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EXTENDING MEANS OF TWO VARIABLES TO SEVERAL VARIABLES

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ABSTRACT. We present a method, based on series expansions and symmetric polynomials, by which a mean of two variables can be extended to several variables. We apply it mainly to the logarithmic mean.

Key words and phrases: Means, Logarithmic mean, Divided differences, Series expansions, Symmetric polynomials.

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1. INTRODUCTION

Throughout this paper, $n \geq 2$ is an integer and $x_1, \dots, x_n$ are positive real numbers.

The logarithmic mean of x_1 and x_2 is defined by

$$\begin{aligned} L(x_1, x_2) &= \frac{x_1 - x_2}{\ln x_1 - \ln x_2} \quad \text{if } x_1 \neq x_2, \\ L(x_1, x_1) &= x_1. \end{aligned} \tag{1.1}$$

There are several ways to extend this to n variables. Bullen ([1, p. 391]) writes that perhaps the most natural extension is due to Pittenger [13]. Based on an integral, it is

$$L(x_1, \dots, x_n) = \left[(n-1) \sum_{i=1}^{n} \frac{x_i^{n-2} \ln x_i}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (\ln x_i - \ln x_j)} \right]^{-1} \tag{1.2}$$

if all the x_i's are unequal. Bullen ([1, p. 392]) also writes that another natural extension has been given by Neuman [9]. Based on the integral (6.3), it is

$$L(x_1, \dots, x_n) = (n-1)! \sum_{i=1}^{n} \frac{x_i}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (\ln x_i - \ln x_j)} \tag{1.3}$$

if all the x_i's are unequal. It is obviously different from (1.2).

If some of the x_i's are equal, then (1.2) and (1.3) are defined by continuity.

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Mustonen [6] gave (1.3) in 1976 but published it only recently [7] in the home page of his statistical data processing system, not in a journal. We will present his method. It is based on a series expansion and supports the notion that (1.3) is the most natural extension of (1.1).

In general, we call a continuous real function μ of two positive (or nonnegative) variables a mean if, for all $x_1, x_2, c > 0$ (or $x_1, x_2, c \geq 0$),

(i_1) $\mu(x_1, x_2) = \mu(x_2, x_1)$,
(i_2) $\mu(x_1, x_1) = x_1$,
(i_3) $\mu(cx_1, cx_2) = c\mu(x_1, x_2)$,
(i_4) $x_1 \leq y_1, x_2 \leq y_2 \Rightarrow \mu(x_1, x_2) \leq \mu(y_1, y_2)$,
(i_5) $\min(x_1, x_2) \leq \mu(x_1, x_2) \leq \max(x_1, x_2)$.

Axiomatization of means is widely studied, see e.g. [1] and references therein.

2. POLYNOMIALS CORRESPONDING TO A MEAN

To extend the arithmetic and geometric means

$$A(x_1, x_2) = \frac{x_1 + x_2}{2}, \quad G(x_1, x_2) = (x_1 x_2)^{\frac{1}{2}}$$

to n variables is trivial, but to visualize our method, it may be instructive.

Substituting

$$x_1 = \mathrm{e}^{u_1},\ x_2 = \mathrm{e}^{u_2}, \tag{2.1}$$

we have

$$\begin{aligned} A(x_1, x_2) &= \tilde{A}(u_1, u_2) \\ &= \frac{1}{2}\left(\mathrm{e}^{u_1} + \mathrm{e}^{u_2}\right) \\ &= \frac{1}{2}\left(1 + u_1 + \frac{u_1^2}{2!} + \cdots + 1 + u_2 + \frac{u_2^2}{2!} + \cdots\right) \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{u_1^2 + u_2^2}{2} + \frac{1}{3!} \cdot \frac{u_1^3 + u_2^3}{2} + \cdots, \end{aligned} \tag{2.2}$$

$$\begin{aligned} G(x_1, x_2) &= \tilde{G}(u_1, u_2) \\ &= \left(\mathrm{e}^{u_1}\mathrm{e}^{u_2}\right)^{\frac{1}{2}} \\ &= \mathrm{e}^{\frac{u_1 + u_2}{2}} \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!}\left(\frac{u_1 + u_2}{2}\right)^2 + \cdots \\ &= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{(u_1 + u_2)^2}{2^2} + \frac{1}{3!} \cdot \frac{(u_1 + u_2)^3}{2^3} + \cdots, \end{aligned} \tag{2.3}$$

$$\begin{aligned} L(x_1, x_2) &= \tilde{L}(u_1, u_2) \\ &= \frac{\mathrm{e}^{u_1} - \mathrm{e}^{u_2}}{u_1 - u_2} \\ &= \left(1 + u_1 + \frac{u_1^2}{2!} + \cdots - 1 - u_2 - \frac{u_2^2}{2!} - \cdots\right)(u_1 - u_2)^{-1} \end{aligned} \tag{2.4}$$

$$= \left(u_1 - u_2 + \frac{u_1^2 - u_2^2}{2!} + \frac{u_1^3 - u_2^3}{3!} + \cdots\right)(u_1 - u_2)^{-1}$$
$$= 1 + \frac{u_1 + u_2}{2} + \frac{1}{2!} \cdot \frac{u_1^2 + u_1u_2 + u_2^2}{3} + \frac{1}{3!} \cdot \frac{u_1^3 + u_1^2u_2 + u_1u_2^2 + u_2^3}{4} + \cdots .$$

All these expansions are of the form

$$1 + P_1(u_1, u_2) + \frac{1}{2!}\, P_2(u_1, u_2) + \frac{1}{3!}\, P_3(u_1, u_2) + \cdots , \tag{2.5}$$

where the P_m's are symmetric homogeneous polynomials of degree m. In all of them,

$$P_1(u_1, u_2) = \frac{u_1 + u_2}{2} = A(u_1, u_2).$$

The coefficients of

$$P_m(u_1, u_2) = b_0u_1^m + b_1u_1^{m-1}u_2 + \cdots + b_mu_2^m \tag{2.6}$$

are nonnegative numbers with sum 1. They are for A

$$b_0 = \frac{1}{2},\ b_1 = \cdots = b_{m-1} = 0,\ b_m = \frac{1}{2},$$

for G

$$b_k = \binom{m}{k} 2^{-m} \quad (0 \le k \le m),$$

and for L

$$b_0 = \cdots = b_m = \frac{1}{m+1}.$$

Let μ be a mean of two variables. Assume that it has a valid expansion (2.5). Fix $m \ge 2$, and denote by $P_m[\mu]$ the polynomial (2.6). Its coefficients define a discrete random variable, denoted by $X_m[\mu]$, whose value is k $(0 \le k \le m)$ with probability b_k. In particular, $X_m[A]$ is distributed uniformly over $\{0, m\}$, and $X_m[G]$ binomially and $X_m[L]$ uniformly over $\{0, \dots, m\}$. Their variances satisfy

$$\operatorname{Var} X_m[G] \le \operatorname{Var} X_m[L] \le \operatorname{Var} X_m[A],$$

which is an interesting reminiscent of

$$G(x_1, x_2) \le L(x_1, x_2) \le A(x_1, x_2). \tag{2.7}$$

Let $u_1, u_2 \ge 0$, then (2.7) holds in fact termwise:

$$P_m[G](u_1, u_2) \le P_m[L](u_1, u_2) \le P_m[A](u_1, u_2) \tag{2.8}$$

for all $m \ge 1$. The functions

$$R_m[\mu](u_1, u_2) = (P_m[\mu](u_1, u_2))^{\frac{1}{m}}$$

are means. In particular, for A they are moment means

$$R_m[A](u_1, u_2) = \left(\frac{u_1^m + u_2^m}{2}\right)^{\frac{1}{m}} = M_m(u_1, u_2),$$

for G all of them are equal to the arithmetic mean

$$R_m[G](u_1, u_2) = \frac{u_1 + u_2}{2} = A(u_1, u_2),$$

and for L they are special cases of complete symmetric polynomial means and Stolarsky means (see e.g. [1, pp. 341, 393])

$$R_m[L](u_1,u_2) = \left[\frac{u_1^{m+1}-u_2^{m+1}}{(m+1)(u_1-u_2)}\right]^{\frac{1}{m}} = \left(\frac{u_1^m + u_1^{m-1}u_2 + \cdots + u_2^m}{m+1}\right)^{\frac{1}{m}}.$$

Since the $P_m[\mu]$'s are symmetric and homogeneous polynomials of two variables, they can be extended to n variables. Thus μ can also be likewise extended.

3. TRIVIAL EXTENSIONS: A AND G

Consider first A. By (2.2),

$$P_m[A](u_1,u_2) = \frac{u_1^m + u_2^m}{2}.$$

To extend it to n variables is actually as trivial as to extend A directly. We obtain

$$P_m[A](u_1,\dots,u_n) = \frac{u_1^m + \cdots + u_n^m}{n},$$

and so

$$\begin{aligned} A(x_1,\dots,x_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} P_m[A](u_1,\dots,u_n) \\ &= \frac{1}{n}\left(\sum_{m=0}^{\infty}\frac{u_1^m}{m!} + \cdots + \sum_{m=0}^{\infty}\frac{u_n^m}{m!}\right) \\ &= \frac{1}{n}\left(\mathrm{e}^{u_1} + \cdots + \mathrm{e}^{u_n}\right) = \frac{x_1 + \cdots + x_n}{n}. \end{aligned}$$

Next, study G. By (2.3),

$$P_m[G](u_1,u_2) = \left(\frac{u_1+u_2}{2}\right)^m,$$

which can be immediately extended to

$$P_m[G](u_1,\dots,u_n) = \left(\frac{u_1+\cdots+u_n}{n}\right)^m,$$

and so

$$\begin{aligned} G(x_1,\dots,x_n) &= \sum_{m=0}^{\infty} \frac{1}{m!} P_m[G](u_1,\dots,u_n) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{u_1+\cdots+u_n}{n}\right)^m \\ &= \mathrm{e}^{\frac{u_1+\cdots+u_n}{n}} = \left(\mathrm{e}^{u_1}\cdots\mathrm{e}^{u_n}\right)^{\frac{1}{n}} = (x_1\cdots x_n)^{\frac{1}{n}}. \end{aligned}$$

We present a "termwise" (cf. (2.8)) proof of the geometric-arithmetic mean inequality

$$G(x_1,\dots,x_n) \le A(x_1,\dots,x_n). \tag{3.1}$$

We can assume that $u_1,\dots,u_n \ge 0$; if not, consider $cG \le cA$ for a suitable $c > 0$. Let $m \ge 1$. Then

$$P_m[G](u_1,\dots,u_n) \le P_m[A](u_1,\dots,u_n) \tag{3.2}$$

or equivalently

$$R_m[G](u_1,\ldots,u_n) \le R_m[A](u_1,\ldots,u_n), \tag{3.3}$$

since

$$\frac{u_1+\cdots+u_n}{n} \le \left(\frac{u_1^m+\cdots+u_n^m}{n}\right)^{\frac{1}{m}}$$

by Schlömilch's inequality (see e.g. [1, p. 203]). Therefore (3.1) follows.

4. EXTENDING L

Let $1 \le m \le n$. The mth *complete symmetric polynomial* of $u_1,\ldots,u_n \ge 0$ (see e.g. [1, p. 341]) is defined by

$$C_m(u_1,\ldots,u_n) = \sum_{i_1+\cdots+i_n=m} {u_1}^{i_1}\cdots {u_n}^{i_n}.$$

(Here $i_1,\ldots,i_n \ge 0$, and we define $0^0 = 1$.)

Let us now study L. Denote $Q_m = P_m[L]$. By (2.4),

$$Q_m(u_1,u_2) = \frac{u_1^m + u_1^{m-1}u_2 + \cdots + u_2^m}{m+1}.$$

This can be easily extended to

$$Q_m(u_1,\ldots,u_n) = \binom{n+m-1}{m}^{-1} C_m(u_1,\ldots,u_n). \tag{4.1}$$

The corresponding mean,

$$R_m[L](u_1,\ldots,u_n) = Q_m(u_1,\ldots,u_n)^{\frac{1}{m}},$$

is called [1] the mth *complete symmetric polynomial mean* of $u_1,\ldots,u_n$.

Thus we extend

$$L(x_1,\ldots,x_n) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} Q_m(u_1,\ldots,u_n). \tag{4.2}$$

We compute this explicitly. Fix $m \ge 2$. Assume that $u_1,\ldots,u_n \ge 0$ are all unequal. We claim that if $2 \le n \le m+1$, then $C_{m-n+1}(u_1,\ldots,u_n)$ is the $(n-1)$th divided difference of the function $f(u) = u^m$ with arguments $u_1,\ldots,u_n$. In other words,

$$C_{m-n+1}(u_1,\ldots,u_n) = \frac{C_{m-n+2}(u_2,\ldots,u_n) - C_{m-n+2}(u_1,\ldots,u_{n-1})}{u_n - u_1}. \tag{4.3}$$

(For $n=2$, we have simply $C_{m-1}(u_1,u_2) = \frac{u_2^m - u_1^m}{u_2 - u_1}$.)

To prove this, note that for $k \ge 1$

$$\begin{aligned} C_k(u_1,\ldots,u_n) &= u_n^k + u_n^{k-1}C_1(u_1,\ldots,u_{n-1}) \\ &\quad + \cdots + u_n C_{k-1}(u_1,\ldots,u_{n-1}) + C_k(u_1,\ldots,u_{n-1}) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} C_k(u_1,\ldots,u_n) &= C_k(u_1,u_n) + C_{k-1}(u_1,u_n)C_1(u_2,\ldots,u_{n-1}) \\ &\quad + \cdots + C_1(u_1,u_n)C_{k-1}(u_2,\ldots,u_{n-1}) + C_k(u_2,\ldots,u_{n-1}). \end{aligned}$$

Hence,

$$\begin{aligned}
&C_{m-n+2}(u_2,\dots,u_n) - C_{m-n+2}(u_1,\dots,u_{n-1})\\
&\qquad = C_{m-n+2}(u_2,\dots,u_n) - C_{m-n+2}(u_2,\dots,u_{n-1},u_1)\\
&\qquad = u_n^{m-n+2} + u_n^{m-n+1}C_1(u_2,\dots,u_{n-1}) + \cdots + C_{m-n+2}(u_2,\dots,u_{n-1})\\
&\qquad\qquad - u_1^{m-n+2} - u_1^{m-n+1}C_1(u_2,\dots,u_{n-1}) - \cdots - C_{m-n+2}(u_2,\dots,u_{n-1})\\
&\qquad = (u_n^{m-n+2} - u_1^{m-n+2}) + (u_n^{m-n+1} - u_1^{m-n+1})C_1(u_2,\dots,u_{n-1}) + \cdots\\
&\qquad\qquad + (u_n - u_1)C_{m-n+1}(u_2,\dots,u_{n-1})\\
&\qquad = (u_n - u_1)\Big[C_{m-n+1}(u_1,u_n) + C_{m-n}(u_1,u_n)C_1(u_2,\dots,u_{n-1}) + \cdots\\
&\qquad\qquad + C_{m-n+1}(u_2,\dots,u_{n-1})\Big]\\
&\qquad = (u_n - u_1)C_{m-n+1}(u_1,\dots,u_n),
\end{aligned}$$

and (4.3) follows.

By a well-known formula of divided differences (see e.g. [4, p. 148]), we now have

$$C_{m-n+1}(u_1,\dots,u_n) = \sum_{i=1}^{n} \frac{u_i^m}{U_i},$$

where

$$U_i = \prod_{\substack{j=1\\ j\neq i}}^{n} (u_i - u_j).$$

Therefore, since

$$\frac{1}{(m-n+1)!}\binom{n+(m-n+1)-1}{m-n+1}^{-1} = \frac{(n-1)!}{m!},$$

we obtain

$$\begin{aligned}
\frac{1}{(m-n+1)!}\,Q_{m-n+1}(u_1,\dots,u_n) &= \frac{(n-1)!}{m!}\,C_{m-n+1}(u_1,\dots,u_n)\\
&= \frac{(n-1)!}{m!}\sum_{i=1}^{n}\frac{u_i^m}{U_i}.
\end{aligned}$$

Hence, and because the mth divided difference of the function $f(u) = u^m$ is 1 if $m = n-1$ and 0 if $m \le n-2$, we have

$$\begin{aligned}
L(x_1,\dots,x_n) &= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\,Q_k(u_1,\dots,u_n)\\
&= 1 + \sum_{m=n}^{\infty}\frac{1}{(m-n+1)!}\,Q_{m-n+1}(u_1,\dots,u_n)\\
&= 1 + (n-1)!\sum_{m=n}^{\infty}\frac{1}{m!}\sum_{i=1}^{n}\frac{u_i^m}{U_i}\\
&= (n-1)!\sum_{m=n-1}^{\infty}\frac{1}{m!}\sum_{i=1}^{n}\frac{u_i^m}{U_i}
\end{aligned}$$

$$
\begin{aligned}
&= (n-1)! \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i=1}^{n} \frac{u_i^m}{U_i} \\
&= (n-1)! \sum_{i=1}^{n} \frac{1}{U_i} \sum_{m=0}^{\infty} \frac{u_i^m}{m!} \\
&= (n-1)! \sum_{i=1}^{n} \frac{\mathrm{e}^{u_i}}{U_i} \\
&= (n-1)! \sum_{i=1}^{n} \frac{\mathrm{e}^{u_i}}{\prod_{\substack{j=1\\ j\neq i}}^{n} (u_i - u_j)} \\
&= (n-1)! \sum_{i=1}^{n} \frac{x_i}{\prod_{\substack{j=1\\ j\neq i}}^{n} (\ln x_i - \ln x_j)}.
\end{aligned}
$$

Thus (1.3) is found.

5. NUMERICAL COMPUTATION OF L

Mustonen [7] noted that, in computing L numerically, the explicit formula (1.3) is very unstable. He programmed a fast and stable algorithm based on (4.1), (4.2), and (4.4). His experiments lead to a conjecture that, denoting $G_n = G(1,\dots,n)$ and $L_n = L(1,\dots,n)$, we have

$$\lim_{n\to\infty} (G_{n+1} - G_n) = \lim_{n\to\infty} (L_{n+1} - L_n) = \frac{1}{\mathrm{e}}$$

and

$$\lim_{n\to\infty} \frac{G_n}{n} = \lim_{n\to\infty} \frac{L_n}{n} = \frac{1}{\mathrm{e}}.$$

For G_n, these limit conjectures can be proved by using Stirling's formula. For L_n, they remain open.

6. INEQUALITY $G \le L \le A$

It is natural to ask, whether

$$G(x_1,\dots,x_n) \le L(x_1,\dots,x_n) \le A(x_1,\dots,x_n) \tag{6.1}$$

is generally valid.

For $n = 2$, this inequality is old (see e.g. [1, pp. 168-169]). Carlson [2] (see also [1, p. 388]) sharpened the first part and Lin [5] (see also [1, p. 389]) the second:

$$(G(x_1,x_2)M_{1/2}(x_1,x_2))^{\frac{1}{2}} \le L(x_1,x_2) \le M_{1/3}(x_1,x_2). \tag{6.2}$$

Neuman [9] defined (as a special case of [9, Eq. (2.3)])

$$L(x_1,\dots,x_n) = \int_{E_{n-1}} \left(\exp \sum_{i=1}^{n} u_i \ln x_i \right) \mathrm{d}u, \tag{6.3}$$

where $u_1 + \cdots + u_n = 1$,

$$E_{n-1} = \{(u_1,\dots,u_{n-1}) \,|\, u_1,\dots,u_{n-1} \ge 0,\ u_1 + \cdots + u_{n-1} \le 1\},$$

and $\mathrm{d}u = \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1}$. He ([9], Theorem 1 and the last formula) proved (6.1) and reduced (6.3) into (1.3).

Pečarić and Šimić [12] tied Neuman's approach to a wider context. As a special case ([12, Remark 5.4]), they obtained (1.3).

Let V denote the Vandermonde determinant and let V_i denote its subdeterminant obtained by omitting its last row and ith column. Xiao and Zhang [14] (unaware of [9]) defined

$$L(x_1,\dots,x_n)=\frac{(n-1)!}{V(\ln x_1,\dots,\ln x_n)}\sum_{i=1}^{n}(-1)^{n+i}x_iV_i(\ln x_1,\dots,\ln x_n),$$

which in fact equals to (1.3). Also they proved (6.1).

We conjecture that (6.2) can be extended to

$$(G(x_1,\dots,x_n)M_{1/2}(x_1,\dots,x_n))^{\frac{1}{2}}\le L(x_1,\dots,x_n)\le M_{1/3}(x_1,\dots,x_n).$$

7. INEQUALITIES $P_m[G]\le P_m[L]\le P_m[A]$

In view of (3.2) and (3.3), it is now natural to ask, whether (6.1) can be strengthened to hold termwise. In other words: Do we have

$$P_m[G]\le P_m[L]\le P_m[A]$$

or equivalently

$$R_m[G]\le R_m[L]\le R_m[A],$$

that is

$$\frac{u_1+\cdots+u_n}{n}\le Q_m(u_1,\dots,u_n)^{\frac{1}{m}}\le\left(\frac{u_1^m+\cdots+u_n^m}{n}\right)^{\frac{1}{m}} \tag{7.1}$$

for all $u_1,\dots,u_n\ge 0$, $m\ge 1$?

Fix $u_1,\dots,u_n$ and denote $q_m=Q_m(u_1,\dots,u_n)^{\frac{1}{m}}$. Neuman ([8, Corollary 3.2]; see also [1, pp. 342-343]) proved that

$$k\le m\Rightarrow q_k\le q_m. \tag{7.2}$$

The first part of (7.1), $q_1\le q_m$, is therefore true. We conjecture that the second part is also true.

DeTemple and Robertson [3] gave an elementary proof of (7.2) for $n=2$, but Neuman's proof for general n is advanced, applying B-splines.

Mustonen [7] gave an elementary proof of (7.1) for $n=2$.

8. OTHER MEANS

Pečarić and Šimić [12] (see also [1, p. 393]) studied a very large class of means, called *Stolarsky-Tobey means*, which includes all the ordinary means as special cases. They first defined these means for two variables and then, applying certain integrals, extended them to n variables. It might be of interest to apply our method to all these extensions, but we take only a small step towards this direction.

Let r and s be unequal nonzero real numbers. (Actually [12] allows $s=0$ and [1] allows $r=0$, both of which are obviously incorrect.) Consider ([12, Eq. (6)]) the mean

$$E_{r,s}(x_1,x_2)=\left(\frac{r}{s}\cdot\frac{x_1^s-x_2^s}{x_1^r-x_2^r}\right)^{\frac{1}{s-r}}, \tag{8.1}$$

where $x_1\ne x_2$. Assuming that $s\ne -r,-2r,\dots,-(n-2)r$, this can be extended ([12, Theorem 5.2(i)]) to

$$E_{r,s}(x_1,\dots,x_n)=\left[\frac{(n-1)!\,r^{n-1}}{s(s+r)\cdots(s+(n-2)r)}\sum_{i=1}^{n}\frac{x_i^{s+(n-2)r}}{\prod_{\substack{j=1\\ j\ne i}}^{n}(x_i^r-x_j^r)}\right]^{\frac{1}{s-r}}, \tag{8.2}$$

where all the x_i's are unequal.

To extend (8.1) by our method, we simply note that

$$\begin{aligned} E_{r,s}(x_1, x_2) &= \left[\frac{x_1^s - x_2^s}{s(\ln x_1 - \ln x_2)} \Big/ \frac{x_1^r - x_2^r}{r(\ln x_1 - \ln x_2)} \right]^{\frac{1}{s-r}} \\ &= \left(\frac{L(x_1^s, x_2^s)}{L(x_1^r, x_2^r)} \right)^{\frac{1}{s-r}}, \end{aligned}$$

which can be immediately extended to

$$\begin{aligned} E_{r,s}(x_1, \dots, x_n) & \\ &= \left(\frac{L(x_1^s, \dots, x_n^s)}{L(x_1^r, \dots, x_n^r)} \right)^{\frac{1}{s-r}} \\ &= \left\{ \sum_{i=1}^n \frac{x_i^s}{\prod_{\substack{j=1 \\ j\neq i}}^n [s(\ln x_i - \ln x_j)]} \Big/ \sum_{i=1}^n \frac{x_i^r}{\prod_{\substack{j=1 \\ j\neq i}}^n [r(\ln x_i - \ln x_j)]} \right\}^{\frac{1}{s-r}} \\ &= \left[\left(\frac{r}{s} \right)^{n-1} \sum_{i=1}^n \frac{x_i^s}{\prod_{\substack{j=1 \\ j\neq i}}^n (\ln x_i - \ln x_j)} \Big/ \sum_{i=1}^n \frac{x_i^r}{\prod_{\substack{j=1 \\ j\neq i}}^n (\ln x_i - \ln x_j)} \right]^{\frac{1}{s-r}}. \end{aligned} \tag{8.3}$$

This is obviously different from (8.2).

Unfortunately the problem of whether (8.3) indeed is a mean, i.e., whether it lies between the smallest and largest x_i, remains open.

ADDENDUM

Neuman ([10, Theorem 6.2]) proved the second part of (7.1) and [11] showed that (8.3) is a mean.

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