# Corecursion and Coinduction

Gabriel Field

15/Aug/2025

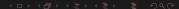
### Goals

### Actual goals:

- Introduce the conatural numbers
- ullet Introduce **corecursion** to specifying functions into  $\mathrm{co}\mathbb{N}$
- Introduce **coinduction** to reason about corecursive functions
- **Generalise** to other corecursive structures
- Give useful examples of coinductive proofs

#### Notable omissions:

- Exhaustive examples (no time)
- Solid background theory (too much category theory)
- Initial algebras and terminal coalgebras in categories other than Set (prerequisites not met)



# Outline

1 Vanilla Induction

**2** Vanilla Corecursion

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(2) Vanilla Corecursion

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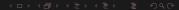
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#### Recursion Principle

Given a set X, a constant  $x_0 \in X$ , and an operation  $f: X \to X$ , there is a unique map  $u: \mathbb{N} \to X$  with

$$u: 0 \mapsto x_0$$
  
 $u: \operatorname{succ}(n) \mapsto f(u(n))$ 

We'll write  $u := iterate(f, x_0)$ .



# Recursion from N

### Example: Powers

The function  $u:\mathbb{N}\to\mathbb{R},\ u:n\mapsto 3^n$  is defined recursively by

$$u: 0 \mapsto 1$$
  $u: \operatorname{succ}(n) \mapsto 3^n \cdot n$   
 $3^0 := 1$   $3^{\operatorname{succ}(n)} := 3^n \cdot n$ 

### Example: Addition on $\mathbb{N}$

The function  $(-+-): \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  is defined recursively by

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

### Theorem: Simple induction

Let  $lhs, rhs : \mathbb{N} \to X$  be two functions out of  $\mathbb{N}$ . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$ then lhs = rhs.

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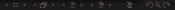
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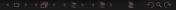
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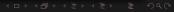
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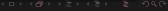
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The **predecessor** operation is

$$\operatorname{pred}: \operatorname{co}\mathbb{N} \longrightarrow \{\operatorname{no}\} \sqcup \operatorname{co}\mathbb{N}$$
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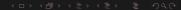
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 $\mathbb{N}$  comes with  $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{\mathrm{(0,succ)}} \mathbb{N}$ .  $\mathrm{co}\mathbb{N}$  comes with  $\mathrm{co}\mathbb{N} \xrightarrow{\mathrm{pred}} \{\mathrm{no}\} \sqcup \mathbb{N}$ .



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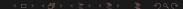
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- pred :  $coN \rightarrow coN$ ; pred(0) fails.



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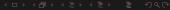
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#### Lemma: Pred-wait

Let  $f: X \nrightarrow X$  be a partial function and  $x \in X$ .

- If f(x) = no, then pred(wait(f)(x)) = no.
- If  $f(x) = x' \neq \text{no}$ , then pred(wait(f)(x)) = wait(f)(x').

Abstract nonsense: pred is the terminal partial endo-function.



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- The lemma says

$$\operatorname{pred}(\operatorname{wait}(\operatorname{tail})([])) = \operatorname{no}$$
$$\operatorname{pred}(\operatorname{wait}(\operatorname{tail})([x_0, x_1, \dots])) = \operatorname{wait}(\operatorname{tail})([x_1, \dots])$$

