Homework 2

1. Consider the following relation on the set $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq 0\}$: $(x,y) \in R \iff y/x > 0$. Prove that R is an equivalence relation. Describe the equivalence classes of R, in particular, how many are they?

2. Consider $X = \mathbb{R}^2 - \{(0,0)\}$ - the real plane without the point zero. Define the relation $(x,y)\, R\, (x',y') \iff \exists a \in \mathbb{R}^\times$ such that $x' = ax$ and $y' = ay$. Prove that R is an equivalence relation. Can you describe the geometry of the set of equivalence classes (what its shape?).

3. Consider the following relation on $\mathbb{Z} \times \mathbb{Z}$: $(x,y) \sim (x',y') \iff xy' = x'y$.

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Consider the set of equivalence classes $\mathbb{Z} \times \mathbb{Z}/\sim$, denote the equivalence class of (x,y) by $\overline{(x,y)}$. Define the following binary operations between equivalence classes:

   $$\begin{aligned} \overline{(x,y)} \cdot \overline{(x',y')} &= \overline{(xx',yy')}, \\ \overline{(x,y)} + \overline{(x',y')} &= \overline{(xy'+x'y,yy')}. \end{aligned}$$

   Prove that these operations are well defined, that is does not depend on a choice of a representative. Do you recognize the arithmetic system $\mathbb{Z} \times \mathbb{Z}/\sim$ you just constructed?

4. Consider the residue ring $\mathbb{Z}_n$ for some fixed $n \geq 2$. Formulate and prove the following list of properties for the operations $+$ and $\cdot$ defined in class: Addition: associativity, commutativity, existence of neutral element, existence of additive inverse; Multiplication: associativity, commutativity, existence of neutral and finally, distributivity.

5. Compute explicitly the addition and multiplication tables for $\mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6$.

6. Let $\sim$ be an equivalence class on X, let $P = P_\sim$ be the associated partition and finally let $\approx$ denote the equivalence relation associated to P. Prove that $\approx = \sim$, that is $x \sim y$ if and only if $x \approx y$.

7. Opposite direction. let P be a partition of a set X. Let $\sim = \sim_P$ be the associated equivalence relation and finally let $Q = P_\sim$ be the partition associated to $\sim$. Prove that $P = Q$.

8. Let $(G,*)$ be a finite group and $H \subset G$ be a subgroup. Define the following relation on G: $g_1 \sim g_2 \Leftrightarrow \exists h \in H$ such that $g_1 = g_2 h$. Prove that $\sim$ is an equivalence relation. Consider the group $(\mathbb{Z},+)$ and the subgroup $E_n \subset \mathbb{Z}$, $E_n = \{x \in \mathbb{Z} : n|x\}$. Describe $\mathbb{Z}/E_n$.

9. Let $X = \{1,2,3\}$. How many relations exists on X? How many equivalence relation exists on X?