Homework 8

1. **General matrices.** Let V, W be a pair of finite dimensional vector spaces over a field F. Let $S = (v_1, .., v_N), T = (w_1, .., w_M)$ be ordered bases of V and W respectively. Given an operator $A : V \to W$ one can associate to it a matrix $M_A \in Mat_{M\times N}(F)$ with respect to the ordered bases S and T. Compute M_A in the following cases:

   (a) $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, $S = ((1,0),(0,1))$, $T = ((1,0),(0,1))$, $A(x,y) = (x+y, x-y)$.

   (b) $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, $S = ((1,0),(0,1))$, $T = ((1,1),(0,1))$, $A(x,y) = (x+y, x-y)$.

   (c) $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, $S = ((1,1),(0,1))$, $T = ((1,0),(0,1))$, $A(x,y) = (x+y, x-y)$.

   (d) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $S = ((1,0,0),(0,1,0),(0,0,1))$, $T = ((1,0),(0,1))$, $A(x,y,z) = (x,y)$.

   (e) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $S = ((1,1,1),(1,1,0),(0,1,1))$, $T = ((1,1),(-1,1))$, $A(x,y,z) = (x,y)$.

   (f) $V = \mathbb{R}^3$, $W = \mathbb{R}^1$, $S = ((1,0,0),(0,1,0),(0,0,1))$, $T = (1)$, $A(x,y,z) = x+y+z$.

   (g) $V = \mathbb{R}^3$, $W = \mathbb{R}^1$, $S = ((1,1,1),(0,1,1),(1,0,1))$, $T = (1)$, $A(x,y,z) = x+y+z$.

   In all the above examples, verify first that S, T are indeed bases of the corresponding vector spaces

2. Let $T : V \to W$, $S_1, S_2 : U \to V$ be linear transformations. Prove that $T \circ (S_1 + S_2) = T \circ S_1 + T \circ S_2$. Similarly, let $T : V \to W$, $S_1, S_2 : W \to U$, prove that $(S_1 + S_2) \circ T = S_1 \circ T + S_2 \circ T$.

3. Let $X = \{1,2,3\}$, let $\phi : X \to X$ be given by

$$\begin{aligned} \phi(1) &= 2, \\ \phi(2) &= 3, \\ \phi(3) &= 1. \end{aligned}$$

   Let $V = F(X)$ for some field F. Consider the ordered sets of vectors $B_1 = (\delta_1, \delta_2, \delta_3)$, $B_2 = (\delta_1 + \delta_2, \delta_2 + \delta_3, \delta_3 + \delta_1)$. Where δ_i is the skycraper function at i.

   (a) Prove that B_1, B_2 are bases (disregard the order).

   (b) Compute the transition matrices $M_{B_1 \to B_2}$ and $M_{B_2 \to B_1}$ and verify that indeed $M_{B_1 \to B_2} \cdot M_{B_2 \to B_1} = I$.

(c) Consider the linear transformation $\phi^* : V \to V$. Compute the matrices M_1,M_2 associated to ϕ^* with respect to B_1 and B_2 respectively. Verify that

$$\begin{aligned} M_1 &= M_{B_2 \to B_1} \cdot M_2 \cdot M_{B_1 \to B_2}, \\ M_2 &= M_{B_1 \to B_2} \cdot M_1 \cdot M_{B_2 \to B_1}. \end{aligned}$$

(d) Prove that ϕ^* is invertible and exhibit its inverse. Try to prove invertability once without using matrices and once with.

4. **Transition matrices.** Compute the transition matrices $M_{B_1 \to B_2}$ and $M_{B_2 \to B_1}$, associated with the following ordered bases:

   (a) $V = \mathbb{R}^2$, $B_1 = ((1,0),(0,1))$, $B_2 = ((1,1),(1,-1))$.

   (b) $V = \mathbb{R}^3$, $B_1 = ((1,1,1),(1,1,0),(0,1,1))$, $B_2 = ((1,1,-1),(0,1,0),(1,1,0))$.

   (c) $V = \mathbb{R}_{\leq n}[x]$, $B_1 = \left(1, x, x^2, .., x^n\right)$, $B_2 = \left(1, x, x^2, .., x^{n-1}, 1 + x + x^2 + .. + x^n\right)$.

5. **Composition of operators vs multiplication of matrices.** Let V be a finite dimensional vector space over a field F. Let $S = (v_1, .., v_N)$ be an ordered basis of V. Given an operator $A : V \to V$ one can associate to it a matrix $M_A \in Mat_{N \times N}(F)$ with respect to the ordered basis S. We denote by $A \circ B$ composition of operators and by $M_A \cdot M_B$ multiplication of matrices. Verify by direct computation that $M_{A \circ B} = M_A \cdot M_B$ in the following concrete scenarios:

   (a) $V = \mathbb{R}^2$, $B = ((1,0),(0,1))$, $A(x,y) = (x+y, x-y)$ and $B(x,y) = (2x, x+y)$.

   (b) $V = \mathbb{R}^3$, $B = ((1,0,0),(0,1,0),(0,0,1))$, $A(x,y,z) = (x+y, x-y, z+x)$ and $B(x,y) = (x, x+y, x+y+z)$.

   (c) $V = \mathbb{R}^3$, $B = ((1,1,1),(1,1,0),(0,1,1))$, $A(x,y,z) = (x+y, x-y, z+x)$ and $B(x,y) = (x, x+y, x+y+z)$.

   (d) $V = \mathbb{R}^3$, $B = ((1,1,1),(-1,1,1),(1,1,-1))$, $A(x,y,z) = (x,x,x)$ and $B(x,y,z) = (y,y,y)$.

   In each of the above cases, verify first that B is indeed a basis.

6. **Addition of operators vs addition of matrices.** Repeat exercise 1, but now show $M_{A+B} = M_A + M_B$.

7. **Inverse operators and inverse matrices.** Recall the definitions:

   Definition 1 *An operator $A : V \to V$ is called invertible if there exists an operator $B : V \to V$ s.t*

$$\begin{aligned} A \circ B &= Id_V \\ B \circ A &= Id_V \end{aligned}$$

Definition 2 *A matrix $M \in Mat_{N\times N}(F)$ is called invertible if there exists a matrix $N \in Mat_{N\times N}(F)$ s.t*

$$\begin{aligned} M \cdot N &= I_N \\ N \cdot M &= I_N, \end{aligned}$$

where $I_N \in Mat_{N\times N}(F)$ is the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Let V be a finite dimensional vector space over F. Let S be an ordered basis of V. Let $A : V \to V$ be an invertible operator. Prove that the matrix M_A (taken with respect to the ordered basis S) is invertible. Hint: Show $M_{A^{-1}} = (M_A)^{-1}$.

(b) Compute the inverse operators A^{-1}

i. $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x, y) = (x, x + y)$.

ii. $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x, y) = (x - y, x + y)$.

iii. $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(x, y, z) = (x - y, x + y, x + y + z)$.
In all the examples above, you should better verify first that A is indeed invertible by checking for example $ker A = \{0\}$.

(c) Compute the inverse matrices $(M_A)^{-1}$

i. $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x, y) = (x, x + y)$ and $S = ((1, 0), (0, 1))$.

ii. $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x, y) = (x - y, x + y)$ and $S = ((1, 1), (-1, 1))$.

iii. $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(x, y, z) = (x - y, x + y, x + y + z)$ and $S = ((1, 1, 1), (0, 1, 1), (0, 0, 1))$

8. Let V be a finite dimensional vector space over a field F. Let $U \subset V$ be a subspace. Prove that U is finite dimensional and moreover $\dim U \leq \dim V$.

9. Consider the vector space of all polynomials $V = \mathbb{R}[x]$. Let $T : V \to V$ be the map defined by $T(p) = x \cdot p$.

(a) Prove that T is a linear transformation.

(b) Prove that T is injective, that is $\ker T = \{0\}$.

(c) Prove that T is not surjective.

(d) Explain why b,c do not contradict the statement we proved in class about injectivity $\Leftrightarrow$surjectivity of a linear transformation.