

# The Riemann-Roch theorem

Following some unpublished results by A. Grothendieck

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**!TODO!**

**Translator's note.** What follows is a translation of the paper “Le théorème de Riemann-Roch”, Armand Borel and Jean-Pierre Serre. *Bulletin de la S.M.F.*, tome 86 (1958), p. 97–136.

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## Introduction

What follows constitutes the notes from a seminar that took place in Princeton in the autumn of 1957 on the works of Grothendieck; the new results that are included are due to Grothendieck; our contribution is solely of an editorial nature.

The “Riemann-Roch theorem” of which we speak here holds true for (non-singular) algebraic varieties over a field of arbitrary characteristic; in the classical case, where the base field is  $\mathbb{C}$ , this theorem encapsulates, as a particular example, the result proven a few years ago by Hirzebruch [10].

The full statement and proof of the Riemann-Roch theorem can be found in sections 7 to 16, with the last section being devoted to an application. Sections 1 to 6 contain some preliminaries on coherent algebraic sheaves [12]. The terminology that we follow is the same as in [12], up to one difference: to conform with a custom which is becoming more and more widespread, we use the word “morphism” instead of “regular maps”.

## 1 Auxiliary results on sheaves

(All the varieties considered below are algebraic varieties over an algebraically closed field  $k$  of arbitrary characteristic. Unless otherwise mentioned, all the sheaves considered are coherent algebraic sheaves.)

**Proposition 1.** *Let  $U$  be an open subset of a variety  $V$ , and let  $\mathcal{F}$  be a coherent sheaf on  $V$  and  $\mathcal{G}$  a coherent subsheaf of  $\mathcal{F}|_U$  (the restriction of  $\mathcal{F}$  to  $U$ ). Then there exists a coherent sheaf  $\mathcal{G}' \subset \mathcal{F}$  such that  $\mathcal{G}'|_U = \mathcal{G}$ .*

(In fact, the proof will show that there exists a *largest* such sheaf having this property.)

*Proof.* For every open subset  $W \subset V$ , we define  $\mathcal{G}'_W$  as the set of sections of  $\mathcal{F}$  over  $W$  that belong to  $\mathcal{G}$  over  $U \cap W$ . Everything reduces to showing that the sheaf  $\mathcal{G}'$  associated to this presheaf is coherent. Since this is a local question, we can suppose that  $V$  is an affine variety. Let  $A$  be its coordinate ring. There exist elements  $f_i \in A$  such that  $U = \bigcup V_{f_i}$ , where  $V_{f_i} = U_i$  denotes the set of points of  $V$  where  $f_i \neq 0$ . If, in the definition of  $\mathcal{G}'$ , we replace the open subset  $U$  by the open subset  $U_i$ , then we obtain a sheaf  $\mathcal{G}'_i \subset \mathcal{F}$ , and it is clear that  $\mathcal{G}' = \bigcap \mathcal{G}'_i$ . By known results on coherent sheaves [12, p. 209], it suffices to show that the  $\mathcal{G}'_i$  are coherent. We can thus restrict to considering the case where  $V$  is affine, and where  $U = V_f$ , for some  $f \in A$ . In this case, the sheaf  $\mathcal{F}$  is defined by an  $A$ -module  $M$ , and the subsheaf  $\mathcal{G}$  of  $\mathcal{F}|_U$  is defined by a submodule  $N$  of  $M_f = M \otimes_A A_f$ . Let  $N'$  be the inverse image of  $N$  in  $M$  under the canonical map  $M \rightarrow M_f$ . The module  $N'$  then corresponds to a coherent subsheaf of  $\mathcal{F}$ , and we can immediately verify (by taking the  $V_{f'}$  to be the  $W$ , for example) that this sheaf is exactly  $\mathcal{G}'$ , which finishes the proof.  $\square$

**Lemma 1.** *Let  $U$  be an open subset of an affine variety  $V$ , and let  $\mathcal{F}$  be a (coherent) sheaf on  $U$ . Then  $\mathcal{F}$  is generated by its sections (over  $U$ ).*

*Proof.* Let  $x \in U$ , and let  $f$  be a regular function on  $V$ , zero on  $V \setminus U$  and non-zero at  $x$ . We have  $V_f \subset U \subset V$ . Since  $V_f$  is affine, we know [12] that  $\mathcal{F}_x$  is generated by its sections over  $V_f$ , and it thus suffices to prove that these sections can be extended to  $U$ , after multiplying by a suitable power of  $f$ . This follows from the more general following lemma:  $\square$

**Lemma 2.** *Let  $X$  be a variety,  $f$  a regular function on  $X$ ,  $\mathcal{F}$  a sheaf on  $X$ , and  $s$  a section of  $\mathcal{F}$  over  $U = X_f$ . Then there exists an integer  $n > 0$  such that  $f^n s$  can be extended to a section of  $\mathcal{F}$  over  $X$ .*

*Proof.* We can cover  $X$  by finitely many affine opens  $X_i$ . By applying [12, lemma 1, p. 247] (or by arguing directly, as in proposition 1), we see that there exists an integer  $n$  and sections  $s_i$  of  $\mathcal{F}$  over the  $X_i$  that extend  $f^n s$  over  $X_i \cap U$ . Since the  $s_i - s_j$  are zero on  $X_i \cap X_j \cap U$ , there exists an integer  $m$  such that  $f^m(s_i - s_j) = 0$  on  $X_i \cap X_j$  ([12, p. 235], or arguing directly), and  $m$  can be chosen independent of the pair  $(i, j)$ . The  $f^m s_i$  then define a section  $s'$  of  $\mathcal{F}$  over  $X$  that indeed extend  $f^{n+m} s$ .  $\square$

**Proposition 2.** *If  $U$  is an open subset of a variety  $V$ , then every sheaf  $\mathcal{F}$  over  $U$  can be extended to  $V$ .*

*Proof.* We show that, if  $U \neq V$ , we can extend  $\mathcal{F}$  to an open subset  $U' \supset U$ , with  $U' \neq U$ ; from that fact that every (strictly) increasing chain of open subsets stabilises, this will imply the proposition. Let  $x \in V \setminus U$ , and let  $W$  be an affine open that contains  $x$ ; let  $U' = W \cup U$ .  $\square$

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