The Riemann-Roch theorem

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(Following some unpublished results by A. Grothendieck)

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Translator's note.

What follows is a translation, by Timothy Hosgood[1], of the French paper:

Borel, Armand; Serre, Jean-Pierre. Le théorème de Riemann-Roch. *Bulletin de la Société Mathématique de France*, Volume 86 (1958), pp. 97-136. DOI: 10.24033/bsmf.1500.

Contents

Introduction

What follows constitutes the notes from a seminar that took place in Princeton in the autumn of 1957 on the works of Grothendieck; the new results that are included are due to Grothendieck; our contribution is solely of an editorial nature.

The "Riemann-Roch theorem" of which we speak here holds true for (non-singular) algebraic varieties over a field of arbitrary characteristic; in the classical case, where the base field is $\mathbb{C}$, this theorem encapsulates, as a particular example, the result proven a few years ago by Hirzebruch [10].

The full statement and proof of the Riemann-Roch theorem can be found in sections 7 to 16, with the last section being devoted to an application. Sections 1 to 6 contain some preliminaries on coherent algebraic sheaves [12]. The terminology that we follow is the same as in [12], up to one difference: to conform with a custom which is becoming more and more widespread, we use the word "morphism" instead of "regular maps".

1 Supplementary results about sheaves

(All the varieties considered below are algebraic varieties over an algebraically closed field k of arbitrary characteristic. Unless otherwise mentioned, all the sheaves considered are coherent algebraic sheaves.)

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Proposition 1. *Let U be an open subset of a variety V, and let $\mathscr{F}$ be a coherent sheaf on V and $\mathscr{G}$ a coherent subsheaf of $\mathscr{F}|U$ (the restriction of $\mathscr{F}$ to U). Then there exists a coherent sheaf $\mathscr{G}' \subset \mathscr{F}$ such that $\mathscr{G}'|U = \mathscr{G}$.*

(In fact, the proof will show that there exists a *largest* such sheaf having this property.)

Proof. For every open subset $W \subset V$, we define $\mathscr{G}'_W$ as the set of sections of $\mathscr{F}$ over W that belong to $\mathscr{G}$ over $U \cap W$. Everything reduces to showing that the sheaf $\mathscr{G}'$ associated to this presheaf is coherent. Since this is a local questions, we can suppose that V is an affine variety. Let A be its coordinate ring. There exist elements $f_i \in A$ such that $U = \bigcup V_{f_i}$, where $V_{f_i} = U_i$ denotes the set of points of V where $f_i \neq 0$. If, in the definition of $\mathscr{G}'$, we replace the open subset U by the open subset U_i, then we obtain a sheaf $\mathscr{G}'_i \subset \mathscr{F}$, and it is clear that $\mathscr{G}' = \bigcap \mathscr{G}'_i$. By known results on coherent sheaves [12, p. 209], it suffices to show that the $\mathscr{G}'_i$ are coherent. We can thus restrict to considering the case where V is affine, and where $U = V_f$, for some $f \in A$. In this case, the sheaf $\mathscr{F}$ is defined by an A-module M, and the subsheaf $\mathscr{G}$ of $\mathscr{F}|U$ is defined by a submodule N of $M_f = M \otimes_A A_f$. Let N' be the inverse image of N in M under the canonical map $M \to M_f$. The module N' then corresponds to a coherent subsheaf of $\mathscr{F}$, and we can immediately verify (by taking the $V_{f'}$ to be the W, for example) that this sheaf is exactly $\mathscr{G}'$, which finishes the proof. $\square$

Lemma 1. *Let U be an open subset of an affine variety V, and let $\mathscr{F}$ be a (coherent) sheaf on U. Then $\mathscr{F}$ is generated by its sections (over U).*

Proof. Let $x \in U$, and let f be a regular function on V, zero on $V \setminus U$ and non-zero at x. We have $V_f \subset U \subset V$. Since V_f is affine, we know [12] that $\mathscr{F}_x$ is generated by its sections over V_f, and it thus suffices to prove that these sections can be extended to U, after multiplying by a suitable power of f. This follows from the more general following lemma: $\square$

Lemma 2. *Let X be a variety, f a regular function on X, $\mathscr{F}$ a sheaf on X, and s a section of $\mathscr{F}$ over $U = X_f$. Then there exists an integer $n > 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ over X.*

Proof. We can cover X by finitely many affine opens X_i. By applying [12, lemma 1, p. 247] (or by arguing directly, as in **??** 1), we see that there exists an integer n and sections s_i of $\mathscr{F}$ over the X_i that extend $f^n s$ over $X_i \cap U$. Since the $s_i - s_j$ are zero on $X_i \cap X_j \cap U$, there exists an integer m such that $f^m(s_i - s_j) = 0$ on $X_i \cap X_j$ ([12, p. 235], or arguing directly), and m can be chosen independent of the pair (i, j). The $f^m s_i$ then define a section s' of $\mathscr{F}$ over X that indeed extend $f^{n+m} s$. $\square$

Proposition 2. *If U is an open subset of a variety V, then every sheaf $\mathscr{F}$ over U can be extended to V.*

Proof. We show that, if $U \neq V$, we can extend $\mathscr{F}$ to an open subset $U' \supset U$, with $U' \neq U$; from that fact that every (strictly) increasing chain of open subsets stabilises, this will imply the proposition. Let $x \in V \setminus U$, and let W be an affine open that contains x; let $U' = W \cup U$. We are thus led to extending the sheaf $\mathscr{F}|W \cap U$ to W, or, in other words, we can restrict to proving the proposition in the specific case where V is affine. In this case, **??** 1 shows that $\mathscr{F}$ is generated by its sections, i.e. it is of the form $\mathscr{L}/\mathscr{R}$, where $\mathscr{L}$ is the direct sum of the sheaves $\mathscr{O}_U$. The sheaf $\mathscr{L}$ can be extended in the obvious way to V, and, by **??** 1, there exists a subsheaf $\mathscr{R}'$ of $\mathscr{L}$ on V whose restriction to U is $\mathscr{R}$. The sheaf $\mathscr{L}/\mathscr{R}'$ is then the desired extension. $\square$

Remark. **??** 1 and **??** 2 correspond to the geometric fact that the closure of any algebraic subvariety of U is an algebraic subvariety of V. These propositions do not extend *as is* to the "analytic" case. The most we can hope for (by results of Rothstein) is that they still hold true if we make certain restrictions on the dimensions of $V \setminus U$ and the varieties appearing in the local primary decomposition of the sheaf $\mathscr{F}$.

2 Proper maps of quasi-projective varieties

A variety X is said to be *quasi-projective* is it is isomorphic to a locally closed subvariety of a projective space. It is said to be *projective* if it is isomorphic to a closed subvariety of a projective space. *From here on in, all the varieties considered are assumed to be quasi-projective.*

Lemma 3. *Let P be a projective space, U an arbitrary variety, and G a closed subset of $P \times U$. Then the projection of G in U is closed.*

This is a translation into geometric language of the well-known fact that a projective space is a "complete" variety, in the sense of Weil. We briefly recall the principal of the proof:

Proof. Since the question is local with respect to U, we can assume that U is affine, and even that U is as affine open of the space k^n. We can also assume that G is irreducible. So we choose projective coordinates x_i in P such that G meets the set $P_0 \times U$ of points where $x_0 \neq 0$. If A denotes the coordinate ring of U, then the coordinate ring of the affine variety $P_0 \times U$ is $A[x_i/x_0] = B_0$; the set G defines (and is defined by) a prime ideal $\mathfrak{p}$ of B_0. If $\mathfrak{p}'$ denotes $A \cap \mathfrak{p}$, then the prime ideal $\mathfrak{p}'$ corresponds to the closure of the projection G' of G in U. A point in this closure is thus a homomorphism $f: A \to k$ (where k denotes the base field) that is zero on $\mathfrak{p}'$; this point is the image of a point in G that lies in $P_0 \times U$ if and only if f can be extended to a homomorphism $g: B_0 \to k$ that is zero on $\mathfrak{p}$. So let L be the field of functions of G; the field L contains $A/\mathfrak{p}'$ as a subring. By the theorem of extension of specialisations, there exists a valuation v of L, with values in k, that extends f. Let Φ be the place associated to this valuation. If $v(x_i/x_0) \geqslant 0$ for all i, then the place Φ is finite over the x_i/x_0, and thus induces, on $B_0/\mathfrak{p} \subset L$, a homomorphism g that extends f. If $v(x_i/x_0) < 0$ for some i, then we replace x_0 by the x_i that gives the smallest possible value of $v(x_i/x_0)$, and we are then back in the previous case. $\square$

If $f: X \to Y$ is a morphism, then we write G_f to denote its graph. It is trivial that G_f is closed in $X \times Y$.

Lemma 4. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms, where X and Y are subvarieties of projective spaces P and P' (respectively). Suppose that G_f is closed in $P \times Y$, and that G_g is closed in $P' \times Z$. Then G_{gf} is closed in $P \times Z$.*

Proof. We have $G_f \subset P \times Y = P \times G_g \subset P \times P' \times Z$, and since each one is closed in the next, we see that G_f can be identified with a closed subset of $P \times P' \times Z$. Since G_{gf} is exactly the projection of G_f to the factor $P \times Z$, the lemma follows from **??** 3. $\square$

Lemma 5. *Let $f : X \to Y$ be a morphism, and let $X \subset P$ and $X \subset P'$ be embeddings of X into projective spaces. If G_f is closed in $P \times Y$, then it is also closed in $P' \times Y$.*

Proof. We apply **??** 4 to the morphisms $X \xrightarrow{i} X \xrightarrow{f} Y$, where i denotes the identity morphism. Everything then reduces to showing that the graph G_i of i in $P \times X$ is closed, which follows from the fact that it is given by the intersection of $P \times X$ with the diagonal of $P \times P$. □

?? 5 justifies the following definition:

Definition. A map $f : X \to Y$ is said to be *proper* if it is a morphism and if its graph G_f is closed in $P \times Y$, where P is a projective space containing X.

We can give a definition of proper maps that is analogous to the definition of complete varieties:

Proposition 3. *For a morphism $f : X \to Y$ to be proper, it is necessary and sufficient, for every variety Z, and every closed subset T of $X \times Z$, for the image of T in $Y \times Z$ to be closed.*

Proof. Let P be a projective space inside which X can be embedded; since G_f is closed in $P \times Y$, the product $G_f \times Z$ is closed in $P \times Y \times Z$, and so T can be embedded as a closed subset into $P \times Y \times Z$. Applying **??** 3, we see that the projection of T to $Y \times Z$ (which is exactly $(f \times 1)(T)$) is closed. Conversely, suppose that this property holds true, and apply it to $Z = P$, with the set T being the diagonal of $X \times X$, embedded into $X \times P$. The image of T in $Y \times Z = Y \times P$ is then exactly G_f, which is indeed closed. □

Proposition 4. *(i) The identity morphism $i : X \to X$ is proper.*

(ii) The composition of two proper maps is proper.

(iii) The direct product of two proper maps is proper.

(iv) The image of a closed subset by a proper map is a closed subset.

(v) An injection $Y \to X$ is proper if and only if Y is closed in X.

(vi) Every morphism from a projective variety is proper.

(vii) A projection $Y \times Z \to Y$ is proper if and only if Z is projective (assuming the variety Y to be non-empty).

Proof. We indicate, as an example, how to prove *(vii)* (since the other claims are even easier to prove). If Z is projective, then we apply the criteria of **??** 3; so let Z' be an arbitrary variety, and T a closed subset of $Y \times Z \times Z'$; we need to show that the projection of T in $Y \times Z'$ is closed, which follows from **??** 3. Conversely, if $Y \times Z \to Y$ is proper, then the composition $Z \to Y \times Z \to Y$ is proper. Since the image of this map is a point, we immediately deduce that Z is projective (by returning to the definition). □

Corollary 5. *For a morphism $f : X \to Y$ to be proper, it is necessary and sufficient for it to factor as $X \to P \times Y \to Y$, where $X \to P \times Y$ is an injection into a closed subvariety, and $P \times Y \to Y$ is the projection onto the second factor (where P denotes some projective space).*

Proof. By the definition of a proper map, this condition is necessary (if we take P to be a projective space into which we can embed X). It is sufficient by *(ii)*, *(v)*, and *(vii)*. $\square$

Proposition 5. *Suppose that the base field k is the field of complex numbers. For a morphism $f : X \to Y$ to be proper (in the above sense), it is necessary and sufficient for it to be proper (in the topological sense) when we endow X and Y with the "usual" topology.*

Proof. Suppose that f is proper in the algebraic sense, and let K be a compact subset of Y (for the usual topology). Suppose that X is embedded into some projective space P. Since P is compact, we know that $f^{-1}(K) = G_f \cap (P \times K)$ is compact, which shows that f is proper in the topological sense. Conversely, assume that this condition is satisfied, and aim to prove that the condition of **??** 3 is satisfied: the image of T in $Y \times Z$ is closed for the usual topology, and thus also for the Zariski topology [13, proposition 7, p. 12]. $\square$

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