Measuring Boltzmann's Constant by Observing Brownian Motion

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Abstract

In this experiment, we determined the numerical value of Boltzmann's constant by observing the Brownian motion of small particles. We measured the particles' displacements at different times and used relations derived by Einstein to determine Boltzmann's constant. Using this technique, we measured a value $k_B = 1.21 \times 10^{-23} \pm 0.02 \times 10^{-23}$ J/K. This value does not agree with the accepted value of Boltzmann's constant, 1.38×10^{-23}J/K.

1 Introduction

1.1 Physics Motivation

Brownian motion refers to the random motion of small particles suspended in a fluid. This motion was first observed by the botanist Robert Brown in 1827 when he was observing pollen grains. [1] Brownian motion eluded explanation until 1905, when Albert Einstein used the atomic theory of matter to describe the motion as being the result of the molecules of the suspending fluid colliding with the suspended particles. [2] Einstein's theory made several predictions about the nature of the motion, relating the motion of the particles to constants like Avogadro's number and Boltzmann's constant. [2] These predictions were experimentally verified in 1909 by Jean Perrin, providing stong support for the atomic theory of matter. [3] Brownian motion and the development of the theory that describes it has provided a glimpse into the world at atomic scales, and has proven to be important in the development of our understanding of nature.

1.2 Theoretical background

In Albert Einstein's 1905 paper, *On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat*, he used the atomic theory of matter to attempt to describe the motion of small particles suspended in a fluid. Einstein first used the atomic theory to show that the diffusion coefficient, D, of a spherical particle of radius P in a fluid with a coefficient of viscosity μ is given by

$$D = \frac{RT}{N_A}\frac{1}{6\pi\mu P} \tag{1}$$

where R is the ideal gas constant, T is the temperature, and N_A is Avogadro's number. [2] He then went on to show that the mean squared-displacement, $\langle x^2 \rangle$ of a particle undergoing Brownian motion in one dimension is given by

$$\langle x^2 \rangle = 2Dt \tag{2}$$

where t is the time that has elapsed since the particle began Brownian motion. Substituting the expression for D given by equation 1 into equation 2 and using the definition of Boltzmann's constant, $k_B = \frac{R}{N_A}$, we obtain

$$\langle x^2 \rangle = \frac{k_B T}{3\pi\mu P}t \tag{3}$$

For a particle moving in two dimensions, x and y, the mean squared displacement is $R^2 = \langle x^2 \rangle + \langle y^2 \rangle$. But the motions in the x and y directions are the same on average, so $\langle x^2 \rangle = \langle y^2 \rangle$. Therefore $R^2 = 2\langle x^2 \rangle$ and so we obtain

$$\langle R^2 \rangle = \frac{2k_B T}{3\pi\mu P}t \tag{4}$$

This expression relates Boltzmann's constant to the mean squared displacement of particles, an easily observable quantity. This is a historically important relation, because it allowed for a method of experimentally determining the value of k_B, and thereby determining the value of N_A. The determination of these values allowed for an estimate of the size of atoms, which was an important step in the development of the atomic theory of matter. [1] [2] Experimental work carried out by Jean Perrin in 1909 confirmed the predictions of Einstein's theory, which provided the greatest support of the atomic theory to date and led to its widespread acceptance. [3]

1.3 Our approach

In our observation of Brownian motion, we aimed to use equation 4 to experimentally determine a value of Boltzmann's constant. To do this, we placed a suspension of small particles under a microscope and took videos of the motion of the particles. Then, using image analysis software, we determined the squared displacement of several different particles at different elapsed times. We then averaged these squared displacements, plotted them against the elapsed time, and fit a line to the data. Then we used the slope of this fitted line, along with known values of the size of the particles and coefficient of viscosity of the fluid in the relation given by equation 4 to find a value of Boltzmann's constant.

2 Experimental setup

2.1 Apparatus

The solution of suspended particles that we observed in this experiment was a solution of 1.02 ± 0.01 μm diameter latex particles in distilled water. We diluted the solution to a concentration that allowed for the observation of several particles in the field of view of the microscope, but prevented the particles from colliding with each other frequently. To prepare the solution for observation, we stuck a ring shaped reinforcement label in the middle of a flat microscope slide. The reinforcement label was used to hold the drop of solution in place and prevent fluid flow. We then took a flat glass cover slip and placed a drop of the solution on the center of the slip. We then inverted the slip and placed it on the slide such that the drop of solution went into the hole of the reinforcement label. A diagram of the slide setup can be found in Figure 1.

Next we placed the slide under the microscope and observed the solution with the 40X objective lens. We ensured that there was no net flow of the particles in the solution so that the motion we observed was due to the random Brownian motion and not the flow of the suspending fluid. If the sample seemed to have no net flow then we used a digital camera connected to the microscope to take videos of the motion at several different locations on the slide.

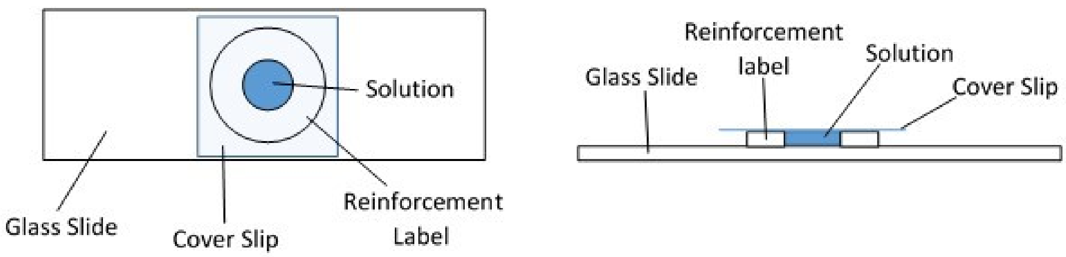

Figure 1: A schematic diagram of the experimental setup.
The left image is a top-down view of the slide and the right image is a side on view. The reinforcement label was used to prevent the solution from flowing around on the slide.

2.2 Data Collection

Using the digital camera, we took 10 six second videos of the particles at different locations on the slide undergoing Brownian motion. We then imported the videos into *Mathematica* and used its image enhancement features to increase the contrast between the particles and the background so the particles were easier to observe. A typical still from a video is shown in Figure 2. We then used *Mathematica's* image coordinate tool to get the coordinates of the particles (in pixels) in the first frame of each video, and used its ImageFeatureTrack function to record the positions of the particles in the subsequent frames. Using this process, we tracked the position of 92 particles over six seconds of 30 frames per second video. We obtained x and y coordinates for each particle for each of the 180 frames.

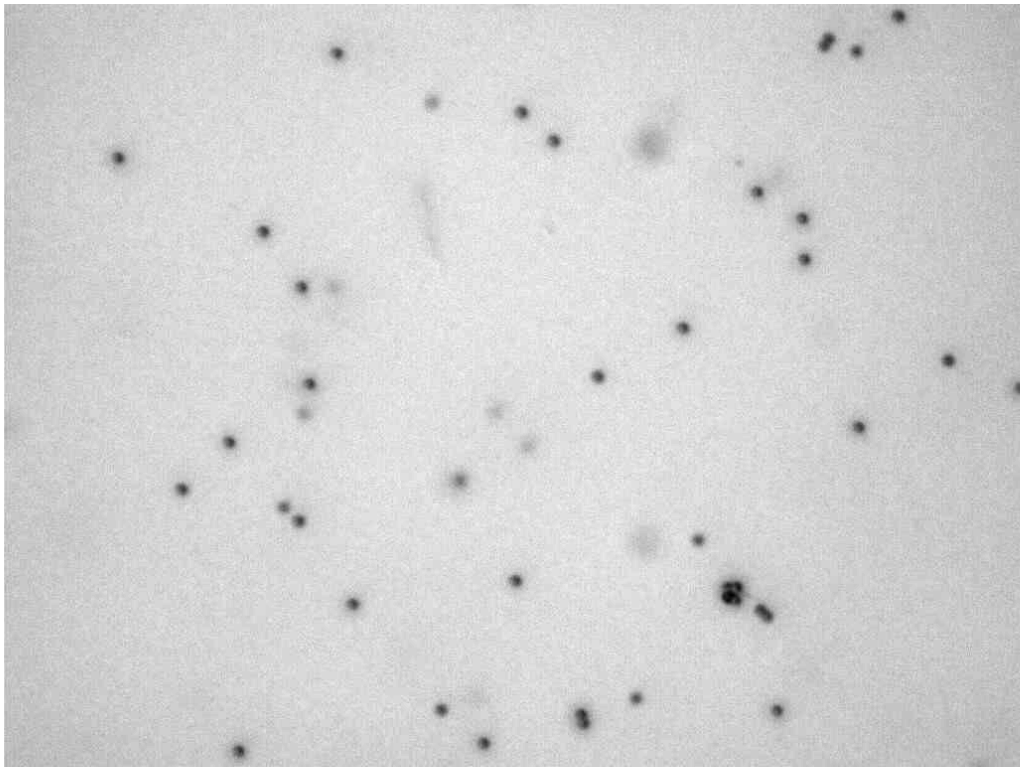

Figure 2: A still from one of the videos showing the appearance of the particles under the microscope after enhancing the image in *Mathematica*.

3 Data Analysis and Results

To process the data into a form that can be used with equation 4, we began by calculating the squared displacement of each particle in each frame from its initial position. This was done using the standard Pythagorean relation between the x and y coordinates and distance. We estimated an uncertainty in the coordinate measurements equal to the radius of the particles, 0.51 μm. This estimate was made because the ImageFeatureTrack function could track the position of any point within the projection of the sphere into the x-y plane. We propagated this error to obtain an estimate of the uncertainty in each squared displacement value.

We then averaged over all particles to obtain an average squared displacement for each frame. Standard error propagation was used to estimate the uncertainty in each average squared displacement. We then converted the displacements from pixels to micrometers. We did this by taking a picture of a calibration wheel under the microscope at the same magnification that we observed the particles at, and then determined the length of the lines on the calibration wheel in pixels. Using the known length of the lines on the wheel, we were able to determine a conversion factor of 0.12 ± 0.01 µm/pixel, with an uncertainty estimated by finding the standard deviation of multiple measurements of the length of the calibration line.

We then plotted the mean squared displacements against the elapsed time. The elapsed time was calculated for each frame using the equation $t = (\frac{1}{30}\ \mathrm{s})(n-1)$ where n is the frame number, because the camera recorded at a frame rate of 30 frames per second. The error in the elapsed time was assumed to be negligible compared to the error in the squared displacement. We then obtained a best fit line to the data using a least-squares regression. We found the equation of the best fit line to be

$$\langle R^2 \rangle = (1.47 \times 10^{-12} \pm 0.02 \times 10^{-12}\ \mathrm{m}^2/\mathrm{s})t + (-4 \times 10^{-14} \pm 2 \times 10^{-14}\ \mathrm{m}^2) \quad (5)$$

where the uncertainties were calculated using the standard formulas for uncertainty in least-squares regression parameters. A plot of our data is shown in Figure 3.

Equating the slope of the best fit line with the coefficient of t given by equation 4 and solving for k_B using the values $\mu = 1.002 \times 10^{-3}$ Pa·s, $2P = D = 1.02 \pm 0.01$ µm, and $T = 293 \pm 1$ K gives a value of $1.21 \times 10^{-23} \pm 0.02 \times 10^{-23}$ J/K for Boltzmann's constant. This value does not agree with the accepted value of Boltzmann's constant, which is 1.38×10^{-23} J/K. The disagreement between our measured value and the accepted value is likely due to systematic error in the tracking of the particles. It is possible that the ImageFeatureTrack function of *Mathematica* mistook stationary smudges and dust particles on the microscope lens for particles at times. This would lower the average squared displacement since the imperfections of the lens do not move. This would result in a smaller slope in the fitted line and therefore a lower value for Boltzmann's constant than expected. The uncertainty in our value was calculated under the assumption of random error, so although the uncertainty due to random error is small, systematic effects make the value unacceptable.

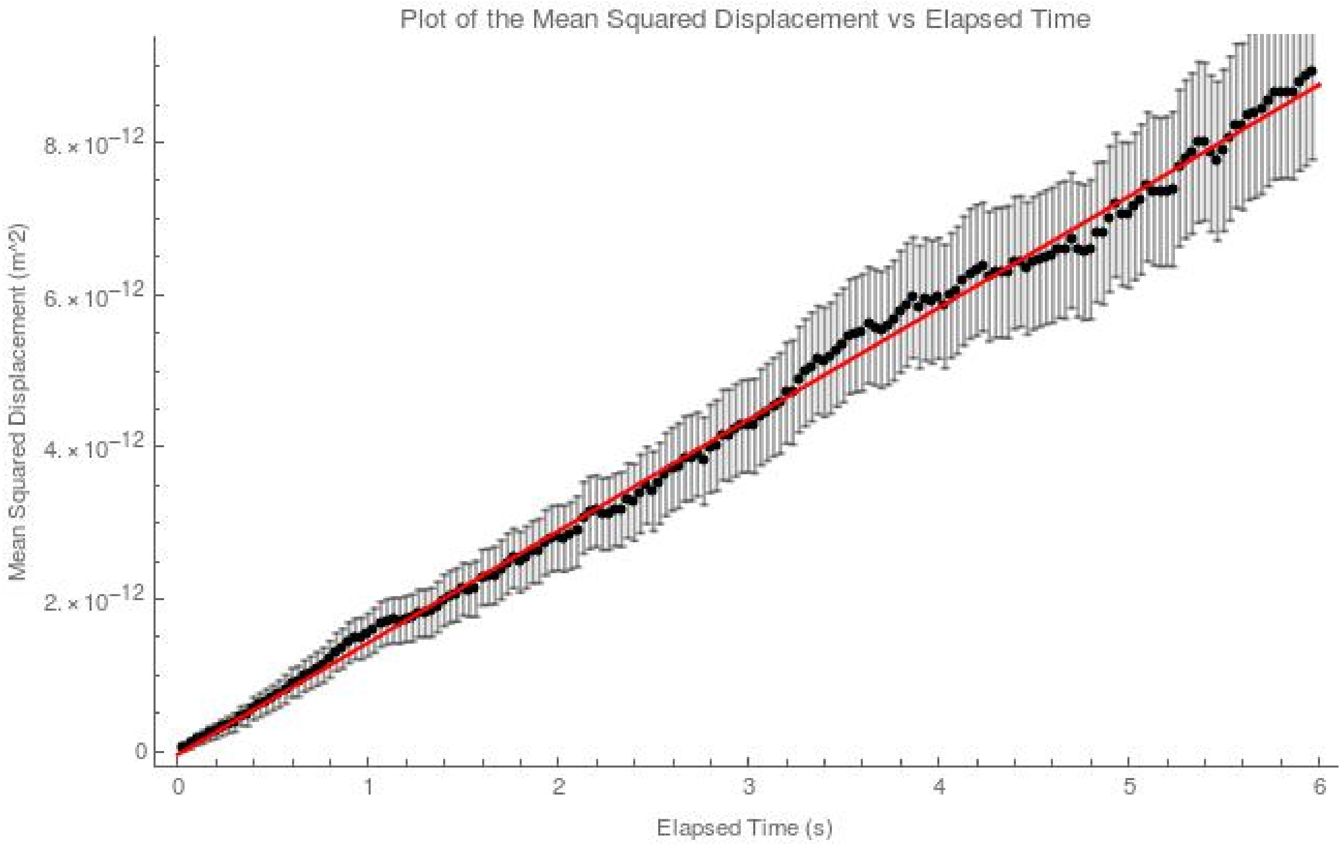

Figure 3: A graph of the data points we obtained, including error bars, and the best-fit line with equation $\langle R^2 \rangle = (1.47 \times 10^{-12} \pm 0.02 \times 10^{-12}\ \mathrm{m}^2/\mathrm{s})t + (-4 \times 10^{-14} \pm 2 \times 10^{-14}\ \mathrm{m}^2)$.

4 Summary and conclusions

Our experiment measured Boltzmann's constant by directly measuring the mean squared displacement of particles undergoing Brownian motion. We fit a line to the data points we obtained and used the fit parameters, along with relations derived by Einstein, to determine a value for Boltzmann's constant. [2] The value we obtained, $1.21 \times 10^{-23} \pm 0.02 \times 10^{-23}$ J/K, was not consistent with the accepted value of 1.38×10^{-23} J/K. This discrepancy was likely due to systematic error, and more in-depth experimentation would be necessary to improve on the measurement.

References

[1] R. Feynman, R. Leighton, M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley, 1963.

[2] A. Einstein, "On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat", *Annalen der Physik*, **17**, 549, (1905)

[3] J. Perrin, "Brownian Movement and Molecular Reality", *Annales de Chimie et de Physique*, **8**, (1909)