Problem Set # 12, Due: Wednesday April 19 by 11:00am

PHY 362K - Quantum Mechanics II, UT Austin, Spring 2017
(Dated: April 26, 2017)

Quantum electrodynamics and light-matter interactions

I. CAVITY QED

Consider a superconducting qubit inside an optical cavity. We can view the superconducting qubit as an artificial "atom" which has different discrete energy levels. Suppose that the optical cavity selects out a single photon mode with frequency ω, which we can describe as a harmonic oscillator with frequency ω, and corresponding creation and annihilation operators: $\hat{a}, \hat{a}^\dagger$ respectively. Suppose that only two energy levels of the qubit have energy difference that is close to the frequency of the cavity mode, so that we can ignore all the states of except these two levels. Then, we can view the superconducting qubit as a two-level system, or effective spin-1/2 degree of freedom, described by Pauli operators $\hat{\sigma}^{x,y,z}$.

With these assumptions and using the dipole approximation, we can model coupled cavity-qubit system by the Hamiltonian:

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\varepsilon_Q}{2}\sigma^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right) \tag{1}$$

where ω is the frequency of the cavity photon, ε_Q is the energy splitting between the two levels of the qubit (in the absence of the cavity), and g is a coupling constant describing the interaction of the qubit and the cavity. This model is known as the Jaynes-Cummings Model.

1. Solve for the eigenstates and energies of this model using the following steps:

   (a) Show that the number of photons, $\hat{n}_\gamma = \hat{a}^\dagger\hat{a}$, plus the total spins of the qubits, σ^z is conserved (i.e. that $\hat{N} = \hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}^z$ commutes with the $\hat{H}$). This means we can simultaneously diagonalize $\hat{N}$ and $\hat{H}$, i.e. break $\hat{H}$ into blocks with fixed eigenvalue of $\hat{N}$.

   Solution: We will show that $\hat{N}\hat{H} = \hat{H}\hat{N}$. To begin, consider,

$$\begin{aligned}\hat{H}\hat{N} &= \left(\hbar\omega\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)\right)\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}^z\right)\\ &= \hbar\omega\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z\hat{a}^\dagger\hat{a} + g\hat{\sigma}^+\hat{a}\hat{a}^\dagger\hat{a} + g\hat{\sigma}^-\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega\hat{a}^\dagger\hat{a}\hat{\sigma}^z + \frac{\epsilon_q}{4}\hat{\sigma}^z\hat{\sigma}^z + \frac{1}{2}g\hat{\sigma}^+\hat{a}\hat{\sigma}^z + \frac{1}{2}g\hat{\sigma}^-\hat{a}^\dagger\hat{\sigma}^z\end{aligned}$$

   We will now consider some terms in this expression separately. Recall that $\left[\hat{a}^\dagger, \hat{a}\right] = -1, [\hat{\sigma}^x, \hat{\sigma}^y] = 2i\hat{\sigma}^z, [\hat{\sigma}^x, \hat{\sigma}^z] = -2i\hat{\sigma}^y, [\hat{\sigma}^y, \hat{\sigma}^z] = 2i\hat{\sigma}^x$. Now consider,

$$\begin{aligned}g\hat{\sigma}^+\hat{a}\hat{a}^\dagger\hat{a} &= g\hat{\sigma}^+\left(\hat{a}^\dagger\hat{a} + 1\right)\hat{a}\\ &= g\hat{\sigma}^+\hat{a}^\dagger\hat{a}\hat{a} + g\hat{\sigma}^+\hat{a}\\ &= (\hat{a}^\dagger\hat{a})g\hat{\sigma}^+\hat{a} + g\hat{\sigma}^+\hat{a}\end{aligned}$$

   and also,

$$\begin{aligned}g\hat{\sigma}^-\hat{a}^\dagger\hat{a}^\dagger\hat{a} &= g\hat{\sigma}^-\hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)\\ &= (\hat{a}^\dagger\hat{a})g\hat{\sigma}^-\hat{a}^\dagger - g\hat{\sigma}^-\hat{a}^\dagger\end{aligned}$$

and also,

$$\begin{aligned}\frac{1}{2}g\hat{\sigma}^+\hat{a}\hat{\sigma}^z &= \frac{1}{2}g(\hat{\sigma}^x + i\hat{\sigma}^y)\hat{\sigma}^z\hat{a}\\ &= \frac{1}{2}g\left(\hat{\sigma}^z\hat{\sigma}^x - 2i\hat{\sigma}^y + i(\hat{\sigma}^z\hat{\sigma}^y + 2i\hat{\sigma}^x)\right)\hat{a}\\ &= \frac{1}{2}g\hat{\sigma}^z(\hat{\sigma}^x + i\hat{\sigma}^y)\hat{a} + \frac{1}{2}g\left(-2i\hat{\sigma}^y - 2\hat{\sigma}^x\right)\hat{a}\\ &= \frac{1}{2}\hat{\sigma}^z g\hat{\sigma}^+\hat{a} - g\hat{\sigma}^+\hat{a}\end{aligned}$$

finally,

$$\begin{aligned}\frac{1}{2}g\hat{\sigma}^-\hat{a}^\dagger\hat{\sigma}^z &= \frac{1}{2}g(\hat{\sigma}^x - i\hat{\sigma}^y)\hat{\sigma}^z\hat{a}^\dagger\\ &= \frac{1}{2}g\left(\hat{\sigma}^z\hat{\sigma}^x - 2i\hat{\sigma}^y - i(\hat{\sigma}^z\hat{\sigma}^y + 2i\hat{\sigma}^x)\right)\hat{a}^\dagger\\ &= \frac{1}{2}g\hat{\sigma}^z(\hat{\sigma}^x - i\hat{\sigma}^y)\hat{a}^\dagger + \frac{1}{2}g\left(-2i\hat{\sigma}^y + 2\hat{\sigma}^x\right)\hat{a}^\dagger\\ &= \frac{1}{2}\hat{\sigma}^z g\hat{\sigma}^-\hat{a}^\dagger + g\hat{\sigma}^-\hat{a}^\dagger\end{aligned}$$

Now, using these results in the above expression for $\hat{H}\hat{N}$ and the fact that the creation and annihilation operators commute with the Pauli matrices, we find

$$\begin{aligned}\hat{H}\hat{N} &= \hbar\omega\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger\hat{a})g\hat{\sigma}^+\hat{a} + g\hat{\sigma}^+\hat{a} + (\hat{a}^\dagger\hat{a})g\hat{\sigma}^-\hat{a}^\dagger - g\hat{\sigma}^-\hat{a}^\dagger + \frac{1}{2}\hbar\omega\hat{a}^\dagger\hat{a}\hat{\sigma}^z + \frac{\epsilon_q}{4}\hat{\sigma}^z\hat{\sigma}^z\\ &\qquad + \frac{1}{2}\hat{\sigma}^z g\hat{\sigma}^+\hat{a} - g\hat{\sigma}^+\hat{a} + \frac{1}{2}\hat{\sigma}^z g\hat{\sigma}^-\hat{a}^\dagger + g\hat{\sigma}^-\hat{a}^\dagger\\ &= (\hat{a}^\dagger\hat{a})\hbar\omega\hat{a}^\dagger\hat{a} + (\hat{a}^\dagger\hat{a})\frac{\epsilon_q}{2}\hat{\sigma}^z + (\hat{a}^\dagger\hat{a})g\hat{\sigma}^+\hat{a} + (\hat{a}^\dagger\hat{a})g\hat{\sigma}^-\hat{a}^\dagger + \left(\frac{1}{2}\hat{\sigma}^z\right)\hbar\omega\hat{a}^\dagger\hat{a} + \left(\frac{1}{2}\hat{\sigma}^z\right)\frac{\epsilon_q}{2}\hat{\sigma}^z\\ &\qquad + \left(\frac{1}{2}\hat{\sigma}^z\right)g\hat{\sigma}^+\hat{a} + \left(\frac{1}{2}\hat{\sigma}^z\right)g\hat{\sigma}^-\hat{a}^\dagger\\ &= \hat{a}^\dagger\hat{a}\left(\hbar\omega\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)\right) + \frac{1}{2}\hat{\sigma}^z\left(\hbar\omega\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)\right)\\ &= \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}^z\right)\left(\hbar\omega\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)\right)\\ &= \hat{N}\hat{H}\end{aligned}$$

Therefore $\hat{N}$ and $\hat{H}$ commute.

(b) Compute the matrix elements of $\hat{H}$ in the basis of product states: $|\sigma, n\rangle$, where $\sigma^z|\sigma, n\rangle = \sigma|\sigma, n\rangle$, $\sigma \in \{-1, +1\}$, is the state of the qubit, and n is the number of cavity photons.

Solution: The matrix elements are:

$$\begin{aligned}\langle\sigma', n'|\hat{H}|\sigma, n\rangle &= \langle\sigma', n'|\left(\hbar\omega\hat{a}^\dagger\hat{a} + \frac{\epsilon_q}{2}\hat{\sigma}^z + g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)\right)|\sigma, n\rangle\\ &= \hbar\omega n\langle\sigma', n'|\sigma, n\rangle + \frac{\epsilon_q}{2}\sigma\langle\sigma', n'|\sigma, n\rangle + g\sqrt{n}\langle\sigma', n'|\hat{\sigma}^+|\sigma, n-1\rangle + g\sqrt{n+1}\langle\sigma', n'|\hat{\sigma}^-|\sigma, n+1\rangle\\ &= \left(\hbar\omega n + \frac{\epsilon_q}{2}\sigma\right)\delta_{\sigma,\sigma'}\delta_{n,n'} + g\sqrt{n}\delta_{\sigma,-1}\langle\sigma', n'|1, n-1\rangle + g\sqrt{n+1}\delta_{\sigma,1}\langle\sigma', n'|-1, n+1\rangle\\ &= \left(\hbar\omega n + \frac{\epsilon_q}{2}\sigma\right)\delta_{\sigma,\sigma'}\delta_{n,n'} + g\left(\sqrt{n}\delta_{\sigma,-1}\delta_{\sigma',1}\delta_{n',n-1} + \sqrt{n+1}\delta_{\sigma,1}\delta_{\sigma',-1}\delta_{n',n+1}\right)\end{aligned}$$

(c) Diagonalize $\hat{H}$ to find the energy levels and eigenstates.

Solution: We can write the Hamiltonian as a block matrix, as follows:

$$\begin{pmatrix} \begin{pmatrix} \langle -1,0|\hat{H}|-1,0\rangle & \langle -1,0|\hat{H}|1,0\rangle \\ \langle 1,0|\hat{H}|-1,0\rangle & \langle 1,0|\hat{H}|1,0\rangle \end{pmatrix} & \begin{pmatrix} \langle -1,0|\hat{H}|-1,1\rangle & \langle -1,0|\hat{H}|1,1\rangle \\ \langle 1,0|\hat{H}|-1,1\rangle & \langle 1,0|\hat{H}|1,1\rangle \end{pmatrix} & \cdots \\ \begin{pmatrix} \langle -1,1|\hat{H}|-1,0\rangle & \langle -1,1|\hat{H}|1,0\rangle \\ \langle 1,1|\hat{H}|-1,0\rangle & \langle 1,1|\hat{H}|1,0\rangle \end{pmatrix} & \begin{pmatrix} \langle -1,1|\hat{H}|-1,1\rangle & \langle -1,1|\hat{H}|1,0\rangle \\ \langle 1,1|\hat{H}|-1,1\rangle & \langle 1,1|\hat{H}|1,0\rangle \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Or, using the result found in part b,

$$\begin{pmatrix} -\frac{\epsilon_Q}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{\epsilon_Q}{2} & g & 0 & 0 & 0 & \cdots \\ 0 & g & \hbar\omega - \frac{\epsilon_Q}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \hbar\omega + \frac{\epsilon_Q}{2} & \sqrt{2}g & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{2}g & 2\hbar\omega - \frac{\epsilon_Q}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

We can see that in order to diagonalize this Hamiltonian, we need to diagonalize 2x2 blocks of the form

$$\begin{pmatrix} \hbar\omega n - \epsilon_Q/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & \hbar\omega(n+1) - \epsilon_Q/2 \end{pmatrix}$$

we can use Mathematica to find the eigenvectors and eigenvalues of this matrix. Doing so gives the eigenvectors

$$|\psi_{n,-}\rangle = \left(\frac{\epsilon_Q - \hbar\omega - \sqrt{4g^2(n+1) + (\epsilon_Q - \hbar\omega)^2}}{2g\sqrt{n+1}}\right)|n,-1\rangle + |n+1,1\rangle$$

$$\text{with eigenvalue} \qquad \lambda_- = \frac{1}{2}\left(\hbar\omega(2n+1) - \sqrt{4g^2(n+1) + (\epsilon_Q - \hbar\omega)^2}\right)$$

and

$$|\psi_{n,+}\rangle = \left(\frac{\epsilon_Q - \hbar\omega + \sqrt{4g^2(n+1) + (\epsilon_Q - \hbar\omega)^2}}{2g\sqrt{n+1}}\right)|n,-1\rangle + |n+1,1\rangle$$

$$\text{with eigenvalue} \qquad \lambda_- = \frac{1}{2}\left(\hbar\omega(2n+1) + \sqrt{4g^2(n+1) + (\epsilon_Q - \hbar\omega)^2}\right)$$

So we can see that the normalized eigenstates of the system are given by $|\psi_{n,\pm}\rangle = \frac{1}{\sqrt{\alpha_{n,\pm}^2+1}}\left(\alpha_{n,\pm}|n,-1\rangle + |n+1,1\rangle\right)$, $n \geq 1$, where $\alpha_{n,\pm} = \frac{\epsilon_Q - \hbar\omega \pm \sqrt{4g^2(n+1)+(\epsilon_Q-\hbar\omega)^2}}{2g\sqrt{n+1}}$ and $|\psi_{0,\pm}\rangle = |0,\pm 1\rangle$ and that the corresponding energies are $E_{n,\pm} = \frac{1}{2}\left(\hbar\omega(2n+1) \pm \sqrt{4g^2(n+1) + (\epsilon_Q - \hbar\omega)^2}\right)$ for $n \geq 1$ and $E_{0,\pm} = \pm\frac{\epsilon_Q}{2}$.

(d) Sketch the lowest several energy levels of the system as a function of the coupling, g.

Solution:

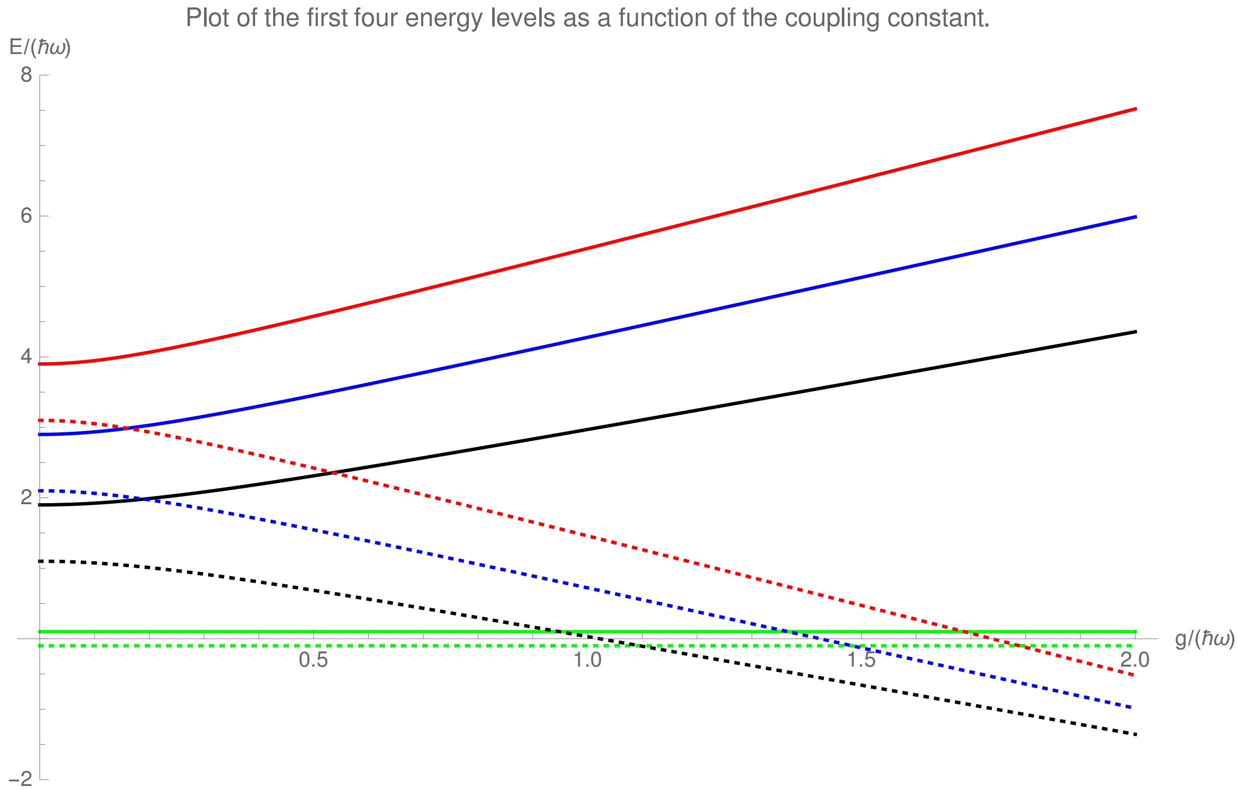

FIG. 1: Plot of the first four energy levels of the system as a function of the coupling constant, g, with $\epsilon_Q = 0.2\hbar\omega$. The colors differentiate the different values of n, with green corresponding to $n = 0$, black to $n = 1$, blue to $n = 2$, and red to $n = 3$. The dotted lines represent the $E_{n,-}$ energy for the appropriate n, while the solid lines represent the $E_{n,+}$ energy.

(e) Compute and sketch the expected number of cavity photons in the ground-state as a function of the coupling g.

The results of parts c and d show that the ground state depends on the value of the coupling constant, g. As g gets larger, the energies $E_{n,-}$ decrease without bound and will start to dip below the previous ground state energy. If we denote the ground state (which depends on g) as $|G(g)\rangle$, then the expected number of cavity photons in the ground state, N, is given by

$$N = \langle G(g)|\hat{a}^\dagger \hat{a}|G(g)\rangle$$

Now, the $\psi_{0,-}$ state is the ground state until $E_{1,-} = E_{0,-} \implies g = \frac{1}{2\sqrt{2}}\sqrt{(\epsilon_Q + 3\hbar\omega)^2 - (\epsilon_Q - \hbar\omega)^2}$, then the ground state is $\psi_{1,-}$ until $E_{1,-} = E_{2,-} \implies g = \hbar\omega\sqrt{5 + \sqrt{25 - 2\frac{\epsilon_Q}{\hbar\omega} + \left(\frac{\epsilon_Q}{\hbar\omega}\right)^2}}$.

Therefore, for $0 \leq g \leq \frac{1}{2\sqrt{2}}\sqrt{(\epsilon_Q + 3\hbar\omega)^2 - (\epsilon_Q - \hbar\omega)^2}$, we have

$$N = \langle 0, -1|\hat{a}^\dagger \hat{a}|0, -1\rangle = 0$$

And for $\frac{1}{2\sqrt{2}}\sqrt{(\epsilon_Q+3\hbar\omega)^2-(\epsilon_Q-\hbar\omega)^2} < g \leq \hbar\omega\sqrt{5+\sqrt{25-2\frac{\epsilon_Q}{\hbar\omega}+\left(\frac{\epsilon_Q}{\hbar\omega}\right)^2}}$, we have

$$
\begin{aligned}
N = \langle\psi_{1,-}|\hat{a}^\dagger\hat{a}|\psi_{1,-}\rangle &= \frac{1}{\alpha_{1,-}^2+1}\left(\alpha_{1,-}^*\langle 1,-1| + \langle 2,-1|\right)\hat{a}^\dagger\hat{a}\left(\alpha_{1,-}|1,-1\rangle + |2,-1\rangle\right) \\
&= \frac{1}{\alpha_{1,-}^2+1}\left(\alpha_{1,-}^*\langle 1,-1| + \langle 2,-1|\right)\left(\alpha_{1,-}|1,-1\rangle + 2|2,-1\rangle\right) \\
&= \frac{1}{\alpha_{1,-}^2+1} \\
&= \frac{1}{\left(\frac{\epsilon_Q-\hbar\omega-\sqrt{8g^2+(\epsilon_Q-\hbar\omega)^2}}{2g\sqrt{2}}\right)^2+1}
\end{aligned}
$$

We can plot this expected value as a function of g and obtain the following plot.

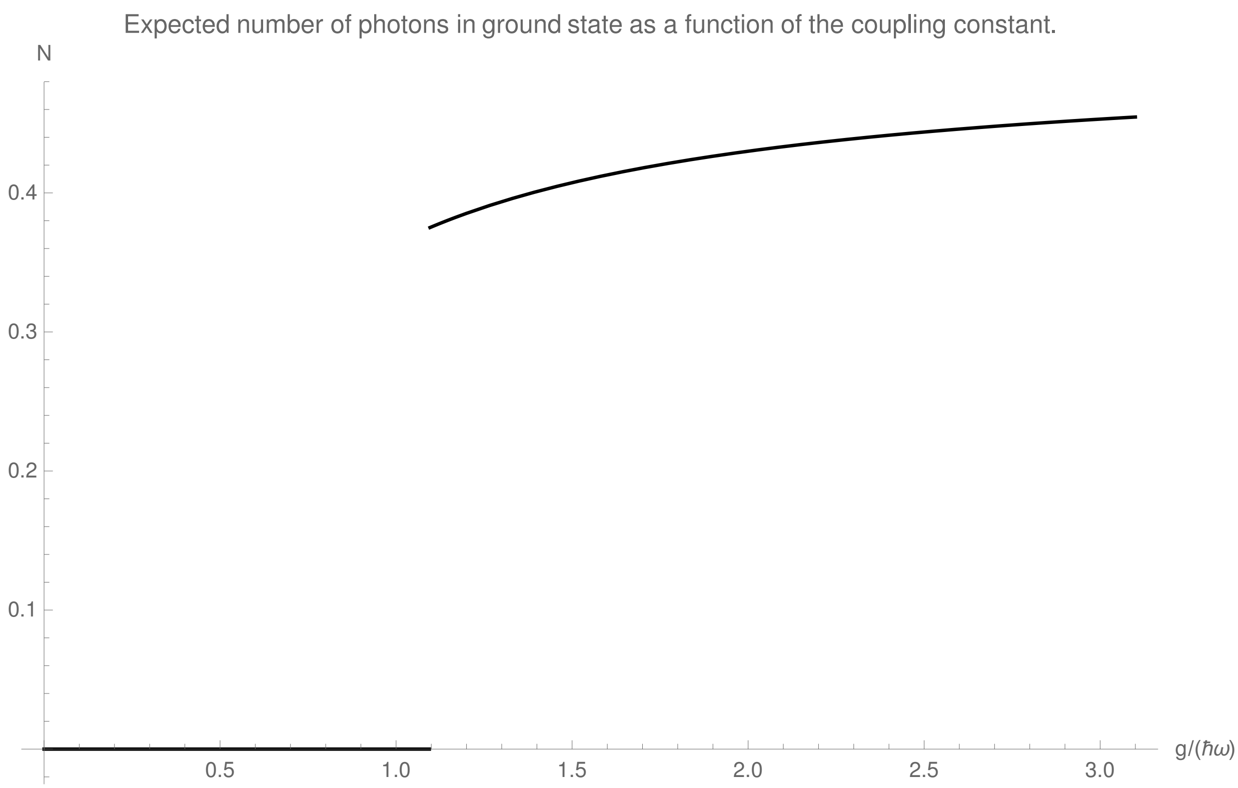

FIG. 2: Plot of the expected number of photons in the ground state as a function of the coupling constant, g, with $\epsilon_Q = 0.2\hbar\omega$.

2. Next, consider two qubits coupled to the same cavity mode:

$$
\hat{H}_2 = \hbar\omega\hat{a}^\dagger\hat{a} + \sum_{i=1,2}\left[\frac{\varepsilon_{Q,i}}{2}\sigma_i^z + g\left(\hat{\sigma}_i^+\hat{a} + \hat{\sigma}_i^-\hat{a}^\dagger\right)\right] \tag{2}
$$

   (a) First consider the case where the two qubit energies are equal: $\varepsilon_{Q,1} = \varepsilon_{Q,2} = \varepsilon_0$. Suppose we start with a state where qubit #1 is excited (↑) and qubit # 2 is in its ground state (↓). What is the rate for the second order process where the excitation of qubit #1 transfers to qubit #2 via a virtual excitation of the cavity photon? Sketch your answer as a function of $\varepsilon_0 - \Delta$ labeling notable features and asymptotic functional forms.

Solution: We will solve this problem by considering the intermediate states of this transition process and using 2nd order Fermi's golden rule. First, assume there are n photons initially in the cavity. The initial state of the first cubit is $|\uparrow\rangle$, and the initial state of the second cubit is $|\downarrow\rangle$, and the initial state of the field is $|n\rangle$. So the initial state of the whole system is $|i\rangle = |n\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle$, which we will denote as $|i\rangle = |n, \uparrow, \downarrow\rangle$. The final state of the system is then $|f\rangle = |n\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle = |n, \downarrow, \uparrow\rangle$.

There are two possible intermediate states for this process, one is $|m_1\rangle = |n+1, \downarrow, \downarrow\rangle$, which corresponds to the first cubit emitting a photon into the cavity, and the other is $|m_2\rangle = |n-1, \uparrow, \uparrow\rangle$, which corresponds to the second cubit absorbing a photon from the cavity.

Let $\hat{H}_{int} = g\left(\hat{\sigma}_1^+ \hat{a} + \hat{\sigma}_1^- \hat{a}^\dagger + \hat{\sigma}_2^+ \hat{a} + \hat{\sigma}_2^- \hat{a}^\dagger\right)$ be the interaction portion of the hamiltonian. Then Fermi's golden rule says

$$\Gamma = \frac{2\pi}{\hbar} \left| \sum_{j=1,2} \frac{\langle f|\hat{H}_{int}|m_j\rangle\langle m_j|\hat{H}_{int}|i\rangle}{E_i - E_{m_j}} \right|^2 \delta(E_f - E_i)$$

The initial energy of the system is $E_i = -\frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} + n\hbar\omega = n\hbar\omega$. The energies of the two intermediate states are $E_{m_1} = -\frac{\epsilon_0}{2} - \frac{\epsilon_0}{2} + (n+1)\hbar\omega = (n+1)\hbar\omega - \epsilon_0$ and $E_{m_2} = \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} + (n-1)\hbar\omega = (n-1)\hbar\omega + \epsilon_0$. Finally, the energy of the final state is $E_i = \frac{\epsilon_0}{2} - \frac{\epsilon_0}{2} + n\hbar\omega = n\hbar\omega$. We will now compute the matrix elements needed for Fermi's golden rule.

$$\begin{aligned}
\langle f|\hat{H}_{int}|m_1\rangle &= g\langle n, \downarrow, \uparrow| \left(\hat{\sigma}_1^+ \hat{a} + \hat{\sigma}_1^- \hat{a}^\dagger + \hat{\sigma}_2^+ \hat{a} + \hat{\sigma}_2^- \hat{a}^\dagger\right) |n+1, \downarrow, \downarrow\rangle \\
&= g\langle n, \downarrow, \uparrow| \left(\sqrt{n+1}|n, \uparrow, \downarrow\rangle + 0 + \sqrt{n+1}|n, \downarrow, \uparrow\rangle + 0\right) \\
&= g\sqrt{n+1}
\end{aligned}$$

$$\begin{aligned}
\langle f|\hat{H}_{int}|m_2\rangle &= g\langle n, \downarrow, \uparrow| \left(\hat{\sigma}_1^+ \hat{a} + \hat{\sigma}_1^- \hat{a}^\dagger + \hat{\sigma}_2^+ \hat{a} + \hat{\sigma}_2^- \hat{a}^\dagger\right) |n-1, \uparrow, \uparrow\rangle \\
&= g\langle n, \downarrow, \uparrow| \left(0 + \sqrt{n}|n, \downarrow, \uparrow\rangle + 0 + \sqrt{n}|n, \uparrow, \downarrow\rangle\right) \\
&= g\sqrt{n}
\end{aligned}$$

$$\begin{aligned}
\langle m_1|\hat{H}_{int}|i\rangle &= g\langle n+1, \downarrow, \downarrow| \left(\hat{\sigma}_1^+ \hat{a} + \hat{\sigma}_1^- \hat{a}^\dagger + \hat{\sigma}_2^+ \hat{a} + \hat{\sigma}_2^- \hat{a}^\dagger\right) |n, \uparrow, \downarrow\rangle \\
&= g\langle n+1, \downarrow, \downarrow| \left(0 + \sqrt{n+1}|n+1, \downarrow, \downarrow\rangle + \sqrt{n}|n-1, \uparrow, \uparrow\rangle + 0\right) \\
&= g\sqrt{n+1}
\end{aligned}$$

$$\begin{aligned}
\langle m_2|\hat{H}_{int}|i\rangle &= g\langle n-1, \uparrow, \uparrow| \left(\hat{\sigma}_1^+ \hat{a} + \hat{\sigma}_1^- \hat{a}^\dagger + \hat{\sigma}_2^+ \hat{a} + \hat{\sigma}_2^- \hat{a}^\dagger\right) |n, \uparrow, \downarrow\rangle \\
&= g\langle n-1, \uparrow, \uparrow| \left(0 + \sqrt{n+1}|n+1, \downarrow, \downarrow\rangle + \sqrt{n}|n-1, \uparrow, \uparrow\rangle + 0\right) \\
&= g\sqrt{n}
\end{aligned}$$

Since, the final and initial energies of the system are the same in this process, we can neglect the energy conserving delta function in Fermi's golden rule. Then using the expressions for the matrix elements that we just calculated, we find

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar}\left|\frac{g\sqrt{n+1}g\sqrt{n+1}}{n\hbar\omega-((n+1)\hbar\omega-\epsilon_0)}+\frac{g\sqrt{n}g\sqrt{n}}{n\hbar\omega-((n-1)\hbar\omega+\epsilon_0)}\right|^2\\ &= \frac{2\pi}{\hbar}g^4\left|\frac{n+1}{\epsilon_0-\hbar\omega}+\frac{n}{\hbar\omega-\epsilon_0}\right|^2\\ &= \frac{2\pi}{\hbar}g^4\left|\frac{1}{\epsilon_0-\hbar\omega}\right|^2\\ &= \frac{2\pi g^4}{\hbar(\epsilon_0-\hbar\omega)^2}\end{aligned}$$

we can plot this transition rate as a function of ϵ_0 as follows

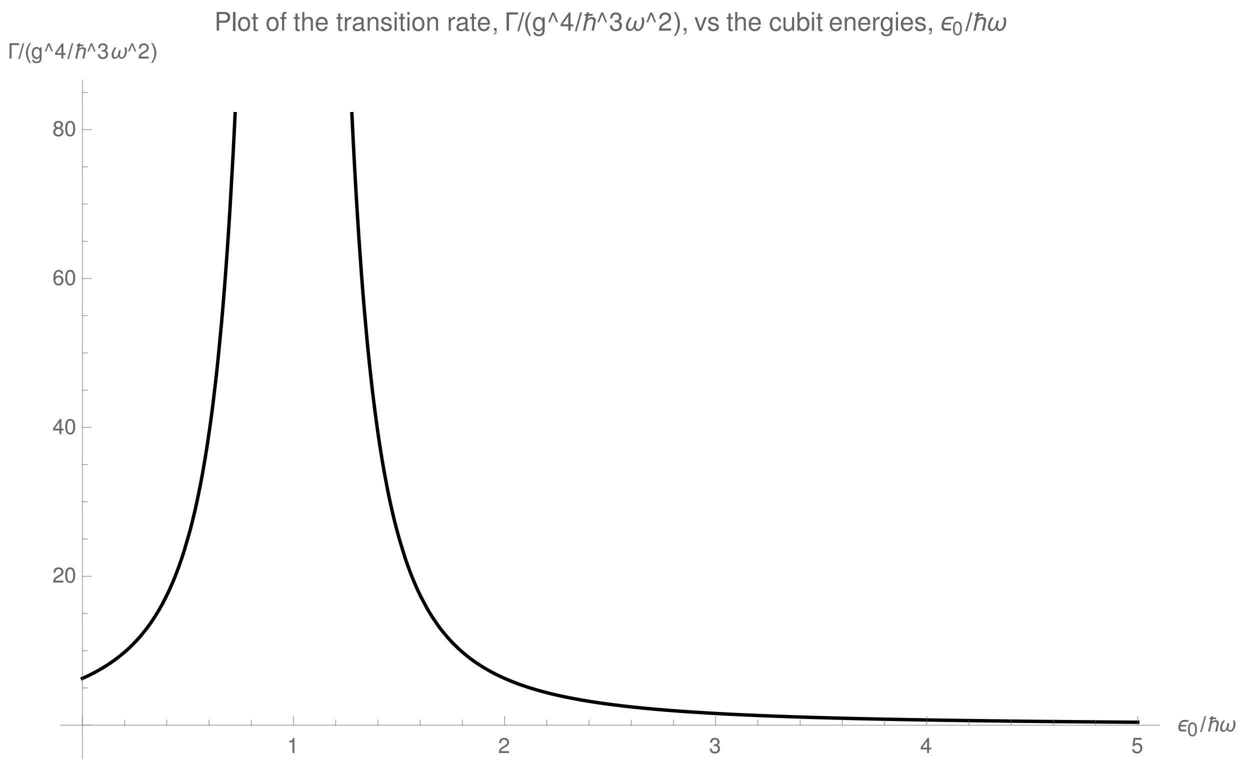

FIG. 3: Plot of the ratio of the transition rate to $\frac{g^4}{\hbar^3\omega^2}$ as a function of $\frac{\epsilon_0}{\hbar\omega}$. Note that the transition rate increases dramatically when the cubit energies approach $\hbar\omega$. The transition rate decreases to zero as the cubit energies get farther away from $\hbar\omega$. This is reminiscent of resonance in driven oscilatory systems.

II. SCATTERING OF LIGHT BY AN ELECTRON (THOMAS CROSS-SECTION)

Using the Born approximation, compute the scattering cross section for the following scattering process: Suppose we initially have N photon's with wave-vector $\vec{k} \sim \vec{e}_z$, and polarization $\vec{e}_{\vec{k},\lambda} \perp \vec{e}_z$, and an electron at rest (zero momentum). What is the cross section (using the Born approximation) for one of these photons to scatter into a different wave-vector $\vec{k'}$ and polarization $\vec{e}_{k',\lambda'}$, with the the electron recoiling with momentum $\vec{p}$.

At first order in perturbation theory, this process occurs through the interaction term:

$$\hat{H}^{(2)}_{\text{int}} = \int d^3r \frac{e^2}{2mc^2}\hat{\rho}(\vec{r})|\hat{\vec{A}}(\vec{r})|^2 \tag{3}$$

1. Compute the matrix element between the initial and final states of the electron and photon. Show that the matrix element is zero unless the total momentum of the electron plus photon system is conserved (the momentum of a photon with wave-vector $\vec{k}$ is $\hbar\vec{k}$).

   Solution: The initial state of the electron and photons is $|\vec{0}\rangle \otimes |N_{\vec{k},\lambda}\rangle$, where $|N_{\vec{k},\lambda}\rangle$ represents the state of N photons with wavevector $\vec{k}$ and polarization $e_{\vec{k},\lambda}$. Using this same notation, the final state of the electron and photons is $|\vec{p}\rangle \otimes |(N-1)_{\vec{k},\lambda}; 1_{\vec{k'},\lambda'}\rangle$. The matrix element of the interaction between these two states is given by

$$\begin{aligned}
\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle &= \langle f| \left(\int d^3r \frac{e^2}{2mc^2}\hat{\rho}(\vec{r})|\hat{\vec{A}}(\vec{r})|^2\right)|i\rangle \\
&= \frac{e^2}{2mc^2}\int d^3r \left[\langle f|\hat{\rho}(\vec{r})|\hat{\vec{A}}(\vec{r})|^2|i\rangle\right]
\end{aligned}$$

   Now, the $\hat{\rho}(r)$ operator only acts on the electron state, while the $|\hat{A}(\vec{r})|^2$ operator only acts on the photon state, so

$$\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle = \frac{e^2}{2mc^2}\int d^3r \left[\langle\vec{p}|\hat{\rho}(\vec{r})|\vec{0}\rangle\langle(N-1)_{\vec{k},\lambda}; 1_{\vec{k'},\lambda'}||\hat{A}(\vec{r})|^2|N_{\vec{k},\lambda}\rangle\right]$$

   Now, $\hat{\rho}(\vec{r}) = \delta(\vec{r} - \hat{\vec{r}}_e)$, and so we can compute its matrix element with the electron states as follows

$$\begin{aligned}
\langle\vec{p}|\hat{\rho}(\vec{r})|\vec{0}\rangle &= \left(\int d^3r_1 \frac{1}{\sqrt{L^3}}e^{i\vec{p}\cdot\vec{r_1}/\hbar}\langle r_1|\right)\left(\delta(\vec{r} - \hat{\vec{r}}_e)\right)\left(\int d^3r_2 \frac{1}{\sqrt{L^3}}e^{-i\vec{0}\cdot\vec{r_2}/\hbar}|r_2\rangle\right) \\
&= \frac{1}{L^3}\int d^3r_1 \int d^3r_2 e^{i\vec{p}\cdot\vec{r_1}/\hbar}\langle r_1|\delta(\vec{r} - \hat{\vec{r}}_e)|r_2\rangle
\end{aligned}$$

   But for any function of an operator, $f(\hat{x})$, when f acts on an eigenstate of $\hat{x}$, we can replace $\hat{x}$ with its eigenvalue x, i.e. $f(\hat{x})|x\rangle = f(x)|x\rangle$. Applying this to the delta function, we obtain

$$\begin{aligned}
\langle\vec{p}|\hat{\rho}(\vec{r})|\vec{0}\rangle &= \frac{1}{L^3}\int d^3r_1 \int d^3r_2 e^{i\vec{p}\cdot\vec{r_1}/\hbar}\langle r_1|\delta(\vec{r} - \vec{r_2})|r_2\rangle \\
&= \frac{1}{L^3}\int d^3r_1 \int d^3r_2 e^{i\vec{p}\cdot\vec{r_1}/\hbar}\delta(\vec{r} - \vec{r_2})\langle r_1|r_2\rangle \\
&= \frac{1}{L^3}\int d^3r_1 \int d^3r_2 e^{i\vec{p}\cdot\vec{r_1}/\hbar}\delta(\vec{r} - \vec{r_2})\delta(\vec{r_1} - \vec{r_2}) \\
&= \frac{1}{L^3}\int d^3r_1 e^{i\vec{p}\cdot\vec{r_1}/\hbar}\delta(\vec{r} - \vec{r_1}) \\
&= \frac{1}{L^3}e^{i\vec{p}\cdot\vec{r}/\hbar}
\end{aligned}$$

Using this result, we find

$$\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle = \frac{e^2}{2mc^2}\int d^3r \left[\frac{1}{L^3}e^{i\vec{p}\cdot\vec{r}/\hbar}\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}||\hat{A}(\vec{r})|^2|N_{\vec{k},\lambda}\rangle\right]$$

We must now compute the matrix element of $|\hat{A}(\vec{r})|^2$ with the photon states. We know that $\hat{A}(\vec{r}) = \sum_{\vec{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_k}}\vec{e}_{\vec{k},\lambda}\left(\hat{a}_{\vec{k},\lambda}\frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{L^3}} + \hat{a}^\dagger_{\vec{k},\lambda}\frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{L^3}}\right)$ and so

$$\begin{aligned}
\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}||\hat{A}(\vec{r})|^2|N_{\vec{k},\lambda}\rangle &= \langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}|\left(\sum_{\vec{k}_1,\lambda_1}\sqrt{\frac{2\pi\hbar c^2}{\omega_{k_1}}}\vec{e}_{\vec{k}_1,\lambda_1}\left(\hat{a}_{\vec{k}_1,\lambda_1}\frac{e^{i\vec{k}_1\cdot\vec{r}}}{\sqrt{L^3}} + \hat{a}^\dagger_{\vec{k}_1,\lambda_1}\frac{e^{-i\vec{k}_1\cdot\vec{r}}}{\sqrt{L^3}}\right)\right)\\
&\quad\cdot\left(\sum_{\vec{k}_2,\lambda_2}\sqrt{\frac{2\pi\hbar c^2}{\omega_{k_2}}}\vec{e}_{\vec{k}_2,\lambda_2}\left(\hat{a}_{\vec{k}_2,\lambda_2}\frac{e^{i\vec{k}_2\cdot\vec{r}}}{\sqrt{L^3}} + \hat{a}^\dagger_{\vec{k}_2,\lambda_2}\frac{e^{-i\vec{k}_2\cdot\vec{r}}}{\sqrt{L^3}}\right)\right)|N_{\vec{k},\lambda}\rangle\\
&= \frac{2\pi\hbar c^2}{L^3}\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}|\sum_{\vec{k}_1,\lambda_1}\sum_{\vec{k}_2,\lambda_2}\sqrt{\frac{1}{\omega_{k_1}\omega_{k_2}}}\vec{e}_{\vec{k}_1,\lambda_1}\cdot\vec{e}_{\vec{k}_2,\lambda_2}\\
&\quad\left(\hat{a}_{\vec{k}_1,\lambda_1}e^{i\vec{k}_1\cdot\vec{r}} + \hat{a}^\dagger_{\vec{k}_1,\lambda_1}e^{-i\vec{k}_1\cdot\vec{r}}\right)\left(\hat{a}_{\vec{k}_2,\lambda_2}e^{i\vec{k}_2\cdot\vec{r}} + \hat{a}^\dagger_{\vec{k}_2,\lambda_2}e^{-i\vec{k}_2\cdot\vec{r}}\right)|N_{\vec{k},\lambda}\rangle
\end{aligned}$$

Now, the only combinations of creation and annihilation operators that will result in a nonzero inner product between the two photon states are $\hat{a}_{\vec{k}_1,\lambda_1}\hat{a}^\dagger_{\vec{k}_2,\lambda_2}$ and $\hat{a}^\dagger_{\vec{k}_1,\lambda_1}\hat{a}_{\vec{k}_2,\lambda_2}$, and so the matrix element reduces to

$$\begin{aligned}
\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}||\hat{A}(\vec{r})|^2|N_{\vec{k},\lambda}\rangle &= \frac{2\pi\hbar c^2}{L^3}\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}|\sum_{\vec{k}_1,\lambda_1}\sum_{\vec{k}_2,\lambda_2}\sqrt{\frac{1}{\omega_{k_1}\omega_{k_2}}}\vec{e}_{\vec{k}_1,\lambda_1}\cdot\vec{e}_{\vec{k}_2,\lambda_2}\\
&\quad\left(\hat{a}_{\vec{k}_1,\lambda_1}\hat{a}^\dagger_{\vec{k}_2,\lambda_2}e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} + \hat{a}^\dagger_{\vec{k}_1,\lambda_1}\hat{a}_{\vec{k}_2,\lambda_2}e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{r}}\right)|N_{\vec{k},\lambda}\rangle
\end{aligned}$$

But the only values of k_1 and k_2 that will result in a nonzero contribution to the sum are k and k', and the only values of λ_1 and λ_2 that will result in a nonzero contribution to the sum are λ and λ', so we can reduce this expression to

$$\begin{aligned}
\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}||\hat{A}(\vec{r})|^2|N_{\vec{k},\lambda}\rangle &= \frac{2\pi\hbar c^2}{L^3}\langle (N-1)_{\vec{k},\lambda};1_{\vec{k}',\lambda'}|\sqrt{\frac{1}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\\
&\quad\left(\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k}',\lambda'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} + \hat{a}^\dagger_{\vec{k}',\lambda'}\hat{a}_{\vec{k},\lambda}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\right)|N_{\vec{k},\lambda}\rangle\\
&= \frac{4\pi\hbar c^2}{L^3}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}
\end{aligned}$$

Substituting this into the expression for the total matrix element, we obtain

$$\begin{aligned}
\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle &= \frac{e^2}{2mc^2}\int d^3r\left[\frac{1}{L^3}e^{i\vec{p}\cdot\vec{r}/\hbar}\left(\frac{4\pi\hbar c^2}{L^3}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\right)\right]\\
&= \frac{2\pi\hbar e^2}{mL^6}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\int d^3r\left(e^{i(\vec{p}/\hbar+(\vec{k}-\vec{k}'))\cdot\vec{r}}\right)
\end{aligned}$$

or using the fact that $\int d^3 r e^{i\vec{k}\cdot\vec{r}} = 0$ if $\vec{p}/\hbar \neq \vec{k}' - \vec{k}$, and $\int d^3 r e^{i\vec{k}\cdot\vec{r}} = L^3$ if $\vec{p}/\hbar = \vec{k}' - \vec{k}$, which implies that $\int d^3 r e^{i\vec{k}\cdot\vec{r}} = L^3 \delta_{\vec{p}/\hbar, \vec{k}'-\vec{k}}$. Note that this must be a Kronecker delta because we are working in the finite volume approximation so $\vec{p}$ can only take on discrete values.

$$\boxed{\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle = \frac{2\pi\hbar e^2}{mL^3}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k \omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\delta_{\vec{p}/\hbar,\vec{k}'-\vec{k}}}$$

Therefore, we can see that the matrix element is only nonzero when $\vec{p} = \hbar\vec{k}' - \hbar\vec{k}$ i.e. when the momentum of the system is conserved.

2. What is the flux of the incoming photons?

   Solution: Consider an element of area, dA, oriented such that the photons strike the area perpendicularly. Now, since the photons travel at the speed of light, all photons within a distance cdt will pass through the area in a time dt. It follows that all photons in the volume $cdAdt$ will pass through the element of area. There are $N_{\vec{k},\lambda}$ photons in a volume L^3, so the density of photons is $\frac{N_{\vec{k},\lambda}}{L^3}$. Therefore, the number of photons that pass through the area element in a time dt is $\frac{N_{\vec{k},\lambda}}{L^3}cdAdt$. So the flux of photons, Φ_p, is this number per area per unit time, or $\boxed{\Phi_p = \frac{cN_{\vec{k},\lambda}}{L^3}}$.

3. Compute the differential cross section $\frac{d\sigma_{\lambda,\lambda'}}{d\Omega'}$ for the scattered photon to be detected at a polarization sensitive detector at direction $\vec{k}'$, with polarization λ' (don't forget to sum over all the possible final states of the electron). Please express your final answer in terms of the polarization vectors: $\vec{e}_{\lambda/\lambda'}$, the initial and final frequency of the photon, (ω, ω'), the fine structure constant, $\alpha = \frac{e^2}{\hbar c}$, and the "classical radius" of the atom $r_0 = \frac{e^2}{mc^2}$ (which is the length-scale at which the Coulomb interaction between charges is equal to the rest mass energy of the electron).

   Solution: The differential cross section is given by

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{2\pi}{\hbar}\frac{1}{\Phi_p}\sum_{\vec{p},\vec{k}'\in d\Omega}|\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle|^2\delta(E_f - E_i)$$

but the final energy of the system is $E_f = \frac{p^2}{2m} + (N_{\vec{k},\lambda} - 1)\hbar c|\vec{k}| + \hbar c|\vec{k}'|$ and the initial energy of the system is $E_i = N_{\vec{k},\lambda}\hbar c|\vec{k}|$, so the expression for the differential cross section becomes

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{2\pi}{\hbar}\frac{1}{\Phi_p}\sum_{\vec{p},\vec{k}'\in d\Omega}|\langle f|\hat{H}^{(2)}_{\text{int}}|i\rangle|^2\delta\left(\frac{p^2}{2m} + \hbar c(|\vec{k}'| - |\vec{k}|)\right)$$

or, plugging in our expression for the matrix element,

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{2\pi}{\hbar}\frac{1}{\Phi_p}\sum_{\vec{p},\vec{k}'\in d\Omega}\left|\frac{2\pi\hbar e^2}{mL^3}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\delta_{\vec{p}/\hbar,\vec{k}'-\vec{k}}\right|^2\delta\left(\frac{p^2}{2m} + \hbar c(|\vec{k}'| - |\vec{k}|)\right)$$

We can now do the sum over all $\vec{p}$, but the kronecker delta in the sum restricts $\vec{p} = \hbar(\vec{k}' - \vec{k})$ and so we have

$$\begin{aligned}
\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} &= \frac{2\pi}{\hbar}\frac{1}{\Phi_p}\sum_{\vec{k}'\in d\Omega}\left|\frac{2\pi\hbar e^2}{mL^3}\sqrt{\frac{N_{\vec{k},\lambda}}{\omega_k\omega_{k'}}}\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)\\
&= \frac{2\pi}{\hbar}\frac{1}{\Phi_p}\frac{4\pi^2\hbar^2 e^4}{m^2L^6}N_{\vec{k},\lambda}\sum_{\vec{k}'\in d\Omega}\frac{1}{\omega_{\vec{k}}\omega_{\vec{k}'}}\left|\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)\\
&= \frac{8\pi^3\hbar e^4}{m^2L^6}N_{\vec{k},\lambda}\frac{1}{\Phi_p}\sum_{\vec{k}'\in d\Omega}\frac{1}{c^2kk'}\left|\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)
\end{aligned}$$

but we found in the last problem that $\Phi_p = \frac{cN_{\vec{k},\lambda}}{L^3}$, and so we have

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{8\pi^3\hbar e^4}{m^2L^3c^3}\sum_{\vec{k}'\in d\Omega}\frac{1}{kk'}\left|\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)$$

or using the fact that $\frac{1}{L^3}\sum_{\vec{k}} = \frac{1}{(2\pi)^3}\int d^3k'$, we have

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{8\pi^3\hbar e^4}{m^2c^3}\frac{1}{(2\pi)^3}\int_{d\Omega}d^3k'\frac{1}{kk'}\left|\vec{e}_{\vec{k},\lambda}\cdot\vec{e}_{\vec{k}',\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)$$

If we choose a spherical coordinate system such that the z direction coincides with the direction of $\vec{k}$, then we can rewrite the integral as follows $\int_{d\Omega} d^3k' = \int_{-\infty}^{\infty}\int_{d\Omega}(k')^2 d\Omega dk'$. However, our angular integral is over an infinitesimal solid angle, $d\Omega$, so we can ignore the angular integral and our expression becomes

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{\hbar e^4}{m^2c^3}\int_{-\infty}^{\infty}dk'(k')^2\frac{1}{kk'}\left|\vec{e}_{\vec{k}'\lambda}\cdot\vec{e}_{\lambda'}\right|^2\delta\left(\frac{\hbar^2|\vec{k}'-\vec{k}|^2}{2m}+\hbar c(|\vec{k}'|-|\vec{k}|)\right)$$

also, in this coordinate system we can write $|\vec{k}'-\vec{k}|^2 = (k')^2 + k^2 - 2kk'\cos(\theta)$. And so,

$$\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_{\lambda}\cdot\vec{e}_{\lambda'}\right|^2\int_0^{\infty}dk'\frac{k'}{k}\delta\left(\frac{\hbar^2}{2m}\left((k')^2+k^2-2kk'\cos(\theta)\right)+\hbar c(k'-k)\right)$$

Note that we can take the polarization vectors out of the integral because this is an integral over the magnitude of $\vec{k}'$, its direction was determined by the angular region that we integrated over, and the polarization vectors only depend on the directions of $\vec{k}$ and $\vec{k}'$, not their magnitudes.

Now, we know that for two functions f and g, the following result holds, $\int g(x)\delta(f(x))dx = \sum_{x_0}\frac{1}{|f'(x_0)|}g(x_0)$, where x_0 are the zeroes of the function f and $f' = \frac{df}{dx}$. In our case, we have $f(k') = \frac{\hbar^2}{2m}\left((k')^2+k^2-2kk'\cos(\theta)\right)+\hbar c(k'-k) = \frac{\hbar^2}{2m}(k')^2+(\hbar c-\frac{\hbar^2}{m}k\cos(\theta))k'+\frac{\hbar^2}{2m}k^2-\hbar ck$, and so $f'(k') = \frac{\hbar^2}{m}(k'-k\cos(\theta))+\hbar c$. Also, the zeroes of f are given by the quadratic formula to be

$$
\begin{aligned}
k'_{0,\pm} &= \frac{-(\hbar c - \frac{\hbar^2}{m}k\cos(\theta)) \pm \sqrt{(\hbar c - \frac{\hbar^2}{m}k\cos(\theta))^2 - 4\left(\frac{\hbar^2}{2m}\right)\left(\frac{\hbar^2}{2m}k^2 - \hbar ck\right)}}{2\left(\frac{\hbar^2}{2m}\right)} \\
&= \frac{-(\hbar c - \frac{\hbar^2}{m}k\cos(\theta)) \pm \sqrt{\hbar^2c^2 + \frac{2\hbar^3 c}{m}(2-\cos(\theta))k + \frac{\hbar^4}{m^2}(\cos^2(\theta)-1)k^2}}{\frac{\hbar^2}{m}} \\
&= k\cos(\theta) - \frac{mc}{\hbar} \pm \frac{m}{\hbar}\sqrt{c^2 + \frac{2\hbar c}{m}(2-\cos(\theta))k + \frac{\hbar^2}{m^2}(\cos^2(\theta)-1)k^2}
\end{aligned}
$$

Using this result, we find that the differential cross-section is

$$
\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2 \sum_{k'_{0,\pm}} \frac{k'}{k}\frac{1}{\frac{\hbar^2}{m}(k' - k\cos(\theta)) + \hbar c}
$$

We can see that

$$
\beta_\pm = \frac{\hbar^2}{m}(k'_{0,\pm} - k\cos(\theta)) + \hbar c = \pm\hbar\sqrt{c^2 + \frac{2\hbar c}{m}(2-\cos(\theta))k + \frac{\hbar^2}{m^2}(\cos^2(\theta)-1)k^2}
$$

and so we obtain

$$
\begin{aligned}
\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} &= \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2 \left[\frac{k'_{0,+}}{k}\frac{1}{\beta_+} + \frac{k'_{0,-}}{k}\frac{1}{\beta_-}\right] \\
&= \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2 \left[\frac{k'_{0,+}}{k}\frac{1}{\beta_+} - \frac{k'_{0,-}}{k}\frac{1}{\beta_+}\right] \\
&= \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2 \left[\frac{k'_{0,+} - k'_{0,-}}{k\beta_+}\right] \\
&= \frac{\hbar e^4}{m^2c^3}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2 \frac{2\frac{m}{\hbar}\sqrt{c^2 + \frac{2\hbar c}{m}(2-\cos(\theta))k + \frac{\hbar^2}{m^2}(\cos^2(\theta)-1)k^2}}{k\hbar\sqrt{c^2 + \frac{2\hbar c}{m}(2-\cos(\theta))k + \frac{\hbar^2}{m^2}(\cos^2(\theta)-1)k^2}} \\
&= \frac{2e^4}{\hbar mc^3k}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2
\end{aligned}
$$

Or, in terms of the quantities α and r_0,

$$
\boxed{\frac{d\sigma_{\lambda,\lambda'}}{d\Omega} = \frac{2\alpha r_0}{k}\left|\vec{e}_\lambda \cdot \vec{e}_{\lambda'}\right|^2}
$$

4. Write down an expression for the 2nd order Fermi-golden rule rate for the same scattering process, acting twice with the linear coupling $H^{(1)}_{\text{int}} = \frac{e}{c}\int d^3r\hat{\vec{j}}(\vec{r}) \cdot \hat{\vec{A}}(\vec{r})$. Estimate the matrix elements and show that this second order process is negligible for states where the photon energy is much less than the rest mass of the electron $\hbar ck \ll mc^2$ (i.e. when the electron is moving much slower than the speed of light before and after scattering).

   Solution: In order to perform calculations using the 2nd ordre Fermi golden rule, we must consider the possible intermediate states of the system that can occur when acting with the

Hamiltonian a single time. We can see that these intermediate states are when the electron absorbs or emits a photon

$$|M_{1,\vec{p}'}\rangle = |\vec{p}'\rangle \otimes |(N-1)_{\vec{k},\lambda}\rangle$$
$$|M_{2,\vec{p}'}\rangle = |\vec{p}'\rangle \otimes |N_{\vec{k},\lambda}, 1_{\vec{k}',\lambda}\rangle$$

Using these intermediate states, Fermi's 2nd order golden rule says

$$\Gamma = \frac{2\pi}{\hbar} \sum_f \left| \sum_j \sum_{\vec{p}'} \frac{\langle f|\hat{H}^{(1)}_{\text{int}}|M_{j,\vec{p}'}\rangle\langle M_{j,\vec{p}'}|\hat{H}^{(1)}_{\text{int}}|i\rangle}{E_i - E_{M_{j,\vec{p}'}}} \right|^2 \delta(E_f - E_i)$$

Now, we know that $\hat{\vec{A}}$ is of the order $\sqrt{\frac{\hbar c^2}{\omega_k}}$ and that $\hat{\vec{j}}$ is of the order $\frac{p'}{m}$. Therefore, it follows that the matrix element product found in the sum, $\langle f|\hat{H}^{(1)}_{\text{int}}|M_{j,\vec{p}'}\rangle\langle M_{j,\vec{p}'}|\hat{H}^{(1)}_{\text{int}}|i\rangle$, is of the order $\frac{\hbar c^2}{\omega_k}\left(\frac{p'}{m}\right)^2\left(\frac{e}{c}\right)^2 = \frac{\hbar c^4 e^2}{\omega_k}\left(\frac{p'}{mc^2}\right)^2$. Now, the electron starts at rest, so the momentum it obtains, p', is purely from interaction with the photons, therefore this momentum must be of the same order as the photon momentum, $\hbar k$. If $\hbar ck \ll mc^2$, then by this argument we must have $\frac{p'}{mc^2} \ll 1$. Since this momentum to rest energy ratio appears to the second power in the matrix element product, it follows that $\langle f|\hat{H}^{(1)}_{\text{int}}|M_{j,\vec{p}'}\rangle\langle M_{j,\vec{p}'}|\hat{H}^{(1)}_{\text{int}}|i\rangle \ll 1$, and so we can neglect this second order effect in this limit.

III. ATOMIC FINE STRUCTURE

Consider treating the leading order relativistic correction to the kinetic energy of an electron:

$$\delta\hat{H} = -\frac{\hat{p}^4}{8m_e^3c^2} \tag{4}$$

as a perturbation to the solution to the non-relativistic Hydrogen atom.

1. What are the selection rules for matrix elements of $\delta\hat{H}$?

Solution: We can see that since $\hat{p}$ is raised to an even power in $\delta\hat{H}$, the perturbation is unchanged when acted on by the parity operator, so $\mathcal{P}\delta\hat{H}\mathcal{P}^\dagger = \delta\hat{H}$, where $\mathcal{P}$ is the parity operator. In addition, we know that $\mathcal{P}|nlm\rangle = (-1)^n|nlm\rangle$. Therefore,

$$\begin{aligned}\langle n'l'm'|\delta\hat{H}|nlm\rangle &= \langle n'l'm'|\mathcal{P}^\dagger\mathcal{P}\delta\hat{H}\mathcal{P}^\dagger\mathcal{P}|nlm\rangle \\ &= (-1)^{n'+n}\langle n'l'm'|\delta\hat{H}|nlm\rangle\end{aligned}$$

It follows that if n' and n sum to an odd number, then $\langle n'l'm'|\delta\hat{H}|nlm\rangle = -\langle n'l'm'|\delta\hat{H}|nlm\rangle \implies \langle n'l'm'|\delta\hat{H}|nlm\rangle = 0$. So for the matrix element to be nonzero, we must have $n' = n$ modulo 2.

Also, $\hat{p}^4$ is rotationally symmetric, so $R_z(\theta)\delta\hat{H}R_z^\dagger(\theta) = \delta\hat{H}$, where $R_z(\theta)$ is the rotation operator about the z axis by an angle θ. We also know that $R_z(\theta)|nlm\rangle = e^{im}|nlm\rangle$, therefore

$$\begin{aligned}\langle n'l'm'|\delta\hat{H}|nlm\rangle &= \langle n'l'm'|R_z^\dagger(\theta)R_z(\theta)\delta\hat{H}R_z^\dagger(\theta)R_z(\theta)|nlm\rangle \\ &= e^{i(m'+m)}\langle n'l'm'|\delta\hat{H}|nlm\rangle\end{aligned}$$

If m' and m do not sum to zero, then we have $\langle n'l'm'|\delta\hat{H}|nlm\rangle = e^{i\phi}\langle n'l'm'|\delta\hat{H}|nlm\rangle$, where $\phi \neq 0$, which implies that $\langle n'l'm'|\delta\hat{H}|nlm\rangle = 0$. So for the matrix element to be nonzero, we must have $m' = -m$.

Combining these results, we see that selection rules only allow nonzero matrix elemetns between states where $n' = n + 2k, \quad k \in \mathbb{Z}$ and $m' = -m$.

2. Compute the first order energy shifts for the Hydrogen atom levels from $\delta\hat{H}$. Write your answer in terms of the rest mass energy, $E_0 = m_e c^2$, the fine structure constant, $\alpha = \frac{e^2}{\hbar c}$, and the quantum numbers n, l, m of the Hydrogen atom states.

   The following formula for the matrix elements of $\hat{p}^4$ in the nlm state of hydrogen will be useful:

$$\langle nlm|\hat{p}^4|nlm\rangle = 4m_e^2\left(\frac{e^2}{a_0 n^2}\right)^2\left(\frac{4n}{l+1/2} - 3\right) \tag{5}$$

   where $a_0 = \frac{\hbar^2}{m_e e^2}$ is the Bohr radius.

   Solution: First order perturbation theory tells us that the first order energy shifts of the the Hydrogen atom levels caused by $\delta\hat{H}$ are given by $\langle nlm|\delta\hat{H}|nlm\rangle$. We can calculate this matrix element as follows

$$\begin{aligned}
\langle nlm|\delta\hat{H}|nlm\rangle &= -\frac{1}{8m_e^3c^2}\langle nlm|\hat{p}^4|nlm\rangle \\
&= -\frac{1}{8m_e^3c^2}\left(4m_e^2\left(\frac{e^2}{a_0 n^2}\right)^2\left(\frac{4n}{l+1/2} - 3\right)\right) \\
&= -\frac{e^4(m_e e^2)^2}{2\hbar^4c^2m_e n^2}\left(\frac{4n}{l+1/2} - 3\right) \\
&= -\frac{e^8 m_e}{2\hbar^4c^2n^2}\left(\frac{4n}{l+1/2} - 3\right) \\
&= -\frac{1}{2n^2}\left(\frac{e^2}{\hbar c}\right)^4(m_e c^2)\left(\frac{4n}{l+1/2} - 3\right) \\
&= -\frac{\alpha^4 E_0}{2n^2}\left(\frac{4n}{l+1/2} - 3\right)
\end{aligned}$$

Therefore the first order energy shifts are

$$\boxed{E_n^1 = -\frac{\alpha^4 E_0}{2n^2}\left(\frac{4n}{l+1/2} - 3\right)}$$