Problem Set # 9, Due: Wednesday Mar 29 by 11:00am

PHY 362K - Quantum Mechanics II, UT Austin, Spring 2017
(Dated: March 29, 2017)

Coulomb potential scattering in the Born approximation

I. BORN APPROXIMATION

Last week, we derived an expression (the Born approximation) for the differential scattering cross section for a massive particle in a wave-packet state, scattering in $3d$ off of a potential, $V(r)$. Here's a summary of the derivation. Note: there was an error in the lecture notes from class regarding factors of volume, that I will correct below (I apologize for this):

A. Notes on Derivation of Born Approximation

Suppose the potential quickly fades to zero away from the origin. Then, from the point of view of the particle, the potential appears to be time-dependent, turning on only as the wave-packet collides with the potential. If the typical wave-vectors in the wave-packet are long compared to the lengthscale over which $V(r)$ varies, this turn on is slow, which we can model as a time dependent potential: $V(r) \to V(r,t) = e^{i\eta t}V(r)$, with η small.

Then, we can use Fermi's golden rule to compute the rate at which the particle scatters into a small solid-angle element $\Delta\Omega$:

$$\Gamma_{\Delta\Omega} = \frac{2\pi}{\hbar}\sum_{k'\in\Delta\Omega}|\langle k'|\hat{V}|k\rangle|^2\delta(\varepsilon_{k'}-\varepsilon_k) \tag{1}$$

where we have assumed that the wave-packet consists of a superposition of plane wave states close to a incoming wave-vector $\vec{k}$. Here, $|k\rangle = \int d^3r\frac{1}{\sqrt{L^3}}e^{i\vec{k}\cdot\vec{r}}|r\rangle$ is a plane wave state with energy $\varepsilon_k = \frac{\hbar^2k^2}{2m}$, and L^3 is the spatial volume of the system.

We can write the matrix element as, $\langle k'|\hat{V}|k\rangle = \frac{1}{L^3}\int d^3re^{i(k-k')\cdot r}V(r) \equiv \frac{1}{L^3}\tilde{V}_{k'-k}$, where $\tilde{V}_q$ is the Fourier transform of $V(r)$ with wave-vector q. With this notation, Eq. 1 becomes:

$$\Gamma_{\Delta\Omega} = \frac{2\pi}{\hbar}\frac{1}{(L^3)^2}\sum_{k'\in\Delta\Omega}|\tilde{V}_{k'-k}|^2\delta(\varepsilon_{k'}-\varepsilon_k) \tag{2}$$

For $L\to\infty$, we can convert the discrete sum over the final momentum, to a continuous integral: $\frac{1}{L^3}\sum_{k'\in\Delta\Omega} \to \int_{k'\in\Delta\Omega}\frac{d^3k'}{(2\pi)^3}$, then we are left with:

$$\Gamma_{\Delta\Omega} = \left(\frac{1}{L^3}\right)\frac{2\pi}{\hbar}\int_{k'\in\Delta\Omega}\frac{d^3k_f}{(2\pi)^3}|\tilde{V}_{k_i-k_f}|^2\delta(\varepsilon_{k'}-\varepsilon_k) \tag{3}$$

(Note the overall factor of $1/L^3$ out front, which was missing from the derivation in class).

In a typical scattering experiment, we typically consider a beam of many incoming particles all scattering off the same potential. Then, it is convenient to re-write the scattering rate as the differential cross section, by dividing by the flux of incoming particles, where the flux is defined as the density of particles times their velocity. Here, there is only one incoming particle in a volume L^3, so its density is $\rho = \frac{1}{L^3}$, and its velocity is $v = \frac{\hbar k}{m}$, so the Flux $= \frac{m}{L^3\hbar k}$.

Dividing the rate by the flux and also dividing by the solid angle $\Delta\Omega$ we obtain the differential cross section:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\frac{2\pi m}{\hbar^2 k}\int_{k'\in\Delta\Omega}\frac{d^3k'}{(2\pi)^3}\delta(\varepsilon_k' - \varepsilon_k) \\ &= \frac{2\pi m}{\hbar k}\frac{1}{(2\pi)^4}\int_0^\infty dk'\ (k')^2|\tilde{V}_{k'-k}|^2\underbrace{\frac{\int_{\theta,\varphi\in\Delta\Omega}\sin\theta d\theta d\varphi}{\Delta\Omega}}_{=1}\delta(\varepsilon_k' - \varepsilon_k) \\ &= \frac{m^2}{4\pi^2\hbar^4}|V_{k'-k}|^2 \qquad (4)\end{aligned}$$

Where, outgoing wave-vector of the scattered particle: $\vec{k'}$ has amplitude $k' = k$ due to energy conservation, and the direction of $\vec{k'}$ points at an angle θ with respect the axis of the initial wave-vector: $\vec{k}$, and is rotated by an azimuthal angle φ in the plane perpendicular to $\vec{k}$ (see, e.g. Fig. 11.3 of Griffiths).

Notice that, for spherical symmetric potentials, the Fourier transform $V_{\vec{k'}-\vec{k}}$ depends only on:

$$|\vec{k'} - \vec{k}| = 2k\sin\frac{\theta}{2} \qquad (5)$$

(see e.g. Fig. 11.11 in Griffiths), so that the differential cross section is depends only on k and θ but is independent of the azimuthal angle φ.

B. Homework problem

1. Show that the total cross-section: $\sigma = \int \sin\theta d\theta d\varphi \frac{d\sigma}{d\Omega}(\theta,\varphi)$ has units of area, which we can loosely interpret as the effective "cross sectional area" of the scatterer (hence the name "cross-section").

   Solution: We know that $\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r\right|^2$. Now, $V(\vec{r})$ has units of energy, therefore $\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r$ has units of energy times length cubed. We will denote the units of a quantity x as $[x]$. Using this notation, $\left[\left|\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r\right|^2\right] = E^2L^6$ where E represents energy and L represents length. Also, $[m] = M$, where M represents mass and $[\hbar] = ET$ where T represents time. Therefore,

   $$\begin{aligned}\left[\frac{d\sigma}{d\Omega}\right] &= \left[\frac{m^2}{4\pi^2\hbar^4}\left|\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r\right|^2\right] = \frac{M^2}{E^4T^4}E^2L^6 \\ &= \frac{M^2L^6}{E^2T^4}\end{aligned}$$

   We know that the dimension of energy can be written in terms of M, L and T as $E = \frac{ML^2}{T^2}$, using this result, we obtain

   $$\begin{aligned}\left[\frac{d\sigma}{d\Omega}\right] &= \left[\frac{m^2}{4\pi^2\hbar^4}\left|\int e^{i(\vec{k'}-\vec{k})\cdot\vec{r}}V(\vec{r})d^3r\right|^2\right] = \frac{M^2L^6}{\left(\frac{ML^2}{T^2}\right)^2T^4} \\ &= L^2\end{aligned}$$

   L^2 is a unit of area, therefore $\frac{d\sigma}{d\Omega}$ has units of area. In the expression $\sigma = \int \sin\theta d\theta d\varphi \frac{d\sigma}{d\Omega}(\theta,\varphi)$, $\frac{d\sigma}{d\Omega}$ is the only dimensionful quantity, so σ, the total cross section, has units of area as well.

2. Compute the differential scattering cross section, $\frac{d\sigma}{d\Omega}$, for a particle with charge q_1 (e.g. an electron) to scatter from a fixed target (e.g. a Gold nucleus) with charge q_2 fixed at the origin. To do this, we will need a trick. Instead of the Coulomb potential for the charges: $V_{\rm C}(r) = \frac{q_1 q_2}{r}$, let us consider the "Yukawa" potential: $V_{\rm Y}(r) = \frac{q_1 q_2 e^{-\mu r}}{r}$. Compute the differential cross section for the Yukawa potential, and then take $\mu \to 0$ to recover the answer for the Coulomb potential.

Solution: To find the differential cross-section, we must first find the Fourier transform of the Yukawa potential, $\tilde{V}_{k'-k} = \int e^{i(\vec{k}'-\vec{k})\vec{r}} e^{-\mu r} d^3r$. We will use a coordinate system in which $\vec{k}' - \vec{k}$ points in the $\hat{z}$ direction, so $\vec{k}' - \vec{k} = |\vec{k}' - \vec{k}|\hat{z}$. Then the expression for the Fourier transform becomes

$$\tilde{V}_{k'-k} = q_1 q_2 \int e^{i|\vec{k}'-\vec{k}|z} \frac{1}{r} e^{-\mu r} d^3 r$$

If we use spherical coordinates to evaluate this integral, the expression becomes

$$\begin{aligned}
\tilde{V}_{k'-k} &= q_1 q_2 \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{i|\vec{k}'-\vec{k}|r\cos\theta} \frac{1}{r} e^{-\mu r} r^2 \sin\theta d\phi d\theta dr \\
&= 2\pi q_1 q_2 \int_0^\infty \int_0^\pi e^{i|\vec{k}'-\vec{k}|r\cos\theta} e^{-\mu r} r \sin\theta d\theta dr
\end{aligned}$$

Now, letting $u = \cos\theta$, we can rewrite the above expression as

$$\begin{aligned}
\tilde{V}_{Y,k'-k} &= 2\pi q_1 q_2 \int_0^\infty \int_{-1}^1 e^{i|\vec{k}'-\vec{k}|ru} e^{-\mu r} r du dr \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \int_0^\infty \frac{1}{r} r e^{-\mu r} \left. e^{i|\vec{k}'-\vec{k}|ru} \right|_{u=-1}^{u=1} dr \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \int_0^\infty e^{(i|\vec{k}'-\vec{k}|-\mu)r} - e^{-(i|\vec{k}'-\vec{k}|+\mu)r} dr \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \left[\frac{1}{i|\vec{k}'-\vec{k}|-\mu} e^{(i|\vec{k}'-\vec{k}|-\mu)r} + \frac{1}{i|\vec{k}'-\vec{k}|+\mu} e^{-(i|\vec{k}'-\vec{k}|+\mu)r} \right]_{r=0}^{r=\infty} \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \left[-\frac{1}{i|\vec{k}'-\vec{k}|-\mu} - \frac{1}{i|\vec{k}'-\vec{k}|+\mu} \right] \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \left[\frac{-i|\vec{k}'-\vec{k}| - \mu - i|\vec{k}'-\vec{k}| + \mu}{-\mu^2 - |\vec{k}'-\vec{k}|^2} \right] \\
&= \frac{2\pi q_1 q_2}{i|\vec{k}'-\vec{k}|} \left[\frac{2i|\vec{k}'-\vec{k}|}{\mu^2 + |\vec{k}'-\vec{k}|^2} \right] \\
&= \frac{4\pi q_1 q_2}{\mu^2 + |\vec{k}'-\vec{k}|^2}
\end{aligned}$$

Now, letting $\mu \to 0$, we obtain the fourier transform of the Coulomb potential $\tilde{V_{k'-k}} = \frac{4\pi q_1 q_2}{|\vec{k}'-\vec{k}|^2}$. Now using the given expression for the differential cross-section, we have

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2 \hbar^4} \left(\frac{4\pi q_1 q_2}{|\vec{k}'-\vec{k}|^2} \right)^2 = \frac{4m^2 q_1^2 q_2^2}{\hbar^4 |\vec{k}'-\vec{k}|^4}$$

or, using $|\vec{k}' - \vec{k}| = 2k \sin\frac{\theta}{2}$, we have

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2 q_1^2 q_2^2}{4\hbar^4 k^4 \sin^4\left(\frac{\theta}{2}\right)}} \tag{6}$$

3. What is the total cross-section $\sigma = \int \sin\theta d\theta d\varphi \frac{d\sigma}{d\Omega}(\theta, \varphi)$ for the Coulomb potential? How should we interpret this result? (consider question #1)

Solution: Substituting the expression for the differential cross section found in the previous problem into the expression for the total cross-section, we obtain

$$\begin{aligned}
\sigma &= \int_0^\pi \int_0^{2\pi} \frac{m^2 q_1^2 q_2^2}{4\hbar^4 k^4 \sin^4\left(\frac{\theta}{2}\right)} \sin\theta d\phi d\theta \\
&= \frac{\pi m^2 q_1^2 q_2^2}{2\hbar^4 k^4} \int_0^\pi \frac{\sin\theta}{\sin^4(\theta/2)} d\theta \\
&= \frac{\pi m^2 q_1^2 q_2^2}{2\hbar^4 k^4} \int_0^\pi \frac{\frac{1}{2}\sin(\theta/2)\cos(\theta/2)}{\sin^4(\theta/2)} d\theta \\
&= \frac{\pi m^2 q_1^2 q_2^2}{2\hbar^4 k^4} \int_0^\pi \frac{\frac{1}{2}\cos(\theta/2)}{\sin^3(\theta/2)} d\theta
\end{aligned}$$

now, letting $u = \sin(\theta/2)$, we can write this integral as

$$\sigma = \frac{\pi m^2 q_1^2 q_2^2}{2\hbar^4 k^4} \int_0^1 \frac{1}{u^3} du$$

This integral diverges, so we cannot calculate the total cross-section for the Coulomb potential using the first order Born approximation. This is because for the Born approximation to be accurate, the potential must be very well localized around the scatterer. However, the $1/r$ dependence of the Coulomb potential does not fall off rapidly enough for the Born approximation to be accurate. We can interpret this as the scatterer having an "infinite cross-section" because the potential tails off slowly and the scattering particle will always interact with the scatterer a significant amount even if it is very far way.