MAT 115B Homework 1

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January 15, 2025

1.

Let $n = \prod_{i=1}^{k} p_i^{r_i}$, then $\phi(n) = \prod_{i=1}^{k} p_i^{r_i - 1}(p_i - 1)$. Then we have that $p^{r_i-1} | n-1$ and $p^{r_i-1}|n$ for all i. Therefore, $r_i = 1$ for all i, so n is a product of distinct primes.

2.

$$\begin{aligned}
64 &= 2^6 \\
&\implies \sigma(64) = \frac{2^7-1}{2-1} = 127 \\
&\implies \tau(64) = 6+1 = 7 \\
105 &= 3 \cdot 5 \cdot 7 \\
&\implies \sigma(105) = \frac{3^2-1}{3-1} \cdot \frac{5^2-1}{5-1} \cdot \frac{7^2-1}{7-1} = 192 \\
&\implies \tau(105) = 2 \cdot 2 \cdot 2 = 8 \\
2592 &= 2^5 \cdot 3^4 \\
&\implies \sigma(2592) = \frac{2^6-1}{2-1} \cdot \frac{3^5-1}{3-1} = 7623 \\
&\implies \tau(2592) = 6 \cdot 5 = 30 \\
4851 &= 3^2 \cdot 7^2 \cdot 11 \\
&\implies \sigma(4851) = \frac{3^3-1}{3-1} \cdot \frac{7^3-1}{7-1} \cdot \frac{11^2-1}{11-1} = 8892 \\
&\implies \tau(4851) = (2+1)(2+1)(1+1) = 18 \\
111111 &= 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \\
&\implies \sigma(111111) = \frac{3^2-1}{3-1} \cdot \frac{7^2-1}{7-1} \cdot \frac{11^2-1}{11-1} \cdot \frac{13^2-1}{13-1} \cdot \frac{37^2-1}{37-1} \\
&\implies \tau(111111) = (1+1)(1+1)(1+1)(1+1)(1+1) = 32 \\
15! &= 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \\
&\implies \sigma(15!) = \frac{2^{12}-1}{2-1} \cdot \frac{3^7-1}{3-1} \cdot \frac{5^4-1}{5-1} \cdot \frac{7^3-1}{7-1} \cdot \frac{11^2-1}{11-1} \cdot \frac{13^2-1}{13-1} = 6686252969760 \\
&\implies \tau(15!) = (11+1)(6+1)(3+1)(2+1)(1+1)(1+1) = 4032
\end{aligned}$$

3.

a. $\tau(n)$ is odd when the powers in the prime factorization of n are all even.

b. $\sigma(n)$ is odd when $n = x^2$ or $n = 2x^2$ for some integer $x \neq 0$.

4.

If there exists an n such that $\tau(n) = k$, we can replace the primes in the prime factorization to obtain another n' where $\tau(n') = k$. We have that $\tau(2^{k-1}) = k$, therefore, $\tau(n) = k$ has infinitely many solutions because there are infinitely many primes.

5.

a. There are at most $\sqrt{n}$ divisors a where $a \leq \frac{n}{a}$. Each of these divisors a is paired off with another divisor $\frac{n}{a}$ where $a \cdot \frac{n}{a} = n$. Therefore, there are at most $2\sqrt{n}$ divisors, so $\tau(n) \leq 2\sqrt{n}$.

b. If $a|n$ then $2^a - 1|2^n - 1$ because $2^n - 1$ can be written $2^{ka} - 1$. Therefore, $\tau(n) \leq \tau(2^n - 1)$.

6.

a.

$$\sum_{d|n,d>0} \frac{1}{d} = \sum_{d|n,d>0} \frac{\frac{n}{d}}{n} = \frac{1}{n} \sum_{d|n,d>0} \frac{n}{d} = \frac{1}{n}\sigma(n) = \frac{\sigma(n)}{n}.$$

b.

$$\sum_{d|n,d>0} \frac{1}{d} = \sum_{d|n,d>0} \frac{\frac{n}{d}}{n} = \frac{1}{n} \sum_{d|n,d>0} \frac{n}{d} = \frac{1}{n}\sigma(n) = \frac{\sigma(n)}{n} = \frac{2n}{n} = 2.$$

7.

a.

$$\sigma(16) = 1 + 2 + 4 + 8 + 16 = 31$$
$$\sigma(31) = 1 + 31 = 32$$

b.

$$\sigma(2^{p-1}) = \sum i = 0^{p-1} 2^i = 2^p - 1$$
$$\sigma(2^p - 1) = 1 + 2^p - 1 = 2^p$$

c. We have that $\sigma(2^a) = 2^{a+1} - 1$, then, $\sigma(2^{a+1} - 1) = \sum_{d|2^{a+1}-1}$. Both 1 and $2^{a+1} - 1$ are divisors, so $\sigma(2^{a+1} - 1) \geq 2^{a+1}$. If there are any more divisors of $2^{a+1} - 1$, then $\sigma(2^{a+1} - 1) > 2^{a+1}$. Therefore, for 2^a to be superperfect, $2^{a+1} - 1$ must be a Mersenne prime.

8.

This homework was a 5/10 on difficulty. I spent 4 hours on this homework.