Homework 6

1. Verify which of the following sets are linear independent, if so prove it, otherwise explain why not.

   (a) $V = \mathbb{R}^2$. $S = \{(1,0),(0,1)\}$; $S = \{(1,1),(1,-1)\}$; $S = \{(1,0),(1,1),(-1,1)\}$.

   (b) $V = \mathbb{R}^3$. $S = \{(1,0,0),(0,1,0),(0,0,1)\}$; $S = \{(1,0,0),(0,1,1),(0,0,1)\}$; $S = \{(1,1,1),(0,1,1),(2,0,1)\}$.

   (c) $V = \mathbb{R}^n$. $S = \{e_1, e_2, .., e_n\}$ where $e_i = \underbrace{(0,..0,1,0..,0)}_{i\text{th}}$; $S = \{e_1, .., e_{n-1}, e_1 + ... + e_n\}$; $S = \{e_1, e_2 - e_1, e_3 - e_2, ..., e_n - e_{n-1}\}$.

   (d) X a finite set, $V = F(X)$ where F is an arbitrary field. $S = \{\delta_x : x \in X\}$; $U \subset X$ a subset, $S_U = \{\delta_x : x \in U\}$; $S = \{f_x : x \in X\}$ where
   $$f_x(y) = \begin{cases} 1 & y \neq x \\ 0 & \text{otherwise} \end{cases}.$$

   (e) $V = \mathbb{R}[x]$. $S = \{1, x, x^2, ..x^d\}$ for some positive number d; $S = \{1, x, x^2, ..x^{d-1}, 1 + x + x^2 + ..x^d\}$ for some positive number d.

2. Verify which of the sets in question 1is a spanning set, if so prove it, if not explain why, that is exhibit a vector that cannot be written as a linear combination of the elements in the set.

3. Verify which of the sets in question 1 is a basis, if so prove it, if not explain why not.

4. Let V be a vector space over a field F of dimension N. Let $S = \{v_1, .., v_N\}$ be a spanning set (consisting of N elements). Prove that S must be linear independent and therefore a basis of V.

5. Write in a nice form the proof (given in class) that any finite dimensional vector space admits a basis.

6. Prove that the vector space of polynomials $\mathbb{R}[x]$ is not finite dimensional.

7. Let $B = \{v_1, ..., v_n\}$ be a basis of a vector space V over F. Prove that any vector $v \in V$ can be written as a linear combination
   $$v = \sum_{i=1}^{n} \alpha_i v_i$$
   in a **unique** way.

8. Let V be a vector space over a field F of dimension n. Let $S \subset V$ be a linear independent set. Prove that $\#S \leq n$. Prove that if $\#S = n$ then S is a basis. Conclude that a basis is a mximal linear independent set.

9. Let V be a finite dimensional vector space over F. Prove that any subspace of V is finite dimensional.

10. Let V be a finite dimensional vector space of dimension n. Let $B = \{v_1, .., v_m\}$ $m < n$. Prove that B can be completed to a basis. Is this completion unique?