

# Partial Differential Equations

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Chapter 1 - Heat Equation</b>	<b>3</b>
<b>3</b>	<b>Seperation of Variables</b>	<b>5</b>
3.1	Introduction and Heat Equation . . . . .	5
3.2	Other Examples . . . . .	6
<b>4</b>	<b>Chapter 3 - Fourier Series Theory</b>	<b>8</b>

Primary Textbook:

Teacher:

# 1 Introduction

Needed - Partial Derivatives - Ordinary Differential Equations - Green Theorem, Divergence, Etc  
 - Complex Numbers -  $z + \bar{w} = \bar{z} + w$ ,  $z\bar{w} = \bar{z} * \bar{w}$  -  $z * \bar{w} = \overline{z * w}$  -  $z * \bar{w} = |z|^2 = a^2 + b^2$  -  $z^{(-1)} = \bar{w} / (|z|^2)$ ,  $z! = 0$  -  $\mathbb{C}$  is a field (closure under  $+$ ,  $*$ , associative for both, distributive, identity for both, inverses except 0 for both) -  $zw = \bar{z}w = \bar{z}w$ , s.t. product of unit vectors is a unit vector - Conservation Laws and Flows, for some body bound by  $\partial R$ , flows have a flux -  $M_r = \int \int_R \rho(\vec{v})dV$ ,  $E_R(t) = \int \int_R e(\vec{v}, t)dV$ ,  $Q_R(t) = \int \int_R Q(\vec{v}, t)dV$

- For  $f(x(t), y(t, s))$ ,  $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$  -  $\int_a^b \frac{\partial f}{\partial x} dx = f(b, y) - f(a, y) + c(y)$  for  $f(x, y)$

## 2 Chapter 1 - Heat Equation

1. The analysis of a physical problem requires three stages, formulation, solution, and interpretation
2. For some one dimensional rod of constant cross-section and length  $L$ , the thermal energy density is defined by  $e(x, t)$ , assumed to be constant across a cross-section, such that for some cross-section, the heat energy  $E = e(x, t)A\delta x$ 
  - (a) It is assumed that heat energy change with respect to time ( $\frac{\partial}{\partial t}(e(x, t)A\delta x)$ ) is equal to the energy flowing across boundaries combined with the energy generated inside
  - (b) Heat flux is defined as the energy flowing to the right per unit time per unit surface area,  $\phi(x, t)$ , such that  $\phi < 0$  means it is flowing to the left
  - (c) Heat energy generated per unit volume per unit time is denoted as  $Q(x, t)$ , such that the conservation of heat energy can be written as  $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$  for some slice
    - i. Alternatively, it can be written not approximating for a small slice then taking the limit, such that  $\frac{d}{dt} \int_a^b e dx = \phi(a, t) - \phi(b, t) + \int_a^b Q dx$
    - ii. This is found to also be equal to  $\int_a^b \frac{\partial e}{\partial t} dx$  if  $a, b$  are constants and  $e$  is continuous  
 \*\*\*HOW\*\*\*
    - iii. It is also noted that  $\phi(a, t) - \phi(b, t) = -\int_a^b \frac{\partial \phi}{\partial x} dx$  if  $\phi$  is continuous differentiable, such that  $\int_a^b (e_t + \phi_x - Q) dx = 0$ , or  $e_t = -\phi_x + Q$ , equal to the differential form above assuming continuity, such that the integral form is more general
  - (d) Temperature is defined as  $u(x, t)$  with  $c(u)$  as the specific heat, or the heat energy per unit mass to raise the temperature one unit for some material, approximately constant over small temperature intervals
    - i. As a result,  $e(x, t) = c(x)\rho(x)u(x, t)$ , where  $\rho(x)$  is the mass density of the tube, giving the relationship between thermal density and temperature, able to be substituted into the equation
  - (e) This provides the relationship between temperature and flux, but does not give a conversion between, found to be  $\phi = -K_0 \frac{\partial u}{\partial x}$ , called Fourier's Law of Heat Conduction
    - i. This is found by the facts that heat goes from hotter to lower, does not flow if temperature is equal, higher differences cause more flow, and the flow will be based on materials
    - ii.  $K_0$  is the ability of a material to conduct heat, called the thermal conductivity, such that for heterogeneous materials, it is a function of  $x$ , and varies with temperature, though is generally constant in some range

- iii. Thus, for constant  $c, \rho, K_0$ , the heat equation is found to be  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ , where  $k = \frac{K_0}{c\rho}$ , called thermal diffusivity
- (f) If the heat energy is originally isolated into one location, it describes the spreading of it, or the diffusion, such that it is also called the diffusion equation
  - i. Similarly, for chemical diffusion,  $u(x, t)$  is the density/concentration of the chemical, gaining Fick's Law of Diffusion, analogous to Fourier's Law
- 3. For PDEs, the number of initial conditions equal to higher derivative of the spacial or temporal factor must be given, for 1D heat equation, generally the initial boundary conditions
  - (a) For a prescribed fluid bath reservoir temperature at one end, the condition is such that  $u(0, t) = u_B(t)$
  - (b) The flux can also be prescribed, such as if the boundary is insulated  $\frac{\partial u}{\partial x}(0, t) = 0$ , such that flux is also 0 at that boundary
  - (c) Newton's Law of Cooling is used if the rod is in contact with a moving fluid, such that heat will continuously move to/from the air, found to be proportional to the temperature difference between the external temperature and the rod at that location
    - i. Thus, at the boundary, it is written as  $-K_0(0) \frac{\partial u}{\partial x}(0, t) = -H(u(0, t) - u_B(t))$ , where  $H$  is the heat transfer coefficient
    - ii. The heat transfer coefficient represent the degree of insulation of the boundary, such that 0 is complete insulation, to infinity for uninsulated
- 4. Steady initial conditions are those that do not depend on time, while equilibrium/steady-state solutions are solutions that do not depend on time, such that for the heat equation,  $\frac{d^2 u}{dx^2} = 0$ 
  - (a) As a result, for steady boundary temperatures,  $u(x) = T_1 + \frac{T_2 - T_1}{L}x$ , such that for some initial state it will eventually reach the steady state solution, while for insulated edges, the steady solution is a constant
    - i. To get a specific constant, some initial function of temperature at the initial time is given,  $f(x)$ , such that  $u(x) = C_2 = \frac{1}{L} \int_0^L f(x) dx$ , such that it is the average of the initial temperature distribution
- 5. This equation is able to be extended to higher dimensions by the  $E = \int \int_R c\rho u dV$  and heat flux is defined as a vector, positive for outward rather than right, using the outward normal vector  $\vec{n}$ 
  - (a) Thus, the conservation law can be written by  $\frac{d}{dt} \int \int_R c\rho u dV = - \oint_{\partial R} \phi \cdot \vec{n} dS + \int \int_R Q dV$ 
    - i. The divergence theorem states that  $\int \int_R \nabla \cdot \vec{A} dV = \oint_{\partial R} \vec{A} \cdot \vec{n} dS$
  - (b) As a result, by the same reasoning as for 1D,  $c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0$  and  $\phi = -K_0 \nabla u$ , combined for Fourier's Law of Conduction
  - (c) For  $Q = 0$ ,  $\frac{\partial u}{\partial t} = k \nabla \cdot \nabla u = k \nabla^2 u$ , where  $\nabla^2 u$  is the Laplacian of  $u$
  - (d) For the boundary conditions, the boundary can have a known constant temperature, or be partially insulated, such that  $\nabla u \cdot \vec{n} = 0$  (directional derivative outward at the boundary is 0)
    - i. Newton's Law of Cooling can also apply, such that  $-K_0 \nabla u \cdot \vec{n} = H(u - u_b)$
  - (e) The steady state solution is such that  $\nabla^2 u = \frac{-Q}{K_0}$ , called Poisson's equation, such that if  $Q = 0$ ,  $\nabla^2 u = 0$ , called Laplace's/the potential equation
- 6. For cylindrical coordinates, the Laplacian is shown to be  $\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$ 
  - (a) Situations where  $u$  is constant for  $\theta$  are said to be circularly or axially symmetric
  - (b) Spherical coordinates are written  $(p, \theta, \phi)$ , where  $0 \leq \phi \leq \pi$ , such that  $x = p \sin(\phi) \cos(\theta)$ ,  $y =$

$$p \sin(\phi) \sin(\theta), z = p \cos(\phi)$$

$$\text{i. Thus, } \nabla^2 u = \frac{1}{p^2} \frac{\partial}{\partial p} (p^2 \frac{\partial u}{\partial p}) + \frac{1}{p^2} \sin \phi \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial u}{\partial \phi}) + \frac{1}{p^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}$$

### 3 Separation of Variables

#### 3.1 Introduction and Heat Equation

1. The method of separation of variables is used when the partial differential equation and boundary conditions are linear and homogeneous
  - (a) Linear operators are those that satisfy the linearity property,  $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$ 
    - i. The heat operator is a linear operator,  $L(u) = \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2}$
    - ii. Linear equations are those of the form  $L(u) = f$ , where  $f$  is a known function and  $L$  is a linear operator
  - (b) Homogeneous equations are those of the form  $L(u) = 0$ , such that if  $L$  is a linear operator, it is a linear homogeneous equation, such that  $u = 0$  is always a solution, called the trivial solution
  - (c) Linear homogeneous equations have the principle of superposition, such that if  $u_1, u_2$  are solutions, then all linear combinations of them are also solutions
    - i. Linear homogeneous properties also must be tested for the boundary conditions
2. For the 1D homogeneous heat equation with zero temperatures at both ends, and initial condition  $u(x, 0) = f(x)$ , it acts as a linear homogeneous partial differential with linear homogeneous boundary conditions
  - (a) Separation of variables attempts to find solutions of  $u(x, t) = \phi(x)G(t)$  (product form), ignoring the initial conditions, due to generally not satisfying in this form
    - i. As a result, by the boundary conditions, either  $u$  is the trivial solution, or  $\phi(0) = 0, \phi(L) = 0$ , providing new boundary condition forms
  - (b) This can be converted to the form  $\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2}$ , such that for each variable to be independent, rather than a function of the other, it is set equal to the separation constant,  $-\lambda$ , forming 2 ODEs
    - i. Thus,  $G(t) = ce^{-\lambda kt}$ , such that physically,  $\lambda \geq 0$ , due to otherwise the temperature increasing exponentially
3. The spacial ODE for the 1D heat equation with 0 temperature boundaries is a boundary value problem, rather than an IVP, not automatically providing a unique solution, allowing nontrivial solutions to be found
  - (a) The values such that there is a nontrivial solution are eigenvalues, where the nontrivial  $\phi(x)$  is called the eigenfunction of the value
  - (b) It is assumed that  $\lambda$  is real, where the solutions are of the form  $\phi = e^{rx}, r^2 = -\lambda$ 
    - i. If  $\lambda > 0$ ,  $r = \pm i\sqrt{\lambda}$ , such that the solutions oscillate by each of the respective components as separate solutions, such that  $\phi = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ , though any linearly independent solution can be used
      - A. Thus, by the boundary conditions,  $c_1 = 0$ , such that  $\phi(x) = c_2 \sin(\frac{n\pi x}{L})$ , where  $n$  is any positive integer, giving eigenvalues/eigenfunctions, denoted  $\phi_n(x)$  for each respective  $n$
    - ii. If  $\lambda = 0$ ,  $\phi = c_1 + c_2 x$ , such that  $c_1 = c_2 = 0$  by the boundary conditions, such that

- it is the trivial solution, such that  $\lambda = 0$  is not an eigenvalue for the heat equation
- iii. If  $\lambda < 0$ ,  $\phi = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ , this can be written in terms of the hyperbolic functions for simplicity
- These are written as  $\cosh(x) = \frac{e^x + e^{-x}}{2}$  and  $\sinh(x) = \frac{e^x - e^{-x}}{2}$ , where  $\cosh'(x) = \sinh(x)$  and  $\sinh'(x) = \cosh(x)$
  - Thus,  $\phi = c'_1 \cosh(\sqrt{s}x) + c'_2 \sinh(\sqrt{s}x)$
  - The boundary conditions produce only the trivial solution for this form of the equation, such that there are no negative eigenvalues
- (c) As a result,  $u(x, t) = B \sin(\frac{n\pi x}{L}) e^{-k(\frac{n\pi}{L})^2 t}$ , such that as  $t \rightarrow \infty$ ,  $u(x, t) = 0$
- This solution can be used to satisfy an IVP assuming the initial condition is of the correct format,  $u(x, 0) = B \sin(\frac{n\pi x}{L})$  for some  $n$
  - Similarly, for initial conditions which are the sum of this form, the solution can be found as the sum of the respective solutions for each initial condition
4. Since the linear combination of solutions is a solution, the sum of each solution for  $n$  is a solution, with possibly different amplitudes for each
- Thus,  $u(x, t) = \sum_{n=1}^M B_n \sin(\frac{n\pi x}{L}) e^{-k(\frac{n\pi}{L})^2 t}$  with initial condition,  $u(x, 0) = \sum_{n=1}^M B_n \sin(\frac{n\pi x}{L})$
  - This is useful, due to the infinite sum of sine curves of this form being a type of Fourier series, such that any function  $f(x)$  can be approximated as it as the series approaches infinity
    - It is noted that  $\int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) dx = 0$  if  $m \neq n$  and  $= \frac{L}{2}$  if  $m = n$
    - Thus, it can be multiplied by an additional term, such that for some initial condition  $f(x)$ ,  $f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$ ,  $\int_0^L f(x) \sin(\frac{m\pi x}{L}) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) dx$ , such that the series reduces to  $B_m \int_0^L \sin^2(\frac{m\pi x}{L}) dx$
    - Thus,  $B_m = \frac{2}{L} \int_0^L f(x) \sin(\frac{m\pi x}{L}) dx$ , such that the initial condition function can be plugged in for  $f(x)$  to give the coefficients for the solution
  - The fact that the integral over some number of complete half-periods of the square of a sinusoidal function is half the interval is needed often in periodic calculations
5. Orthogonal functions over some interval are those whose product integrated over the interval is 0, such that a set of functions where each is orthogonal to each other function in the set is an orthogonal set of functions
- Thus, the set of functions,  $f(x) = \sin(\frac{n\pi x}{L})$  is an orthogonal set, shown to fit the orthogonal condition
6. Thus, the basic process for separation of variables is to make sure both the equation and boundary conditions are linear, homogeneous, ignore the nonzero initial condition, introduce the separation constant
- Then, the constants are determined as eigenvalues of a boundary value problem, the other differential equations are solved, the solutions are combined, initial condition is applied, and coefficients are found using the orthogonality of the eigenfunctions
  - It can then be approximated as the earlier terms often as the time increases, moving towards equilibrium solution

### 3.2 Other Examples

- For heat conduction in 1D with insulated ends, such that there is no internal generation of heat or outside sources, it is found to have both a zero and positive solution, such that  $u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L}) e^{-(\frac{n\pi}{L})^2 kt}$

- (a) This is able to be rewritten  $u(x, t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}$
  - (b) Thus, the initial condition is valid for this solution if  $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$  for  $0 \leq x \leq L$
  - (c) The orthogonality relations,  $\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = L$  if  $m = n = 0$ ,  $= \frac{L}{2}$  if  $n = m \neq 0$ , and  $= 0$  if  $n \neq m$ , such that they are an orthogonal set of functions
    - i. As a result,  $A_0 = \frac{1}{L} \int_0^L f(x) dx$  and  $A_m = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx$ , where  $m \geq 1$
  - (d) The steady-state solution as a result is the constant solution, due to the remainder decaying over time
2. For a circular 1D wire bound at the ends of length  $2L$ , such that both the temperature and the flux are equal at both ends, with no internal heat sources, the boundary conditions are called mixed type due to involving both boundaries and periodic, due to applying over the entire  $x$  axis, as the  $x$  values repeat
    - (a) It is found to be valid for  $\lambda \geq 0$ , such that  $u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n\pi^2}{L^2} kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n\pi^2}{L^2} kt}$
    - (b) The orthogonality relations state for  $-L$  to  $L$ ,  $\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$  if  $n \neq m$ ,  $L$  if  $n = m \neq 0$ ,  $2L$  if  $n = m = 0$ ,  $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0$  if  $n \neq m$ ,  $L$  if  $n = m \neq 0$ , and  $\int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$ 
      - i. As a result,  $a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$ ,  $a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx$ , and  $b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$
  3. For some rectangle, such that each edge has a nonhomogeneous boundary condition, with the temperature of the rectangle being based on the Laplacian, it is solved by dividing into four problems, each with a single nonhomogeneous condition and three homogeneous conditions with maximum  $x$  of  $L$ , maximum  $y$  of  $H$ 
    - (a) For the boundary problem such that  $u_1(0, y) = g_1(y)$ , using the product solution,  $u_1(0, y) = h(x)\phi(y)$ , such that  $\frac{1}{h} \frac{d^2 h}{dx^2} = \frac{-1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda$
    - (b) Since the  $x$  component is not a BVP due to not having two homogeneous boundary conditions, using  $y$  to determine eigenvalues of  $\lambda = \left(\frac{n\pi}{H}\right)^2$ , with  $\phi(y) = \sin(\lambda y)$ 
      - i. The eigenvalues are then used to produce an ODE to solve for the  $x$  component, giving a hyperbolic linear combination, neither ideal for the  $h(L) = 0$  boundary condition
        - A. Since the differential equation remains the same for a translation, called invariant on translation, the linear combination is able to be shifted  $L$ , such that it is made a function of  $(x - L)$  instead of  $x$
  4. For some circle under the Laplace equation with a single nonhomogeneous boundary condition on the radius, additional conditions must be determined to allow it to be solved
    - (a) Additional boundaries are found by the fact that the center of the circle must be bounded, such that  $|u(0, \theta)| < \infty$ , and due to being a circle, there are periodicity conditions such that  $u(r, -\pi) = u(r, \pi)$ ,  $\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$ 
      - i. These conditions are homogeneous, such that they act as the three homogeneous conditions for separation of variables
      - ii. The periodic boundary conditions are the forms of the periodicity conditions applying purely to the  $\theta$  component of the product solution
        - A. As a result, the eigenvalue problem can be solved to find that  $\lambda = n^2$ , for  $n \geq 0$ , with  $\phi(\theta) = A \sin(n\theta) + B \cos(n\theta)$
      - iii. The radial component is then found to be valid for the equation  $r^2 \frac{d^2 G}{dr^2} + r \frac{dG}{dr} - n^2 G = 0$

0, such that it is a Cauchy-Euler/equidimensional equation, of the form  $r^p$ , such that  $G(r) = Cr^n + Dr^{-n}$  for  $n \neq 0$ ,  $G(r) = C + D\ln(r)$  for  $n = 0$

A. Since it cannot approach infinity as  $r \rightarrow 0$ , D must be equal to 0, such that it remains finite as n approaches infinity

- iv. Thus,  $u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$ , such that for boundary condition at  $r = a$ ,  $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$ ,  $A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) d\theta$ ,  $B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) d\theta$ ,
- (b) As a result, the Laplace Equation is found to have the properties that any point inside a circle is equal to the average of the values of the border, such that the maximum and minimum must lie on the border in the steady state
  - i. It is also found that as a result of these characteristics, the Laplacian is well-posed (such that a small change in the boundary conditions leads to a small change in the solution), and that the solution is unique
- (c) The solvability condition for Laplace's equation is found that if the heat flux is specified,  $0 = \int \nabla^2 u dx dy = \oint \nabla u \cdot \vec{n} ds$  by the Divergence Theorem or there is no solution, called the solvability/compatibility condition
  - i. This is due to going against the steady state assumption, due to a change in time of the thermal energy

## 4 Chapter 3 - Fourier Series Theory

1. The Fourier series is defined for  $f(x)$  over the interval  $-L \leq x \leq L$  as  $a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$ , with Fourier coefficients  $a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$ ,  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{n\pi x}{L}) dx$ ,  $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{n\pi x}{L}) dx$ 
  - (a) Thus, the Fourier series only exists if the coefficients exist for the functions, and cannot be assumed to be precisely equal to  $f(x)$ , such that it is written for Fourier series  $g(x)$  as  $f(x) g(x)$
2. The Convergence Theorem for Fourier Series states that if  $f(x)$  is piecewise smooth on the interval  $-L \leq x \leq L$ , the series converges to the periodic extension of  $f(x)$  where the extension is continuous, and to the average of the two limits at jump discontinuities
  - (a) Piecewise smooth is defined as continuous in the function and its first derivative except with possible jump discontinuities, in which both the left and right limit exist, but are unequal
  - (b) Periodic extension of a function is the function drawn over a  $2L$  period, then made periodic and repeating
  - (c) As a result, Fourier series can be drawn by drawing  $f(x)$  over the period, then making the periodic extension, with an  $x$  to mark the average of two values at jump discontinuities