

# Differential Equations

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# 1 Introduction

## 1.1 Definitions

**Differential equations** are relations between one or more unknown functions and a finite amount of their derivatives, along with a certain number of independent variables. Most are not solvable, and do not have approximations for more than a minute region of the function.

**Linear differential equations** are defined as equations with algebraic functions as the multiplier of each function derivative and each derivative with a degree/power of 1, such that all linear equations are able to be solved.

**First order differential equations** means that the highest derivative in the equation is the first derivative, and is extended as such.

**Coupled systems** are those defined as requiring being solved together, such as

$$X' = aX + bY, Y' = cX + dY,$$

rather than equations containing only their own function, which would be uncoupled.

**Ordinary Differential Equations** are those where all unknown functions depend on the same, single independent variable, while **Partial Differential Equations** are those where the functions have multiple independent variables. In addition though, vector function differential equations can fall into either category, action more as a system of differential equations, rather than a single one. **Standard form** is written in the format of  $G(t, x, x', \dots, x^{(n)}) = 0$ , where  $x = x(t)$ , the unknown function.

Finally, in general, either an initial condition is set allowing it to be solved for a single solution, or a family of solution must be considered as the answer.

**Vector fields** are used to determine the trajectory of a function based on the initial condition, such that in some region D, for every point  $\vec{x}$ , we attach the vector  $\vec{F}(\vec{x})$  to the point, represented as a vector from  $\vec{x}$ .

## 1.2 Example

For some identically restricted, coupled functions,

$$R' = aR + bJ, J' = bR + aJ,$$

where  $R(0) = R_0$  and  $J(0) = J_0$ , where  $a < 0, b > 0$ , if  $|a| > |b|$ , we can graph the phase plane of R on the x-axis, J on the y-axis, such that for all possible functions, it moves toward the stable node (0, 0). These stable nodes don't need to be a point, but rather can be a curve of some sort. If  $|a| < |b|$ , if the initial points are above  $R = -J$ , it moves in a parabolic fashion until asymptotic to  $R = J$  on the positive side. If below  $R = -J$ , it moves similarly, except asymptotic to  $R = J$  on the negative side. Finally, if  $|a| = |b|$ , the function moves infinitely in a circle around the origin, with the radius determined by the initial point.

This is a form of phase plane analysis, finding the long term behavior, rather than a particular solution, used in cases where the specific solution cannot be found. Algorithms can also be used to approximate the solution at specific points.

## 2 Basic Existence and Uniqueness Theorems

### 2.1 Flow Theorem

#### 2.1.1 Theorem

Let  $\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$  be a vector field (the assignment of a vector to each point within some subset of space) defined on some closed, bounded region  $D$  in  $\mathbb{R}^n$ . Also assume  $\vec{F}$  is  $C^1$  and let  $\vec{p}$  be a specific point interior to  $D$ . Then  $\exists$  a function  $\vec{\sigma}(t)$  from some time interval  $t$ ,  $(-\epsilon, \epsilon)$  with  $\epsilon > 0$  into  $D$ , such that  $\vec{\sigma}(0) = \vec{p}$  and  $\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$ .

Thus, for any point on the interval  $t$ , the velocity of  $\vec{\sigma}$  is the vector field, such that we call  $\vec{\sigma}$  the flow for  $\vec{F}$  on that interval, starting at  $\vec{p}$ , which we call the initial condition.

The flow is unique, such that for any two flows for  $\vec{F}$  starting at the same initial condition, they must agree whenever both are defined.

This can easily be extended to higher derivatives, such that it is written as a system of two differential equations, each going one derivative, though an initial condition is needed for both of the systems.

#### 2.1.2 Application

For some  $n$ th-order differential equation,  $f(t)x^{(n)} = F(t, x, x', x'', \dots, x^{(n-1)})$ , written in **singular standard form**, assuming there is no singularity (value at which  $f(t_0) = 0$ , such that the equation becomes algebraic at that point), and  $f(t)$  is continuous, such that it is of constant sign by the Intermediate Value Theorem on the interval, we divide by  $f(t)$  to put the equation explicitly equal to the  $n$ th derivative term, called the **regular standard form case**. It must also be an initial value problem, such that initial values for each derivative and the function,  $x(t)$ , itself, are given for some initial time,  $t_0$ . It should be noted that  $t$  and  $x$  can be vectors/vector-valued functions as well, and this would still be valid.

Theorem: There is a unique solution,  $\sigma = x(t)$  defined in some time interval  $(t_0 - \epsilon, t_0 + \epsilon)$ , where  $\epsilon > 0$ .

Proof: Let  $x_0 = t, x_1 = x_1(t) = x(t), x_2 = x'(t), \dots, x_n = x^{(n-1)}(t)$ , such that  $x'_0 = 1, x'_1 = x_2, x'_2 = x_3, \dots, x'_n = x^{(n)}$ . It should also be noted that  $x'_0 = x_1 = F_0(x_0, x_1, \dots, x_n), x'_1 = F_1(x_0, x_1, \dots, x_n)$ , and so forth, removing the time dependence, such that each is purely in terms of  $\vec{x} = (x_0, x_1, \dots, x_n)$ .

Thus, the flow theorem is able to apply, assuming that the  $F_d(\vec{x})$  functions are all  $C^1$ . Since each of the functions are either constant or continuous, with the exception of  $x'_n$ , we must show that  $F$  is  $C^1$  for there to be a unique solution.

#### 2.1.3 Proof

We prove this theorem in 2 dimensions, but the proof will be general enough for  $n$ -dimensions.

Let us start with vector field  $\vec{F}(x, y) = (A(x, y), B(x, y))$  where  $A$  and  $B$  are defined on a closed, bounded region  $D$  and are  $C^1$ . The system differential equation we are trying to solve is:

$$\begin{cases} x' &= A(x, y) \\ y' &= B(x, y) \end{cases}$$

where  $x = x(t)$  and  $y = y(t)$ . The system can also be expressed as a single vector function  $\vec{x}' = \vec{F}(\vec{x})$ . Now we need the initial condition, let  $\vec{x}(0) = \vec{p}$ .

First let us take the difference between two values of  $A$ :

where  $x^*$  is some  $x$  between  $x_1$  and  $x_2$ , and  $y^*$  is some  $y$  between  $y_1$  and  $y_2$ .

Since  $A$  and  $B$  are continuous themselves on  $D$ , by the EVT we can find an upper bound  $M$  for  $|A|$  and  $|B|$  on  $D$ :

$$M = \max(\max_{(x,y) \in D} |A(x, y)|, \max_{(x,y) \in D} |B(x, y)|)$$

The initial point  $(p, q)$  is assumed to be on the *interior* of  $D$ . So we can surround the point  $p$  with a rectangle with corners defined by  $(p - r, q - s)$ ,  $(p + r, q - s)$ ,  $(p + r, q + s)$ ,  $(p - r, q + s)$  for some  $r, s > 0$  such that the rectangle lies within  $D$ . Now draw two lines through  $(p, q)$ , one with slope  $M$  and one with slope  $-M$ .

Define  $h := \min(r, \frac{s}{M}) > 0$

### 3 First Order Scalar Ordinary Differential Equations

#### 3.1 Seperable Linear Equations

**Multiplicatively seperable functions** are those of the format  $F(t, x) = f(t)g(x)$ , such that **Seperable ODEs** are those of the form,  $x'(t) = f(t)g(x)$ .

For any function in regular standard form, a vector field can be drawn with  $t$  on the  $x$ -axis and  $x(t)$  on the  $y$ -axis, such that there is a grid of vectors, with the slope of the unknown function drawn at each point, allowing the trajectory from each initial point to be drawn.

Thus, for a seperable differential equation, it can be drawn at any point on the  $x$ -axis, such that  $g(x)$  is the single slope function at that  $t$ .

First, we can divide by  $g(x)$ , such that  $f(t) = \frac{x'(t)}{g(x(t))}$ , assuming that  $g(x(t))$  is non-zero within some region,  $I$ . We then integrate with respect to some variable,  $\tau$ , such that  $\int_a^t \frac{x'(\tau)}{g(x(\tau))} d\tau = \int_a^t f(\tau) d\tau$ , leaving it in terms of the integral in case it is some nonstandard function, such that it cannot be differentiated precisely. Then, we let  $u = x(\tau)$ , such that  $\int_{x(a)}^{x(t)} \frac{du}{g(u)} = \int_a^t f(\tau) d\tau = F(t)$ . Then,  $F'(t) = f(t)$  and  $G(x(t)) = F(t) + c$  by the Fundamental Theorem of Calculus, where  $c \in \mathbb{R}$  (some constant). Thus,  $x(t) = G^{-1}(F(t) + c)$ .