Corecursion and Coinduction

Gabriel Field

15/Aug/2025

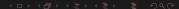
Goals

Actual goals:

- Introduce the conatural numbers
- ullet Introduce **corecursion** to specifying functions into $\mathrm{co}\mathbb{N}$
- Introduce **coinduction** to reason about corecursive functions
- **Generalise** to other corecursive structures
- Give useful examples of coinductive proofs

Notable omissions:

- Exhaustive examples (no time)
- Solid background theory (too much category theory)
- Initial algebras and terminal coalgebras in categories other than Set (prerequisites not met)



Outline

- Vanilla Induction
- 2 Vanilla Corecursion
- 3 Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- 3 Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

The Natural Numbers

Coinduction \leftarrow Corecursion \leftarrow Recursion \leftarrow N.

The Natural Numbers

Coinduction \leftarrow Corecursion \leftarrow Recursion \leftarrow N.

Loose Definition

The set of **natural numbers** is $\mathbb{N} := \{0, 1, \ldots\}$, "freely generated" by the constant 0 and the successor operation $\mathrm{succ}: x \mapsto x + 1$.

The Natural Numbers

Coinduction \leftarrow Corecursion \leftarrow Recursion \leftarrow N.

Loose Definition

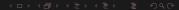
The set of **natural numbers** is $\mathbb{N} := \{0, 1, \ldots\}$, "freely generated" by the constant 0 and the successor operation $\mathrm{succ}: x \mapsto x + 1$.

Recursion Principle

Given a set X, a constant $x_0 \in X$, and an operation $f: X \to X$, there is a unique map $u: \mathbb{N} \to X$ with

$$u: 0 \mapsto x_0$$

 $u: \operatorname{succ}(n) \mapsto f(u(n))$



Recursion from N

Example: Powers

The function $u:\mathbb{N}\to\mathbb{R},\ u:n\mapsto 3^n$ is defined recursively by

$$u: 0 \mapsto 1$$
 $u: \operatorname{succ}(n) \mapsto 3^n \cdot n$
 $3^0 := 1$ $3^{\operatorname{succ}(n)} := 3^n \cdot n$

Example: Addition on \mathbb{N}

The function $(-+-): \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$ then lhs = rhs.

Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

then lhs = rhs.

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Theorem: Simple induction

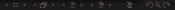
Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

then lhs = rhs.

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Example:
$$\forall n \in \mathbb{N}, n+0=n$$



Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

then lhs = rhs.

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Example: $\forall n \in \mathbb{N}, n+0=n$

Let $lhs: n \mapsto n + 0$ and $rhs: n \mapsto n$. Then,

Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

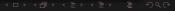
then lhs = rhs.

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Example: $\forall n \in \mathbb{N}, n+0=n$

Let $lhs : n \mapsto n + 0$ and $rhs : n \mapsto n$. Then,

• lhs(0) = 0 + 0 = 0 = rhs(0);



Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

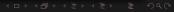
then lhs = rhs.

$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Example: $\forall n \in \mathbb{N}, n+0=n$

Let $lhs : n \mapsto n + 0$ and $rhs : n \mapsto n$. Then,

- lhs(0) = 0 + 0 = 0 = rhs(0);
- lhs(succ(n)) = succ(n) + 0 = succ(n + 0) = succ(n) = rhs(succ(n));



Theorem: Simple induction

Let $lhs, rhs : \mathbb{N} \to X$ be two functions out of \mathbb{N} . If

- lhs(0) = rhs(0), and
- $(\forall n \in \mathbb{N}, \, \text{lhs}(n) = \text{rhs}(n) \implies \text{lhs}(\text{succ}(n)) = \text{rhs}(\text{succ}(n)))$

then lhs = rhs.

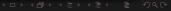
$$0 + y := y \qquad \operatorname{succ}(x) + y := \operatorname{succ}(x + y)$$

Example: $\forall n \in \mathbb{N}, n+0=n$

Let $lhs: n \mapsto n + 0$ and $rhs: n \mapsto n$. Then,

- lhs(0) = 0 + 0 = 0 = rhs(0);
- lhs(succ(n)) = succ(n) + 0 = succ(n + 0) = succ(n) = rhs(succ(n));

so lhs = rhs.



Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- 3 Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

The Conatural Numbers

- \mathbb{N} \rightarrow Recursion \rightarrow Induction
- $co\mathbb{N} \to Corecursion \to Coinduction$

The Conatural Numbers

- $\mathbb{N} \to \mathsf{Recursion} \to \mathsf{Induction}$
- $co\mathbb{N} \to \mathsf{Corecursion} \to \mathsf{Coinduction}$

Definition: Conatural numbers, Predecessor

The set of **conatural numbers** is $co\mathbb{N} := \mathbb{N} \sqcup \{\infty\}$.

The **predecessor** operation is the bijection

$$\begin{array}{c} \operatorname{pred}: \operatorname{co}\mathbb{N} \longrightarrow \{\operatorname{no}\} \sqcup \operatorname{co}\mathbb{N} \\ 0 \longmapsto \operatorname{no} \\ \operatorname{succ}(n) \longmapsto n \\ \infty \longmapsto \infty \end{array}$$

The Conatural Numbers

- $\mathbb{N} \to \mathsf{Recursion} \to \mathsf{Induction}$
- $co\mathbb{N} \to Corecursion \to Coinduction$

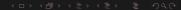
Definition: Conatural numbers, Predecessor

The set of **conatural numbers** is $co\mathbb{N} := \mathbb{N} \sqcup \{\infty\}$.

The **predecessor** operation is the bijection

$$\operatorname{pred}: \operatorname{co}\mathbb{N} \longrightarrow \{\operatorname{no}\} \sqcup \operatorname{co}\mathbb{N}$$
$$0 \longmapsto \operatorname{no}$$
$$\operatorname{succ}(n) \longmapsto n$$
$$\infty \longmapsto \infty$$

 \mathbb{N} comes with $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{\mathrm{pred}} \mathbb{N}$. \subset $\mathrm{co}\mathbb{N}$ comes with $\mathrm{co}\mathbb{N} \xrightarrow{\mathrm{pred}} \{\mathrm{no}\} \sqcup \mathbb{N}$.



Definition: Partial function

A partial function $X \nrightarrow Y$ is a function $X \to \{no\} \sqcup Y$.

Examples:

Definition: Partial function

A partial function $X \nrightarrow Y$ is a function $X \to \{no\} \sqcup Y$.

Examples:

• $f: x \mapsto 1/x : \mathbb{R} \to \mathbb{R};$ f(0) = no.

Definition: Partial function

A partial function $X \rightarrow Y$ is a function $X \rightarrow \{no\} \sqcup Y$.

Examples:

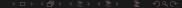
- $f: x \mapsto 1/x : \mathbb{R} \to \mathbb{R}$; f(0) = no.
- "Get input from user": ComputerState → String;
 Fails when the user doesn't give input.

Definition: Partial function

A partial function $X \rightarrow Y$ is a function $X \rightarrow \{no\} \sqcup Y$.

Examples:

- $f: x \mapsto 1/x : \mathbb{R} \to \mathbb{R}$; f(0) = no.
- "Get input from user": ComputerState → String;
 Fails when the user doesn't give input.
- pred : $coN \rightarrow coN$; pred(0) fails.



Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to \mathrm{co}\mathbb{N}$ by

Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to \mathrm{co}\mathbb{N}$ by

• $u(x) = 0 \iff f(x) = \text{no}$

Corecursion principle

Let $f: X \to X$ be a partial function. Construct $u: X \to co\mathbb{N}$ by

- $u(x) = 0 \iff f(x) = \text{no}$
- $u(x) = 1 \iff f(x) \neq \text{no and } f(f(x)) = \text{no}$

Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to co\mathbb{N}$ by

- $u(x) = 0 \iff f(x) = \text{no}$
- $u(x) = 1 \iff f(x) \neq \text{no and } f(f(x)) = \text{no}$
- $u(x) = 2 \iff f(x) \neq \text{no and } f(f(x)) \neq \text{no and } f^3(x) = \text{no}$
- etc.

Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to co\mathbb{N}$ by

- $u(x) = 0 \iff f(x) = \text{no}$
- $u(x) = 1 \iff f(x) \neq \text{no and } f(f(x)) = \text{no}$
- $u(x) = 2 \iff f(x) \neq \text{no and } f(f(x)) \neq \text{no and } f^3(x) = \text{no}$
- etc.
- $u(x) = \infty \iff f(x) \neq \text{no and } f^2(x) \neq \text{no and } \cdots$

Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to co\mathbb{N}$ by

- $u(x) = 0 \iff f(x) = \text{no}$
- $u(x) = 1 \iff f(x) \neq \text{no and } f(f(x)) = \text{no}$
- $u(x) = 2 \iff f(x) \neq \text{no and } f(f(x)) \neq \text{no and } f^3(x) = \text{no}$
- etc.
- $u(x) = \infty \iff f(x) \neq \text{no and } f^2(x) \neq \text{no and } \cdots$

Write u := wait(f).

Corecursion principle

Let $f: X \nrightarrow X$ be a partial function. Construct $u: X \to co\mathbb{N}$ by

- $u(x) = 0 \iff f(x) = \text{no}$
- $u(x) = 1 \iff f(x) \neq \text{no and } f(f(x)) = \text{no}$
- $u(x) = 2 \iff f(x) \neq \text{no and } f(f(x)) \neq \text{no and } f^3(x) = \text{no}$
- etc.
- $u(x) = \infty \iff f(x) \neq \text{no and } f^2(x) \neq \text{no and } \cdots$

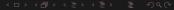
Write u := wait(f).

Lemma: Pred-wait

Let $f: X \nrightarrow X$ be a partial function and $x \in X$.

- If f(x) = no, then pred(wait(f)(x)) = no.
- If $f(x) = x' \neq \text{no}$, then pred(wait(f)(x)) = wait(f)(x').

Abstract nonsense: pred is the terminal partial endo-function.



Example:

• Let $c: \mathbb{Z}_{>0} \nrightarrow \mathbb{Z}_{>0}$ by

$$c(1) := no;$$
 $c(n) := n/2$ if n is even; $c(n) := 3n + 1$ otherwise

Example:

• Let $c: \mathbb{Z}_{>0} \nrightarrow \mathbb{Z}_{>0}$ by

$$c(1) := \mathrm{no}; \ \ c(n) := n/2 \ \mathrm{if} \ n \ \mathrm{is} \ \mathrm{even}; \ \ c(n) := 3n+1 \ \mathrm{otherwise}$$

• wait(c)(x) is the length of $x \stackrel{c}{\mapsto} \cdots \stackrel{c}{\mapsto} 1$, possibly ∞ .

Example:

• Let $c: \mathbb{Z}_{>0} \nrightarrow \mathbb{Z}_{>0}$ by

$$c(1):=\mathrm{no}; \quad c(n):=n/2 \text{ if } n \text{ is even}; \quad c(n):=3n+1 \text{ otherwise}$$

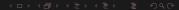
- wait(c)(x) is the length of $x \stackrel{c}{\mapsto} \cdots \stackrel{c}{\mapsto} 1$, possibly ∞ .
- E.g. $5 \stackrel{c}{\mapsto} 16 \stackrel{c}{\mapsto} 8 \stackrel{c}{\mapsto} 4 \stackrel{c}{\mapsto} 2 \stackrel{c}{\mapsto} 1 \stackrel{c}{\mapsto} \text{no, so wait}(c)(5) = 5.$

Example:

• Let $c: \mathbb{Z}_{>0} \nrightarrow \mathbb{Z}_{>0}$ by

$$c(1):=\mathrm{no}; \quad c(n):=n/2 \text{ if } n \text{ is even}; \quad c(n):=3n+1 \text{ otherwise}$$

- wait(c)(x) is the length of $x \stackrel{c}{\mapsto} \cdots \stackrel{c}{\mapsto} 1$, possibly ∞ .
- E.g. $5 \stackrel{c}{\mapsto} 16 \stackrel{c}{\mapsto} 8 \stackrel{c}{\mapsto} 4 \stackrel{c}{\mapsto} 2 \stackrel{c}{\mapsto} 1 \stackrel{c}{\mapsto} \text{no, so wait}(c)(5) = 5.$
- The Collatz conjecture asks whether wait $(c)(n) \neq \infty$ for all $n \in \mathbb{Z}_{>0}$.



Definition: Addition in coN

The function $(-+-): co\mathbb{N} \times co\mathbb{N} \to co\mathbb{N}$ is defined by

$$(-+-) := wait(f)$$

where $f: \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N}$ is given by

$$f:(x,y)\mapsto \begin{cases} \text{no} & \text{if } \operatorname{pred}(x)=\operatorname{pred}(y)=\text{no}\\ (x,y') & \text{if } \operatorname{pred}(x)=\text{no and } \operatorname{pred}(y)=y'\neq \text{no}\\ (x',y) & \text{if } \operatorname{pred}(x)=x'\neq \text{no} \end{cases}$$

Definition: Addition in coN

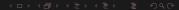
The function $(-+-): co\mathbb{N} \times co\mathbb{N} \to co\mathbb{N}$ is defined by

$$(-+-) := wait(f)$$

where $f: \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N}$ is given by

$$f:(x,y)\mapsto \begin{cases} \text{no} & \text{if } \operatorname{pred}(x)=\operatorname{pred}(y)=\text{no}\\ (x,y') & \text{if } \operatorname{pred}(x)=\text{no and } \operatorname{pred}(y)=y'\neq \text{no}\\ (x',y) & \text{if } \operatorname{pred}(x)=x'\neq \text{no} \end{cases}$$

By the pred-wait lemma,



Corecursion into coN

Definition: Addition in coN

The function $(-+-): co\mathbb{N} \times co\mathbb{N} \to co\mathbb{N}$ is defined by

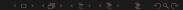
$$(-+-) := wait(f)$$

where $f: co\mathbb{N} \times co\mathbb{N} \nrightarrow co\mathbb{N} \times co\mathbb{N}$ is given by

$$f:(x,y)\mapsto \begin{cases} \text{no} & \text{if } \operatorname{pred}(x)=\operatorname{pred}(y)=\text{no}\\ (x,y') & \text{if } \operatorname{pred}(x)=\text{no and } \operatorname{pred}(y)=y'\neq \text{no}\\ (x',y) & \text{if } \operatorname{pred}(x)=x'\neq \text{no} \end{cases}$$

By the pred-wait lemma,

•
$$\operatorname{pred}(0+0) = \operatorname{no}$$
, so $0+0=0$;



Corecursion into coN

Definition: Addition in coN

The function $(-+-): co\mathbb{N} \times co\mathbb{N} \to co\mathbb{N}$ is defined by

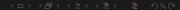
$$(-+-) := wait(f)$$

where $f: \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N}$ is given by

$$f:(x,y)\mapsto \begin{cases} \text{no} & \text{if } \operatorname{pred}(x)=\operatorname{pred}(y)=\text{no}\\ (x,y') & \text{if } \operatorname{pred}(x)=\text{no and } \operatorname{pred}(y)=y'\neq \text{no}\\ (x',y) & \text{if } \operatorname{pred}(x)=x'\neq \text{no} \end{cases}$$

By the pred-wait lemma,

- $\operatorname{pred}(0+0) = \overline{\text{no, so } 0+0=0}$;
- pred(0 + y) = pred(y), so 0 + y = y;



Corecursion into coN

Definition: Addition in coN

The function $(-+-): co\mathbb{N} \times co\mathbb{N} \to co\mathbb{N}$ is defined by

$$(-+-) := wait(f)$$

where $f: \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N} \times \mathrm{co}\mathbb{N}$ is given by

$$f:(x,y)\mapsto \begin{cases} \text{no} & \text{if } \operatorname{pred}(x)=\operatorname{pred}(y)=\text{no}\\ (x,y') & \text{if } \operatorname{pred}(x)=\text{no and } \operatorname{pred}(y)=y'\neq \text{no}\\ (x',y) & \text{if } \operatorname{pred}(x)=x'\neq \text{no} \end{cases}$$

By the pred-wait lemma,

- pred(0+0) = no, so 0+0=0;
- pred(0 + y) = pred(y), so 0 + y = y;
- $\operatorname{pred}(x+y) = \operatorname{pred}(x) + y$ whenever $\operatorname{pred}(x) \neq \operatorname{no}$.



Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- 3 Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

Given $f,g:X\to Y$, let $Y_{/f=g}$ be the quotient of Y by the smallest equivalence relation \sim with $f(x)\sim g(x)$ for each $x\in X$.

Given $f,g:X\to Y$, let $Y_{/f=g}$ be the quotient of Y by the smallest equivalence relation \sim with $f(x)\sim g(x)$ for each $x\in X$.

So,
$$[f(x)] = [g(x)]$$
 in $Y_{/f=g}$.

Given $f,g:X\to Y$, let $Y_{/f=g}$ be the quotient of Y by the smallest equivalence relation \sim with $f(x)\sim g(x)$ for each $x\in X$.

So,
$$[f(x)] = [g(x)]$$
 in $Y_{/f=g}$.

Theorem: Simple coinduction

Let $lhs, rhs : X \to co\mathbb{N}$ be two functions into $co\mathbb{N}$.

Let $[-]:\mathrm{co}\mathbb{N}\to\mathrm{co}\mathbb{N}_{/\mathrm{lhs=rhs}}$ be the quotient map.

lf

$$\forall x \in X, \quad [\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$$

(with
$$[no] := no$$
)

Given $f,g:X\to Y$, let $Y_{/f=g}$ be the quotient of Y by the smallest equivalence relation \sim with $f(x)\sim g(x)$ for each $x\in X$.

So,
$$[f(x)] = [g(x)]$$
 in $Y_{/f=g}$.

Theorem: Simple coinduction

Let $lhs, rhs : X \to co\mathbb{N}$ be two functions into $co\mathbb{N}$.

Let $[-]: \mathrm{co}\mathbb{N} \to \mathrm{co}\mathbb{N}_{/\mathrm{lhs=rhs}}$ be the quotient map.

lf

$$\forall x \in X, \quad [\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$$

(with [no] := no), then lhs = rhs.

Coinduction into coN: proof sketch

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof sketch.

It follows that $[\operatorname{pred}^n(\operatorname{lhs}(x))] = [\operatorname{pred}^n(\operatorname{rhs}(x))]$ for all n (with $\operatorname{pred}(\operatorname{no}) := \operatorname{no}$).

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof sketch.

It follows that $[\operatorname{pred}^n(\operatorname{lhs}(x))] = [\operatorname{pred}^n(\operatorname{rhs}(x))]$ for all n (with $\operatorname{pred}(\operatorname{no}) := \operatorname{no}$).

• If $lhs(x) = n \in \mathbb{N}$, then

$$[\operatorname{pred}^{n+1}(\operatorname{rhs}(x))] = [\operatorname{pred}^{n+1}(\operatorname{lhs}(x))] = [\operatorname{no}] = \operatorname{no}$$

so $\operatorname{pred}^{n+1}(\operatorname{rhs}(x)) = \operatorname{no}$, and hence $\operatorname{rhs}(x) = n = \operatorname{lhs}(x)$.

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof sketch.

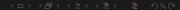
It follows that $[\operatorname{pred}^n(\operatorname{lhs}(x))] = [\operatorname{pred}^n(\operatorname{rhs}(x))]$ for all n (with $\operatorname{pred}(\operatorname{no}) := \operatorname{no}$).

• If $lhs(x) = n \in \mathbb{N}$, then

$$[\operatorname{pred}^{n+1}(\operatorname{rhs}(x))] = [\operatorname{pred}^{n+1}(\operatorname{lhs}(x))] = [\operatorname{no}] = \operatorname{no}$$

so $\operatorname{pred}^{n+1}(\operatorname{rhs}(x)) = \operatorname{no}$, and hence $\operatorname{rhs}(x) = n = \operatorname{lhs}(x)$.

• If $rhs(x) \in \mathbb{N}$, do the same thing.



Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof sketch.

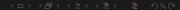
It follows that $[\operatorname{pred}^n(\operatorname{lhs}(x))] = [\operatorname{pred}^n(\operatorname{rhs}(x))]$ for all n (with $\operatorname{pred}(\operatorname{no}) := \operatorname{no}$).

• If $lhs(x) = n \in \mathbb{N}$, then

$$[\operatorname{pred}^{n+1}(\operatorname{rhs}(x))] = [\operatorname{pred}^{n+1}(\operatorname{lhs}(x))] = [\operatorname{no}] = \operatorname{no}$$

so $\operatorname{pred}^{n+1}(\operatorname{rhs}(x)) = \operatorname{no}$, and hence $\operatorname{rhs}(x) = n = \operatorname{lhs}(x)$.

- If $rhs(x) \in \mathbb{N}$, do the same thing.
- The only remaining case is $lhs(x) = \infty = rhs(x)$.



Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof sketch.

It follows that $[\operatorname{pred}^n(\operatorname{lhs}(x))] = [\operatorname{pred}^n(\operatorname{rhs}(x))]$ for all n (with $\operatorname{pred}(\operatorname{no}) := \operatorname{no}$).

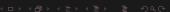
• If $lhs(x) = n \in \mathbb{N}$, then

$$[\operatorname{pred}^{n+1}(\operatorname{rhs}(x))] = [\operatorname{pred}^{n+1}(\operatorname{lhs}(x))] = [\operatorname{no}] = \operatorname{no}$$

so $\operatorname{pred}^{n+1}(\operatorname{rhs}(x)) = \operatorname{no}$, and hence $\operatorname{rhs}(x) = n = \operatorname{lhs}(x)$.

- If $rhs(x) \in \mathbb{N}$, do the same thing.
- The only remaining case is $lhs(x) = \infty = rhs(x)$.

Therefore, lhs = rhs.



Coinduction into coN: $\forall x \in coN$, x + 0 = x

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- (*) $\operatorname{pred}(x+0) = \operatorname{no} \text{ whenever } \operatorname{pred}(x) = \operatorname{no}$
- (**) $\operatorname{pred}(x+y) = \operatorname{pred}(x) + y$ whenever $\operatorname{pred}(x) \neq \operatorname{no}$

Proof that $\forall x \in \text{co}\mathbb{N}, \ x + 0 = x$.

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

(*)
$$\operatorname{pred}(x+0) = \operatorname{no} \text{ whenever } \operatorname{pred}(x) = \operatorname{no}$$

Proof that $\forall x \in \text{coN}, x + 0 = x$.

Set $lhs: x \mapsto x + 0$ and $rhs: x \mapsto x$.

Coinduction into coN: $\forall x \in coN$, x + 0 = x

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

(*)
$$\operatorname{pred}(x+0) = \operatorname{no} \text{ whenever } \operatorname{pred}(x) = \operatorname{no}$$

Proof that $\forall x \in \text{co}\mathbb{N}, x + 0 = x$.

Set $lhs: x \mapsto x + 0$ and $rhs: x \mapsto x$.

• If pred(x) = no, then

$$[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(x+0)]$$

$$= [\operatorname{no}]$$

$$= [\operatorname{pred}(x)]$$

$$[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$$

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

(**)
$$\operatorname{pred}(x+y) = \operatorname{pred}(x) + y$$
 whenever $\operatorname{pred}(x) \neq \operatorname{no}(x)$

Proof that $\forall x \in \text{co}\mathbb{N}, x + 0 = x$.

Set $lhs: x \mapsto x + 0$ and $rhs: x \mapsto x$.

• If instead $\operatorname{pred}(x) \neq \operatorname{no}$, then

$$[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(x+0)]$$

$$= [\operatorname{pred}(x)+0]$$

$$= [\operatorname{lhs}(\operatorname{pred}(x))]$$

$$= [\operatorname{rhs}(\operatorname{pred}(x))]$$

$$= [\operatorname{pred}(x)]$$

$$[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$$

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof that $\forall x \in \text{co}\mathbb{N}, \ x+0=x$. Set $\text{lhs}: x \mapsto x+0$ and $\text{rhs}: x \mapsto x$.

• The theorem asserts lhs = rhs.



Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- $\operatorname{pred}(a+b) = \operatorname{pred}(b)$ whenever $\operatorname{pred}(a) = \operatorname{no}$
- $\operatorname{pred}(a+b) = \operatorname{pred}(a)$ whenever $\operatorname{pred}(b) = \operatorname{no}$
- $\operatorname{pred}(a+b) = \operatorname{pred}(a) + b$ whenever $\operatorname{pred}(a) \neq \operatorname{no}$
- $\operatorname{pred}(a+b) = a + \operatorname{pred}(b)$ whenever $\operatorname{pred}(b) \neq \operatorname{no}$ (possible)

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

- Set $X := \operatorname{co}\mathbb{N}^2$.
- Set lhs(x, y) := x + y.
- Set rhs(x, y) := y + x.

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- $\operatorname{pred}(a+b) = \operatorname{pred}(b)$ whenever $\operatorname{pred}(a) = \operatorname{no}$
- $\operatorname{pred}(a+b) = \operatorname{pred}(a)$ whenever $\operatorname{pred}(b) = \operatorname{no}$

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

• If pred(x) = no, then

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- $\operatorname{pred}(a+b) = \operatorname{pred}(b)$ whenever $\operatorname{pred}(a) = \operatorname{no}$
- $\operatorname{pred}(a+b) = \operatorname{pred}(a)$ whenever $\operatorname{pred}(b) = \operatorname{no}$

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

• If pred(x) = no, then

$$[\operatorname{pred}(\operatorname{lhs}(x,y))] = [\operatorname{pred}(x+y)]$$
$$= [\operatorname{pred}(y)]$$
$$= [\operatorname{pred}(y+x)]$$
$$[\operatorname{pred}(\operatorname{lhs}(x,y))] = [\operatorname{pred}(\operatorname{rhs}(x,y))]$$

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- $\operatorname{pred}(a+b) = \operatorname{pred}(a) + b$ whenever $\operatorname{pred}(a) \neq \operatorname{no}$
- $\operatorname{pred}(a+b) = a + \operatorname{pred}(b)$ whenever $\operatorname{pred}(b) \neq \operatorname{no}$

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

• If $pred(x) \neq no$, then

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

- $\operatorname{pred}(a+b) = \operatorname{pred}(a) + b$ whenever $\operatorname{pred}(a) \neq \operatorname{no}$
- $\operatorname{pred}(a+b) = a + \operatorname{pred}(b)$ whenever $\operatorname{pred}(b) \neq \operatorname{no}$

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

• If $pred(x) \neq no$, then

$$\begin{aligned} [\operatorname{pred}(\operatorname{lhs}(x,y))] &= [\operatorname{pred}(x+y)] \\ &= [\operatorname{pred}(x)+y] \\ &= [\operatorname{lhs}(\operatorname{pred}(x),y)] \\ &= [\operatorname{rhs}(\operatorname{pred}(x),y)] \\ &= [y+\operatorname{pred}(x)] \\ &= [\operatorname{pred}(y+x)] = [\operatorname{pred}(\operatorname{rhs}(x,y))] \end{aligned}$$

Theorem: Simple coinduction

If $\forall x \in X$, $[\operatorname{pred}(\operatorname{lhs}(x))] = [\operatorname{pred}(\operatorname{rhs}(x))]$ in $\operatorname{coN}_{/\operatorname{lhs}=\operatorname{rhs}}$, then $\operatorname{lhs} = \operatorname{rhs}$.

Proof that $\forall x, y \in \text{co}\mathbb{N}, x + y = y + x$.

• The theorem asserts lhs = rhs.



Other properties of $co\mathbb{N}$

These can be proven coinductively:

- $\operatorname{pred}(x+y) = x + \operatorname{pred}(y)$ whenever $\operatorname{pred}(y) \neq \operatorname{no}$.
- + is associative.
- If x, y are (finite or infinite) sequences, then length(x + + y) = length(x) + length(y).

Other properties of $co\mathbb{N}$

Coinductive proofs only involve ≤ 2 cases:

- pred(something) = no
- $\operatorname{pred}(\operatorname{something}) \neq \operatorname{no}$

and sometimes no splits at all!

Other properties of $co\mathbb{N}$

Coinductive proofs only involve ≤ 2 cases:

- pred(something) = no
- $\operatorname{pred}(\operatorname{something}) \neq \operatorname{no}$

and sometimes no splits at all!

Contrast against inductive-plus-infinity proofs with 3 cases:

- (something) = 0
- (something) = succ(something else)
- (something) = ∞

Why induct when you could coinduct?

Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- **3** Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

Lists

Let A be a set.

Definition: Lists on A

A **list** on A is a finite-length sequence of elements in A.

Lists

Let A be a set.

Definition: Lists on A

A **list** on A is a finite-length sequence of elements in A.

$$List(A) := \{[], [a_0], [a_0, a_1], \dots \mid a_0, a_1, \dots \in A\}$$

Lists

Let A be a set.

Definition: Lists on A

A **list** on A is a finite-length sequence of elements in A.

$$List(A) := \{[], [a_0], [a_0, a_1], \dots \mid a_0, a_1, \dots \in A\}$$

 $\operatorname{List}(A)$ comes with a constant $[] \in \operatorname{List}(A)$, and an operation

cons :
$$A \times \text{List}(A) \longrightarrow \text{List}(A)$$

 $(a_0, [a_1, \ldots]) \longmapsto [a_0, a_1, \ldots]$

Colists

Let \overline{A} be a set.

Definition: Colists on A

A **colist** on A is a finite-or-infinite sequence of elements in A.

Colists

Let A be a set.

Definition: Colists on A

A **colist** on A is a finite-or-infinite sequence of elements in A.

$$coList(A) := List(A) \sqcup \{[a_0, a_1, \ldots] \mid a_0, a_1, \ldots \in A\}$$

Colists

Let A be a set.

Definition: Colists on A

A **colist** on A is a finite-or-infinite sequence of elements in A.

$$coList(A) := List(A) \sqcup \{[a_0, a_1, \ldots] \mid a_0, a_1, \ldots \in A\}$$

coList(A) comes with an operation

$$\operatorname{pop} : \operatorname{coList}(A) \longrightarrow \{\operatorname{no}\} \sqcup (A \times \operatorname{coList}(A))$$
$$[] \longmapsto \operatorname{no}$$
$$[a_0, a_1, \ldots] \longmapsto (a_0, [a_1, \ldots])$$

Recursion/Corecursion

Recursion principle

To define a function $u: \operatorname{List}(A) \to X$, it is enough to specify u([]) and to specify $u(\cos(a, \operatorname{as}))$ in terms of a and $u(\operatorname{as})$.

Recursion/Corecursion

Recursion principle

To define a function $u: \operatorname{List}(A) \to X$, it is enough to specify u([]) and to specify $u(\cos(a, \operatorname{as}))$ in terms of a and $u(\operatorname{as})$.

Corecursion principle

To define a function $u: X \to \operatorname{coList}(A)$, it is enough to specify a partial function $f: X \nrightarrow A \times X$.

Recursion/Corecursion

Recursion principle

To define a function $u: \operatorname{List}(A) \to X$, it is enough to specify u([]) and to specify $u(\cos(a, \operatorname{as}))$ in terms of a and $u(\operatorname{as})$.

Corecursion principle

To define a function $u: X \to \operatorname{coList}(A)$, it is enough to specify a partial function $f: X \nrightarrow A \times X$.

- pop(u(x)) = no whenever f(x) = no
- pop(u(x)) = (a, u(x')) whenever f(x) = (a, x')

Recursion/Corecursion

Recursion principle

To define a function $u: \operatorname{List}(A) \to X$, it is enough to specify u([]) and to specify $u(\cos(a, \operatorname{as}))$ in terms of a and $u(\operatorname{as})$.

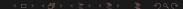
Corecursion principle

To define a function $u: X \to \operatorname{coList}(A)$, it is enough to specify a partial function $f: X \nrightarrow A \times X$.

- pop(u(x)) = no whenever f(x) = no
- pop(u(x)) = (a, u(x')) whenever f(x) = (a, x')

Think of $u(x) = [a_0, a_1, \ldots]$ as tracing the execution

$$x \xrightarrow{f} (a_0, x') \xrightarrow{f} (a_0, (a_1, x'')) \xrightarrow{f} \cdots$$



Induction principle

Let lhs, rhs : List(A) $\rightarrow X$. If

- lhs([]) = rhs([])
- lhs(cons(a, as)) = rhs(cons(a, as)) whenever lhs(as) = rhs(as)

then lhs = rhs.

Induction principle

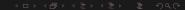
Let lhs, rhs : List(A) $\rightarrow X$. If

- lhs([]) = rhs([])
- lhs(cons(a, as)) = rhs(cons(a, as)) whenever lhs(as) = rhs(as)

then lhs = rhs.

Example theorems:

- Concatenation of lists is associative
- Length of concatenation is sum of lengths (induct from List)
- List is a functor
- List has the "list comprehension" applicative from Haskell



Coinduction principle

Let $lhs, rhs : X \to coList(A)$.

Let $[-] : \operatorname{coList}(a) \to \operatorname{coList}(A)_{/\operatorname{lhs=rhs}}$ be the quotient map.

lf

$$\forall x \in X, [\text{pop}(\text{lhs}(x))] = [\text{pop}(\text{rhs}(x))]$$

(with [no] := no), then lhs = rhs.

Coinduction principle

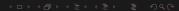
Let $\mathrm{lhs}, \mathrm{rhs}: X \to \mathrm{coList}(A)$. Let $[-]: \mathrm{coList}(a) \to \mathrm{coList}(A)_{/\mathrm{lhs}=\mathrm{rhs}}$ be the quotient map. If

$$\forall x \in X, [\text{pop}(\text{lhs}(x))] = [\text{pop}(\text{rhs}(x))]$$

(with [no] := no), then lhs = rhs.

Example theorems:

- Concatenation of colists is associative
- ullet Length of concatenation is sum of lengths (coinduct into $\mathrm{co}\mathbb{N}$)
- coList is a functor
- coList has the "ZipList" applicative from Haskell



Every (non-dependent) inductive data you can make in Haskell/Lean/Agda/etc. has a coinductive version.

• $\mathbb{N} \leadsto \mathrm{co}\mathbb{N}$

- $\mathbb{N} \leadsto \mathrm{co}\mathbb{N}$
- $List \leadsto coList$

- $\mathbb{N} \leadsto \mathrm{co}\mathbb{N}$
- List \rightsquigarrow coList
- Finite-size binary trees → potentially infinite binary trees

- $\mathbb{N} \leadsto \operatorname{co}\mathbb{N}$
- List → coList
- Finite-size binary trees → potentially infinite binary trees
- Etc.

Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- **3** Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

All of the inductive data types are **initial algebras**.

Initial algebras

Let $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor.

All of the inductive data types are **initial algebras**.

Initial algebras

Let $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor.

An **algebra** for T is an arrow $a \xrightarrow{\alpha} T(a)$ in \mathcal{C} .

All of the inductive data types are **initial algebras**.

Initial algebras

Let $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor.

An **algebra** for T is an arrow $a \xrightarrow{\alpha} T(a)$ in \mathfrak{C} .

A **morphism** from $a \xrightarrow{\alpha} T(a)$ to $b \xrightarrow{\beta} T(b)$ is an arrow $f: a \to b$ in $\mathfrak C$ such that this commutes:

$$\begin{array}{ccc} a & \stackrel{\alpha}{\longrightarrow} T(a) \\ f \downarrow & & \downarrow T(f) \\ b & \stackrel{\beta}{\longrightarrow} T(b) \end{array}$$

All of the inductive data types are **initial algebras**.

Initial algebras

Let $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor.

An **algebra** for T is an arrow $a \xrightarrow{\alpha} T(a)$ in \mathfrak{C} .

A morphism from $a \xrightarrow{\alpha} T(a)$ to $b \xrightarrow{\beta} T(b)$ is an arrow $f: a \to b$ in ${\mathfrak C}$ such that this commutes:

$$\begin{array}{ccc} a & \stackrel{\alpha}{\longrightarrow} T(a) \\ f \downarrow & & \downarrow T(f) \\ b & \stackrel{\beta}{\longrightarrow} T(b) \end{array}$$

With identity and composite arrows as in \mathcal{C} , these form a category.

All of the inductive data types are **initial algebras**.

Initial algebras

Let $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor.

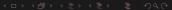
An **algebra** for T is an arrow $a \xrightarrow{\alpha} T(a)$ in \mathfrak{C} .

A **morphism** from $a \xrightarrow{\alpha} T(a)$ to $b \xrightarrow{\beta} T(b)$ is an arrow $f: a \to b$ in ${\mathfrak C}$ such that this commutes:

$$\begin{array}{ccc} a & \stackrel{\alpha}{\longrightarrow} T(a) \\ f \downarrow & & \downarrow^{T(f)} \\ b & \stackrel{\beta}{\longrightarrow} T(b) \end{array}$$

With identity and composite arrows as in $\ensuremath{\mathfrak{C}}$, these form a category.

An **initial algebra** for T is an initial object in this category.



All of the inductive data types are **initial algebras**.

All of the inductive data types are initial algebras.

Recursion

An **initial algebra** for T is an initial object $i \xrightarrow{\operatorname{cons}} T(i)$ in the category of T-algebras.

All of the inductive data types are initial algebras.

Recursion

An **initial algebra** for T is an initial object $i \xrightarrow{\operatorname{cons}} T(i)$ in the category of T-algebras.

The recursion principle is its universal property.

Induction

Let $i \xrightarrow{\text{cons}} T(i)$ be an initial algebra.

Let $f, g: i \to a$ in \mathcal{C} .

Suppose the equaliser $\iota: e \rightarrowtail i$ of f and g exists in \mathcal{C} , and that

 $f \circ \cos \circ T(\iota) = g \circ \cos \circ T(\iota)$:

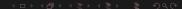
$$T(e) \xrightarrow{T(\iota)} e$$

$$T(i) \xrightarrow{\text{cons}} i$$

$$f \downarrow g$$

$$a$$

Then, f = g.



Corecursion/Coinduction

Turn all the arrows around.

Corecursion/Coinduction

Turn all the arrows around.

Coinduction

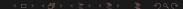
Let $T(j) \xrightarrow{\text{des}} j$ be an initial algebra.

Let $f,g:a\to j$ in \mathcal{C} .

Suppose the coequaliser $\pi:j \twoheadrightarrow e$ of f and g exists in \mathcal{C} , and that this commutes:

$$\begin{array}{c}
a \\
f \downarrow g \\
j \xrightarrow{\text{des}} T(j) \\
\pi \downarrow & \downarrow T(\pi) \\
e \xrightarrow{} T(e)
\end{array}$$

Then, f = g.



 $T:\mathbf{Set}\to\mathbf{Set}$ given by $X\mapsto \{\mathrm{no}\}\sqcup X$ has:

- Initial algebra $\{no\} \sqcup \mathbb{N} \xrightarrow{(0,succ)} \mathbb{N}$.
- Terminal coalgebra $\mathbb{N} \xrightarrow{\operatorname{pred}} \{\operatorname{no}\} \sqcup \mathbb{N}$.

 $T:\mathbf{Set}\to\mathbf{Set}$ given by $X\mapsto \{\mathrm{no}\}\sqcup X$ has:

- Initial algebra $\{no\} \sqcup \mathbb{N} \xrightarrow{(0,succ)} \mathbb{N}$.
- Terminal coalgebra $\mathbb{N} \xrightarrow{\operatorname{pred}} \{\operatorname{no}\} \sqcup \mathbb{N}$.

Instead regard $T : \mathbf{Top} \to \mathbf{Top}$.

 $T:\mathbf{Set}\to\mathbf{Set}$ given by $X\mapsto \{\mathrm{no}\}\sqcup X$ has:

- Initial algebra $\{no\} \sqcup \mathbb{N} \xrightarrow{(0, \text{succ})} \mathbb{N}$.
- Terminal coalgebra $\mathbb{N} \xrightarrow{\operatorname{pred}} \{\operatorname{no}\} \sqcup \mathbb{N}$.

Instead regard $T : \mathbf{Top} \to \mathbf{Top}$.

ullet The initial algebra exists, and is $\mathbb N$ with discrete topology.

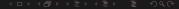
 $T: \mathbf{Set} \to \mathbf{Set}$ given by $X \mapsto \{\mathrm{no}\} \sqcup X$ has:

- Initial algebra $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{(0,\mathrm{succ})} \mathbb{N}$.
- Terminal coalgebra $\mathbb{N} \xrightarrow{\operatorname{pred}} \{\operatorname{no}\} \sqcup \mathbb{N}$.

Instead regard $T : \mathbf{Top} \to \mathbf{Top}$.

- The initial algebra exists, and is N with discrete topology.
- ullet The terminal coalgebra exists, and is $\mathrm{co}\mathbb{N}$ with

$$(0,1,2,\dots) \xrightarrow{\mathsf{converges}} \infty$$



Outline

- 1 Vanilla Induction
- 2 Vanilla Corecursion
- 3 Vanilla Coinduction
- 4 Structural Induction and Co-structural(?) Coinduction
- 5 Abstract Nonsense
- 6 Outroduction

Thx 4 watch

```
If you want more, I've written an article at https:
//emptylimit.github.io/discussion/coinduction.html!
```

- [1] nLab authors. bisimulation. URL: https://ncatlab.org/nlab/show/bisimulation.
- [2] nLab authors. coinduction. URL: https://ncatlab.org/nlab/show/coinduction.
- [3] nLab authors. corecursion. URL: https://ncatlab.org/nlab/show/corecursion.
- [4] nLab authors. inductive type. URL: https://ncatlab.org/nlab/show/inductive+type.
- [5] nLab authors. terminal coalgebra for an endofunctor. URL: https://ncatlab.org/nlab/show/terminal+coalgebra+for+an+endofunctor.

