MAT 115B Number Theory **Winter 2025**

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Lecture 4

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1 Lecture 3 Recap

Definition 4.1. For $n = p_1^{r_1} \dots p_k^{r_k}$, $\mu(n)$ is the Möbius function, defined as

$$\mu(1) = 1,$$
$$\mu(n) = \begin{cases} (-1)^k & \text{if } \forall i, r_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Essentially, $\mu(n) = 0$ if it is divisible by a square and $(-1)^k$ otherwise. We showed that $\mu(n)$ is multiplicative.

Example 4.2. Now we ask, what is the value of $f(x) = \sum_{d|x, d\geq 1} \mu(d)$? We have that

$$\begin{aligned} f(15) &= \mu(1) + \mu(3) + \mu(5) + \mu(15) \\ &= 1 - 1 - 1 + 1 = 0, \\ f(8) &= \mu(1) + \mu(2) + \mu(4) + \mu(8) \\ &= 1 - 1 - 1 + 1 = 0, \\ f(12) &= \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) \\ &= 1 - 1 - 1 - 0 + 1 + 0 = 0, \\ f(1) &= \mu(1) = 1. \end{aligned}$$

Consider the function

$$\sum_{d|n, d\geq 1} \mu(d)\nu(\frac{n}{d}),$$

then

$$\begin{aligned} F(5) &= \mu(1)\nu(5) + \mu(5)\nu(1) \\ &= 1 \cdot 2 + (-1) \cdot 1 = 1, \\ F(8) &= \mu(1)\nu(8) + \mu(2)\nu(4) + \mu(4)\nu(2) + \mu(8)\nu(1) \\ &= 1 \cdot 4 - 1 \cdot 3 + 0 + 0 = 1. \end{aligned}$$

2 Möbius Inversion

Remark 4.3. $\{d|n\} = \{\frac{n}{d}\ \forall d|n\}$

Fact 4.4. $f(n) = \sum_{d|n, d\geq 1 \mu(d)} = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ otherwise} \end{cases}$

Theorem 4.5 (Moebius Inversion (MI))**.** *Let f, g be arithmetic functions. Then*

$$f(n) = \sum_{d|n, d\geq 1} g(d) \Longleftrightarrow g(n) = \sum_{d|n, d\geq 1} \mu(d) f\left(\frac{n}{d}\right).$$

Proof. ($\Longrightarrow$) Assume $g(n) = \sum_{d|n, d\geq 1} f(d)$, then

$$\begin{aligned} \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{d'|d} g(d') \\ &= \sum_{d'|d|n} g(d') \mu\left(\frac{n}{d}\right) \end{aligned} \tag{4.3}$$

Note that $d'|d \Rightarrow \frac{n}{d}|\frac{n}{d'}$, so we can rewrite the sum as

$$\sum_{d'|d|n} g(d') \mu\left(\frac{n}{d}\right) = \sum_{d'|n} g(d') \sum_{m|\frac{n}{d'}} \mu(m)$$

Finally, terms of the inner sum are all 0 except when $d' = n$, so we have

$$\sum_{d'|n} g(d') \sum_{m|\frac{n}{d'}} \mu(m) = g(n)\mu(1) = g(n).$$

($\Longleftarrow$) *To be written.*

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