Homework 11

1. Let U, V be finite dimensional vector spaces. Let $T : U \to V$. Consider the induced transformation $\widetilde{T} = Bil\,(T) : Bil\,(V) \to Bil\,(U)$.

   (a) Prove that if $b \in S^2\,(V)$ then $\widetilde{T}\,(b)$ is symmetric, namely $\widetilde{T}\,(b) \in S^2\,(U)$. Conclude that $\widetilde{T}$ yields a linear transformation $S^2\,(T) : S^2\,(V) \to S^2\,(U)$.

   (b) Prove that if $b \in A^2\,(V)$ then $\widetilde{T}\,(b)$ is anti-symmetric, namely $\widetilde{T}\,(b) \in A^2\,(U)$. Conclude that $\widetilde{T}$ yields a linear transformation $A^2\,(T) : A^2\,(V) \to A^2\,(U)$.

2. Let V be a vector space, $\dim V = n$. Let $\{v_1, ..., v_n\}$ be a basis for V and $\{v_1^*, ..., v_n^*\}$ the corresponding dual basis.

   (a) Prove that $B_1 = \left\{v_i^* \otimes v_j^* : 1 \leq i, j \leq n\right\}$ is a basis f $Bil\,(V)$.

   (b) Prove that $B_2 = \left\{v_i^* \otimes v_j^* + v_j^* \otimes v_i^* : 1 \leq i \leq j \leq n\right\}$ is a basis $S^2\,(V)$. Explain why it consists of $n \cdot (n+1)/2$ vectors.

   (c) Prove that $B_3 = \left\{v_i^* \otimes v_j^* - v_j^* \otimes v_i^* : 1 \leq i < j \leq n\right\}$ is a basis $A^2\,(V)$. Explain why it consists of $n \cdot (n-1)/2$ vectors.

3. Let V be a vector space, $\dim V = n$. Let $\{v_1, ..., v_n\}$ be a basis for V and $\{v_1^*, ..., v_n^*\}$ the corresponding dual basis. Consider a bilinear form $b \in Bil\,(V)$. Consider its expression as a linear combination of the elements of the basis B_1 (from question 1): $b = \sum\limits_{i,j=1}^{n} x_{i,j} v_i^* \otimes v_j^*$. Prove that

$$x_{i,j} = b\,(v_i, v_j)\,.$$

4. Let V be a vector space, $\dim V = n$. Let $(v_1, ..., v_n)$ be an ordered basis for V and $(v_1^*, ..., v_n^*)$ the corresponding dual basis. Let $T : V \to V$ be a linear transformation presented by a matrix $M = (a_{ij})$ with respect to the basis $(v_1, ..., v_n)$.

   (a) Consider $\widetilde{T} = Bil\,(T) : Bil\,(V) \to Bil\,(V)$. Consider the basis B_1 with the lexicographic order. Compute the matrix $\widetilde{M}$ of $\widetilde{T}$ with respect to this basis. Compute $tr\left(\widetilde{M}\right)$.

   (b) Consider $\widetilde{T} = S^2\,(T) : S^2\,(V) \to S^2\,(V)$. Consider the basis B_2 with the lexicographic order. Compute the matrix $\widetilde{M}$ of $\widetilde{T}$ with respect to this basis. Compute $tr\left(\widetilde{M}\right)$.

   (c) Consider $\widetilde{T} = A^2\,(T) : A^2\,(V) \to A^2\,(V)$. Consider the basis B_3 with the lexicographic order. Compute the matrix $\widetilde{M}$ of $\widetilde{T}$ with respect to this basis. Compute $tr\left(\widetilde{M}\right)$.

(d) Prove that if $\dim V = 2$ then $A^2(T) = \det(M)\, Id$. Conclude that the determinant does not depend on the choice of a basis.

5. Consider the composition

$$U \xrightarrow{T} V \xrightarrow{S} W.$$

   (a) Prove that $Bil(S \circ T) = Bil(T) \circ Bil(S)$.

   (b) Prove that $S^2(S \circ T) = S^2(T) \circ S^2(S)$.

   (c) Prove that $A^2(S \circ T) = A^2(T) \circ A^2(S)$.

   (d) Conclude that in case $\dim V = 2$ and $T, S : V \to V$ then $\det(S \circ T) = \det(S)\det(T)$.

6. Consider the composition $U \xrightarrow{T} V \xrightarrow{S} W$. Prove that $(S \circ T)^* = T^* \circ S^*$.

7. Consider the map $b : F^n \times F^n \to F$, given by $b(\overrightarrow{x}, \overrightarrow{y}) = \sum\limits_{i=1}^{n} x_i y_i$.

   (a) Prove that b is a bilinear form.

   (b) Prove that b is symmetric.

   (c) Prove that b is non-degenerate.

8. Consider the map $b : F^n \times F^n \to F$, given by $b(\overrightarrow{x}, \overrightarrow{y}) = \sum\limits_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$.

   (a) Prove that b is bilinear.

   (b) Prove that b is anti-symmetric.

   (c) Prove that b is non-degenerate.

9. Prove that any bilinear from b on F^n can be written as $b(\overrightarrow{x}, \overrightarrow{y}) = \sum\limits_{i,j=1}^{n} b_{ij} x_i y_j$, for some $b_{ij} \in F$. Prove that b is symmetric iff $b_{ij} = b_{ji}$, for every $i, j = 1, .., n$. Prove that b is anti-symmetric iff $b_{ij} = -b_{ji}$, for every $i, j = 1, .., n$.