

Where to from here?

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Outline

- 1 Categorical Duality
- 2 A brief introduction to cohomology
- 3 A brief introduction to higher homotopy theory
- 4 Outroduction

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Refresher: Categories

Definition: Categories

Examples:

Refresher: Functors

Definition: Functors

Examples:

Duality: The opposite category

Definition: Opposite category

Examples:

Working in \mathcal{C}^{op} rather than \mathcal{C} is called **dualising**.

Duality: The logical view

Principle: Categorical duality

In any statement of the form “for all categories \mathcal{C} , ...”, we can replace \mathcal{C} with \mathcal{C}^{op} , and re-interpret the statement in \mathcal{C} .

Example: Initial and terminal objects

Duality: Examples

Definition: Isomorphism. *For all categories \mathcal{C} and objects $x, y \in \mathcal{C}$, an **isomorphism** $x \simeq_{\mathcal{C}} y$ in \mathcal{C} is...*

Question: What is the dual concept to isomorphisms?

Duality: Examples

Result: Essential uniqueness of initial objects.

Dual result:

Contravariant functors

Definition: Functor. *For all categories \mathcal{C}, \mathcal{D} , a **functor** $\mathcal{C} \rightarrow \mathcal{D}$ is*

Dual definition: Contravariant functor¹.

Key example: Representable functors.

¹To self: cod

Contravariant functors

Result: Functors preserve isomorphisms.

Dual result:

Contravariant functors

Example application of result:

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Axioms of homology theory (as in [2])

Axioms for homology. A (reduced) homology theory \tilde{h} consists of:

²The LHS and RHS are functors from the category of CW pairs (A, X) ; the left is a composite of the “take the quotient” functor with \tilde{h}_n , and the right is a composite of the “forget X ” functor with \tilde{h}_{n-1} .

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- For each $n \in \mathbb{Z}$, a functor $\tilde{h}_n : \text{CWCmplx} \rightarrow \text{Ab}$,
- For each $n \in \mathbb{Z}$, a natural transformation $\partial_n : \tilde{h}_n(X/A) \rightarrow \tilde{h}_{n-1}(A)$ (natural in CW pairs (X, A))²,

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- $f \simeq_{\text{htpy}} g$ implies $\tilde{h}_n(f) = \tilde{h}_n(g)$,
- For each CW pair (X, A) , we have an exact sequence

$$\dots \xrightarrow{\partial_{n+1}} \tilde{h}_n(A) \xrightarrow{\tilde{h}_n(\text{in.})} \tilde{h}_n(X) \xrightarrow{\tilde{h}_n(\text{qt.})} \tilde{h}_n(X/A) \xrightarrow{\partial_n} \dots$$

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- For any $\{X_\alpha\}_{\alpha \in \mathcal{A}}$, the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha \in \mathcal{A}} X_\alpha$ give

$$\bigoplus_{\alpha \in \mathcal{A}} \tilde{h}_n(X_\alpha) \simeq_{\text{Ab}} \tilde{h}_n \left(\bigvee_{\alpha \in \mathcal{A}} X_\alpha \right)$$

²The LHS and RHS are functors from the category of CW pairs (A, X) ; the left is a composite of the “take the quotient” functor with \tilde{h}_n , and the right is a composite of the “forget X ” functor with \tilde{h}_{n-1} .

Going to cohomology

“Taking the homology” yields functors $H_n : \text{ChCmplx} \rightarrow \text{Ab}$.

³See [2, p. 202]. Admittedly, I haven’t checked. Life busy :/

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What if we replace Ab with Ab^{op} ?

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Going to cohomology

“Taking the homology” yields functors $H_n : \text{ChCmplx} \rightarrow \text{Ab}$.

What if we replace Ab with Ab^{op} ?

- “Cohomology functors” $H^n : \text{ChCmplx} \rightarrow \text{Ab}^{\text{op}}$.
- Axiom list with Ab replaced by Ab^{op} should be satisfied³.

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Cohomology of cochain complexes

Fix $G \in \mathbf{Ab}$.

Definition: Cochain group functor

Cohomology of cochain complexes

Fix $G \in \mathbf{Ab}$ and $n \in \mathbb{Z}$.

Definition: Cohomology group functors $H^n(-; G)$

Cohomology of cochain complexes

As a highly nontrivial offhand remark...

Theorem: universal coefficient [2, Theorem 3.2]

For all chain complexes C of free Abelian groups, there is a split exact sequence

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \rightarrow \text{Ab}(H_n(C), G) \rightarrow 0$$

...so the two ways we could have dualised:

- $C_n \rightsquigarrow \text{Ab}(C_n, G) \rightsquigarrow H^n(C; G)$
- $C_n \rightsquigarrow H_n(C) \rightsquigarrow \text{Ab}(H_n(C), G)$

are related.

Cohomology of spaces

Fix $G \in \mathbf{Ab}$ and $n \in \mathbb{Z}$.

Definition: Cohomology group functors $H^n(-; G) : \mathbf{Top} \rightarrow \mathbf{Ab}$

Cohomology of spaces

We recover [2, pp. 199–204]:

- Reduced groups
- Relative groups
- LES of a pair
- Homotopy invariance
- Excision
- Simplicial cohomology
- Cellular cohomology
- Mayer-Vietoris sequences

Cohomology of spaces

But also...

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We gain ring structure on $H^*(X; G)$

Cohomology of spaces

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We gain ring structure on $H^*(X; G)$

This is a “graded ring” $H^*(X; G) = \bigoplus_{n \geq 0} H^n(X; G)$ with product

$$H^i(X; G) \times H^j(X; G) \rightarrow H^{i+j}(X; G)$$

This product is called the **cup product** [2, p. 206].

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The idea

Fix $X \in \text{Top}_*$ ⁴.

The fundamental group $\pi_1(X)$ analyses maps of the form

$$\begin{aligned}\mathbb{I} &\longrightarrow X \\ \partial\mathbb{I} &\longrightarrow \{*\}\end{aligned}$$

up to homotopy.

⁴Basepoint always named $*$ and often omitted.

The idea

Fix $X \in \text{Top}_*$ ⁴.

The homotopy groups $\pi_n(X)$ analyse maps of the form

$$\begin{aligned}\mathbb{I}^n &\longrightarrow X \\ \partial\mathbb{I}^n &\longrightarrow \{*\}\end{aligned}$$

up to homotopy.

⁴Basepoint always named $*$ and often omitted.

The higher homotopy groups

Fix $X \in \text{Top}_*$ and $n \in \mathbb{Z}_{\geq 0}$.

Definition: The homotopy “group” $\pi_n(X)$

The higher homotopy groups

Fix $X \in \text{Top}_*$.

Example: $\pi_0(X)$

The higher homotopy groups

Fix $X \in \text{Top}_*$ and $n \in \mathbb{Z}$. Assume $n \geq 1$.

Definition: Group structure on $\pi_n(X)$

Why didn't we see these earlier?

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$\pi_n(X)$ is hard to calculate.

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Example: Homotopy groups $\pi_n(\mathbb{S}^2)$ for $n \in \{1, \dots, 12\}$:

0 \mathbb{Z} \mathbb{Z} \mathbb{Z}_2 \mathbb{Z}_2 \mathbb{Z}_{12}

\mathbb{Z}_2 \mathbb{Z}_2 \mathbb{Z}_3 \mathbb{Z}_{15} \mathbb{Z}_2 $\mathbb{Z}_2 \times \mathbb{Z}_2$

This mess is part of the motivation for homology theory.

Slight peeks of regularity

Earlier, we saw $\pi_n(\mathbb{S}^2)$ was Abelian for $n \geq 2$.

Result: For all $X \in \text{Top}_*$ and $n \geq 2$, $\pi_n(X)$ is Abelian.

Proof coming soon (rest of this talk)...

The higher homotopy groups are Abelian

Fix $X \in \text{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.

Observation: There are multiple group structures⁵ on $\pi_n(X)$

⁵To self: interchange

The higher homotopy groups are Abelian

Fix $X \in \text{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.

Our group structures fit together by

$$(a +_1 b) +_2 (x +_1 y) = (a +_2 x) +_1 (b +_2 y)$$

Result: Eckmann-Hilton

Corollary: $\pi_n(X)$ is Abelian.

The Eckmann-Hilton argument

Result: Eckmann-Hilton. Suppose monoids $(A, \bullet, 1)$ and $(A, \circ, 1)$ defined on the same set A satisfy the *interchange law*

$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ and the monoid is Abelian.

Proof: (1/3)

The Eckmann-Hilton argument

Result: Eckmann-Hilton. Suppose monoids $(A, \bullet, 1)$ and $(A, \circ, 1)$ defined on the same set A satisfy the *interchange law*

$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ and the monoid is Abelian.

Proof: (2/3)

The Eckmann-Hilton argument

Result: Eckmann-Hilton. Suppose monoids $(A, \bullet, 1)$ and $(A, \circ, 1)$ defined on the same set A satisfy the *interchange law*

$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ and the monoid is Abelian.

Proof: (3/3)

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 - Theorems for free!
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 - “Dualise homology”
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- Categorical duality
 - Theorems for free!
 - Definitions for free!
- Basic idea of cohomology theory
 - “Dualise homology”
 - Maps pointing in the correct direction for a graded ring
- Basic idea of higher homotopy theory
 - Higher homotopy groups
 - They’re Abelian

Thanks for watching!

Good luck with final talks!

- [1] nLab Authors. *Category Theory in Context*. Literally just to look up citation year of the book! URL: <https://ncatlab.org/nlab/show/Category+Theory+in+Context>.
- [2] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://books.google.com.au/books?id=BjKs86kosqgC>.
- [3] E. Riehl. *Category Theory in Context*. 2017. URL: <https://emilyriehl.github.io/files/context.pdf>.