Homework 12

1. Let V be a f.d vector space over $\mathbb{R}$. Let $(-,-) : V \times V \to \mathbb{R}$ be an Euclidian inner product. Let $\|-\| : V \to \mathbb{R}^{\geq 0}$ be the associated norm.

   (a) Prove $\|-\|$ satisfy the triangle inequality. Write the complete proof of Cauchy Schwartz inequality.

   (b) based on a. prove $|\|u\| - \|v\|| \leq |\, \|u - v\|$, for every $u, v \in V$.

   (c) Prove that $\|u + v\| = \|u\| + \|v\|$ iff $(u, v) = 0$.

2. Let V be a f.d vector space over $\mathbb{R}$. Let $(-,-) : V \times V \to \mathbb{R}$ be an Euclidian inner product. Let $U \subset V$ be a subvector space. Consider the orthogonal projections: $P_U, P_{U^\perp} : V \to V$.

   (a) Prove that $\left(U^\perp\right)^\perp = U$.

   (b) Prove that $P_U \circ P_U = P_U$ and that $P_{U^\perp} \circ P_{U^\perp} = P_{U^\perp}$.

   (c) Prove that $P_U \circ P_{U^\perp} = P_{U^\perp} \circ P_U = 0$.

   (d) Prove that $P_U + P_{U^\perp} = Id$.

   (e) Prove that $\ker P_U = U^\perp$ and $\ker P_{U^\perp} = P_U$.

3. Consider $\mathbb{R}^2$ with the standard bilinear form. Let $U = \mathbb{R}(1, 1)$.

   (a) Compute $P_U(x, y)$.

   (b) Prove by explicit computation that $P_U(x, y)$ is the vector in U of minimal distance to (x, y).

4. Let V be a f.d vector space over a field F. Let $b : V \times V \to F$ be a bilinear form. Let $U \subset V$ be a subvector space.

   (a) Prove that $\dim U + \dim U^\perp \geq \dim V$ and equality holds iff b is non-degenerate.

   (b) Prove that $U \subset \left(U^\perp\right)^\perp$.

   (c) Prove that if b is non-degenerate then $U = \left(U^\perp\right)^\perp$.
   Here orthogonal complement is w.r.t b.

5. Compute the orthogonal projections $P_U, P_{U^\perp} : V \to V$ in the following cases:

   (a) Arbitrary f.d V with some Euclidian inner product and $U = \{0\}$, $U = V$.

   (b) $V = \mathbb{R}^2$ with the standard inner product, $U = \mathbb{R}(0, 1)$; $U = \mathbb{R}(1, 0)$; $U = \mathbb{R}(1, 1)$.

   (c) $V = \mathbb{R}^3$ with the standard inner product, $U = \mathbb{R}(1, 0, 0) + \mathbb{R}(0, 1, 0)$; $U = \mathbb{R}(1, 1, 0) + \mathbb{R}(1, -1, 1)$.

(d) $V=\mathbb{R}^n$ with the standard inner product; $U=\left\{(x_1,...,x_n) : \sum\limits_{i=1}^{n} x_i = 0\right\}$.

(e) $V=\mathbb{R}(X)$, X a finite set with the inner product $(f,g)=\sum_{x\in X} f(x)\, g(x)$. $S\subset X$ a subset and $U=\{f\in V : f(x)=0 \text{ for every } x\in S\}$.

6. Apply the Grahm Schmidt orthogonalization procedure in order to compute the associated orthonormal basis in the following cases.

   (a) $V=\mathbb{R}^2$, $B=\{(1,0),1,-1\}$; $B=\{(1,1),(2,-2)\}$.

   (b) $V=\mathbb{R}^3$, $B=\{(1,0,0),(0,1,0),(1,1,1)\}$; $B=\{(1,1,0);(1,0,1);(1,1,1)\}$.

   (c) $V=\mathbb{R}^n$, $B=\{e_1,e_2-e_1,e_3-e_2,...,e_n-e_{n-1}\}$; $B=\{e_1,e_2,..,e_{n-1},e_1+..+e_n\}$.

   (d) $V=\mathbb{R}(X)$, $X=\{x_1,..,x_n\}$. $B=\left\{\delta_{x_1},..,\delta_{x_{n-1}},\delta_{x_n}+\delta_{x_1}\right\}$.

7. Let V be a f.d vector space over $\mathbb{R}$. Let $(-,-):V\times V\to\mathbb{R}$ be an Euclidian inner product. Consider a pair of non-zero vectors v_1,v_2 of unit norm. Let $U_i=\mathbb{R}v_i$, for $i=1,2$. Prove that

$$Tr\left(P_{U_1}\circ P_{U_2}\right)=(v_1,v_2).$$

8. Let V be a f.d vector space over a field F. Let $b:V\times V\to F$ be a bilinear form. Let $U\subset V$ be a subvector space. Prove that

$$Tr\left(P_U\right)=\dim U.$$

9. Consider the following bilinear form on $\mathbb{R}^2$:

$$b\left((x,y),(x',y')\right)=xx'+2xy'+2yx'+5yy'.$$

   (a) Prove that b is Euclidian form.

   (b) Let $U=\mathbb{R}(1,1)$. Compute $U^{\perp}$ with respect to b. Compute $P_U,P_{U^{\perp}}$.