1)$ and $\vec{u}_2 = \vec{g}(\vec{x}_2)$. However, then we know that $\vec{f}(\vec{g}(\vec{x}_1)) = \vec{f}(\vec{g}(\vec{x}_2))$, which implies that $x_1 = x_2$, which implies that $\vec{g}(\vec{x}_1) = \vec{g}(\vec{x}_2)$, which means $\vec{u}_1 = \vec{u}_2$. Therefore $\vec{f}$ is one-to-one.

Implications

Let $\vec{f} : D \subseteq \mathbb{R}^d_{\vec{u}} \to \mathbb{R}^d_{\vec{x}}$ be a C^1 transformation such that $\partial\vec{f}(\vec{u}) \neq 0$ for all $\vec{u} \in D^\circ$. Let K be a compact subset of D°, Then $\vec{f}(K^\circ) = [\vec{f}(K)]^\circ$. In other words, the interior of the pre-image maps onto the interior of the image.

Moreover, if $\vec{f}$ is also one-to-one on some open superset U of K, then $\vec{f}(\partial K) = \partial\{\vec{f}(K)\}$.

3.5 Lagrange Multiplier

3.5.1 Level-Sets as Manifolds

Definitions

Level Sets

Let $g : D \subseteq \mathbb{R}^d \to \mathbb{R}$ be C^1 on D°. The Level-set at level $c \in g(D)^\circ$ of the function g, written as $L_c g := \{\vec{x} \in D \mid g(\vec{x}) = c\}$. Note that $L_c g \neq \emptyset$ and $L_c g \neq \{\vec{x}_0\}$.

K-Patch

A K-patch in d-space is a set $P \subseteq \mathbb{R}^d$ which is the C^1 and one-to-one image of some open connected set in $\mathbb{R}^k (k \leq d)$ by a mapping of "full rank", i.e. $P = \vec{\phi}(U)$ where $U \subseteq \mathbb{R}^k$ is open and connected, $\vec{\phi}$ is C^1 and one-to-one and $\{\partial_{u_1}\vec{\phi}, \partial_{u_2}\vec{\phi}, \ldots, \partial_{u_k}\vec{\phi}\}$ are linearly independent for all $\vec{u} \in U$.

In layman terms, the geometric transformation is one that bends the surface in $\mathbb{R}^k$ "smoothly," it is a "bending" of the original set without any sharp edges or elimination of dimensions. In other way, the dimensionality is reserved. Which is why $\vec{\phi}$ must be C^1 and one-to-one.

The function $\vec{\phi}$ is known as a parameterization of the set P because as the argument transverses U, $\vec{\phi}$ traces out P. Note that our "full rank" condition, which is that $\{\partial_{u_1}\vec{\phi}, \partial_{u_2}\vec{\phi}, \ldots, \partial_{u_k}\vec{\phi}\}$ are linearly independent. This is because in the domain, pick an arbitrary point $\vec{p}$ and move in the u_1 direction, since $\vec{\phi}$ is C^1 and one-to-one, this movement in the domain will trace out $\vec{\phi}(\vec{p})$ in the co-domain. The tangent vector to $\vec{\phi}(\vec{p})$ is $\partial_{u_1}\vec{\phi}(\vec{p})$. If we move along u_2 direction in the domain of $\vec{p}$, the co-domain tangent velocity would be $\partial_{u_2}\vec{\phi}(\vec{p})$, which should be linearly independent from $\partial_{u_2}\vec{\phi}(\vec{p})$. In other words, independent movements from the domain maps onto independent movements in the co-domain. This prevents foldings and crimping.

It so happens that if P has a parameterization, it has an infinitely many of them.

K-Manifold

A K-manifold in $\mathbb{R}^d$ is a set $M \subseteq \mathbb{R}^d$ that can be completely covered by countably many (possibly overlapping) K-patches.

Level-Set Theorem

Let $g : D \subseteq \mathbb{R}^d \to \mathbb{R}$ be C^1, and choose $c \in g(D)^\circ$. Then $L_c g = \{\vec{x} \in D \mid g(\vec{x}) = c\}$ is a $(d-1)$-manifold in $\mathbb{R}^d$ provided that $\vec{\nabla} g \neq \vec{0}$ for any point $\vec{x} \in L_c g$.

Proof:

We know that $g(x_1, x_2, \ldots, x_d) = c$, now let's pick $\vec{p} \in L_c g$, we know that $g(p_1, p_2, \ldots, p_d) = c$ and that $\partial_{x_1} g(\vec{p}), \partial_{x_2} g(\vec{p}), , \partial_{x_d} g(\vec{p})$ cannot all be 0 (as $\vec{\nabla} g(\vec{p}) \neq 0$), therefore $\exists\ j \in \{1, 2, \ldots, d\}$: $\partial_{x_j} g(\vec{p}) \neq 0$. To be definite, let $j = d$ (wlog). By the Implicit Function Theorem, we get the equivalency between $g(\vec{x}) = c$ and $x_d = \phi(x_1, x_2, \ldots, x_{d-1})$ in a neighborhood $U \times I \subseteq \mathbb{R}^{d-1} \times \mathbb{R}$ of $\vec{p}$. In this neighborhood, we can define a parameterization function $\vec{\phi}(\vec{u}) : U \subseteq \mathbb{R}^{d-1} \to \mathbb{R}^d$ as:

$$\vec{\phi}(u_1, u_2, \ldots, u_{d-1}) = (u_1, u_2, \ldots, u_{d-1}, \phi(u_1, u_2, \ldots, u_{d-1}))$$

Therefore the range, $\vec{\phi}(U) = L_c g \cap N_{\vec{p}}$ where $N_{\vec{p}}$ is some open neighborhood of $\vec{p}$ in $\mathbb{R}^d$.

We now seek to show that $\vec{\phi}(U)$ is a $(d-1)$-patch in $\mathbb{R}^d$. We have to prove:

1. $\vec{\phi}$ is one-to-one
2. $\vec{\phi}$ is C^1 on U
3. $\{\partial_{u_1}\vec{\phi}, \ldots, \partial_{u_{d-1}}\vec{\phi}\}$ are linearly independent

Condition 2 is trivial, because $\vec{\phi} = Id \times \phi$, and since both the identity function and ϕ are C^1, $\vec{\phi}$ must be C^1.

Condition 1 is also not hard to see. suppose $\vec{\phi}(\vec{u}_1) = \vec{\phi}(\vec{u}_2)$, we then know that the first $d-1$ coordinates for $\vec{u}_1$ and $\vec{u}_2$ are the same, but $\vec{u}_1$ and $\vec{u}_2$ are both of dimension $d-1$. Therefore $\vec{u}_1 = \vec{u}_2$.

Condition 3, or the "full rank" condition is a bit harder to prove, first we need the Jacobian Matrix:

$$D\vec{\phi}(\vec{u}) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \frac{\partial \phi}{\partial u_1} & \frac{\partial \phi}{\partial u_2} & \frac{\partial \phi}{\partial u_3} & \dots & \frac{\partial \phi}{\partial u_{d-1}} \end{bmatrix}$$

From here is is kind of obvious that the columns are linearly independent. Therefore the "full rank" condition is fulfilled.

Therefore this is the correct parameterization. Therefore the level set is indeed a manifold.

Applications

Find a local extremum of $f(x_1, \dots, x_d)$ subject to the constraint $g(x_1, \dots, x_d) = c$ where f, g are C^1.

This has tremendous application in engineering, economics, etc.