$f_b(z) = (z+b)/(1+b)$. In the case $k = 3$, our corresponding surfaces are circular cones with vertex $(0,0,-b)$ and containing the point $(1,0,1)$. This determines the mean

$$M_{f_b} = \frac{\sum \frac{z_i}{z_i + b}}{\sum \frac{1}{z_i + b}}.$$

For $b = 0$ this is the harmonic mean and $\lim_{b\to\infty} M_{f_b}$ is the arithmetic mean. A similar example using the function $f(z) = (1+b)/(z+b)$ yields for $b = 0$ the contraharmonic mean and $\lim_{b\to\infty} M_{f_b}$ is the arithmetic mean.

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GINI MEANS

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Corrado Gini (1883–1965), a prolific author who founded the journal of statistics, *Metron*, defined an interesting class of means. For data $X = \langle x_i | i = 1,2,\dots,n\rangle$, the Gini family of means is defined, for parameters r and s, as

$$G(r,s;X) = (\Sigma x_i^{r+s}/\Sigma x_i^s)^{1/r}\ [6].$$

The family includes the so-called power means $P(r;X) = (\Sigma x_i^r/n)^{1/r}$ (for $s = 0$), the contraharmonic mean (for $r = s = 1$), and the self-weighting means $\Sigma x_i^{1+s}/\Sigma x_i^s$ (for $r = 1$). We call the last self-weighting because they are of the form of a weighted average $\Sigma x_i^s x_i/\Sigma x_i^s$ where the weights x_i^s are generated by the data. In Sections 1 and 2, we discuss some fundamental properties of Gini means and their relation to power means. In Sections 3 and 4, we provide a simple geometric construction for some of the Gini means including the self-weighting means for integer s. We conclude, in Section 5, with a simple construction for five power means.

1. Some properties of the Gini means. The Gini means $G(r,s;X)$ for $x_i > 0$ share some of the standard properties which are associated with averages. For example, $\min\{x_i\} \leqslant G(r,s;X) \leqslant \max\{x_i\}$, that is, they satisfy the betweenness property. Also, if all $x_i = a$, then $G(r,s;X) = a$, the identity property. $G(r,s;tX) = tG(r,s;X)$; so, they are homogeneous of degree one. The Gini means are symmetric in that, if Y is a permutation of X, then $G(r,s;X) = G(r,s;Y)$.

The Gini means may differ from ordinary averages in several respects. Most of the ordinary

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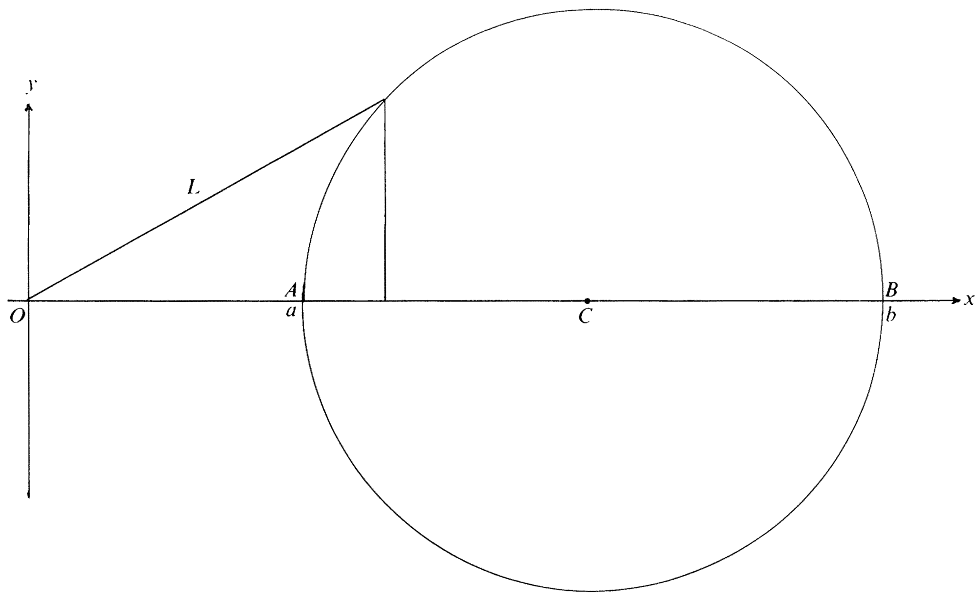

FIG. 1. $x^2 + y^2 - (a + b)x + ab = 0$.

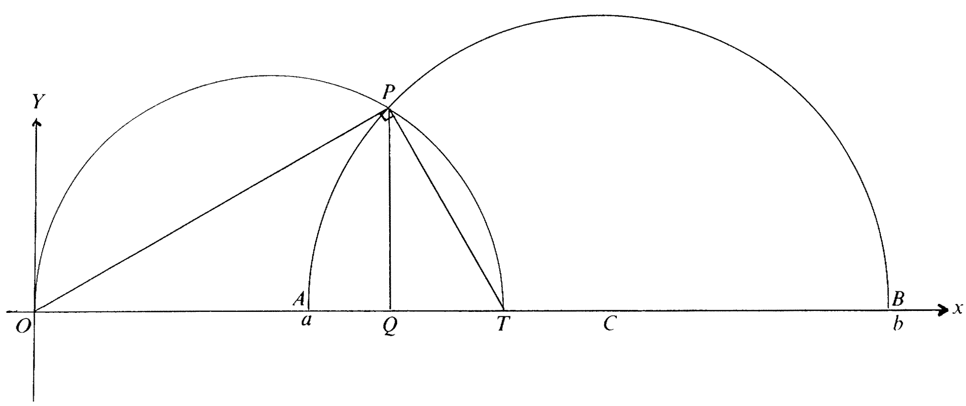

FIG. 2. Construction of T from Q and Q from T.

averages satisfy the substitution rule: if several data values are replaced by their average, the overall average is unchanged. In general, the Gini means do not satisfy this rule. For example, for the contraharmonic mean of $1, 2, 3$, we have $G(1,1;1,2,3) = 7/3$; the average for the data $1, 2$ is $G(1,1;1,2) = 5/3$, and $G(1,1;5/3,5/3,3) = 131/57 \neq 7/3$.

Commonly used averages are monotonic functions of the data values: the power means increase with each x_i, and the median is not decreasing. It is known that for any number of positive data, the Gini means with $r = 1$ are increasing for $-1 \leqslant s \leqslant 0$ and for certain parts of their domain for other s [**1**].

For the contraharmonic mean $(a^2 + b^2)/(a + b)$, let b be fixed and let a vary. For $0 < a \leqslant b$, one has a convex function of a. As a increases from 0 to $(\sqrt{2} - 1)b$, the mean decreases from b to $2(\sqrt{2} - 1)b \approx 0.828\ b$. Then, as a increases from $(\sqrt{2} - 1)b$ to b, the mean increases to b. Thus, not only is the mean not monotonic, but also the Gini means can be very insensitive to certain data.

2. Relationship with the power means. The classical power means are $P(r;\,X) = (\Sigma x_i^r/n)^{1/r}$ [8]. The power means are monotonic in each variable. As mentioned, the Gini means include the power means $G(r,0;\,X) = P(r;\,X)$. The lack of monotonicity exhibited in Section 1 shows that not all Gini means are power means.

It is well known that the power means with $r = 0$ are the unique numbers c which minimize $\Sigma(x_i^r - c^r)^2$ [1]. That the Gini means $G(r,s;\,X)$ are the unique numbers c which minimize $\Sigma x_i^s(x_i^r - c^r)^2$ can be shown by differentiating with respect to c.

The subset of Gini means with $r = 1$, that is, our $G(1,s;\,X)$, includes for $s = 0$ the arithmetic mean $P(1;\,X)$ and for $s = -1$ the harmonic mean $P(-1;\,X)$. These two are the only means common to the two classes $G(1,s;\,X)$ and $P(r;\,X)$ [7], [11]. For data sets consisting of just two values, $G(1,-\frac{1}{2};\,a,b) = P(0;\,a,b)$, the geometric mean which is defined as $\lim_{r\to 0} P(r;\,X)$.

In the next section, we provide a geometric construction for $G(1,s;\,X)$ with s any integer.

3. A geometric construction for some Gini means. We provide a geometric construction for the self-weighting Gini means ($r = 1$)

$$G(1,s;\,a,b) = (a^{s+1} + b^{s+1})/(a^s + b^s),$$

where s is any integer, by extending a construction of Schoenberg [14]. For fixed a, b, let us simply denote this by $G(s)$. Accordingly, let a, b be unequal positive real numbers with $b > a$ ($a = b$ is a trivial case) and consider the circle on AB as diameter where A and B are the points $(a,0)$ and $(b,0)$ in a cartesian plane (Fig. 1). The circle has center $C(\frac{1}{2}(a+b),0)$, radius $\frac{1}{2}(b-a)$, and its equation is

$$\left(x - \frac{1}{2}(a+b)\right)^2 + y^2 = \left(\frac{1}{2}(b-a)\right)^2$$

or

$$x^2 + y^2 - (a+b)x + ab = 0.$$

For the point (x, y) on the circle, the length L of the hypotenuse of the triangle with vertices $(0,0)$, (x,y), and $(x,0)$ is given by

$$L^2 = x^2 + y^2 = (a+b)x - ab.$$

Referring to Fig. 2, suppose, at any point Q on the x-axis between A and B, a line perpendicular to the x-axis is constructed to intersect the circle at P, and the line through P, perpendicular to OP, is constructed to intersect the x-axis at a new point T. In semicircle APB, PB is perpendicular to PA. Since the inclination of OP is not as great as that of AP, then PT, which is perpendicular to OP, must place T on the x-axis between A and B. Conversely, for any point T on the x-axis between A and B, the circle with diameter OT can be constructed to intersect the original circle at P, from which a perpendicular to the x-axis gives Q. Hence, from the position of either Q or T, the other can be constructed.

From the proportional sides of similar right triangles OPQ and OPT, we have

$$OQ \cdot OT = OP^2. \tag{1}$$

For a point $P(x,y)$ on the circle with diameter AB we have

$$OP^2 = x^2 + y^2 = (a+b)x - ab = (a+b)OQ - ab,$$

and we obtain

$$OQ \cdot OT = (a+b)OQ - ab.$$

Now, if OQ happened to have the value $G(s)$, it would follow that

$$OT = (a+b) - \frac{ab}{OQ} = a + b - ab\left(\frac{a^s + b^s}{a^{s+1} + b^{s+1}}\right)$$

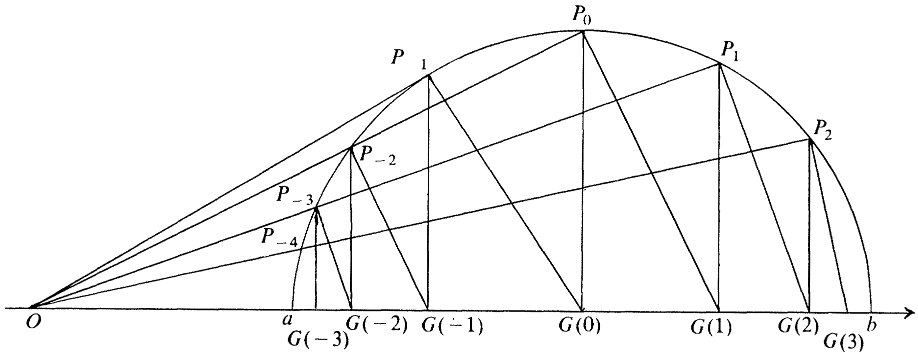

FIG. 3. Iterating to obtain the Gini means $G(s)$ for s between -3 and 3.

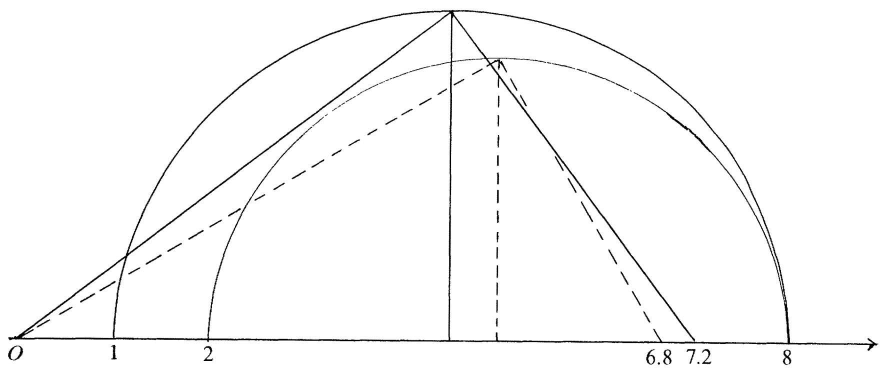

FIG. 4. Increasing a data value may decrease a Gini mean.

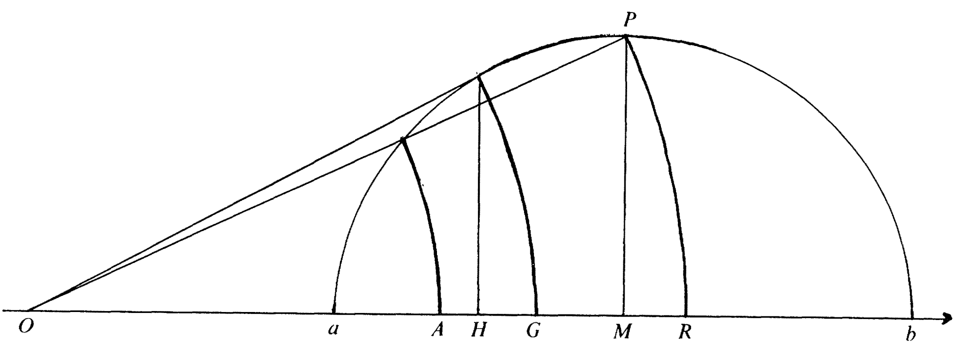

FIG. 5. The power means $((a^r + b^r)/2)^{1/r}$ along the horizontal axis for $r = -2, -1, 0, 1, 2$.

$$= \frac{a^{s+2} + b^{s+2}}{a^{s+1} + b^{s+1}} = G(s+1).$$

Conversely, for $OT = G(s)$, we obtain $OQ = G(s-1)$.

Observe that for Q at the center C, we have $OQ = \frac{1}{2}(a+b) = G(0)$. Therefore, starting with Q at C and iterating our construction for T, we obtain points on the x-axis whose abscissae are $G(1), G(2), G(3), \ldots$; similarly, beginning with T at C and iterating the reverse construction for the point Q, we obtain points having abscissae $G(-1), G(-2), G(-3), \ldots$. See Fig. 3.

For an integer s, then, we have that $G(s) < G(s+1)$ and that $G(s)$ is a value between a and

b, that is, $G(s)$ has the betweenness property. More generally for data set X, the class of Gini means $G(1, s; X)$ is an increasing function of s. This can also be shown using calculus methods **[1]**. Note, in addition, that the class of power means $P(r; X) = G(r, 0; X)$ is an increasing function of r **[8]**.

The symmetry of Fig. 3 leads to an unexpected relationship; it suggests $G(s) + G(-s) = 2G(0)$ which quickly can be confirmed algebraically.

The construction above also yields interesting results when applied to other choices for OQ. For example, if $OQ = ta + (1 - t)b$, then

$$OT = \frac{ta^2 + (1 - t)b^2}{ta + (1 - t)b},$$

a weighted Gini mean.

4. Further constructions. In Fig. 3 the distances OP_s, which we denote by $L(s)$, are Gini means, too. Equation (1) reveals that $L(s)$ is the geometric mean of $G(s)$ and $G(s + 1)$, which gives

$$L(s) = G(2, s; a, b) = \left(\frac{a^{s+2} + b^{s+2}}{a^s + b^s}\right)^{1/2}.$$

Because the geometric mean satisfies the betweenness property, $G(s)$ and $L(s)$ interlace, that is,

$$(2) \qquad \cdots < L(s - 1) < G(s) < L(s) < G(s + 1) < L(s + 1) < \cdots$$

and $a < L(s) < b$.

The lack of monotonicity can be displayed by the construction. In Fig. 4 it is shown that $G(1, 1; 2, 8) = 6.8 < 7.2 \approx G(1, 1; 1, 8)$. Increasing the first data value from 1 to 2 decreases the mean in this example.

5. An interesting special case. Five power means are produced by the Gini-mean construction. As in Section 3, $G(0) = M$, the arithmetic mean, and $G(-1) = H$, the harmonic mean. As in Section 4, $L(0) = R$, the root-mean-square, $L(-1) = G$, the geometric mean, and $L(-2) = A$, the harmonic-root-mean, $(\Sigma x_i^{-2}/n)^{-1/2}$. The interlacing property (set $s = -1$ in (2)) gives $A < H < G < M < R$. Fig. 5 displays all five means. For constructions with a similar goal, see **[2]**, **[3]**, **[4]**, **[5]**, **[9]**, **[10]**, **[12]**, **[13]**, **[14]**, and **[15]**.

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