Partial Differential Equations

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Contents

<u>Primary Textbook</u>:
<u>Teacher</u>:

1 Introduction

Needed - Partial Derivatives - Ordinary Differential Equations - Green Theorem, Divergence, Etc - Complex Numbers - $z \bar{+} w = \bar{z} + \bar{w}, z\bar{w} = \bar{z} * \bar{w}$ - $z * bar(z) = |z|^2 = a^2 + b^2$ - $z^(- 1) = bar(z)/(|z|^2), z! = 0$ - C is a field (closure under +, *, assocative for both, distributive, identity for both, inverses except 0 for both) - —zw— = —z——w—, s.t. product of unit vectors is a unit vector - Conservation Laws and Flows, for some body bound by by ∂R, flows have a flux - $M_r = \int\int\int_R = \rho(\vec{v})dV, E_R(t) = \int\int\int_R e(\vec{v}, t)dV, Q_R(t) = \int\int\int_R Q(\vec{v}, t)dV$

- For f(x(t), y(t, s)), $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$ - $\int_a^b \frac{\partial f}{\partial x} dx = f(b, y) - f(a, y) + c(y)$ for f(x, y)

2 Chapter 1 - Heat Equation

1. The analysis of a physical problem requires three stages, formulation, solution, and interpretation
2. For some one dimensional rod of constant cross-section and length L, the thermal energy density is defined by $e(x, t)$, assumed to be constant across a cross-section, such that for some cross-section, the heat energy $E = e(x, t)A\delta x$
   (a) It is assumed that heat energy change with respect to time ($\frac{\partial}{\partial t}(e(x, t)A\delta x)$ is equal to the energy flowing across boundaries combined with the energy generated inside
   (b) Heat flux is defined as the energy flowing to the right per unit time per unit surface area, $\phi(x, t)$, such that $\phi < 0$ means it is flowing to the left
   (c) Heat energy generated per unit volume per unit time is denoted as $Q(x, t)$, such that the conservation of heat energy can be written as $\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$ for some slice
      i. Alternatively, it can be written not approximating for a small slice then taking the limit, such that $\frac{d}{dt}\int_a^b e dx = \phi(a, t) - \phi(b, t) + \int_a^b Q dx$
      ii. This is found to also be equal to $\int_a^b \frac{\partial e}{\partial t} dx$ if a, b are constants and e is continuous *****HOW*****
      iii. It is also noted that $\phi(a, t) - \phi(b, t) = -\int_a^b \frac{\partial \phi}{\partial x} dx$ if ϕ is continuous differentiable, such that $\int_a^b (e_t + \phi_x - Q)dx = 0$, or $e_t = -\phi_x + Q$, equal to the differential form above assuming continuity, such that the integral form is more general
   (d) Temperature is defined as $u(x, t)$ with c(u) as the specific heat, or the heat energy per unit mass to raise the temperature one unit for some material, approximately constant over small temperature intervals
      i. As a result, $e(x, t) = c(x)\rho(x)u(x, t)$, where $\rho(x)$ is the mass density of the tube, giving the relationship between thermal density and temperature, able to be substituted into the equation
   (e) This provides the relationship between temperature and flux, but does not give a conversion between, found to be $\phi = -K_0\frac{\partial u}{\partial x}$, called Fourier's Law of Heat Conduction
      i. This is found by the facts that heat goes from hotter to lower, does not flow if temperature is equal, higher differences cause more flow, and the flow will based on materials
      ii. K_0 is the ability of a material to conduct heat, called the thermal conductivity, such that for heterogeneous materials, it is a function of x, and varies with temperature, though is generally constant in some range

iii. Thus, for constant c, ρ, K_0, the heat equation is found to be $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, where $k = \frac{K_0}{c\rho}$, called thermal diffusivity

(f) If the heat energy is originally isolated into one location, it describes the spreading of it, or the diffusion, such that it is also called the diffusion equation

i. Similarly, for chemical diffusion, u(x, t) is the density/concentration of the chemical, gaining Fick's Law of Diffusion, analogous to Fourier's Law

3. For PDEs, the number of initial conditions equal to higher derivative of the spacial or temporal factor must be given, for 1D heat equation, generally the initial boundary conditions

(a) For a prescribed fluid bath reservoir temperature at one end, the condition is such that $u(0, t) = u_B(t)$

(b) The flux can also be prescribed, such as if the boundary is insulated $\frac{\partial u}{\partial x}(0, t) = 0$, such that flux is also 0 at that boundary

(c) Newton's Law of Cooling is used if the rod is in contact with a mooving fluid, such that heat will continuously move to/from the air, found to be proportional to the temperature difference between the external temperature and the rod at that location

i. Thus, at the boundary, it is written as $-K_0(0)\frac{\partial u}{\partial x}(0, t) = -H(u(0, t) - u_B(t))$, where H is the heat transfer coefficient

ii. The heat transfer coefficient represent the degree of insulation of the boundary, such that 0 is complete insulation, to infinity for uninsulated

4. Steady initial conditions are those that do not depend on time, while equilibrium/steady-state solutions are solutions that do not depend on time, such that for the heat equation, $\frac{d^2 u}{dx^2} = 0$

(a) As a result, for steady boundary temperatures, $u(x) = T_1 + \frac{T_2 - T_1}{L}x$, such that for some initial stateit will eventually reach the steady state solution, while for insultated edges, the steady solution is a constant

i. To get a specific constant, some initial function of temperature at the initial time is given, f(x), such that $u(x) = C_2 = \frac{1}{L}\int_0^L f(x)dx$, such that it is the average of the initial temperature distribution

5. This equation is able to be extended to higher dimensions by the $E = \int\int\int_R c\rho u dV$ and heat flux is defined as a vector, positive for outward rather than right, using the outward normal vector $\vec{n}$

(a) Thus, the conservation law can be written by $\frac{d}{dt}\int\int\int_R c\rho u dV = -\oiint_{\partial R} \phi \cdot \vec{n} dS + \int\int\int_R Q dV$

i. The divergence theorem states that $\int\int\int_R \nabla \cdot \vec{A} dV = \oiint_{\partial R} \vec{A} \cdot \vec{n} dS$

(b) As a result, by the same reasoning as for 1D, $c\rho\frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0$ and $\phi = -K_0 \nabla u$, combined for Fourier's Law of Conduction

(c) For Q = 0, $\frac{\partial u}{\partial t} = k\nabla \cdot \nabla u = k\nabla^2 u$, where $\nabla^2 u$ is the Laplacian of u

(d) For the boundary conditions, the boundary can have a known constant temperature, or be partially insulated, such that $\nabla u \cdot \vec{n} = 0$ (directional derivative outward at the boundary is 0)

i. Newton's Law of Cooling can also apply, such that $-K_0 \nabla u \cdot \vec{n} = H(u - u_b)$

(e) The steady state solution is such that $\nabla^2 u = \frac{-Q}{K_0}$, called Poisson's equation, such that if Q = 0, $\nabla^2 u = 0$, called Laplace's/the potential equation

6. For cylindrical coordinates, the Laplacian is shown to be $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$

(a) Situations where u is constant for θ are said to be circularly or axially symmetric

(b) Spherical coordinates are written (p, θ, ϕ), where $0 \leq \phi \leq \pi$, such that $x = psin(\phi)cos(\theta), y =$

$psin(\phi)sin(\theta), z = pcos(\phi)$

i. Thus, $\nabla^2 u = \frac{1}{p^2}\frac{\partial}{\partial p}(p^2\frac{\partial u}{\partial p}) + \frac{1}{p^2}sin\phi\frac{\partial}{\partial \phi}(sin\phi\frac{\partial u}{\partial \phi}) + \frac{1}{p^2 sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$

3 Seperation of Variables

1. The method of seperation of variables is used when the partial differential equation and boundary conditions are linear and homogeneous
   (a) Linear operators are those that satisfy the linearity property, $L(c_1u_1+c_2u_2) = c_1L(u_1)+c_2L(u_2)$
      i. The heat operator is a linear operator, $L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}$
      ii. Linear equations are those of the form L(u) = f, where f is a known function and L is a linear operator
   (b) Homogeneous equations are those of the form L(u) = 0, such that if L is a linear operator, it is a linear homogeneous equation, such that u = 0 is always a solution, called the trivial solution
   (c) Linear homogeneous equations have the principle of superpoosition, such that if u_1, u_2 are solutions, then all linear combinations of them are also solutions
      i. Linear homogeneous properties also must be tested for the boundary conditions
2. For the 1D homogeneous heat equation with zero temperatures at both ends, and initial condition u(x, 0) = f(x), it acts as a linear homogeneous partial differential with linear homogeneous boundary conditions
   (a) Separation of variables attempts to find solutions of $u(x,t) = \phi(x)G(t)$ (product form), ignoring the initial conditions, due to generally not satisfying in this form
      i. As a result, by the boundary conditions, either u is the trivial solution, or $\phi(0) = 0, \phi(L) = 0$, providing new boundary condition forms
   (b) This can be converted to the form $\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}$, such that for each variable to be independent, rather than a function of the other, it is set equal to the separation constant, $-\lambda$, forming 2 ODEs
      i. Thus, $G(t) = ce^{-\lambda kt}$, such that physically, $-\lambda \geq 0$, due to otherwise the temperature increasing exponentially
3. The spacial ODE for the 1D heat equation with 0 temperature boundaries is a boundary value problem, rather than an IVP, not automatically providing a unique solution, allowing nontrivial solutions to be found
   (a) The values such that there is a nontrivial solution are eigenvalues, where the nontrivial $\phi(x)$ is called the eigenfunction of the value
   (b) It is assumed that λ is real, where the solutions are of the form $\phi = e^{rx}, r^2 = -\lambda$
      i. If $\lambda > 0$, $r = \pm i\sqrt{\lambda}$, such that the solutons oscillate by each of the respective components as separate solutions, such that $\phi = c_1cos(\sqrt{\lambda}x) + c_2sin(\sqrt{\lambda}x)$, though any linearly independent solution can be used
         A. Thus, by the boundary conditions, $c_1 = 0$, such that $\phi(x) = c_2sin(\frac{n\pi x}{L})$, where n is any positive integer, giving eigenvalues/eigenfunctions, denoted $\phi_n(x)$ for each respective n
      ii. If $\lambda = 0$, $\phi = c_1 + c_2x$, such that $c_1 = c_2 = 0$ by the boundary conditions, such that it is the trivial solution, such that $\lambda = 0$ is not an eigenvalue for the heat equation
      iii. If $\lambda < 0$, $\phi = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$, this can be written in terms of the hyperbolic

functions for simplicity

A. These are written as $cosh(x) = \frac{e^x+e^{-x}}{2}$ and $sinh(x) = \frac{e^x-e^{-x}}{2}$, where $cosh'(x) = sinh(x)$ and $sinh'(x) = cosh(x)$
B. Thus, $\phi = c_1' cosh(\sqrt{s}x) + c_2' sinh(\sqrt{s}x)$
C. The boundary conditions produce only the trivial solution for this form of the equation, such that there are no negative eigenvalues