Problem Set # 11, Due: Wednesday April 12 by 11:00am

PHY 362K - Quantum Mechanics II, UT Austin, Spring 2017
(Dated: April 12, 2017)

Field theory of scalar particles

I. FLUCTUATIONS OF A QUANTUM STRING

Consider a quantum string a.k.a. massless scalar field in 1d, described by the Hamiltonian:

$$H = \frac{1}{2}\int dx \left[\hat{\Pi}(x)^2 + c^2\left(\frac{\partial\hat{\phi}}{\partial x}\right)^2\right] \tag{1}$$

Note: if you compare to lectures I have rescaled the fields $\Pi \to \sqrt{\rho}\Pi$, and $\phi \to \phi/\sqrt{\rho}$ to absorb the overall factor of ρ. This rescaling does not alter the commutation relations of ϕ, Π, or the normal mode raising/lowering operators), and also does not effect the excitation energies, $\hbar c|k|$. All expressions from class can be used by just setting $\rho = 1$ everywhere, which makes life a bit more convenient.

A. Problem

Denoting the ground-state ("vacuum") of the string as: $|Vac.\rangle$. Compute the fluctuations in the the difference of the displacement of the string at position 0 and position x along the string:

$$C(x) = \langle Vac.|\left(\hat{\phi}(x) - \hat{\phi}(0)\right)^2|Vac.\rangle \tag{2}$$

To compute this, remember that continuum description of the string breaks down for very short distances on the scale of the spacing, a, between the atoms making up the string. This means that momentum sums/integrals should be restricted to $|k| < 1/a$.
You may find the asymptotic form of this cosine integral useful:

$$\lim_{a\to 0}\int_0^{1/a} dk\frac{(1-\cos kx)}{k} \approx \log\frac{x}{a} \tag{3}$$

Solution: We will begin by expanding the operator $\left(\hat{\phi}(x) - \hat{\phi}(0)\right)^2 = \hat{\phi}^2(x) - \hat{\phi}(0)\hat{\phi}(x) - \hat{\phi}(x)\hat{\phi}(0) + \hat{\phi}^2(0)$. Now we will consider

$$\begin{aligned}
\hat{\phi}^2(x)|Vac\rangle &= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(e^{ikx}\hat{a}_k + e^{-ikx}\hat{a}_k^\dagger\right)\left(e^{ik'x}\hat{a}_{k'} + e^{-ik'x}\hat{a}_{k'}^\dagger\right)|Vac\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(e^{ikx}\hat{a}_k + e^{-ikx}\hat{a}_k^\dagger\right)e^{-ik'x}|k'\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left[e^{i(k-k')x}\delta_{k,k'}|Vac\rangle + e^{-i(k+k')x}|k,k'\rangle\right]
\end{aligned}$$

It then follows that

$$\langle Vac|\hat{\phi}^2(x)|Vac\rangle = \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{i(k-k')x}\delta_{k,k'} = \sum_k \frac{\hbar}{2L\omega_k}$$

We will now consider,

$$\begin{aligned}
\hat{\phi}(0)\hat{\phi}(x)|Vac\rangle &= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(\hat{a}_k + \hat{a}_k^\dagger\right)\left(e^{ik'x}\hat{a}_{k'} + e^{-ik'x}\hat{a}_{k'}^\dagger\right)|Vac\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(\hat{a}_k + \hat{a}_k^\dagger\right)e^{-ik'x}|k'\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left[e^{-ik'x}\delta_{k,k'}|Vac\rangle + e^{-ik'x}|k,k'\rangle\right]
\end{aligned}$$

and so

$$\langle Vac|\hat{\phi}(0)\hat{\phi}(x)|Vac\rangle = \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{-ik'x}\delta_{k,k'} = \sum_k \frac{\hbar}{2L\omega_k}e^{-ikx}$$

Similarly,

$$\begin{aligned}
\hat{\phi}(x)\hat{\phi}(0)|Vac\rangle &= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(e^{ikx}\hat{a}_k + e^{-ikx}\hat{a}_k^\dagger\right)\left(\hat{a}_{k'} + \hat{a}_{k'}^\dagger\right)|Vac\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(e^{ikx}\hat{a}_k + e^{-ikx}\hat{a}_k^\dagger\right)|k'\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left[e^{ikx}\delta_{k,k'}|Vac\rangle + e^{-ikx}|k,k'\rangle\right]
\end{aligned}$$

and so

$$\langle Vac|\hat{\phi}(x)\hat{\phi}(0)|Vac\rangle = \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{ikx}\delta_{k,k'} = \sum_k \frac{\hbar}{2L\omega_k}e^{ikx}$$

Finally,

$$\begin{aligned}
\hat{\phi}(0)\hat{\phi}(0)|Vac\rangle &= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(\hat{a}_k + \hat{a}_k^\dagger\right)\left(\hat{a}_{k'} + \hat{a}_{k'}^\dagger\right)|Vac\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left(\hat{a}_k + \hat{a}_k^\dagger\right)|k'\rangle \\
&= \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\left[\delta_{k,k'}|Vac\rangle + |k,k'\rangle\right]
\end{aligned}$$

and so,

$$\langle Vac|\hat{\phi}(x)\hat{\phi}(0)|Vac\rangle = \frac{\hbar}{2L}\sum_k\sum_{k'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}\delta_{k,k'} = \sum_k \frac{\hbar}{2L\omega_k}$$

Therefore,

$$\begin{aligned}\langle Vac|\left(\hat{\phi}(x)-\hat{\phi}(0)\right)^2|Vac\rangle &= \langle Vac|\hat{\phi}^2(x)-\hat{\phi}(0)\hat{\phi}(x)-\hat{\phi}(x)\hat{\phi}(0)+\hat{\phi}^2(0)|Vac\rangle \\ &= \sum_k \frac{\hbar}{2L\omega_k}\left(1-e^{ikx}-e^{-ikx}+1\right) \\ &= \sum_k \frac{\hbar}{L\omega_k}\left(1-\cos(kx)\right)\end{aligned}$$

However, we know that $\omega_k = c|k|$, and also we can replace $\frac{1}{L}\sum_k$ with $\frac{1}{2\pi}\int dk$. Making these substitutions, we find

$$\langle Vac|\left(\hat{\phi}(x)-\hat{\phi}(0)\right)^2|Vac\rangle = \frac{\hbar}{2\pi}\int \frac{1-\cos(kx)}{c|k|}dk$$

However, the integrand is an even function so we can write

$$\langle Vac|\left(\hat{\phi}(x)-\hat{\phi}(0)\right)^2|Vac\rangle = \frac{\hbar}{2\pi c}\left(2\int_{k>0}\frac{1-\cos(kx)}{k}dk\right)$$

However, we must restrict our integral to values of k with $|k| < \frac{1}{a}$. We can then take the limit as $a \to 0$ to obtain the desired answer

$$\begin{aligned}\langle Vac|\left(\hat{\phi}(x)-\hat{\phi}(0)\right)^2|Vac\rangle &= \lim_{a\to 0}\frac{\hbar}{\pi c}\int_0^{1/a}\frac{1-\cos(kx)}{k}dk \\ &= \frac{\hbar}{\pi c}\log\left(\frac{x}{a}\right)\end{aligned}$$

B. Commentary

Notice that the expected deviations in the string displacement grow without bound as we look at further and further separated points so that the displacement of the string at one point and another a distance x away becomes less and less correlated as we increase x. This feature is special to 1d systems (in $d > 1$ spatial dimensions, $\lim_{x\to\infty} C(x) \approx \text{constant} + 1/x^{d-1}$, which tends towards a finite constant). Since the quantum string describes vibrations of atoms in a crystal lattice, this calculation shows that quantum fluctuations in the crystal lattice grow without bound in 1d and end up destroying the crystalline arrangement of atoms. Whereas, in 2d and 3d materials, one can have long-range crystalline ordering of atoms with finite quantum fluctuations. A final remark is that these results all have to do with zero-point fluctuations of the quantum ground-state. At finite temperature, there are also thermally excited vibrations. These thermal fluctuations end up destroying crystalline order also in 2d materials at long distances. These results are all collectively known as the "Mermin-Wagner" theorem, which constrains whether certain types of phases (e.g. crystals, magnets, superfluids, and superconductors) based on the dimensionality of the system.

II. MOMENTUM OF PHONONS

Show that the operator:

$$\hat{p} = \sum_k \hbar k \hat{n}_k \tag{4}$$

generates translations, i.e. defining an operator $\hat{T}_\alpha = e^{-i\alpha\hat{p}/\hbar}$, show that:

$$\begin{aligned}\hat{T}^\dagger_\alpha\hat{\phi}(x)\hat{T}_\alpha &= \hat{\phi}(x-\alpha)\\ \hat{T}^\dagger_\alpha\hat{\Pi}(x)\hat{T}_\alpha &= \hat{\Pi}(x-\alpha)\end{aligned} \tag{5}$$

Solution: Consider an arbitrary state $|\{n_k\}\rangle$, we will begin by calculating $\hat{\phi}(x-\alpha)|\{n_k\}\rangle$.

$$\begin{aligned}\hat{\phi}(x-\alpha)|\{n_k\}\rangle &= \sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'(x-\alpha)}\hat{a}_{k'} + e^{-ik'(x-\alpha)}\hat{a}^\dagger_{k'}\right)|\{n_k\}\rangle\\ &= \sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle + e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\end{aligned} \tag{6}$$

Now, recall that $\hat{n}_{k'}|\{n_k\}\rangle = n_{k'}|\{n_k\}\rangle$, so it follows that $\hat{p}|\{n_{k'}\}\rangle = \sum_{k'}\hbar k'\hat{n}_{k'}|\{n_k\}\rangle = p|\{n_k\}\rangle$ where $p = \sum_{k'}\hbar k' n_{k'}$. Using this result we can compute

$$\begin{aligned}\hat{T}^\dagger_\alpha\hat{\phi}(x)\hat{T}_\alpha|\{n_k\}\rangle &= \hat{T}^\dagger_\alpha\hat{\phi}(x)e^{-i\alpha\hat{p}/\hbar}|\{n_k\}\rangle\\ &= e^{-i\alpha p/\hbar}\hat{T}^\dagger_\alpha\hat{\phi}(x)|\{n_k\}\rangle\end{aligned}$$

we can now use the result of equation 6 with $\alpha = 0$ to write

$$\hat{T}^\dagger_\alpha\hat{\phi}(x)\hat{T}_\alpha|\{n_k\}\rangle = e^{-i\alpha p/\hbar}\hat{T}^\dagger_\alpha\left(\sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'x}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle + e^{-ik'x}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right)$$

Now, $\hat{n}^\dagger_k = \hat{n_k}$, therefore $\hat{p}^\dagger = \hat{p}$, and so $\hat{T}^\dagger_\alpha = e^{i\alpha\hat{p}/\hbar}$. Also,

$$\begin{aligned}\hat{p}|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle &= \left(\sum_{k\neq k'}\hbar k n_k + \hbar k'(n_{k'}\pm 1)\right)|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle = \left(\sum_k \hbar k n_k \pm \hbar k'\right)|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle\\ &= (p\pm\hbar k')|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle\end{aligned}$$

And so,

$$\begin{aligned}\hat{T}^\dagger_\alpha\hat{\phi}(x)\hat{T}_\alpha|\{n_k\}\rangle &= e^{-i\alpha p/\hbar}\left(\sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'x}\sqrt{n_{k'}}e^{i\alpha\hat{p}/\hbar}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right.\right.\\ &\qquad\left.\left.+e^{-ik'x}\sqrt{n_{k'}+1}e^{i\alpha\hat{p}/\hbar}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right.\\ &= e^{-i\alpha p/\hbar}\left(\sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'x}\sqrt{n_{k'}}e^{i\alpha(p-\hbar k')/\hbar}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right.\right.\\ &\qquad\left.\left.+e^{-ik'x}\sqrt{n_{k'}+1}e^{i\alpha(p+\hbar k')/\hbar}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right.\\ &= e^{-i\alpha p/\hbar}e^{i\alpha p/\hbar}\sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right.\\ &\qquad\left.+\ e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\\ &= \sum_{k'}\sqrt{\frac{\hbar}{2L\omega_{k'}}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle + e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\end{aligned}$$

Therefore, $\hat{T}_\alpha^\dagger \hat{\phi}(x)\hat{T}_\alpha|\{n_k\}\rangle = \hat{\phi}(x-\alpha)|\{n_k\}\rangle$ and since $|\{n_k\}\rangle$ was an arbitrary state, it follows that $\hat{T}_\alpha^\dagger \hat{\phi}(x)\hat{T}_\alpha = \hat{\phi}(x-\alpha)$.

Again, consider an arbitrary state $|\{n_k\}\rangle$, we will begin by calculating $\hat{\Pi}(x-\alpha)|\{n_k\}\rangle$.

$$\begin{aligned}\hat{\Pi}(x-\alpha)|\{n_k\}\rangle &= \sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'(x-\alpha)}\hat{a}_{k'} - e^{-ik'(x-\alpha)}\hat{a}_{k'}^\dagger\right)|\{n_k\}\rangle \\ &= \sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle - e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\end{aligned} \tag{7}$$

Now, recall that $\hat{n}_{k'}|\{n_k\}\rangle = n_{k'}|\{n_k\}\rangle$, so it follows that $\hat{p}|\{n_{k'}\}\rangle = \sum_{k'}\hbar k'\hat{n}_{k'}|\{n_k\}\rangle = p|\{n_k\}\rangle$ where $p = \sum_{k'}\hbar k' n_{k'}$. Using this result we can compute

$$\begin{aligned}\hat{T}_\alpha^\dagger\hat{\Pi}(x)\hat{T}_\alpha|\{n_k\}\rangle &= \hat{T}_\alpha^\dagger\hat{\Pi}(x)e^{-i\alpha\hat{p}/\hbar}|\{n_k\}\rangle \\ &= e^{-i\alpha p/\hbar}\hat{T}_\alpha^\dagger\hat{\Pi}(x)|\{n_k\}\rangle\end{aligned}$$

we can now use the result of equation 7 with $\alpha = 0$ to write

$$\hat{T}_\alpha^\dagger\hat{\Pi}(x)\hat{T}_\alpha|\{n_k\}\rangle = e^{-i\alpha p/\hbar}\hat{T}_\alpha^\dagger\left(\sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'x}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle - e^{-ik'x}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right)$$

Now, $\hat{n}_k^\dagger = \hat{n_k}$, therefore $\hat{p}^\dagger = \hat{p}$, and so $\hat{T}_\alpha^\dagger = e^{i\alpha\hat{p}/\hbar}$. Also,

$$\begin{aligned}\hat{p}|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle &= \left(\sum_{k\neq k'}\hbar k n_k + \hbar k'(n_{k'}\pm 1)\right)|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle = \left(\sum_k \hbar k n_k \pm \hbar k'\right)|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle \\ &= (p\pm\hbar k')|\{n_{k\neq k'}\}, n_{k'}\pm 1\rangle\end{aligned}$$

And so,

$$\begin{aligned}\hat{T}_\alpha^\dagger\hat{\Pi}(x)\hat{T}_\alpha|\{n_k\}\rangle &= e^{-i\alpha p/\hbar}\left(\sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'x}\sqrt{n_{k'}}e^{i\alpha\hat{p}/\hbar}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right.\right. \\ &\qquad \left.\left. -e^{-ik'x}\sqrt{n_{k'}+1}e^{i\alpha\hat{p}/\hbar}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right) \\ &= e^{-i\alpha p/\hbar}\left(\sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'x}\sqrt{n_{k'}}e^{i\alpha(p-\hbar k')/\hbar}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right.\right. \\ &\qquad \left.\left. -e^{-ik'x}\sqrt{n_{k'}+1}e^{i\alpha(p+\hbar k')/\hbar}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\right) \\ &= e^{-i\alpha p/\hbar}e^{i\alpha p/\hbar}\sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle\right. \\ &\qquad \left. -\ e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right) \\ &= \sum_{k'}\sqrt{\frac{\hbar\omega_{k'}}{2L}}\left(e^{ik'(x-\alpha)}\sqrt{n_{k'}}|\{n_{k\neq k'}\}, n_{k'}-1\rangle - e^{-ik'(x-\alpha)}\sqrt{n_{k'}+1}|\{n_{k\neq k'}\}, n_{k'}+1\rangle\right)\end{aligned}$$

Therefore, $\hat{T}_\alpha^\dagger\hat{\Pi}(x)\hat{T}_\alpha|\{n_k\}\rangle = \hat{\Pi}(x-\alpha)|\{n_k\}\rangle$ and since $|\{n_k\}\rangle$ was an arbitrary state, it follows that $\hat{T}_\alpha^\dagger\hat{\Pi}(x)\hat{T}_\alpha = \hat{\Pi}(x-\alpha)$.

III. DECAY OF A MASSIVE PARTICLE

Consider a 1D massive scalar field, φ coupled to a different, massless scalar field, ϕ, described by the Hamiltonian:

$$
\begin{aligned}
H &= H_\varphi + H_\phi + V_{\text{int}} \\
H_\varphi &= \frac{1}{2}\int dx \left[\hat{\Pi}_\varphi(x)^2 + c^2\left(\frac{\partial\hat{\varphi}}{\partial x}\right)^2 + m^2c^4\hat{\varphi}^2(x)\right] \\
H_\phi &= \frac{1}{2}\int dx \left[\hat{\Pi}_\phi(x)^2 + c^2\left(\frac{\partial\hat{\phi}}{\partial x}\right)^2\right] \\
V_{\text{int}} &= g\int dx\hat{\varphi}(x)\left(\hat{\phi}(x)\right)^2
\end{aligned}
\tag{8}
$$

Here, $\Pi_\varphi(x)$ is the conjugate variable to $\varphi(x)$ (i.e. $[\hat{\Pi}_\varphi(x), \varphi(x')] = -i\hbar\delta(x-x')$), and Π_ϕ is the conjugate variable to ϕ, and V_{int} represents an interaction energy that depends on the joint configuration of φ and ϕ.

1. Diagonalize the non-interacting part of the Hamiltonian: $H_0 = H_\varphi + H_\phi$ by going to the basis of normal modes (Fourier transform the fields) and introducing creation and annihilation operators (a.k.a. harmonic oscillator raising and lowering operators) for each mode. Show that the "phonons" of the massive scalar field, φ, have the energy of a massive relativistic particle.

   Solution: We will begin by Fourier transforming the non-interacting terms of the hamiltonian, $\hat{H}_0 = \hat{H}_\varphi + \hat{H}_\phi$. The fourier transform of an operator $\hat{O}(x)$ is

   $$\hat{O}(x) = \sum_k \frac{e^{ikx}}{\sqrt{L}}\hat{O}(k)$$

   Using this result we can write

$$
\begin{aligned}
\hat{H}_0 &= \frac{1}{2}\int\left(\hat{\Pi}_\phi(x)^2 + c^2\left(\frac{\partial\hat{\phi}}{\partial x}\right)^2 + \hat{\Pi}_\varphi(x)^2 + c^2\left(\frac{\partial\hat{\varphi}}{\partial x}\right)^2 + m^2c^4\hat{\varphi}(x)^2\right)dx \\
&= \frac{1}{2L}\sum_{k,k'}\int e^{i(k+k')x}\left(\hat{\Pi}_\varphi(k)\hat{\Pi}_\varphi(k') + \hat{\Pi}_\phi(k)\hat{\Pi}_\phi(k') - c^2kk'\hat{\varphi}(k)\hat{\varphi}(k') - c^2kk'\hat{\phi}(k)\hat{\phi}(k') + m^2c^4\hat{\varphi}(k)\hat{\varphi}(k')\right)dx \\
&= \frac{1}{2L}\sum_{k,k'}\delta_{k,-k'}\left(\hat{\Pi}_\varphi(k)\hat{\Pi}_\varphi(k') + \hat{\Pi}_\phi(k)\hat{\Pi}_\phi(k') - c^2kk'\hat{\varphi}(k)\hat{\varphi}(k') - c^2kk'\hat{\phi}(k)\hat{\phi}(k') + m^2c^4\hat{\varphi}(k)\hat{\varphi}(k')\right) \\
&= \frac{1}{2L}\sum_{k}\left(\hat{\Pi}_\varphi(k)\hat{\Pi}_\varphi(-k) + \hat{\Pi}_\phi(k)\hat{\Pi}_\phi(-k) - c^2kk'\hat{\varphi}(k)\hat{\varphi}(-k) - c^2kk'\hat{\phi}(k)\hat{\phi}(-k) + m^2c^4\hat{\varphi}(k)\hat{\varphi}(-k)\right)
\end{aligned}
$$

   Now, let $\omega_k = ck$ and $\tilde{\omega}_k = \sqrt{c^2k^2 + m^2c^4}$. Then introduce the operators

   $$\hat{a}_\phi(k) = \sqrt{\frac{\omega_k}{2\hbar}}\left(\hat{\phi}(k) + \frac{i}{\omega_k}\hat{\Pi}_\phi(k)\right) \qquad \text{and} \qquad \hat{a}^\dagger_\phi(k) = \sqrt{\frac{\omega_k}{2\hbar}}\left(\hat{\phi}(-k) - \frac{i}{\omega_k}\hat{\Pi}_\phi(-k)\right)$$

$$\hat{a}_\varphi(k) = \sqrt{\frac{\tilde{\omega}_k}{2\hbar}}\left(\hat{\varphi}(k) + \frac{i}{\tilde{\omega}_k}\hat{\Pi}_\varphi(k)\right) \qquad \text{and} \qquad \hat{a}^\dagger_\varphi(k) = \sqrt{\frac{\tilde{\omega}_k}{2\hbar}}\left(\hat{\varphi}(-k) - \frac{i}{\tilde{\omega}_k}\hat{\Pi}_\varphi(-k)\right)$$

Now, we note that

$$\hat{a}^\dagger_\phi(k)\hat{a}_\phi(k) = \frac{\omega_k}{2\hbar}\left(\hat{\phi}(-k)\hat{\phi}(k) + \frac{1}{\omega_k^2}\hat{\Pi}(k)\hat{\Pi}_\phi(-k) - \frac{i}{\omega_k}\hat{\Pi}_\phi(-k)\hat{\phi}(k) + \frac{i}{\omega_k}\hat{\phi}(-k)\hat{\Pi}_\phi(k)\right)$$

Now consider the last two terms in the above expression. Since these two terms will only appear for us inside the sum over all k in the hamiltonian, and they are symmetric under $k \to -k$. Therefore, for our purposes, we can write

$$\begin{aligned} -\frac{i}{\omega_k}\hat{\Pi}_\phi(-k)\hat{\phi}(k) + \frac{i}{\omega_k}\hat{\phi}(-k)\hat{\Pi}_\phi(k) &= -\frac{i}{\omega_k}\left[\hat{\Pi}_\phi(-k), \hat{\phi}(k)\right] \\ &= \frac{i}{\omega_k} i\hbar\delta_{-k,-k} \\ &= -\frac{\hbar}{\omega_k} \end{aligned}$$

Using this result, we find

$$\hat{a}^\dagger_\phi(k)\hat{a}_\phi(k) = \frac{\omega_k}{2\hbar}\left(\hat{\phi}(-k)\hat{\phi}(k) + \frac{1}{\omega_k^2}\hat{\Pi}(k)\hat{\Pi}_\phi(-k) - \frac{\hbar}{\omega_k}\right)$$

which then implies

$$\omega_k^2\hat{\phi}(-k)\hat{\phi}(k) + \hat{\Pi}_\phi(k)\hat{\Pi}_\phi(-k) = 2\hbar\omega_k\hat{a}^\dagger_\phi(k)\hat{a}_\phi(k) + \hbar\omega_k$$

A similar result holds for the massive scalar field, φ. Plugging these expressions into the hamiltonian gives

$$\hat{H}_0 = \frac{1}{L}\sum_k\left(\hbar\omega_k\hat{a}^\dagger_\phi(k)\hat{a}_\phi(k) + \frac{1}{2}\hbar\omega_k + \hbar\tilde{\omega_k}\hat{a}^\dagger_\varphi(k)\hat{a}_\varphi(k) + \frac{1}{2}\hbar\tilde{\omega}_k\right)$$

If we now consider the part of the Hamiltonian that corresponds to the massive scalar field,

$$\hat{H}_\varphi = \frac{1}{L}\sum_k\left(\hbar\tilde{\omega_k}\hat{a}^\dagger_\varphi(k)\hat{a}_\varphi(k) + \frac{1}{2}\hbar\tilde{\omega}_k\right)$$

we see that the energy above the zero point energy is given by summing the number of particles in the state k times their energy $\hbar\tilde{\omega}_k$. This energy can be written in terms of m and c as

$$\hbar\tilde{\omega}_k = \hbar\sqrt{c^2k^2 + m^2c^4}$$

This is the energy of a massive relativistic particle, so the phonons of the massive scalar field have the energy of a massive particle.

2. Write the interaction term in terms of these creation and annihilation operators.

   Solution: We can use the creation and annihilation operators to rewrite the field operators as

$$\hat{\phi}(k) = \sqrt{\frac{\hbar}{2\omega_k}}\left(\hat{a}_\phi(k) + \hat{a}_\phi^\dagger(-k)\right)$$

$$\hat{\varphi}(k) = \sqrt{\frac{\hbar}{2\tilde{\omega_k}}}\left(\hat{a}_\varphi(k) + \hat{a}_\varphi^\dagger(-k)\right)$$

   The interaction potential is given by

$$\hat{V}_{int} = g\int \hat{\varphi}(x)\left(\hat{\phi}(x)\right)^2 dx$$

   or, taking the Fourier transform,

$$\begin{aligned}
\hat{V}_{int} &= \frac{g}{L^{3/2}}\sum_{k,k',k''}\hat{\varphi}(k'')\hat{\phi}(k)\hat{\phi}(k')\int e^{i(k+k'+k'')x}dx \\
&= \frac{g}{L^{3/2}}\sum_{k,k',k''}\hat{\varphi}(k'')\hat{\phi}(k)\hat{\phi}(k')\delta_{k',-(k+k'')} \\
&= \frac{g}{L^{3/2}}\sum_{k,k'}\hat{\varphi}(k'')\hat{\phi}(k)\hat{\phi}(-k-k'')
\end{aligned}$$

   Substituting the expressions for the field operators in terms of the creation and annihilation operators, we find

$$\hat{V}_{int} = \frac{\hbar^{3/2}g}{2\sqrt{2}L^{3/2}}\sum_{k,k'}\frac{1}{\sqrt{\omega_k\omega_{k'}\tilde{\omega}_{-k-k'}}}\left(\hat{a}_\varphi(-k-k') + \hat{a}_\varphi^\dagger(k+k')\right)\left(\hat{a}_\phi(k) + \hat{a}_\phi^\dagger(-k)\right)\left(\hat{a}_\phi(k' + \hat{a}_\phi^\dagger(-k')\right)$$

3. Treating the interaction term as a perturbation, using Fermi's golden rule, find the rate at which a massive particle excitation of the φ field that is initially at rest (zero momentum) decays into two massless particles.

   Solution: Our initial state, $|i\rangle = |1_0\rangle_\varphi$ can be written in terms of the vacuum state of the massive scalar field, $|Vac\rangle_\varphi$, as $|i\rangle = \hat{a}_\varphi^\dagger(0)|Vac\rangle_\varphi$. The final state, $|f\rangle$ consists of two massless particles, but to conseve momentum, these particles must have opposite momenta. (i.e. if one particle has wave number k, the other must have wavenumber $-k$). Therefore we can write the final state in terms of the vacuum state of the massless field as $|f\rangle = |1_k, 1_{-k}\rangle = \hat{a}_\phi^\dagger(k)\hat{a}_\phi^\dagger(-k)|Vac\rangle_\phi$.

   Now, the initial energy of the system is mc^2, and the final energy of the system is $2\hbar\omega_k$. Therefore, the transition rate is given by Fermi's golden rule to be

$$\Gamma = \frac{2\pi}{\hbar}\sum_k |\langle f|\hat{V}_{int}|i\rangle|^2\delta(mc^2 - 2\hbar\omega_k)$$

If we expand the expression for the interaction potential in terms of the creation and anihilation operators found in the last problem, we can write the matrix element as

$$\begin{aligned}\langle f|\hat{V}_{int}|i\rangle = \langle f|&\hat{a}_\varphi(k')\hat{a}_\phi(k)\hat{a}_\phi(-k-k') + \hat{a}_\varphi(k')\hat{a}^\dagger_\phi(-k)\hat{a}_\phi(-k-k')\\ &+ \hat{a}_\varphi(k')\hat{a}_\phi(k)\hat{a}^\dagger_\phi(k+k') + \hat{a}_\varphi(k')\hat{a}^\dagger_\phi(-k)\hat{a}^\dagger_\phi(k+k') + \hat{a}^\dagger_\varphi(-k')\hat{a}_\phi(k)\hat{a}_\phi(-k-k')\\ &+ \hat{a}^\dagger_\varphi(-k')\hat{a}^\dagger_\phi(-k)\hat{a}_\phi(-k-k') + \hat{a}^\dagger_\varphi(-k')\hat{a}_\phi(k)\hat{a}^\dagger_\phi(k+k') + \hat{a}^\dagger_\varphi(-k')\hat{a}^\dagger_\phi(-k)\hat{a}^\dagger_\phi(k+k')|i\rangle\end{aligned}$$

However, many of these terms involve applying an annihilation operator to the vacuum. Also, some terms do not produce 2 excitations in the massless field, which causes the inner product with the final state to be zero, so the matrix element reduces to

$$\begin{aligned}\langle f|\hat{V}_{int}|i\rangle &= \langle f|(\hat{a}_\varphi(k')\hat{a}^\dagger_\phi(k)\hat{a}^\dagger_\phi(k+k') + \hat{a}^\dagger_\varphi(-k')\hat{a}^\dagger_\phi(-k)\hat{a}^\dagger_\phi(k+k'))|i\rangle\\ &= \langle f|\left(\sqrt{k+k'+1}\hat{a}_\varphi(k')\hat{a}^\dagger_\phi(-k)|1_0\rangle_\varphi\otimes|1_{k+k'}\rangle_\phi + \sqrt{k+k'+1}\hat{a}^\dagger_\varphi(-k')\hat{a}^\dagger_\phi(-k)|1_0\rangle_\varphi\otimes|1_{k+k'}\rangle_\phi\right)\\ &= \sqrt{k+k'+1}\delta_{q,0}\left(\sqrt{2}\delta_{k,0}|2_0\rangle_\phi + (1-\delta_{k,0})|1_k,1_{-k}\rangle_\phi\right)\otimes|Vac\rangle_\varphi\end{aligned}$$

Now, when we use this matrix element in fermi's golden rule, the delta function will not allow $k=0$, since the initial energy is non-zero, therefore, for our purposes, the matrix element is

$$\langle f|\hat{V}_{int}|i\rangle = \langle f|\sqrt{k+1}(1-\delta_{k,0})|1_k,1_{-k}\rangle_\phi\otimes|Vac\rangle_\varphi$$

Or, replacing the constants that were neglected,

$$\langle f|\hat{V}_{int}|i\rangle = g\left(\frac{\hbar}{2L}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\tilde{\omega}_0\omega_k^2}}\sqrt{k+1}$$

Then substituting into Fermi's golden rule and changing to an integral we obtain

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar}\sum_k g^2\left(\frac{\hbar}{2L}\right)^3\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt{\tilde{\omega}_0\omega_k^2}}\sqrt{k+1}\right)^2\delta(mc^2-2\hbar\omega_k)\\ &= \frac{2\pi}{\hbar}g^2\left(\frac{\hbar}{2}\right)^3\int\frac{dk^3}{(2\pi)^3}\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt{\tilde{\omega}_0\omega_k^2}}\sqrt{k+1}\right)^2 2\hbar c\delta(\frac{mc}{2\hbar}-k)\\ &= \frac{\hbar^3 c}{16\pi^2 mc}\left(1+\frac{2\hbar}{mc}\right)\end{aligned}$$