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Fundamentals of SIGLA, an Interval Computing System over the Completed Set of Intervals*

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Abstract — Zusammenfassung

Fundamentals of SIGLA, an Interval Computing System over the Completed Set of Intervals. The SIGLA System has been designed to obtain an Interval Computing System working with the whole surrounding structure of the standard interval set $I(R)$ (see [9]), that is, the lattice completion of $I(R)$ with the ordinary algebraic operations extended onto it. A statistical component associating intervals to Bounded Fuzzy Distributions is added. Rounding semantics on $I^*(R)$ is considered, and bounding theorems directing programming are given. Improper intervals are shown to mean "control-band-like variables", in contrast to the classical "autonomous random variables" represented by proper intervals. An outline of the SIGLA language is given.

Grundlagen von SIGLA, einem Programmsystem für Intervalle über der verbandstheoretischen Vervollständigung der Intervallmenge. Das SIGLA-System wurde entworfen, um ein Programmsystem für Intervalle zu erhalten, das mit den erweiterten Strukturen der Intervallmenge $I(R)$ versehen ist (siehe [9]), d. h. mit der verbandstheoretischen Vervollständigung von $I(R)$ und den darauf erweiterten algebraischen Operationen. Eine statistische Komponente, die Intervallen beschränkte Fuzzy-Distributionen zuordnet, wurde hinzugefügt. Die Regeln für das gerundete Rechnen von I^* (R) werden betrachtet und Inklusions-Theoreme (für die Programmierung) werden angegeben. Es wird gezeigt, daß im Gegensatz zu den eigentlichen Intervallen, die durch autonome Zufallsvariable dargestellt werden, die uneigentlichen Intervalle „control-band-like variables" entsprechen. Ein Auszug der SIGLA-Sprache wird angegeben.

1. The I* (R) Set

The structure over $I(R):=\{[a,b]\mid a,b\in R; a\leq b\}$ determined by $\subseteq$, $\vee$, $\wedge$ (see [9], [15]) is completed to $I^*(R):=\{[a,b]\mid a,b,\in R\}$ by dropping the condition $a\leq b$. That way, Improper Intervals $[a,b]$, $a>b$, are added to $I(R)$. Now, if $\{A_i\mid i\in I\}$ is a bounded family of $I^*(R)$ elements, the operations

$$\vee\ \{A_i\mid i\in I\}:=\sup_{\subseteq}\ \{A_i\mid i\in I\}=[\inf_{\leq}\ \{\underline{a}_i\mid i\in I\},\ \sup_{\leq}\ \{\bar{a}_i\mid i\in I\}],$$

$$\wedge\ \{A_i\mid i\in I\}:=\inf_{\subseteq}\ \{A_i\mid i\in I\}=[\sup_{\leq}\ \{\underline{a}_i\mid i\in I\},\ \inf_{\leq}\ \{\bar{a}_i\mid i\in I\}],$$

are ever defined on $I^*(R)$, from R being a conditionally complete lattice for the $\leq$-order relation.

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Addition and multiplication by real numbers can be defined upon $I^*(R)$ by

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] := [\underline{a} + \underline{b}, \bar{a} + \bar{b}];$$
$$\lambda [a, b] := \textit{if } \lambda \in R_+ \textit{ then } [\lambda a, \lambda b], \textit{ else } [\lambda b, \lambda a]; \text{ (see [7])}.$$

Then, the equation $A + X = B$ has ever a unique solution over $I^*(R)$, and the additive inverse of any element $A \in I^*(R)$, opp A, exists and holds

$$\begin{aligned}
&A + \operatorname{opp} A = \underset{\cdot}{0} \Rightarrow \operatorname{opp} [a, b] = [-a, -b];\\
&\operatorname{opp} (\operatorname{opp} A) = A;\\
&\operatorname{opp} (-A) = -\operatorname{opp} A;\\
&(\lambda \in R),\ \operatorname{opp} (\lambda A) = \lambda \operatorname{opp} A;\\
&\operatorname{opp} \underset{\cdot}{A} = -\underset{\cdot}{A}.
\end{aligned}$$

Let $A = [a, b] \in I^*(R)$. From the definitions (see [12])

$$\begin{aligned}
&\sigma :\in \{a, b \mid |\sigma| = \max(|a|, |b|)\};\\
&\iota :\in \{a, b\} \setminus \{\sigma\};\\
&\chi := \textit{if } \sigma \neq 0 \textit{ then } \iota/\sigma, \textit{ else} \text{ not defined}.
\end{aligned}$$

We find that proper intervals may be written $A = \underset{\cdot}{a} \vee \underset{\cdot}{b} = \underset{\cdot}{\sigma} \vee \underset{\cdot}{\iota} = \sigma \cdot (\underset{\cdot}{1} \vee \underset{\cdot}{\chi})$, and improper intervals $A = \underset{\cdot}{a} \wedge \underset{\cdot}{b} = \underset{\cdot}{\sigma} \wedge \underset{\cdot}{\iota} = \sigma \cdot (\underset{\cdot}{1} \wedge \underset{\cdot}{\chi})$. These reduced expressions for the elements of $I^*(R)$, suggest the element-to-element symmetry dual $[a, b] := [b, a]$. Then, if we write duality upon the trees of the lattice expressions by dual $f(\vee, \wedge; x_1, \ldots, x_n) = f(\wedge, \vee; x_1, \ldots, x_n)$, we can state the fundamental result: for $A_1, \ldots, A_n, A \in I^*(R)$,

$$A = f(\vee, \wedge; A_1, \ldots, A_n) \Leftrightarrow \operatorname{dual} A = f(\wedge, \vee; \operatorname{dual} A_1, \ldots, \operatorname{dual} A_n).$$

Now, if $\omega, \bar{\omega} \in \{\vee, \wedge\}$ with $\bar{\omega} = \operatorname{dual} \omega$ the expression of the basic symmetries $-()$, opp $()$, dual $()$ in the reduced form is

$$\begin{aligned}
A &= \quad [a, b] \quad = \quad \sigma\, \omega\, \iota \quad = \sigma (1\, \omega\, \chi);\\
-A &= [-b, -a] = (-\sigma)\, \omega\, (-\iota) = (-\sigma)(1\, \omega\, \chi);\\
\operatorname{opp} A &= [-a, -b] = (-\sigma)\, \bar{\omega}\, (-\iota) = (-\sigma)(1\, \bar{\omega}\, \chi);\\
-\operatorname{opp} A &= \quad [b, a] \quad = \quad \sigma\, \bar{\omega}\, \iota \quad = \sigma (1\, \bar{\omega}\, \chi);
\end{aligned}$$

The set id $()$, $-()$, opp $()$, dual $()$ multiply according the classical quaternion structure.

Properties:

1. The basic symmetries are commutable among them and with the product by real numbers.

   Inclusion properties:

2. $A \subseteq B \Rightarrow$ a) $-A \subseteq -B$;
   b) $\operatorname{dual} A \supseteq \operatorname{dual} B$;
   c) $\operatorname{opp} A \supseteq \operatorname{opp} B$;
   d) $\lambda A \subseteq \lambda B,\ \lambda \in R$.

3. $A_1 \subseteq A_2\ \&\ B_1 \subseteq B_2 \Rightarrow$ a) $A_1 + B_1 \subseteq A_2 + B_2$;
   b) $A_1 \vee B_1 \subseteq A_2 \vee B_2$;
   c) $A_1 \wedge B_1 \subseteq A_2 \wedge B_2$.

$\leq$-properties:

Definition: $A \leq B \Leftrightarrow \underline{a} \leq \underline{b}$ & $\bar{a} \leq \bar{b}$, extended from $I(R)$ alike to the inclusion relation.

4. $A \leq B \Rightarrow$ a) $-A \geq -B$;
   b) $\operatorname{dual} A \leq \operatorname{dual} B$;
   c) $\operatorname{opp} A \geq \operatorname{opp} B$;
   d) $\lambda \in R_{+}$, $\lambda A \leq \lambda B$.

5. $A_1 \leq A_2$ & $B_1 \leq B_2 \Rightarrow$ a) $A_1 + B_1 \leq A_2 + B_2$;
   b) $A_1 \vee B_1 \leq A_2 \vee B_2$;
   c) $A_1 \wedge B_1 \leq A_2 \wedge B_2$.

Distributivity properties:

6. The basic symmetries and the product by real numbers, are distributive for the addition.

7. Let $\omega, \bar{\omega} \in \{\vee, \wedge\}$, $\bar{\omega} = \operatorname{dual} \omega$ and $\lambda \in R$;
   a) $-(A \,\omega\, B) = (-A)\,\omega\,(-B)$;
   b) $\operatorname{dual}(A \,\omega\, B) = \operatorname{dual} A \,\bar{\omega}\, \operatorname{dual} B$;
   c) $\operatorname{opp}(A \,\omega\, B) = \operatorname{opp} A \,\bar{\omega}\, \operatorname{opp} B$;
   d) $\lambda(A \,\omega\, B) = (\lambda A)\,\omega\,(\lambda B)$.

8. From the addition isotony on R, the dual properties follow:
   a) $A + (B \vee C) = (A + B) \vee (A + C)$;
   b) $A + (B \wedge C) = (A + B) \wedge (A + C)$.

9. From being $(R, \leq)$ a distributive lattice, it follows that $(I^*(R), \subseteq)$ is distributive too:
   a) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$;
   b) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.

Remarks:

10. The inner (algebraic) difference (solution of the equation $A + X = B$) is naturally defined on $I^*(R)$ by

$$A \div B := A - \operatorname{dual} B, \text{ with } A \div A = \underset{\cdot}{0}.$$

That allows to look at $I^*(R)$ as the displacements set of $I(R)$.

11. By extending to $I^*(R)$ the ordinary definition of the distance on $I(R)$,

$$\rho(A, B) := \max(|\underline{a} - \underline{b}|, |\bar{a} - \bar{b}|),$$

the ε-neighbourhood of any $A \in I^*(R)$ is the A-translation of the ε-neighbourhood of $\underset{\cdot}{0}$.

12. The property of being $(R, \leq)$ a conditionally complete lattice is transported by the definitions to $(I^*(R), \subseteq)$. In the same way, the continuity properties of the arithmetical and inf, sup operations on R, yield continuity properties for the related operations on $I^*(R)$ (see [8]). From this, each one of the finite distributive laws aforementioned has its corresponding general distri-

butive law for bounded families of $I^*(R)$. For instance, aside of $A+(B \vee C) = (A+B) \vee (A+C)$ it holds $A + \vee\{B_i \mid i \in I\} = \vee\{A+B_i \mid i \in I\}$.

13. From the general distributivity of the addition on $I^*(R)$, it is obtained

$$A+B = \underset{a \in \operatorname{prop} A}{\Omega^A} \quad \underset{b \in \operatorname{prop} B}{\Omega^B} \quad (\underset{.}{a}+\underset{.}{b}) = \underset{b \in \operatorname{prop} B}{\Omega^B} \quad \underset{a \in \operatorname{prop} A}{\Omega^A} \quad (\underset{.}{a}+\underset{.}{b})$$

where $\Omega^x := \mathit{if}\ X$ proper *then* $\vee$, *else* $\wedge$;
and prop $X := \mathit{if}\ X$ proper *then* X, *else* dual X.

2. Multiplication on I* (R)

The lattice expression 1.13 for the addition has been obtained using its algebraic definition over $I^*(R)$. For the multiplication we take as definition the parallel formula

$$A^* B := \underset{a \in \operatorname{prop} A}{\Omega^A} \quad \underset{b \in \operatorname{prop} B}{\Omega^B} \quad (\underset{.}{a}^* \underset{.}{b}), \text{ with } \underset{.}{a}^* \underset{.}{b} := (a \underset{.}{} b).$$

To compute the multiplication table we benefit from the general distributivity of the multiplication by real numbers, and from the simplification that the reduced expression of intervals yields. This leads to

$$A^* B = \sigma_A \sigma_B \underset{\chi_A \le \xi \le 1}{\Omega^A} \quad \underset{\chi_B \le \eta \le 1}{\Omega^B} \quad (\xi \underset{.}{} \eta),$$

and if $A = \sigma_A(1\ \omega_A\ \chi_A)$, $B = \sigma_B(1\ \omega_B\ \chi_B)$, with $\omega_A, \omega_B \in \{\vee, \wedge\}$, assuming $\chi_A \le \chi_B$:

$$\begin{aligned} A^* B = & \mathit{if}\ \omega_A = \omega_B,\ 0 \le \chi_A \le 1 && \mathit{then}\ \sigma_A \sigma_B (1\ \omega_A\ \chi_A \cdot \chi_B); \\ & \mathit{if}\ \omega_A = \omega_B,\ -1 \le \chi_A \le 0 && \mathit{then}\ \sigma_A \sigma_B (1\ \omega_A\ \chi_A); \\ & \mathit{if}\ \omega_A \ne \omega_B,\ 0 \le \chi_A \le 1 && \mathit{then}\ \sigma_A \iota_B (1\ \omega_A\ \chi_A \cdot \chi_B^{-1}); \\ & \mathit{if}\ \omega_A \ne \omega_B,\ -1 \le \chi_A \le 0 < \chi_B \le 1 && \mathit{then}\ \sigma_A \iota_B (1\ \omega_A\ \chi_A); \\ & \mathit{if}\ \omega_A \ne \omega_B,\ -1 \le \chi_A \le \chi_B \le 0 && \mathit{then}\ \underset{.}{0}. \end{aligned}$$

Properties:

1. $A^* B = B^* A$;
2. $(A^* B)^* C = A^*(B^* C)$;
3. dual A^* dual $B =$ dual $(A^* B)$;
4. $(-1)(A^* B) = (-A)^* B = A^*(-B)$;
5. $\lambda \in R$, $\lambda(A^* B) = (\lambda A)^* B = A^*(\lambda B)$;
6. $(-A)^*(-B) = A^* B$;
7. opp A^* opp $B =$ dual $(A^* B)$;
8. [inf, sup] multiplication rule:
$[\underline{a}, \bar{a}]^* [\underline{b}, \bar{b}] = \mathit{if}\ \omega_A = \omega_B\ \mathit{then}\ \operatorname{ext}_A(\underline{a}\,\underline{b}, \underline{a}\,\bar{b}, \bar{a}\,\underline{b}, \bar{a}\,\bar{b})$,
else (*let* $\chi_A \le \chi_B$) $\operatorname{int}_A(\underline{a}\,\underline{b}, \underline{a}\,\bar{b}, \bar{a}\,\underline{b}, \bar{a}\,\bar{b})$;
where $\operatorname{ext}_A/\operatorname{int}_A$ mean the two $\le$-external/internal values of the argument set, connected by ω_A (see [7]).

9. $A' \subseteq A$ and $B' \subseteq B \Rightarrow A' * B' \subseteq A * B$;

10. $\underset{.}{0} \leq A' \leq A$ and $\underset{.}{0} \leq B' \leq B \Rightarrow A' * B' \leq A * B$;

11. Multiplication distributivity for the lattice operations:
    a) $A * (B \vee C) = (A * B) \vee (A * C)$;
    b) $A * (B \wedge C) = (A * B) \wedge (A * C)$.

12. Multiplication subdistributivity for the addition:
    Case A proper, $A * (B + C) \subseteq A * B + A * C$;
    Case A improper, $A * (B + C) \supseteq A * B + A * C$.

13. Distributivity domains:
    Let $U = A * (B_1 + \ldots + B_n)$ and $V = A * B_1 + \ldots + A * B_n$.

Every $A \in I^* (R)$ determines on $I^* (R)$ the A-distributive domains $D_r (A), r = 1, 2, 3, 4$, such that if $(\exists\, r)\, (\forall\, i, i = 1, \ldots, n)\; B_i \in D_r (A)$, then $U = V$.

The $D_r (A)$ domains are defined as follows:

$$D_1 (A) := \{B \mid \sigma_B \geq 0, 0 \leq \chi_B \leq 1\} \cup \{C \mid \sigma_C \geq 0, \omega_C = \omega_A, \chi_A \leq \chi_C \leq 0\};$$

$$D_2 (A) := \{B \mid \sigma_B \leq 0, 0 \leq \chi_B \leq 1\} \cup \{C \mid \sigma_C \leq 0, \omega_C = \omega_A, \chi_A \leq \chi_C \leq 0\};$$

$$\begin{aligned} D_3 (A) := &\mathit{case}\ \chi_A = -1, \omega_A = \vee & &\mathit{then}\ \emptyset, \\ &\mathit{case}\ \omega_A = \vee, & &\mathit{then}\ \{B \mid \omega_B = \vee, -1 \leq \chi_B \leq \min(0, \chi_A)\}, \\ &\mathit{case}\ \omega_A = \wedge, & &\mathit{then}\ \{B \mid \omega_B = \vee, -1 \leq \chi_B \leq 0\}; \end{aligned}$$

$$\begin{aligned} D_4 (A) := &\mathit{case}\ \chi_A = -1, \omega_A = \wedge & &\mathit{then}\ \emptyset, \\ &\mathit{case}\ \omega_A = \wedge, & &\mathit{then}\ \{B \mid \omega_B = \wedge, -1 \leq \chi_B \leq \min(0, \chi_A)\}, \\ &\mathit{case}\ \omega_A = \vee, & &\mathit{then}\ \{B \mid \omega_B = \wedge, -1 \leq \chi_B \leq 0\}. \end{aligned}$$

The intersection of $D_1 (A)$, $D_2 (A)$ and $D_3 (A)$ with $I (R)$ for A proper, agrees with the known distributivity domains on $I (R)$ (see [13]). For $A = \underset{.}{a}$ $(\chi_A = 1)$ it is always $U = V$. The remaining situations are shown in Figs. 1 to 3.

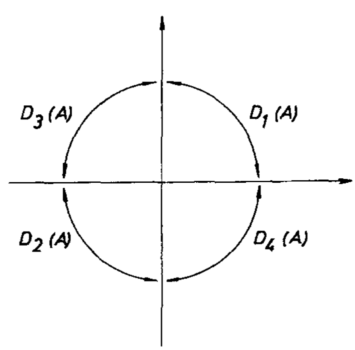

Fig. 1. Distributivity domains for $0 \leq \chi_A < 1$

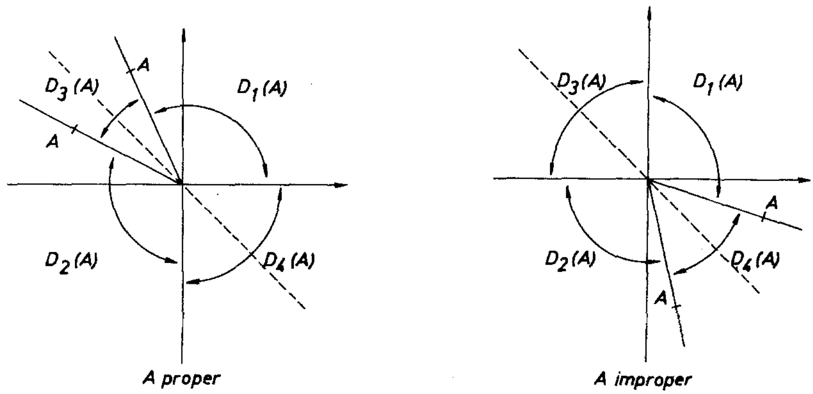

Fig. 2. Distributivity domains for $-1<\chi_A<0$

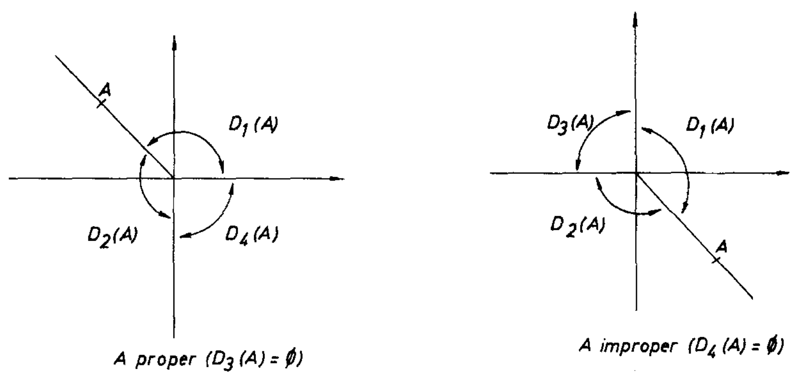

Fig. 3. Distributivity domains for $\chi_A=-1$

3. Division on I* (R)

The solution of the equation $B{*}\,X=A$ may be written $A/.\,B$.

From the fact that $B{*}\,X=1$ has a unique solution if and only if $B\neq_{*}0$, we have

$$1/.\,B=\iota_B^{-1}\,(1\;\bar{\omega}_B\;\chi_B),\text{ for } 0<\chi_B \text{ and is not defined elsewhere.}$$

The algebraic division on $I^*(R)$ may therefore be given in the following way:

$A/.\,B=$ *case* $0<\chi_B$ *then* $A{*}\,(1/.\,B)$,
case $-1\leq\chi_A\leq\chi_B\leq 0$, $\omega_A=\omega_B$ and $\sigma_B\neq 0$ *then* A/σ_B,
otherwise not defined.

(The case $\chi_A=\chi_B\leq 0$ is chosen from continuity requirements.)

Set inverse:

$1/B:=\mathrm{dual}\,(1/.\,B)$ when $0<\chi_B$ and not defined otherwise, is a theorem for $B\in I\,(R)$, and a definition on $I^*(R)$.

Set division:

$A/B := A * (1/B)$ when $0 < \chi_B$ and not defined otherwise.

When the involved operations are defined, the properties of the following list hold.

Algebraic inverse:

1. $1/.\ (1/.\ A) = A$;
2. $\text{dual}\,(1/.\ A) = 1/.\ (\text{dual}\ A) = 1/A$;
3. $-(1/.\ A) = 1/.\ (-A)$;
4. $\text{opp}\,(1/.\ A) = 1/.\ (\text{opp}\ A)$.

   Set inverse:

5. $\text{dual}\,(1/A) = 1/\text{dual}\ A = 1/.\ A$;
6. $-(1/A) = 1/(-A)$;
7. $\text{opp}\,(1/A) = 1/\text{opp}\ A$;
8. $1/(A/B) = B/A$.

   Inclusion properties:

9. $A' \subseteq A \Rightarrow$ a) $1/A' \subseteq 1/A$;
   b) $1/.\ A' \supseteq 1/.\ A$.
10. $A' \subseteq A\ \&\ B' \subseteq B \Rightarrow$ a) $A'/B' \subseteq A/B$;
    b) $A'/.\ B \subseteq A/.\ B$;
    c) $A/.\ B' \supseteq A/.\ B$.

4. Fuzzy Component

A statistical parametrization is adjoined to $I^*(R)$, making intervals into Bounded Fuzzy Distributions. That way, $X \in I(R)$ is looked at as a bounded distribution $\hat{X}$ with $E(\hat{X}) = \bar{x}$, $E((\hat{X} - \bar{x})^2) = s^2$ (see [5], [6]). Arithmetical set operations on $I(R)$ are matched to corresponding arithmetical set operations on $\{(\bar{x}, s^2)\}$, with the known results of the probability theory:

$$\bar{x}(-\hat{X}) = -\bar{x}(\hat{X}),$$
$$s^2(-\hat{X}) = s^2(\hat{X});$$
$$\bar{x}(\hat{X} + \hat{Y}) = \bar{x}(\hat{X}) + \bar{x}(\hat{Y}),$$
$$s^2(\hat{X} + \hat{Y}) = s^2(\hat{X}) + s^2(\hat{Y});$$
$$\bar{x}(\hat{X} * \hat{Y}) = \bar{x}(\hat{X}) * \bar{x}(\hat{Y}),$$
$$s^2(\hat{X} * \hat{Y}) = (\bar{x}^2(\hat{X}) + |s^2(\hat{X})|/2)\, s^2(\hat{Y}) + (\bar{x}^2(\hat{Y}) + |s^2(\hat{Y})|/2)\, s^2(\hat{X}).$$

The correspondence $I(R) \leftrightarrow \{(\bar{x}, s^2)\}$ is extended to $I^*(R) \leftrightarrow \{(\bar{x}, s^2)\}$, keeping the general properties of symmetries and operations on $I^*(R)$. That leads to keep the previous formulae and to the new ones:

$$\bar{x}\,(\text{opp}\,\hat{X}) = -\bar{x}\,(\hat{X}),$$
$$s^2\,(\text{opp}\,\hat{X}) = -s^2\,(\hat{X});$$
$$\bar{x}\,(\text{dual}\,\hat{X}) = \bar{x}\,(\hat{X}),$$
$$s^2\,(\text{dual}\,\hat{X}) = -s^2\,(\hat{X});$$
$$\bar{x}\,(1/.\;\hat{X}) = 1/\bar{x}\,(\hat{X}),$$
$$s^2\,(1/.\;\hat{X}) = -s^2\,(\hat{X})/\bar{x}^4\,(\hat{X}), \text{ when defined.}$$

The set division would be distribution dependent. Accordingly, we keep for the statistical components to the $I^*(R)$ definition $1/X = \text{dual}\,(1/.\;X)$, which amounts to adopt the linear distribution for the set model.

Lattice operations are solved by an interval-algebra way using a fuzzy membership function resulting from the Tchebycheff unequality and consistent with a general principle about the maximum variance associated with an interval, which will be also applied to the events where the statistical information is lost.

To weigh the importance of the fuzzy components, two reasons may be accounted for:

1. Interval digital solutions, and sometimes real interval solutions too, are only inclusion bounds of the real, "exact", solution; may be very pessimistic.

   The fuzzy components give an approached delimitation of the most probable region for the "exact", or model theoretical, solution.

2. Arithmetical operations do actually concentrate the most probable region, and that fact cannot be accounted by an only interval information.

   The fuzzy model allows the alternative pivoting between the probability and the interval models, according to the leading positional or combinatorial character of each operation.

5. Instrumental Tools for Digitalization

The SIGLA System is not implemented on the support of a bounds arithmetic. It is adopted a one sided radial arithmetic with:

an only pivot, σ, that's the element of greater modulus from the interval;

the amplitude of the interval, with the sign related with the interval being proper or improper.

The amplitude parameter is defined from the fundamental parameters σ, ι:

$$\lambda_A := \textit{if } A \text{ proper } \textit{then } \operatorname{sign}(\sigma_A)\cdot|\sigma_A - \iota_A|, \quad \textit{else } -\operatorname{sign}(\sigma_A)\cdot|\sigma_A - \iota_A|.$$

From that definition, an interval of $I^*(R)$ will be referred to by the following notation:

$$A = \langle\sigma, \lambda\rangle := \sigma + \lambda\,[-\operatorname{sign}(\sigma\,\lambda), 0],$$

with $0 \le |\lambda| \le |2\,\sigma|$.

That choice subordinates the overall absolute precision to the precision of the magnitude, and the pair σ, λ is structurally equivalent to the pair σ, χ, as $|\lambda/\sigma|$

conveys the information of the parameter $\chi = 1 - |\lambda/\sigma|$. Furthermore, the parameter λ is better for the software arithmetic because it induces shorter computation sequences than χ, and directly associates with the fuzzy components to define the shape of the fuzzy distribution.

Let us list some generally relevant properties of the $\langle\sigma, \lambda\rangle$ representation:

a) Basic symmetries:

$$-\langle\sigma, \lambda\rangle = \langle -\sigma, -\lambda\rangle;$$
$$\text{opp}\,\langle\sigma, \lambda\rangle = \langle -\sigma, \lambda\rangle;$$
$$\text{dual}\,\langle\sigma, \lambda\rangle = \langle\sigma, -\lambda\rangle.$$

b) Addition properties:

1. $|\lambda_{A+B}| = \mathit{if}\ \omega_A = \omega_B\ \mathit{then}\ |\lambda_A| + |\lambda_B|,$
   $\mathit{else}\ ||\lambda_A| - |\lambda_B||.$

2. $|\sigma_{A+B}| = \mathit{case}\ \omega_A = \omega_B\ \&\ \sigma_A\sigma_B \geq 0\ \mathit{then}\ |\sigma_A| + |\sigma_B|;$
   $\mathit{case}\ \omega_A \neq \omega_B\ \&\ \sigma_A\sigma_B < 0\ \mathit{then}\ \max(|\sigma_A + \sigma_B|, |\iota_A + \iota_B|);$
   $\mathit{otherwise}\ \max(|\sigma_A + \iota_B|, |\sigma_B + \iota_A|).$

3. Bounds for $\chi(A+B)$:
   Case $\omega_A = \omega_B\ \&\ \sigma_A\sigma_B \geq 0$:
   $\chi \in [\min(\chi_A, \chi_B), \max(\chi_A, \chi_B)];$
   $|\lambda/\sigma| \in [\min(|\lambda_A/\sigma_A|, |\lambda_B/\sigma_B|), \max(|\lambda_A/\sigma_A|, |\lambda_B/\sigma_B|)];$
   Case $\omega_A = \omega_B\ \&\ \sigma_A\sigma_B < 0$:
   $\chi \leq \max(\chi_A, \chi_B);$
   $|\lambda/\sigma| \geq \min(|\lambda_A/\sigma_A|, |\lambda_B/\sigma_B|);$
   Case $\omega_A \neq \omega_B\ \&\ \sigma_A\sigma_B > 0$:
   $\chi \geq \min(\chi_A, \chi_B);$
   $|\lambda/\sigma| \leq \max(|\lambda_A/\sigma_A|, |\lambda_B/\sigma_B|);$
   Case $\omega_A \neq \omega_B\ \&\ \sigma_A\sigma_B < 0$:
   $\chi \in [-1, 1];$
   $|\lambda/\sigma| \in [0, 2].$

c) Operation tables: They are obtained in a straightforward way from the reduced representation tables (see [4]), splitting cases so as to make sure the correct programming of the rounding options.

The operational support of the $I^*(R)$ digitalization, will be the $\langle\sigma, \lambda\rangle$ representation, with no systematic radial to bounds reduction. That choice leads, with every elementary digital operation, to a widening of the order of the digital step of the resulting σ plus a fraction of the digital step of the resulting λ. The variance of this digital widening is of the order the square power of the digital step of λ. That way, the overall behaviour for small amplitude intervals, is very uniform.

The problem of implementing order and lattice operations, which need having actual bounds for the intervals involved, is solved by the $\langle\sigma, \lambda\rangle$ to extended format mapping. Extended format is not to be confused with a double length one.

The digital handling of the σ and λ parameters proceeds up- or down-rounding according the interval being proper or improper, with the parameter λ being compensated for the σ correction. This is done in order to obtain a rounded interval which includes in the sense of $I^*(R)$, the exact one. This rounding method may lead to improper-to-proper digital transitions. This exceptional event will be dealt with in the implementation, with a condition checking procedure.

The $\langle \sigma, \lambda \rangle$ digital domain is delimitated so as to make possible for every item the exactness of the extended mapping of its bounds and, too, the expressability of its statistical parameters.

Relative variance shows up a determinism degree of an interval result. Since point-wise intervals do not have any relative variance defined, no point-wise interval result is admitted unless it arises from one of the following events:

a) programmer's specification,
b) aknowledged optimal operation with point-wise operands.

Any point-wise item otherwise arisen, is coerted to a minimal amplitude proper interval.

Any digital arithmetic has its own structurally built up repertory of optimal events. Only some choice has to be made about acknowledging all of them, some of them, or none of them. The choice for the SIGLA System Controlled Arithmetic has been to correct the rough float machine arithmetic so as to reach the general rule:

if aknowledged optimal event
  then $x\,\omega_D\,y = x\,\omega\,y$, dev$=0$;
  else $x\,\omega_+\,y - x\,\omega_-\,y = \text{dstep}_-\,(x\,\omega_-\,y)$, dev$=\max(x\,\omega_D\,y - x\,\omega\,y)$;
    ($\omega \in \{+, —, *, /\}$; ω_D digital operation);

and, aside of that, the choice of the aknowledged minimal events will be the minimal one keeping the function of the most important elements for the algebra of the arithmetical system. That's to say:

$x+0,\ x+(-x)$;
$x*0,\ x*(\pm base^k)$;
$0/x,\ \ x/(\pm x),\ x/(\pm base^k)$;
(underflow event is resolved upon the rounding option).

An important remark for the controlled addition is that, when an optimal event arises, the exact result is assigned to the downrounded option, $(x+y)_-$, when the operands are of the same sign. In case the operands being of opposed sign, it is assigned to the uprounded option, $(x+y)_+$.

The float extended format to compute exact interval bounds from the σ, λ parameters consists of the two components e_1, e_2 which hold the exact relation $x+y=e_1+e_2$ and the condition

$$\begin{aligned}
&\textit{if } \operatorname{sign}(x)\cdot\operatorname{sign}(y)\geq 0\\
&\quad \textit{then } e_1=(x+y)_-,\\
&\qquad\quad |e_2|<\operatorname{dstep}_-(|e_1|),\\
&\qquad\quad \operatorname{sign}(e_1)\cdot\operatorname{sign}(e_2)\geq 0;\\
&\quad \textit{else } e_1=(x+y)_+,\\
&\qquad\quad |e_2|<\operatorname{dstep}_+(|e_1|),\\
&\qquad\quad \operatorname{sign}(e_1)\cdot\operatorname{sign}(e_2)\leq 0.
\end{aligned}$$

The second component, e_2, of the float extended format is computed by:

$$\begin{aligned}
&(\textit{let } |x|\geq|y|)\\
&e_2:=\textit{case}\ \ |x|=|y|\ \&\ \operatorname{sign}(x)\cdot\operatorname{sign}(y)=-1\ \textit{or}\ x\cdot y=0\\
&\qquad\qquad \textit{then } 0;\\
&\qquad\ \ \textit{case}\ \ \operatorname{sign}(x)\cdot\operatorname{sign}(y)>0\\
&\qquad\qquad \textit{then } \big(y-((x+y)_- - x)_+\big)_+;\\
&\qquad\ \ \textit{case}\ \ \operatorname{sign}(x)\cdot\operatorname{sign}(y)<0\ \&\ |x|\neq|y|\\
&\qquad\qquad \textit{then } \big(y-((x+y)_+ - x)_+\big)_+.
\end{aligned}$$

6. Rounding Semantics on I* (R) and Bounding Theorems on I* (M)

Every element $A\in I^*(R)$ defines a section $S(A)$ over $I(R)$ as it is shown in the Fig. 4, according to the expression $S(A):=\{X\in I(R)\mid X\supseteq A\}$.

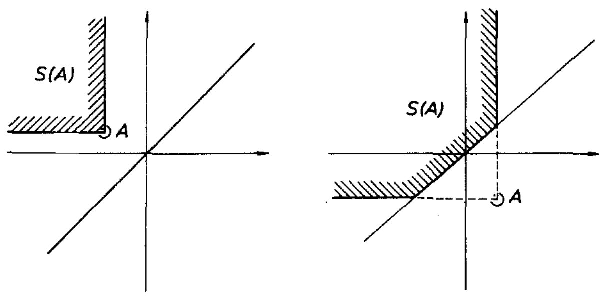

Fig. 4

Let $\mathfrak{S}:=\{S(A)\mid A\in I^*(R)\}$. Now, $I^*(R)$ may be interpreted trough the correspondence

$$\mathfrak{S}\leftrightarrow I^*(R).$$

The corresponding elements of $\mathfrak{S}$ and $I^*(R)$ hold the fundamental property:

$$(A,B\in I^*(R)),\quad A\supseteq B\Leftrightarrow S(A)\subseteq S(B).$$

Furthermore, if ω is a set operation $(+,\ -,\ *,\ /,\ \vee,\ \wedge)$ and $A,B,C\in I^*(R)$, then

$$A\,\omega\, B=C\Rightarrow S(A)\,\omega\, S(B)\subseteq S(C),$$

and

$$S(C)=\cap\,\{S\in\mathfrak{S}\mid S\supseteq S(A)\,\omega\, S(B)\}.$$

Therefore, as a computation should result in a subset of the set of solutions, but never in a superset, the actual rounding normative must be as follows:

Let $A, B \in I^*(M)$,
let $A\,\omega\,B = C$ on $I^*(R)$ and $A\,\bar{\omega}\,B = \bar{C}$ on $I^*(M)$;
then, as $S(C)$ is the actual set of solutions, it must be
$S(\bar{C}) \subseteq S(C)$ on $\mathfrak{S} \Leftrightarrow A\,\bar{\omega}\,B \supseteq A\,\omega\,B$ on $I^*(R)$.

From this semantical starting point the four bounding theorems are obtained, which govern the programming on $I^*(M)$ to solve problems defined in the context of $I^*(R)$.

In the inference and use of these theorems, the following known properties are important:

a) $A \subseteq B \Leftrightarrow \text{dual}\, A \supseteq \text{dual}\, B$;
b) $\text{dual}\,(A\,\omega\,B) = \text{dual}\, A\,(\text{dual}\,\omega)\,\text{dual}\, B$;
c) every elementary interval operation is selfdual, except $\vee$ and $\wedge$;
d) every operation not involving duality (interval set operations), is isotonic.

1st bounding theorem on I* (M):

Let $A, B \in I^*(M)$ and $\bar{\omega} = \text{digital}\,(\omega)$; then

$$A\,\bar{\omega}\,B \supseteq A\,\omega\,B \supseteq \text{dual}\,(\text{dual}\, A\ \overline{\text{dual}\,\omega}\ \text{dual}\, B),$$

when both digital operations exist.

(It is easily obtained from: $\text{dual}\, A\ \overline{\text{dual}\,\omega}\ \text{dual}\, B \supseteq \text{dual}\, A\ (\text{dual}\,\omega)\ \text{dual}\, B$.)

2nd bounding theorem on I* (M):

Let

$$F\,[(\omega_1, \ldots, \omega_m), (A_1, \ldots, A_n)]$$

be a rational function on $I^*(R)$; let

$$D\,(F\,[(\omega), (A)]) = F\,[(\bar{\omega}), (\bar{A})]$$

be its associated digital function, with $\bar{A}_i \supseteq A_i$; then if $\omega_1, \ldots, \omega_m$ are $\subseteq$-isotonic operations

$F\,[(\bar{\omega}), (\bar{A})] \supseteq F\,[(\omega), (A)] \supseteq \text{dual}\, F\,[(\overline{\text{dual}\,\omega}), (\overline{\text{dual}\, A})]$,

when both groups of digital functions exist.

(It follows by induction from $\overline{\text{dual}\, A_i} \supseteq \text{dual}\, A_i$ and from the 1st theorem.)

3rd bounding theorem on I* (M) (Twin process):

Definitions:

$F\,[(\omega), (A); (\Omega), (A)]$	Piecewise rational function on $I^*(R)$;
$DF := F\,[(\bar{\omega}), (\bar{A}); (\bar{\Omega}), (\bar{A})]$	Digital associated function;
$AF := F\,[(\omega), (A); (\bar{\Omega}), (\bar{A})]$	Algorithmic associated function;
$TF := F\,[(\overline{\text{dual}\,\omega}), (\overline{\text{dual}\, A}); (\bar{\Omega}), (\bar{A})]$	Twin digital function.

(The first set of arguments are the operators and operands of the component algebraic expressions. The second set, (Ω) and (A), are relational operators and the operands of the branching expressions.)

Theorem: *If $\omega_1, \ldots, \omega_m$ are $\subseteq$-isotonic operations, then*

$$DF \supseteq AF \supseteq \text{dual } TF,$$

when DF and TF do exist.

Corollary (General Twin process):

Let $A_1, \ldots, A_n$ proper, and let be B such that $A \supseteq \bar{B} \supseteq \overline{\text{dual } A}$; let

$$GTF := F\,[(\overline{\text{dual } \omega}), (\bar{B}); (\bar{\Omega}), (\bar{A})];$$

then in case DF does exist, GTF exists too and

$$DF \supseteq AF \supseteq \text{dual } GTF.$$

(The theorem arises from applying the second theorem to the execution-time branches of the process F.)

Remarks:

a) While the bounding theorems are digital arithmetic assertions, the relationship between F and AF ought to be analytically established.

b) $A\,\bar{\omega}\,B$ cannot be expressed, generally, as $\rho_\omega(A, B)$ where ρ is a rounding function. But it is $A\,\bar{\omega}\,B = A\,\omega\,B + \Delta$, $\Delta \supseteq \underset{\cdot}{0}$, enough to validate these bounding theorems.

4th bounding theorem on I* (M) (Dual Reduced form):

Let $\omega_1, \ldots, \omega_m$ be $\subseteq$-isotonic operations; then

$$F\,[(\omega, \text{dual}), (A)] \equiv F_1\,[(\omega), (A) \parallel (\text{dual } A)]$$

and

$$F_1\,[(\bar{\omega}), (\bar{A}) \parallel (\overline{\text{dual } A})] \supseteq F\,[(\omega, \text{dual}), (A)] \supseteq \text{dual } F_1\,[(\overline{\text{dual } \omega}), (\overline{\text{dual } A}) \parallel (\bar{A})],$$

when both digital functions do exist.

Remarks:

c) From the distributivity laws of the dual operator, any algebraic expression on $I^*(R)$ is equivalent to its dual-reduced form, built up with only isotonic operators and dual operator afecting only single variables.

d) Any $I^*(M)$ algebraic expression should be dual-reduced, to benefit from the 4th bounding theorem.

7. An Example on the Semantics of the SIGLA Algebraic System

An elementary example is given to show up the use of the $I^*(R)$ model, in mapping quantitative situations, where the deterministic kind and degree of magnitudes have a meaning, aside of their bounds and the concentration trends inside bounds. Also, it is exemplified how the bounding theorems are used to solve the numerical problem.

Let be the following elementary electrical circuit:

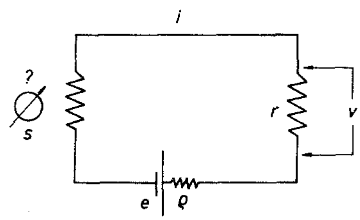

Fig. 5

Let's admit the equation $e=(\rho+r+s)\,i$ as the effective model for this circuit. The goal is to keep $v=r\,i=e\,r/(\rho+r+s)$ such that $\underset{\cdot}{v}\subseteq V$, given the random bounded variables (proper intervals) $E\supseteq\underset{\cdot}{e}$, $R\supseteq\underset{\cdot}{r}$, $R0\supseteq\rho$, $V\supseteq\underset{\cdot}{v}$.

We have to determine the value of the control variable S, and from the computation we shall obtain an interval value which will show the interval of values proper $(S)\supseteq\underset{\cdot}{s}$, and the deterministic type of S: if S may be an ordinary random variable (proper interval) or a control band (improper interval).

The $I^*(R)$ representation of this problem is the expression:

$$\underset{\cdot}{v}\subseteq E^*\,R/(R+R0+S)\subseteq V$$

with the context restrictions E, R, $R0$, V, $S>\underset{\cdot}{0}$. Then we must solve S from the equation $V=E^*\,R/(R+R0+S)$. The $I^*(R)$ algebra is used to solve S:

$1/V=(R+R0+S)/(E^*\,R)$, (from $V, A, B\neq_{\#}\underset{\cdot}{0}$ and the property $1/(A/B)=B/A$);
$\operatorname{dual}(E^*\,R)/V=R+R0+S$, (from $1/.\ \big(1/(E^*\,R)\big)=\operatorname{dual}(E^*\,R)$, when $E^*\,R\neq_{\#}\underset{\cdot}{0}$);
$S=\operatorname{dual}(E^*\,R)/V-\operatorname{dual}(R+R0)$, (from adding $\operatorname{opp}(R+R0)$ to the previous equality).

As the expression giving V is isotonic on its elementary arguments, we must compute an $S^*\subseteq S$ to guarantee the value of the expression

$$E^*\,R/(R+R0+S)$$

being included in V. Then, applying the 2nd bounding theorem, the dual of the associated digital function of

$$E^*\,R/\operatorname{dual}V-(R+R0)=\operatorname{dual}S$$

will be computed to obtain $S^*=\operatorname{dual}\overline{(\operatorname{dual}S)}\subseteq S$. From that, a V_1 interval will result such that

$$V\supseteq V_1:=E^*\,R/(R+R0+S^*).$$

To verify this property, we shall compute

$$V_2:=\operatorname{digital}\big(E^*\,R/(R+R0+S^*)\big),$$

and

$$V_0:=\operatorname{dual}\big(\operatorname{digital}\big(\operatorname{dual}E^*\operatorname{dual}R/(\operatorname{dual}R+\operatorname{dual}R0+\operatorname{dual}S^*)\big)\big),$$

which is a $I^*(M)$ right expression, according the 4th bounding theorem.

From the 2nd bounding theorem it holds

$$V_0 \subseteq V_1 \subseteq V_2.$$

The following observation is still worth: it is a known result, the advantage of monotony properties to compute interval bounds of functions while eliminating the dependency widening phenomenon (see [10]). The $I^*(R)$ algebra allows, under certain monotony conditions, to eliminate the widening phenomenon in a single computation run.

In our example, $v = e \cdot r_1/(r_2 + \rho + s)$ is an unconditionally $\leq$-isotonic function for $r_1 = r_2 = r$, unconditionally $\leq$-isotonic for r_1, and unconditionally $\leq$-antitonic for r_2 in the domain $\rho + s > 0$. This allows to apply the dependency widening phenomenon elimination technique which consists in substituting

$$V = E^* \, R/(\text{dual}\, R + R0 + S^*)$$

for the corresponding previous expression, and modifying accordingly the subsequent formulae.

	Example 1	Example 2	Example 3
Input:			
E	[10.00, 10.00]	[9.00, 11.00]	[9.00, 11.00]
R	[3.00, 3.00]	[2.00, 4.00]	[2.00, 4.00]
$R0$	[2.00, 2.00]	[1.50, 2.50]	[1.50, 2.50]
V	[3.00, 3.00]	[2.00, 4.00]	[2.00, 8.00]
Output:			
S^*	[5.02, 4.99]	[7.50, 2.50]	[2.00, 2.50]
V_0	[3.03, 2.97]	[2.01, 3.96]	[2.01, 7.97]
V_2	[2.97, 3.03]	[1.99, 4.01]	[1.99, 8.01]
Output with reduced dependency widening:			
S^*		[5.50, 4.50]	[0.00, 4.50]
V_0		[2.01, 3.96]	[2.01, 7.97]
V_2		[1.99, 4.01]	[1.99, 8.01]

Fig. 6

8. Analytical Ground of the I* (R) Semantics

Let $f(x, y)$ a rational function (with the operators $+, -, *, /$), continuous over $(X \times \text{proper}(Y))$ with $x \in X$, $y \in \text{prop}(Y)$, X the vector of arguments with proper values, Y the same with improper values.

Theorem: (see [4])

$$F = \bigvee_X \bigwedge_{\text{pr}(Y)} f(x, y) \text{ holds:}$$

if F proper *then* $(\forall x \in X)(\exists y \in \text{pr}(Y))\, f(x, y) \in F$;
if F improper *then* $(\forall x \in X)(\forall \alpha \in F)(\exists y \in \text{pr}(Y))\, f(x, y) = \alpha \in \text{pr}(F)$.

Theorem (*commutation of* $\vee$ *and* $\wedge$):

Given the interval function $f(X, a, b)$ *with* $f(x, a, b)$ *continuous over*

$$(\text{prop}(X) \times A \times \text{prop}(B))$$

it holds

$$\bigwedge_{\text{prop}(B)} \bigvee_{A} f(X, a, b) \supseteq \bigvee_{A} \bigwedge_{\text{prop}(B)} f(X, a, b).$$

Remark: Aside of commutative cases like those defining multiplication and addition on $I^*(R)$, there are noncommutative ones as it may be seen for the function $f = a^2(2 + a\,b)$ with $A = [-1, 1]$ and $B = [1, -1]$, where the left side value is $[0, 2]$ and the right side is $[0, 1]$.

Remark: According to the use of the expression "united extension" on $I(R)$ (see [8]), it may be applied, if $f(x, y)$ is a real function on $(X \times \text{prop}(Y))$, to the interval function

$$f^*(X, Y) := \bigvee_{X} \bigwedge_{\text{prop}(Y)} f(x, y).$$

Theorem: *If* $f(x, y)$ *is a single-incidence rational function, continuous over* $(X \times \text{prop}(Y))$, *then*

$$\bigwedge_{\text{prop}(Y)} \bigvee_{X} f(x, y) = f^*(X, Y) = f(X, Y)$$

where $f(X, Y)$ *is the corresponding rational interval function.*

Remark: If $f(X, Y)$ is multi-incident in X, and single-incident in Y, then $f(X, Y) \supseteq f^*(X, Y)$. On the reverse hypotheses, $f(X, Y) \subseteq f^*(X, Y)$.

Theorem (*reduction of the dependency widening*):

Let $f(X, A)$ *a rational interval function, multi-incident in* A. *Let* $f(X, A', A'')$ *its expression with explicit reference to the incidences* $A' = (A, \ldots, A)$, $A'' = (A, \ldots, A)$ *of the parameter* A. *Let* $f(x', a', a'')$, *with* $a' = (a'_1, \ldots, a'_p)$, $a'' = (a''_1, \ldots, a''_q)$, *be continuous over* $(\text{prop}(X) \times \text{prop}(A') \times \text{prop}(A''))$.

Suppose $f(x, a', a'')$ *is unconditionally* $\leq$*-isotonic for any component of* a' *and unconditionally* $\leq$*-antitonic for any component of* a'' *over*

$$(\text{prop}(X) \times \text{prop}(A') \times \text{prop}(A'')).$$

Then:

a) *in case* $f(x, a)$ *is unconditionally* $\leq$*-isotonic for* a *over* $D = (\text{prop}(X) \times \text{prop}(A))$, *if* A *is proper*

$$f(X, A) \supseteq f(X, A', \text{dual}\, A'') = \bigvee_{a \in A} f(X, a)$$

and if A *is improper*

$$f(X, A) \subseteq f(X, A', \text{dual}\, A'') = \bigwedge_{a \in \text{dual}\, A} f(X, a);$$

b) in case $f(x, a)$ is unconditionally $\leq$-antitonic for a over D, if A is proper

$$f(X, A) \supseteq f(X, \operatorname{dual} A', A'') = \bigvee_{a \in A} f(X, a)$$

and if A improper

$$f(X, A) \subseteq f(X, \operatorname{dual} A', A'') = \bigwedge_{a \in \operatorname{dual} A} f(X, a).$$

9. Language SIGLA-PL/1(0) Description (Experimental Level)

(1) SIGLA is an extension of PL/1. SIGLA statements are marked by the label *:

(2) SIGLA statements are:
  a) *: DECLARE (identifier, ...) INTERVAL;
  b) *: DECLARE entryname ENTRY (..., INTERVAL, ...), ...;
  c) *: ... intervalvariable = (* arithmetical expression with interval variables or coercible to *);
  d) *: ... (* intervalvariable ⟨relationaloperator⟩ intervalvariable *) ...; (logical expressions must be in logical variable context).

(3) Arithmetical operators:
$-()$, dual (), propin ( ), abrng (), $+$, $-$, *, /, '+' (join), '*' (meet).

(4) Relational operators:
$=$, $<=$, $>=$, $<$, $>$, '$<=$' (subset), '$>=$', '$<$' (strict subset), '$>$', '$=*$' (incidence), '$<*$' (less or incident), '$>*$'.

(5) SIGLA functions (are not to be declared):
  a) Boolean: isprin () (is proper interval?)
     isprdi () (is proper distribution?).
  b) Point-wise: sig (), lam ().
  c) Minimal digital interval: jot (), rlam () (λ/σ), inf (), sup ().
  d) Statistical: amean () ($\sigma - \bar{x}$), rmean () (amean/λ), xmean () ($\bar{x}$), avar () (s^2), rvar () (sign ($\sigma\,\lambda$)* avar/$|\lambda/2|^2$).

(6) SIGLA subroutines (are not to be declared). Compose operations: sgfpt (x, $\underset{\cdot}{x}$), sgrcd (sig, lam, amean, avar, interval), sgslc (sig, lam, rmean, rvar, interval), sgslcpr (sig, lam, rmean, rvar, interval), sgcm (inf, sup, rmean, rvar, interval), sgcmpr (inf, sup, rmean, rvar, interval).

(7) Input-output oriented subroutines (deal with hexadecimal floating point format): hextchr (floatvariable, N, NDEC, outputcharacter), chrthex (inputcharacter, floatvariable).

(8) The (* ... *) delimit the SIGLA interval context.

(9) Algebraic inverse operators have to be dealt with via dual operator.

A sample program is shown in Fig. 7.

```
SAMPLE:PROCEDURE OPTIONS(MAIN);       /* CIRCUIT PROBLEM  */
*: DECLARE (E,R,RR,V,SA,VO,V2,ZERO) INTERVAL;
                                      /* SIGLA STATEMENTS ARE HEADED BY
                                         AN  *:  LABEL  */
DCL INPUT(4,2) BIN FLOAT(53);
GET LIST(INPUT) COPY;                 /* READ E,R,RR,V, ARGUMENTS */
CALL SGCMPR(INPUT(1,1),INPUT(1,2),0.5,1.0,E);
CALL SGCMPR(INPUT(2,1),INPUT(2,2),0.5,1.0,R);
CALL SGCMPR(INPUT(3,1),INPUT(3,2),0.5,1.0,RR);
CALL SGCMPR(INPUT(4,1),INPUT(4,2),0.5,1.0,V);
                                      /* BUILD UP INTERVALS */
                                      /* SGCMPR COMPENSATES FOR MACHINE
                                         APPROXIMATED INPUT */
                                      /* SGCMPR NOT DECLARED, AS IT IS A
                                         SIGLA PROCEDURE */
CALL SGFPT(0,ZERO);                   /* BUILDS UP A POINT-WISE INTERVAL */
*: IF ¬((* E>=ZERO *) & (* R>=ZERO *) & (* RR>=ZERO *) & (* V>=ZERO *))
       THEN SIGNAL ERROR;
                                      /* SIGLA EXPRESSIONS ARE ENCLOSED
                                         BY  (* ... *)   */
*: SA=(* DUAL(E*R/DUAL(V)-(R+RR)) *);
*: V2=(* E*R/(R+RR+SA) *);
*: VO=(* DUAL(DUAL(E)*DUAL(R)/(DUAL(R)+DUAL(RR)+DUAL(SA))) *);
                                      /* TWIN EXPRESSION OF V2  */
CALL OUTINTV(SA);
CALL OUTINTV(VO);
CALL OUTINTV(V2);                     /* INTERVAL OUTPUT */
END SAMPLE;
```

Fig. 7

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