Where to from here?

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Outline

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Refresher: Categories

Definition: Categories

Examples:

Refresher: Functors

Definition: Functors

Examples:

Duality: The opposite category

Definition: Opposite category

Examples:

Working in $\mathcal{C}^{\mathrm{op}}$ rather than $\mathcal{C}$ is called dualising.

Duality: The logical view

Principle: Categorical duality

In any statement of the form "for all categories $\mathcal{C}$, ...", we can replace $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, and re-interpret the statement in $\mathcal{C}$.

Example: Initial and terminal objects

Duality: Examples

Definition: Isomorphism. *For all categories $\mathcal{C}$ and objects $x, y \in \mathcal{C}$, an* **isomorphism** $x \simeq_{\mathcal{C}} y$ *in $\mathcal{C}$ is. . .*

Question: What is the dual concept to isomorphisms?

Duality: Examples

Result: Essential uniqueness of initial objects.

Dual result:

Contravariant functors

Definition: Functor. *For all categories $\mathcal{C}, \mathcal{D}$, a* **functor** $\mathcal{C} \to \mathcal{D}$ *is*

Dual definition: Contravariant functor[1].

Key example: Representable functors.

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[1] To self: cod

Contravariant functors

Result: Functors preserve isomorphisms.

Dual result:

Contravariant functors

Example application of result:

Outline

Axioms of homology theory (as in [2])

Axioms for homology. A (reduced) homology theory $\tilde{h}$ consists of:

[2] The LHS and RHS are functors from the category of CW pairs (A, X); the left is a composite of the "take the quotient" functor with $\tilde{h}_n$, and the right is a composite of the "forget X" functor with $\tilde{h}_{n-1}$.

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$$\cdots \xrightarrow{\partial_{n+1}} \tilde{h}_n(A) \xrightarrow{\tilde{h}_n(\text{in.})} \tilde{h}_n(X) \xrightarrow{\tilde{h}_n(\text{qt.})} \tilde{h}_n(X/A) \xrightarrow{\partial_n} \cdots$$

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- For any $\{X_\alpha\}_{\alpha \in \mathcal{A}}$, the inclusions $X_\alpha \hookrightarrow \bigvee_{\alpha \in \mathcal{A}} X_\alpha$ give

$$\bigoplus_{\alpha \in \mathcal{A}} \tilde{h}_n(X_\alpha) \simeq_{\mathrm{Ab}} \tilde{h}_n\left(\bigvee_{\alpha \in \mathcal{A}} X_\alpha\right)$$

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Going to cohomology

"Taking the homology" yields functors $H_n : \mathrm{ChCmplx} \to \mathrm{Ab}$.

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What if we replace Ab with $\mathrm{Ab}^{\mathrm{op}}$?

- "Cohomology functors" $H^n : \mathrm{ChCmplx} \to \mathrm{Ab}^{\mathrm{op}}$.
- Axiom list with Ab replaced by $\mathrm{Ab}^{\mathrm{op}}$ should be satisfied[3].

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Cohomology of cochain complexes

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$$\mathrm{ChCmplx} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{Ab}$$

- Cohomology group functors $H^n(-; G)$

$$\mathrm{Top} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{ChCmplx} \longrightarrow \mathrm{Ab}$$

Cohomology of spaces

We recover [2, pp. 199–204]:

- Reduced groups
- Relative groups
- LES of a pair
- Homotopy invariance
- Excision
- Simplicial cohomology
- Cellular cohomology
- Mayer-Vietoris sequences

Cohomology of spaces

But also...

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This is a "graded ring" $H^*(X;G) = \bigoplus_{n\geq 0} H^n(X;G)$ with product

$$H^i(X;G) \times H^j(X;G) \to H^{i+j}(X;G)$$

This product is called the cup product [2, p. 206].

Outline

The idea

Fix $X \in \mathrm{Top}_*$[4].
The fundamental group $\pi_1(X)$ analyses maps of the form

$$\mathbb{I} \longrightarrow X$$
$$\partial\mathbb{I} \longrightarrow \{*\}$$

up to homotopy.

[4] Basepoint always named $*$ and often omitted.

The idea

Fix $X \in \mathrm{Top}_*$[4].
The homotopy groups $\pi_n(X)$ analyse maps of the form

$$\begin{aligned} \mathbb{I}^n &\longrightarrow X \\ \partial\mathbb{I}^n &\longrightarrow \{*\} \end{aligned}$$

up to homotopy.

[4] Basepoint always named $*$ and often omitted.

The higher homotopy groups

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 0}$.

Definition: The homotopy "group" $\pi_n(X)$

The higher homotopy groups

Fix $X \in \mathrm{Top}_*$.
Example: $\pi_0(X)$

The higher homotopy groups

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}$. Assume $n \geq 1$.
Definition: Group structure on $\pi_n(X)$

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Example: Homotopy groups $\pi_n(\mathbb{S}^2)$ for $n \in \{1, \ldots, 12\}$:

0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

This mess is part of the motivation for homology theory.

Slight peeks of regularity

Earlier, we saw $\pi_n(\mathbb{S}^2)$ was Abelian for $n \geq 2$.

Result: For all $X \in \mathrm{Top}_*$ and $n \geq 2$, $\pi_n(X)$ is Abelian.

Proof coming soon (rest of this talk)...

The higher homotopy groups are Abelian

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.
Observation: There are multiple group structures[5] on $\pi_n(X)$

[5] To self: interchange

The higher homotopy groups are Abelian

Fix $X \in \mathrm{Top}_*$ and $n \in \mathbb{Z}_{\geq 2}$.
Our group structures fit together by

$$(a +_1 b) +_2 (x +_1 y) = (a +_2 x) +_1 (b +_2 y)$$

Result: Eckmann-Hilton

Corollary: $\pi_n(X)$ is Abelian.

The Eckmann-Hilton argument

Result: Eckmann-Hilton. *Suppose monoids* $(A, \bullet, 1)$ *and* $(A, \circ, 1)$ *defined on the same set* A *satisfy the interchange law*

$$(a \bullet b) \circ (x \bullet y) = (a \circ x) \bullet (b \circ y)$$

Then, $(\bullet, 1) = (\circ, 1)$ *and the monoid is Abelian.*

Proof: (1/3)

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Proof: (2/3)

The Eckmann-Hilton argument

Result: Eckmann-Hilton. *Suppose monoids* $(A, \bullet, 1)$ *and* $(A, \circ, 1)$ *defined on the same set* A *satisfy the interchange law*

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Then, $(\bullet, 1) = (\circ, 1)$ *and the monoid is Abelian.*
Proof: (3/3)

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- Categorical duality
  - Theorems for free!
  - Definitions for free!

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- Categorical duality
    - Theorems for free!
    - Definitions for free!

- Basic idea of cohomology theory
    - "Dualise homology"
    - Maps pointing in the correct direction for a graded ring

- Basic idea of higher homotopy theory
    - Higher homotopy groups
    - They're Abelian

Thanks for watching!

Good luck with final talks!

[1] nLab Authors. *Category Theory in Context*. Literally just to look up citation year of the book! URL: https://ncatlab.org/nlab/show/Category+Theory+in+Context.

[2] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: https://books.google.com.au/books?id=BjKs86kosqgC.

[3] E. Riehl. *Category Theory in Context*. 2017. URL: https://emilyriehl.github.io/files/context.pdf.