Where to from here?

Gabriel Field

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Outline

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Refresher: Categories

Reminder: *A* **category** *is a directed multigraph with a composition operation and identity arrows.*

Examples:

Refresher: Functors

Reminder: *A* **functor** $F : \mathcal{C} \to \mathcal{D}$ *is a structure-preserving map of categories.*

Examples:

Definition: Opposite category

Examples:

Working in $\mathcal{C}^{\mathrm{op}}$ rather than $\mathcal{C}$ is called dualising.

Duality: The logical view

Principle: Categorical duality

In any statement of the form "for all categories $\mathcal{C}$, …", we can replace $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, and re-interpret the statement in $\mathcal{C}$.

Definition: *For all categories $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, an* **isomorphism** $x \simeq_{\mathcal{C}} y$ *in $\mathcal{C}$ is an arrow $f : x \to y$ such that…*

Question: What is the dual concept to isomorphisms?

Contravariant functors

Definition: *A* **functor** $F : \mathcal{C} \to \mathcal{D}$ *is a structure-preserving map*

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\begin{tikzcd} x \arrow[d, "f"'] & \mapsto & Fx \arrow[d, "Ff"] \\ y & & Fy \end{tikzcd} \qquad F(g \circ f) = Fg \circ Ff \qquad F1_x = 1_{Fx}$$

Dual definition: Contravariant functor[1].

[1] To self: cod

Contravariant functors

Key example: Representable functors. For $z \in \mathcal{C}$,

Contravariant functors

Result: **Functors preserve isomorphisms**. *For all* $\mathcal{C} \xrightarrow{F} \mathcal{D}$ *and* $f : x \simeq_{\mathcal{C}} y$, *we have* $Ff : Fx \simeq_{\mathcal{D}} Fy$.

Dual result:

Contravariant functors

Example application: Isomorphic groups have isomorphic duals

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Axioms of homology theory (as in [2])

Axioms for homology. A (reduced) homology theory $\tilde{h}$ consists of:

[2] The LHS and RHS are functors from the category of CW pairs (A, X); the left is a composite of the "take the quotient" functor with $\tilde{h}_n$, and the right is a composite of the "forget X" functor with $\tilde{h}_{n-1}$.

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- For any $\{X_\alpha\}_{\alpha \in \mathcal{A}}$, the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha \in \mathcal{A}} X_\alpha$ give

$$\bigoplus_{\alpha \in \mathcal{A}} \tilde{h}_n(X_\alpha) \simeq_{\mathrm{Ab}} \tilde{h}_n\left(\bigvee_{\alpha \in \mathcal{A}} X_\alpha\right)$$

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Going to cohomology

"Taking the homology" yields functors $H_n : \mathrm{ChCmplx} \to \mathrm{Ab}$.

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- "Cohomology functors" $H^n : \mathrm{ChCmplx} \to \mathrm{Ab}^{\mathrm{op}}$.
- Axiom list with Ab replaced by $\mathrm{Ab}^{\mathrm{op}}$ should be satisfied[3].

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$$\mathrm{ChCmplx} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{Ab}$$

- Cohomology group functors $H^n(-; G)$

$$\mathrm{Top} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{Ab}$$

Cohomology of spaces

We recover [2, pp. 199–204]:

- Reduced groups
- Relative groups
- LES of a pair
- Homotopy invariance
- Excision
- Simplicial cohomology
- Cellular cohomology
- Mayer-Vietoris sequences

But also. . .

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This is a "graded ring" $H^*(X;G) = \bigoplus_{n\geq 0} H^n(X;G)$ with product

$$H^i(X;G) \times H^j(X;G) \to H^{i+j}(X;G)$$

This product is called the cup product [2, p. 206].

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The idea

Fix $X \in \mathrm{Top}_*$[4].
The fundamental group $\pi_1(X)$ analyses maps of the form

$$\begin{aligned} \mathbb{I} &\longrightarrow X \\ \partial\mathbb{I} &\longrightarrow \{*\} \end{aligned}$$

up to homotopy.

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The idea

Fix $X \in \mathrm{Top}_*$[4].
The homotopy groups $\pi_n(X)$ analyse maps of the form

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The higher homotopy groups

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 0}$.
Definition: The homotopy "group" $\pi_n(X)$

$$\pi_n(X) := \{f : \mathbb{I}^n \to X \mid f(\partial \mathbb{I}^n) \subseteq \{*\}\}_{/\text{htpy}}$$

The higher homotopy groups

Fix $X \in \mathrm{Top}_*$.

Example: $\pi_0(X)$

The higher homotopy groups

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}$. Assume $n \geq 1$.

Definition: Group structure on $\pi_n(X)$

$$f + g := (t_1, t_2, \ldots, t_n) \mapsto \begin{cases} f(2t_1, t_2, \ldots, t_n) & \text{if } t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{else} \end{cases}$$

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Example: Homotopy groups $\pi_n(\mathbb{S}^2)$ for $n \in \{1, \ldots, 12\}$:

0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

This mess is part of the motivation for homology theory.

Slight peeks of regularity

Earlier, we saw $\pi_n(\mathbb{S}^2)$ was Abelian for $n \geq 2$.

Result: For all $X \in \mathrm{Top}_*$ and $n \geq 2$, $\pi_n(X)$ is Abelian.

Proof coming soon (rest of this talk)...

The higher homotopy groups are Abelian

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.
Observation: There are multiple group structures[5] on $\pi_n(X)$

[5] To self: interchange

The higher homotopy groups are Abelian

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$$f +_2 g := (t_1, t_2, \dots, t_n) \mapsto \begin{cases} f(t_1, 2t_2, \dots, t_n) & \text{if } t_2 \leq 1/2 \\ g(t_1, 2t_2 - 1, \dots, t_n) & \text{else} \end{cases}$$

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The higher homotopy groups are Abelian

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.
Our group structures fit together by

$$(a +_1 b) +_2 (x +_1 y) = (a +_2 x) +_1 (b +_2 y)$$

Result: Eckmann-Hilton

Corollary: $\pi_n(X)$ is Abelian.

The Eckmann-Hilton argument

Result: Eckmann-Hilton. *Suppose monoids* $(A, \bullet, 1)$ *and* $(A, \circ, 1)$ *defined on the same set* A *satisfy the interchange law*

$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ *and the monoid is Abelian.*

Proof: (1/3)

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$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ *and the monoid is Abelian.*

Proof: (2/3)

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- Categorical duality
  - Theorems for free!
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- Basic idea of cohomology theory
  - "Dualise homology"
  - Maps pointing in the correct direction for a graded ring
- Basic idea of higher homotopy theory
  - Higher homotopy groups
  - They're Abelian

Thanks for watching!

Good luck with final talks!

[1] nLab Authors. *Category Theory in Context*. Literally just to look up citation year of the book! URL: https://ncatlab.org/nlab/show/Category+Theory+in+Context.

[2] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com.au/books?id=BjKs86kosqgC.

[3] E. Riehl. *Category Theory in Context*. 2017. URL: https://emilyriehl.github.io/files/context.pdf.