Homework 3

1. Consider the group $\left(\mathbb{R}^2, +\right)$. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y, 2x - y)$. Prove that f is a homomorphism. Prove that f is an isomorphism. Is $f(x, y) = (x + y, 1)$ a homomorphism? Is $f(x, y) = (x + y, 0)$ a homomorphism?

2. Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that $\operatorname{Im} f$ is a subgroup of G_2.

3. Let $f : G_1 \to G_2$ be a homomorphism. Prove that $f\left(g^{-1}\right) = f(g)^{-1}$, that is, f sends the inverse of an element in G_1 to the inverse of its image in G_2.

4. Let $(R, +, \cdot, 0, 1)$ be a ring. Let $a, b \in R$ and consider their additive inverses $-a$ and $-b$. Prove that $(-a) \cdot (-b) = a \cdot b$.

5. Let X, Y be two finite sets and $\varphi : X \to Y$ a function. Consider the map $\varphi^* : \mathbb{R}(Y) \to \mathbb{R}(X)$ given by: $\varphi^*(f)(x) = f(\varphi(x))$. Prove that φ^* is a homomorphism of rings. Prove that φ^* is an ismorphism of rings iff φ is bijective.

6. Consider the set $\mathbb{R}^2$. Define the following operations: $+ : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$; $\cdot : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} (x, y) + (x', y') &= (x + x', y + y'), \\ (x, y) \cdot (x', y') &= (xx' - yy', yx + xy'). \end{aligned}$$

   (a) Prove that $(1, 0)$ is a unit with respect to $\cdot$.

   (b) Prove that every $(x, y) \neq (0, 0)$ admits an inverse with respect to $\cdot$.

   (c) Prove that $\left(\mathbb{R}^2, +, \cdot, (0, 0), (1, 0)\right)$ form a field. Can you recognize this field.

   (d) Consider the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(x, y) = (x, -y)$. Prove that φ is an isomorphism.

7. Let $(G, *)$ be a finite group and $H \subset G$ a subgroup. Consider the equivalence relation on G: $x \sim y$ iff there exists $h \in H$ such that $y = x * h$. Let P be an equivalence class.

   (a) Prove that $\#P = \#H$.

   (b) Prove that $\#H$ divides $\#G$.

8. (*difficult) a prime number. Consider the group $\mathbb{Z}_p^\times = \mathbb{Z}_p - \{0\}$. Define the map (Legendre character): $\sigma : \mathbb{Z}_p^\times \to \{-1, 1\}$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x = y^2 \\ -1 & \text{otherwise} \end{cases}.$$

(a) Prove that σ is a homomorphism. (hint: use exersice 7).

(b) Prove that σ taks the same number of 1 and -1. (hint: use exersice 7).