Homework 7

1. **Addition and Mutilplication of matrices.**

   (a) Multiply the following matrices.

   i. $F = \mathbb{R}$. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$, $A \cdot B =?$ and $B \cdot A =?$.

   ii. $F = \mathbb{R}$. $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A \cdot A =?$.

   iii. $F = \mathbb{R}$. $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $A \cdot A =?$, $A \cdot A \cdot A =?$, $A \cdot A \cdot A \cdot A =?$. In general $A^N =?$

   iv. $F = \mathbb{Z}_3$. $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A^N =?$

   v. $F = \mathbb{Z}_5$. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A \cdot B =?$ and $B \cdot A =?$

   vi. $F = \mathbb{Z}_2$. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A \cdot B =?$.

   vii. $F = \mathbb{R}$. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0.5 & 1 \\ 1.5 & 1 \\ 2.5 & 1 \end{pmatrix}$, $A \cdot B =?$. Can you multiply $B \cdot A$?

   (b) In a. i-vi write the addition of the corresponding matrices, that is compute $A + B =?$.

   (c) Consider the set $M_{2\times 2}(\mathbb{R})$ with the operation of multiplication, is it true that for every $A, B \in M_{2\times 2}(\mathbb{R})$ we have

   $$A \cdot B = B \cdot A.$$

   If the statement is correct, prove it. If not exhibit a counter example.

   (d) The same like in c. but for $M_{1\times 1}(\mathbb{R})$.

2. **Multiplication of a vector by a matrix.**

   (a) Let F be a field. Consider the vector space $M_{n\times 1}(F)$ of n coloumn vectors.

i. Let $A \in M_{n\times n}(F)$ be a square n by n matrix. The matrix A defines a map $A : M_{n\times 1}(F) \to M_{n\times 1}(F)$ given by matrix multiplication

$$v \longrightarrow A \cdot v,$$

show the map A is linear.

ii. More generally, Let $A \in M_{m\times n}(F)$ be a m by n matrix. The matrix A defines a map $A : M_{n\times 1}(F) \to M_{m\times 1}(F)$ given by matrix multiplication

$$v \longrightarrow A \cdot v,$$

show the map A is linear.

(b) Apply the following multiplications.

i. $F = \mathbb{R}$. $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

ii. $F = \mathbb{R}$. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

iii. $F = \mathbb{Z}_5$. $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(c) Same as a. but for $F = \mathbb{Z}_3$.

3. **Ordered basis.** Let V be a f.d vector space over a field F. Let $S = (v_1, .., v_N)$ be an ordered basis. Consider the coordinate map

$$\varphi_S : V \to F^N,$$

defined by $v \mapsto (\alpha_1, .., \alpha_N)$ where the N-tuple $(\alpha_1, .., \alpha_N)$ is defined uniquely by $v = \sum_{i=1}^{N} \alpha_i v_i$.

(a) Show that φ_S is a linear map.

(b) Show that φ_S is an isomorphism. Hint: since the dimension of V and F^N is the same, it is enough to show $Ker(\varphi_S) = \{0\}$, that is φ_S is injective.

(c) Compute the inverse map $\varphi_S^{-1} : F^N \longrightarrow V$.

(d) Let $V = \mathbb{R}^3$. Consider the following ordered bases:

$$\begin{aligned} S &= (e_1, e_2, e_3) \text{ the standard basis.} \\ T &= (u_1 = (1,1,0), u_2 = (0,1,1), u_3 = (1,0,1)). \end{aligned}$$

i. Compute $\varphi_S : \mathbb{R}^3 \to \mathbb{R}^3$ and $\varphi_T : \mathbb{R}^3 \to \mathbb{R}^3$.

ii. Compute $\varphi_T \circ \varphi_S^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$.

4. **Matrices in $\mathbb{R}^N$.** Recall the standard ordered basis of $\mathbb{R}^N$ is $S = (e_1, e_2, .., e_N)$, where $e_i = (0, 0, .., \underset{i}{1}, 0, .., 0)$. Write the corresponding matrices, in terms of the standard basis of $\mathbb{R}^N$, of the following operators.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + y, x - y)$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x, 0)$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x, x)$.

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x, x, y)$.

   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x - y, x + y, z + x + y)$.

   (f) $T : \mathbb{R}^4 \to \mathbb{R}^4$, $T(x, y, z, t) = (x + y, x + z, y + t, a + y + z + t)$.

   (g) $T : \mathbb{R}^N \to \mathbb{R}^N$, $T(x_1, .., x_N) = (x_1 - x_N, x_2 - x_N, ..., x_{N-1} - x_N, x_N)$.

5. **Matrices continue.** Write the corresponding matrices of the following operators.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + y, x - y)$ in terms of the ordered basis $S = ((1, 1), (0, 1))$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x + y, x - y, x + y + z)$, in terms of the ordered basis $S = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y) = (x + y, x - y, x + y + z)$, in terms of the ordered basis $S = ((1, 1, 1), (0, 1, 1), (1, 0, 1))$.

   (d) $T : \mathbb{R}^4 \to \mathbb{R}^4$, $T(x, y, z, t) = (x+y, x+z, y+t, a+y+z+t)$, in terms of the ordered basis $S = ((1, 1, 1, 1), (1, -1, 1, 1), (1, 1, -1, 1), (1, 1, 1, -1))$. In each case, prove that the set S is indeed a basis, that is linear independent and spanning.

6. **Associativity of matrix multiplication.**

   (a) Let $A, B, C \in Mat_{n \times n}(F)$. Prove that $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

   (b) Let $A \in Mat_{m \times n}(F)$, $B \in Mat_{n \times k}(F)$, $C \in Mat_{k \times l}(F)$. Prove that $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

   (c) Consider the set $Mat_{n \times n}(F)$ equipped with the operations of addition and multiplication of matrices. Prove it is a ring: in your answer exhibit units with respect to addition and multiplication. Is it a commutative ring, that is, a ring where multiplication is commutative?