Homework 10

1. Let V_i $i = 1, .., N$ be a collection of vector spaces over a field F. Consider the Cartesian product $V = V_1 \times .. \times V_N$ equipped with the natural projections $p_i : V \to V_i$.

   (a) Prove that p_i is linear and compute its kernel.

   (b) Let U be a vector space and $T_i : U \to V_i$ linear transformations. Prove there exists a unique $T : U \to V$ such that $p_i \circ T = T_i$.

2. Let V be a finite dimensional vector space over a field F. Let $B = \{v_1, .., v_n\}$ be a basis and consider the dual basis $B^* = \{v_1^*, ..., v_n^*\}$.

   (a) Let $v \in V$. Prove that
   $$v = \sum_{i=1}^{n} v_i^* (v) \, v_i.$$

   (b) Let $\alpha \in V^*$. Prove that
   $$v = \sum_{i=1}^{n} \alpha (v_i) \, v_i^*.$$

3. Let V be a vector space. Consider the vector space $W = (V^*)^* = Hom\,(Hom\,(V, F)\,, F)$ (the dual of the dual). Consider the map $\theta : V \to (V^*)^*$ given by: $\theta\,(v)\,(\alpha) = \alpha\,(v)$.

   (a) Prove that θ is a linear transformation.

   (b) Prove that θ is injective.

   (c) Prove that if V is finite dimensional then θ is bijective.

4. Let V be a f.d vector space. Let $T : V \to V$ be a linear map.

   (a) Consider the dual map $T^* : V^* \to V^*$. Write the definition of T^* and prove it is linear.

   (b) Consider an ordered basis $B = (v_1, ..., v_n)$ and its dual basis B^*. Let M_T be the matrix associated with T with respect to B. Let M_{T^*} be the matrix associated with T^* with respect to B^*. Prove that $M_{T^*} = M_T^t$. that is $b_{ij} = a_{ji}$.

   (c) Consider the dual of the dual $T^{**} : V^{**} \to V^{**}$. Prove that under the isomorphism θ from question 4, $T^{**} = T$.

5. Let V be a vector space over F finite dimensional. Let $U \subset V$ and consider the orthogonal complement $U^\perp \subset V^*$

   (a) Write the complete proof that $\dim U + \dim U^\perp = \dim V$.

(b) Consider the vector space $\left(U^{\perp}\right)^{\perp} \subset (V^{*})^{*}$. Prove that $\theta : V \to (V^{*})^{*}$ maps U isomorphically onto $\left(U^{\perp}\right)^{\perp}$. Hint: its enough to prove that θ maps U injectively into $\left(U^{\perp}\right)^{\perp}$ and that $\dim U = \dim \left(U^{\perp}\right)^{\perp}$.

6. Consider the operation of transpose: $(-)^{t} : Mat_{m\times n}(F) \to Mat_{n\times m}(F)$.

   (a) Prove that $(-)^{t}$ is a linear transformation. Moreover, prove that $(-)^{t}$ is an isomorphism.

   (b) Consider the case $n = m$. Prove that $(A \cdot B)^{t} = B^{t} \cdot A^{t}$.

   (c) Let $A \in GL_{n}(F)$. Prove that $\left(A^{-1}\right)^{t} = \left(A^{t}\right)^{-1}$. Conclude that A is invertible iff A^{t} is invertible.

   (d) Prove that if $A, B \in Mat_{n\times n}$ are similar then A^{t}, B^{t} are similar.