

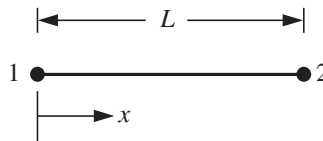
# Appendix B

## Shape Functions and Element Node Numbering

### 1D Elements

#### 2-node rod

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L}$$



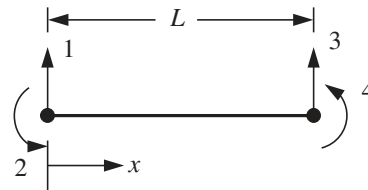
#### 2-node beam

$$N_1 = \frac{1}{L^3}(L^3 - 3Lx^2 + 2x^3)$$

$$N_2 = \frac{1}{L^2}(L^2x - 2Lx^2 + x^3)$$

$$N_3 = \frac{1}{L^3}(3Lx^2 - 2x^3)$$

$$N_4 = \frac{1}{L^2}(x^3 - Lx^2)$$



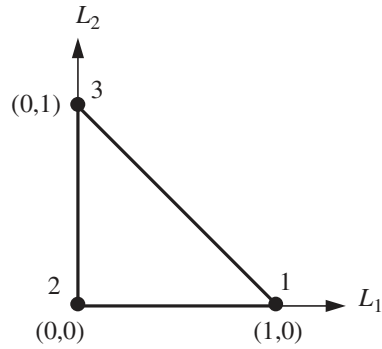
## 2D Elements

### 3-node triangle

$$N_1 = L_1$$

$$N_2 = (1 - L_1 - L_2)$$

$$N_3 = L_2$$



### 6-node triangle

$$N_1 = (2L_1 - 1)L_1$$

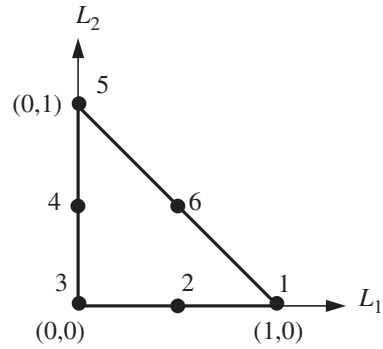
$$N_2 = 4(1 - L_1 - L_2)L_1$$

$$N_3 = (2(1 - L_1 - L_2) - 1)(1 - L_1 - L_2)$$

$$N_4 = 4L_2(1 - L_1 - L_2)$$

$$N_5 = (2L_2 - 1)L_2$$

$$N_6 = 4L_1L_2$$



### 10-node triangle

$$N_1 = \frac{1}{2}(3L_1 - 1)(3L_1 - 2)L_1$$

$$N_2 = -\frac{9}{2}(L_2 + L_1 - 1)(3L_1 - 1)L_1$$

$$N_3 = \frac{9}{2}(3L_2 + 3L_1 - 2)(L_2 + L_1 - 1)L_1$$

$$N_4 = -\frac{1}{2}(3L_2 + 3L_1 - 1)(3L_2 + 3L_1 - 2)(L_2 + L_1 - 1)$$

$$N_5 = \frac{9}{2}(3L_2 + 3L_1 - 2)(L_2 + L_1 - 1)L_2$$

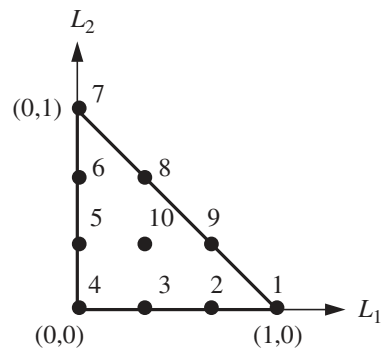
$$N_6 = -\frac{9}{2}(3L_2 - 1)(L_2 + L_1 - 1)L_2$$

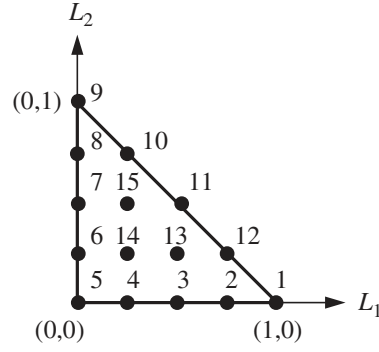
$$N_7 = \frac{1}{2}(3L_2 - 1)(3L_2 - 2)L_2$$

$$N_8 = \frac{9}{2}(3L_2 - 1)L_2L_1$$

$$N_9 = \frac{9}{2}(3L_1 - 1)L_2L_1$$

$$N_{10} = -27(L_2 + L_1 - 1)L_2L_1$$



**15-node triangle**

$$N_1 = \frac{32}{3} L_1 (L_1 - \frac{1}{4}) (L_1 - \frac{1}{2}) (L_1 - \frac{3}{4})$$

$$N_2 = \frac{128}{3} L_1 (1 - L_1 - L_2) (L_1 - \frac{1}{4}) (L_1 - \frac{1}{2})$$

$$N_3 = 64 L_1 (1 - L_1 - L_2) (L_1 - \frac{1}{4}) (1 - L_2 - L_1 - \frac{1}{4})$$

$$N_4 = \frac{128}{3} L_1 (1 - L_1 - L_2) (1 - L_2 - L_1 - \frac{1}{4}) (1 - L_2 - L_1 - \frac{1}{2})$$

$$N_5 = \frac{32}{3} (1 - L_1 - L_2) (1 - L_2 - L_1 - \frac{1}{4}) (1 - L_2 - L_1 - \frac{1}{2}) (1 - L_2 - L_1 - \frac{3}{4})$$

$$N_6 = \frac{128}{3} (1 - L_1 - L_2) L_2 (1 - L_2 - L_1 - \frac{1}{4}) (1 - L_2 - L_1 - \frac{1}{2})$$

$$N_7 = 64 (1 - L_1 - L_2) L_2 (L_2 - \frac{1}{4}) (1 - L_1 - L_2 - \frac{1}{4})$$

$$N_8 = \frac{128}{3} (1 - L_1 - L_2) L_2 (L_2 - \frac{1}{4}) (L_2 - \frac{1}{2})$$

$$N_9 = \frac{32}{3} L_2 (L_2 - \frac{1}{4}) (L_2 - \frac{1}{2}) (L_2 - \frac{3}{4})$$

$$N_{10} = \frac{128}{3} L_2 L_1 (L_2 - \frac{1}{4}) (L_2 - \frac{1}{2})$$

$$N_{11} = 64 L_2 L_1 (L_2 - \frac{1}{4}) (L_1 - \frac{1}{4})$$

$$N_{12} = \frac{128}{3} L_2 L_1 (L_1 - \frac{1}{4}) (L_1 - \frac{1}{2})$$

$$N_{13} = 128 L_2 L_1 (1 - L_1 - L_2) (L_1 - \frac{1}{4})$$

$$N_{14} = 128 L_2 L_1 (1 - L_1 - L_2) (1 - L_2 - L_1 - \frac{1}{4})$$

$$N_{15} = 128 L_2 L_1 (L_2 - \frac{1}{4}) (1 - L_1 - L_2)$$

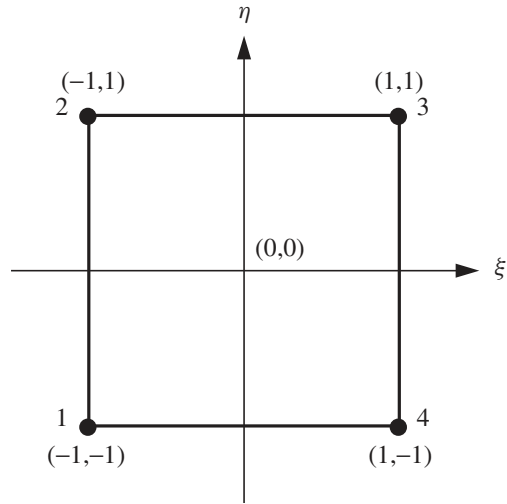
**4-node quadrilateral**

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$N_2 = \frac{1}{4}(1 - \xi)(1 + \eta)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 + \xi)(1 - \eta)$$

**8-node quadrilateral**

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1)$$

$$N_2 = \frac{1}{2}(1 - \xi)(1 - \eta^2)$$

$$N_3 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1)$$

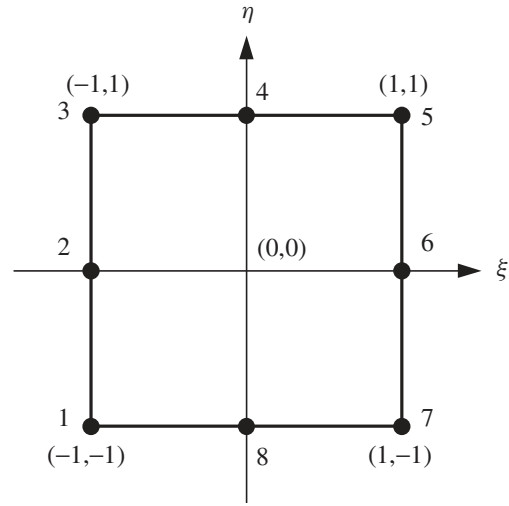
$$N_4 = \frac{1}{2}(1 - \xi^2)(1 + \eta)$$

$$N_5 = \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta - 1)$$

$$N_6 = \frac{1}{2}(1 + \xi)(1 - \eta^2)$$

$$N_7 = \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1)$$

$$N_8 = \frac{1}{2}(1 - \xi^2)(1 - \eta)$$

**9-node quadrilateral**

$$N_1 = \frac{1}{4}\xi(\xi - 1)\eta(\eta - 1)$$

$$N_2 = -\frac{1}{2}\xi(\xi - 1)(\eta + 1)(\eta - 1)$$

$$N_3 = \frac{1}{4}\xi(\xi - 1)\eta(\eta + 1)$$

$$N_4 = -\frac{1}{2}(\xi + 1)(\xi - 1)\eta(\eta + 1)$$

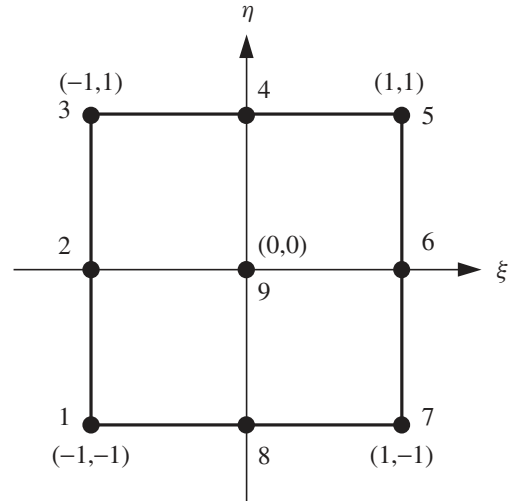
$$N_5 = \frac{1}{4}\xi(\xi + 1)\eta(\eta + 1)$$

$$N_6 = -\frac{1}{2}\xi(\xi + 1)(\eta + 1)(\eta - 1)$$

$$N_7 = \frac{1}{4}\xi(\xi + 1)\eta(\eta - 1)$$

$$N_8 = -\frac{1}{2}(\xi + 1)(\xi - 1)\eta(\eta - 1)$$

$$N_9 = (\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)$$



### 3D Elements

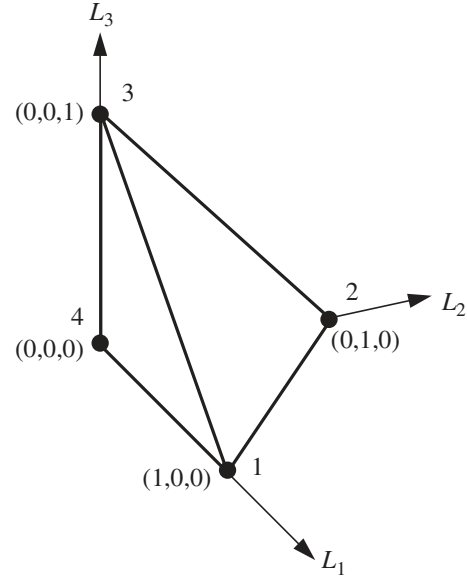
#### 4-node tetrahedron

$$N_1 = L_1$$

$$N_2 = L_2$$

$$N_3 = L_3$$

$$N_4 = (1 - L_1 - L_2 - L_3)$$



#### 8-node hexahedron

$$N_1 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta)$$

$$N_2 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \zeta)$$

$$N_3 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \zeta)$$

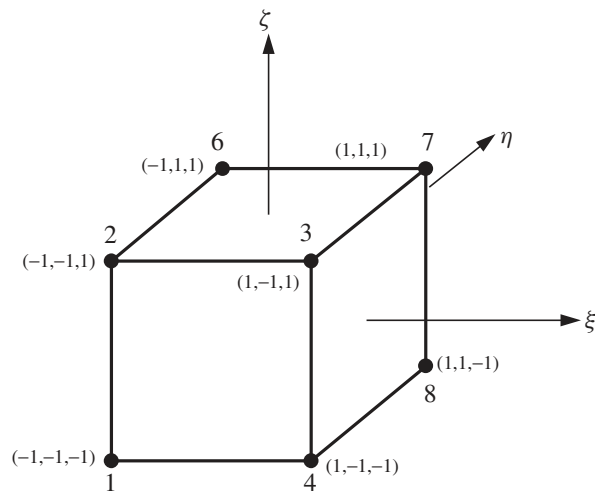
$$N_4 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \zeta)$$

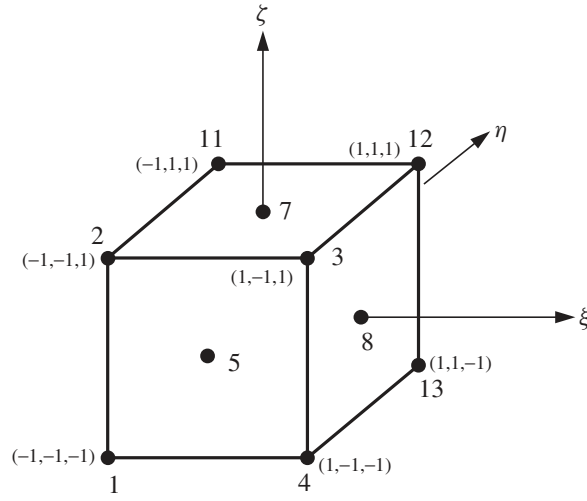
$$N_5 = \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \zeta)$$

$$N_6 = \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \zeta)$$

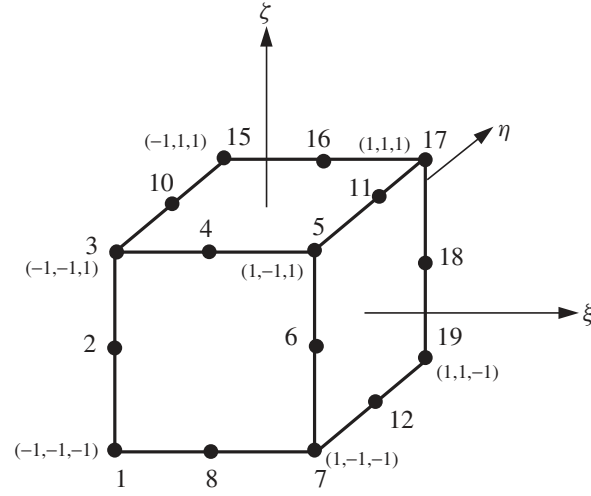
$$N_7 = \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \zeta)$$

$$N_8 = \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \zeta)$$



**14-node hexahedron (Type 6)**

$$\begin{aligned}
 N_1 &= \frac{1}{8}(\xi\eta + \xi\zeta + 2\xi + \eta\zeta + 2\eta + 2\zeta + 2)(\xi - 1)(\eta - 1)(\zeta - 1) \\
 N_2 &= -\frac{1}{8}(\xi\eta - \xi\zeta + 2\xi - \eta\zeta + 2\eta - 2\zeta + 2)(\xi - 1)(\eta - 1)(\zeta + 1) \\
 N_3 &= -\frac{1}{8}(\xi\eta - \xi\zeta + 2\xi + \eta\zeta - 2\eta + 2\zeta - 2)(\xi + 1)(\eta - 1)(\zeta + 1) \\
 N_4 &= \frac{1}{8}(\xi\eta + \xi\zeta + 2\xi - \eta\zeta - 2\eta - 2\zeta - 2)(\xi + 1)(\eta - 1)(\zeta - 1) \\
 N_5 &= -\frac{1}{2}(\xi + 1)(\xi - 1)(\eta - 1)(\zeta + 1)(\zeta - 1) \\
 N_6 &= -\frac{1}{2}(\xi - 1)(\eta + 1)(\eta - 1)(\zeta + 1)(\zeta - 1) \\
 N_7 &= \frac{1}{2}(\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)(\zeta + 1) \\
 N_8 &= \frac{1}{2}(\xi + 1)(\eta + 1)(\eta - 1)(\zeta + 1)(\zeta - 1) \\
 N_9 &= -\frac{1}{2}(\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)(\zeta - 1) \\
 N_{10} &= \frac{1}{8}(\xi\eta - \xi\zeta - 2\xi + \eta\zeta + 2\eta - 2\zeta - 2)(\xi - 1)(\eta + 1)(\zeta - 1) \\
 N_{11} &= -\frac{1}{8}(\xi\eta + \xi\zeta - 2\xi - \eta\zeta + 2\eta + 2\zeta - 2)(\xi - 1)(\eta + 1)(\zeta + 1) \\
 N_{12} &= -\frac{1}{8}(\xi\eta + \xi\zeta - 2\xi + \eta\zeta - 2\eta - 2\zeta + 2)(\xi + 1)(\eta + 1)(\zeta + 1) \\
 N_{13} &= \frac{1}{8}(\xi\eta - \xi\zeta - 2\xi - \eta\zeta - 2\eta + 2\zeta + 2)(\xi + 1)(\eta + 1)(\zeta - 1) \\
 N_{14} &= \frac{1}{2}(\xi + 1)(\xi - 1)(\eta + 1)(\zeta + 1)(\zeta - 1)
 \end{aligned}$$

**20-node hexahedron**

$$N_1 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta)(-\xi - \eta - \zeta - 2)$$

$$N_2 = \frac{1}{4}(1 - \xi)(1 - \eta)(1 - \zeta^2)$$

$$N_3 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \zeta)(-\xi - \eta + \zeta - 2)$$

$$N_4 = \frac{1}{4}(1 - \xi^2)(1 - \eta)(1 + \zeta)$$

$$N_5 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \zeta)(\xi - \eta + \zeta - 2)$$

$$N_6 = \frac{1}{4}(1 + \xi)(1 - \eta)(1 - \zeta^2)$$

$$N_7 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \zeta)(\xi - \eta - \zeta - 2)$$

$$N_8 = \frac{1}{4}(1 - \xi^2)(1 - \eta)(1 - \zeta)$$

$$N_9 = \frac{1}{4}(1 - \xi)(1 - \eta^2)(1 - \zeta)$$

$$N_{10} = \frac{1}{4}(1 - \xi)(1 - \eta^2)(1 + \zeta)$$

$$N_{11} = \frac{1}{4}(1 + \xi)(1 - \eta^2)(1 + \zeta)$$

$$N_{12} = \frac{1}{4}(1 + \xi)(1 - \eta^2)(1 - \zeta)$$

$$N_{13} = \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \zeta)(-\xi + \eta - \zeta - 2)$$

$$N_{14} = \frac{1}{4}(1 - \xi)(1 + \eta)(1 - \zeta^2)$$

$$N_{15} = \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \zeta)(-\xi + \eta + \zeta - 2)$$

$$N_{16} = \frac{1}{4}(1 - \xi^2)(1 + \eta)(1 + \zeta)$$

$$N_{17} = \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \zeta)(\xi + \eta + \zeta - 2)$$

$$N_{18} = \frac{1}{4}(1 + \xi)(1 + \eta)(1 - \zeta^2)$$

$$N_{19} = \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \zeta)(\xi + \eta - \zeta - 2)$$

$$N_{20} = \frac{1}{4}(1 - \xi^2)(1 + \eta)(1 - \zeta)$$