Homework 4

1. Let G be a group and $H_1, H_2 \subset G$ be two subgroups. Prove that $H_1 \cap H_2 \subset G$ is also a subgroup.

2. Verify explicitly the axioms of a vector space over a field for the following examples that were presented in class. Before you verify the axioms, write explicitly the operations of addition and multiplication by scalar for each example.

   (a) $(\mathbb{R}^n, +, \alpha)$ as presented in class.

   (b) $(F(X), +, \alpha)$ where X be a set and F a field; $F(X)$ is the set of F valued functions on X with the operation $+ : F(X) \times F(X) \to F(X)$ of addition of functions and $\alpha : F \times F(X) \to F(X)$ of multiplication by scalar.

   (c) $(\mathbb{R}[x], +, \alpha)$ where $\mathbb{R}[x]$ is the set of polynomials in one variable with coefficients in $\mathbb{R}$.

   (d) $V = \{0\}$ with $0 + 0 = 0$ and $\lambda \cdot 0 = 0$ for every $\lambda \in F$ where F is an arbitrary field.

   (e) $V = \mathbb{R}$ and $F = \mathbb{Q}$.

3. Let V be a vector space over a field F. Let $\mathbf{0} \in V$ be the zero vector.

   (a) Prove that $\lambda \cdot \mathbf{0} = \mathbf{0}$ for every $\lambda \in F$.

   (b) Prove that $0 \cdot v = \mathbf{0}$ for every $v \in V$.

   (c) prove that $(-1) \cdot v = -v$ for every $v \in V$.

4. Verify for the following maps if it is a linear transformation. If yes prove it, if not explain which property is not satisfied.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (x + y, 2x - y)$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (x + y, 2x - 5)$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (y, x)$.

   (d) $T : \mathbb{Q}^3 \to \mathbb{Q}^3$; $T(x, y, z) = (x + y + z, z - y, 2 \cdot x + 5 \cdot y - z)$.

   (e) Let F be a field. $T : F^n \to F$; $T(x_1, ..., x_2) = x_1 \cdot x_2 \cdot .. \cdot x_n$.

   (f) Let $\varphi : X \to Y$ a map of sets and F a field. Let $\varphi^* : F(Y) \to F(X)$ be the map sending a function $f \in F(Y)$ a function $\varphi^*(f) \in F(X)$ given by $\varphi^*(f)(x) = f(\varphi(x))$ for every $x \in X$.

   (g) $D : \mathbb{R}[x] \to \mathbb{R}[x]$ defined as follows: for $p(x) = a_n x^n + a_{n-1} x^{n-1} + .. + a_1 x + a_0$, $D(p) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + .. + 1 a_1$. (This map is called derivative).

   (h) Generalize example g for polynomials in one variable with coefficients in an arbitrary (infinite) field F.