# Multivariable Calculus

Avery Karlin

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Teacher: Stern

# 1 Multivariable Calculus Basics

Multivariable calculus is the study of the analogs, if any, of the fudemental theorem of calculus to higher dimensions, and any restrictions that may exist on higher dimensions

Let R be a simple closed curve, where simple means that each point is crossed by the curve at most once and closed means that the curve does not have a unqiue starting point and end point.

When integrating in 2 dimensions, we also need to pick an "interval." In this case, $R$, the bounded region, would be an interval (analogous to $(a, b)$ in single variable calculus) and $C$ would be the boundary (analogous to $a, b$).

If we were to integrate a function $f(x, y)$ over $R$, it is denoted by:

$$\iint_R f(x,y)dA_{xy}$$

The $dA_{xy}$ is the area element, an infinitesimally small piece of area, analogous to $dx$, the length element in single variable calculus.

Note that in single variable calculus, there is an implied orientation, going left to right is the positive "direction," In multivariable calculus, it is accepted that the positive direction for the curve to go in is the counterclockwise direction, such that the opposite direction adds a negative.

## 1.1 Green's Theorem

This is one of the FTC's generalization to higher dimensions. The Green's Theorem works with functions that take in 2 variables.

Suppose there exists $f(x, y)$ and $g(x, y)$, and a region $R$ bounded by a positively oriented, simple closed curve $C$. Then Green's theorem states that:

$$\iint_R (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dA_{xy} = \int_C f dx + g dy = \int_C (f x'(t) + g y'(t))dt$$

## 1.2 Generalized FTC

The goal of this course in multivariable calculus is to reach the following conclusion:

For some function $\omega$ evaluated over the region $M$, where $\partial M$ is the boundry of $M$:

$$\int_M d\omega = \int_{\partial M} \omega$$

# 2 Real Number Set

## 2.1 Definitions of Number Structures

### 2.1.1 $\mathbb{N}$

We can define the natural number system by sets, like the following:

$$0 = \emptyset$$

And from there we introduce a succession operation:

$$n + 1 = n \cup \{n\}$$

So for example, $1 = \{0\} = \{\emptyset\}$, $1 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$, etc.
Set theory is the deepest concept in mathematics, such that it must be assumed as postulates.

### 2.1.2 $\mathbb{Z}$

Positive integers are defined as an ordered pair of natural numbers, for example, $2 = (2, 0)$, and $-2 = (0, 2)$

### 2.1.3 $\mathbb{Q}$

Rationals are defined as an infinite set of ordered pairs of integers. A rational number $q = \frac{n}{m}$, then $q = \{(n, m), (2n, 2m), (3n, 3m) \dots (-2n, -2m), (-3, -3m) \dots\}$

### 2.1.4 $\mathbb{R}$

There are several definitions of the real number system

- Each real number can be thouhgt of as an infinite sequence in the following format:

  $$(s, N, d_1, d_2, d_3 \dots)$$

  Where $s = \pm 1$, $N \in \mathbb{N}$, and $d_n \in 0, 1, 2, 3, \dots 8, 9$. It is also not the case that $d_n = d_{n+1} = \dots = 0$. If there is a terminal decimal, we express it as 9 repeated.

- Each real number forms a subdivision of $\mathbb{Q}$ into two disjoint sets that cover the entirety of $\mathbb{Q}$, one of which lies entirely to the left of the other.

## 2.2 Basic Structure of $\mathbb{R}^1$

$\mathbb{R}$ is an instance of many kinds of mathematical structures, such as:

- Field
  - A set closed under addition and multiplication (results are within set)
  - Obeys all associated laws of addition and multiplication
- Ordered Set
  - The set has an ordering that reflect the operations in the field (such that x, y ¿ 0, then x + y and xy ¿ 0)
  - This structure is what allows for comparisons, like the $<$ function
- Metric Space
  - There exists a standard distance operation between numbers
  - In $\mathbb{R}$, $dist(x, y) := |x - y|$ (:= means "is defined as")
- Vector Space
  - Elements can be thought of as vectors from the origin
- Geometric Space
  - This structure means that you can measure angle in a meaningful way
  - In $\mathbb{R}^1$, the angle measure can be either 0 or $\pi$
  - In higher dimensions there are more angles possible

## 2.3 Properties of $\mathbb{R}^n$

In higher dimensions, several of the properties of $\mathbb{R}$ no longer hold. $\mathbb{R}^n, n > 1$ is not a field, and not an ordered set, but it is a vector, metric, and geometric space.

## 2.4 Basic Axioms for $\mathbb{R}^1$

$\mathbb{R}$ is a field under $+$ and $\cdot$ $(x, y \in \mathbb{R})$

1. Additive closure: $x + y \in \mathbb{R}$
2. Associative Property of Addition: $x + (y + z) = (x + y) + z$
3. Communicative Property of Addition: $x + y = y + x$
4. 0 is the identity element of addition: $x + 0 = x$
5. Every element has an additive inverse: $x + (-x) = 0$
6. Multiplicative closure: $xy \in \mathbb{R}$
7. Associative Property of Multiplication: $x(yz) = (xy)z$
8. Communicative Property of Multiplication: $xy = yx$
9. 1 is the identity element of multiplication: $x(1) = x$
10. Every element (except 0) has a multiplicative inverse: $x \cdot \frac{1}{x} = 1$
    - Theorem: $x(0) = 0$
    - Proof: $x(0) = x(0 + 0)$, then we apply the distributive law, and get $x(0) = x(0) + x(0)$. Now we add $-x(0)$ to both side, and we get: $0 = x(0)$
11. Distributive Law: $x(y + z) = xy + xz$

$\mathbb{R}$ is an ordered field and has a proper subset (aka not the entire set) $\mathbb{R}^+$ (the positives) such that:

1. $\mathbb{R}^+$ is closed under $+$ and $\cdot$
2. $1 \in \mathbb{R}^+, 0 \notin \mathbb{R}^+$
3. **Trichotomy Property**: for any $x \in \mathbb{R}$, $x$ is either 0, $\in \mathbb{R}^+$ or $\notin \mathbb{R}^+$
4. Definition of $<$ and $>$:
    - $x < y$ means $y - x \in \mathbb{R}^+$
    - $x > y$ means $y < x$

### 2.5 The Seperation Axiom

The main difference between the $\mathbb{Q}$ and the $\mathbb{R}$ is the separation axiom which the rationals do not have. If $\mathcal{A} \subseteq \mathbb{R}$ and $\mathcal{B} \subseteq \mathbb{R}$, and:

1. $\mathcal{A} \cap \mathcal{B} = \emptyset$
2. $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$
3. $\mathcal{A} < \mathcal{B}$ "$\mathcal{A}$ is to the left of $\mathcal{B}$"
    - $\forall$ (for all) $a \in \mathcal{A}, \forall b \in \mathcal{B}, a < b$

#### 2.5.1 Proof of Irrationals

$$\mathcal{A} = \mathbb{Q}^- \cup \{0\} \cup \{q \in \mathbb{Q}^+ | q^2 < 2\}$$

$$\mathcal{B} = \{q \in \mathbb{Q}^+ | q^2 \geq 2\}$$

Then, $\exists$ (there exists at least 1) $c \in \mathbb{R}$ such that $\mathcal{A} \leq c \leq \mathcal{B}$

We know that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$ because 0 is in $\mathcal{A}$ and 2 is in $\mathcal{B}$. We also know that $\mathcal{A} \cup \mathcal{B} = \mathbb{Q}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. As well as the fact that $\mathcal{A} < \mathcal{B}$ (all elements of A are less than all elements of B.

Now, if we want to find the boundary element, $q_0$, which separates $\mathcal{A}$ and $\mathcal{B}$. We know that $\mathcal{A} \leq q_0 \leq \mathcal{B}$. So that must mean $q_0 = \sqrt{2}$. However, $\sqrt{2} \notin \mathbb{Q}$. Therefore, we know that the set of rational numbers do not follow the separation axiom.

## 3 Sequence Theorems

### 3.1 The Least Upper Bound Theorem

If $\mathcal{A} \subseteq \mathbb{R}$ is non-empty, and is <u>bounded above</u> (So $\exists b_1 \in \mathbb{R}$ such that $\mathcal{A} < b_1$), then $\mathcal{A}$ has a <u>least upper bound</u>, i.e. a number $b_0 \in \mathbb{R}$ such that $\mathcal{A} \leq b_0$ and for any $b$ with $\mathcal{A} \leq b$, $b_0 \leq b$

$$\mathcal{A} \subseteq \mathbb{R}, \mathcal{A} \neq \emptyset, (\exists\ b_1 \in \mathbb{R} : \mathcal{A} \leq b_1) \rightarrow [\exists\ b_0 \in \mathbb{R} : \mathcal{A} < b_0, \forall b \in \mathbb{R}(\mathcal{A} \leq b \rightarrow b_0 \leq b)]$$

$b_0$ is known as the least possible upper bound, or the *supremum* of $\mathcal{A}$, we write $b_0 = \sup\ \mathcal{A}$.

Similarly, for any non-empty set $\mathcal{A}$ bounded below, it has a <u>greatest lower bound</u>, $\inf \mathcal{A}$, called the *infimum* of $\mathcal{A}$

#### 3.1.1 Proof

Define $\mathcal{B}$ to be the set of all upper bounds of $\mathcal{A}$. Let $\mathcal{C} = \mathbb{R} \setminus \mathcal{B}$. Clearly $\mathcal{B}$ is nonempty; also $\mathcal{C}$ is non-empty because it contains $x_0 - 1$, where $x_0 \in \mathcal{A}$. By the way in which we defined $\mathcal{C}$, $\mathcal{B} \cap \mathcal{C} = \emptyset$. Pick any $c \in \mathcal{C}$ and $b \in \mathcal{B}$. By thedefinition of $\mathcal{C}$, $\exists\ x_1 \in \mathcal{A} : c < x_1$. But $x_1 \leq b$ by the definition of $\mathcal{B}$. Therefore $\mathcal{C} < \mathcal{B}$. By the separation postulate, $\exists\ b_0 \in \mathbb{R} : \mathcal{C} \leq b_0 \leq \mathcal{B}$. Note that $\mathcal{A} \setminus \{b_0\} \subseteq \mathcal{C}$. Thus, $b_0$ is an upper bound for $\mathcal{A}$. Morever, it is the least upper bound because $b_0 \leq \mathcal{B}$.

## 3.2 Bounded Monotone Sequence Theorem

For any sequence $\{a_n\}$

1. If $a_n \leq a_{n+1}$ for all $n \geq 1$, and $\exists\ b \in \mathbb{R}$ such that $a_n \leq b$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n$ exists and is less than or equal to $b$

2. If $a_n \geq a_{n+1}$ for all $n \geq 1$, and $\exists\ b \in \mathbb{R}$ such that $a_n \geq b$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n$ exists and is greater than or equal to $b$

### 3.2.1 Proof

We first convert the sequence $\{a_n\}$, which is bounded by $b$ into the set $\mathcal{A} = \{a_n | n \geq 1\}$. We know that $\mathcal{A} \neq \emptyset$ because the sequence has some terms. We also know that $\mathcal{A}$ is bounded above by $b$; $\mathcal{A} < b$

By the Least Upper Bound Theorem, $\exists\ b_0 = sup\mathcal{A}$. We now show that $b_0 = \lim_{n\to\infty} a_n$. By the definition of limits, to say $b_0 = \lim_{n\to\infty} a_n$ means to say $\forall \epsilon > 0, \exists\ N > 0, \forall n \geq N$, $|a_n - b_0| < \epsilon$

If we look at the number $b_0 - \epsilon$, it is not an upper bound on $\mathcal{A}$ because $b_0$ is the least upper bound and $\epsilon > 0$. Therefore, $\exists\ a_N > b_0 - \epsilon$. Since $\{a_n\}$ is increasing, $\forall n > N$, $a_n > b_0 - \epsilon$. If we rearrange the terms, we get $b_0 - a_n < \epsilon$. Therefore, $b_0$ (which exists by the least upper bound theorem) is the limit of $a_n$ as $n \to \infty$.

## 3.3 Archimedean Property

For any positive numbers $x$ and $y$, it is possible to find some $n \in \mathbb{N}$ such that $nx > y$.

$$\forall x, y > 0, \exists\ n \in \mathbb{N} : nx > y$$

### 3.3.1 Proof

Assume that $\neg\exists\ n : nx > y$, this is logically equivalent to $\forall n : nx \leq y$. Let $\mathcal{C} = \{nx | n \in \mathbb{N}\}$. Then $\mathcal{C} \leq y$, let $c = \sup \mathcal{C}$. We claim that $\exists N : c - \frac{1}{2}x < Nx \leq c$. This is true because if such $N$ does not exist, then $c - \frac{1}{2}x$ would be an upper bound, but $c$ is the least upper bound, so such $N$ must exist. Now we've established the existence of $N$, let us consider $(N+1)x$. $(N+1)x = Nx + x > (c - \frac{1}{2}x) + x = c + \frac{1}{2}x > c$. But $(N+1)x \in \mathcal{C}$, so it should be $< c$. We have a contradiction. This shows that the original assumption is false, so $\forall x, y > 0, \exists n \in \mathbb{N} : nx > y$

### 3.3.2 Consequences

This property can be used to show that $\lim_{n\to\infty} \frac{1}{n} = 0$. If we consider the definition of limits, the statement is equivalent to saying that $\forall \epsilon \in \mathbb{R} > 0, \exists N \in \mathbb{N}, \forall n > N, n \in \mathbb{R}, \frac{1}{n} < \epsilon$. If we rearrange the term, we get that we need to show $1 < \epsilon N$ for any $\epsilon$. This is true because of the Archimedean Property. $\forall n > N$, since $\epsilon > 0$, $1 < \epsilon N < \epsilon n$. Therefore we know the limit is truely 0.

## 3.4 Sunrise Lemma

The Sunrise Lemma states for any sequence $(a_n)_n^\infty = 1$ in $\mathbb{R}$, $\exists$ monotone subsequence $(a_{n_k})_{k=1}^\infty$, where $(n_k)_{k=1}^\infty$ is a strictly increasing sequence in $\mathbb{N}$ and $n_k \geq k$ for all $k \in \mathbb{N}$

Vistas are points in a sequence, $a_n$, where $N \in \mathbb{N}$, such that $a_N > a_n$ for all $n > N$

This means that for any sequence, there exists a subset of points within, such that within that sequence, the sequence is monotone

### 3.4.1 Well-Ordering Property

For any set $A \subseteq \mathbb{N}, A \neq \emptyset, min(A)$ exists

### 3.4.2 Proof

Case I: The set V of vistas, is infinite, such that $n_1 = min(v)$ and $n_k = min(V \cap n_{k-1}^{\infty}$, where $k \geq 2$, then $a_{n_k}$ is strictly decreasing
Case II:

$$n_1 = \begin{cases} 1 & if v = \emptyset \\ 1 + max(v) & if v \neq \emptyset \end{cases}$$

$n_k = choice\{n > n_{k-1} | a_n \geq a_{n_{k-1}}$, thus $n_k \neq \emptyset$ because V is finite, thus $a_{n_k}$ is increasing

## 3.5 Bolzano-Weierstrass Theorem

Every bounded sequence in $\mathbb{R}$ has at least one convergent subsequence

### 3.5.1 Proof

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence. For any monotone sequence $(a_{n_k})_{k=1}^{\infty}$, then $(a_{n_k})_{k=1}^{\infty}$ is both bounded and mono0tone, so it converges.

# 4 Extreme Value Theorem

For some function $f : [a, b] \to \mathbb{R}$ (for the codomain, not the range) that is continuous ($f(x_0) = \lim_{x \to x_o} f(x)$ for any x $\in (a, b)$, $f(a) = \lim_{x \to a^+} f(x)$, $f(b) = \lim_{x \to b^-} f(x)$), then $\exists c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$.

## 4.1 Proof

### 4.1.1 $\exists M > 0$ such that $f(x) \leq < M \forall x \in [a, b]$ (f(x) is bounded above)

Lets assume that f is not bounded above, such that for any $n \in mathbbN, \exists x_n \in [a, b]$ such that $f(x_n) > n$. The sequence $(x_n)_{n=1}^{\infty}$ is bounded between a and b, such that it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ converging to some point $k \lim_{k \to \infty} x_{n_k}$, by the below claim. Claim: $t \in [a, b]$. Let $t > b$, then $\epsilon > 0$ such that $[a, b] \cap (t - \epsilon, t + \epsilon) = \emptyset$. But $\exists N$ such that $x_N \in (t - \epsilon, t + \epsilon)$ and $x_N \in [a, b]$, which is a contradiction.

By the previous claim, $lim_{k \to \infty} f(x_{n_k}) = f(t)$ by the assumed continuity of f. Thus, $\exists$ K such that for all $k \geq K, f(x_{n_k}) < f(t) + 1 \in \mathbb{R}$. On the other hand, $f(x_{n_k}) > n_k > f(t) + 1$ when k is sufficiently large. Thus, there is a contradiction, and f(x) is bounded from above. This can be reversed to show it is bounded from below, as well.

### 4.1.2 The function reaches the supremum and infimum

We now know $R := f([a, b]) = \{f(x) | x \in [a, b]\}$ is bounded, such that S := sup(R) and I := inf(R). By the definition of infimum and supremum, $\exists (y_n)_{n=1}^{\infty}$ such that $y_n \in R \forall n$, and $\lim_{n \to \infty} y_n = S$. Since $y_n \in R, \exists x_n [a, b]$ such that $f(x_n) = y_n$. Now $(x_n)_{n=1}^{\infty}$ is bounded between a and b so it has a convergent subsequence, $(x_{n_k})_{k=1}^{\infty}$, converging to $t \in [a, b]$. Also, by continuity of f, $\lim_{k \to \infty} f(x_{n_k}) = f(t)$. Thus, f(t) = S. This can be reversed to apply to the infimum.

# 5 EVT in Higher Dimensions

## 5.1 Higher Dimensions

### 5.1.1 Cartesian Product

The Cartesian Plane represents the set $\mathbb{R}^2 := \{(x, y) + | + x, y \in (R)\}$. This is known as the **Cartesian Product** of $\mathbb{R}$ with itself. The Cartesian Product of two sets $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \times \mathcal{T} := \{(s, t) + | + s \in \mathcal{S}, t \in \mathcal{T}\}$.

### 5.1.2 Boundary

Given $D \subseteq \mathbb{R}^2, D \neq \emptyset$. We say $(a, b) \in \partial D$, i.e. $(a, b)$ is on a boundry point of $D$ if $\forall \epsilon > 0$, there are points $(x, y) \in D$ and $(u, v) \in D^c$ ($D^c := \mathbb{R}^2 \setminus D$) such that $dist(x, y+; +a, b) < \epsilon$ and $dist(u, v+; +a, b) < \epsilon$. Since the definition is symmetrical, $\partial D^c = \partial D$.

### 5.1.3 Ordered Pair

The ordered pair $(a, b)$ can be thought of as a set, but a set is inheritly unordered. To express the order, we can do the following: $(a, b) = \{\{a\}, \{a, b\}\}$. Now we know that $a$ is the first element because it appears in both subsets. We can then expand this into higher dimensions like the following: $(a, b, c) = ((a, b), c)$. Note that this means that $((a, b), c) \neq (a, (b, c))$. But this does not matter to us. **Fundamental Postulate of Ordered Pairs**: $(a_1, a_2, a_3, \dots, a_n) = (b_1, b_2, b_3, \dots, b_n)$ if and only if $a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_n = b_n$.

### 5.1.4 Distance

For our purpose, we define the distance function in 2D as:

$$dist(x, y + +; +a, b) := \sqrt{(x-a)^2 + (y-b)^2}$$

We use the Euclidean distance because it is preserved under many transformations. **Minkowski Distance** This is another distance formula, but under this, only reflection preserves distance.

$$((x-a)^p + (y-b)^p)^{\frac{1}{p}} + + + + (p > 1)$$

## 5.2 Functions in Higher Dimensions

The domain is a subset of $\mathbb{R}^{\ltimes}$. Let the function of f(x, y) be an ordered pair within some curve, such that $(x, y) \in D$. Thus, the range of f, G $= x, y, f(x, y) | (x, y) \in D \subsetneq R^{n+1}$. Functions are defined as one-to-one if for f(x, y), $(x, y), (u, v) \in D, f(x, y) = f(u, v)$ iff (if and only if) x = u $\wedge$ (and) y = v (such that for every z value, there is only one point that will create it.

## 5.3 Metric Topology in $\mathbb{R}^{\ltimes}$

### 5.3.1 Continuity

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, $D \neq \emptyset$. Let $(a, b) \in D$. We say that $f$ is continuous at $(a, b)$ if:

$$f(a, b) = \lim_{\substack{(x,y)\to(a,b) \\ (x,y)\in D}} f(x, y)$$

Or in other terms:

$$\forall\epsilon > 0, \exists + \delta > 0, \forall(x, y) \in D : dist(x, y+; +a, b) < \delta \to |f(x, y) - f(a, b)| < \epsilon$$

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^{\not\vDash}, D \neq \emptyset is continuous iff is continuous at (a, b) for all (a, b) \in D.$$

### 5.3.2 Directional Limits

Let $D \subseteq \mathbb{R} and a \in D \cup \partial D$ (the boundry, both already included in D, and not), $\lim_{x\to a^+} f(x) = L$ means $\forall\epsilon > 0, \exists\delta(\epsilon) > 0 : \forall x \in D \cap (a, \infty), |x - a| < \delta(\epsilon) \Rightarrow |f(x) - L| < \epsilon$. The limit only exists if both directions equal the same value.

### 5.3.3 Limits in Higher Dimensions

The same theory can be applied to higher dimensions, such that if two arbitrary approaches are not the same, it doesn't exist, but if several approaches yield the same result, the definition of a limit is used. Polar coordinate substitutions can be used to give format to directions of approach.

Due to difficulty defining approaching through lines, it is said that $(x, y) \to (a, b) iff dist(x, y : a, b) \to 0$.

Let $f : D \to \mathbb{R}, D \subseteq \mathbb{R}^{\not\vDash}, D \neq \emptyset, and (a, b) \in D \cup \delta D$. Then, $L = \lim_{(x,y)\in D\to(a,b)} f(x) iff \forall\epsilon > 0, \exists\delta > 0, \forall(x, y) \in D : dist(x, y : a, b) < \delta \to |f(x, y) - L| < \epsilon$. As a corollary, when (a, b) is on the boundry, the approach can only be from the domain.

## 5.4 Properties of Domain

For the extreme value theorem to apply to a domain, the set must be compact, such that it must be bounded and closed over limits. On the $\mathbb{R}$ dimension, this applies to all closed intervals, as well as the empty set, though functions except the empty set cannot accept it as a domain.

### 5.4.1 Bounded

$If D \subseteq (R^2)$ is bounded if $\exists M > 0 : D \subseteq [-M, M]x[-M, M]$.

### 5.4.2 Closed

The term closed is used to apply to sets which are closed under limits. On $mathbbR$, if $x_n \in [a, b] for \forall n \in mathbbN, and x_n \to x \in \mathbb{R}, then x \in [a, b]$.

$D \subseteq \mathbb{R}^{\not\vDash} is closed iff for any points (x_n, y_n) \in D(for all n \in \mathbb{N}, if (x_n, y_n) \to (x, y) \in \mathbb{R}^{\not\vDash}, then (x, y) \in D. (x_n, y_n) \to (a, b) as n \to \infty$ means $d_n = \sqrt[2]{(x_n - a)^2 + (y_n - b)^2} \to 0 as n \to \infty$.

This applies the definition of limits to sequences, such that $(x_n, y_n) \to (a, b) if dist(x_n, y_n : a, b) \to 0 as n \to \infty$.

Thus, $D \subseteq \mathbb{R}^{\not\vDash}$ is closed if for any sequence $((x_n, y_n))_{n=1}^{\infty}$ in D that converges, the limit poiint (a, b) of the sequence also lies in D.

### 5.4.3 Open Set Theorem

$D \subseteq \mathbb{R}^{\not\vDash}$ is open if $\forall(a, b) \in D, \exists r > 0 :$ the disk of radius r, $B_r(a, b) \subseteq D$.

It follows that for any open set, the complement set within the space is a closed set.