Homework 5

1. Let $\varphi : X \to Y$ be a map between sets. Let F be a field. Consider the linear transformation of pull back: $\varphi^* : F(Y) \to F(X)$.

   (a) Prove that φ^* is injective iff φ is surjective.

   (b) Prove that φ^* is surjective iff φ is injective.

   (c) Conclude that φ^* is an isomorphism iff φ is bijective. Assuming φ is bijective. Prove that the inverse transformation $(\varphi^*)^{-1} : F(X) \to F(Y)$ is given by $(\varphi^*)^{-1} = \left(\varphi^{-1}\right)^*$.

   (d) Assume φ is arbitrary. Describe $Ker(\varphi^*) \subset F(Y)$. Describe $\operatorname{Im}(\varphi^*) \subset F(X)$.

2. Let V be a vector space over F.

   (a) Let $T, S : V \to V$ be invertible transformations (isomorphisms). Prove that $T \circ S$ is also invertible. Prove that the inverse function T^{-1} is a liner transformation as well.

   (b) Consider the set $GL(V)$ of all invertible transformations $T : V \to V$, with the operation $\circ$ of composition. Prove that $(GL(V), \circ)$ is a group. Write the axioms and explain why is holds.

   (c) Is the group $(GL(V), \circ)$ commutative? if yes prove it, if no give a counter example.

3. Prove that the following are invertible transformations and compute their inverse:

   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y, x)$.

   (b) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y, x)$.

   (c) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + y, 3x - y)$.

   (d) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + y + z, x - y, y + z)$.

   (e) $f : F^n \to F^n$, $f(x_1, ..., x_n) = (x_1, .., x_{n-1}, x_1 + x_2 + ..., x_n)$.

   (f) $T : \mathbb{R}[x] \to \mathbb{R}[x]$, given by $T(p(x)) = p(x + 5)$.
   In all these examples, prove first that these are linear transformations.

4. Let $\theta \in \mathbb{R}$ and consider the function $T_\theta(x, y) = (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y)$.

   (a) Prove that T_θ is a linear transformation. Convince yourself that T_θ is rotation anticlockwise by angle theta.

   (b) Prove that $T_{\theta_1+\theta_2} = T_{\theta_1} \circ T_{\theta 2}$. Prove that $T_0 = Id$.

   (c) Prove that T_θ is invertible.

(d) Consider the map $T : \mathbb{R} \to GL\left(\mathbb{R}^2\right)$ given by $T(\theta) = T_\theta$. Prove that T is a homomorphism of groups.

5. Consider the finite plane $V = \mathbb{F}_p^2$, where p is a prime number greater than 2.

   (a) Compute the nuber of lines in V. Prove your answer.

   (b) Calculate the nuber of lines in $\mathbb{F}_p^3$.

   (c) Calculate the number of subplanes in $\mathbb{F}_p^3$. (Hint: related to b.)

6. Consider $\mathbb{R}^2$. Prove that the set $S = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$ form a basis.