Corecursion and Coinduction

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15/Aug/2025

Goals

Actual goals:

- Introduce the **conatural numbers**
- Introduce **corecursion** to specifying functions into $\mathrm{co}\mathbb{N}$
- Introduce **coinduction** to reason about corecursive functions
- **Generalise** to other corecursive structures
- Give **useful examples** of coinductive proofs

Notable omissions:

- Exhaustive examples (no time)
- Solid background theory (too much category theory)
- Initial algebras and terminal coalgebras in categories other than $\mathbf{Set}$ (prerequisites not met)

Outline

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Coinduction $\leftarrow$ Corecursion $\leftarrow$ Recursion $\leftarrow \mathbb{N}$.

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Loose Definition

The set of **natural numbers** is $\mathbb{N} := \{0, 1, \dots\}$, "freely generated" by the constant 0 and the successor operation $\mathrm{succ} : x \mapsto x + 1$.

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Recursion Principle

Given a set X, a constant $x_0 \in X$, and an operation $f : X \to X$, there is a unique map $u : \mathbb{N} \to X$ with

$$u : 0 \mapsto x_0$$
$$u : \mathrm{succ}(n) \mapsto f(u(n))$$

We'll write $u := \mathrm{iterate}(f, x_0)$.

Recursion from $\mathbb{N}$

Example: Powers

The function $u : \mathbb{N} \to \mathbb{R}$, $u : n \mapsto 3^n$ is defined recursively by

$$u : 0 \mapsto 1 \qquad u : \mathrm{succ}(n) \mapsto 3^n \cdot n$$

$$3^0 := 1 \qquad 3^{\mathrm{succ}(n)} := 3^n \cdot n$$

Example: Addition on $\mathbb{N}$

The function $(- + -) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$0 + y := y \qquad \mathrm{succ}(x) + y := \mathrm{succ}(x + y)$$

Induction from $\mathbb{N}$

Theorem: Simple induction

Let $\mathrm{lhs}, \mathrm{rhs} : \mathbb{N} \to X$ be two functions out of $\mathbb{N}$. If

- $\mathrm{lhs}(0) = \mathrm{rhs}(0)$, and
- $(\forall n \in \mathbb{N}, \mathrm{lhs}(n) = \mathrm{rhs}(n) \implies \mathrm{lhs}(\mathrm{succ}(n)) = \mathrm{rhs}(\mathrm{succ}(n)))$

then $\mathrm{lhs} = \mathrm{rhs}$.

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Definition: Conatural numbers, Predecessor

The set of **conatural numbers** is $\mathrm{co}\mathbb{N} := \mathbb{N} \sqcup \{\infty\}$.
The **predecessor** operation is

$$\begin{aligned} \mathrm{pred} : \mathrm{co}\mathbb{N} &\longrightarrow \{\mathrm{no}\} \sqcup \mathrm{co}\mathbb{N} \\ 0 &\longmapsto \mathrm{no} \\ \mathrm{succ}(n) &\longmapsto n \\ \infty &\longmapsto \infty \end{aligned}$$

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$\mathbb{N}$ comes with $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{(0,\mathrm{succ})} \mathbb{N}$.

$\mathrm{co}\mathbb{N}$ comes with $\mathrm{co}\mathbb{N} \xrightarrow{\mathrm{pred}} \{\mathrm{no}\} \sqcup \mathbb{N}$.

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  Fails when the user doesn't give input.

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- $\mathrm{pred} : \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N}$;
  $\mathrm{pred}(0)$ fails.