Problem Set # 6, Due: Wednesday Mar 1 by 11:00am

PHY 362K - Quantum Mechanics II, UT Austin, Spring 2017
(Dated: March 1, 2017)

Path integrals, charged particles, and more practice with perturbation theory

I. PERTURBED PARTICLE ON A RING

Consider a mass m particle that constrained to move on a 1D ring of radius R, with a delta function potential of strength g. I.e. parameterizing the angle of the particle on the ring by $\hat{\phi}$, the Hamiltonian in the ϕ-eigenbasis is:

$$\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2} + g\delta(\phi) \tag{1}$$

where ϕ is the eigenvalue of $\hat{\phi}$. (The first term is just the analog of writing $\frac{\hat{p}^2}{2m}$ in the x-eigenbasis as $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ and changing: $dx \to Rd\phi$).

1. What are the energy levels, degeneracies, and corresponding eigenstates for $g = 0$?

   Solution: When $g = 0$, the Hamiltonian becomes $\hat{H}_0 = -\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\phi^2}$. Using this Hamiltonian, the time independent Schrodinger equation becomes

   $$-\frac{\hbar^2}{2mR^2}\frac{\partial^2\psi}{\partial\phi^2} = E\psi$$

   If we define $k = \frac{R}{\hbar}\sqrt{2mE}$, then we can rewrite this equation as

   $$\frac{\partial^2\psi}{\partial\phi^2} = -k^2\psi$$

   This differential equation has two solutions for every $k \geq 0$, $\psi_+(\phi) = Ae^{ik\phi}$ and $\psi_-(\phi) = Ae^{-ik\phi}$. Because the particle in this problem is constrained to move on a ring, we must have $\psi_\pm(\phi) = \psi_\pm(\phi + 2\pi)$. Applying this condition gives

   $$Ae^{\pm ik\phi} = e^{\pm ik(\phi+2\pi)} \implies e^{2\pi i} = 1 \implies k \in \mathbb{Z}$$

   Therefore k must be an integer, so the allowed energies are quantized and are

   $$\boxed{E_k = \frac{\hbar^2 k^2}{2mR^2}} \tag{2}$$

   Note that the solutions ψ_+ and ψ_- have the same energy since $k^2 = (-k^2)$. It follows that there is a twofold energy degeneracy for every value of $k \geq 1$.

   If we now require our wave function to be normalized, then we have

   $$\int_0^{2\pi} \left(A^* e^{\mp ik\phi}\right)\left(Ae^{\pm ik\phi}\right) d\phi = 1 \implies A^2 \int_0^{2\pi} d\phi = 1 \implies A = \frac{1}{\sqrt{2\pi}}$$

So the normalized eigenstates of the unperturbed Hamiltonian are

$$\boxed{|k_+\rangle = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{ik\phi}|\phi\rangle d\phi \qquad \text{and} \qquad |k_-\rangle = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\phi}|\phi\rangle d\phi} \tag{3}$$

2. When can we use perturbation theory to get a good approximation for $g \neq 0$?

   Solution: The natural energy scale of the problem is given by $\frac{\hbar^2}{2mR^2}$. g has units of energy, so we can say that perturbation theory can be used to give a good approximation when $g \ll \frac{\hbar^2}{2mR^2}$.

3. For $g > 0$ (with g sufficiently small to use perturbation theory), perturbatively compute the eigenstates (to first order in g and eigen-energies (to second order in g).

   Solution: We know that $|k_+\rangle$ and $|k_-\rangle$ are degenerate states. We will begin by computing the matrix representing the perturbation $g\delta(\phi)$ in the degenerate subspace spanned by $|k_+\rangle$ and $|k_-\rangle$. This matrix is given by

$$\begin{pmatrix} \langle k_+|g\delta(\phi)|k_+\rangle & \langle k_+|g\delta(\phi)|k_-\rangle \\ \langle k_-|g\delta(\phi)|k_+\rangle & \langle k_-|g\delta(\phi)|k_-\rangle \end{pmatrix}$$

We will first calculate $\langle k_\pm|g\delta(\phi)|k_\pm\rangle$

$$\begin{aligned}\langle k_\pm|g\delta(\phi)|k_\pm\rangle &= \frac{g}{2\pi}\left(\int_0^{2\pi}\langle\phi'|e^{\mp ik\phi'}d\phi'\right)\delta(\phi)\left(\int_0^{2\pi} e^{\pm ik\phi}|\phi\rangle d\phi\right) \\ &= \frac{g}{2\pi}\int_0^{2\pi}\delta(\phi)\int_0^{2\pi}\langle\phi'|\phi\rangle d\phi' d\phi \\ &= \frac{g}{2\pi}\int_0^{2\pi}\delta(\phi)d\phi \\ &= \frac{g}{2\pi}\end{aligned}$$

We will now calculate $\langle k_\pm|g\delta(\phi)|k_\mp\rangle$

$$\begin{aligned}\langle k_\pm|g\delta(\phi)|k_\mp\rangle &= \frac{g}{2\pi}\left(\int_0^{2\pi}\langle\phi'|e^{\mp ik\phi'}d\phi'\right)\delta(\phi)\left(\int_0^{2\pi} e^{\mp ik\phi}|\phi\rangle d\phi\right) \\ &= \frac{g}{2\pi}\int_0^{2\pi}\delta(\phi)\int_0^{2\pi} e^{\mp ik(\phi+\phi')}\langle\phi'|\phi\rangle d\phi' d\phi \\ &= \frac{g}{2\pi}\int_0^{2\pi}\delta(\phi)e^{\mp 2ik\phi}d\phi \\ &= \frac{g}{2\pi}\end{aligned}$$

So the matrix representing $g\delta(\phi)$ in the degenerate subspace is

$$\frac{g}{2\pi}\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}$$

By inspection, we can see that this matrix has the eigenvectors $|1_k^0\rangle = \frac{1}{\sqrt{2}}(|k_+\rangle + |k_-\rangle)$ with eigenvalue $\frac{g}{\pi}$ and $|2_k^0\rangle = \frac{1}{\sqrt{2}}(|k_+\rangle - |k_-\rangle)$ with eigenvalue 0.

It follows that the first order energy corrections are $E_{1_k}^1 = \langle 1_k^0|g\delta(\phi)|1_k^0\rangle = \frac{g}{\pi}$ and $E_{2_k}^1 = \langle 2_k^0|g\delta(\phi)|2_k^0\rangle = 0$.

The first order eigenstate corrections are given by

$$
\begin{aligned}
|1_k^1\rangle &= \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{\langle 1_n^0|g\delta(\phi)|1_k^0\rangle}{E_{1_k}^0 - E_{1_n}^0}|1_n^0\rangle + \sum_{\substack{n=1\\n\neq k}}^{\infty} \frac{\langle 2_n^0|g\delta(\phi)|1_k^0\rangle}{E_{1_k}^0 - E_{2_n}^0}|2_n^0\rangle \\
|2_k^1\rangle &= \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{\langle 1_n^0|g\delta(\phi)|2_k^0\rangle}{E_{2_k}^0 - E_{1_n}^0}|1_n^0\rangle + \sum_{\substack{n=1\\n\neq k}}^{\infty} \frac{\langle 2_n^0|g\delta(\phi)|2_k^0\rangle}{E_{2_k}^0 - E_{2_n}^0}|2_n^0\rangle
\end{aligned}
$$

Now,

$$
\begin{aligned}
\langle 1_n^0|g\delta(\phi)|1_k^0\rangle &= \frac{1}{2}((\langle n_+| + \langle n_-|)g\delta(\phi)(|k_+\rangle + |k_-\rangle)) \\
&= \frac{1}{2}(\langle n_+|g\delta(\phi)|k_+\rangle + \langle n_+|g\delta(\phi)|k_-\rangle + \langle n_-|g\delta(\phi)|k_+\rangle + \langle n_-|g\delta(\phi)|k_-\rangle)
\end{aligned}
$$

but for any $a, b \in \mathbb{Z}$, we have $\int_0^{2\pi} e^{i(a+b)\phi}\delta(\phi)d\phi = e^{i(a+b)(0)} = 1$ so it follows that $\langle n_+|g\delta(\phi)|k_+\rangle = \langle n_+|g\delta(\phi)|k_-\rangle = \langle n_-|g\delta(\phi)|k_+\rangle = \langle n_-|g\delta(\phi)|k_-\rangle = \frac{g}{2\pi}$. Therefore,

$$
\begin{aligned}
\langle 1_n^0|g\delta(\phi)|1_k^0\rangle &= \frac{1}{2}\left(\frac{g}{2\pi} + \frac{g}{2\pi} + \frac{g}{2\pi} + \frac{g}{2\pi}\right) \\
&= \frac{g}{\pi}
\end{aligned}
$$

and,

$$
\begin{aligned}
\langle 2_n^0|g\delta(\phi)|1_k^0\rangle &= \frac{1}{2}((\langle n_+| - \langle n_-|)g\delta(\phi)(|k_+\rangle + |k_-\rangle)) \\
&= \frac{1}{2}(\langle n_+|g\delta(\phi)|k_+\rangle + \langle n_+|g\delta(\phi)|k_-\rangle - \langle n_-|g\delta(\phi)|k_+\rangle - \langle n_-|g\delta(\phi)|k_-\rangle) \\
&= \frac{1}{2}\left(\frac{g}{2\pi} + \frac{g}{2\pi} - \frac{g}{2\pi} - \frac{g}{2\pi}\right) \\
&= 0
\end{aligned}
$$

similar calculations show that $\langle 1_n|g\delta(\phi)|2_k\rangle = \langle 2_n|g\delta(\phi)|2_k\rangle = 0$. Using these results in the expressions for the first order eigenstate corrections, we find

$$
\begin{aligned}
|1_k^1\rangle &= \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{\langle 1_n^0|g\delta(\phi)|1_k^0\rangle}{E_{1_k}^0 - E_{1_n}^0}|1_n^0\rangle + 0 \\
&= \frac{g}{\pi}\sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{1}{\frac{\hbar^2}{2mR^2}(k^2 - n^2)}|1_n^0\rangle \\
&= \frac{2mR^2 g}{\pi\hbar^2}\sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{1}{k^2 - n^2}|1_n^0\rangle \\
|2_k^1\rangle &= 0 + 0 = 0
\end{aligned}
$$

The second order energy corrections are given by

$$E^2_{1_k} = \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{|\langle 1^0_n|g\delta(\phi)|1^0_k\rangle|^2}{E^0_{1_k} - E^0_{1_n}} + \sum_{\substack{n=1\\n\neq k}}^{\infty} \frac{|\langle 2^0_n|g\delta(\phi)|1^0_k\rangle|^2}{E^0_{1_k} - E^0_{2_n}}$$
$$E^2_{2_k} = \sum_{\substack{n=0\\n\neq k}}^{\infty} \frac{|\langle 1^0_n|g\delta(\phi)|2^0_k\rangle|^2}{E^0_{2_k} - E^0_{1_n}} + \sum_{\substack{n=1\\n\neq k}}^{\infty} \frac{|\langle 2^0_n|g\delta(\phi)|2^0_k\rangle|^2}{E^0_{2_k} - E^0_{2_n}}$$

applying the results we obtained earlier, we find

$$E^2_{1_k} = \frac{g^2}{\pi^2}\frac{2mR^2}{\hbar^2}\sum_{\substack{n=0\\n\neq k}}^{\infty}\frac{1}{k^2-n^2} + 0$$
$$E^2_{2_k} = 0 + 0 = 0$$

So to first order in g, the energy eigenstates are

$$\begin{aligned} |1_k\rangle &= \frac{1}{\sqrt{2}}\left(|k_+\rangle + |k_-\rangle\right) + \frac{\sqrt{2}mR^2 g}{\pi\hbar^2}\sum_{\substack{n=0\\n\neq k}}^{\infty}\frac{1}{k^2-n^2}(|n_+\rangle + |n_-\rangle) \\ |2_k\rangle &= \frac{1}{\sqrt{2}}(|k_+\rangle - |k_-\rangle) \end{aligned} \tag{4}$$

and to second order in g, the corresponding energies are

$$\begin{aligned} E_{1_k} &= \frac{\hbar^2k^2}{2mR^2} + \frac{g}{\pi} + \frac{2mR^2g^2}{\pi^2\hbar^2}\sum_{\substack{n=0\\n\neq k}}^{\infty}\frac{1}{k^2-n^2} \\ E_{2_k} &= \frac{\hbar^2k^2}{2mR^2} \end{aligned} \tag{5}$$

II. DOUBLE SLIT INTERFERENCE VIA A SUM OVER PATHS

Consider a free particle of mass m moving in the xy-plane initially (time $t = 0$) at position $(x, y) = (0, 0)$. Suppose we place a thin barrier extending from $y = (-\infty, +\infty)$ a distance $x = \frac{L}{2}$ that blocks the particle from passing, except at two point-like openings at $y = \pm a$. Suppose we place a screen of detectors at $x = L$ extending from $y = (-\infty, \infty)$, and see if the particle ends up at some position $(x = L, y)$ at a later time τ.

For this problem, it may be useful to note that the propagator for a free particle (no barrier) in d-spatial dimensions is:

$$U_{\text{free}}(x_f, t_f; x_0, t_0) = \left(\frac{m}{2\pi i\hbar(t_f - t_0)}\right)^{d/2} e^{-\frac{i}{\hbar}\left(\frac{1}{2}m\frac{|\vec{x}_f - \vec{x}_0|^2}{(t_f - t_0)}\right)} \tag{6}$$

1. Compute and sketch $\frac{P(y,\tau)}{P(y=0,\tau)}$, where $P(y, \tau)$ is the probability for the particle to be measured in a detector at position $(x = L, y)$ at time τ, by summing over the different paths by which the particle can propagate through either hole in the barrier.
   You may assume that the size of the experiment is sufficiently large compared to the de

Broglie wavelength of the particle, so that the classical path(s) dominate the sum over paths, i.e. that the particle arrives at the barrier in time $t/2$, and only passes through the barrier once (rather than following a crazy quantum path where it circles around the slits many times).

Solution: The total propagator for the particle to reach a point (L, y) on the screen, $U((L, y), \tau; (0, 0), 0)$, can be found by summing the propagators for the particle to go through either of the two slits and reach (L, y). The propagator to the slit at $(L/2, a)$ is given by equation 6 to be

$$U_a = \frac{m}{2\pi i\hbar(\tau/2 - 0)} e^{-\frac{i}{\hbar}\left(\frac{m|(L/2)\hat{x}+a\hat{y}|^2}{2(\tau/2-0)}\right)} = \frac{m}{\pi i\hbar\tau} e^{-\frac{i}{\hbar}\left(\frac{m(L^2/4+a^2)}{\tau}\right)}$$

The propagator from the slit at $(L/2, a)$ to the point (L, y) on the screen is given by equation 6 to be

$$U_y = \frac{m}{2\pi i\hbar(\tau - \tau/2)} e^{-\frac{i}{\hbar}\left(\frac{m|(L/2)\hat{x}+(y-a)\hat{y}|^2}{2(\tau-\tau/2)}\right)} = \frac{m}{\pi i\hbar\tau} e^{-\frac{i}{\hbar}\left(\frac{m(L^2/4+(y-a)^2)}{\tau}\right)}$$

Therefore the propagator for the path through the slit at $(L/2, a)$ is

$$U_1 = U_a U_y = \left(\frac{m}{\pi i\hbar\tau}\right)^2 e^{-\frac{im}{\hbar\tau}(L^2/2+a^2+(y-a)^2)}$$

The propagator to the slit at $(L/2, -a)$ is given by equation 6 to be

$$U_{-a} = \frac{m}{2\pi i\hbar(\tau/2 - 0)} e^{-\frac{i}{\hbar}\left(\frac{m|(L/2)\hat{x}-a\hat{y}|^2}{2(\tau/2-0)}\right)} = \frac{m}{\pi i\hbar\tau} e^{-\frac{i}{\hbar}\left(\frac{m(L^2/4+a^2)}{\tau}\right)}$$

The propagator from the slit at $(L/2, -a)$ to the point (L, y) on the screen is given by equation 6 to be

$$U_{-y} = \frac{m}{2\pi i\hbar(\tau - \tau/2)} e^{-\frac{i}{\hbar}\left(\frac{m|(L/2)\hat{x}+(y+a)\hat{y}|^2}{2(\tau-\tau/2)}\right)} = \frac{m}{\pi i\hbar\tau} e^{-\frac{i}{\hbar}\left(\frac{m(L^2/4+(y+a)^2)}{\tau}\right)}$$

Therefore the propagator for the path through the slit at $(L/2, -a)$ is

$$U_2 = U_{-a} U_{-y} = \left(\frac{m}{\pi i\hbar\tau}\right)^2 e^{-\frac{im}{\hbar\tau}(L^2/2+a^2+(y+a)^2)}$$

Let $A = \left(\frac{m}{\pi\hbar\tau}\right)^2$, $\quad \gamma_+ = \frac{m}{\hbar\tau}(L^2/2 + a^2 + (y+a)^2)$, and $\gamma_- = \frac{m}{\hbar\tau}(L^2/2 + a^2 + (y-a)^2)$. Now the total propagator to the point (L, y) is given by

$$U = U_1 + U_2 = -A\left(e^{-i\gamma_+} + e^{-i\gamma_-}\right)$$

Therefore the probability for the particle to be detected at a point (L, y) on the screen at a time τ is

$$P(y, \tau) = |U|^2 = A^2\left(e^{i\gamma_+} + e^{i\gamma_-}\right)\left(e^{-i\gamma_+} + e^{-i\gamma_-}\right) = A^2\left(1 + 1 + e^{i(\gamma_+-\gamma_-)} + e^{i(\gamma_--\gamma_+)}\right)$$

Recall that $e^{ix} + e^{-ix} = 2\cos(x)$. Using this result we find

$$P(y, \tau) = 2A^2\left(1 + \cos(\gamma_+ - \gamma_-)\right)$$

or in terms of the original physical parameters,

$$P(y,\tau) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4 \left(1+\cos\left(\frac{m}{\hbar\tau}(y+a)^2-(y-a)^2\right)\right) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4\left(1+\cos\left(\frac{4may}{\hbar\tau}\right)\right)$$

It follows that $P(0,\tau) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4(1+1) = 4\left(\frac{m}{\pi\hbar\tau}\right)$. Therefore,

$$\boxed{\mathcal{R}(y,\tau) = \frac{P(y,\tau)}{P(0,\tau)} = \frac{1}{2}\left(1+\cos\left(\frac{4may}{\hbar\tau}\right)\right) = \cos^2\left(\frac{2may}{\hbar\tau}\right)} \tag{7}$$

We can plot this in terms of the dimensionless parameter $\xi = \frac{may}{\hbar\tau}$ to obtain

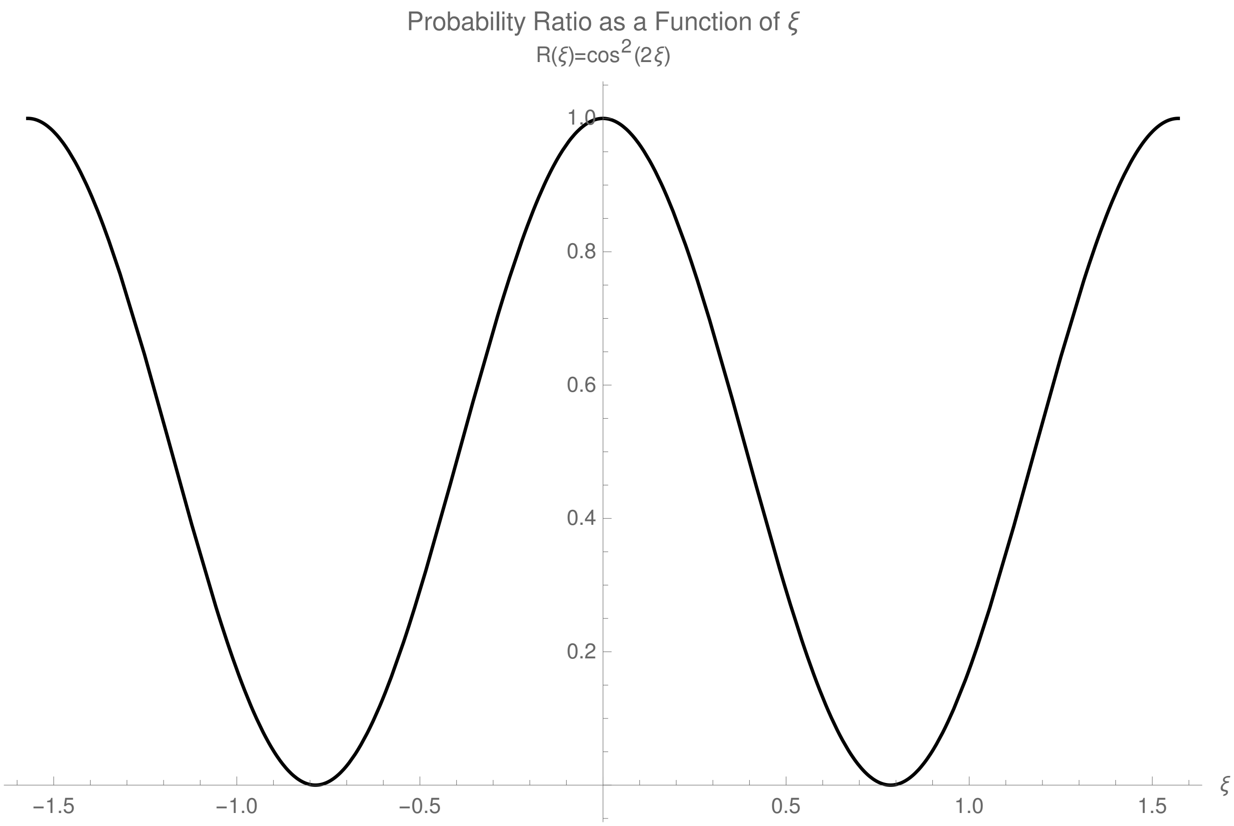

FIG. 1: Plot of the probability ratio $\mathcal{R}(y,\tau) = \frac{P(y,\tau)}{P(0,\tau)} = \cos^2\left(\frac{2may}{\hbar\tau}\right)$ as a function of the dimensionless parameter $\xi = \frac{may}{\hbar\tau}$ on the interval $[-\pi/2, \pi/2]$.

2. Suppose the particle is an electron with charge $-e$, and we now add a vanishingly thin solenoid with magnetic flux Φ_B into the barrier at $x = L/2$, somewhere between the two slits ($-a < y < a$). Find $\frac{P(y,\tau,B)}{P(y=0,\tau,B=0)}$ (i.e. the probability normalized by the corresponding results at $y = 0$ and zero magnetic field) as a function of the magnetic flux Φ_B, and sketch/plot the result for a few different values of Φ_B.

   Solution: As shown in class, when a solenoid is inserted between the two slits, the propagators for the paths going through the two slits pick up a phase $e^{i\frac{q}{\hbar}\phi}$, where q is the charge of the particle and $\phi = \int_P \vec{A} \cdot d\vec{l}$ (where P is the path taken by the particle and $\vec{A}$ is the magnetic vector potential). Therefore, the total propagator to the point (L, y) on the screen when a solenoid is placed between the slits is given by

$$U' = U_1 e^{i\frac{q}{\hbar}\phi_1} + U_2 e^{i\frac{q}{\hbar}\phi_2} = -A\left(e^{-i\gamma_+} e^{i\frac{q}{\hbar}\phi_1} + e^{-i\gamma_-} e^{i\frac{q}{\hbar}\phi_2}\right)$$

where ϕ_1 and ϕ_2 are the line integrals of $\vec{A}$ over the paths going through the slits at $y = a$ and $y = -a$, respectively, and $A, \gamma_\pm$ are defined as before.

It follows that the probability for the particle to be detected at the point (L, y) on the screen when a magnetic field B is in the solenoid is given by

$$\begin{aligned}
P(y, \tau, B) = |U'|^2 &= A^2\left(e^{i\gamma_+} e^{-i\frac{q}{\hbar}\phi_1} + e^{i\gamma_-} e^{-i\frac{q}{\hbar}\phi_2}\right)\left(e^{-i\gamma_+} e^{i\frac{q}{\hbar}\phi_1} + e^{-i\gamma_-} e^{i\frac{q}{\hbar}\phi_2}\right) \\
&= A^2\left(1 + 1 + e^{i(\gamma_+ - \gamma_- + \frac{q}{\hbar}(\phi_2 - \phi_1))} + e^{-i(\gamma_+ - \gamma_- + \frac{q}{\hbar}(\phi_2 - \phi_1))}\right) \\
&= 2A^2\left(1 + \cos(\gamma_+ - \gamma_- + \frac{q}{\hbar}(\phi_2 - \phi_1)\right)
\end{aligned}$$

In terms of the original physical parameters, and letting $q = -e$, we find

$$P(y, \tau, B) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4\left(1 + \cos\left(\frac{4may}{\hbar\tau} + \frac{e}{\hbar}(\phi_1 - \phi_2)\right)\right)$$

Now consider $\phi_1 - \phi_2$. In terms of the line integrals, we have

$$\begin{aligned}
\phi_1 - \phi_2 &= \int_{P1} \vec{A} \cdot d\vec{l} - \int_{P2} \vec{A} \cdot d\vec{l} \\
&= \int_{P1} \vec{A} \cdot d\vec{l} + \int_{-P2} \vec{A} \cdot d\vec{l} \\
&= \oint_{\partial C} \vec{A} \cdot d\vec{l}
\end{aligned}$$

where $P1$ and $P2$ are paths going from the origin to the point (L, y) through the slit at $y = a$ and $y = -a$ respectively and ∂C is a closed curve surrounding the region C that contains the solenoid. But using Stokes' theorem and the definition of the vector potential, we have

$$\begin{aligned}
\phi_1 - \phi_2 &= \oint_{\partial C} \vec{A} \cdot d\vec{l} \\
&= \int_C \left(\nabla \times \vec{A}\right) \cdot d\vec{a} \\
&= \int_C \vec{B} \cdot d\vec{a} \\
&= \Phi_B
\end{aligned}$$

where $B = \nabla \times \vec{A}$ is the magnetic field and Φ_B is the magnetic flux through the solenoid. Using this result, we have

$$P(y, \tau, B) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4\left(1 + \cos\left(\frac{4may}{\hbar\tau} + \frac{e}{\hbar}\Phi_B\right)\right)$$

It follows that $P(0,\tau,0) = 2\left(\frac{m}{\pi\hbar\tau}\right)^4(1+1) = 4\left(\frac{m}{\pi\hbar\tau}\right)^4$ and therefore

$$\boxed{\mathcal{R}'(y,\tau,B) = \frac{P(y,\tau,B)}{P(0,\tau,0)} = \frac{1}{2}\left(1+\cos\left(\frac{4may}{\hbar\tau}+\frac{e}{\hbar}\Phi_B\right)\right)} \tag{8}$$

We can plot this in terms of the dimensionless parameters $\xi = \frac{may}{\hbar\tau}$ and $\nu = \frac{e}{\hbar}\Phi_B$ to obtain:

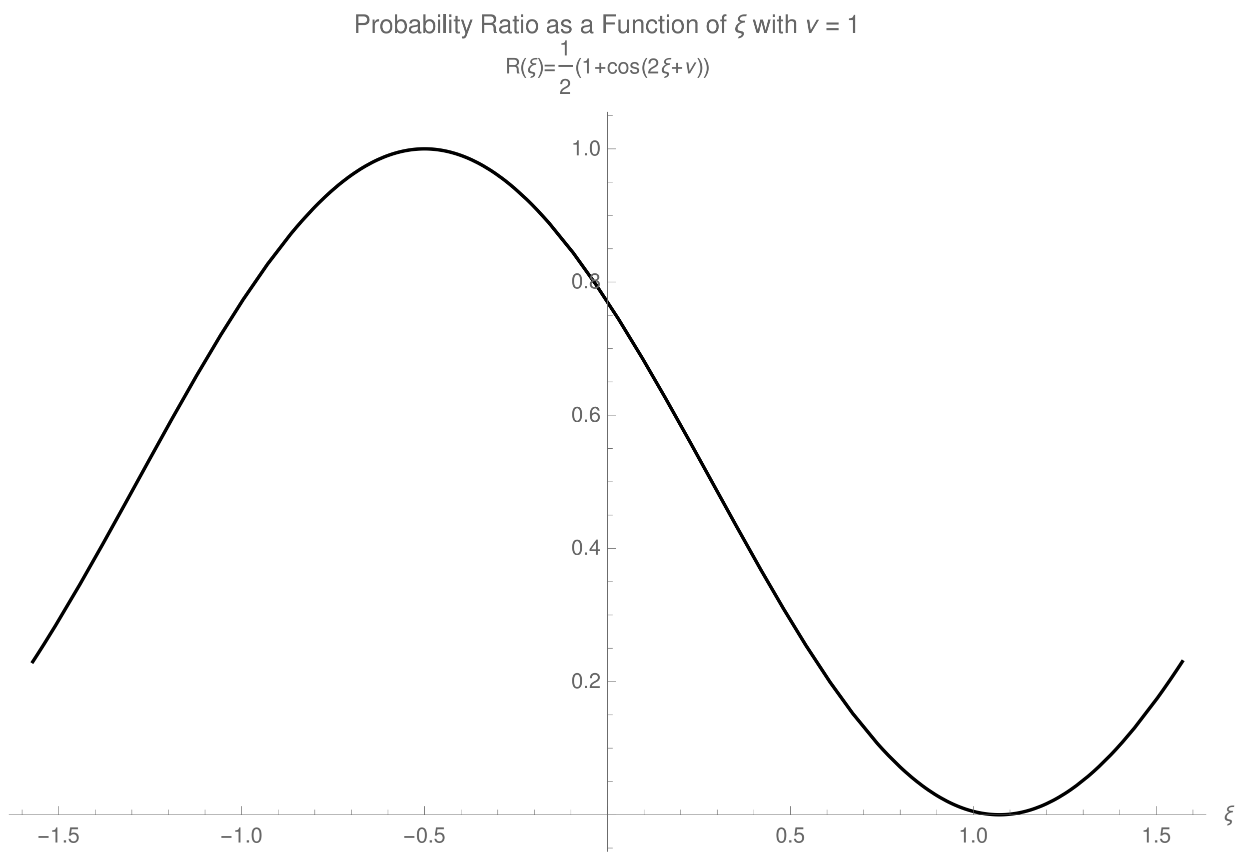

FIG. 2: Plot of the probability ratio $\mathcal{R}'(y,\tau,B) = \frac{P(y,\tau,B)}{P(0,\tau,0)} = \frac{1}{2}\left(\cos\left(\frac{2may}{\hbar\tau}+\frac{e}{\hbar}\Phi_B\right)\right)$ as a function of the dimensionless parameter $\xi = \frac{may}{\hbar\tau}$ with $\nu = \frac{e}{\hbar}\Phi_B = 1$ on the interval $[-\pi/2, \pi/2]$.

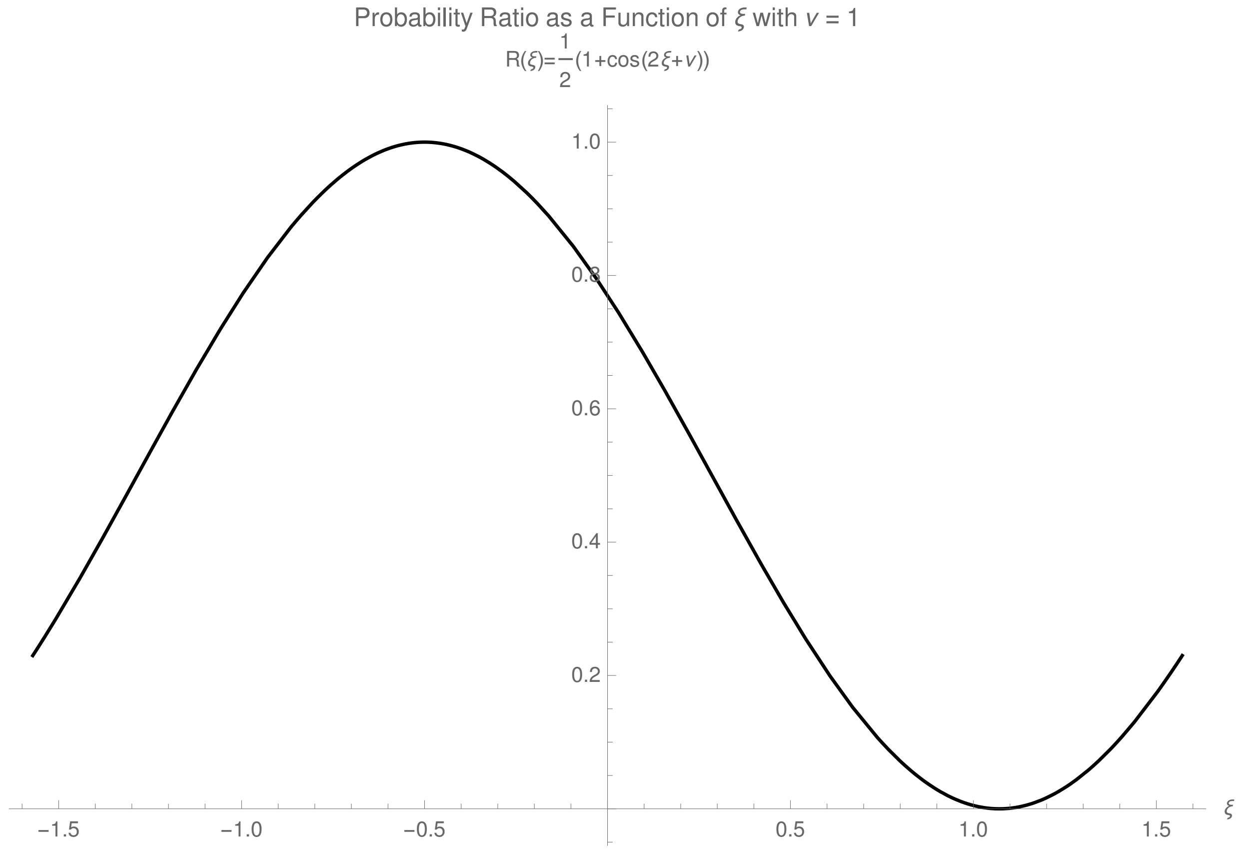

FIG. 3: Plot of the probability ratio $\mathcal{R}'(y,\tau,B) = \frac{P(y,\tau,B)}{P(0,\tau,0)} = \frac{1}{2}\left(\cos\left(\frac{2may}{\hbar\tau} + \frac{e}{\hbar}\Phi_B\right)\right)$ as a function of the dimensionless parameter $\xi = \frac{may}{\hbar\tau}$ with $\nu = \frac{e}{\hbar}\Phi_B = -1$ on the interval $[-\pi/2, \pi/2]$.

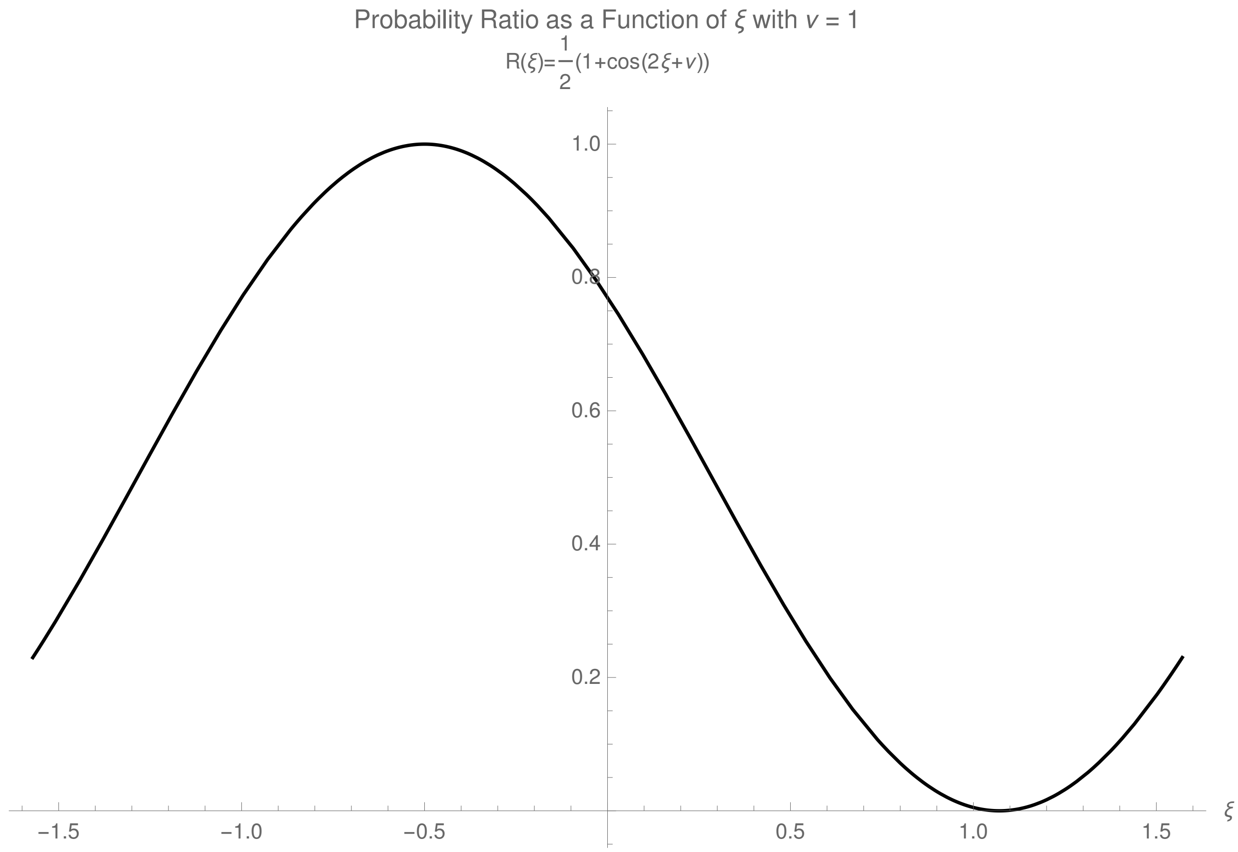

FIG. 4: Plot of the probability ratio $\mathcal{R}'(y,\tau,B) = \frac{P(y,\tau,B)}{P(0,\tau,0)} = \frac{1}{2}\left(\cos\left(\frac{2may}{\hbar\tau} + \frac{e}{\hbar}\Phi_B\right)\right)$ as a function of the dimensionless parameter $\xi = \frac{may}{\hbar\tau}$ with $\nu = \frac{e}{\hbar}\Phi_B = \frac{\pi}{2}$ on the interval $[-\pi/2, \pi/2]$.