MAT 115B Number Theory	**Winter 2025**
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Lecture 2

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Each block has n different residues mod n in $\{0, \ldots, n-1\}$.

Fact 2.1. *If $y \equiv x \mod n$, then $\gcd(n,x) = \gcd(n,y)$. Then, the number of prime numbers in each block relatively prime to n is $\varphi(n)$.*

There are $\varphi(n)$ relatively prime numbers in each block and $\varphi(m)$ blocks, so $\varphi(mn) = \varphi(m)\varphi(n)$.

Exercise 2.2. $\varphi(p^\ell) = p^\ell - p^{\ell-1}$. Then, if $n = p_1^{r_1} \cdots p_1^{r_1}$, $\varphi(n) = \prod(p_i^{r_i} - p_i^{r_i-1}) \Rightarrow \varphi(n) = n \cdot \prod(1 - \frac{1}{p_i})$.

Theorem 2.3. *Suppose f is a multiplicative arithmetic function, define*

$$F(x) = \sum_{d|x, d\geq 1} f(x),$$

then $F(x)$ is multiplciative.

Corollary 2.4. *$\nu(x)$ is multiplicative.*

Proof. $\nu(x) = \sum_{d|x,d\geq 1} 1$, $f(x) = 1$ is trivially multiplicative, so by 2.3, $\nu(x)$ is multiplicative. □

Corollary 2.5. *$\sigma(x)$ is multiplicative.*

Proof. To be written. □

Proof of Theorem 2.3. Let $(m,n) = 1$, note that for all $d|mn$, there exists $d_1|m$, $d_2|n$ where $d_1 d_2 = d$ and $(d_1, d_2) = 1$. Then, we have that

$$\begin{aligned}
F(mn) &= \sum_{d|mn} f(d) \\
&= \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2) \\
&= \sum_{d_1|m} \sum_{d_2|n} f(d_1) f(d_2) \\
&= \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2) \\
&= F(m)F(n).
\end{aligned}$$

□

Formula 2.6. $\nu(p^r) = r + 1 \Rightarrow$ if $n = p_1^{r_1} \cdots p_k^{r_k}, \nu(n) = \prod(r_i + 1)$.

Remark 2.7. $(y^n - 1) = (y-1)(y^{n-1} + \cdots + y + 1)$.

Formula 2.8. $\sigma(p^r) = 1 + p + \cdots + p^r = \frac{p^{r_i+1}-1}{p-1} \Rightarrow$ if $n = p_1^{r_1} \cdots p_k^{r_k}, \sigma(n) = \prod \frac{p_i^{r_i}-1}{p_i-1}$.

Definition 2.9. $n \geq 1$ is perfect if $\sigma(n) = 2n$.

Some examples are as follows: $6 = 1+2+3, 28 = 1+2+4+7+14, 496 = \sum_{d|496} d$. Note that we can rewrite each of these using the form $n = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime, using $p = 2, 3, 5$ respectively.

Remark 2.10. $2^p - 1$ is prime $\Rightarrow p$ is prime.

Proof. Assume towards a contradiction that p is not prime, then $p = ab$ where $a, b > 1$. By 2.7, $2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + \cdots + 1)$, which is a contradiction. □

Theorem 2.11. *A number of the form $n = 2^{p-1}(2^p - 1)$ is perfect if $2^p - 1$ is prime.*

Proof.

$$\begin{aligned}
\sigma(n) &= \sigma(2^{p-1}(2^p - 1)) \\
&= \sigma(2^{p-1})\sigma(2^p - 1) \\
&= \frac{2^p - 1}{2 - 1}(2^p - 1 + 1) \\
&= (2^p - 1)(2^p) \\
&= 2 \cdot 2^{p-1}(2^p - 1) \\
&= 2n.
\end{aligned}$$

□