Corecursion and Coinduction

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15/Aug/2025

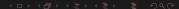
Goals

Actual goals:

- Introduce the conatural numbers
- ullet Introduce **corecursion** to specifying functions into $\mathrm{co}\mathbb{N}$
- Introduce **coinduction** to reason about corecursive functions
- **Generalise** to other corecursive structures
- Give useful examples of coinductive proofs

Notable omissions:

- Exhaustive examples (no time)
- Solid background theory (too much category theory)
- Initial algebras and terminal coalgebras in categories other than Set (prerequisites not met)



Outline

1 Vanilla Induction

2 Vanilla Corecursion

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(2) Vanilla Corecursion

The Natural Numbers

Coinduction \leftarrow Corecursion \leftarrow Recursion \leftarrow N.

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Loose Definition

The set of **natural numbers** is $\mathbb{N} := \{0, 1, \ldots\}$, "freely generated" by the constant 0 and the successor operation $\mathrm{succ}: x \mapsto x + 1$.

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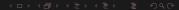
Recursion Principle

Given a set X, a constant $x_0 \in X$, and an operation $f: X \to X$, there is a unique map $u: \mathbb{N} \to X$ with

$$u: 0 \mapsto x_0$$

 $u: \operatorname{succ}(n) \mapsto f(u(n))$

We'll write $u := iterate(f, x_0)$.



Recursion from N

Example: Powers

The function $u:\mathbb{N}\to\mathbb{R},\ u:n\mapsto 3^n$ is defined recursively by

$$u: 0 \mapsto 1$$
 $u: \operatorname{succ}(n) \mapsto 3^n \cdot n$
 $3^0 := 1$ $3^{\operatorname{succ}(n)} := 3^n \cdot n$

Example: Addition on \mathbb{N}

The function $(-+-): \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$ then lhs = rhs.

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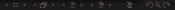
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$$\forall n \in \mathbb{N}, n+0=n$$



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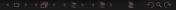
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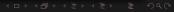
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Example: $\forall n \in \mathbb{N}, n+0=n$

Let $lhs: n \mapsto n + 0$ and $rhs: n \mapsto n$. Then,

- lhs(0) = 0 + 0 = 0 = rhs(0);
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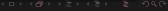
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so lhs = rhs.



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- $co\mathbb{N} \to Corecursion \to Coinduction$

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Definition: Conatural numbers, Predecessor

The set of **conatural numbers** is $co\mathbb{N} := \mathbb{N} \sqcup \{\infty\}$.

The **predecessor** operation is

$$\operatorname{pred}: \operatorname{co}\mathbb{N} \longrightarrow \{\operatorname{no}\} \sqcup \operatorname{co}\mathbb{N}$$
$$0 \longmapsto \operatorname{no}$$
$$\operatorname{succ}(n) \longmapsto n$$
$$\infty \longmapsto \infty$$

The Conatural Numbers

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- $co\mathbb{N} \to \mathsf{Corecursion} \to \mathsf{Coinduction}$

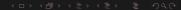
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$$0 \longmapsto \operatorname{no}$$
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$$\infty \longmapsto \infty$$

 \mathbb{N} comes with $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{\mathrm{(0,succ)}} \mathbb{N}$. $\mathrm{co}\mathbb{N}$ comes with $\mathrm{co}\mathbb{N} \xrightarrow{\mathrm{pred}} \{\mathrm{no}\} \sqcup \mathbb{N}$.



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A partial function $X \nrightarrow Y$ is a function $X \to \{no\} \sqcup Y$.

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Examples:

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Examples:

- $f: x \mapsto 1/x : \mathbb{R} \nrightarrow \mathbb{R}$; f(0) = no.
- "Get input from user": ComputerState → String;
 Fails when the user doesn't give input.

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Examples:

- $f: x \mapsto 1/x : \mathbb{R} \to \mathbb{R}$; f(0) = no.
- "Get input from user": ComputerState → String;
 Fails when the user doesn't give input.
- pred : $coN \rightarrow coN$; pred(0) fails.

