PHY373 Homework Assignment 5

NOTE: This assignment has two pages.

1) (20 pts.) A few weeks ago, in class, we dealt with the wavefunction

$$\Psi(x) = \sqrt{\frac{12}{L^3}} \times \begin{cases} x & \text{for } 0 < x < \frac{L}{2}, \\ (L-x) & \text{for } \frac{L}{2} < x < L, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

when discussing the infinite square well. In words, this is a wavefunction that increases linearly from zero until the mid-point of the well, and then decreases linearly back to zero. When I discussed this wavefunction in class, I calculated the c_n, from which I then calculated the expectation value of the energy

$$\langle E \rangle = \sum |c_n|^2 E_n = \frac{12}{\pi^2} E_1, \tag{2}$$

where E_1 is the ground state energy. Now I ask you to reproduce this result in a different way, namely by using

$$\langle E \rangle = \int \Psi^*(x) \hat{H} \Psi(x)\, dx. \tag{3}$$

You may remember that there is something funny about $\Psi(x)$: since its second derivative vanishes *almost* everywhere, at first sight it seems to have $\langle E \rangle = 0$. But of course, by now you know that kinks hold surprises, and this wavefunction has three of them, at the ends and at the mid-point of the well. Which, if any, of the three contribute to $\langle E \rangle$? Reproduce our original result.

Hint: Your life will be much simpler if you remember that the Heaviside step function defined as

$$\Theta(x - x_0) = \begin{cases} 0 & \text{for } x < x_0, \\ 1 & \text{for } x \geq x_0, \end{cases} \tag{4}$$

has the property that $\frac{d\Theta(x-x_0)}{dx} = \delta(x - x_0)$.

2) (30 pts.) In terms of the classical turning point $x = \pm a$, the ground state wavefunction $\psi_0(x)$ of a harmonic oscillator is given by

$$\psi_0(x) = \sqrt[4]{\frac{1}{\pi a^2}}\, e^{-\frac{x^2}{2a^2}} .$$

a) Calculate the Fourier transform $\tilde{\psi}_0(k)$. Actually, you have already done this calculation in homework assignment 1 (problem 5), so you can just use that result, provided that you make the appropriate substitutions.

b) A particle is in this ground state when at $t = 0$ the potential *suddenly* vanishes ($V(x) = 0$ for $t \geq 0$), leaving behind a free particle with $\Psi(x, 0) = \psi_0(x)$. Using your answer to part a, and the time dependence of plane waves, write down $\Psi(x, t)$ for $t > 0$ (in the form of an integral over k). Write out the k-dependence of $\omega(k)$ explicitly.

c) Now perform the integral over k, focusing only on the x-dependence of the result. In other words, if there are multiplicative factors that are x-independent, you may neglect them (even if they depend on t). The best way to evaluate the integral is to complete the square in the exponent. By now you should be familiar with the integral of a Gaussian

$$\int_{-\infty}^{+\infty} d\xi\, e^{-\alpha \xi^2} = \sqrt{\frac{\pi}{\alpha}},$$

which holds even for complex values of α, as long as $\mathrm{Re}(\alpha) > 0$. If you have done this right, $|\Psi(x,t)|^2$ should still have an x-dependence of the form $e^{-\frac{x^2}{\sigma^2(t)}}$. Show that in the limit $t \gg \frac{ma^2}{\hbar}$, $\sigma(t)$ has a time dependence of the form $\sigma(t) \propto t^{\beta}$, where I ask you to determine β.

Comment: Now you know how quickly a free wave packet spreads out. In other words, you confine a particle to an interval of size a and then let it go, the answer to this problem tells you how quickly its wavefunction will expand afterwards.

3) (50 pts.) Consider the potential

$$V(x) = \alpha\, \delta(x) + V_0\, \Theta(x),$$

where $\Theta(x)$ is the Heaviside step function (see above). This is a combination of the delta-function and barrier potentials for which we calculated the reflection and transmission coefficients in class. Take the wavefunction of the form

$$\Psi(x) = \begin{cases} e^{ikx} + r\, e^{-ikx} & \text{for } x < 0, \\ t\, e^{ik'x} & \text{for } x > 0. \end{cases}$$

a) For a particle of energy $E > V_0$, solve for k and k'.

b) Write down the boundary conditions for Ψ and $\frac{d\Psi}{dx}$ at $x = 0$. For the latter, use the same trick as we did in class for the delta-function potential. Express everything in terms of the dimensionless strength of the delta-function potential $\beta = \frac{m\alpha}{\hbar^2 k}$ and the dimensionless ratio $v = \sqrt{\frac{E-V_0}{E}}$ which measures the strength of the step-function barrier. Note that $0 < v < 1$, with $v = 1$ corresponding to no barrier, and $v = 0$ corresponding to a barrier as high as the energy of the particle so that the particle has no kinetic energy left at $x > 0$.

c) Solve for r and t in terms of β and v.

d) Calculate R and T. Remember that since $k \neq k'$, T is not simply $|t|^2$ (use the probability current to get the correct T, as we discussed in class for the barrier case). Check that your answer satisfies $R+T = 1$. Show that in the limit $\beta \to 0$ you recover the answer we derived for the barrier potential, and in the limit $v \to 1$ you recover the answer we derived for the delta-function potential.

e) Show that for any value of β, the transmission coefficient is maximized when $v = 1$, namely when the barrier vanishes, and that for any value of v, the transmission coefficient is maximized when $\beta = 0$, that is when the delta-function disappears.