The Riemann-Roch theorem

Following some unpublished results by A. Grothendieck

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1958

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Translator's note. What follows is a translation of the paper "Le théorème de Riemann-Roch", Armand Borel and Jean-Pierre Serre. *Bulletin de la S.M.F.*, tome 86 (1958), p. 97–136.

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Introduction

What follows constitutes the notes from a seminar that took place in Princeton in the autumn of 1957 on the works of Grothendieck; the new results that are included are due to Grothendieck; our contribution is solely of an editorial nature.

The "Riemann-Roch theorem" of which we speak here holds true for (non-singular) algebraic varieties over a field of arbitrary characteristic; in the classical case, where the base field is $\mathbb{C}$, this theorem encapsulates, as a particular example, the result proven a few years ago by Hirzebruch [10].

The full statement and proof of the Riemann-Roch theorem can be found in sections 7 to 16, with the last section being devoted to an application. Sections 1 to 6 contain some preliminaries on coherent algebraic sheaves [12]. The terminology that we follow is the same as in [12], up to one difference: to conform with a custom which is becoming more and more widespread, we use the word "morphism" instead of "regular maps".

1 Auxiliary results on sheaves

(All the varieties considered below are algebraic varieties over an algebraically closed field k of arbitrary characteristic. Unless otherwise mentioned, all the sheaves considered are coherent algebraic sheaves.)

Proposition 1. *Let U be an open subset of a variety V, and let $\mathscr{F}$ be a coherent sheaf on V and $\mathscr{G}$ a coherent subsheaf of $\mathscr{F}|U$ (the restriction of $\mathscr{F}$ to U). Then there exists a coherent sheaf $\mathscr{G}' \subset \mathscr{F}$ such that $\mathscr{G}'|U = \mathscr{G}$.*

(In fact, the proof will show that there exists a *largest* such sheaf having this property.)

Proof. For every open subset $W \subset V$, we define $\mathscr{G}'_W$ as the set of sections of $\mathscr{F}$ over W that belong to $\mathscr{G}$ over $U \cap W$. Everything reduces to showing that the sheaf $\mathscr{G}'$ associated to this presheaf is coherent. Since this is a local questions, we can suppose that V is an affine variety. Let A be its coordinate ring. There exist elements $f_i \in A$ such that $U = \bigcup V_{f_i}$, where $V_{f_i} = U_i$ denotes the set of points of V where $f_i \neq 0$. If, in the definition of $\mathscr{G}'$, we replace the open subset U by the open subset U_i, then we obtain a sheaf $\mathscr{G}'_i \subset \mathscr{F}$, and it is clear that $\mathscr{G}' = \bigcap \mathscr{G}'_i$. By known results on coherent sheaves [12, p. 209], it suffices to show that the $\mathscr{G}'_i$ are coherent. We can thus restrict to considering the case where V is affine, and where $U = V_f$, for some $f \in A$. In this case, the sheaf $\mathscr{F}$ is defined by an A-module M, and the subsheaf $\mathscr{G}$ of $\mathscr{F}|U$ is defined by a submodule N of $M_f = M \otimes_A A_f$. Let N' be the inverse image of N in M under the canonical map $M \to M_f$. The module N' then corresponds to a coherent subsheaf of $\mathscr{F}$, and we can immediately verify (by taking the $V_{f'}$ to be the W, for example) that this sheaf is exactly $\mathscr{G}'$, which finishes the proof. □

Lemma 1. *Let U be an open subset of an affine variety V, and let $\mathscr{F}$ be a (coherent) sheaf on U. Then $\mathscr{F}$ is generated by its sections (over U).*

Proof. Let $x \in U$, and let f be a regular function on V, zero on $V \setminus U$ and non-zero at x. We have $V_f \subset U \subset V$. Since V_f is affine, we know [12] that $\mathscr{F}_x$ is generated by its sections over V_f, and it thus suffices to prove that these sections can be extended to U, after multiplying by a suitable power of f. This follows from the more general following lemma: □

Lemma 2. *Let X be a variety, f a regular function on X, $\mathscr{F}$ a sheaf on X, and s a section of $\mathscr{F}$ over $U = X_f$. Then there exists an integer $n > 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ over X.*

Proof. We can cover X by finitely many affine opens X_i. By applying [12, lemma 1, p. 247] (or by arguing directly, as in **??** 1), we see that there exists an integer n and sections s_i of $\mathscr{F}$ over the X_i that extend $f^n s$ over $X_i \cap U$. Since the $s_i - s_j$ are zero on $X_i \cap X_j \cap U$, there exists an integer m such that $f^m(s_i - s_j) = 0$ on $X_i \cap X_j$ ([12, p. 235], or arguing directly), and m can be chosen independent of the pair (i, j). The $f^m s_i$ then define a section s' of $\mathscr{F}$ over X that indeed extend $f^{n+m} s$. □

Proposition 2. *If U is an open subset of a variety V, then every sheaf $\mathscr{F}$ over U can be extended to V.*

Proof. We show that, if $U \neq V$, we can extend $\mathscr{F}$ to an open subset $U' \supset U$, with $U' \neq U$; from that fact that every (strictly) increasing chain of open subsets stabilises, this will imply the proposition. Let $x \in V \setminus U$, and let W be an affine open that contains x; let $U' = W \cup U$. We are thus led to extending the sheaf $\mathscr{F}|W \cap U$ to W, or, in other words, we can restrict to proving the proposition in the specific case where V is affine. In this case, **??** 1 shows that $\mathscr{F}$ is generated by its sections, i.e. it is of the form $\mathscr{L}/\mathscr{R}$, where $\mathscr{L}$ is the direct sum of the sheaves $\mathscr{O}_U$. The sheaf $\mathscr{L}$ can be extended in the obvious way to V, and, by **??** 1, there exists a subsheaf $\mathscr{R}'$ of $\mathscr{L}$ on V whose restriction to U is $\mathscr{R}$. The sheaf $\mathscr{L}/\mathscr{R}'$ is then the desired extension. $\square$

Remark.

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