MAT 115B Homework 1

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1.

For each distinct prime number dividing n, we can choose to either include or exclude the prime in a product starting with the value 1. Doing every combination of this generates every single divisor of n that is square free. There are $2^{\omega(n)}$ combinations, and $|\mu(d)| = 1$ when d is square free. Therefore, $\sum_{d|n} |\mu(d)| = 2^{\omega(n)}$.

2.

Let $S = \{p_1, p_2, \ldots, p_m\}$ Consider a term in the expanded form of $\prod_i = 1^m(1 - f(p_i))$. It has the form $f(p_{t_1})f(p_{t_2}) \cdots f(p_{t_\ell})(-1)^\ell = f(p_{t_1}p_{t_2} \cdots p_{t_\ell})(-1)^\ell$ for some subset of prime divisors $S' = \{p_{t_1}, p_{t_2}, \ldots, p_{t_\ell}\} \subseteq S$. If we take the product over all elements in S' we get $d = p_{t_1}p_{t_2} \cdots p_{t_\ell}$. Over the expanded form of the original product, we choose either 1 or $-f(p_i)$ for each factor and this gives every combination of including or excluding each prime, namely, all subsets of the S. This gives all divisors d of n which are square free because each prime can only be used once. Finally, $(-1)^\ell = \mu(d)$ because d is prime free for each of the terms in the product. Therefore, the terms of expanded form of $\prod_{i=1}^m(1 - f(p_i))$ are $f(d)\mu(d)$ for each square free $d|n$, which is $\sum_{d|n} \mu(d)f(d)$.

3.

By Möbius inversion, $g(n) = \sum_{d|n} \mu(d)f\left(\frac{n}{d}\right)$, so

$$\begin{aligned}
g(12) &= \mu(1)f(12) + \mu(2)f(6) + \mu(3)f(4) + \mu(4)f(3) + \mu(6)f(2) + \mu(12)f(1) \\
&= 8 + (-1)4 + (-1)\frac{8}{3} + 0 + (1)\frac{4}{3} + 0 \\
&= 8 - 4 - \frac{4}{3} \\
&= \frac{8}{3}.
\end{aligned}$$

4.

Let $n = p_1^{r_1}p_2^{r_2} \cdots p_k^{r_k}$. We will first prove that $\sum_{d|n} \Lambda(d) = \ln n$,

$$\begin{aligned}
\sum_{d|n} \Lambda(d) &= \sum_{i=1}^{k} \sum_{j=1}^{r_i} \Lambda(p_i^{r_i}) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r_i} \ln p_i \\
&= \sum_{i=1}^{k} r_i \ln p_i \\
&= \sum_{i=1}^{k} \ln p_i^{r_i} \\
&= \ln(p_1^{r_1 \cdots p_k^{r_k}}) \\
&= \ln n.
\end{aligned}$$

Now, we will prove that $\Lambda(n) = -\sum_{d|n} \mu(d) \ln d$. By Möbius inversion,

$$\begin{aligned}
\Lambda(n) &= \sum_{d|n} \mu(d) \ln \left(\frac{n}{d}\right) \\
&= \left(\sum_{d|n} \mu(d)\right) \ln n + \sum_{d|n} \mu(d) \ln \left(\frac{1}{d}\right)
\end{aligned}$$

By a theorem proved in class we can simplify this to

$$\begin{aligned}
\left(\sum_{d|n} \mu(d)\right) \ln n + \sum_{d|n} \mu(d) \ln \left(\frac{1}{d}\right) &= 0 \ln n + \sum_{d|n} \mu(d) \ln \left(\frac{1}{d}\right) \\
\Lambda(n) &= -\sum_{d|n} \mu(d) \ln d
\end{aligned}$$

Therefore, $\Lambda(n) = -\sum_{d|n} \mu(d) \ln d$.

5.

Let a, b be coprime. By Möbius inversion, F is multiplicative, and $c|a, d|b \implies \gcd(c, d) = 1$, we have that

$$
\begin{aligned}
f(a)f(b) &= \left(\sum_{c|a} \mu(c)F(\frac{a}{c})\right) \left(\sum_{d|b} \mu(d)F(\frac{b}{d})\right) \\
&= \sum_{c|a} \sum_{d|b} \mu(c)\mu(d)F(\frac{a}{c})F(\frac{b}{d}) \\
&= \sum_{c|a} \sum_{d|b} \mu(cd)F(\frac{ab}{cd}) \\
&= \sum_{e|ab} \mu(e)F(\frac{ab}{e}) \\
&= f(ab).
\end{aligned}
$$

Therefore, if F is multiplicative, then f is multiplicative.

6.

This homework was around 4 hours to complete with a 7/10 difficulty level.