Homework 9

1. Prove that the trace map $Tr : Mat_{n\times n}(F) \to F$ is a linear transformation. Describe the kernel of Tr. Compute the dimension of $\ker Tr$ (explain your computation).

2. Compute the determinants of the following matrices according to the two definitions we gave in class and verify that you get the same answer:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \text{ in } \mathbb{R} \text{ and in } \mathbb{F}_5,\ A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \\
A &= \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \text{ in } \mathbb{F}_3. \\
A &= \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & . & * & * \\ 0 & 0 & \lambda_{n-1} & * \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}, \text{ where } * \text{ denotes any element in } F.
\end{aligned}
$$

3. Consider the set $X = \{1, 2, 3, 4\}$. Consider the map $\phi : X \to X$ given by: $\phi(1) = 2, \phi(2) = 3, \phi(3) = 4, \phi(4) = 1$. Compute $Tr(\phi^*)$ and $\det(\phi^*)$. In your computation you will have to choose a basis, present ϕ^* as a matrix and compute the trace and the determinant of this matrix. Describe your full computation.

4. for a set X, can you formulate a condition on a map $\phi : X \to X$ which is sufficient and necessary for $Tr(\phi^*) = 0$.

5. Define the set $SL_n(F) = \{A \in Mat_{n\times n}(F) : \det A = 1\}$. Prove that $SL_n(F)$ is a group with respect to the operation of multiplication of matrices. This group is called the *special linear group.* Is the set $\{A \in Mat_{n\times n}(F) : \det A = a\}$ for $a \neq 1$ a group? if yes prove it, if no explain why.

6. Consider the permutation group on n symbols S_n. Define the sign map $sign : S_n \to \{+1, -1\}$ and prove that it satisfies

$$
\begin{aligned}
sign(Id) &= 1, \\
sign(\sigma \circ \tau) &= sign(\sigma) sign(\tau), \text{ for every } \sigma, \tau \in S_n.
\end{aligned}
$$

   Your task is to write a comprehensive version of the proof that we presented in class.

7. Consider the determinant $\det : Mat_{2\times 2}(F) \to F$, F is a field. Prove the multiplicativity property in this particular case, namely, show that $\det(A \cdot B) = \det(A)\det(B)$, for every $A, B \in Mat_{2\times 2}(F)$.

8. (This question is slightly more involved) Prove that the two definitions of determinants we gave in class coincide . Try to do it first for matrices of order 2 and 3 and then try to generalize your answer.