mathematics

MDPI

Article

Stolarsky Means in Many Variables

Slavko Simić [1,2,*] and Bandar Bin-Mohsin [3]

1 Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City 758307, Vietnam
2 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 758307, Vietnam
3 Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia; balmohsen@ksu.edu.sa
* Correspondence: slavkosimic@tdtu.edu.vn or ssimic@turing.mi.sanu.ac.rs

Received: 14 July 2020; Accepted: 6 August 2020; Published: 8 August 2020

check for updates

Abstract: We give in this article two possible explicit extensions of Stolarsky means to the multi-variable case. They attain all main properties of Stolarsky means and coincide with them in the case of two variables.

Keywords: extended mean values; logarithmic convexity; multi-variable case

MSC: 26A51; 60E15

1. Introduction

There is a huge number of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean values, defined for positive variables $x, y; x \neq y$, as

$$E_{r,s}(x,y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, \quad rs(r-s) \neq 0.$$

Those means can be continuously extended on the domain

$$\{(r,s;x,y)|r,s \in \mathbb{R}; x,y \in \mathbb{R}_+\}$$

by the following

$$E_{r,s}(x,y) = \begin{cases} \left(\frac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r-s) \neq 0; \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0; \\ \left(\frac{x^s - y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0, r = 0; \\ \sqrt{xy}, & r = s = 0; \\ x, & y = x > 0, \end{cases}$$

and in this form has been introduced by Keneth Stolarsky in [1].

Most of the classical two variable means are just special cases of the class E. For example, $E_{1,2} = \frac{x+y}{2}$ is the arithmetic mean, $E_{-r,r} = E_{0,0} = \sqrt{xy}$ is the geometric mean, $E_{0,1} = \frac{x-y}{\log x - \log y}$ is the logarithmic mean, $E_{1,1} = (x^x/y^y)^{\frac{1}{x-y}}/e$ is the identric mean, etc. More generally, the r-th power mean $\left(\frac{x^r+y^r}{2}\right)^{1/r}$ is equal to $E_{r,2r}$ ([2]).

Characteristic properties of Stolarsky means are:

1. Symmetry in variables, $E_{r,s}(x,y) = E_{r,s}(y,x)$;
2. Symmetry in parameters, $E_{r,s}(x,y) = E_{s,r}(x,y)$;
3. Means $E_{r,s}(x,y)$ are homogeneous of order one i.e, $E_{r,s}(tx,ty) = tE_{s,r}(x,y), t > 0$.

4. Means $E_{r,s}(x,y)$ are monotone increasing in both parameters r and s.

By two articles ([3,4]) published in Amer.Math. Monthly, this class of means attains popularity in a wide audience. As a result, great number of papers are produced investigating its most subtle properties. In this sense we quote here papers [5,6]. A comparison of Stolarsky and Gini means is given in [7–9], weighted variants in [10,11]. F. Qi in [12] find intervals of r, s where these means are logarithmically convex/concave, etc.

Furthermore, there are several papers attempting to define an extension of the class E to $n,\ n > 2$ variables. Unfortunately, this is done in a highly implicit mode ([5,6,13–15]).

Here is an illustration of this point; J. Merikoski ([13]) has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables

$$E_{r,s}(X) := \Big[\frac{L(X^s)}{L(X^r)}\Big]^{\frac{1}{s-r}}, r \neq s,$$

where $X = (x_1, \cdots, x_n)$ is an n-tuple of positive numbers and

$$L(X^s) := (n-1)! \int_{I_{n-1}} \prod_{i=1}^{n} x_i^{su_i} du_1 \cdots du_{n-1}.$$

The symbol I_{n-1} stands for the Euclidean simplex which is defined by

$$I_{n-1} := \{(u_1, \cdots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1;\ u_1 + \cdots + u_{n-1} \leq 1\}.$$

In this article we shall expose two possible explicit formulae of Stolarsky means in $n \geq 2$ variables which preserve its main properties and coincide for $n = 2$.

The first one is given by the following

Let $X_n = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$. Then,

$$e_{r,s}(X_n) = e_{r,s}(x_1, x_2, ..., x_n) := \Big(\frac{r^2}{s^2}\frac{x_1^{ns} + x_2^{ns} + ... + x_n^{ns} - n(x_1x_2...x_n)^s}{x_1^{nr} + x_2^{nr} + ... + x_n^{nr} - n(x_1x_2...x_n)^r}\Big)^{\frac{1}{n(s-r)}}, rs(s-r) \neq 0,$$

represents an extension of Stolarsky means to the multi-variable case.

Remark 1. *We assume that there exist $j, k; 1 \leq j < k \leq n$, such that $x_j \neq x_k$.*

It is of interest to examine the inner structure of those means. For example, applying the formula

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)[(x-y)^2 + (y-z)^2 + (z-x)^2],$$

we obtain that

$$e_{r,s}(x_1, x_2, x_3) = (A_{r,s}(x_1, x_2, x_3))^{1/3}(B_{r,s}(x_1, x_2, x_3))^{2/3},$$

where

$$A_{r,s}(x_1, x_2, x_3) := \Big(\frac{x_1^s + x_2^s + x_3^s}{x_1^r + x_2^r + x_3^r}\Big)^{1/(s-r)}$$

is the well-known Gini mean, and

$$B_{r,s}(x_1, x_2, x_3) := \Big(\frac{r^2}{s^2}\frac{(x_1^s - x_2^s)^2 + (x_2^s - x_3^s)^2 + (x_3^s - x_1^s)^2}{(x_1^r - x_2^r)^2 + (x_2^r - x_3^r)^2 + (x_3^r - x_1^r)^2}\Big)^{1/(2(s-r))}$$

is the new mean in 3 variables which coincides with the Stolarsky mean $E_{r,s}(x_1, x_2)$ whenever $x_3 = x_1$ or $x_3 = x_2$.

This notion leads to the second, more general representation of Stolarsky means in many variables. Let $A_n = (a_1, a_2, ..., a_n), X_n = (x_1, x_2, ..., x_n), Y_n = (y_1, y_2, ..., y_n); A_n, X_n, Y_n \in \mathbb{R}^n_+$.

Then

$$E_{r,s}(A_n; X_n, Y_n) := \Big(\frac{r^2}{s^2}\frac{a_1(x_1^s - y_1^s)^2 + a_2(x_2^s - y_2^s)^2 + \cdots + a_n(x_n^s - y_n^s)^2}{a_1(x_1^r - y_1^r)^2 + a_2(x_2^r - y_2^r)^2 + \cdots + a_n(x_n^r - y_n^r)^2}\Big)^{\frac{1}{2(s-r)}},$$

represents another multi-variable variant of Stolarsky means.

It will be shown in the sequel that both means $e_{r,s}(X_n)$ and $E_{r,s}(A_n; X_n, Y_n)$ are monotone increasing in parameters r and s. An intriguing task is to determine some necessary and sufficient conditions for their monotonicity in n. Although the solution is relatively simple in the second case and reduces to the monotonicity of sequences X_n and Y_n (independently of A_n), this question is much more complicated for the means $e_{r,s}(X_n)$.

For example, means $e_{0,0}(X_n)$ are monotone increasing/decreasing in n if and only if $x_n \gtrless g(z(X_{n-1}), e_{0,0}(X_{n-1})), n \geq 3$, where $z(X_n)$ is the geometric mean of numbers X_n and $g(z_n, e_{0,0}(X_n)) := z_n(e_{0,0}(X_n)/z_n)^{3(n+1)/(n+2)}$.

2. Results and Proofs

Recall that the Jensen functional $J_n(p, x; f)$ is defined on an interval $I \subseteq \mathbb{R}$ by

$$J_n(p, x; f) := \sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i),$$

where $f : I \to \mathbb{R},\ x = (x_1, x_2, \cdots, x_n) \in I^n$ and $p = \{p_i\}_1^n$ is a positive weight sequence.

Another well known assertion is the following

Jensen's inequality *If f is twice continuously differentiable and $f'' \geq 0$ on an interval I, then f is convex on I and the inequality*

$$J_n(p, x; f) = \sum_1^n p_i f(x_i) - f(\sum_1^n p_i x_i) \geq 0$$

holds for each $x := (x_1, ..., x_n) \in I^n$ and any positive weight sequence $p := \{p_i\}_1^n$ with $\sum_1^n p_i = 1$.

The next two properties of Jensen functionals will be of importance in the sequel.

Theorem 1. *([16,17]) Let $f, g : I \to \mathbb{R}$ be twice continuously differentiable functions. Assume that g is strictly convex and ϕ is a continuous and strictly monotone function on I.*

Then the expression

$$\phi^{-1}\Big(\frac{J_n(p, x; f)}{J_n(p, x; g)}\Big),\ \ (n \geq 2),$$

represents a mean value of the numbers $x_1, \cdots, x_n \in I$, that is

$$\min\{x_1, \cdots, x_n\} \leq \phi^{-1}\Big(\frac{J_n(p, x; f)}{J_n(p, x; g)}\Big) \leq \max\{x_1, \cdots, x_n\},$$

if and only if the relation

$$f''(t) = \phi(t) g''(t)$$

holds for each $t \in I$.

Theorem 2. *([18]) Let f_s be a twice continuously differentiable function on the interval $J := (c,d)$ for each parameter $s \in I := (a,b)$. If $s \to f_s''(x)$ is log-convex on I for each $x \in J$, then the expression*

$$s \to \Phi_f(w,x;s) := \sum_1^n w_i f_s(x_i) - f_s(\sum_1^n w_i x_i),$$

is log-convex on I for each $x := (x_1, ..., x_n) \in J^n$, where $w = \{w_i\}_1^n$ is any positive weight sequence.

Lemma 1. *A function F is convex on an interval I if and only if the ratio*

$$\frac{F(s) - F(r)}{s - r}$$

is monotone increasing in both r and s for $r, s \in I$.

In the following two theorems we shall prove that our expressions $e_{r,s}(X_n)$ and $E_{r,s}(A_n; X_n, Y_n)$, extended to the whole (r,s) plane, are actually means which preserve all main properties of the ordinary Stolarsky means and coincide with them for $n = 2$.

Theorem 3. *Let,*

$$e_{r,s}(x_1, x_2, ..., x_n) = \begin{cases} \left(\frac{r^2(\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s)}{s^2(\sum_1^n x_i^{nr} - n(\prod_1^n x_i)^r)}\right)^{1/(n(s-r))} & , rs(s-r) \neq 0; \\ \left(\frac{2}{ns^2}\frac{\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s}{n\sum_1^n \log^2 x_i - (\sum_1^n \log x_i)^2}\right)^{1/(ns)} & , r = 0, s \neq 0; \\ \exp\left(\frac{-2}{ns} + \frac{\sum_1^n x_i^{ns}\log x_i - (\sum_1^n \log x_i)(\prod_1^n x_i)^s}{\sum_1^n x_i^{ns} - n(\prod_1^n x_i)^s}\right) & , r = s \neq 0; \\ \exp\left(\frac{n^2\sum_1^n \log^3 x_i - (\sum_1^n \log x_i)^3}{3n(n\sum_1^n \log^2 x_i - (\sum_1^n \log x_i)^2)}\right) & , r = s = 0. \end{cases}$$

Then

1. *Expressions $e_{r,s}(X_n)$ are means, that is,*

$$\min\{x_1, x_2, ..., x_n\} \leq e_{r,s}(x_1, x_2, ..., x_n) \leq \max\{x_1, x_2, ..., x_n\}.$$

2. *$e_{r,s}(X_n)$ are symmetric in parameters r and s i.e., $e_{r,s}(X_n) = e_{s,r}(X_n)$.*
3. *$e_{r,s}(X_n)$ are symmetric in all variables.*
4. *$e_{r,s}(X_n)$ are homogeneous of order one.*
5. *$e_{r,s}(X_n)$ are monotone increasing in both parameters r and s.*
6. *$e_{r,s}(x_1, x_2) = E_{r,s}(x_1, x_2)$.*

Proof. Note that the properties 2, 3 and 4 are evident and can be proved directly.

We apply Theorem A for the proof of Property 1.

Namely, choose that $g = f_r(y)$ and

$$f = f_s(y) := \begin{cases} (e^{sy} - sy - 1)/s^2 & , s \neq 0; \\ y^2/2 & , s = 0. \end{cases}$$

The conditions of Theorem A are fulfilled with

$$f''(y) = e^{sy},\ g''(y) = e^{ry},\ \phi(y) = e^{(s-r)y},\ \phi^{-1}(y) = \frac{1}{s-r}\log y,$$

for $r \neq s$.

Therefore, with $p_i = 1/n$, we obtain

$$\min\{y_i\}_1^n \le \frac{1}{s-r}\log\Big(\frac{r^2}{s^2}\frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{\sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}}\Big) \le \max\{y_i\},$$

that is,

$$e^{\min\{y_i\}_1^n} \le \Big(\frac{r^2}{s^2}\frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{\sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}}\Big)^{1/(s-r)} \le e^{\max\{y_i\}_1^n}.$$

In the case $r = 0, s \neq 0$, we have

$$f''(y) = e^{sy},\ g''(y) = 1,\ \phi(y) = e^{sy},\ \phi^{-1}(y) = \frac{1}{s}\log y.$$

Hence,

$$e^{\min\{y_i\}_1^n} \le \Big(\frac{2n}{s^2}\frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{n\sum_1^n y_i^2 - (\sum_1^n y_i)^2}\Big)^{1/s} \le e^{\max\{y_i\}_1^n}.$$

Now, change of variables $e^{y_i} = x_i,\ s \to ns,\ r \to nr$, evidently leads to the desired results. □

For the proof of Property 5. we shall use Theorem B.

By the function $f_s(y)$ defined above, we have that $f_s''(y) = e^{sy}$ is log-convex for $s \in \mathbb{R}$.

Hence, by Theorem B we obtain that the form

$$F(s) = \frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{ns^2},$$

is log-convex on $\mathbb{R}$.

Since a positive function is log-convex on I if its logarithm is convex on I, applying Lemma 1 we have that the form

$$\frac{\log F(s) - \log F(r)}{s-r} = \log\Big(\frac{r^2}{s^2}\frac{\sum_1^n e^{sy_i} - ne^{(\sum_1^n y_i)s/n}}{\sum_1^n e^{ry_i} - ne^{(\sum_1^n y_i)r/n}}\Big)^{1/(s-r)},$$

is monotone increasing in both r and s.

The same change of variables $e^{y_i} = x_i,\ s \to ns,\ r \to nr$, proves the validity of Property 5. □

Finally, for the Property 6. of Theorem 3, we have

$$e_{r,s}(x_1,x_2) = \Big(\frac{r^2}{s^2}\frac{x_1^{2s} + x_2^{2s} - 2(x_1x_2)^s}{x_1^{2r} + x_2^{2r} - 2(x_1x_2)^r}\Big)^{1/(2(s-r))}$$

$$= \Big(\frac{r^2}{s^2}\frac{(x_1^s - x_2^s)^2}{(x_1^r - x_2^r)^2}\Big)^{1/(2(s-r))} = \Big|\frac{r}{s}\frac{(x_1^s - x_2^s)}{(x_1^r - x_2^r)}\Big|^{1/(s-r)} = E_{r,s}(x_1,x_2).$$

□

Theorem 4. *Let,*

$$E_{r,s}(A_n;X_n,Y_n) = \begin{cases} \Big(\frac{r^2(\sum_1^n a_i(x_i^s - y_i^s)^2}{s^2(\sum_1^n a_i(x_i^r - y_i^r)^2}\Big)^{1/(2(s-r))} & ,rs(s-r) \neq 0; \\ \Big(\frac{\sum_1^n a_i(x_i^s - y_i^s)^2}{s^2\sum_1^n a_i(\log x_i - \log y_i)^2}\Big)^{1/(2s)} & ,r = 0, s \neq 0; \\ \exp\Big(\frac{-1}{s} + \frac{\sum_1^n a_i(x_i^s - y_i^s)(x_i^s\log x_i - y_i^s\log y_i)}{\sum_1^n a_i(x_i^s - y_i^s)^2}\Big) & ,r = s \neq 0; \\ \exp\Big(\frac{\sum_1^n a_i(\log x_i - \log y_i)(\log^2 x_i - \log^2 y_i)}{2\sum_1^n a_i(\log x_i - \log y_i)^2}\Big) & ,r = s = 0. \end{cases}$$

Then

1. *Functions $E_{r,s}(A_n; X_n, Y_n)$ are means.*
2. *Means $E_{r,s}(A_n; X_n, Y_n)$ are symmetric in parameters r and s.*
3. *Means $E_{r,s}(A_n; X_n, Y_n)$ are symmetric in variables, that is, $E^n_{r,s}(A_n; X_n, Y_n) = E^n_{r,s}(A_n; Y_n, X_n)$.*
4. *Means $E_{r,s}(A_n; X_n, Y_n)$ are homogeneous of order one.*
5. *$E_{r,s}(A_n; X_n, Y_n)$ are monotone increasing in both parameters r and s.*
6. *$E_{r,s}(a_1; x_1, y_1) = E_{r,s}(x_1, y_1)$.*

Remark 2. *We assume that there exists i, $1 \le i \le n$, such that $x_i \ne y_i$.*

Proof. Properties 2, 3, 4 and 6 are self-evident. For the rest of the proof we can assume that $x_i > y_i, i = 1, 2, ..., n$. Otherwise, we put $x_i \in Y_n, y_i \in X_n$.

Furthermore, because of symmetry, we take $s \ge r$.

To prove Property 1, note that from the definition of Stolarsky means, for $s > r \ne 0$ and each $i = 1, 2, ..., n$, the bounds

$$y_i \le \Big(\frac{r(x_i^s - y_i^s)}{s(x_i^r - y_i^r)}\Big)^{1/(s-r)} \le x_i.$$

are known.

Hence,

$$(s(x_i^r - y_i^r))^2 y_i^{2(s-r)} \le (r(x_i^s - y_i^s))^2 \le (s(x_i^r - y_i^r))^2 x_i^{2(s-r)},$$

and

$$s^2 \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 y_i^{2(s-r)} \le r^2 \sum_{i=1}^n a_i (x_i^s - y_i^s)^2 \le s^2 \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 x_i^{2(s-r)},$$

wherefrom one easily obtains that

$$s^2 (\min\{y_i\})^{2(s-r)} \sum_{i=1}^n a_i (x_i^r - y_i^r)^2 \le r^2 \sum_{i=1}^n a_i (x_i^s - y_i^s)^2 \le s^2 (\max\{x_i\})^{2(s-r)} \sum_{i=1}^n a_i (x_i^r - y_i^r)^2,$$

i.e.,

$$\min\{y_i\} \le \Big(\frac{r^2(\sum_1^n a_i (x_i^s - y_i^s)^2}{s^2(\sum_1^n a_i (x_i^r - y_i^r)^2}\Big)^{1/2(s-r)} \le \max\{x_i\}, \ \ i = 1, 2, ..., n.$$

The other cases follow simultaneously as a results of limit processes inside the definite fixed bounds.

For example, for $r = s = 0$, we have

$$\begin{aligned} E_{0,0}(A_n; X_n, Y_n) &= \exp\Big(\frac{\sum_1^n a_i (\log x_i - \log y_i)(\log^2 x_i - \log^2 y_i)}{2\sum_1^n a_i (\log x_i - \log y_i)^2}\Big) \\ &= \exp\Big(\frac{\sum_1^n a_i (\log x_i - \log y_i)^2 \log(\sqrt{x_i y_i})}{\sum_1^n a_i (\log x_i - \log y_i)^2}\Big), \end{aligned}$$

and applying the inequality

$$\min\{y_i\} \le y_i \le \sqrt{x_i y_i} \le x_i \le \max\{x_i\},$$

we obtain

$$\min\{y_i\} \le E_{0,0}(A_n; X_n, Y_n) \le \max\{x_i\}, \ \ i = 1, 2, ..., n.$$

Therefore, the fact that expressions $E_{r,s}(A_n; X_n, Y_n)$ are means is proved. □

For the proof of Property 5., let us recall some basic facts from Convexity Theory.

A function f is convex on an interval I if it is continuous on I and it is Jensen convex on I, that is for all $x, y \in I$,

$$\frac{f(x)+f(y)}{2} \geq f\Big(\frac{x+y}{2}\Big).$$

Lemma 2. *A positive function g is log-convex on an interval I if it is continuous on I and the inequality*

$$\alpha^2 g(s) + 2\alpha\beta g(\frac{s+t}{2}) + \beta^2 g(t) \geq 0,$$

holds for all $\alpha, \beta \in \mathbb{R}$ and $s, t \in I$.

Proof. The above inequality holds for all $\alpha, \beta \in \mathbb{R}$ if and only if

$$g(s)g(t) \geq g^2(\frac{s+t}{2}),$$

that is

$$\frac{\log g(s) + \log g(t)}{2} \geq \log g\Big(\frac{s+t}{2}\Big).$$

This means that $\log \circ g$ is convex in the Jensen sense, and hence the continuity of g implies that it is log-convex.

□

Lemma 3. *Let the function $h(x, y; s), x > y > 0$, be defined as*

$$h(s) = h(x, y; s) := \begin{cases} \frac{x^s - y^s}{s} & , s \neq 0; \\ \log(x/y) & , s = 0. \end{cases}$$

Then $h(s)$ is log-convex on $s \in \mathbb{R}$.

Proof. Indeed, $h(s)$ is continuous on $s \in \mathbb{R}$ and the inequality

$$\alpha^2 h(s) + 2\alpha\beta h(\frac{s+t}{2}) + \beta^2 h(t) \geq 0$$

holds, because

$$\alpha^2 h(s) + 2\alpha\beta h(\frac{s+t}{2}) + \beta^2 h(t)$$

$$= \alpha^2 \int_y^x u^{s-1} du + 2\alpha\beta \int_y^x u^{\frac{s+t}{2}-1} du + \beta^2 \int_y^x u^{t-1} du$$

$$= \int_y^x (\alpha u^{s/2} + \beta u^{t/2})^2 u^{-1} du.$$

Therefore Lemma 2 can be applied. □

Lemma 4. *If, for positive u, v, w, the inequality*

$$\alpha^2 u + 2\alpha\beta v + \beta^2 w \geq 0,$$

holds for each $\alpha, \beta \in \mathbb{R}$, then also

$$\alpha^2 u^p + 2\alpha\beta v^p + \beta^2 w^p \geq 0,$$

holds for each $p > 0$.

Proof. Obvious.
□

Now we are enabled to prove Property 5. of Theorem 4. For this cause, denote

$$g_n(s) = \sum_1^n a_i h_i^2(s),$$

where $h_i(s) := h(x_i, y_i; s)$ and $a_i,\ i = 1, 2, ..., n$ are positive numbers.

By Lemmas 2 , 3 and 4, we see that $g_n(s)$ is log-convex in $s, s \in \mathbb{R}$, since

$$\alpha^2 g_n(s) + 2\alpha\beta g_n(\frac{s+t}{2}) + \beta^2 g_n(t) = \sum_1^n a_i(\alpha^2 h_i^2(s) + 2\alpha\beta h_i^2(\frac{s+t}{2}) + \beta^2 h_i^2(t)) \geq 0.$$

Therefore the function $F(s) = \log g_n(s)$ is convex and, applying Lemma 1, we obtain that

$$\frac{\log g_n(s) - \log g_n(r)}{s-r} = \log\Big(\frac{g_n(s)}{g_n(r)}\Big)^{\frac{1}{s-r}} = 2\log E_{r,s}(A_n; X_n, Y_n),$$

is monotone increasing in both r and s, which is equivalent with the Property 5 in the case $s > r \neq 0$.

By continuity, the proof of other cases follows immediately. For example, since for any $\epsilon > 0$ we have

$$E_{r+\epsilon, s+\epsilon}(A_n; X_n, Y_n) \geq E_{r,s}(A_n; X_n, Y_n),$$

letting $r \to s$, we obtain

$$E_{s+\epsilon, s+\epsilon}(A_n; X_n, Y_n) \geq E_{s,s}(A_n; X_n, Y_n),$$

that is, $E_{s,s}(A_n; X_n, Y_n)$ is monotone increasing in s.
□

Our task in the sequel is to investigate under what conditions the means $e_{r,s}(X_n)$ and $E_{r,s}(A_n; X_n, Y_n)$ are monotone increasing/decreasing in n.

For this cause we need the following two lemmas.

Lemma 5. *Stolarsky means $E_{r,s}(x, y)$ are monotone increasing in both variables x and y.*

This is the well-known assertion ([1]).

Lemma 6. *For two given sequences $\{u_n\}$ and $\{v_n\}$ of positive numbers, denote*

$$w_n := \frac{u_n}{v_n};\ \ W_n := \frac{u_1 + u_2 + ... + u_n}{v_1 + v_2 + ... + v_n}.$$

If the sequence w_n is monotone decreasing/increasing, then the sequence W_n is also monotone decreasing/increasing.

Proof. Let w_n be a decreasing sequence. The other case can be treated similarly.

We prove firstly that $v_{n+1}\sum_1^n u_i \geq u_{n+1}\sum_1^n v_i$.

Indeed,

$$v_{n+1}\sum_1^n u_i = v_{n+1}\sum_1^n v_i w_i \geq v_{n+1}w_{n+1}\sum_1^n v_i = u_{n+1}\sum_1^n v_i.$$

Hence,

$$\sum_1^n u_i \sum_1^{n+1} v_i = \sum_1^n u_i \sum_1^n v_i + v_{n+1}\sum_1^n u_i \geq \sum_1^n u_i \sum_1^n v_i + u_{n+1}\sum_1^n v_i = \sum_1^n v_i \sum_1^{n+1} u_i,$$

i.e., $W_n \geq W_{n+1}$.

□

Theorem 5. *If both sequences $\{X_n\}$ and $\{Y_n\}$ are monotone decreasing (increasing), then means $E_{r,s}(A_n; X_n, Y_n)$ are monotone decreasing (increasing)in n.*

Proof. We shall prove the "decreasing" part of Theorem 5. The proof of the other part is analogous.

Hence, we assume that both sequences $\{X_n\}$ and $\{Y_n\}$ are monotone decreasing. In the case $s > r \neq 0$, denote

$$u_n := a_n r^2 (x_n^s - y_n^s)^2;\ v_n := a_n s^2 (x_n^r - y_n^r)^2.$$

By Lemma 5, we have

$$w_n = \frac{u_n}{v_n} = \frac{r^2(x_n^s - y_n^s)^2}{s^2(x_n^r - y_n^r)^2} \geq \frac{r^2(x_{n+1}^s - y_{n+1}^s)^2}{s^2(x_{n+1}^r - y_{n+1}^r)^2} = w_{n+1}.$$

Therefore the sequence w_n is monotone decreasing and, by Lemma 6, this implies $W_n \geq W_{n+1}$, that is,

$$\frac{r^2}{s^2}\frac{a_1(x_1^s - y_1^s)^2 + a_2(x_2^s - y_2^s)^2 + \cdots + a_n(x_n^s - y_n^s)^2}{a_1(x_1^r - y_1^r)^2 + a_2(x_2^r - y_2^r)^2 + \cdots + a_n(x_n^r - y_n^r)^2}$$
$$\geq \frac{r^2}{s^2}\frac{a_1(x_1^s - y_1^s)^2 + a_2(x_2^s - y_2^s)^2 + \cdots + a_n(x_n^s - y_n^s)^2 + a_{n+1}(x_{n+1}^s - y_{n+1}^s)^2}{a_1(x_1^r - y_1^r)^2 + a_2(x_2^r - y_2^r)^2 + \cdots + a_n(x_n^r - y_n^r)^2 + a_{n+1}(x_{n+1}^r - y_{n+1}^r)^2}.$$

Since $s > r$, this is equivalent to $E_{r,s}(A_n; X_n, Y_n) \geq E_{r,s}(A_{n+1}; X_{n+1}, Y_{n+1})$.

In the cases $r = s \neq 0, s > 0 = r$ and $r = s = 0$ one should take

$$u_n = a_n(x_n^s - y_n^s)(x_n^s \log x_i - y_n^s \log y_i),\ v_n = a_n(x_n^s - y_n^s)^2;$$
$$u_n = a_n(x_n^s - y_n^s)^2,\ v_n = s^2 a_n(\log x_n - \log y_n)^2;$$
$$u_n = a_n(\log x_n - \log y_n)(\log^2 x_n - \log^2 y_n),\ v_n = 2a_n(\log x_n - \log y_n)^2,$$

respectively, and proceed as above.

□

On the other hand, the problem of monotonicity in n for means $e_{r,s}(X_n)$ seems significantly harder. We are able to solve it only in the simplest case $r = s = 0$.

Theorem 6. *The means $e_{0,0}(X_n)$ are monotone increasing/decreasing in n if and only if*

$$x_n \gtrless z(X_{n-1})(e_{0,0}(X_{n-1})/z(X_{n-1}))^{3n/(n+1)},\ n \geq 3,$$

where $z(X_n)$ denotes the geometric mean of numbers X_n.

Proof. We have

$$\log e_{0,0}(X_n) = \frac{b_n(X_n)}{3a_n(X_n)},$$

with

$$a_n(X_n) = \frac{1}{n}\sum_1^n \log^2 x_i - \Big(\frac{1}{n}\sum_1^n \log x_i\Big)^2 = \frac{1}{n}\sum_1^n \log^2(x_i/z(X_n)) \geq 0,$$

and

$$b_n(X_n) = \frac{1}{n}\sum_1^n \log^3 x_i - \Big(\frac{1}{n}\sum_1^n \log x_i\Big)^3.$$

Note that for $x_n = z(X_{n-1})$ we have $e_{0,0}(X_n) = e_{0,0}(X_{n-1})$. Therefore by Taylor expansion around this point, we obtain

$$a_n(X_n) = \frac{n-1}{n}[a_{n-1}(X_{n-1}) + \frac{1}{n}\log^2(x_n/z(X_{n-1}))],$$

and

$$b_n(X_n) = \frac{n-1}{n}[b_{n-1}(X_{n-1}) + \frac{3\log z(X_{n-1})}{n}\log^2(x_n/z(X_{n-1})) + \frac{n+1}{n^2}\log^3(x_n/z(X_{n-1}))].$$

Since $b_{n-1}(X_{n-1})/3a_{n-1}(X_{n-1}) = \log e_{0,0}(X_{n-1})$, we finally get

$$\log e_{0,0}(X_n) - \log e_{0,0}(X_{n-1}) = \frac{b_n(X_n)}{3a_n(X_n)} - \frac{b_{n-1}(X_{n-1})}{3a_{n-1}(X_{n-1})}$$

$$= \frac{\log^2(x_n/z(X_{n-1}))}{3na_n(X_n)}[3\log z(X_{n-1}) + \frac{n+1}{n}\log(x_n/z(X_{n-1})) - b_{n-1}(X_{n-1})/a_{n-1}(X_{n-1})],$$

and the proof follows.

□

3. Conclusions

In this article we give two explicit generalizations of Stolarsky means to the multi-variable case and proved that they preserve all main properties of the original means. Let us note that other subtle properties are not equally transposed. For example, log-convexity of $E_{r,s}(x_1, x_2)$ entirely depends on parameters r, s ([12]), but in the case of means $B_{r,s}(x_1, x_2, x_3)$, mentioned in the Introduction, it also depends on $x_3 \lesseqgtr \sqrt{x_1 x_2}$.

Furthermore, many open questions can be proposed. For example, is monotone increase of the sequences $\{X_n\}$ and $\{Y_n\}$ necessary for $E_{r,s}(A_n; X_n, Y_n)$ to be increasing in n?

Or, is the monotonicity in variables possible for the means $e_{r,s}(X_n)$ only if $n = 2$ or $n = 3$?

Author Contributions: Theoretical part, S.S.; numerical part with examples, B.B.-M. All authors have read and agreed to the published version of the manuscript.

Funding: Bandar Bin-Mohsin is supported by Researchers Supporting Project number (RSP-2020/158), King Saud University, Riyadh, Saudi Arabia.

Acknowledgments: The authors are grateful to the referees for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interests.

References

1. Stolarsky, K.B. Generalizations of the logarithmic mean. *Math. Mag.* **1995**, *48*, 87–92.

2. Simic, S. Another converse of Jensen's inequality. http://vuir.vu.edu.au/17742/1/simic.pdf (accessed on 20 March 2020).
3. Stolarsky, K.B. The power and generalized logarithmic means. *Amer. Math. Mon.* **1980**, *87*, 545–548.
4. Leach E.B.; Sholander, M.C. Extended mean values. *Amer. Math. Mon.* **1978**, *85*, 84–90.
5. Leach, E.B.; Sholander, M.C. Multi-variable extended mean values. *J. Math. Anal. Appl.* **1984**, *104*, 390–407.
6. Páles, Z. Inequalities for differences of powers. *J. Math. Anal. Appl.* **1988**, *131*, 271–281.
7. Neuman E.; Páles, Z. On comparison of Stolarsky and Gini means. *J. Math. Anal. Appl.* **2003**, *278*, 274–284.
8. Neuman E.; Sándor, J. Inequalities involving Stolarsky and Gini means. *Math. Pannonica* **2003**, *14*, 29–44.
9. Czinder, P.; Páles, Z. An extension of the Hermite-Hadamard inequality and an application for Gini and Stolarsky means. *J. Ineq. Pure Appl. Math.* **2004**, *5*, 42.
10. Witkowski, A. Weighted extended mean values. *Colloq. Math.* **2004**, *100*, 111–117.
11. Witkowski, A. Convexity of weighted Stolarsky means. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 73.
12. Qi, F. Logarithmic convexity of extended mean values. *Proc. Amer. Math. Soc.* **2001**, *130*, 1787–1796.
13. Merikoski, J. K. Extending means of two variables to several variables. *J. Ineq. Pure Appl. Math.* **2004**, *5*, 65.
14. Pečarić, J.; Šimić, V. Stolarsky-Tobey mean in n variables. *Math. Ineq. Appl.* **1999**, *2*, 325–334.
15. Horvath, I.; Pečarić, J. New versions of weighted multidimensional functional and Stolarsky means. *Acta Math. Hung.* **2015**, *147*, 81–96.
16. Simic, S. On a class of mean values including Jensen functionals. *Publ. Math. Debrecen* **2008**, *73*, 461–469.
17. Simic, S. An extension of Stolarsky means to the multivariable case. *Int. J. Math. Math. Sci.* **2009**, 32857.
18. Simic, S. On certain new inequalities in Information Theory. *Acta Math. Hungar.* **2009**, *124*, 353–361.