Problem Set # 8, Due: Wednesday Mar 22 by 11:00am

PHY 362K - Quantum Mechanics II, UT Austin, Spring 2017
(Dated: March 22, 2017)

Time-dependent perturbation theory, Fermi's Golden rule

I. KICKED HARMONIC OSCILLATOR

Consider a 1D harmonic oscillator, perturbed by a linear potential that we turn on with a Gaussian time-profile:

$$\hat{H}(t) = \underbrace{\frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega^2\hat{x}^2}_{\hat{H}_0} + \underbrace{g(t)\hat{x}}_{\hat{V}(t)}$$
$$g(t) = g_0\frac{e^{-t^2/2\tau^2}}{\sqrt{2\pi\tau^2}} \tag{1}$$

1. If the system is initially in the ground-state of $\hat{H}_0$ for $t \ll -\tau$, what is the probability (to first order in $\hat{V}(t)$) that it ends up in the first excited state of $\hat{H}_0$ a long time later ($t \gg \tau$)?

   Solution: Denote the ground state of $\hat{H}_0$ as $|0\rangle$ and the first excited state of $\hat{H}_0$ as $|1\rangle$. After a time t, the particle is in the state $|\psi\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle$. It is a standard result in time-dependent perturbation theory that these coefficients are given to first order by

$$c_n = \frac{1}{i\hbar}\int_{t_1}^{t_2} \langle n|\hat{V}|k\rangle e^{i\frac{E_n - E_k}{\hbar}t'}dt'$$

   where t_1 is the initial time, $|k\rangle$ is the initial state, t_2 is the final time and $|n\rangle$ is the final state. For this problem, $t_1 = -t$, $t_2 = t$, $n = 1$ and $k = 0$. Therefore, we have

$$c_1 = \frac{1}{i\hbar}\int_{-t}^{t} \langle 1|g_0\frac{e^{-t'^2/2\tau^2}}{\sqrt{2\pi\tau^2}}\hat{x}|0\rangle e^{i\frac{E_1 - E_0}{\hbar}t'}dt'$$

   but $E_1 = \frac{3}{2}\hbar\omega$ and $E_0 = \frac{1}{2}\hbar\omega$, so,

$$c_1 = \frac{1}{i\hbar}\frac{g_0}{\sqrt{2\pi\tau^2}}\int_{-t}^{t} e^{-\frac{t'^2}{2\tau^2}}\langle 1|\hat{x}|0\rangle e^{i\omega t'}dt'$$

   However, we can rewrite $\hat{x}$ in terms of the creation and annihilation operators as $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$. Using this result we have,

$$c_1 = \frac{g_0}{i}\frac{1}{\sqrt{4\pi\hbar m\omega\tau^2}}\int_{-t}^{t} e^{-\frac{t'^2}{2\tau^2}}\langle 1|\left(\hat{a} + \hat{a}^\dagger\right)|0\rangle e^{i\omega t'}dt'$$

   but we know that $\hat{a}|0\rangle = 0$ and $\hat{a}^\dagger|0\rangle = |1\rangle$, so we can write

$$
\begin{aligned}
c_1 &= \frac{g_0}{i}\frac{1}{\sqrt{4\pi\hbar m\omega\tau^2}}\int_{-t}^{t} e^{-\frac{t'^2}{2\tau^2}+i\omega t'}\langle 1|1\rangle dt' \\
&= -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}\int_{-t}^{t} e^{-\frac{1}{2\tau^2}\left(t'^2-2i\omega\tau^2 t'\right)}dt' \\
&= -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}\int_{-t}^{t} e^{-\frac{1}{2\tau^2}\left(t'^2-2i\omega\tau^2 t'-\tau^4\omega^2+\tau^4\omega^2\right)}dt' \\
&= -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}\int_{-t}^{t} e^{-\frac{1}{2\tau^2}\left(t'-i\omega\tau^2\right)^2-\frac{1}{2}\omega^2\tau^2}dt' \\
&= -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}e^{-\frac{1}{2}\omega^2\tau^2}\int_{-t}^{t} e^{-\frac{1}{2\tau^2}(t'-i\omega\tau^2)^2}dt'
\end{aligned}
$$

Now, since we have assumed that $|t| \gg \tau$ and the integral is a gaussian integral, we can approximate the limits of the integral as being infinite, we can also change variables to $u = \sqrt{\frac{1}{2\tau^2}}(t' - i\omega\tau^2)$. Making this approximation and change of variables we have

$$c_1 = -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}e^{-\frac{1}{2}\omega^2\tau^2}\sqrt{2\tau^2}\int_{-\infty}^{\infty} e^{-u^2}du$$

Using the result $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$, we have

$$c_1 = -\frac{ig_0}{\sqrt{2\hbar m\omega}}e^{-\frac{1}{2}\omega^2\tau^2}$$

Now, the probability for the system to be found in the first excited state is given by $P_1 = |c_1|^2$, therefore the desired probability to first order in $\hat{V}$ is

$$\boxed{P_1 = \frac{g_0^2}{2\hbar m\omega}e^{-\omega^2\tau^2}} \tag{2}$$

2. Sketch this the transition probability as a function of time (you don't need to precisely compute it, a rough sketch will suffice, however, please indicate the approximate functional in the extreme limits $t \gg \tau$ and $t \ll -\tau$).

   Solution: In the previous problem, we obtained an expression for c_1 in terms of the initial and final times. This expression can be written as

$$c_1 = -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}e^{-\frac{1}{2}\omega^2\tau^2}\int_{t_1}^{t_2} e^{-\frac{1}{2\tau^2}(t'-i\omega\tau^2)^2}dt'$$

   Now if we assume that the system was initialized in the ground state at time $-t$ (i.e. $t_1 = -t$), and call the final time t, we can write this coefficient as a function of t as

$$c_1(t) = -\frac{ig_0}{\sqrt{4\pi\hbar m\omega\tau^2}}e^{-\frac{1}{2}\omega^2\tau^2}\int_{-t}^{t} e^{-\frac{1}{2\tau^2}(t'-i\omega\tau^2)^2}dt'$$

   and therefore the transition probability as a function of t is given by

$$
\begin{aligned}
P_1(t) = |c_1(t)|^2 &= \frac{g_0^2}{4\pi\hbar m\omega\tau^2}e^{-\omega^2\tau^2}\left|\int_{-t}^{t} e^{-\frac{1}{2\tau^2}(t'-i\omega\tau^2)^2}dt'\right|^2 \\
&= \frac{g_0^2}{4\pi\hbar m\omega\tau^2}e^{-\omega^2\tau^2}\left|\int_{-t}^{t} e^{-\frac{1}{2}(t'/\tau-i\omega\tau)^2}dt'\right|^2
\end{aligned}
$$

or, letting $\xi = \frac{1}{\sqrt{2}}\left(t'/\tau - i\omega\tau\right)$, we have

$$
\begin{aligned}
P_1(t/\tau) &= \frac{g_0^2}{4\pi\hbar m\omega\tau^2} e^{-\omega^2\tau^2} \left| \sqrt{2}\tau \int_{\frac{1}{\sqrt{2}}(-t/\tau - i\omega\tau)}^{\frac{1}{\sqrt{2}}(t/\tau - i\omega\tau)} e^{-\xi^2} d\xi \right|^2 \\
&= \frac{g_0^2}{2\pi\hbar m\omega} e^{-\omega^2\tau^2} \left| \int_{\frac{1}{\sqrt{2}}(-t/\tau - i\omega\tau)}^{\frac{1}{\sqrt{2}}(t/\tau - i\omega\tau)} e^{-\xi^2} d\xi \right|^2
\end{aligned}
$$

Using Mathematica, we can plot this transition probability for several values of $\omega\tau$.

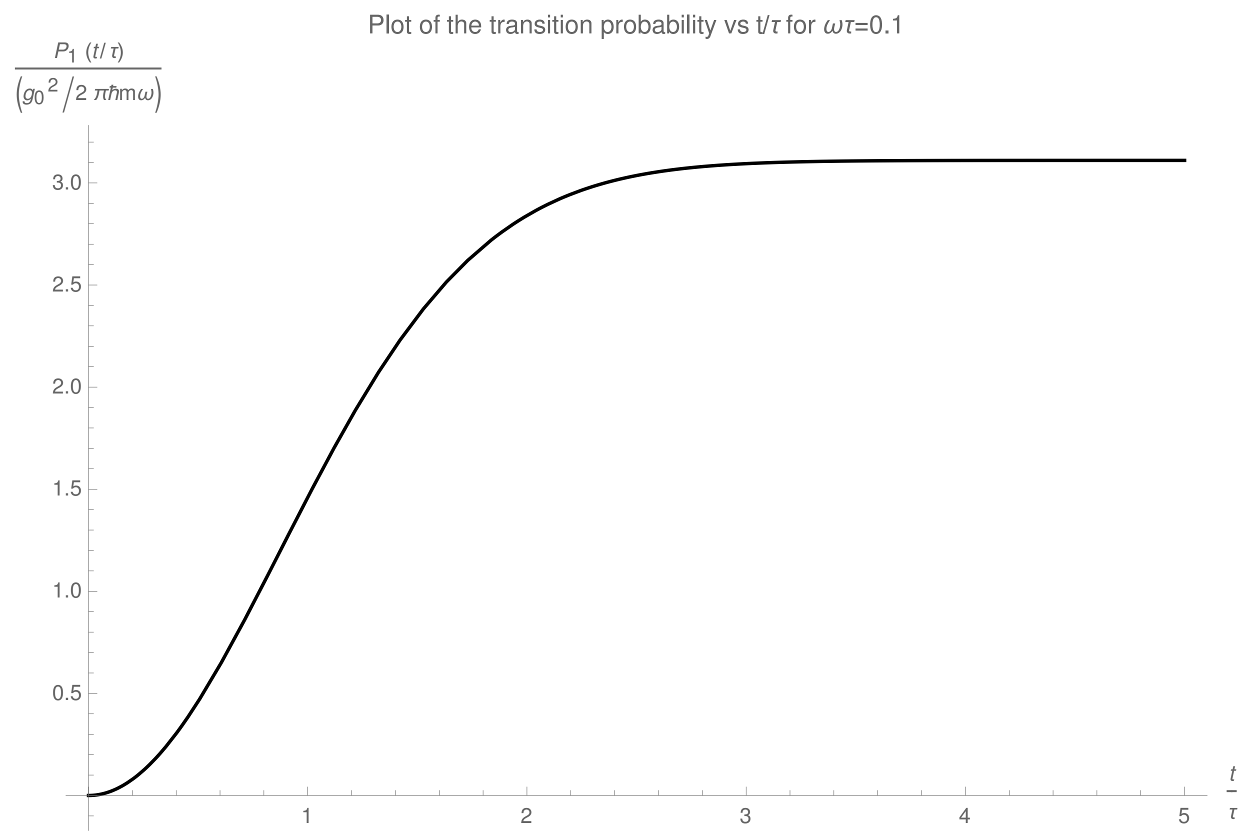

FIG. 1: Plot of the ratio of the transition probability, $P_1(t/\tau)$ to $\frac{g_0^2}{2\pi\hbar m\omega}$ vs t/τ with $\omega\tau = 0.1$.

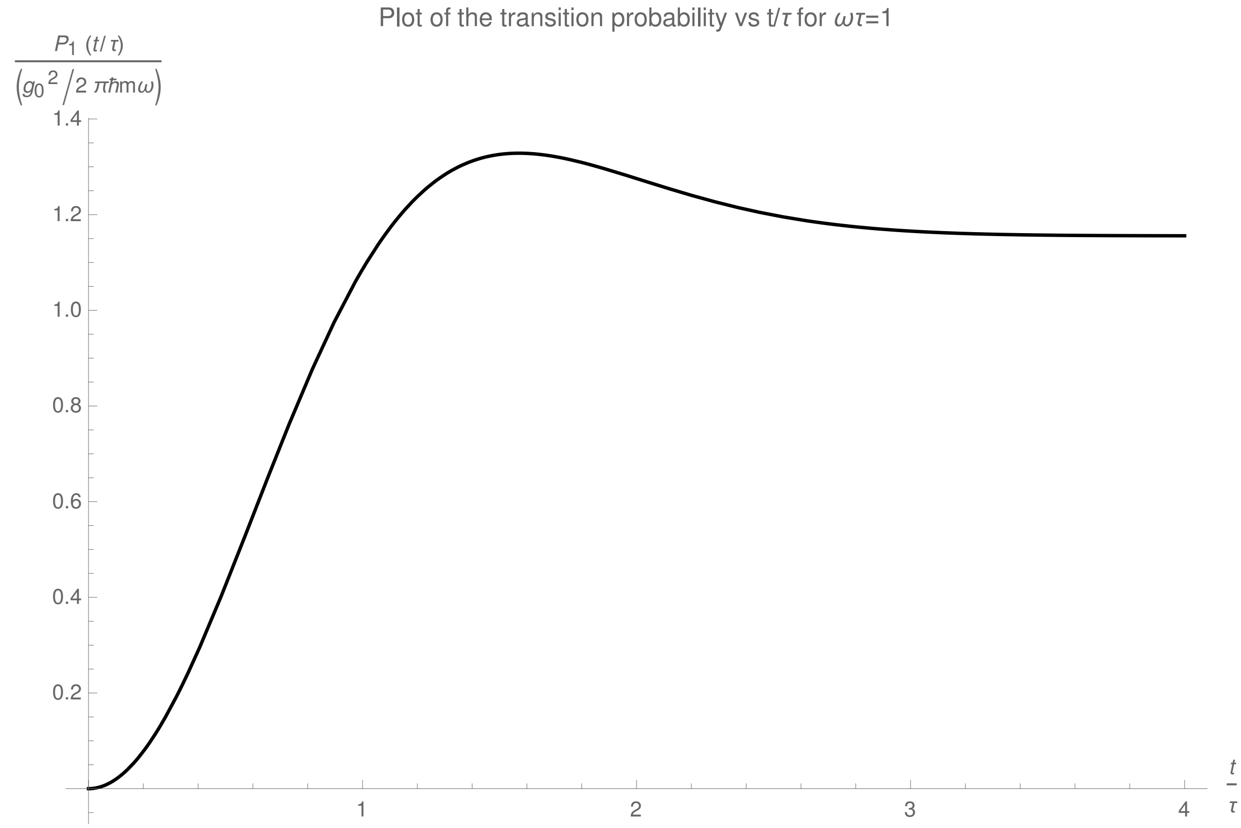

FIG. 2: Plot of the ratio of the transition probability, $P_1(t/\tau)$ to $\frac{g_0^2}{2\pi\hbar m\omega}$ vs t/τ with $\omega\tau = 1$.

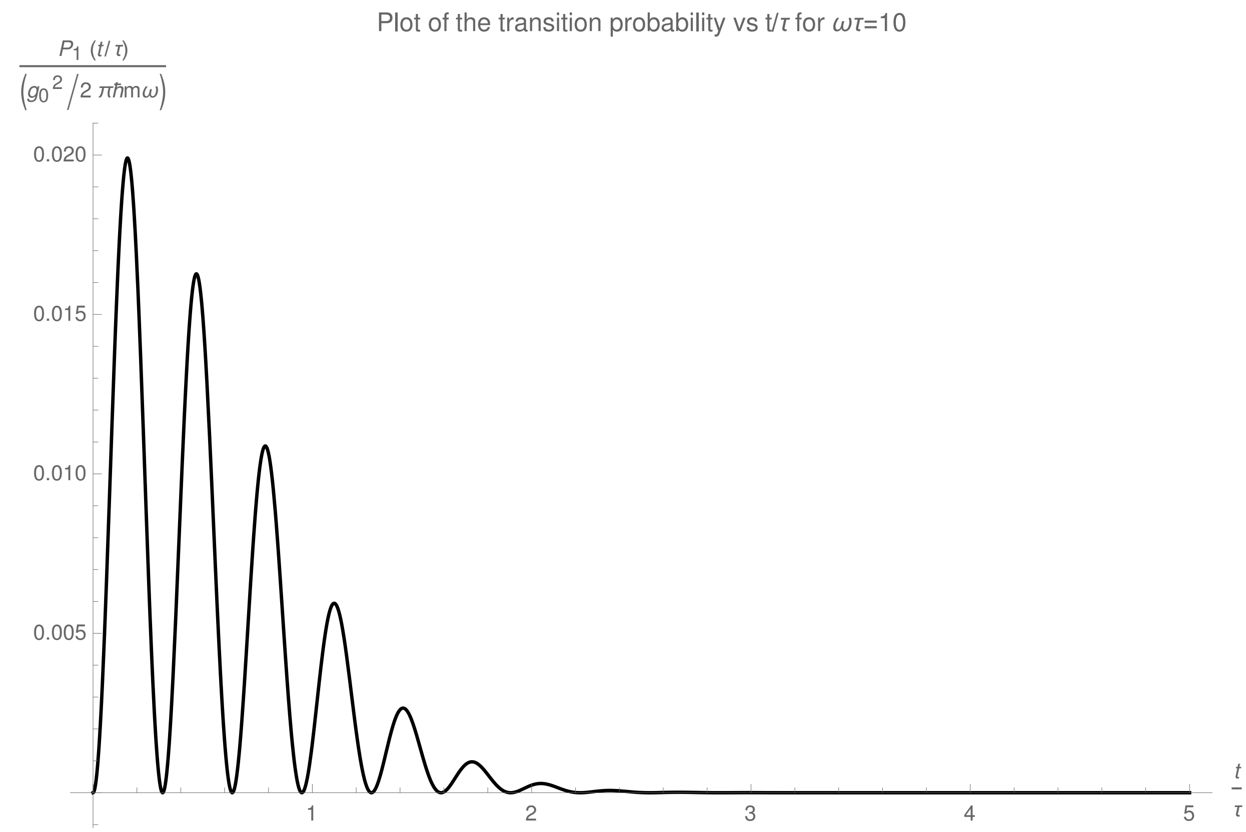

FIG. 3: Plot of the ratio of the transition probability, $P_1(t/\tau)$ to $\frac{g_0^2}{2\pi\hbar m\omega}$ vs t/τ with $\omega\tau = 10$.

In the limit when $t \gg \tau$ and $-t \ll \tau$, we can approximate the lower and upper limits of integration as $-\infty$ and ∞ respectively, since the integral is a Gaussian. In this limit, we recover the answer from part 1, which has the functional form of a constant.