# Corecursion and Coinduction

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15/Aug/2025

# Goals

Actual goals:

- Introduce the **conatural numbers**
- Introduce **corecursion** to specifying functions into $\mathrm{co}\mathbb{N}$
- Introduce **coinduction** to reason about corecursive functions
- **Generalise** to other corecursive structures
- Give **useful examples** of coinductive proofs

Notable omissions:

- Exhaustive examples (no time)
- Solid background theory (too much category theory)
- Initial algebras and terminal coalgebras in categories other than $\mathbf{Set}$ (prerequisites not met)

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The set of **natural numbers** is $\mathbb{N} := \{0, 1, \ldots\}$, "freely generated" by the constant $0$ and the successor operation $\mathrm{succ} : x \mapsto x + 1$.

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## Recursion Principle

Given a set $X$, a constant $x_0 \in X$, and an operation $f : X \to X$, there is a unique map $u : \mathbb{N} \to X$ with

$$u : 0 \mapsto x_0$$
$$u : \mathrm{succ}(n) \mapsto f(u(n))$$

We'll write $u := \mathrm{iterate}(f, x_0)$.

# Recursion from $\mathbb{N}$

**Example: Powers**

The function $u : \mathbb{N} \to \mathbb{R}$, $u : n \mapsto 3^n$ is defined recursively by

$$u : 0 \mapsto 1 \qquad u : \mathrm{succ}(n) \mapsto 3^n \cdot n$$

$$3^0 := 1 \qquad 3^{\mathrm{succ}(n)} := 3^n \cdot n$$

**Example: Addition on $\mathbb{N}$**

The function $(- + -) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined recursively by

$$0 + y := y \qquad \mathrm{succ}(x) + y := \mathrm{succ}(x + y)$$

# Induction from $\mathbb{N}$

**Theorem: Simple induction**

Let $\mathrm{lhs}, \mathrm{rhs} : \mathbb{N} \to X$ be two functions out of $\mathbb{N}$. If

- $\mathrm{lhs}(0) = \mathrm{rhs}(0)$, and
- $(\forall n \in \mathbb{N}, \mathrm{lhs}(n) = \mathrm{rhs}(n) \implies \mathrm{lhs}(\mathrm{succ}(n)) = \mathrm{rhs}(\mathrm{succ}(n)))$

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so $\mathrm{lhs} = \mathrm{rhs}$.

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1. Vanilla Induction
2. Vanilla Corecursion

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**Definition: Conatural numbers, Predecessor**

The set of **conatural numbers** is $\mathrm{co}\mathbb{N} := \mathbb{N} \sqcup \{\infty\}$.
The **predecessor** operation is

$$\begin{aligned} \mathrm{pred} : \mathrm{co}\mathbb{N} &\longrightarrow \{\mathrm{no}\} \sqcup \mathrm{co}\mathbb{N} \\ 0 &\longmapsto \mathrm{no} \\ \mathrm{succ}(n) &\longmapsto n \\ \infty &\longmapsto \infty \end{aligned}$$

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$\mathbb{N}$ comes with $\{\mathrm{no}\} \sqcup \mathbb{N} \xrightarrow{(0,\mathrm{succ})} \mathbb{N}$.

$\mathrm{co}\mathbb{N}$ comes with $\mathrm{co}\mathbb{N} \xrightarrow{\mathrm{pred}} \{\mathrm{no}\} \sqcup \mathbb{N}$.

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- "Get input from user" : $\texttt{ComputerState} \nrightarrow \texttt{String}$;
  Fails when the user doesn't give input.
- $\mathrm{pred} : \mathrm{co}\mathbb{N} \nrightarrow \mathrm{co}\mathbb{N}$;
  $\mathrm{pred}(0)$ fails.

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- etc.

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## Lemma: Pred-wait

Let $f : X \nrightarrow X$ be a partial function and $x \in X$.

- If $f(x) = \mathrm{no}$, then $\mathrm{pred}(\mathrm{wait}(f)(x)) = \mathrm{no}$.
- If $f(x) = x' \neq \mathrm{no}$, then $\mathrm{pred}(\mathrm{wait}(f)(x)) = \mathrm{wait}(f)(x')$.

Abstract nonsense: $\mathrm{pred}$ is the terminal partial endo-function.

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$$\mathrm{tail}([x_0, x_1, \ldots]) = [x_1, \ldots] \qquad \mathrm{tail}([]) = \mathrm{no}$$

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- Then, $\mathrm{wait}(\mathrm{tail})(x)$ is the length of $x$, possibly $= \infty$.
- The lemma says

$$\mathrm{pred}(\mathrm{wait}(\mathrm{tail})([])) = \mathrm{no}$$
$$\mathrm{pred}(\mathrm{wait}(\mathrm{tail})([x_0, x_1, \ldots])) = \mathrm{wait}(\mathrm{tail})([x_1, \ldots])$$