

Project 3 Mathematics 512

Instructor: Ricardo Mancera Fall 2024

Due date: Wednesday October 9th

1.

Let W_t be a standard Wiener process, that is the drift parameter is zero and the Variance parameter $\sigma^2 = 1$. Suppose that we divide the interval $[0,2]$ into L subintervals $[t_i, t_{i+1}]$, with $t_i = i\delta t$ and $\delta t = 2/L$. Let $W_i = W(t_i)$ and $\delta W_i = W_{i+1} - W_i$. Verify numerically that

- a) $\sum_{i=0}^{L-1} |\delta W_i|$ is unbounded as δt goes to zero
- b) $\sum_{i=0}^{L-1} \delta W_i^2$ converges to 2 in probability as δt goes to zero

2.

Evaluate numerically the stochastic integrals

- a) Itô $\int_0^2 W(t) dW(t)$
- b) Stratonovich $\int_0^2 W(t) \circ dW(t)$
- c) $E \left[\int_0^2 W(t) dW(t) \right]$
- d) $E \left\{ \left[\int_0^2 W(t) dW(t) \right]^2 \right\}$
- e) $E \left[\int_0^2 W^2(t) dt \right]$
- f) For $t \in [0,2]$ evaluate $\int_0^t W(t) dW(t)$, $\int_0^t W(t) \circ dW(t)$ and $\frac{1}{2} \int_0^t dt$ what do you observe.

3.

Consider the following SDE:

$$dX(t) = \mu X(t) dt + \sigma X(t) dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

Where $t \in [0,1]$.

- a) Show that the Euler Maruyama method has weak order of convergence equal to one. That is $|E[X_1] - E[X(1)]| = C\Delta t$. Here $X(1)$ is the exact solution at time 1 and X_1 is the computed solution at time 1.
- b) Show that the Euler Maruyama method has strong order of convergence equal to one half. That is $E|X_1 - X(1)| = C\Delta t^{0.5}$. Here $X(1)$ is the exact solution at time 1 and X_1 is the computed solution at time 1.

4.

Consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 3, \quad \mu = 2, \quad \sigma = 0.10$$

a) Simulate (over the interval $[0,20]$) this stochastic process using an implicit method of the form

$$X_{n+1} = X_n + (1 - \theta)\Delta t f(X_n) + \theta\Delta t f(X_{n+1}) + \sqrt{\Delta t}\alpha_n g(X_n)$$

b) Compare with the analytical solution.

c) For what values of μ and σ is the SDE mean-square stable.

d) For what values of θ is the implicit method mean-square stable.

e) For what values of μ and σ is the SDE asymptotically stable.

f) For what values of θ is the Implicit method asymptotically stable.

Verify your results using numerical simulations.

5.

Consider the following SDE:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \quad X(0) = 2, \quad \mu = 0.2, \quad \sigma = 0.15$$

Let $a = 0.5$ and $b = 3$.

Compute the mean exit time function $v(x)$ for $x \in [0.5, 3]$