MAT 115B Number Theory **Winter 2025**

January 15, 2025

Lecture 5

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Theorem 5.1 (Gauss)**.** $\sum_{d|n} \varphi(d) = n$.

Proof. For every $d|n$, define $S_d = \{m|1 \leq m \leq n \text{ s.t. } (m,n) = d\}$. Then $\bigcup_{d|n} S_d = \{1 \ldots n\}$ and the set of S_d's partition $\{1 \ldots n\}$. Note that $(m,n) = d \iff (m/d, n/d) = 1 \Rightarrow S_d = \{kd|1 \leq k \leq n/d \text{ s.t. } (k, n/d) = 1\}$. By definition, $|S_d| = \varphi(n)$. Thus,

$$\begin{aligned} n &= \sum_{d|n} |S_d| \\ &= \sum_{d|n} \varphi(n/d) \\ n &= \sum_{d|n} \varphi(d) \end{aligned}$$

$\square$

Convolution of Functions

The convolution of functions defined over arithmetic functions is as follows:

Definition 5.2 (Convolution of arithmetic functions)**.** Let f, g be arithmetic functions. The convolution of f and g is defined as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) = (g * f)(n)$$